Population Lorenz-Monotonic Allocation Schemes for TU-games

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Abstract

Sprumont (1990) introduces Population Monotonic Allocation Schemes (PMAS) and proves that every assignment game with at least two sellers and two buyers, where each buyer-seller pair derives a positive gain from trade, lacks a PMAS. In particular glove games lacks PMAS. We propose a new cooperative TU-game concept, Population Lorenz-Monotonic Allocation Schemes (PLMAS), which relaxes some population monotonicity conditions by requiring that the payoff vector of any coalition is Lorenz dominated by the corresponding restricted payoff vector of larger coalitions. We show that every TU-game having a PLMAS is totally balanced, but the converse is not true in general. We obtain a class of games having a PLMAS, but no PMAS in general. Furthermore, we prove the existence of PLMAS for every glove game and for every assignment game with at most five players.

Additionally, we also introduce two new notions, PLMAS-extendability and PLMAS-exactness, and discuss their relationships with the convexity of the game.

Keywords: Convex cooperative games, glove games, Population Monotonic Allocation Schemes, Lorenz domination.

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1 Introduction

Cooperation is not without conflict. This tension results from the negotiation process on how to share the collective earnings. In some cases, cooperation is enforced; that is, economic gains are only achieved when all of us cooperate. The construction and maintenance of public goods generally require the cooperation of many economic agents. These situations represent instances where cooperation is not a choice but a necessity, as the achievement of societal goals relies on it. In other cases, cooperation may be voluntary and agents increase their benefits simply by cooperating. For instance, this is the case when several agents jointly invest their funds, getting a higher yield the larger is the amount invested. In such instances, cooperation becomes a mutually beneficial strategy, where individuals recognize that their collective efforts yield greater rewards, aligning their individual interests with the broader societal benefit.

In this study, we examine a society represented as a coalition of n individuals engaged in a cooperative problem. To model this scenario, we employ a cooperative game with transferable utility and its characteristic function, which delineates the gains of the entire society and the potential gains of subcoalitions of individuals. The central issue is how to distribute the overall gain among the n agents.

In some societies, social equity is considered a desirable goal and a social value in itself. At this point, individual interests, where maximizing personal earnings is paramount, may enter into conflict with the societal objective of promoting an equitable distribution. How to reconcile these two contrasting positions?

Consider, for example, an egalitarian and equitable extreme distribution, such as the equal division of gains. From an egalitarian perspective, there is no better distribution. However, from the cooperation standpoint, this distribution may conflict with the individual or coalitional interests that would require a distribution within the core of the cooperative game associated.

At this point, let us remark that each individual may have their own interpretation of the egalitarian concept. Within the context of cooperative game theory, various solutions related to egalitarianism have already been proposed: the equal division core (Selten, 1972), the constrained egalitarian solution (Dutta and Ray, 1989), the strong constrained egalitarian solution (Dutta and Ray, 1991), the stable egalitarian set (Arin and Iñarra, 2001 and 2002), and the split-off set (Branzei et al., 2006).

A standard way to compare distributions that allocate the same total amount (with same efficiency) among agents, and assess their quality concerning the equality of their payoffs, is to use the Lorenz criterion (Lorenz, 1905). For our purposes, one allocation of gains Lorenz dominates another one when, upon arranging the payoffs to agents from smallest to largest, the cumulative sums in the first allocation exceed those in the second. The Lorenz criterion, or Lorenz dominance, has been used in various economic contexts to provide some justification for certain distributions over others. As a result, there is a vast body of literature on the application of Lorenz dominance: in bankruptcy problems (see Miras et al., 2023; Thomson, 2019 and 2012; Bosmans and Lauwers, 2011), in taxation analysis (see Ju and Moreno-Ternero, 2008; Moreno-Ternero and Villar, 2006; Mitra and Ok,1997; Eichorn et al., 1984; Jakobsson, 1976), and in general cooperative games (see Sánchez-Soriano et al., 2010; and Arin and Feltkamp, 2002). Furthermore, experiments have been carried out with the result that agents prefer allocations where payoffs between agents do no differ too much, reinforcing from a positive point of view egalitarian criteria (see Traub et al., 2003). In this line, we can quote Moulin (1988), page 24:

> *Experimental evidence strongly supports egalitarianism when utilities are perceived as representing objective needs; see Yaari and Bar-Hillel (1984), where utilities are measured by the amount of certain vitamins metabolized by the agents. When utilities represent different tastes, the experimental outcome is much less easy to read.*

The Lorenz domination has been extended and used to compare vectors with different efficiency. In Moulin (1988, p. 48), Arin and Feltkamp[1] (2002) or Marshall[2] et al. (2011), an allocation of gains Lorenz dominates another one with smaller efficiency when, upon arranging the payoffs to agents from smallest to largest, the cumulative sums in the first allocation exceed those in the second. This way, Lorenz domination can be interpreted as a social welfare ordering that compares two allocation vectors with different efficiency where the more preferred vector correspond to a society with a larger welfare, e.g. see Endriss et al. (2006).

Within the context of cooperatives games, Sprumont (1990) introduces Population Monotonic Allocation Schemes (PMAS). An allocation scheme proposes an allocation for each subcoalition of agents. These allocations serve to justify the final allocation for the whole society showing that the more agents join a coalition (and thus the population grows) the larger can be the payoffs to agents. As a consequence of the definition of a PMAS, it is evident that not only individual payoffs, but social welfare of any subgroup of agents increases, leading to an allocation in the core of the associated cooperative game. However, not for all games with a non-empty core we can describe such an allocation scheme. For instance, as Sprumont remarks, an assignment game with at least two sellers and two buyers, where each buyer-seller pair derives a positive gain from trade, lacks a PMAS.

In this paper we propose a new cooperative TU-game concept, namely *Population Lorenz-Monotonic Allocation Scheme* (PLMAS). This concept encompasses the concept of PMAS and proposes an allocation scheme such that if new agents join a coalition the initial group of agents becomes socially better (in the Lorenz sense). As we have commented, societal interests may enter in conflict with individual or coalitional interests. However, a non-negligible consequence of adopting a PLMAS is that the final allocation proposed for the whole society turns out to be an allocation in the core of the game. This way PLMAS could be interpreted as a method to select a Pareto efficient allocation in the core of the game.

The remainder of the paper is organized as follows. In Section 2, we define the main concepts of cooperative games. In Section 3, we introduce the concept of PLMAS and Lorenz-monotonic core (the set of all the PLMAS). We show that this set can be discrete, and thus a non-convex set (see Example 1), which makes it different from the

[1] For a formal definition, see page 872.

[2] In this compilation book, the extended Lorenz relationship is named weak-majorization. See page 12, definition A.2.

case of PMAS. In Proposition 1, we point out that a PMAS can be reinterpreted as a PLMAS, and thus the individual incentive point of view makes the allocation compatible with the social point of view. However, the converse is not true. In fact, in Example 4, we exhibit a four-person game with PLMAS, but without PMAS, demonstrating that there are cases where the social point of view is appropriate to justify allocations. Indeed, in Theorem 1, we introduce a sufficient condition for having PLMAS that includes games with no PMAS. In Theorem 2, we discuss the case of glove games, a particular case of assignment games, and show that even though they do not have PMAS, any core allocation can be supported by a PLMAS.

In Section 4, we discuss concepts related to PLMAS. We state that convex games are PLMAS-extendable (Theorem 3) and are the unique class of games that are PLMAS-exact (Theorem 4). In Section 5 we conclude.

2 Notations

A cooperative game with transferable utility (a *game*) is a pair (N, v) (in short v), where $N = \{1, 2, \cdots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function with $v(\varnothing) = 0$. A subset S of N, $S \in 2^N$, is a coalition of players,$|S|$ denotes its cardinality, and $v(S)$ is interpreted as the worth of coalition S. We denote by $P(N) = \{S \subseteq N \mid S \neq \varnothing\}$ the set of nonempty coalitions of N. Given $S \in P(N)$, we denote by (S, v_S) the subgame of (N, v) related to coalition S (i.e. $v_S(R) = v(R)$ for all $R \subseteq S$).

A *payoff allocation* is a vector $z = (z_i)_{i \in N} \in \mathbb{R}^N$, where z_i is the payoff to player i, and for $S \in P(N)$ we write $z(S) = \sum_{i \in S} z_i$, $z(\varnothing) = 0$ and $z|_S = (z_i)_{i \in S}$. The *core* of a game (N, v) is the set $C(N, v) = \left\{z \in \mathbb{R}^N \mid z(N) = v(N),\ z(S) \geq v(S) \, \forall S \in P(N)\right\}$.

A game (N, v) is *balanced* if it has a nonempty core, it is *totally balanced* if the subgame (S, v_S) is balanced for all $S \in P(N)$, and it is *convex* (Shapley, 1971) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

A *Population Monotonic Allocation Scheme (PMAS)* of a game (N, v) (Sprumont, 1990) is a vector $x = \left(x^S\right)_{S \in P(N)}$, where $x^S = \left(x_i^S\right)_{i \in S} \in \mathbb{R}^S$, that satisfies the following conditions:

(i) Efficiency in each coalition: $\sum_{i \in S} x_i^S = v(S)$ for all $S \in P(N)$.

(ii) Monotonicity: $x^S \leq x^T\big|_S$ ($x_i^S \leq x_i^T$ for all $i \in S$) for all $S, T \in P(N),\ S \subseteq T$.

We also use the notation $x = \left(x_i^S\right)_{S \in P(N),\ i \in S}$ to describe a PMAS. The above definition implies that a PMAS x selects a core allocation $x^S = \left(x_i^S\right)_{i \in S} \in C(S, v_S)$ for every subgame (S, v_S) in such a way that the payoff to any player cannot decrease when the coalition to which he/she belongs becomes larger. Thus every game having a PMAS is totally balanced. Sprumont shows that all convex games have a PMAS.

The *monotonic core* of a game $v \in G^N$, denoted by $MC(N, v)$, is the set of all its PMAS (Moulin, 1990). This set always coincides with the core of a certain game associated to the initial game (Getán *et al.*, 2009).

PMAS-extendability

A balanced game (N,v) is *core-extendable* (Kikuta and Shapley, 1986) when for every $S \in P(N)$ and $y \in C(S,v_S)$ there exists $z \in C(N,v)$ such that $z_i = y_i$ for all $i \in S$. Each convex game is core-extendable, but the converse is not necessarily true (Sharkey, 1982; Kikuta and Shapley, 1986).

A game (N,v) is *PMAS-extendable* (Getán *et al.*, 2014) if for every $S \in P(N)$ and for every $y = \left(y^R\right)_{R\in P(S)} \in MC(S,v_S)$ there exists $x = \left(x^R\right)_{R\in P(N)} \in MC(N,v)$ such that $y^R = x^R$ for all $R \in P(S)$. Notice that every PMAS-extendable game has at least one PMAS. Moreover, we know that a game (N,v) is convex if and only if it is PMAS-extendable (Getán *et al.*, 2014). In particular, every PMAS-extendable game is core-extendable.

PMAS-exactness

A game (N,v) is called *exact* (Schmeidler, 1972) if for every $S \in P(N)$ there exists $z \in C(N,v)$ with $z(S) = v(S)$. It is evident that all exact games are totally balanced. Additionally, it is easy to observe that every convex game is exact. However, in general, the converse statement does not hold.

A game (N,v) is *PMAS-exact* (Getán *et al.*, 2014) when for every $S \in P(N)$ there exists $x = (x^R)_{R\in P(N)} \in MC(N,v)$ such that $x^N(S) = v(S)$. It is important to note that every PMAS-exact game is also exact, and any subgame of a PMAS-exact game is also PMAS-exact. Moreover, it is known that a game (N,v) is convex if and only if it is PMAS-exact (Getán *et al.*, 2014).

Lorenz domination

A standard of fairness is the one provided by the *Lorenz domination criterion* (Lorenz, 1905). To define it, consider a fixed population of individuals denoted as $N = \{1,2,\dots,n\}$. Given a vector $x = (x_1,\cdots,x_n) \in \mathbb{R}^N$, we can interpret x_i as the income of individual $i \in N$ and we can order the individuals from the poorest to the richest to obtain $x_{(1)} \leq \dots \leq x_{(n)}$. Now, given $x = (x_1,\dots,x_n) \in \mathbb{R}^N$ and $y = (y_1,\dots,y_n) \in \mathbb{R}^N$, we say that y *weakly Lorenz dominates* x, and we denote it by $x \preccurlyeq_{\mathcal{L}} y$ or by $y \succcurlyeq_{\mathcal{L}} x$, if:

$$\begin{array}{rcl} x_{(1)} & \leq & y_{(1)}, \\ x_{(1)} + x_{(2)} & \leq & y_{(1)} + y_{(2)}, \\ \cdots & \cdots & \cdots \\ x_{(1)} + x_{(2)} + \cdots + x_{(n)} & \leq & y_{(1)} + y_{(2)} + \cdots + y_{(n)}. \end{array}$$

An equivalent way to express the Lorenz domination criterion is by means of a function $\varphi : \mathbb{R}^N \to \mathbb{R}^n$ $(n = |N|)$, defined as follows. Let $x \in \mathbb{R}^N$ and $1 \leq k \leq n$, then we define the function $\varphi_k(x)$ as

$$\varphi_k(x) = \min\{x(S) | S \subseteq N \text{ and } |S| = k\} = x_{(1)} + \cdots + x_{(k)}.$$

For $x, y \in \mathbb{R}^N$, we have that $x \preccurlyeq_{\mathcal{L}} y$ if $\varphi_k(x) \leq \varphi_k(y)$ for all $k = 1,\dots,n$. It is said that y *Lorenz dominates* x, denoted by $x \prec_{\mathcal{L}} y$, if $x \preccurlyeq_{\mathcal{L}} y$ and $\varphi(x) \neq \varphi(y)$ (i.e. $\varphi_k(x) \neq \varphi_k(y)$ for some $k = 1,\dots,n$).

The relation $\preccurlyeq_{\mathcal{L}}$ is a preorder on $\mathbb{R}^N$ but not a partial order, as it satisfies the following properties:

(i) Reflexivity: $x \preccurlyeq_{\mathcal{L}} x$ for all $x \in \mathbb{R}^N$.
(ii) Transitivity: For $x, y, z \in \mathbb{R}^N$ with $x \preccurlyeq_{\mathcal{L}} y$ and $y \preccurlyeq_{\mathcal{L}} z$ we have $x \preccurlyeq_{\mathcal{L}} z$.
(iii) Non anti symmetry[3]: For $x, y \in \mathbb{R}^N$ we have

$$\begin{aligned}
&x \preccurlyeq_{\mathcal{L}} y \text{ and } y \preccurlyeq_{\mathcal{L}} x \\
&\Longleftrightarrow x_{(k)} = y_{(k)} \text{ for all } k = 1, \ldots, n \\
&\Longleftrightarrow x = y\Pi \text{ for some permutation matrix } \Pi.
\end{aligned}$$

However, the relation $\preccurlyeq_{\mathcal{L}}$ is a partial order on the commutative monoid $\mathcal{D} = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^N \mid x_1 \leq \ldots \leq x_n\}$. Moreover, $\preccurlyeq_{\mathcal{L}}$ is compatible with the sum "+" of $\mathcal{D}$:

$$x \preccurlyeq_{\mathcal{L}} y \Longrightarrow x + z \preccurlyeq_{\mathcal{L}} y + z \text{ for all } x, y, z \in \mathcal{D}.$$

Notice that for $x, y \in \mathbb{R}^N$ we have the implications:

$$x \leq y \quad \Rightarrow \quad x \preccurlyeq_{\mathcal{L}} y \quad \Rightarrow \quad x(N) \leq y(N) \tag{1}$$

where $x \leq y$ means $x_i \leq y_i$ for all $i \in N$. Moreover, notice that the egalitarian allocation $\alpha := \left(\frac{\nu}{n}, \ldots, \frac{\nu}{n}\right) \in \mathbb{R}^N$, where $\nu \in \mathbb{R}$, satisfies

$$\alpha \succcurlyeq_{\mathcal{L}} x \quad \text{ for all } x \in \mathbb{R}^N \text{ with } x(N) = \nu; \tag{2}$$

in others words, the egalitarian allocation $\alpha = \left(\frac{\nu}{n}, \ldots, \frac{\nu}{n}\right) \in \mathbb{R}^N$ Lorenz dominates any efficient allocation $x \in \mathbb{R}^N$ with $x(N) = \nu$.

3 Population Lorenz-Monotonic Allocation Schemes

In this section, we use the Lorenz domination criterion to introduce a new concept for a cooperative game. This concept aims to mimic and generalize the notion of PMAS.

Definition 1. *Let (N, v) be a cooperative game. We say that a vector $x = \left(x^S\right)_{S \in P(N)}$, where each $x^S = \left(x_i^S\right)_{i \in S} \in \mathbb{R}^S$, is a* Population Lorenz-Monotonic Allocation Scheme (PLMAS) *if it satisfies the following conditions:*

(i) Efficiency in each coalition: for all $S \in P(N)$, $\sum_{i \in S} x_i^S = v(S)$.

(ii) Lorenz-monotonicity: for all $S, T \in P(N)$, $S \subseteq T$,

$$x^S \preccurlyeq_{\mathcal{L}} x^T\big|_S \ (i.e.\ \varphi_k\left(x^S\right) \leq \varphi_k\left(x^T\big|_S\right) \text{ for all } k = 1, \cdots, s).$$

Notice that, by (1), the Lorenz-monotonicity condition relaxes the monotonicity condition of Sprumont. After providing the definition of PLMAS, we present several results regarding PLMAS for general cooperative games.

[3] A square matrix Π is said to be a *permutation matrix* if each row and column has a single unit entry, and all other entries are zero (Marshall et al., 2011)

The set of PLMAS of the game (N, v) is denoted by

$$\mathcal{L}MC(N, v) = \{x \mid x \text{ is a PLMAS of } (N, v)\},$$

and its projection to $\mathbb{R}^N$ is denoted by

$$\mathcal{L}MC^N(N, v) = \left\{x^N \mid x = \left(x^S\right)_{S \in P(N)} \in \mathcal{L}MC(N, v)\right\}.$$

Notice that the set $\mathcal{L}MC(N, v)$ is compact, but is not convex in general, as illustrated in the following example where the $\mathcal{L}MC(N, v)$ is a discrete set. This is a significant difference between PLMAS and PMAS, which makes it difficult to state a general existence theorem for PLMAS.

Example 1. *Consider the three-player game* (N, v) *defined by:*

$$v(S) = \begin{cases} 1 & \textit{if } S = \{1, 2\}, \{1, 3\} \textit{ or } N, \\ 0 & \textit{otherwise}, \end{cases}$$

for all $S \subseteq N$*. Then* $|\mathcal{L}MC(N, v)| = 4$*, since any* $x \in \mathcal{L}MC(N, v)$ *can be described as follows:*

$$\begin{aligned} &x^N = (1, 0, 0); \\ &x^{\{1,2\}} = (1, 0) \textit{ or } (0, 1),\ x^{\{1,3\}} = (1, 0) \textit{ or } (0, 1); \\ &x^{\{2,3\}} = (0, 0); \\ &x^{\{i\}} = (0) \textit{ for all } i \in N. \end{aligned}$$

The four previous possibilities give rise to all PLMAS of (N, v)*.*

We collect some basic facts about PLMAS in the following proposition. Proofs are left to the reader.

Proposition 1. *Let* (N, v) *be a cooperative game.*

(a) If $x = \left(x^S\right)_{S \in P(N)}$ *is a PLMAS of* (N, v)*, then* $x^S \in C(S, v_S)$ *for all* $S \in P(N)$*. In particular,* $\mathcal{L}MC^N(N, v) \subseteq C(N, v)$*.*
(b) Every PMAS of (N, v) *is also a PLMAS, i.e.* $MC(N, v) \subseteq \mathcal{L}MC(N, v)$*.*
(c) If (N, v) *has a PLMAS and* (N, v') *is a game satisfying* $v'(S) = v(S)$ *for all* $S \subseteq N, S \neq N$ *and* $v'(N) \geq v(N)$*, then* (N, v') *also has a PLMAS.*

Part (a) in Proposition 1 states that all cooperative games having a PLMAS are totally balanced. However, it is not true that all totally balanced games have a PLMAS, as shown in the following example. Since every three-player totally balanced game has a PMAS (Sprumont, 1990), we need to consider games with at least four players.

Example 2. *Fix a real number* $a \geq 6$*, and consider the four-player game* (N, v) *defined by:*

$$\begin{aligned}
&v(N) = a,\\
&v(134) = v(234) = 2,\ v(123) = 3,\ v(124) = a,\\
&v(14) = v(24) = 0,\\
&v(12) = v(13) = v(23) = v(34) = 2,\\
&v(i) = 0 \textit{ for all } i \in N.
\end{aligned}$$

It is straightforward to see that this game is totally balanced. Its core is

$$C(N, v) = \{(\alpha, \beta, 0, a - \alpha - \beta) \mid \alpha, \beta \geq 2 \textit{ and } \alpha + \beta \leq a - 2\}.$$

Additionally, we have $C(R, v_R) = \{(1, 1, 1)\}$ for $R = \{1, 2, 3\}$. Moreover, (N, v) lacks a PLMAS. To see this, suppose to the contrary that $x = (x^S)_{S \in P(N)}$ is a PLMAS of (N, v). By part (a) in Proposition 1, we know that $x^N \in C(N, v)$, $x^R \in C(R, v_R)$, for $R = \{1, 2, 3\}$. Therefore, we obtain

$$(1, 1, 1) = x^R \preccurlyeq_{\mathcal{L}} x^N\big|_R = (\alpha, \beta, 0) \textit{ for some } \alpha, \beta \geq 2.$$

This leads to a contradiction since $1 = \varphi_1(x^R) \leq \varphi_1\left(x^N\big|_R\right) = 0$.

Note that for the game (N, v) in Example 2 every game (N, v') such that $v'(S) = v(S)$ for all $S \subseteq N$, $S \neq N$, and $v'(N) \geq v(N)$ lacks a PMAS, as $v(123) + v(134) < v(12) + v(13) + v(34)$ (Norde and Reijnierse, 2002). However, if we take $v'(N) \geq \frac{4}{3}a$ it can be shown that (N, v') has a PLMAS. In fact, we can state a more general result for totally balanced four-player games.

Proposition 2. *Let (N, v) be a totally balanced four-player game. Then there exists a real number $\nu' \geq v(N)$ such that the game (N, v') defined by $v'(S) = v(S)$ for all $S \subseteq N, S \neq N$ and $v'(N) = \nu'$, has a PLMAS.*

Proof. Take $\nu' \in \mathbb{R}$ such that $\nu' \geq \max\left\{\frac{4v(S)}{|S|} \mid S \in P(N)\right\}$. Then, it is straightforward that the egalitarian allocation $\alpha^{v'} = \left(\frac{\nu'}{4}, \frac{\nu'}{4}, \frac{\nu'}{4}, \frac{\nu'}{4}\right)$ is in the core of (N, v'), i.e. $\alpha^{v'} \in C(N, v')$. Hence, define the vector $x = (x^S)_{S \in P(N)}$ satisfying the following properties:

$$\begin{aligned}
&x^N = \alpha^{v'},\\
&x^S \in C(S, v_S) \text{ for all } S \in P(N) \text{ with } |S| = 3,\\
&x^{\{i,j\}} = (v(i), v(ij) - v(i))\\
&\qquad \text{for all } i, j \in N \text{ with } i < j \text{ and } v(i) \leq v(j),\\
&x^{\{i,j\}} = (v(ij) - v(j), v(j))\\
&\qquad \text{for all } i, j \in N \text{ with } i < j \text{ and } v(i) > v(j), \text{ and}\\
&x^{\{i\}} = (v(i)) \text{ for all } i \in N.
\end{aligned}$$

Notice that the vector x^S is fixed for all $S \in P(N)$ with $|S| = 4, 2\, or\, 1$, while x^S is an arbitrary core allocation of the subgame (S, v_S) for all $S \in P(N)$ with $|S| = 3$. We next prove that $x \in \mathcal{L}MC(N, v')$. Indeed, it is clear that x satisfies efficiency in each coalition, $x^S(S) = v(S)$ for all $S \in P(N)$. Given a coalition $S \in P(N)$, $S \neq N$, we have

$$x^S \preccurlyeq_{\mathcal{L}} \left(\frac{v(S)}{s}, \overset{(s)}{\ldots\ldots}, \frac{v(S)}{s}\right)$$

since $x^S(S) = v(S)$ and (2). On the other hand, we have

$$\left(\frac{v(S)}{s}, \overset{(s)}{\ldots\ldots}, \frac{v(S)}{s}\right) \preccurlyeq_{\mathcal{L}} \left(\frac{\nu'}{4}, \overset{(s)}{\ldots\ldots}, \frac{\nu'}{4}\right) = x^N\big|_S$$

by the choice of parameter ν'. Hence, $x^S \preccurlyeq_{\mathcal{L}} x^N\big|_S$. Now, given two coalitions $S, T \in P(N)$ with $S \subseteq T$, $S \neq T$ and $T \neq N$, it is clear that $x^S \preccurlyeq_{\mathcal{L}} x^T\big|_S$. Therefore, we conclude x is a PLMAS of (N, v'). $\square$

We remark that the previous proposition is not valid in the case of games with five or more players as the following example shows.

Example 3. *We consider the extension of the four-player game (N, v) given in Example 2 to a game (M, w), where $M = N \cup N'$, $N' \neq \emptyset$ and $N \cap N' = \emptyset$, defined by*

$$w(S \cup S') := v(S) \quad \textit{for all } S \subseteq N, S' \subseteq N'.$$

Then it is clear that the game (M, w) is totally balanced with $m \geq 5$ players, and every game (M, w') such that $w'(T) = w(T)$ for all $T \subseteq M$, $T \neq M$, and $w'(M) \geq w(M)$ lacks a PLMAS, since as we have seen in Example 2 the subgame (N, v) lacks a PLMAS.

Now we introduce a class of games having PLMAS, but not PMAS in general as Example 4 illustrates.

Definition 2. *A zero-normalized game (N, v) (i.e. $v(i) = 0$ for all $i \in N$) is a* leadership game *if it satisfies the following properties:*

(i) $\frac{v(S)}{s} \leq \frac{v(N)}{n}$ *for all* $S \in P(N)$.
(ii) There exists a family $\{i_T\}_{T \in P(N), T \neq N}$, with each $i_T \in T$, such that:

$$\frac{v(T)}{t-1} \geq \begin{cases} \frac{v(S)}{s-1} & \textit{if } s > 1 \textit{ and } i_T \in S, \\[2ex] \frac{v(S)}{s} & \textit{if } s = 1 \textit{ or } i_T \notin S, \end{cases}$$

for all $S, T \in P(N)$ with $S \subseteq T$, $S \neq T$ and $T \neq N$.

We can interpret these games as follows. Consider a game (N, v) and a coalition of players $T \subseteq N$, where one specific player, denoted as i_T, assumes the role of the leader within the group. Player i_T contributes a unique set of assets, including know-how, capital, networking contacts, and prestige, while also organizing the collaborative efforts of the remaining members. The remaining group members contribute symmetrically through their work efforts. To establish a structured framework, we impose the following conditions:

(i) The equal distribution of the total output should be a core allocation of the game.
(ii) The productivity of each worker diminishes as the workforce size decreases and the leader stays in the coalition, i.e. $\frac{v(T)}{t-1} \geq \frac{v(S)}{s-1}$. On the other hand, if some group of workers $S \subseteq T$ opt to leave the group, the per capita value they could generate by working elsewhere as a collective, $\frac{v(S)}{s}$, is lower than the productivity they achieve under the leadership of i_T.

Theorem 1. *Let (N, v) be a leadership game. Then (N, v) has a PLMAS.*

Proof. First, notice that, by hypothesis, $v(S) \geq 0$ for all $S \in P(N)$. Then, define the vector $x = \left(x^S\right)_{S\in P(N)}$ as follows:

$$x_i^N = \frac{v(N)}{n} \text{ for all } i \in N;$$
$$x_i^S = \begin{cases} \frac{v(S)}{s-1} & \text{if } i \neq i_S, \\ 0 & \text{if } i = i_S, \end{cases}$$

for all $S \in P(N), S \neq N, i \in S$. We next prove that $x \in \mathcal{L}MC(N, v)$. Indeed, it is clear that x satisfies efficiency in each coalition. By part (a) of Lemma 1 below and property *(i)* we have that for each $S \in P(N)$, $S \neq N$ and $|S| \geq 2$ (case $|S| = 1$ is trivial), it holds:

$$x^S = \left(\frac{v(S)}{s-1}, \dots, \frac{v(S)}{s-1}, 0\right) \preccurlyeq_{\mathcal{L}} \left(\frac{v(N)}{n}, \dots, \frac{v(N)}{n}\right) = x^N\big|_S .$$

Now, given two coalitions $S, T \in P(N)$ with $S \subseteq T$, $S \neq T$ and $T \neq N$, we want to argue that $x^S \preccurlyeq_{\mathcal{L}} x^T\big|_S$. If $s = 1$, it is clear since $x^S = (0)$. If $s > 1$ and $i_T \notin S$, then, by part (a) of Lemma 1 below and property *(ii)* we have:

$$x^S = \left(\frac{v(S)}{s-1}, \dots, \frac{v(S)}{s-1}, 0\right) \preccurlyeq_{\mathcal{L}} \left(\frac{v(T)}{t-1}, \dots, \frac{v(T)}{t-1}\right) = x^T\big|_S .$$

Finally, if $s > 1$ and $i_T \in S$, by part (b) of Lemma 1 below and property *(ii)*, we have:

$$x^S = \left(\frac{v(S)}{s-1}, \dots, \frac{v(S)}{s-1}, 0\right) \preccurlyeq_{\mathcal{L}} \left(\frac{v(T)}{t-1}, \dots, \frac{v(T)}{t-1}, 0\right) = x^T\big|_S .$$

Hence, we conclude x is a PLMAS of (N, v). □

Lemma 1. *Let $\nu, \nu' \in \mathbb{R}_+$ and $s \geq 1$ an integer. Then:*

(a) $(\nu', \dots, \nu', 0) \preccurlyeq_{\mathcal{L}} (\nu, \dots, \nu)$ *(in $\mathbb{R}^s$) if and only if $(s-1)\nu' \leq s\nu$.*
(b) $(\nu', \dots, \nu', 0) \preccurlyeq_{\mathcal{L}} (\nu, \dots, \nu, 0)$ *(in $\mathbb{R}^s$) if and only if $s = 1$ or $\nu' \leq \nu$.*

Proof. (a) Let $z := (\nu, \ldots, \nu)$ and let $z' := (\nu', \ldots, \nu', 0) \in \mathbb{R}^S$. Then we have:

$$\begin{aligned} z' \preccurlyeq_{\mathcal{L}} z &\iff \varphi_k(z') \leq \varphi_k(z) \text{ for all } k = 1, \ldots, s \\ &\iff (k-1)\nu' \leq k\nu \text{ for all } k = 1, \ldots, s \\ &\iff (1 - (1/k))\nu' \leq \nu \text{ for all } k = 1, \ldots, s \\ &\iff \max\{(1 - (1/k))\nu' \mid k = 1, \ldots, s\} \leq \nu \\ &\iff (1 - (1/s))\nu' \leq \nu. \end{aligned}$$

(b) This part is straightforward. □

In general, a leadership game does not have a PMAS, as the following example illustrates.

Example 4. *Fix $a, b, c \in \mathbb{R}$ such that $2 \leq a \leq b \leq c$, and consider the four-player game (N, v) defined by:*

$$\begin{aligned} &v(N) = 4c, \\ &v(123) = 2b,\ v(124) = v(234) = 3c,\ v(134) = a, \\ &v(12) = 2b,\ v(13) = a, \\ &v(14) = v(23) = v(24) = 0,\ v(34) = 1, \\ &v(i) = 0 \textit{ for all } i \in N. \end{aligned}$$

Then (N, v) is a leadership game since it is straightforward to check that this game satisfies properties (i) and (ii) of Definition 2 (taking $i_T := \max\{i \mid i \in T\}$ for all $T \in P(N), T \neq N$), and thus by Theorem 1 the game has a PLMAS. However, this game lacks a PMAS since $v(123) + v(134) < v(12) + v(13) + v(34)$ (Norde and Reijnierse, 2002).

In the next theorem we demonstrate the existence of PLMAS for every glove game (Shapley, 1959). In fact, we show that every core allocation in a glove game can be reached by a PLMAS.

Theorem 2. *Let (N, v) be the glove game with respect to the disjoints sets L and R (i.e. $N = L \cup R, L \neq \emptyset, R \neq \emptyset$ and let $v(S) := min\{|S \cap L|, |S \cap R|\}$ for all $S \in P(N)$). Then $\mathcal{L}MC^N(N, v) = C(N, v)$. In particular, any glove game has a PLMAS.*

Proof. We write $y = (y|_{S\cap L}\,; y|_{S\cap R})$ for all $S \in P(N)$ and $y = (y_i)_{i\in S} \in \mathbb{R}^S$. Without loss of generality, let us suppose that $|L| \leq |R|$ and let $z \in C(N, v)$. We next prove that there exists $x = \left(x^S\right)_{S\in P(N)} \in \mathcal{L}MC(N, v)$ such that $x^N = z$. To this aim, for every $S \in P(N)$ we denote $l_S = |S \cap L|$ and $r_S = |S \cap R|$. Then, define x^S as follows:

$$x^S = \begin{cases} z & \text{if } S = N, \\ (0, \overset{(s)}{\ldots\ldots}, 0) & \text{if } S \subseteq L \text{ or } S \subseteq R, \\ (1, \overset{(l_s)}{\ldots\ldots}, 1; 0, \overset{(r_s)}{\ldots\ldots}, 0) & \text{if } S \neq N \text{ and } 1 \leq l_S \leq r_S, \\ (0, \overset{(l_s)}{\ldots\ldots}, 0; 1, \overset{(r_s)}{\ldots\ldots}, 1) & \text{if } 1 \leq r_S < l_S. \end{cases}$$

Notice that $x^N = z$ and $x^S \in C(S, v_S)$ for all $S \in P(N)$, Thus, $x = \left(x^S\right)_{S\in P(N)}$ satisfies efficiency in each coalition.

To prove that x is a PLMAS of (N, v), it remains only to check the Lorenz monotonocity of x. Let $S, T \in P(N)$ be two coalitions such that $S \subseteq T, S \neq T$. We claim that $x^S \preccurlyeq_{\mathcal{L}} x^T\big|_S$. Indeed, to prove it, we need to differentiate between several cases based on the previous definition of x^S:

Case 1. If $S \subseteq L$ or $S \subseteq R$, then it is straightforward since $x^S = (0, \overset{(s)}{\ldots}, 0)$.

Case 2. If $T = N$ and $l = |L| < r = |R|$, then $z = (1, \overset{(l)}{\ldots}, 1; 0, \overset{(r)}{\ldots}, 0)$ and $x^S \preccurlyeq_{\mathcal{L}} z|_S$ since the number of components equal to 1 in x^S is at most the number of components equal to 1 in $z|_S$, which is equal to l_s.

Case 3. If $T = N$ and $l = |L| = r = |R|$, then $z = (\lambda, \overset{(l)}{\ldots}, \lambda; 1-\lambda, \overset{(r)}{\ldots}, 1-\lambda)$ for some $0 \leq \lambda \leq 1$ and we must see that $x^S \preccurlyeq_{\mathcal{L}} z|_S$. Suppose that $1 \leq l_S \leq r_S$ (the other case $l_S > r_S \geq 1$ is similar and it is left to the reader), and thus $x^S = (1, \overset{(l_s)}{\ldots}, 1; 0, \overset{(r_s)}{\ldots}, 0)$. For $k = 1, ..., r_S$, we have $\varphi_k\left(x^S\right) = 0 \leq \varphi_k\left(z|_S\right)$. For $k = r_S + 1, ..., r_S + l_S = s$, we have $\varphi_k\left(x^S\right) = k - r_S \leq \varphi_k\left(z|_S\right)$ since $\varphi_k\left(z|_S\right) = (k - r_S)\lambda + r_S(1-\lambda)$ when $\lambda \geq 1/2$, and $\varphi_k\left(z|_S\right) = l_S\lambda + (k - l_S)(1-\lambda)$ when $\lambda \leq 1/2$.

Case 4. If $T \neq N$, $S \cap L \neq \emptyset$ and $S \cap R \neq \emptyset$, then $x^S \preccurlyeq_{\mathcal{L}} x^T\big|_S$, as the number of components equal to 1 in x^S is at most the number of components equal to 1 in $x^T\big|_S$ since $S \subseteq T$. □

Next we show the existence of PLMAS for every assignment game (Shapley and Shubik, 1971) with at most five players.

The player set is $N = M \cup M'$ where M, M' are two disjoints finite sets with respective cardinality $m, m' \geq 1$, named set of buyers and set of sellers respectively; so $n = m + m'$. Given a matrix $A = (a_{ij})_{i\in M, j\in M'} \in \mathrm{M}_{m\times m'}(\mathbb{R}_+)$, where each entry of the matrix $a_{ij} \geq 0$, we can associate a cooperative game (N, w_A), named *assignment game* defined by the matrix A, defining the worth of any coalition $S \cup S' \subseteq N$, with $S \subseteq M$ and $S' \subseteq M'$, by:

$$w_A(S \cup S') = \max\left\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S, S')\right\},$$

where $\mathcal{M}(S, S')$ is the set of all matchings μ between S and S'; i.e. $\mu \subseteq S \times S'$ is a bijection from $S_0 \subseteq S$ to $S_0' \subseteq S'$ such that $|S_0| = |S_0'| = \min\{s, s'\}$. A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* w.r.t. the matrix A when it satisfies

$$\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij} \text{ for all } \mu' \in \mathcal{M}(M, M').$$

We denote by $\mathcal{M}_A^*$ the set of optimal matchings for the grand coalition. So, we have $\sum_{(i,j)\in\mu} a_{ij} = w_A(N)$ for all $\mu \in \mathcal{M}_A^*$.

Shapley and Shubik (1971) prove that the core of the assignment game (N, w_A) is nonempty and it is enough to impose coalitional rationality for one-player coalitions and mixed-pair coalitions:

$$C(N, w_A) = \left\{ (u; v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'} \mid \begin{array}{l} \sum_{i \in M} u_i + \sum_{j \in M'} v_j = w_A(N), \\ u_i + v_j \geq a_{ij} \; \forall i \in M \, \forall j \in M' \end{array} \right\}.$$

Therefore if $(u; v) \in C(N, w_A)$, then for every optimal matching $\mu \in \mathcal{M}_A^*$ we have:

$$\begin{array}{l} u_i + v_j = a_{ij} \text{ if } (i, j) \in \mu, \\ u_i = 0 \text{ if } i \in M \text{ is not mached by } \mu, \\ v_j = 0 \text{ if } j \in M' \text{ is not mached by } \mu. \end{array}$$

There exists a *sellers-optimal* core allocation, $(\underline{u}; \overline{v}) = \left(\underline{u}^A; \overline{v}^A\right) \in C(N, w_A)$, where each seller attains his maximum core payoff. For every seller $j \in M'$ is

$$\overline{v}_j := w_A(N) - w_A(N \setminus \{j\}),$$

and given an optimal matching $\mu \in \mathcal{M}_A^*$ for every buyer $i \in M$ is

$$\underline{u}_i := \begin{cases} a_{i\mu(i)} + w_A(N \setminus \{\mu(i)\}) - w_A(N) & \text{if } i \text{ is matched by } \mu; \\ 0 & \text{if } i \text{ is not matched by } \mu. \end{cases}$$

A survey on assignment games is given in Núñez and Rafels (2015).

Proposition 3. *Every assignment game with at most five players has a PLMAS.*

Proof. Let (M, M', A) be an assignment game with $1 \leq m \leq m'$ and $m + m' \leq 5$. For a coalition $S \in P(N)$, we denote $s = |S|$, $m_S = |S \cap M|$ as the number of buyers in S, $m'_S = |S \cap M'|$ as the number of sellers in S and $A_S = (a_{ij})_{i \in S \cap M, j \in S \cap M'} \in \mathrm{M}_{m_S \times m'_S}(\mathbb{R}_+)$ denotes the corresponding submatrix of A at S.

If $m = 1$, it is clear that for each $z \in C(N, w_A)$ the vector $x = \left(x^S\right)_{S \in P(N)}$ defined by

$$x^S = \begin{cases} z & \text{if } S = N, \\ (0, \overset{(s)}{\ldots\ldots}, 0) & \text{if } S \subseteq M \text{ or } S \subseteq M', \\ (w_A(S); 0, \overset{(s-1)}{\ldots\ldots}, 0) & \text{if } m_S = 1 \leq m'_S, \end{cases}$$

for all $S \in P(N)$, is a PLMAS of (N, w_A).

If $m = m' = 2$, it is straightforward to check that for each $z \in C(N, w_A)$ the vector $x = \left(x^S\right)_{S \in P(N)}$ defined by

$$x^S = \begin{cases} z & \text{if } S = N, \\ (0, \overset{(s)}{\ldots\ldots}, 0) & \text{if } S \subseteq M \text{ or } S \subseteq M', \\ (w_A(S); 0, \overset{(s-1)}{\ldots\ldots}, 0) & \text{if } m_S = 1 \leq m'_S, \\ (0, 0; w_A(S)) & \text{if } m_S = 2 \text{ and } m'_S = 1, \end{cases}$$

for all $S \in P(N)$, is a PLMAS of (N, w_A).

If $m = 2$ and $m' = 3$, we denote $z = (\underline{u}; \overline{v}) \in C(N, w_A)$ as the sellers-optimal core allocation of (N, w_A). It is straightforward to check that $z(N \setminus \{j\}) = w_A(N \setminus \{j\})$ for all $j \in M'$ and thus we have $z|_S \in C(S, w_{A_S})$ for all $S \subseteq N$ with $m_S = m'_S = 2$. Proceeding as in the proof of Theorem 2, we observe that the vector $x = (x^S)_{S \in P(N)}$ defined by

$$x^S = \begin{cases} z & \text{if } S = N, \\ (0, \overset{(s)}{\ldots}, 0) & \text{if } S \subseteq M \text{ or } S \subseteq M', \\ (w_A(S); 0, \overset{(s-1)}{\ldots}, 0) & \text{if } m_S = 1 \leq m'_S, \\ (0, 0; w_A(S)) & \text{if } m_S = 2 \text{ and } m'_S = 1, \\ z|_S & \text{if } m_S = m'_S = 2, \end{cases}$$

for all $S \in P(N)$, is a PLMAS of (N, w_A). $\square$

We would like to remark that in general ($n \geq 6$), assignment games could lack PLMAS, as the following example illustrates.

Example 5. *Consider the following matrix* $A = \begin{pmatrix} 9 & 7 & 5 & 3 \\ 7 & 5 & 3 & 1 \end{pmatrix}$. *We claim the assignment game* (N, w_A) *relative to* A *lacks of PLMAS. Indeed, let* $M = \{1, 2\}$ *be the set of buyers and* $M' = \{3, 4, 5, 6\}$ *be the set of sellers. If* $x = (x^S)_{S \in P(N)} \in \mathcal{L}MC(N, v)$ *was a PLMAS, then we would necessarily have* $x_5^N = x_6^N = 0$ *and* $\varphi_2(x^R) = 0$ *for* $R := \{1, 2, 5, 6\}$. *However, this is not possible since*

$$C(R, w_{A_R}) = \{(\alpha, \alpha - 2; 5 - \alpha, 3 - \alpha) \mid 2 \leq \alpha \leq 3\}$$

and therefore $\varphi_2(z) > 0$ *for all* $z \in C(R, w_{A_R})$.

Now we show the existence of PLMAS in another interesting model. Shapley and Shubik (1967) introduces a model of a production economy involving a landowner and $m \geq 1$ peasants. The profit that arises if p peasants work for the landowner is denoted by $f(p)$, where $f : \{0, 1, 2, \ldots, m\} \to \mathbb{R}$ is a production function such that:

$f(0) = 0$,
if $0 \leq p_1 < p_2 \leq m$, then $f(p_1) \leq f(p_2)$ (increasing function),
if $0 \leq p_1 < p_2 < p_3 \leq m$, then $f(p_2) - f(p_1) \leq f(p_3) - f(p_2)$ (concavity).

Then, the associated cooperative game between the landowner (player 0) and the m peasants is defined as follows: for any coalition $\varnothing \neq S \subseteq N := \{0, 1, 2, \ldots, m\}$,

$$v(S) := \begin{cases} f(|S| - 1) & \text{if } 0 \in S \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In this model, the marginal productivity of any peasant when working for the landowner is equal to $\Delta := f(m) - f(m-1) \geq 0$. The allocation $z \in \mathbb{R}^N$ such that $z_0 := f(m) - m\Delta$ and $z_i := \Delta$ for all $i = 1, \ldots, m$ is a core allocation since f is a concave function. In next proposition we prove this core allocation z is supported by a PLMAS.

Proposition 4. *Let (N,v) be the cooperative game between a landowner (player 0) and m peasants associated to a increasing concave function $f : N = \{0,1,2,\ldots,m\} \to \mathbb{R}$ with $f(0) = 0$, and defined by (3). Let $z \in \mathbb{R}^N$ the allocation defined by*

$$z_i = \begin{cases} f(m) - m\Delta & \text{if } i = 0, \\ \Delta & \text{if } i = 1,\ldots,m, \end{cases}$$

for all $i \in N$, where $\Delta := f(m) - f(m-1)$. Then $z \in \mathcal{L}MC^N(N,v)$. In particular, (N,v) has a PLMAS.

Proof. Consider the vector $x = \left(x^S\right)_{S \in P(N)}$ defined by:

$$x^S = \begin{cases} (f(m) - m\Delta, \Delta, \overset{(m)}{\ldots\ldots}, \Delta) & \text{if } S = N, \\ (f(|S|-1), 0, \overset{(s-1)}{\ldots\ldots}, 0) & \text{if } 0 \in S \text{ and } S \neq N, \\ (0, \overset{(s)}{\ldots\ldots}, 0) & \text{if } 0 \notin S. \end{cases}$$

Observe that $x^N = z$. We next prove that $x \in \mathcal{L}MC(N,v)$. Indeed, it is straightforward that x satisfies efficiency in each coalition. Now, given two coalitions $S, T \in P(N)$ with $S \subseteq T, S \neq T$, we want to see that $x^S \preccurlyeq_{\mathcal{L}} x^T\big|_S$. If $0 \notin S$, it is clear since $x^S = (0, \overset{(s)}{\ldots\ldots}, 0)$. If $0 \in S$ and $T = N$, then $x^S = (f(|S|-1), 0, \overset{(s-1)}{\ldots\ldots}, 0)$ and we have $x^S \preccurlyeq_{\mathcal{L}} z|_S = x^N\big|_S$ since $z(S) \geq v(S) = x^S(S)$. Finally, if $0 \in S$ and $T \neq N$ then $x^T = (f(|T|-1), 0, \ldots, 0)$ and $x^S \preccurlyeq_{\mathcal{L}} x^T\big|_S$ since f is an increasing function. Hence, we conclude x is a PLMAS of (N,v). $\square$

We finish this section with another interesting example. Moretti and Norde (2021) analyze *weighted multi-glove games.* They generalize the model of glove markets, a two-sector production economy, by introducing several sectors, all of which are necessary to extract some positive profit. Each member of a sector has a certain number of units of an input. The production process requires using one unit of input from each sector to obtain one unit of output.

Formally, given a player set N and a partition of N into k sectors, $P = \{P_1, P_2, \ldots, P_k\}$, each member i is endowed with w_i units of input. The vector $w \in \mathbb{N}^N$ is the vector of inputs. Then, the worth of a coalition $S \subseteq N$, $S \neq \varnothing$ (the amount of output), is given by

$$v^{P,w}(S) = \min\left\{\sum_{i \in S \cap P_r} w_i : r = 1,\ldots,k\right\}.$$

The authors demonstrate that the corresponding game is totally balanced and provide a characterization of when the game admits PMAS. However, in Example 3.6 of their paper (page 728), they present an example of a five-player game with a core element that cannot be extended by a PMAS. The specific game is as follows.

Example 6. *(Example 3.6 of Moretti and Norde (2021)) Let $N = \{1,2,3,4,5\}$ be the set of agents, let $P = \{\{1,2\},\{3,4\},\{5\}\}$ be the partition of N that defines* *three sectors, and let $w = (1,1,1,1,2)$ be the vector of inputs. The allocation $x = (0.5, 0.5, 0.5, 0.5, 0)$ is in the core of the game, but the authors prove that there is no PMAS that extends this core allocation. However, it is easy to check that the following PLMAS extends this allocation:*

$$x^S = \begin{cases} (0.5, 0.5, 0.5, 0.5, 0) & \text{if } S = N, \\ (0, \overset{(s-1)}{\ldots\ldots}, 0, v^{P,w}(S)) & \text{if } 5 \in S \neq N, \\ (0, \ldots \overset{(s)}{\ldots}, 0) & \text{if } 5 \notin S, \end{cases}$$

where $s = |S|$. The proof is left to the reader.

4 PLMAS-extendability and PLMAS-exactness

In this section, we provide a characterization of the convexity of a game in terms of PLMAS. To do this, we introduce two new concepts related to the Lorenz-monotonic core: *PLMAS-extendability* and *PLMAS-exactness*. These notions are inspired by the concepts of PMAS-extendability and PMAS-exactness introduced by Getán *et al.* (2014).

Definition 3. *A game (N, v) is* PLMAS-extendable *if for every $S \in P(N)$ and for every $y = \left(y^R\right)_{R \in P(S)} \in \mathcal{L}MC\left(S, v_S\right)$ there exists $x = \left(x^R\right)_{R \in P(N)} \in \mathcal{L}MC(N, v)$ such that $y^R = x^R$ for all $R \in P(S)$.*
It is worth noting that every PLMAS-extendable game possesses at least one PLMAS, as every game contains subgames with PLMAS. For example, one can consider the restriction of the game to individual coalitions.

The following theorem proves that PLMAS-extendability is implied by the convexity of the game.

Theorem 3. *Let (N, v) be a convex game. Then (N, v) is PLMAS-extendable.*

Proof. To show that (N, v) is PLMAS-extendable we proceed by recurrence. We consider $S \in P(N)$, $j \in N \backslash S$, and $y \in \mathcal{L}MC\left(S, v_S\right)$. Then, we define $x = \left(x^R\right)_{R \in P(S \cup \{j\})}$ as follows: $x^R = y^R$, for $\varnothing \neq R \subseteq S$, and for $R \subseteq S \cup \{j\}$, with $j \in R$,

$$x_i^R = \begin{cases} y_i^{R \backslash \{j\}} & \text{if } i \neq j, \\ v(R) - v(R \backslash \{j\}) & \text{if } i = j, \end{cases} \qquad \text{for all } i \in R.$$

First, by definition we have $\left(x^R\right)_{R \in P(S)} = y$. Let us see that x is a PLMAS of $v_{S \cup \{j\}}$. Notice that for each coalition $R \in P\left(S \cup \{j\}\right)$ we have $x^R(R) = y^R(R) = v(R)$ when $j \notin R$, and $x^R(R) = x^R\left(R \backslash \{j\}\right) + x_j^R = y^{R \backslash \{j\}}\left(R \backslash \{j\}\right) + \left[v\left(R\right) - v\left(R \backslash \{j\}\right)\right] = v(R)$ when $j \in R$. Moreover, we claim that for each $R, T \in P\left(S \cup \{j\}\right)$ such that $R \subseteq T$ the Lorenz-monotonicity property holds, i.e. $x^R \preccurlyeq_{\mathcal{L}} x^T\big|_R$. To prove it we must distinguish different cases:
Case 1. If $j \notin T$, then $j \notin R$ and $x^R = y^R \preccurlyeq_{\mathcal{L}} y^T\big|_R = x^T\big|_R$.
Case 2. If $j \in T$ and $j \notin R$ then $R \subseteq T \backslash \{j\} \subseteq S$ and $x^R = y^R \preccurlyeq_{\mathcal{L}} y^{T \backslash \{j\}}\big|_R = x^T\big|_R$.

Case 3. If $j \in R$ then

$$x^R = \left(y^{R\setminus\{j\}}, v(R) - v(R\setminus\{j\})\right) \preccurlyeq_{\mathcal{L}} \left(y^{T\setminus\{j\}}\Big|_{R\setminus\{j\}}, v(T) - v(T\setminus\{j\})\right) = x^T\big|_R,$$

where the Lorenz domination follows from Lemma 2 below taking into account that $y^{R\setminus\{j\}} \preccurlyeq_{\mathcal{L}} y^{T\setminus\{j\}}\big|_{R\setminus\{j\}}$ and $v(R) - v(R\setminus\{j\}) \le v(T) - v(T\setminus\{j\})$, due to the convexity of the game.

Therefore we conclude $x \in \mathcal{L}MC\left(S \cup \{j\}, v_{S\cup\{j\}}\right)$. □

Lemma 2. *Let $x, y \in \mathbb{R}^N$ with $x \preccurlyeq_{\mathcal{L}} y$ and let $a, b \in \mathbb{R}$ with $a \le b$. Then $(x,a) \preccurlyeq_{\mathcal{L}} (y,b)$.*

Proof. Since $x \preccurlyeq_{\mathcal{L}} y$ it holds that $\varphi_k(x) \le \varphi_k(y)$, for all $k = 1, \ldots, n$. Hence, $\varphi_1(x,a) = \min\{\varphi_1(x), a\} \le \min\{\varphi_1(y), b\} = \varphi_1(y,b)$. Moreover, for all $k = 2, \ldots, n$, we have $\varphi_k(x,a) = \min\{\varphi_k(x), \varphi_{k-1}(x) + a\} \le \min\{\varphi_k(y), \varphi_{k-1}(x) + b\} = \varphi_k(y,b)$. Therefore, we conclude $(x,a) \preccurlyeq_{\mathcal{L}} (y,b)$. □

Since a game is PMAS-extendable if and only if it is convex (Getán *et al.*, 2014) we obtain the following result.

Corollary 1. *Every PMAS-extendable game is PLMAS-extendable.* □

It is generally not true that every PLMAS-extendable game is convex.

Example 7. *Fix a real number a with $1.5 \le a < 2$, and consider the three-player game (N,v) defined by:*

$$\begin{aligned} &v(N) = a, \\ &v(12) = v(13) = v(23) = 1, \\ &v(i) = 0 \text{ for all } i \in N. \end{aligned}$$

Then (N,v) is not convex, but it is totally balanced and its core is

$$C(N,v) = \{(\alpha, \beta, a - \alpha - \beta) \mid \alpha, \beta \le a - 1 \text{ and } \alpha + \beta \ge 1\}.$$

Moreover, (N,v) is PLMAS-extendable. Indeed, first notice that $z := \left(0,0,0; \left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right); \left(\frac{a}{3},\frac{a}{3},\frac{a}{3}\right)\right) \in MC(N,v) \subseteq \mathcal{L}MC(N,v)$. Let $S \in P(N)$ and $y = \left(y^R\right)_{R\in P(S)} \in \mathcal{L}MC(S, v_S)$. We must build an allocation scheme $x = \left(x^R\right)_{R\in P(N)} \in \mathcal{L}MC(N,v)$ such that $x^R = y^R$ for all $R \in P(S)$. If $|S| = 1$, then $S = \{i\}$, $y = (0)$ and $x := z$ extends y. If $|S| = 2$, then $S = \{i,j\}$, $y = (0,0;(\lambda, 1-\lambda))$ for some $0 \le \lambda \le 1$. Without loss of generality we can assume, by symmetry, that $S = \{1,2\}$. Then $x := \left(0,0,0; (\lambda, 1-\lambda), \left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right); \left(\frac{a}{3},\frac{a}{3},\frac{a}{3}\right)\right) \in \mathcal{L}MC(N,v)$ extends y, since $\min\{\lambda, 1-\lambda\} \le \frac{1}{2} \le \frac{a}{3}$ *and so $(\lambda, 1-\lambda) \preccurlyeq_{\mathcal{L}} \left(\frac{a}{3},\frac{a}{3}\right)$. Finally, the case $|S| = 3$ trivially holds.*

Next, we approach the notion of convexity from a different perspective by introducing the concept of PLMAS-exactness. In simple terms, PLMAS-exactness implies that the worth of any coalition of players is achieved in at least one PLMAS of the entire game.

Definition 4. *A game* (N, v) *is* PLMAS-exact *when for every* $S \in P(N)$ *there exists* $x = (x^R)_{R \in P(N)} \in \mathcal{L}MC(N, v)$ *such that* $x^N(S) = v(S)$.

It is evident that a game which is PLMAS-exact is also exact. Furthermore, it can be easily demonstrated that any subgame of a PLMAS-exact game is also PLMAS-exact. Next theorem establishes that PLMAS-exactness is a characterization of the convexity of the game.

Theorem 4. *Let* (N, v) *be a game. The following statements are equivalent:*

(i) (N, v) *is convex.*
(ii) (N, v) *is PLMAS-exact.*

Proof. $(i) \Rightarrow (ii)$ It is known that a game is convex if and only if it is PMAS-exact (Getán *et al.*, 2014). Moreover, any PMAS-exact game is PLMAS-exact since $MC(N, v) \subseteq \mathcal{L}MC(N, v)$ by part (b) of Proposition 1.

$(ii) \Rightarrow (i)$ Assume that (N, v) is PLMAS-exact and let $S, T \subseteq N$. Since the subgame $(S \cup T, v_{S \cup T})$ is PLMAS-exact too, there exists $x = (x^R)_{R \in P(S \cup T)} \in \mathcal{L}MC(S \cup T, v_{S \cup T})$ such that $x^{S \cup T}(S \cap T) = v(S \cap T)$. Therefore, by the second implication in (1), we obtain

$$\begin{aligned} v(S) + v(T) - v(S \cap T) &= x^S(S) + x^T(T) - x^{S \cup T}(S \cap T) \\ &\leq x^{S \cup T}(S) + x^{S \cup T}(T) - x^{S \cup T}(S \cap T) \\ &= x^{S \cup T}(S \cup T) = v(S \cup T). \end{aligned}$$

This proves the convexity of (N, v). □

5 Conclusion

Allocation schemes serve as a means to illustrate the benefits of forming larger coalitions. The PMAS concept primarily emphasizes individual incentives, whereas PLMAS justifies the final allocation from a social standpoint. This concept holds particular relevance in cooperative scenarios where players are substitutable or symmetric, as demonstrated in the case of a production economy or market situation.

For future research, it would be valuable to characterize the games that admit PLMAS and analyze other models where the Lorenz criterion offers fresh perspectives on allocation problems.

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