Universitat de Barcelona

Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

Entanglement, Segre Embeddings and Quantum Teleportation

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Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, January 15, 2025

Contents

Abstract

Quantum entanglement is key to the development of more advanced and efficient technologies. While its nature can be conceptually confusing, it admits ma-thematical descriptions. This work begins with an overview of quantum mechanics fundamentals, followed by a study of methods to characterize and quantify entanglement. Such methods are based on a complex projective geometry approach, with Segre embeddings playing a central role. Once we have gone deeper into this mathematical formalism, we explore how quantum teleportation works. Thanks to some properties that entanglement exhibits, qubits of information can be transmitted from one point to another using only classical bits. Finally, we establish a set of conditions under which a two-qubit state can be utilized for quantum teleportation and, furthermore, qualifies as a maximally entangled state. For other states, we briefly introduce an algorithm for achieving better teleportation results.

Resum

L'entrellaçament quàntic és clau per al desenvolupament de tecnologies més avançades i eficients. Tot i que la seva naturalesa pot resultar conceptualment confusa, admet descripcions matemàtiques. Aquest treball comença amb una visió general dels fonaments de la mecànica quàntica, seguida d'un estudi de mètodes per caracteritzar i quantificar l'entrellaçament. Tals mètodes es basen en un enfocament de geometria projectiva complexa, amb les *Segre embeddings* com a element central. Un cop aprofundit en aquest formalisme matemàtic, explorem com funciona la teleportació quàntica. Gràcies a algunes propietats que exhibeix l'entrellaçament, els qubits d'informació poden ser transmesos de manera segura d'un punt a un altre utilitzant només bits clàssics. Finalment, establim un conjunt de condicions sota les quals un estat de dos qubits pot ser utilitzat per a la teleportació quàntica i, a més, qualifica com un estat màximament entrellaçat. Per a altres estats, fem una introducció breu d'un algorisme per tal d'assolir millors resultats de teleportació.

2020 Mathematics Subject Classification. 14E25, 81P42, 81P48

Acknowledgments

I am very grateful to my advisor Dra. Joana Cirici Núñez for her advices, corrections and guidance throughout this work. Her dedication has been essential for the development of the project.

I would also like to thank Dr. Jordi Salvadó Serra, whose numerous explanations and meetings have been such a great help. His contributions have been fundamental for achieving the results in our Physics Final Degree Project.

Lastly, I would like to express my gratitude to my family, friends and partner for their support and encouragement throughout these years. A special mention goes to my colleague Andreu, whose notes on quantum mechanics have been crucial in clarifying some of its fundamental concepts.

Introduction

Since the 20th century, quantum mechanics has undergone enormous development. This is the fundamental theory that studies the behavior of physical systems, constituted by particles smaller than atoms, such as electrons and photons. One of the most important implications of this field is that some characteristics of such particles are undetermined: they are found to be in a superposition of different values or *states*. However, quantum mechanics can only predict the probability of finding the particle at every different state. For instance, the exact position of the electron in a hydrogen atom cannot be obtained, but one obtains the probability of finding it at different locations.

The features of these particles are described mathematically by *quantum states*. Their formalism depends on the kind of problem under study. Generally, they are defined as wave functions or *vector quantum states*. When the characteristic of the particle under study is more than assumed or is not important, then it is just said *state of the particle*.

One interesting situation is when a particle presents a physical feature that is found in a superposition of only two states. Such characteristic is described by what is known as a *qubit*. This is the fundamental unit of quantum information, consisting on a quantum state superposed in two different states, typically labeled as "0" and "1". For example, the *spin* of a spin-$\frac{1}{2}$ particle, such as an electron, acts as a qubit as it can exist in a superposition of spin-up and spin-down (along the $\hat{z}$ direction, for example):

$$\text{state of the spin} \equiv |\mathcal{S}\rangle = \alpha_1 |\uparrow_{\hat{z}}\rangle + \alpha_2 |\downarrow_{\hat{z}}\rangle\,, \quad \alpha_1, \alpha_2 \in \mathbb{C}$$

where $|\alpha_1|^2$ and $|\alpha_2|^2$ are the probabilities of finding the spin upwards or downwards, respectively.

In this context, when we say that a physical system is formed by n qubits, we refer to a system of n particles where the physical characteristic under study of each one behaves as a qubit. We could have a quantum state of a 2-qubit system of spin-$\frac{1}{2}$ particles:

$$|\mathcal{S}_1\mathcal{S}_2\rangle = \alpha_1 |\uparrow_{\hat{z}}\uparrow_{\hat{z}}\rangle + \alpha_2 |\uparrow_{\hat{z}}\downarrow_{\hat{z}}\rangle + \alpha_3 |\downarrow_{\hat{z}}\uparrow_{\hat{z}}\rangle + \alpha_4 |\downarrow_{\hat{z}}\downarrow_{\hat{z}}\rangle$$

This case give 4-dimensional systems since there are up to four possible states. We will often say for states like this that they are *two-particle states*, or in this case *two-qubit states*.

When preparing a system of several particles, it has been found that there can arise some interesting phenomena between the states of these particles. One of the most well-known is *quantum entanglement*. When two particle states are entangled, both behave as a single, inseparable entity. In other words, the state of each particle cannot be described independently of the state of the other. Consequently, any measurement or transformation applied to the state of one particle instantaneously changes the state of the other one, regardless of the distance between them.

The simplest case for illustrating how entanglement works is with the 2-qubit state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

If a measurement on one of these two qubits gives as an outcome the state labeled as “0” (or “1”), then the state of the other changes instantly to “0” (or “1”). From this state it is impossible to obtain two independent states, one for each particle. On the other hand, the states for which this can be done are called *product* states. An example of this is the 3-qubit state

$$|B\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle) = |0\rangle \otimes |\Phi^+\rangle \,.$$

The concept of entanglement was first introduced by Erwin Schrödinger†[2] in response to the Einstein-Podolsky-Rosen†[3] (EPR) paradox in 1935 [EPR35]. This paper led to the conception of *EPR pairs*: two-qubit states which were later recognized as being *maximally entangled* —the state $|\Phi^+\rangle$ is one of these pairs. In 1964, John Bell formulated Bell's theorem [Bel64], introducing the well-known Bell inequalities. His work tested the validity of the EPR paradox, and after that, EPR pairs were renamed *Bell states*. The presence of quantum entanglement in physical systems was proven theoretical and experimentally. Although it is still an active area of research, it has been recognized its importance and utility in technology and communication. In fact, it is at the forefront of several scientific areas such as quantum information theory, dense coding and quantum computing.

One of the most important implementations of quantum entanglement is to “teleport” one qubit from one point to another without being concerned about distance. Actually, there is no matter teleport but transmission of information. This phenomenon is known as *quantum teleportation (QT)* and a brief explanation of this goes as follow: a Bell state is prepared and the two entangled particles are separated; the qubit wanted to be teleported is coupled to one of these two particles; then a Bell measurement is performed onto the state of the two-coupled qubits, changing to one of the four Bell states; so, the qubit isolated is also modified because of the existing entanglement; after that, the measurement outcome

[2]† Physicist and Nobel Prize-winning Austrain, 1887-1961.

[3]† Albert Einstein, German theoretical physicist and Nobel Prize-winning, 1879-1955.
Boris Yakovlevich Podolsky, Russian-American physicist, 1896-1966.
Nathan Rosen, American-Israeli physicist, 1909-1995.

is sent to the insulated particle via two bits of classical information and, depending on the message, a certain transformation on this qubit is made, leading to a quantum state equivalent to the one desired to transfer. At the end of the process, despite the entangled pair —which could be prepared long ago, without concerning about the teleportation—, a qubit of information has been teleported via the transmission of two bits.

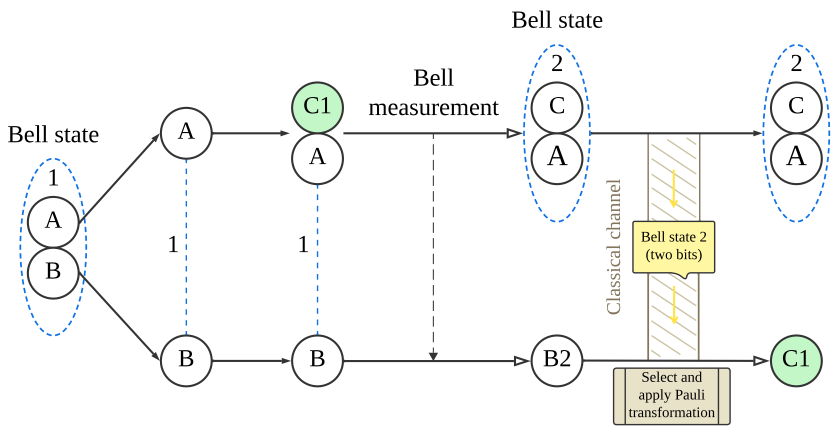


Figure 1: A schematic representation of the global quantum teleportation process.

Among the first papers investigating about (QT) using Bell states we find [BBC+93] published in 1993. After that, some experiments with photons have verified this phenomenon: across the Danube River (600m) in 2004 [UJA+04] and between Las Palmas and Tenerife (143km) in 2012 [MHS+12], among others. One of the latest experiments has been carry out the last December by a research team from the University of Northwestern [TYC+24]. They have achieved performing quantum teleportation with existing fiber optics crowded with classical Internet signals. Such a discovery is a big approach to the settlement of a large-scale quantum network, because it seems that there is no need of a new infrastructure for transmitting quantum information. On the other hand, there are studies related to teleport two qubits instead of only one [YC06]. Furthermore, [DC00] investigates how to teleport one qubit to several different places. All in all, because of the fast information transmission, the versatility of qubits and the message encryption provided by QT, this phenomenon is very useful for doing quantum computing. It is also visualized as an option to establish a better and more secure global-scale internet.

This process seems unnatural and impossible to perform, but it is already a reality. Theoretically, it can be also proven that this actually works. In Chapter 4, one can find a detailed explanation of how it is carried out with the mathematical quantum formalism —see also [Pre01]. On the other hand, a crucial step to achieve QT is when performing a Bell measurement on the 2-qubit state. For this reason, the procedure of this is described as well. Hereafter, we investigate how this process goes for a generic 2-qubit state. We are able to establish some conditions under which a state is *perfect for quantum teleportation (PQT)*.

Quantum mechanics is the new emerging paradigm in the technology field. Although there still is so much to investigate and develop, some quantum phenomenons are already being exploited to achieve more accurate results in the shortest possible time. The main objective of this work is to show that some quantum features and behaviors, such as entanglement and quantum teleportation, can be explained with a complex algebraic-geometrical approach. Extensive research on this topic can be found in [BBC$^+$19]. But before going straight to these mathematical descriptions, in Chapter 1 one has at its disposal some basic concepts about quantum mechanics, such as the description of *Hilbert spaces* $\mathcal{H}$, quantum states $|\varphi\rangle$ and how *measurements of observables* $\mathcal{A}$ work.

The fascinating nature of entanglement and its crucial role in quantum mechanics have been the motivation to delve deeper into this phenomenon in this work. In fact, it is the key for doing quantum teleportation. For this reason, we consider that it is important to give a mathematical characterization of quantum entanglement for *pure* states. In general, an algebraic-geometrical approach to quantum entanglement via a hypercube of Segre embeddings[†4] [CST21] is studied (Chapter 2) and implemented (Chapter 3) on systems of qubits. The *Segre embedding* is a map describing how to take products between projective spaces $\mathbb{P}$:

$$\Xi : \mathbb{P} \times \overset{(k)}{\cdots} \times \mathbb{P} \longrightarrow \mathbb{P}.$$

We settle the relation between quantum states $|\varphi\rangle \in \mathcal{H}$ and projective points in a complex projective space, $[\varphi] \in \mathbb{P}$. Then, there is a clear implementation of the Segre embedding for obtaining the state of a composite system:

$$\Xi([\varphi_1], \dots, [\varphi_k]) \longleftrightarrow |\varphi_1\rangle \otimes \cdots \otimes |\varphi_k\rangle .$$

From this, some interesting results are extracted. For instance, the *Segre variety* is defined as the image of the Segre embedding, $\Sigma = \operatorname{Im}\Xi$. If a projective point lies in Σ, then it can be separated, so the state associated to the former point is a product state.

We dedicate more attention to bipartite-type Segre embeddings between systems of n qubits:

$$\Xi_{n,l} : \mathbb{P}^{2^l-1} \times \mathbb{P}^{2^{n-l}-1} \longrightarrow \mathbb{P}^{2^n-1},$$

[4]† Named after the Italian mathematician Corrado Segre, 1863-1924.

where $1 \leq l \leq n-1$ refers to the system partition. The notation $\mathbb{P}^N$ means that it is N-dimensional. These type of functions are the fundamental objects in the Segre theory, as every Segre embedding is a combination of some bipartite-type ones. Actually, this decomposition is not unique. Something interesting is found when considering all possible decompositions: it leads to the construction of a $(n-1)$-hypercube, where the vertices are products of projective spaces and the edges are maps of the form $\mathbb{I} \times \Xi_{n,l} \times \mathbb{I}$.

The bipartite-type Segre varieties, $\Sigma_{n,l}$, are then enough for characterizing the entanglement of n-qubit systems. In fact, the *Generalized Decomposability theorem* settles that $[\psi]$ lies in at least q different bipartite-type Segre varieties if and only if the state is $(q+1)$-partite (that is, it can be expressed as a product of $q+1$ states). Given the importance of $\Sigma_{n,l}$, it is crucial to describe the structure of these Segre varieties. We proof that they are defined by the zero locus of all 2×2 minors of a matrix constructed using projective coordinates.

The images of bipartite-type Segre maps become essential for defining a family of quantum observables $\{\mathcal{J}_{n,l}\}_l$. Generally, their values range from zero to one. Notably, a state is q-partite if at least $q-1$ observables $\mathcal{J}_{n,l}$ are zero. Therefore, they settle the connection between the geometry from the Segre embedding to quantum entanglement.

A crucial step for being capable to teleport one qubit is to use one maximally entangled state of two qubits. This concept emerged when measurements for quantifying the amount of entanglement on any physical state arose. For pure states, the most widely used measurement of quantum entanglement is given by the so called von Neumann entanglement entropy. According to this, a state of two qubits is maximally entangled if the entropy of entanglement is maximal, with value $\ln 2$. Nevertheless, it is still an open problem and there is not only one way for quantifying entanglement, but in general it is agreed that a good measure must satisfy some properties [HHHH09]. Other commonly methods are exposed in [PV07].

In this work, we will study and adopt an entanglement measure provided in the paper [CST21] already mentioned. This is defined as the average observable of the family $\{\mathcal{J}_{n,l}\}_l$ from before:

$$\mathcal{J}_n = \frac{1}{n-1} \sum_{l=1}^{n-1} \mathcal{J}_{n,l}.$$

It has been observed that $\mathcal{J}_n$ works well for quantifying entanglement when $n \leq 4$. The physical interpretations of its extreme values are

$$\mathcal{J}_n(\psi) = 0 \Leftrightarrow |\psi\rangle \text{ is } (n-1)\text{-partite}, \quad \mathcal{J}_n(\psi) = 1 \Leftrightarrow |\psi\rangle \text{ is maximally entangled.}$$

For a number of qubits greater than four, maximally entangled states are not well defined since it is unknown the maximal value of this function: have been observed states with $\mathcal{J}_n(\psi) > 1$. However, in this work we will only consider states consisting of up to four qubits.

For the ones that are more interested in such teleportation protocol, we suggest taking a look on our Physics Final Degree Project [LSC25]. It is contemplated to carry out the process in question by considering two-qubit states of the form $|\Phi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$ for angles between 0 and π. The main issue is that these states are or are not maximally entangled depending on the angle, according to $\mathcal{J}_2$. For the latter case, it is impossible to perform successful QT. Despite of that, we implement an algorithm called *Multiple Correlated-Try Metropolis* (MCTM) [CL07] with the aim to find the best transformations to apply on the isolated qubit in order to obtain the best possible results. This method is a modification of the Metropolis-Hastings[†5] algorithm [Rob16].

We briefly explain the structure and contents of this project. In Chapter 1, we introduce the principal concepts about Quantum mechanics that will appear along, as well as their mathematic formalism. Later, in Chapter 2 we describe how practical are projective points for representing quantum states. Consequently, we also introduce the fundamentals of the Segre embedding and give important results for the following part. Once provided these features, we implement the Segre theory for studying the separability of n-qubit states in Chapter 3. Moreover, we also define the family of observables $\{\mathcal{J}_{n,l}\}$, the entanglement measure given by $\mathcal{J}_n$ and it is also provided an illustration of the $(n-1)$-hypercube for $2 \leq n \leq 5$. Lastly, in Chapter 4 we explain how quantum teleportation and Bell measurement are carried out. After that, a discussion about performing QT with an arbitrary two-qubit state is given. We will see that this is connected to the work done in our physics project.

[5]† Named after the Greek-American physicist Nicholas Constantine Metropolis (1915-1999) and the Canadian statistician Wilfred Keith Hastings (1930-2016).

Chapter 1

Quantum mechanics

In this first section, the reader not familiarized with quantum mechanics will find explanations about some fundamental concepts that will appear frequently during the next chapters. Specially, we will describe the structure of the Hilbert space $\mathcal{H}$ we will work in and how quantum states may be represented. Later, we will talk about some functions called operators and how they act on states. Two important groups of these are the observables and the unitary transformations. Finally, we will introduce the fundamental quantum element: the qubit.

1.1 The Hilbert space

Definition 1.1. A *Hilbert space* $\mathcal{H}$ is a complex vector space equipped with an inner product denoted as $\langle\,,\rangle$ which satisfies the following properties:

1) $\langle x, y\rangle \in \mathbb{C}$, $\forall x, y \in \mathcal{H}$.

2) It is conjugate symmetric, i.e. $\langle x, y\rangle = \langle y, x\rangle^*$. This implies that $\langle x, x\rangle \in \mathbb{R}$.

3) It is linear in its first argument.

4) The product $\langle x, x\rangle$ is positive definite: $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

This inner product induces a norm in $\mathcal{H}$ defined as $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathcal{H}$.

Hilbert spaces are indispensable for describing the quantum states of mechanical systems:

Definition 1.2. A *(quantum) state* is a vector $x \in \mathcal{H}$ such that $\langle x, x\rangle = 1$.

The nature of the Hilbert space $\mathcal{H}$ depends on the physical system. In this work, we will focus only on describing the spin[1] of fixed particles. The corresponding Hilbert spaces are of finite dimension $N < \infty$ —we will denote it as $\mathcal{H}^N$— and they are just the complex vector space $\mathbb{C}^N$ equipped with the usual inner product. The inner product of two vectors $x, y \in \mathcal{H}^N$ is then $\langle x, y \rangle = x^* \cdot y$. From now on, we will only consider as elements of the Hilbert space the states of this kind.

For quantum states, we use the *bra-ket* notation. On the one hand, the term *ket* is referred for vectors of $\mathcal{H}^N$ written as $|\varphi\rangle$. They are expressed in terms of some vector basis $\{|e_i\rangle\}_{i=1}^N$ where $|e_i\rangle$ also are quantum states (i.e. this basis is orthonormal):

$$|\varphi\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle + \cdots + a_N |e_N\rangle \equiv \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}_{e_i}. \tag{1.1}$$

The coefficients $a_1 \ldots, a_N \in \mathbb{C}$ are the coordinates of the vector in the respective basis and they satisfy $|a_1|^2 + \cdots + |a_N|^2 = 1$. This is called *normalization of the quantum state*. With this notation, $|\varphi\rangle$ refers to a state. An important observation is that states are physically indistinguishable up to a global phase factor with unitary norm. Therefore, we say that two states $|\varphi\rangle$ and $|\psi\rangle$ are *equivalent*, $|\varphi\rangle \sim |\psi\rangle$, (i.e. with the same physical properties) if and only if

$$|\varphi\rangle = e^{i\theta} |\psi\rangle, \quad \text{for some } \theta \in \mathbb{R}. \tag{1.2}$$

On the other hand, the term *bra* $\langle\varphi|$ represents a linear form of the dual Hilbert space $\left(\mathcal{H}^N\right)^*$ and it is expressed as

$$\langle\varphi| = \bar{a}_1 \langle e_1| + \bar{a}_2 \langle e_2| + \cdots + \bar{a}_N \langle e_N| \equiv \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_N \end{pmatrix}_{e_i} = |\varphi\rangle^\dagger, \tag{1.3}$$

where $\bar{a}_i$ is the complex conjugate of a_i and $\{\langle e_i|\}$ is the dual vector basis.

The inner product of two quantum states $|\varphi\rangle, |\psi\rangle \in \mathcal{H}^N$ is written as $\langle\psi \mid \varphi\rangle$ and it can be easily computed using matrix forms:

$$\langle\psi \mid \varphi\rangle = \begin{pmatrix} \bar{b}_1 & \cdots & \bar{b}_N \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \bar{b}_1 a_1 + \cdots + \bar{b}_N a_N \in \mathbb{C}.$$

The coordinates a_i and b_i of $|\varphi\rangle$ and $|\psi\rangle$, respectively, are in the same basis. This can be done as well by using that $\{|e_i\rangle\}$ is an orthonormal basis, i.e. $\langle e_i \mid e_j\rangle = \delta_{ij}$. Note that $\langle e_i \mid \varphi\rangle = a_i$ for all $i = 1, \ldots, N$.

[1] The spin is an intrinsic angular momentum characteristic of elementary particles.

1.2 Composite Hilbert space

Definition 1.3. Let $\mathcal{H}_1^{N_1}$ and $\mathcal{H}_2^{N_2}$ be two Hilbert spaces, with N_1 and N_2 two positive integers. The *composite Hilbert space* of $\mathcal{H}_1^{N_1}$ and $\mathcal{H}_2^{N_2}$ is the Hilbert space given by the tensor product of these vector spaces $\mathcal{H}_1^{N_1} \otimes \mathcal{H}_2^{N_2}$. Its inner product is defined as

$$\langle \langle \varphi_1 | \otimes \langle \varphi_2 | \mid |\psi_1\rangle \otimes |\psi_2\rangle \rangle := \langle \varphi_1 \mid \psi_1 \rangle \langle \varphi_2 \mid \psi_2 \rangle ,$$

for all $|\varphi\rangle_i, |\psi\rangle_i \in \mathcal{H}_i^{N_i}$ with $i = 1, 2$. The operation $\otimes$ is the *Kronecker product*.

If $\{|u_i\rangle\}_{i=1}^{N_1}$ is basis of $\mathcal{H}_1^{N_1}$ and $\{|v_j\rangle\}_{j=1}^{N_2}$ is one of $\mathcal{H}_2^{N_2}$, then a basis of the composite space $\mathcal{H}_1^{N_1} \otimes \mathcal{H}_2^{N_2}$ is formed by the vectors $|u_i\rangle \otimes |v_j\rangle =: |u_i v_j\rangle$. Observe that $\dim(\mathcal{H}_1^{N_1} \otimes \mathcal{H}_2^{N_2}) = N_1 N_2$.

Note that the definition of this inner product is only defined by quantum states of $\mathcal{H}_1 \otimes \mathcal{H}_2$ that factorize —that is states of the form $|\varphi\rangle_1 \otimes |\varphi\rangle_2$ where $|\varphi\rangle_1 \in \mathcal{H}_1$ and $|\varphi\rangle_2 \in \mathcal{H}_2$. Fortunately, we can always use a base of states of $\mathcal{H}_1 \otimes \mathcal{H}_2$ where all of the elements are factorized vectors. So thanks to the linearity of the inner product, we can extend this to every state written in a basis like this.

Definition 1.4. A state $|\varphi\rangle \in \mathcal{H}^N$ is *product* if there exist two smaller Hilbert spaces satisfying $\mathcal{H}_1^{N_1} \otimes \mathcal{H}_2^{N_2} = \mathcal{H}^N$ (where $0 < N_1, N_2 < N$ are two integers) and two states $|\varphi_1\rangle \in \mathcal{H}_1^{N_1}$, $|\varphi_2\rangle \in \mathcal{H}_2^{N_2}$ such that

$$|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle .$$

In other words, a state is product if there exists a composite Hilbert space where it factorizes. In contrast, if one state can not be factorized in any way, then the state is said to be *entangled*.

Composite Hilbert spaces are very important and frequently used in quantum mechanics, because sometimes it is necessary to consider different quantum states of separated systems as a unique state of a single system, or trying to decompose a quantum state into several independent states. These actions are useful for studying some properties of the state, such us its separability or its entanglement.

Given an n-particle state $|\varphi\rangle \in \mathcal{H}^N$, sometimes it is convenient to consider this Hilbert space as a composite of the form $\mathcal{H}_1^{N_1} \otimes \cdots \otimes \mathcal{H}_n^{N_n}$. Each $\mathcal{H}_i^{N_i}$ represents the N_i-dimensional system formed by the i-th particle. In respect of these dimensions, we have $N = N_1 \cdots N_n$. If $\{|u_{j_i}^i\rangle\}_{j_i=1}^{N_i}$ is a basis of $\mathcal{H}_i^{N_i}$ for each $i = 1, \ldots, n$, then $\{|u_{j_1}^1 \cdots u_{j_n}^n\rangle\}$ is one of $\mathcal{H}^N$.

Moreover, it is useful to introduce some *observers* $\mathcal{O}$, each living in one of the Hilbert spaces forming the previous composite system. They are used for emphasizing that only the observer $\mathcal{O}_i$ of $\mathcal{H}_i^{N_i}$ can make transformations and measurements (see the next two sections) on the particle living at this particular Hilbert space. With this new concept, the original state $|\varphi\rangle$ can be written as

$$|\varphi\rangle_{\mathcal{O}_1\cdots\mathcal{O}_n} = \sum_{j_1,\ldots,j_n} a_{j_1\cdots j_n} \, |u^1_{j_1}\cdots u^n_{j_n}\rangle_{\mathcal{O}_1\cdots\mathcal{O}_n}. \tag{1.4}$$

In this context, in quantum mechanics is said that this state $|\varphi\rangle$ is *separable* if there exist states $|\varphi_i\rangle \in \mathcal{H}_i^{N_i}$ for all $i = 1,\ldots,n$ such that

$$|\varphi\rangle_{\mathcal{O}_1\cdots\mathcal{O}_n} = |\varphi_1\rangle_{\mathcal{O}_1} \otimes \cdots \otimes |\varphi_n\rangle_{\mathcal{O}_n}.$$

Note that this is a particular case of product states which can be factorized in terms of the states of each particle.

1.3 Observables, measurements and transformations

Definition 1.5. An *operator* O is a linear map on a Hilbert space $O : \mathcal{H} \longrightarrow \mathcal{H}$.

For any pair of states $|\varphi_1\rangle, |\varphi_2\rangle \in \mathcal{H}$, the function $|\varphi_2\rangle\langle\varphi_1|$ is an operator:

$$\begin{aligned} |\varphi_2\rangle\langle\varphi_1| : \quad \mathcal{H} &\longrightarrow \mathcal{H} \\ |\psi\rangle &\longmapsto |\varphi_2\rangle\langle\varphi_1 \mid \psi\rangle \end{aligned}$$

Given a Hilbert space $\mathcal{H}^N$ and a certain orthonormal basis $\{|e_i\rangle\}$, the set of operators $\{|e_i\rangle\langle e_j|\}$ is a base of the space of the operators in $\mathcal{H}^N$. So for any operator O on $\mathcal{H}$ there exists a set of constants $\{c_{ij}\} \subset \mathbb{C}$ such that $O = \sum_{i,j} c_{ij} |e_i\rangle\langle e_j|$. For this reason, this operator has an associated matrix which in the base $\{|e_i\rangle\langle e_j|\}$ has the form

$$\begin{pmatrix} c_{11} & \cdots & c_{1N} \\ \vdots & \ddots & \vdots \\ c_{N1} & \cdots & c_{NN} \end{pmatrix}.$$

From now on, we will identify operators with their corresponding matrices given in a certain basis. Note that any complex matrix of dimension $N \times N$ defines an operator on $\mathcal{H}^N$.

In bra-ket notation,

$$O|\varphi\rangle := O(|\varphi\rangle) = \begin{pmatrix} c_{11} & \cdots & c_{1N} \\ \vdots & \ddots & \vdots \\ c_{N1} & \cdots & c_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \in \mathcal{H}^N,$$

$$\langle\psi|O|\varphi\rangle := \langle\psi|\,(O|\varphi\rangle) = \begin{pmatrix} \overline{b}_1 & \cdots & \overline{b}_N \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1N} \\ \vdots & \ddots & \vdots \\ c_{N1} & \cdots & c_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \in \mathbb{C}.$$

There is a postulate in quantum mechanics establishing that every *observable* magnitude (such as the spin, the momentum, the energy...) is associated with one linear operator on the Hilbert space which has real eigenvalues.

Definition 1.6. An operator $\mathcal{A} : \mathcal{H}^N \to \mathcal{H}^N$ is called *hermitian* if $\mathcal{A}^\dagger = \mathcal{A}$.

This kind of operators satisfy two interesting properties:

1) Their eigenvalues live in the real space. That is if $\mathcal{A}$ is hermitian and $\mathcal{A} |\varphi\rangle = \lambda |\varphi\rangle$, then $\lambda \in \mathbb{R}$.

2) Given an hermitian operator $\mathcal{A}$, there exists an orthonormal basis of $\mathcal{H}$ formed with eigenvectors of $\mathcal{A}$.

From this, for any hermitian operator exists one orthonormal basis of $\mathcal{H}^N$ such that its associated matrix is diagonal with real numbers:

$$\mathcal{A} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{pmatrix}, \quad \text{where } \lambda_i \in \mathbb{R} \; \forall i = 1, \ldots, s \; \text{ and } \; 1 \leq s \leq N.$$

This is the general case, where we contemplate the possibility of the eigenvalue spectrum of $\mathcal{A}$ being degenerate. In this sense, we have λ_i repeated r_i times, where $1 \leq r_i \leq N$ and $r_1 + \cdots + r_s = N$. The eigenstates (unit eigenvectors) associated to the eigenvalue λ_i are denoted as $|\lambda_i, 1\rangle, \ldots, |\lambda_i, r_i\rangle$ and they form a vector basis of one subspace of $\mathcal{H}^N$ of dimension r_i. Therefore, an orthonormal basis in $\mathcal{H}^N$ formed by the eigenstates of $\mathcal{A}$ is of the form

$$\{|\lambda_1, 1\rangle, \overset{(r_1)}{\ldots}, |\lambda_1, r_1\rangle, |\lambda_2, 1\rangle, \overset{(r_2)}{\ldots}, |\lambda_2, r_2\rangle, \ldots, |\lambda_s, 1\rangle, \overset{(r_s)}{\ldots}, |\lambda_s, r_s\rangle\}.$$

Note that $\langle \lambda_j, m_j \mid \mathcal{A} \mid \lambda_i, m_i \rangle = \lambda_i \delta_{ij} \delta_{m_i m_j} \; \forall i, j = 1, \ldots, s$, where $1 \leq m_{i,j} \leq r_{i,j}$.

These two properties are the reason why hermitian operators correspond to the operators associated to the observables discussed above. Because of this connection, this type of operators are also named *observables*.

An observable has a spectral decomposition of the form $\mathcal{A} = \sum_{i=1}^{s} \lambda_i \Pi_i$ where $\Pi_i = \sum_{l=1}^{r_i} |\lambda_i, l\rangle \langle \lambda_i, l|$ are one kind of observables called *projectors*: hermitian operators such that $\Pi^2 = \Pi$. Given this, there exists a postulate in quantum mechanics called the *Born rule* which states that:

i) The outcome of the measurement of the observable $\mathcal{A}$ on one state $|\varphi\rangle$ will be one of its eigenvalues λ_i.

ii) The probability of measuring λ_i is

$$P_{|\varphi\rangle}(\mathcal{A} : \lambda_i) = \| \Pi_i |\varphi\rangle \|^2 = \langle \varphi \mid \Pi_i \mid \varphi \rangle . \tag{1.5}$$

After an observable measurement, the state of the system always changes following this criteria: if the measurement outcome has been the eigenvalue λ_i, then $|\varphi\rangle$ has become the state

$$|\varphi_i\rangle := \frac{\Pi_i |\varphi\rangle}{\|\Pi_i |\varphi\rangle\|}. \tag{1.6}$$

If we develop this expression we get

$$|\varphi_i\rangle = \frac{\sum_{l=1}^{r_i} |\lambda_i, l\rangle \langle \lambda_i, l \mid \varphi\rangle}{\|\Pi_i |\varphi\rangle\|} = \sum_{l=1}^{r_i} w_l |\lambda_i, l\rangle,$$

where $w_l = \dfrac{\langle \lambda_i, l \mid \varphi\rangle}{\|\Pi_i |\varphi\rangle\|} \in \mathbb{C}$ for all $l = 1, \ldots, r_i$. So, an interesting interpretation of (1.6) is that, after getting the measurement outcome λ_i, the state $|\varphi\rangle$ has been projected to the subspace generated by the eigenstates associated to this eigenvalue. In this respect, the notation $|\varphi\rangle \to |\varphi_i\rangle$ is commonly used. Note that the likelihood that the state $|\varphi\rangle$ will end up being $|\varphi_i\rangle$ is the same as obtaining the measurement λ_i, i.e.

$$P\left(|\varphi\rangle \to |\varphi_i\rangle\right) = P_{|\varphi\rangle}(\mathcal{A} : \lambda_i).$$

If the eigenspace associated to λ_i is one-dimensional and generated by the eigenstate $|\lambda_i\rangle = |\lambda_i, 1\rangle$, then $\Pi_i = |\lambda_i\rangle \langle \lambda_i|$, so $|\varphi_i\rangle = |\lambda_i\rangle$. Hence, the probability is

$$P\left(|\varphi\rangle \to |\lambda_i\rangle\right) = |\langle \lambda_i \mid \varphi\rangle|^2. \tag{1.7}$$

Let us explain how operators are used on a composite Hilbert space. Suppose a state like (1.4) but of two particles, i.e. $|\varphi\rangle_{\mathcal{O}_1\mathcal{O}_2} = \sum_{i,j} c_{ij} |u_i v_j\rangle_{\mathcal{O}_1\mathcal{O}_2} \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Suppose one operator O acting only on the first particle (the one associated to the observer $\mathcal{O}_1$). An operator like this is defined as

$$\begin{aligned} O^{\mathcal{O}_1} := O^{\mathcal{O}_1} \otimes \mathbb{I}^{\mathcal{O}_2} : \quad \mathcal{H}_1 \otimes \mathcal{H}_2 &\longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \\ |\varphi\rangle_{\mathcal{O}_1\mathcal{O}_2} &\longmapsto \sum_{ij} c_{ij} \left(O^{\mathcal{O}_1} |u_i\rangle_{\mathcal{O}_1}\right) \otimes |v_j\rangle_{\mathcal{O}_2}. \end{aligned}$$

where $\mathbb{I}^{\mathcal{O}_2}$ is the identity operator in $\mathcal{H}_2$. Note that there is an abuse of notations, but we could avoid writing the observer in $O \otimes \mathbb{I}$ if we clearly differentiate which observable acts on each particle. For the second particle we would take $O^{\mathcal{O}_2} := \mathbb{I} \otimes O$. If now we consider both operators acting together,

$$O^{\mathcal{O}_1} O^{\mathcal{O}_2} := (O \otimes \mathbb{I})(\mathbb{I} \otimes O) = O \otimes O = O^{\mathcal{O}_1} \otimes O^{\mathcal{O}_2},$$

what we have is

$$\begin{aligned} O^{\mathcal{O}_1} \otimes O^{\mathcal{O}_2} : \quad \mathcal{H}_1 \otimes \mathcal{H}_2 &\longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \\ |\varphi\rangle_{\mathcal{O}_1\mathcal{O}_2} &\longmapsto \sum_{ij} c_{ij} \left(O^{\mathcal{O}_1} |u_i\rangle_{\mathcal{O}_1}\right) \otimes \left(O^{\mathcal{O}_2} |v_j\rangle_{\mathcal{O}_2}\right). \end{aligned} \tag{1.8}$$

An observation about these last operators is that $O^{\mathcal{O}_1} \otimes O^{\mathcal{O}_2} = O^{\mathcal{O}_2} \otimes O^{\mathcal{O}_1}$. When this happens and both are observables, it means that both measurements can be taken simultaneously. However, not every operator acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be factorized. Despite of that, a generic operator can be written as a linear combination of factorizing operators.

Unitary transformations

There are some particular operators $\mathcal{U}$ that preserve the norm, in other words $\|\mathcal{U}x\| = \|x\|$, $\forall x \in \mathcal{H}^N$. They are called *unitary transformations* and the associated complex matrices form the *unitary group* defined as

$$U(N) = \left\{ \mathcal{U} \in \mathcal{M}_N(\mathbb{C}) \mid \mathcal{U}^\dagger = \mathcal{U}^{-1} \right\}.$$

Every unitary matrix $\mathcal{U}$ satisfies that $|\det(\mathcal{U})| = 1$.

For any state $|\varphi\rangle$ one can perform a unitary transformation $\mathcal{U}$ and get a different state $|\widetilde{\varphi}\rangle = \mathcal{U}|\varphi\rangle$. Some examples of these transformations in $\mathcal{H}^2$ are the well-known *Pauli*[†] *operators* (or matrices):

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{1.9}$$

which also are hermitian —therefore observables. Because of that, $\sigma_i^{-1} = \sigma_i^\dagger = \sigma_i$, so these operators satisfy $\sigma_i^2 = \mathbb{I}$.

One matrix from the unitary group $U(2)$ can be written as

$$\mathcal{U} = e^{i\alpha} \begin{pmatrix} e^{i\beta}\cos\theta & e^{i\gamma}\sin\theta \\ -e^{-i\gamma}\sin\theta & e^{-i\beta}\cos\theta \end{pmatrix} \tag{1.10}$$

where $\alpha, \beta, \gamma, \theta \in \mathbb{R}$. Note that there are four degrees of freedom. However, in the context of quantum mechanics, the factor $e^{i\alpha}$ is a global phase and does not affect the physical properties of the system. As a result, these transformations are defined by three degrees of freedom, up to a global phase.

[1]† Named after the physicist Wolfrang Pauli.

1.4 Qubits

A *qubit* is a state of the smallest system one can consider. It consists of a superposition of two states usually labeled as "0" and "1". A general qubit is given by the state

$$|\psi\rangle = a_1 |0\rangle + a_2 |1\rangle$$

where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of a Hilbert space $\mathcal{H}^2$. As we have mentioned before, this state must be normalized, so its coefficients $a_1, a_2 \in \mathbb{C}$ must satisfy $|a_1|^2 + |a_2|^2 = 1$.

This quantum state is called like this because it is interpreted as the quantum version of the *bit*: a classical object that has one of two possible values, commonly represented as "0" and "1" as well. In the same way bits are the fundamental element for classical computation, qubits are for quantum, where many qubits have to coexist and interact. It is therefore interesting to consider a quantum system composed of n qubits living in a Hilbert space $\mathcal{H}^N$, where $N = 2^n$. A general *n-qubit state* has the following form of representation:

$$|\Psi\rangle = a_1 |0\overset{(n)}{\cdots}0\rangle + a_2 |0\overset{(n-1)}{\cdots}01\rangle + \cdots + a_{N-1} |1\overset{(n-1)}{\cdots}10\rangle + a_N |1\overset{(n)}{\cdots}1\rangle$$

where $a_1 \ldots, a_N \in \mathbb{C}$ and $\sum_{i=k}^{N} |a_k|^2 = 1$. We have used the orthonormal basis $|j_1 \cdots j_n\rangle \equiv |j_1\rangle_{\mathcal{O}_1} \otimes \cdots \otimes |j_n\rangle_{\mathcal{O}_n}$, where $j_i \in \{0, 1\}$ represents the state of the j-th qubit for all $i = 1, \ldots, n$. One difference here is that now there are n different particles, so as we explained before it may be convenient using observers $\mathcal{O}_i$, one for each qubit.

One of the most well-known examples of qubits arises from *spin-$\frac{1}{2}$ particles*, such as electrons. These particles have a spin that can exist in a superposition of two states: up $|\uparrow\rangle$ and down $|\downarrow\rangle$, along a chosen direction. More precisely, the particles themselves are not qubits, but their spin states serve as qubits.

Additionally, photons can also function as qubits, making them highly valuable in quantum experiments and technological applications. Specifically, their polarization can exist in a superposition of horizontal $|H\rangle$ and vertical $|V\rangle$ polarization states.

Chapter 2

The Segre embedding

This chapter has been written following the structure and results explained in [CST21]. We will introduce briefly what a complex projective space is and how it is strongly connected with the Hilbert space. Moreover, we will use a map called Segre embedding for taking products between two different projective spaces. After that, we will generalize this map for a set of projective spaces.

2.1 Complex projective space

Denote by $\mathbb{C}^N$ the complex vector space of dimension $N < \infty$.

Definition 2.1. The *complex projective space* $\mathbb{P}^N$ is the quotient space defined as

$$\mathbb{P}^N := \frac{\mathbb{C}^{N+1}\backslash\{0\}}{z \sim \lambda z}$$

where $z \in \mathbb{C}^{N+1}\backslash\{0\}$, $\lambda \in \mathbb{C}\backslash\{0\}$ and $\sim$ is an equivalence relation that identifies proportional vectors in $\mathbb{C}^{N+1}$.

The notation used $\mathbb{P}^N$ also means that the dimension of the projective space is N. The elements of $\mathbb{P}^N$ are called *projective points* and they are represented by the *projective coordinates* $z_0, \dots, z_N \in \mathbb{C}$ using the notation $p = [z_0 : \cdots : z_N]$. Actually, these coordinates are the ones of a vector z from $\mathbb{C}^{N+1}\backslash\{0\}$ in a certain basis. According to the equivalence relation $\sim$ from Definition 2.1, we have that $(z_0, \dots, z_N) \sim (\lambda z_0, \dots, \lambda z_N)$, $\forall \lambda \in \mathbb{C}^*$, so in terms of projective points,

$$p = [z_0 : \cdots : z_N] = [\lambda z_0 : \cdots : \lambda z_N], \quad \forall \lambda \in \mathbb{C}^* \tag{2.1}$$

On the other hand, the complex projective space $\mathbb{P}^N$ may be also understood as the set of lines in $\mathbb{C}^{N+1}$ that go through the origin.

2.2 Projective Hilbert space

Complex algebraic geometry can be implemented for understanding some concepts of quantum mechanics.

Let $N > 1$ be a positive integer and let $\{e_i\}_{i=1}^N$ be the standard basis in $\mathbb{C}^N$. We will consider a Hilbert space $\mathcal{H}^N$ with basis $|e_i\rangle$ and a complex projective space $\mathbb{P}^{N-1}$ with e_i as adapted basis.

Let be $|\varphi\rangle \in \mathcal{H}^N$ one state of an N-dimensional system. This state is of the form (1.1). According to (1.2), $|\varphi\rangle$ is unique up to a global phase, so for any $\theta \in \mathbb{R}$ the state of the form $e^{i\theta}|\varphi\rangle$ represents the same physical state. If we consider the coordinates of any of them, we see that $\{e^{i\theta}a_k\}_{k=1}^N \subset \mathbb{C}$ and there must be at least one nonzero element. Therefore, we have that $[e^{i\theta}a_1 : \cdots : e^{i\theta}a_N]$ is a projective point in $\mathbb{P}^{N-1}$. Consequently, we have the following identity:

$$[e^{i\theta}a_1 : \cdots : e^{i\theta}a_N] = [a_1 : \cdots : a_N], \quad \forall \theta \in \mathbb{R}.$$

In other words, all equivalent physical states in $\mathcal{H}^N$ are represented by the same projective point in $\mathbb{P}^{N-1}$.

On the other hand, let $p = [z_1 : \cdots : z_N]$ be a projective point in $\mathbb{P}^{N-1}$. If we consider the constant $\mathcal{C} = |z_1|^2 + \cdots + |z_N|^2 \in \mathbb{R}^+$, then p can be also expressed like

$$p = \frac{e^{i\theta}}{\sqrt{\mathcal{C}}}[z_1 : \cdots : z_N] = [a_{\theta,1} : \cdots : a_{\theta,N}],$$

where $a_{\theta,k} = e^{i\theta}z_k/\sqrt{\mathcal{C}}$ for all $k = 1\ldots, N$ and θ is some real constant. Note that $a_{\theta,k} \in \mathbb{C}$ with at least one nonzero, according to Definition 2.1. Moreover, they satisfy $|a_{\theta,1}|^2 + \cdots + |a_{\theta,N}|^2 = 1$. For each $\theta \in \mathbb{R}$, the vector $u_\theta = (a_{\theta,1}, \ldots, a_{\theta,N})$ representing p can be considered thus a state $|u_\theta\rangle$ in $\mathcal{H}^N$. Actually, the family of vectors $\{|u_\theta\rangle\}_{\theta\in\mathbb{R}}$ are all the ones representing a unique physical state. In other words, any projective point in $\mathbb{P}^{N-1}$ describes a distinctive mechanical state in $\mathcal{H}^N$.

All in all, one may observe that there is a strong correlation between states in $\mathcal{H}^N$ and projective points in the complex projective space $\mathbb{P}^{N-1}$. From the above discussion we obtain:

Proposition 2.2. *There exists a bijection between the sets of equivalent states in a Hilbert space $\mathcal{H}^N$ and projective points in $\mathbb{P}^{N-1}$:*

$$\begin{array}{ccc} \textit{equivalent states in } \mathcal{H}^N & \longleftrightarrow & \mathbb{P}^{N-1} \\ \left\{ \dfrac{e^{i\theta}}{\sqrt{\mathcal{C}}} \left(a_1 |e_1\rangle + \cdots + a_N |e_N\rangle\right) \,\middle|\, \theta \in \mathbb{R} \right\} & \longleftrightarrow & [a_1 : \cdots : a_N] \end{array}$$

where $\mathcal{C} = |a_1|^2 + \cdots + |a_N|^2$.

Therefore, we can visualize quantum states as projective points. Moreover, one unique projective point represents all possible states (which are infinity of them) describing the same properties of a given system. For this reason, and given that there is no need to compute any normalization constant, using projective points may be more practical than quantum vector states. When we deal with projective points, instead of the Hilbert space we use what is called *projective Hilbert space* $\mathbb{P}$.

Note that, from Proposition 2.2, the correspondence between $\mathcal{H}^N$ and $\mathbb{P}^{N-1}$ is described also by $[\varphi] \leftrightarrow \{|\psi\rangle \mid |\varphi\rangle \sim |\psi\rangle\}$, where $[\varphi] = [a_1 : \cdots : a_N]$ and $|\varphi\rangle = \frac{1}{\sqrt{C}}(a_0 |e_1\rangle + \cdots a_N |e_N\rangle)$. For simplicity, from now on we will write $[\varphi] \leftrightarrow |\varphi\rangle$, where $|\varphi\rangle$ denotes the state representing all equivalent states.

Example 2.3. Here are some examples of well-known states of two and three qubits written as projective points:

a) The four Bell states:
$|\phi_0\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longleftrightarrow [\phi_0] = [1:0:0:1]$
$|\phi_1\rangle := \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \longleftrightarrow [\phi_1] = [0:1:1:0]$
$|\phi_2\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \longleftrightarrow [\phi_2] = [0:1:-1:0]$
$|\phi_3\rangle := \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \longleftrightarrow [\phi_3] = [1:0:0:-1]$

b) $|Sep\rangle := |000\rangle \longleftrightarrow [Sep] = [1:0:0:0:0:0:0:0]$

c) $|B_1\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle) \longleftrightarrow [B_1] = [1:0:0:1:0:0:0:0]$

d) $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \longleftrightarrow [W] = [0:1:1:0:1:0:0:0]$

e) $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \longleftrightarrow [\mathrm{GHZ}] = [1:0:0:0:0:0:0:1]$

In this work, we will often use projective points when studying states as it is more comfortable: there is no need of computing the normalization constant nor considering global phases, as we already noticed just above.

2.3 Segre embedding

We will see that the *Segre embedding* is a map describing how to take products of projective Hilbert spaces $\mathbb{P}$.

Bipartite-type Segre embedding

Consider a state $[\varphi]$ in a composite projective Hilbert space $\mathbb{P}^N = \mathbb{P}^{N_1} \otimes \mathbb{P}^{N_2}$, where $N \geq 3$ and $1 \leq N_1, N_2 < N$. According to the concepts already introduced about composite Hilbert spaces in Section 1.2, there exist two adapted bases u_i and

v_j from two projective Hilbert spaces $\mathbb{P}^{N_1}$ and $\mathbb{P}^{N_2}$, respectively, such that $u_i v_j$ is an adapted basis of $\mathbb{P}^N$, where $1 \leq N_1, N_2 < N$. Therefore, the projective coordinates of our state in this basis are of the form:

$$[\varphi] = [c_{00} : \cdots : c_{0N_2} : \cdots : c_{ij} : \cdots : c_{N_1 0} : \cdots : c_{N_1 N_2}].$$

Note that this is a point with $(N_1+1)(N_2+1)$ projective coordinates, so we have the relation $N = (N_1+1)(N_2+1) - 1$ between the dimensions of the projective spaces. One may wonder if our state is product in these two projective Hilbert spaces. In other words, if there exist two states $[\varphi_1] = [a_0 : \cdots : a_{N_1}] \in \mathbb{P}^{N_1}$ and $[\varphi_2] = [b_0 : \cdots : b_{N_2}] \in \mathbb{P}^{N_2}$ such that, according to (1.4),

$$\begin{aligned} |\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle &\iff [\varphi] = [a_0 b_0 : \cdots : a_0 b_{N_2} : \cdots : a_{N_1} b_0 : \cdots : a_{N_1} b_{N_2}] \\ &\iff c_{ij} = a_i b_j, \quad \forall i = 0, \ldots, N_1 \text{ and } \forall j = 0, \ldots, N_2. \end{aligned}$$

The next definition is motivated on that:

Definition 2.4. Let $\mathbb{P}^{N_1}$ and $\mathbb{P}^{N_2}$ be two projective spaces, where $N_1, N_2 \in \mathbb{Z}^+$. The *(bipartite-type) Segre embedding* Ξ is the map defined as

$$\begin{array}{cccc} \Xi_{N_1,N_2} : & \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} & \longrightarrow & \mathbb{P}^N \\ & ([a_0 : \cdots : a_{N_1}], [b_0 : \cdots : b_{N_2}]) & \longmapsto & [\cdots : a_i b_j : \cdots] \end{array}$$

where $0 \leq i \leq N_1$, $0 \leq j \leq N_2$ and $N := (N_1+1)(N_2+1) - 1$.

Note that for any two states $[\varphi_1] \in \mathbb{P}^{N_1}$ and $[\varphi_2] \in \mathbb{P}^{N_2}$, their Segre image $\Xi_{N_1,N_2}([\varphi_1],[\varphi_2])$ is a state in the projective Hilbert space $\mathbb{P}^N$. More precisely, it is a product state.

This function is injective but not exhaustive. Indeed, the decomposition of one product state into two states given the spaces $\mathbb{P}^{N_1}$ and $\mathbb{P}^{N_2}$ is unique, and on the other hand not every state in $\mathbb{P}^N$ is product.

Definition 2.5. The *(bipartite-type) Segre variety* Σ_{N_1,N_2} of a Segre embedding Ξ_{N_1,N_2} is the image of this map:

$$\Sigma_{N_1,N_2} := \mathrm{Im}\left(\Xi_{N_1,N_2}\right).$$

This can be interpreted as the set of all product states generated by every pair of states formed by one state from $\mathbb{P}^{N_1}$ and another one from $\mathbb{P}^{N_2}$. That is so interesting, because by knowing how this set is constructed, one may easily characterize if one state is a product state in respect of two projective Hilbert spaces.

Proposition 2.6. *Given $N_1, N_2 \in \mathbb{Z}^+$, the Segre variety Σ_{N_1,N_2} can be described as*

$$\Sigma_{N_1,N_2} = \left\{ [\cdots : c_{ij} : \cdots] \in \mathbb{P}^N \quad | \quad c_{ij}c_{kl} = c_{il}c_{kj}, \quad \forall i \neq k, \forall j \neq l \right\}$$
$$= \left\{ \textit{zero locus of all } 2 \times 2 \textit{ minors of } \begin{pmatrix} c_{00} & \cdots & c_{0N_2} \\ \vdots & \ddots & \vdots \\ c_{N_1 0} & \cdots & c_{N_1 N_2} \end{pmatrix} \right\}.$$

Proof. Let N_1 and N_2 be two arbitrary positive integers. Consider the Segre variety $\Sigma_{N_1,N_2} \subset \mathbb{P}^N$, where $N = (N_1+1)(N_2+1) - 1$. Note that the projective coordinates of every point in $\mathbb{P}^N$ can be indexed like $[z_{00} : \cdots : z_{0N_2} : \cdots : z_{N_1 N_2}]$.

Let us take one arbitrary point $p = [c_{00} : \cdots : c_{N_1 N_2}]$ from $\mathbb{P}^N$.

First of all, suppose that $p \in \Sigma_{N_1,N_2}$. By definition of bipartite-type Segre variety, $\exists q_1 = [a_0 : \cdots : a_{N_1}] \in \mathbb{P}^{N_1}$ and $\exists q_2 = [b_0 : \cdots : b_{N_1}] \in \mathbb{P}^{N_2}$ such that

$$p = \Xi_{N_1,N_2}(q_1, q_2) = [a_0 b_0 : \cdots : a_0 b_{N_2} : a_1 b_0 : \cdots : a_{N_1} b_{N_2}].$$

Therefore, we have that $[c_{00} : \cdots : c_{N_1 N_2}] = [a_0 b_0 : \cdots : a_{N_1} b_{N_2}]$. This happens if and only if $(c_{00}, \ldots, c_{N_1 N_2}) = \lambda(a_0 b_0, \ldots, a_{N_1} b_{N_2})$ for some $\lambda \in \mathbb{C}^*$. With this, the projective coordinates of p satisfy

$$c_{ij}c_{kl} - c_{il}c_{kj} = \lambda^2 a_i b_j a_k b_l - \lambda^2 a_i b_l a_k b_j = 0,$$

$\forall i, k = 0, \ldots, N_1$ and $\forall j, l = 0, \ldots, N_2$. In particular, this happens $\forall i \neq k$ and $\forall j \neq l$.

On the other hand, now consider that the projective coordinates of p have the property that $c_{ij}c_{kl} = c_{il}c_{kj}$, $\forall i \neq k$ and $\forall j \neq l$. By definition of projective point, $\exists c_{k'l'} \neq 0$ for some $k' = 0, \ldots, N_1$ and $l' = 0, \ldots, N_2$. Then we have that

$$c_{ij} = \frac{c_{il'}c_{k'j}}{c_{k'l'}}, \quad \forall i \in \{0, \ldots, N_1\} \backslash \{k'\} \text{ and } \forall j \in \{0, \ldots, N_2\} \backslash \{l'\}.$$

After replacing the corresponding projective coordinates of p with this expression and multiplying every one with the constant $c_{k'l'} \in \mathbb{C}^*$, we obtain

$$\begin{aligned} c_{ij} = \frac{c_{il'}c_{k'j}}{c_{k'l'}} &\longrightarrow c_{il'}c_{k'j} \quad \forall i \in \{0, \ldots, N_1\} \backslash \{k'\} \text{ and } \forall j \in \{0, \ldots, N_2\} \backslash \{l'\} \\ c_{il'} &\longrightarrow c_{il'}c_{k'l'} \quad \forall i \in \{0, \ldots, N_1\} \backslash \{k'\} \\ c_{k'j} &\longrightarrow c_{k'l'}c_{k'j} \quad \forall j \in \{0, \ldots, N_2\} \backslash \{l'\} \\ c_{k'l'} &\longrightarrow c_{k'l'}c_{k'l'} \end{aligned}$$

In summary, note that c_{ij} has become $c_{il'}c_{k'j}$, $\forall i, j$. Therefore, we have another possible representation of p:

$$p = [c_{00} : \cdots : c_{ij} : \cdots : c_{N_1 N_2}] = [c_{0l'}c_{k'0} : \cdots : c_{il'}c_{k'j} : \cdots : c_{N_1 l'}c_{k'N_2}]$$

With that in mind, let be $a_r = c_{rl'}$ for all $r = 0, \dots, N_1$ and $b_s = c_{k's}$ for all $s = 0, \dots, N_2$. Note that, as $a_{k'} = b_{l'} = c_{k'l'} \neq 0$, then $q_1 := [a_0 : \cdots : a_{N_1}] \in \mathbb{P}^{N_1}$ and $q_2 := [b_0 : \cdots : b_{N_2}] \in \mathbb{P}^{N_2}$ are well-defined projective points. Using this notation, from the last expression of p we have that $p = [a_0 b_0 : \cdots : a_{N_1} b_{N_2}] = \Xi_{N_1,N_2}(q_1, q_2)$, so $p \in \Sigma_{N_1,N_2}$.

It is easy to check that $c_{ij}c_{kl} = c_{il}c_{kj}$ $\forall i \neq k$ and $\forall j \neq l$ is equivalent to the fact that all 2×2 minors from the matrix

$$\begin{pmatrix} c_{00} & \cdots & c_{0N_2} \\ \vdots & \ddots & \vdots \\ c_{N_1 0} & \cdots & c_{N_1 N_2} \end{pmatrix}$$

are zero. Indeed, given $i \neq k \in \{0, \dots, N_1\}$ and $j \neq l \in \{0, \dots, N_2\}$ arbitrariness integers,

$$c_{ij}c_{kl} - c_{il}c_{kj} = 0 \Leftrightarrow \begin{vmatrix} c_{ij} & c_{il} \\ c_{kj} & c_{kl} \end{vmatrix} = 0,$$

which is clearly a minor from the matrix of above. By changing the indexes we run over all possible 2×2 minors.

$\square$

In respect of the previous proposition, the total number of 2×2 minors existing in that matrix is given by

$$\xi_{N_1,N_2} := \frac{N_1(N_1+1)N_2(N_2+1)}{4}. \tag{2.2}$$

Example 2.7. The simplest Segre variety is

$$\Sigma_{1,1} = \left\{ [c_{0,0} : c_{0,1} : c_{1,0} : c_{1,1}] \in \mathbb{P}^3 \quad | \quad c_{0,0}c_{1,1} = c_{0,1}c_{1,0} \right\}.$$

Another two Segre varieties which we will see very often in the next chapter are the following:

a) $$\Sigma_{1,3} = \left\{ [c_{0,0} : \cdots : c_{1,3}] \in \mathbb{P}^7 | \text{ all } 2 \times 2 \text{ minors of } \begin{pmatrix} c_{0,0} & \cdots & c_{0,3} \\ c_{1,0} & \cdots & c_{1,3} \end{pmatrix} \text{ are zero} \right\}$$

b) $$\Sigma_{3,1} = \left\{ [c_{0,0} : \cdots : c_{3,1}] \in \mathbb{P}^7 | \text{ all } 2 \times 2 \text{ minors of } \begin{pmatrix} c_{0,0} & c_{0,1} \\ c_{1,0} & c_{1,1} \\ c_{2,0} & c_{2,1} \\ c_{3,0} & c_{3,1} \end{pmatrix} \text{ are zero} \right\}$$

Generalized Segre embedding

The Segre embedding from Definition 2.4 can be generalized by considering more than two projective Hilbert spaces.

First of all, let us define a function that sets a relation between the dimensions of the projective spaces involved, like in Definition 2.4 but in a generalized way:

$$\mathcal{N}(N_1, \ldots, N_m) := (N_1 + 1) \cdots (N_m + 1) - 1, \quad \forall m \geq 1. \tag{2.3}$$

Definition 2.8. Let $\mathbb{P}^{N_1}, \ldots, \mathbb{P}^{N_m}$ be $m \geq 3$ projective spaces, where the respective dimensions are $N_1 \ldots, N_m \in \mathbb{Z}^+$. The *generalized Segre embedding* is the map defined as:

$$\begin{array}{cccc} \Xi_{N_1,\ldots,N_m}: & \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m} & \longrightarrow & \mathbb{P}^{\mathcal{N}(N_1,\ldots,N_m)} \\ & \Big([a_0^1 : \cdots : a_{N_1}^1], \ldots, [a_0^m : \cdots : a_{N_m}^m]\Big) & \longmapsto & [\cdots : a_{i_1}^1 \cdots a_{i_m}^m : \cdots] \end{array}$$

where $0 \leq i_j \leq N_j$ for all $j = 1, \ldots, m$ and $\mathcal{N}(N_1, \ldots, N_m)$ is the function given by (2.3).

Definition 2.9. The *generalized Segre variety* $\Sigma_{N_1,\ldots,N_m}$ is the image of the corresponding generalized Segre embedding:

$$\Sigma_{N_1,\ldots,N_m} := \mathrm{Im}\left(\Xi_{N_1,\ldots,N_m}\right).$$

With these definitions, given a state $[\varphi] \in \mathbb{P}^N$ and a Segre embedding $\Xi_{N_1,\ldots,N_m}$ such that $\mathcal{N}(N_1, \ldots, N_m) = N$, if $[\varphi] \in \Sigma_{N_1,\ldots,N_m}$ then there exist m smaller states $[\varphi_1] \in \mathbb{P}^{N_1}, \ldots, [\varphi_m] \in \mathbb{P}^{N_m}$ such that $[\varphi]$ is a product state in terms of these m states:

$$[\varphi] = \Xi_{N_1,\ldots,N_m}\left([\varphi_1], \ldots, [\varphi_m]\right) \longleftrightarrow |\varphi\rangle = |\varphi_1\rangle \otimes \cdots \otimes |\varphi_m\rangle.$$

In contrast with the bipartite-type Segre variety from Definition 2.5, the generalized one can not be described in terms of homogeneous quadratic polynomial equations. Despite of that, from Definition 2.8 it is direct to deduce that generalized Segre maps can be described in terms of some bipartite ones:

Lemma 2.10. *If $\Xi_{N_1,\ldots,N_m} : \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m} \longrightarrow \mathbb{P}^{\mathcal{N}(N_1,\ldots,N_m)}$ is a generalized Segre embedding, then it can be written as a successive composition of $m-1$ functions of the form*

$$\mathbb{I}_{\widetilde{N}_1} \times \cdots \times \mathbb{I}_{\widetilde{N}_{i-1}} \times \Xi_{\widetilde{N}_i,\widetilde{N}_{i+1}} \times \mathbb{I}_{\widetilde{N}_{i+2}} \times \cdots \times \mathbb{I}_{\widetilde{N}_{\widetilde{m}}},$$

where the value of $\widetilde{m}$ is different for each function involved in the composition, going from m to 2. For a function of this form with value $\widetilde{m}$, for all $i = 1, \ldots, \widetilde{m}$:

i) $\mathbb{I}_{\widetilde{N}_i}$ is the identity map of the projective space $\mathbb{P}^{\widetilde{N}_i}$, *ii)* $\Xi_{\widetilde{N}_i,\widetilde{N}_{i+1}} : \mathbb{P}^{\widetilde{N}i} \times \mathbb{P}^{\widetilde{N}_{i+1}} \longrightarrow \mathbb{P}^{\mathcal{N}(\widetilde{N}_i,\widetilde{N}_{i+1})}$ *is the bipartite-type Segre embedding,*

iii) $\min\{N_1,\dots,N_m\} \leq \widetilde{N}_i < \mathcal{N}(N_1,\dots,N_m)$.

For instance, for any generalized Segre embedding with $m = 3$, i.e. Ξ_{N_1,N_2,N_3}, we have the following diagram:

$$\begin{tikzcd}
\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \mathbb{P}^{N_3} \arrow[rr, "\Xi_{N_1,N_2} \times \mathbb{I}_{N_3}"] \arrow[d, "\mathbb{I}_{N_1} \times \Xi_{N_2,N_3}"'] \arrow[drr, dashed, "\Xi_{N_1,N_2,N_3}"] & & \mathbb{P}^{\mathcal{N}(N_1,N_2)} \times \mathbb{P}^{N_3} \arrow[d, "\Xi_{\mathcal{N}(N_1,N_2),N_3}"] \\
\mathbb{P}^{N_1} \times \mathbb{P}^{\mathcal{N}(N_2,N_3)} \arrow[rr, "\Xi_{N_1,\mathcal{N}(N_2,N_3)}"'] & & \mathbb{P}^{\mathcal{N}(N_1,N_2,N_3)}
\end{tikzcd}$$

The dashed arrow represents the generalized Segre map, while the continuous arrows correspond to the maps from Lemma 2.10. Note that these together form a square, i.e. a *2-hypercube* as defined in Definition 3.16. Regarding the dimensions of the projective spaces involved in the diagram, it can be verified that

$$\begin{aligned}
\mathcal{N}\left(\mathcal{N}(N_1,N_2),N_3\right) &= \left(\mathcal{N}(N_1,N_2)+1\right)(N_3+1)-1 \\
&= (N_1+1)(N_2+1)(N_3+1)-1 = \mathcal{N}(N_1,N_2,N_3),
\end{aligned}$$

and similarly for $\mathcal{N}\left(N_1,\mathcal{N}(N_2,N_3)\right)$.

Example 2.11. Let us consider the following generalized Segre embedding:

$$\Xi_{1,1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^7.$$

This one has just two possible decompositions in terms of bipartite-type Segre embeddings:

- On one hand, $\Xi_{1,1,1} = \Xi_{3,1} \circ (\Xi_{1,1} \times \mathbb{I}_1)$, because:

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\Xi_{1,1}\times\mathbb{I}_1} \mathbb{P}^3 \times \mathbb{P}^1 \xrightarrow{\Xi_{3,1}} \mathbb{P}^7$$

- On the other hand, $\Xi_{1,1,1} = \Xi_{1,3} \circ (\mathbb{I}_1 \times \Xi_{1,1})$, because:

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\mathbb{I}_1\times\Xi_{1,1}} \mathbb{P}^1 \times \mathbb{P}^3 \xrightarrow{\Xi_{1,3}} \mathbb{P}^7$$

For a deeper understanding of the result from last Lemma, one may refer to the end of Chapter 3, where several $(n-1)$-*hypercubes* constructed by Segre embeddings of the latter form are presented. The projective spaces involved are of the form $\mathbb{P}^{2^k-1}$, where $1 \leq k \leq n$.

Chapter 3

Characterization of quantum entanglement in qubits

This chapter is based on the work of [CST21] as well. Here, it will be only considered quantum states of particles systems acting as qubits. First of all, we will implement the Segre's theory to this type of states. Thanks to that, we will be able to easily characterize its entanglement, or equivalently its decomposability. We will see in addition a function that quantifies how entangled a state is. Finally, it is shown that all Segre embeddings that may be performed in a n-qubit state can be illustrated with an $(n-1)$-hypercube.

3.1 Segre embedding in qubits

According to the introduction made in Section 1.4 about qubits, it is of relevant importance studying how much entanglement has any state of $n \geq 2$ qubits. Fortunately, in this section we may see that Segre embeddings may be a useful tool for achieving this objective.

Let $|\psi\rangle$ be a state of n particles. As we have already seen, this state lives in a 2^n-dimensional Hilbert space, or equivalently, its projective point $[\psi]$ belongs to a projective Hilbert space $\mathbb{P}^{2^n-1}$. We will use the following notation:

$$M_n := 2^n - 1 = \mathcal{N}(1, \overset{(n)}{\ldots}, 1).$$

The first question one might consider is if $[\psi]$ is a product state. Applying the concepts introduced in Section 2.3, this state is a product state if and only if there exist one state of n_1 particles $[\psi_1] \in \mathbb{P}^{M_{n_1}}$ and another $[\psi_2] \in \mathbb{P}^{M_{n_2}}$ of n_2 particles such that $\Xi_{M_{n_1}, M_{n_2}}([\psi_1], [\psi_2]) = [\psi]$. The positive integers M_{n_1} and M_{n_2}

must satisfy $\mathcal{N}(M_{n_1}, M_{n_2}) = M_n$. Note that

$$\mathcal{N}(M_{n_1}, M_{n_2}) = (M_{n_1}+1)(M_{n_2}+1) - 1 = 2^{n_1+n_2} - 1 = M_n \Leftrightarrow n_1 + n_2 = n.$$

This result aligns with the physical principle that the number of particles must be preserved. Therefore, imposing that $n_2 = n - n_1$ is equivalent to the condition $\mathcal{N}(M_{n_1}, M_{n_2}) = M_n$.

All in all, it is possible to give a more simplest way of characterizing a product state:

Lemma 3.1. *A state $[\psi] \in \mathbb{P}^{M_n}$ of n particles is a product state if and only if exists a positive integer $l \in \{1, \ldots, n-1\}$ such that $[\psi] \in \Sigma_{M_l, M_{n-l}}$.*

Now we can understand why it is called *bipartite-type* Segre embedding: for determining if it is a product state, it is enough by finding one bipartition of the n particles satisfying the previous Lemma. The partitions are indexed by the l parameter and there are a total of $n-1$ different partitions. Therefore, it is important to consider all possible Segre embeddings of the form

$$\Xi_{M_l, M_{n-l}} : \mathbb{P}^{2^l-1} \times \mathbb{P}^{2^{n-l}-1} :\longrightarrow \mathbb{P}^{2^n-1}, \quad \forall l = 1, \ldots, n-1; \tag{3.1}$$

as well as the corresponding Segre varieties

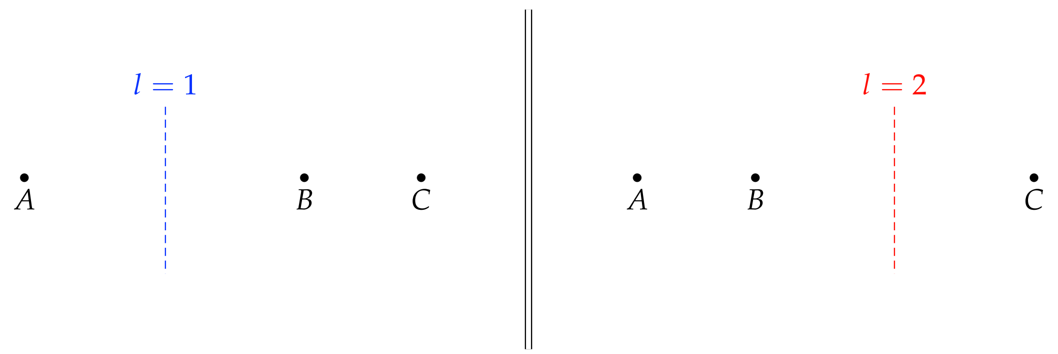
$$\Sigma_{M_l, M_{n-l}} = \mathrm{Im}\left(\Xi_{M_l, M_{n-l}}\right), \quad \forall l = 1, \ldots, n-1; \tag{3.2}$$

which we already know their structure thanks to Proposition 2.6.

Example 3.2. Consider a qubit-system formed by 3 particles in the state

$$|B_1\rangle_{ABC} = \frac{1}{\sqrt{2}}\left(|000\rangle_{ABC} + |011\rangle_{ABC}\right) \longleftrightarrow [B_1]_{ABC} = [1:0:0:1:0:0:0:0],$$

where A, B and C are three observers, one for each particle. There are only two possible partitions:



In the left partition ($l = 1$), we have two different subsystems: the system of A with one particle $\mathbb{P}^1_A$ and the system of B and C with two particles $\mathbb{P}^3_{BC}$. Meanwhile, in the right partition, we get the subsystems $\mathbb{P}^3_{AB}$ and $\mathbb{P}^1_C$.

For checking if the state $[B_1]$ is a product state of one state in $\mathbb{P}_A$ and another of $\mathbb{P}_{BC}$ we may see if $[B_1] \in \Sigma_{1,3} =: \Sigma_{A\otimes BC}$. According to Example 2.7, we know that

$$\Sigma_{A\otimes BC} = \left\{ \text{zero locus of all } 2\times 2 \text{ minors of } \begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} & c_{0,3} \\ c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} \end{pmatrix} \right\}.$$

In our case, we have $c_{0,0} = c_{0,3} = 1$ and the rest of the projective coordinates are zero. So, we have to check if all the 2×2 minors of the following matrix are zero:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that there is a total of 6 minors —a number that can be computed with (2.2)— and all of them are zero. Therefore, $[B_1] \in \Sigma_{A\otimes BC}$ and it is a product state. In fact, we could also noticed that

$$|B_1\rangle_{ABC} = |0\rangle_A \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{BC} + |11\rangle_{BC}\right)$$

On the other hand, by doing the same for the left partition ($l = 2$) we obtain $[B_1] \notin \Sigma_{AB\otimes C}$ because there exists a non-zero minor in the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

One might wonder that there may be more than two partitions, for example in one side the observer B while in the other one A and C. Nevertheless, this partition is not possible in our case, let us see why. In order to perform this partition, we have to consider the order BAC by permuting the particles A and B on the state from before —and therefore permuting the basis of $\mathbb{P}^7$. The result is a different state from $|B_1\rangle$:

$$|B_2\rangle_{ABC} = \frac{1}{\sqrt{2}}\left(|000\rangle_{ABC} + |101\rangle_{ABC}\right) \longleftrightarrow [B_2]_{ABC} = [1:0:0:0:0:1:0:0],$$

where the names of the observers A and B have been exchanged after reordering the basis. Doing the same as before, $[B_2] \notin \Sigma_{A\otimes BC} \cup \Sigma_{AB\otimes C}$.

3.2 Decomposability and separability

Another interesting aspect of study is the possibility of the state $|\psi\rangle$ not only being a product state of two states but of three or more states.

Definition 3.3. Let $|\psi\rangle$ be a state of $n \geq 2$ particles. $|\psi\rangle$ is said to be *q-partite* if there exist q states of n_i particles $|\psi_i\rangle$ such that

$$|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_q\rangle$$

where $2 \leq q \leq n$ and $n_1 + \cdots + n_q = n$.

This definition provides a way for classifying entangled and product states:

$$|\psi\rangle \text{ is a product state} \iff |\psi\rangle \text{ is } q\text{-partite for some } 2 \leq q \leq n.$$

It seems very interesting for characterizing quantum states in this way. Hopefully, Segre embeddings could be a valuable tool for this purpose. Before explaining how this works, it is crucial to introduce some important concepts.

Definition 3.4. Let $[\psi] \in \mathbb{P}^{M_n}$ be an n-particle state, $n \geq 2$ and $2 \leq q \leq n$. $[\psi]$ is *q-decomposable* if there exist q positive integers $n_1, \ldots, n_q$ such that

$$[\psi] \in \Sigma_{M_{n_1},\ldots,M_{n_q}}$$

where $n_1 + \cdots + n_q = n$.

Note that if $[\psi]$ is q-decomposable, then there exist n_i-particle states $[\psi_i] \in \mathbb{P}^{M_{n_i}}$ with $i = 1, \ldots, q$ such that

$$\Xi_{M_{n_1},\ldots,M_{n_q}}\big([\psi_1], \ldots, [\psi_q]\big) = [\psi].$$

In other words, $[\psi]$ can be separated within q states, i.e. $|\psi\rangle$ is q-partite. With this, one can observe that there is a strong correlation between Definitions 3.3 and 3.4:

Lemma 3.5. *Given an integer $2 \leq q \leq n$, one n-particle state $|\psi\rangle$ is q-partite if and only if its corresponding projective point $[\psi]$ is q-decomposable.*

According to this last result, $|\psi\rangle$ is a separable state if and only if $[\psi]$ is n-decomposable. That is if it is a point in the generalized Segre variety of the form $\Sigma_{1,\overset{(n)}{\ldots},1}$.

Lemma 3.6. *For $n \geq 3$, if $[\psi]$ is q-decomposable for some $2 < q \leq n$, then it is $(q-1)$-decomposable.*

Proof. Let $[\psi]$ be q-decomposable. Then, there exist $[\psi_i] \in \mathbb{P}^{M_{n_i}}$ for $i = 1, \ldots, q$ such that

$$[\psi] = \Xi_{M_{n_1},\ldots,M_{n_q}} \left([\psi_1], \ldots, [\psi_{i-1}], [\psi_i], [\psi_{i+1}], [\psi_{i+2}], \ldots, [\psi_q] \right).$$

Because $q > 2$, then $\Xi_{M_{n_1},\ldots,M_{n_q}}$ is one generalized Segre embedding. According to Lemma 2.10, this Segre map can be written as a composition of two functions of the form

$$\begin{tikzcd}
\mathbb{P}^{M_{n_1}} \times \cdots \times \mathbb{P}^{M_{n_q}} \arrow[d, "\mathbb{I}_{M_{n_1}} \times \cdots \times \Xi_{M_{n_i}, M_{n_{i+1}}} \times \cdots \times \mathbb{I}_{M_{n_q}}"'] \arrow[dr, dashed, "\Xi_{M_{n_1},\ldots,M_{n_q}}"] & \\
\mathbb{P}^{M_{n_1}} \times \cdots \times \mathbb{P}^{M_{n_i+n_{i+1}}} \times \cdots \times \mathbb{P}^{M_{n_q}} \arrow[r, "\Xi_{M_{n_1},\ldots,M_{n_i+n_{i+1}},\ldots,M_{n_q}}"'] & \mathbb{P}^{M_n}
\end{tikzcd}$$

for some $i = 1, \ldots, q-1$. Let be $\widetilde{n} = n_i + n_{i+1}$, therefore

$$[\psi] = \Xi_{M_{n_1},\ldots,M_{\widetilde{n}},\ldots,M_{n_q}} \left([\psi_1], \ldots, [\psi_{i-1}], [\widetilde{\psi}], [\psi_{i+2}], \ldots, [\psi_q] \right)$$

Note that $[\psi] \in \Sigma_{M_{n_1},\ldots,M_{\widetilde{n}},\ldots,M_{n_q}}$ and the following $q-1$ positive integers satisfy

$$n_1 + \cdots + n_{i-1} + \widetilde{n} + n_{i+2} + \cdots + n_q = n.$$

$\square$

Lemma 3.7. *If $[\psi]$ is q-decomposable for some $2 \leq q \leq n$, then it is 2-decomposable.*

According to the reciprocal of the last Lemma, if $[\psi]$ is not 2-decomposable, then it is not q-decomposable for all $2 \leq q \leq n$. The next definition is motivated on this result:

Definition 3.8. A state $[\psi]$ is *indecomposable* if it is not 2-decomposable.

Lemma 3.9. *The state $|\psi\rangle$ is entangled if and only if its corresponding projective point $[\psi]$ is indecomposable.*

We will see the decomposability of a state $[\psi]$ can be described only by means of bipartite-type Segre varieties. In [CST21] the reader can find an extended and detailed proof of the following result:

Theorem 3.10. (Generalized Decomposability) *Let $n \geq 2$ and $2 \leq q \leq n$ be two integers. The state $[\psi] \in \mathbb{P}^{M_n}$ is q-decomposable if and only if the state can be separated within $q-1$ different bi-partitions. In other words,*

$$[\psi] \text{ is } q\text{-decomposable} \iff \exists 1 \leq l_1 < \cdots < l_i < \cdots < l_{q-1} \leq n-1 \text{ such that } [\psi] \in \Sigma_{M_{l_i}, M_{n-l_i}}, \quad \forall i = 1, \ldots, q-1.$$

This theorem is very useful as it means that decomposability of states is fully characterized only by Segre varieties of the form (3.2). More specifically, it offers a method for classifying entangled and separable n-particle states:

$$|\psi\rangle \text{ entangled } \iff [\psi] \notin \bigcup_{l=1}^{n-1} \Sigma_{M_l, M_{n-l}}$$

$$|\psi\rangle \text{ separable } \iff [\psi] \in \bigcap_{l=1}^{n-1} \Sigma_{M_l, M_{n-l}}$$

Example 3.11. Some cases for determining the decomposability of states:

a) $[\psi] \in \mathbb{P}^7$ is 2-decomposable $\Leftrightarrow [\psi] \in \Sigma_{1,3} \cup \Sigma_{3,1}$;

b) $[\psi] \in \mathbb{P}^{15}$ is 3-decomposable $\Leftrightarrow [\psi] \in (\Sigma_{1,7} \cap \Sigma_{3,3}) \cup (\Sigma_{1,7} \cap \Sigma_{7,1}) \cup (\Sigma_{3,3} \cap \Sigma_{7,1})$.

It is already possible to give the characterization of separability depending on n. For instance, for smaller states we have

a) 2-particle states are separable $\Leftrightarrow [\psi] \in \Sigma_{1,1}$,

b) 3-particle states are separable $\Leftrightarrow [\psi] \in \Sigma_{1,3} \cap \Sigma_{3,1}$,

c) 4-particle states are separable $\Leftrightarrow [\psi] \in \Sigma_{1,7} \cap \Sigma_{3,3} \cap \Sigma_{7,1}$,

d) 5-particle states are separable $\Leftrightarrow [\psi] \in \Sigma_{1,15} \cap \Sigma_{3,7} \cap \Sigma_{7,3} \cap \Sigma_{15,1}$.

Example 3.12. Let's consider again the 3-particle states

$$[B_1]_{ABC} = [1:0:0:1:0:0:0:0] \quad \text{and} \quad [B_2]_{ABC} = [1:0:0:0:0:1:0:0].$$

We have already seen in Example 3.2 that $[B_1] \in \Sigma_{A \otimes BC} := \Sigma_{1,3}$. Conversely, $[B_1] \notin \Sigma_{AB \otimes C} := \Sigma_{3,1}$. Thus, according to Theorem 3.10 this state is 2-decomposable, or equivalently, it is a 2-partite (*bipartite*) state, meaning that it can be separated within two different states. On the other hand, $[B_2] \notin \Sigma_{A \otimes BC} \cup \Sigma_{AB \otimes C}$, so $[B_2]$ is an entangled state.

3.3 Measurement of entanglement

We have already seen that for characterizing the separability of a n-particle state it is sufficient by analyzing if its corresponding projective point lies in Segre varieties of the form $\Sigma_{M_l,M_{n-l}}$, where $1 \leq l \leq n-1$ and $M_l = 2^l - 1$.

Let $|\psi\rangle$ be an n-particle state and consider the family of observables defined as

$$\mathcal{J}_{n,l}\left(|\psi\rangle\right) := 4 \sum_{i=1}^{\xi_{M_l,M_{n-l}}} \left|\mathcal{M}_i\left(|\psi\rangle\right)\right|^2, \quad \forall 1 \leq l \leq n-1, \tag{3.3}$$

where $\mathcal{M}_i$ are the 2×2 minors determining the zero locus of $\Sigma_{M_l,M_{n-l}}$ defined as functions $\mathcal{H}^{2^n} \to \mathbb{C}$ of the form

$$\mathcal{M}_i\left(\left(\begin{array}{c} x_0 \\ x_1 \\ \vdots \\ x_{M_n} \end{array}\right)\right) := \left|\begin{array}{cc} x_{i_0} & x_{i_1} \\ x_{i_2} & x_{i_3} \end{array}\right| = x_{i_0}x_{i_3} - x_{i_1}x_{i_2}$$

and $\xi_{M_l,M_{n-l}}$ is the total number of these type of minors (take a look at Proposition 2.6). From these set of observables, there is a result —which can be found in [CST21]— that settles a connection between geometry and physics:

Theorem 3.13. *Given an integer* $1 \leq l \leq n-1$,

$$\mathcal{J}_{n,l}\left(|\psi\rangle\right) = 0 \iff [\psi] \in \Sigma_{M_l,M_{n-l}}.$$

Given this, if we desire to know if one n-particle state is separable in a certain partition l, one option is to look if the observable $\mathcal{J}_{n,l}$ acting on the state vanishes.

A particular case of study are the 2-particle states. Their general quantum state is $|\psi\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$ with the corresponding projective point $[\psi] = [a_0 : a_1 : a_2 : a_3]$. These type of states only can be of two types: 2-decomposable or indecomposable (i.e. separable or entangled, respectively) as there is just one possible partition. For characterizing them, it is enough by only considering the Segre variety of the form $\Sigma_{1,1}$. According to Proposition 2.6,

$$[\psi] \in \Sigma_{1,1} \Leftrightarrow a_0a_3 - a_1a_2 = 0 \Leftrightarrow \left|\begin{array}{cc} a_0 & a_1 \\ a_2 & a_3 \end{array}\right| = \mathcal{M}_1\left(|\psi\rangle\right) = 0 \Leftrightarrow \left|\mathcal{M}_1\left(|\psi\rangle\right)\right|^2 = 0$$

$$\Leftrightarrow 0 = 4\sum_{i=1}^{1} \left|\mathcal{M}_i\left(|\psi\rangle\right)\right|^2 = \mathcal{J}_{2,1}\left(|\psi\rangle\right).$$

From this, we easily see that

$$\mathcal{J}_{2,1}\left(|\psi\rangle\right) = 4|a_0a_3 - a_1a_2|^2. \tag{3.4}$$

Instead of the expression of $\mathcal{J}_{n,l}$ from (3.3), it can be shown that the value of this family of observables can be computed in terms of the Pauli matrices from (1.9):

$$\mathcal{J}_{n,l}(|\psi\rangle) = 2 - \left(\frac{1}{2^{l-1}} \sum_{i_1,\ldots,i_l=0}^{3} |\langle\psi|\, \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_l} \otimes \mathbb{I}_{2^{n-l}} \,|\psi\rangle\,|^2 \right). \tag{3.5}$$

The sum runs over all possible sets $\{i_1, \ldots, i_j, \ldots, i_l\}$, where $i_j = 0, 1, 2, 3$. This form may be more comfortable for computing the values of $\{\mathcal{J}_{n,l}\}$ than the former introduced. For a more detailed explanation about these observables consult [CST21].

In general, every function from the family $\{\mathcal{J}_{n,l}\}$ ranges from 0 to 1. For states consisting of three or more particles, its exact value does not provide important physical information. Despite of that, according to Theorem 3.13 it is possible to extract from them interesting equivalences:

i) $\mathcal{J}_{n,l}(|\psi\rangle) = 0 \iff |\psi\rangle$ is a product state and it can be separated in respect of the l system partitioning, so it is at least 2-partite.

ii) $\mathcal{J}_{n,l}(|\psi\rangle) > 0 \iff |\psi\rangle$ is not separable.

On the other hand, instead of considering only one observable $\mathcal{J}_{n,l}$, we may compute the values of all the members of the family $\{\mathcal{J}_{n,l}\}$ on the state. After that, let j be the cardinal of the set

$$\mathrm{j} := |\{l \mid \mathcal{J}_{n,l}(|\psi\rangle) = 0 \text{ and } 1 \leq l \leq n-1\}.$$

Taking a look again to the Generalized Decomposability theorem, we can affirm that $|\psi\rangle$ is $(\mathrm{j}+1)$-partite, or equivalently that $[\psi]$ is $(\mathrm{j}+1)$-decomposable.

Therefore, the separability (or decomposability) of any n-particle state is fully determined by measuring only $(n-1)$ observables. We may define the observable $\mathcal{J}_n$ that calculates the average value of the set $\{\mathcal{J}_{n,l}(|\psi\rangle)\}$ acting on n-particle states:

$$\mathcal{J}_n(|\psi\rangle) := \frac{1}{n-1} \sum_{l=1}^{n-1} \mathcal{J}_{n,l}(|\psi\rangle). \tag{3.6}$$

For states of $n < 5$ particles, the values of this new observable range from 0 to 1. In contrast, this is a good measure of entanglement as it is proven in the work already mentioned. The values of $\mathcal{J}_n$ acting on several states have already been computed and compared to other measurement methods, generally with great success. As bigger they are, more entangled the states. Moreover, it is able to provide a definition of *maximally entangled* which frequently agrees with the literature:

Definition 3.14. A smaller state $|\psi\rangle$, typically of $n \leq 4$ qubits, is *maximal entangled* if $\mathcal{J}_n(|\psi\rangle) = 1$.

All in all, let us make a compilation of the physical results provided by the observable $\mathcal{J}_n$ defined in (3.6):

$$\begin{aligned} \mathcal{J}_n(|\psi\rangle) = 1 &\iff |\psi\rangle \text{ is maximally entangled } (\mathrm{j} = 0) \\ \mathcal{J}_n(|\psi\rangle) = 0 &\iff |\psi\rangle \text{ is separable } (\mathrm{j} = n-1) \\ 0 < \mathcal{J}_n(|\psi\rangle) < 1 &\iff 0 < \mathrm{j} < n-1 \end{aligned}$$

For the latter case, one can observe that the observable $\mathcal{J}_n$ does not give much information about the state: we only know that it is neither maximally entangled nor separable, but it could still be either a product state or an entangled state. In order to characterize if a state is product, we must dispose of the values of each observable from the family $\{\mathcal{J}_{n,l}\}$ and check if some of them vanishes. If it is not, then $|\psi\rangle$ is entangled. Therefore, even product states may have a certain level of entanglement, which might be counter-intuitive.

On the other hand, the fact that for a given n-particle state the value of the observable is $0 < \mathcal{J}_n(|\psi\rangle) < 1$ is interesting because it means that there exists a partition l such that $\mathcal{J}_{n,l}(|\psi\rangle) > 0$. Thus, if we perform such system partition on this state, entanglement phenomenons may arose between the resulting two states —even for product states. This may be interesting when performing quantum teleportation of more than one qubit, where the particles involved must be separated and placed in different locations (see Chapter 4).

To end this section, let us compute the entanglement of the state $|W\rangle$ as an example of how these observables work.

Example 3.15. Let us study the entanglement of the state

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \longleftrightarrow [W] = [0:1:1:0:1:0:0:0]$$

by computing the values of $\mathcal{J}_{3,l}(|W\rangle)$ for the partitions $l = 1$ and $l = 2$ using the expression from (3.5). In respect of the partition $l = 1$,

$$\mathcal{J}_{3,1}(|W\rangle) = 2 - \left(\frac{1}{2^{1-1}}\sum_{i_1=0}^{3} |\langle W|\,\sigma_{i_1} \otimes \mathbb{I}_{2^{3-1}}\,|W\rangle|^2\right) = 2 - \sum_{i=0}^{3} |\langle W|\,\sigma_i \otimes \mathbb{I}_4\,|W\rangle|^2 = \frac{8}{9}.$$

Note that $\mathcal{J}_{3,1}(|W\rangle) > 0$. Indeed, $[W] \notin \Sigma_{M_1,M_2} = \Sigma_{1,3}$ because there exists one non-zero 2×2 minor of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, about the $l = 2$ partition, we get

$$\mathcal{J}_{3,2}(|W\rangle) = 2 - \frac{1}{2}\sum_{i_1,i_2=0}^{3} |\langle W|\,\sigma_{i_1} \otimes \sigma_{i_2} \otimes \mathbb{I}_2\,|W\rangle\,|^2 = \frac{8}{9}.$$

Again, the value of this observable is grater than zero. Furthermore, we can easily observe that the point $[W]$ does not lie in the Segre variety $\Sigma_{M_2,M_1} = \Sigma_{3,1}$ either.

Finally, the amount of entanglement in $|W\rangle$ is

$$\mathcal{J}_3(|W\rangle) = \frac{1}{2}\left(\mathcal{J}_{3,1}(|W\rangle) + \mathcal{J}_{3,2}(|W\rangle)\right) = \frac{8}{9}.$$

In conclusion, the values of $\mathcal{J}_{3,1}$ and $\mathcal{J}_{3,2}$ are grater than zero, so the state $|W\rangle$ is 1-partite (entangled). However, even $8/9$ is so close to one, the state is not maximally entangled.

3.4 Entanglement of small states

In this section, we will summarize the results obtained in this chapter by applying them to states consisting of $n = 2, 3, 4, 5$ particles. Furthermore, we will expose the values of $\{\mathcal{J}_{n,l}\}$ and $\mathcal{J}_n$ for some states of $2 \leq n \leq 4$ qubits. They have been computed according to (3.3) using the Python code in [Lap] Some of these states appear in [YC06] and [CST21].

Before doing so, we will briefly introduce how Segre embeddings on n-qubit systems provide (n-1)-hypercubes, offering a more illustrative way to understand the separability of states.

Definition 3.16. The *k-hypercube* is the set of points in the euclidean space $\mathbb{R}^k$ such that

$$\{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_i \in [0,1] \quad \forall i = 1, \ldots, k\}.$$

In other words, it is the extension of the idea of a square in $\mathbb{R}^2$ to $\mathbb{R}^k$. For example, the square is the 2-hypercube, meanwhile the 3-hypercube and 4-hypercube are the cube and the tesseract, respectively.

For any hypercube of dimension k, there are 2^k vertices, a total of $k2^{k-1}$ edges and k edges concurrent at every single vertex. We will deal with *directed hypercubes*, which all of their edges have an orientation settled.

In our case, if we have an n-qubit system, then an $(n-1)$-dimensional directed hypercube can be constructed as following:

1. Considering as vertices the categorical product of projective Hilbert spaces of the form $\mathbb{P}^{M_{n_1}} \times \cdots \times \mathbb{P}^{M_{n_m}}$, where $1 \leq m \leq n$ and $n_1 + \cdots + n_m = n$.

2. Considering as directed edges the Segre maps of the form given by Lemma 2.10, connecting the latter projective spaces.

In this sense, there is a starting vertex $\mathbb{P}^1 \times \overset{(n)}{\cdots} \times \mathbb{P}^1$ which all the $n-1$ edges concurrent at this point are outgoing represented by the maps $\mathbb{I} \times \Xi_{1,1} \times \mathbb{I}$. Meanwhile, in the finishing one every edge is ingoing and this vertex is $\mathbb{P}^{M_n} = \mathbb{P}^{2^n-1}$. These last edges are described by all possible bipartite-type Segre maps, $\Xi_{M_l,M_{n-l}}$.

Once the $(n-1)$-hypercube is constructed, we obtain a global vision of all the partitions from $\mathbb{P}^{2^n-1}$ to $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ obtained by taking bi-partitions sequentially.

Two-particle states

The general form of the projective point of state formed by two qubits is

$$[\psi] = [a_0 : a_1 : a_2 : a_3] \in \mathbb{P}^3.$$

In order to study its separability, we can focus on the decomposability of $[\psi]$ by considering all possible bipartite-type Segre varieties. Because there is only one possible bi-partition of the system, we only have to check if $[\psi] \in \Sigma_{1,1}$. This can be done by computing the minors given in Example 2.7 or by measuring the observable $\mathcal{J}_{2,1}(|\psi\rangle) = \frac{4}{\mathcal{C}^2}|a_0a_3 - a_1a_2|^2$, where $\mathcal{C} = |a_0|^2 + \cdots + |a_3|^2$. If its value is zero, then $[\psi]$ is 2-decomposable (product, in particular separable). If it is not, $[\psi]$ is indecomposable (entangled). In respect of measuring its entanglement, we can use the observable (3.6) which in this case is the same as $\mathcal{J}_{2,1}$. Thus, the possible results are:

$$\begin{aligned} \mathcal{J}_2(|\psi\rangle) = 0 &\Leftrightarrow |\psi\rangle \text{ is product and also separable,} \\ 0 < \mathcal{J}_2(|\psi\rangle) < 1 &\Leftrightarrow |\psi\rangle \text{ is entangled,} \\ \mathcal{J}_2(|\psi\rangle) = 1 &\Leftrightarrow |\psi\rangle \text{ is maximally entangled.} \end{aligned}$$

In this cases, a part for quantifying the entanglement, $\mathcal{J}_2$ fully determines if the state is product or entangled.

For this states, there is only one Segre embedding $\Xi_{1,1}$ characterizing the separability of states. This one generates a 1-hypercube in $\mathbb{R}^1$, which is simply a segment:

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\Xi_{1,1}} \mathbb{P}^3$$

Two-particle states	$\mathcal{J}_2$
$[\phi_i] \quad \forall i = 0, \ldots, 3$	1
$[1 : \pm 1 : 0 : 0]$	0
$[1 : 1 : 1 : 0]$, $[1 : 0 : 1 : 1]$, $[1 : 1 : 0 : 1]$, $[0 : 1 : 1 : 1]$	4/9
$[\cos\theta : \sin\theta : \pm\sin\theta : \mp\cos\theta] \quad \forall\theta \in [0, \pi)$	1
$[\pm\cos\theta : \mp\sin\theta : \sin\theta : \cos\theta] \quad \forall\theta \in [0, \pi)$	1

Table 3.1: Measurement of entanglement of some 2-particle states.

Three-particle states

The general form of these states is

$$[\psi] = [a_0 : a_1 : a_2 : a_3 : a_4 : a_5 : a_6 : a_7 :] \in \mathbb{P}^7$$

Their decomposability is fully characterized by $\Sigma_{1,3}$ and $\Sigma_{3,1}$:

$[\psi]$ is indecomposable $\Leftrightarrow [\psi] \notin \Sigma_{1,3} \cup \Sigma_{3,1} \Leftrightarrow \mathcal{J}_{3,1}(|\psi\rangle), \mathcal{J}_{3,2}(|\psi\rangle) > 0$,

$[\psi]$ is 2-decomposable $\Leftrightarrow [\psi] \in \Sigma_{1,3} \cup \Sigma_{3,1} \Leftrightarrow \mathcal{J}_{3,1}(|\psi\rangle) = 0$ or $\mathcal{J}_{3,2}(|\psi\rangle) = 0$,

$[\psi]$ is 3-decomposable $\Leftrightarrow [\psi] \in \Sigma_{1,3} \cap \Sigma_{3,1} \Leftrightarrow \mathcal{J}_{3,1}(|\psi\rangle) = \mathcal{J}_{3,2}(|\psi\rangle) = 0$.

For measuring their entanglement, one can use the observable $\mathcal{J}_3 = \frac{1}{2}\left(\mathcal{J}_{3,1} + \mathcal{J}_{3,2}\right)$.

The hypercube related to these states is the two dimensional one, which represents a square in $\mathbb{R}^2$:

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\mathbb{I}_1 \times \Xi_{1,1}"] \arrow[d, "\Xi_{1,1} \times \mathbb{I}_1"'] & \mathbb{P}^1 \times \mathbb{P}^3 \arrow[d, "\Xi_{1,3}"] \\
\mathbb{P}^3 \times \mathbb{P}^1 \arrow[r, "\Xi_{3,1}"'] & \mathbb{P}^7
\end{tikzcd}$$

Three-particle states	$\mathcal{J}_{3,1}$	$\mathcal{J}_{3,2}$	$\mathcal{J}_{3}$
$[Sep] = [1:0:0:0:0:0:0]$	0	0	0
$[B_1] = [1:0:0:1:0:0:0:0]$	0	1	1/2
$[W] = [0:1:1:0:1:0:0:0]$	8/9	8/9	8/9
$[GHZ] = [1:0:0:0:0:0:0:1]$	1	1	1

Table 3.2: Measurement of entanglement of some 3-particle states.

Four-particle states

These states are represented with projective points in $\mathbb{P}^{15}$:

$$[\psi] = [a_0 : \cdots : a_{15}].$$

In this case, there are three Segre varieties that determines the decomposability: $\Sigma_{1,7}, \Sigma_{3,3}$ and $\Sigma_{7,1}$. They are connected with the observables $\mathcal{J}_{4,1}, \mathcal{J}_{4,2}, \mathcal{J}_{4,3}$, respectively. The 3-hypercube (a cube) describes all possible Segre embeddings.

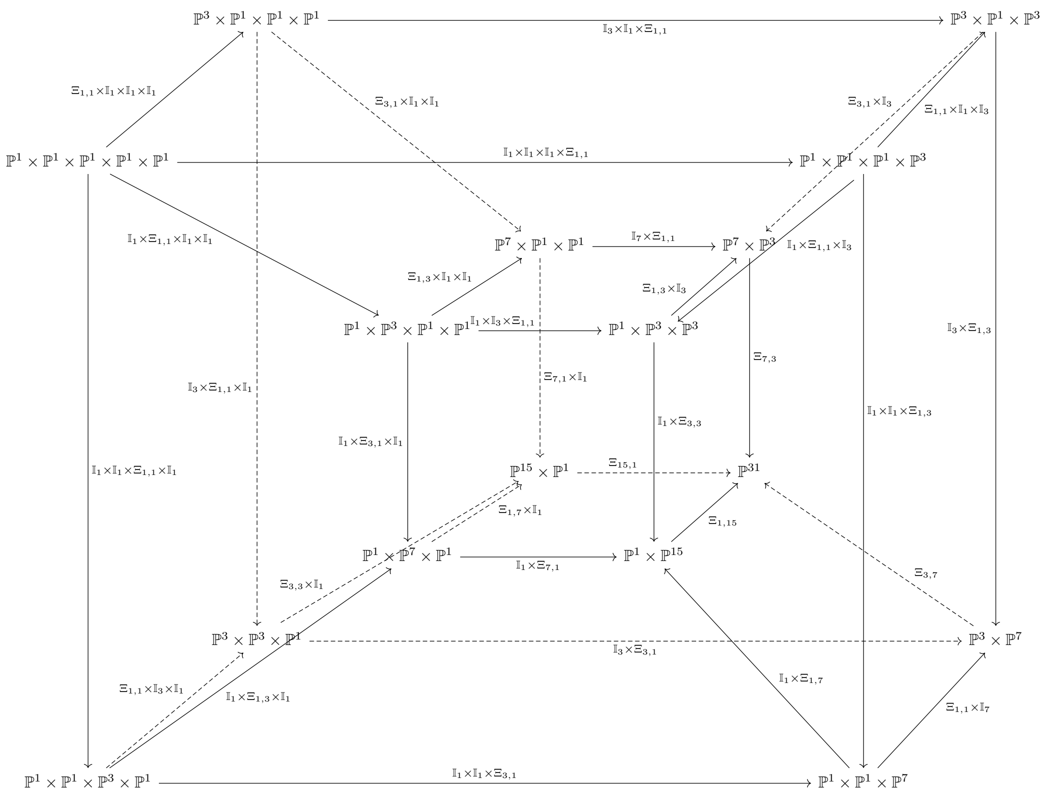
$$\begin{tikzcd}
 & & \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr, "\mathbb{I}_3 \times \Xi_{1,1}"] \arrow[dd, dashed, "\Xi_{3,1} \times \mathbb{I}_1"] & & \mathbb{P}^3 \times \mathbb{P}^3 \arrow[dd, "\Xi_{3,3}"] \\
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[urr, "\Xi_{1,1} \times \mathbb{I}_1 \times \mathbb{I}_1"] \arrow[rrr, "\mathbb{I}_1 \times \mathbb{I}_1 \times \Xi_{1,1}"] \arrow[dd, "\mathbb{I}_1 \times \Xi_{1,1} \times \mathbb{I}_1"'] & & & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \arrow[ur, "\Xi_{1,1} \times \mathbb{I}_3"] \arrow[dd, "\mathbb{I}_1 \times \Xi_{1,3}"] & \\
 & & \mathbb{P}^7 \times \mathbb{P}^1 \arrow[rr, dashed, "\Xi_{7,1}"] & & \mathbb{P}^{15} \\
\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^1 \arrow[urr, dashed, "\Xi_{1,3} \times \mathbb{I}_1"] \arrow[rrr, "\mathbb{I}_1 \times \Xi_{3,1}"] & & & \mathbb{P}^1 \times \mathbb{P}^7 \arrow[ur, "\Xi_{1,7}"'] &
\end{tikzcd}$$

Four-particle states	$\mathcal{J}_{4,1}$	$\mathcal{J}_{4,2}$	$\mathcal{J}_{4,3}$	$\mathcal{J}_{4}$
$[D_{4,1}] = [0:1:1:0:1:0:0:0:1:0:0:0:0:0:0:0]$	3/4	1	3/4	5/6
$[D_{4,2}] = [0:0:0:1:0:1:1:0:0:1:1:0:1:0:0:0]$	1	1	1	1
$[D_{4,3}] = [0:0:0:0:0:0:0:1:0:0:0:1:0:1:1:0]$	3/4	1	3/4	5/6
$[\zeta^0] = [1:0:0:-1:0:-1:1:0:0:0:0:0:0:0:0:0]$	0	1	1	2/3
$[\zeta^1] = [0:0:0:0:0:0:0:0:0:1:1:0:1:0:0:1]$	0	1	1	2/3

Table 3.3: Measurement of entanglement of some 4-particle states.

Five-particle states

Their corresponding projective points are from $\mathbb{P}^{31}$. The bipartite-type Segre varieties we must consider here are $\Sigma_{1,15}, \Sigma_{3,7}, \Sigma_{7,3}$ and $\Sigma_{15,1}$, or the respective observables $\mathcal{J}_{5,1} \ldots, \mathcal{J}_{5,4}$. We only focus on showing one way of drawing a 4-hypercube in $\mathbb{R}^3$. This four-dimensional space is also known as tesseract.



Chapter 4

Quantum teleportation of one qubit

4.1 Problem statement

For explaining quantum teleportation (QT) it is commonly used three people, like in [Pre01], named Alice, Bob and Charlie. One day, Alice and Bob met and prepared one maximally entangled state of two qubits, $|\Phi\rangle$. After that, Alice took one of the particles and Bob grabbed the other one, and then they moved away from each other. For this reason, it is convenient to assign one observer for each particle: we will say that the observer of Alice's particle is labeled with the letter A, meanwhile Bob's observer is B. Therefore, we will represent the maximally entangled state shared between Alice and Bob as $|\Phi\rangle_{AB}$. It is important to remark that, despite the distance between Alice and Bob (which can be from one centimeter to light years), both are still connected by one classical channel[1].

After a certain period of time, Charlie prepares a 1-particle state $|\psi\rangle_C$, i.e. a qubit, where C represents the observer associated to this system. He desires to send his qubit to Bob but he has no communication at all with him. Fortunately, he is able to transfer his qubit to Alice through a preexisting quantum channel, hoping that she might find a way to use the classical channel to relay the qubit to Bob. However, Alice can only send Bob bits of information. Furthermore, she is not able to know Charlie's particle state because if she does any measurement on his qubit, some information about it will be lost. Therefore, how can she manage to teleport one qubit under these conditions?

We will see that Alice may undertake some measurement on the two particles

[1] A classical channel represents a communication medium where classical messages (bits) can be sent from one point to another. With a quantum channel, instead, one can transfer qubits.

she is in possession of. This action, because of the existing entanglement between particles A and B, would change in some way the state of Bob's. The measurement outcome gotten by Alice can be transferred to Bob using bits via the classical channel. Depending on this message, Bob would perform a certain local transformation on his qubit, resulting on an identical state like Charlie's original qubit. For a clearer understanding of this process, refer to the diagram in Figure 1 in the Introduction.

4.2 Using a Bell state

Let us take into account the Bell basis, which is a vector basis of a 4-dimensional Hilbert space generated by the four Bell states $|\phi_0\rangle$, $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$ already introduced in Example 2.3. These are states of two qubits— and all of them have the property that are maximally entangled, i.e. $\mathcal{J}_2(|\phi_i\rangle) = 1\, \forall i = 0, \ldots, 3$.

Taking back to the situation from before, let us suppose that the shared state between Alice and Bob is one of the Bell states, for example

$$|\Phi\rangle_{AB} = |\phi_0\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

At this point, we have two states: $|\Phi\rangle_{AB} \in \mathcal{H}^4_{AB}$ and $|\psi\rangle_C \in \mathcal{H}^2_C$, where the general form of Charlie's qubit is $|\psi\rangle_C = \alpha|0\rangle + \beta|1\rangle$, being α and β complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. We can consider the composite system of the three particles, which its state is $|\Psi\rangle_{CAB} = |\psi\rangle_C \otimes |\Phi\rangle_{AB} \in \mathcal{H}^2_C \otimes \mathcal{H}^4_{AB} = \mathcal{H}^8_{C\otimes AB}$. Let's compute this state:

$$\begin{aligned}
|\Psi\rangle_{CAB} &= \left(\alpha|0\rangle_C + \beta|1\rangle_C\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) \\
&= \frac{1}{\sqrt{2}}\left(\alpha|000\rangle_{CAB} + \alpha|011\rangle_{CAB} + \beta|100\rangle_{CAB} + \beta|111\rangle_{CAB}\right).
\end{aligned}$$

Once Charlie has given Alice his qubit, then she is in possession of both particles A and C. She may perform a measurement on these ones, so it is convenient to write $|\Psi\rangle_{CAB}$ in terms of a basis in $\mathcal{H}^4_{CA} \otimes \mathcal{H}^2_B$. More precisely, she will make a *Bell measurement*: a projection of her two qubits onto one of the four Bell states[2]. For this reason, instead of using the standard basis $\{|000\rangle, |001\rangle, \ldots, |111\rangle\}_{CAB}$, we will change to $\{|\phi_0\rangle|0\rangle, \ldots, |\phi_3\rangle|0\rangle, |\phi_0\rangle|1\rangle, \ldots, |\phi_3\rangle|1\rangle\}_{CAB}$. We can do this since the four Bell states are orthogonal. To achieve this, we must apply the fol-

[2]See Section 4.3 for an explanation of how it is done.

lowing relations:

$$\begin{aligned} |00\rangle &= \frac{1}{\sqrt{2}}\left(|\phi_0\rangle + |\phi_3\rangle\right) & |01\rangle &= \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + |\phi_2\rangle\right) \\ |10\rangle &= \frac{1}{\sqrt{2}}\left(|\phi_1\rangle - |\phi_2\rangle\right) & |11\rangle &= \frac{1}{\sqrt{2}}\left(|\phi_0\rangle - |\phi_3\rangle\right) \end{aligned} \tag{4.1}$$

Therefore, what we have is

$$\begin{aligned} |\Psi\rangle_{BCA} =& \frac{1}{2}\left[|\phi_0\rangle_{CA}\left(\alpha\,|0\rangle_B + \beta\,|1\rangle_B\right) + |\phi_1\rangle_{CA}\left(\beta\,|0\rangle_B + \alpha\,|1\rangle_B\right)\right] \\ &+ \frac{1}{2}\left[|\phi_2\rangle_{CA}\left(-\beta\,|0\rangle_B + \alpha\,|1\rangle_B\right) + |\phi_3\rangle_{CA}\left(\alpha\,|0\rangle_B - \beta\,|1\rangle_B\right)\right]. \end{aligned}$$

Note that if we perform the Pauli transformations (1.9) on $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, the resulting four states are

$$\sigma_0\,|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad \sigma_1\,|\psi\rangle = \begin{pmatrix}\beta\\ \alpha\end{pmatrix}, \quad \sigma_2\,|\psi\rangle = \begin{pmatrix}-i\beta\\ i\alpha\end{pmatrix} \quad \text{and} \quad \sigma_3\,|\psi\rangle = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}.$$

We can apply these results into the last expression of $|\Psi\rangle$ and what we get is

$$\begin{aligned} |\Psi\rangle_{CAB} =& \frac{1}{2}\,|\phi_0\rangle_{CA}\,\sigma_0^B\,|\psi\rangle_B + \frac{1}{2}\,|\phi_1\rangle_{CA}\,\sigma_1^B\,|\psi\rangle_B \\ &+ \frac{1}{2}\,|\phi_2\rangle_{CA}\,(-i\sigma_2^B)\,|\psi\rangle_B + \frac{1}{2}\,|\phi_3\rangle_{CA}\,\sigma_3^B\,|\psi\rangle_B \end{aligned} \tag{4.2}$$

With this final expression, we can see that after Alice projects the state of her two particles $|\widetilde{\Phi}\rangle_{CA}$ onto one of the four Bell states, then Bob's state changes immediately to a state $|\varphi\rangle_B$ depending on her outcome. As described in [YC06], if $|\widetilde{\Phi}\rangle_{CA} \to |\phi_i\rangle_{CA}$ for some $i = 0, \ldots, 3$, then $|\varphi\rangle_B = \dfrac{{}_{CA}\langle\phi_i \mid \Psi\rangle_{CAB}}{\|{}_{CA}\langle\phi_i \mid \Psi\rangle_{CAB}\|}$, so there are four possibilities:

$$\left.\begin{aligned} {}_{CA}\langle\phi_0 \mid \Psi\rangle_{CAB} &= \frac{1}{2}\sigma_0^B\,|\psi\rangle_B \\ {}_{CA}\langle\phi_1 \mid \Psi\rangle_{CAB} &= \frac{1}{2}\sigma_1^B\,|\psi\rangle_B \\ {}_{CA}\langle\phi_2 \mid \Psi\rangle_{CAB} &= \frac{-i}{2}\sigma_2^B\,|\psi\rangle_B \\ {}_{CA}\langle\phi_3 \mid \Psi\rangle_{CAB} &= \frac{1}{2}\sigma_3^B\,|\psi\rangle_B \end{aligned}\right\} \Rightarrow |\varphi\rangle_B = \left\{\begin{array}{rl} \sigma_0^B\,|\psi\rangle_B & \text{if}\quad |\widetilde{\Phi}\rangle_{CA} \to |\phi_0\rangle_{CA} \\ \sigma_1^B\,|\psi\rangle_B & \text{if}\quad |\widetilde{\Phi}\rangle_{CA} \to |\phi_1\rangle_{CA} \\ -i\sigma_2^B\,|\psi\rangle_B & \text{if}\quad |\widetilde{\Phi}\rangle_{CA} \to |\phi_2\rangle_{CA} \\ \sigma_3^B\,|\psi\rangle_B & \text{if}\quad |\widetilde{\Phi}\rangle_{CA} \to |\phi_3\rangle_{CA} \end{array}\right.$$

where $\|{}_{CA}\langle\phi_i \mid \Psi\rangle_{CAB}\| = 1/2$ for all $i = 0, \ldots, 3$. Let $|\varphi_i\rangle_B$ be each of these possible states when $|\widetilde{\Phi}\rangle_{CA} \to |\phi_i\rangle_{CA}$, for each i.

All together, after the Bell measurement performed by Alice, the state of qubits A and C is one Bell state $|\phi_i\rangle_{CA}$, while Bob's state is $|\varphi_i\rangle_B$, for some $i = 0, \ldots, 3$.

Therefore, the global state of the three particles $|\Psi\rangle_{CAB}$ changes to the quantum state $|\widetilde{\Psi}\rangle_{CAB} = |\phi_i\rangle_{CA} |\varphi_i\rangle_B$ with a probability P_i that is computed from (4.2) as

$$P_i := \left| {}_{CAB} \langle \widetilde{\Psi} \mid \Psi \rangle_{CAB} \right|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4}, \quad \forall i = 0, \ldots, 1.$$

Note that all outcomes are equiprobable.

From $|\widetilde{\Psi}\rangle_{CAB}$ we can already extract an interesting aspect: this state is a product of two states $|\phi_i\rangle_{CA}$ and $|\varphi_i\rangle_B$. Therefore, after the measurement performed by Alice, qubits A and B have lost totally the entanglement they had before, becoming two independent qubits. On the other hand, qubits A and C both started being independent states but now they are maximally entangled.

At this point, Bob's qubit has changed to the state $|\varphi\rangle_B = |\varphi_j\rangle_B = e^{i\theta_j}\sigma_j^B |\psi\rangle_B$ for some[3] $j = 0, \ldots, 3$, where $\theta_0 = \theta_1 = \theta_3 = 0$ and $\theta_2 = \frac{3\pi}{2}$. This happened because of the fact that qubits A and B were entangled: if two particles are entangled, a change on one state causes modification on the other one. Note that Bob's qubit is very close to the original state of Charlie's: he only needs to perform the Pauli transformation σ_j to his qubit and this one will be equivalent to $|\psi\rangle$ [4]. The problem, however, is that he is unaware of which of the four possible states his qubit has been transformed into. In particular, he does not even realize that his state has been modified. Despite of that, it is enough for Bob to know what state $|\widetilde{\Phi}\rangle_{CA}$ Alice has obtained:

$$\text{if } |\widetilde{\Phi}\rangle_{CA} = |\phi_j\rangle_{CA} \implies |\varphi\rangle_B = |\varphi_j\rangle_B \quad \text{and} \quad \sigma_j^B |\varphi_j\rangle_B = e^{i\theta_j} |\psi\rangle_B . \tag{4.3}$$

Fortunately, thanks to the existing classical channel, Alice is able to send her outcome to Bob via a classical message: as there are only four possibilities, it is only needed two bits for codifying Alice's measurement. First of all, it is required that Alice and Bob had agreed (before QT) what code to use. For example, they could set up the following one —which is the one that we will always consider in this chapter:

$$''00'' \leftrightarrow |\phi_0\rangle, \; ''01'' \leftrightarrow |\phi_1\rangle, \; ''10'' \leftrightarrow |\phi_2\rangle, \; ''11'' \leftrightarrow |\phi_3\rangle .$$

With all this already explained, let us go back and suppose that, after the Bell measurement, Alice's state is $|\phi_j\rangle_{CA}$ for some $j = 0, 1, 2, 3$. Then, the state of Bob's particle is projected onto $|\varphi_j\rangle_B$. Alice sends a message to Bob with the j-th outcome written in binary. After that, according to (4.3), he performs the Pauli transformation σ_j to his qubit, obtaining a state such that $\sigma_j^B |\varphi_j\rangle_B = e^{i\theta_j} |\psi\rangle_B$. Note

[3]We have changed the index i to j to avoid confusion with the complex number $i = \sqrt{-1}$.

[4]We should recall that $\sigma_i^2 = \mathbb{I}$ for all $i = 0, \ldots, 3$.

that $\sigma_j^B \left|\varphi_j\right\rangle_B \sim \left|\psi\right\rangle_B$. Therefore, at the end of the process Bob is in possession of one quantum state with the same physical properties than $\left|\psi\right\rangle_C$, successfully achieving the objective.

The equivalence between both states from before can be described also in terms of the *fidelity* of the quantum teleportation. This is computed as

$$F = |\left\langle \widetilde{\varphi} \mid \psi \right\rangle|^2, \tag{4.4}$$

where $\left|\widetilde{\varphi}\right\rangle_B$ is Bob's state at the end of the experiment. It is satisfied that

$$F = |\left\langle \widetilde{\varphi} \mid \psi \right\rangle|^2 = 1 \Longleftrightarrow \left|\widetilde{\varphi}\right\rangle = e^{i\theta}\left|\psi\right\rangle \Longleftrightarrow \left|\widetilde{\varphi}\right\rangle \sim \left|\psi\right\rangle. \tag{4.5}$$

The value of the fidelity F also represents the probability of the state $\left|\widetilde{\varphi}\right\rangle$ to be equivalent to $\left|\psi\right\rangle$.

In our case, $\sigma_j^B \left|\varphi_j\right\rangle_B \sim \left|\psi\right\rangle_B$, so effectively

$$F = |\left\langle \varphi_j \mid \sigma_j \mid \psi \right\rangle|^2 = 1.$$

Observations on quantum teleportation

From the theoretical experiment we have already explained, it is possible to get to know some important aspects about QT.

At the beginning of the experiment, we could also use another Bell state instead of $\left|\phi_0\right\rangle$ as the one shared between Alice and Bob. Nevertheless, the algebraic process to follow is always the same. The only thing that might change are the *instructions* Λ that Bob has to follow when applying a transformation on $\left|\varphi\right\rangle_B$. These transformations will be denoted as $\Lambda := \{\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\}$, where $\Lambda_j \in U(2)$, and are used as follow: if he gets the message $''00''$ then he must use Λ_0; if it is $''01''$, then Λ_1, and so on. Therefore, in the case from before, we have that $\Lambda_i = \sigma_i$ for all $i = 0, \ldots, 3$. The instructions corresponding to each Bell state are:

$\left|\boldsymbol{\Phi}\right\rangle_{AB}$	$\boldsymbol{\Lambda_0}$	$\boldsymbol{\Lambda_1}$	$\boldsymbol{\Lambda_2}$	$\boldsymbol{\Lambda_3}$
$\left|\boldsymbol{\phi_0}\right\rangle$	σ_0	σ_1	σ_2	σ_3
$\left|\boldsymbol{\phi_1}\right\rangle$	σ_1	σ_0	σ_3	σ_2
$\left|\boldsymbol{\phi_2}\right\rangle$	σ_2	σ_3	σ_0	σ_1
$\left|\boldsymbol{\phi_3}\right\rangle$	σ_3	σ_2	σ_1	σ_0

Table 4.1: Transformations of the instructions Λ^{ϕ_i}, where $\left|\phi_i\right\rangle$ are the four Bell states.

Secondly, we can already confirm that quantum teleportation always works if the shared state between Alice and Bob is one of Bell's. More generally, it is known that if this state is maximally entangled then the process always goes well, but if it is not one Bell state the algorithm might change.

Moreover, the instructions Λ that Bob follows are always the same even though Charlie's state $|\psi\rangle_C$ is completely arbitrary. Therefore, Λ is fully determined by the shared state $|\Phi\rangle_{AB}$. This makes sense because Alice and Bob remain unaware of Charlie's state at all times, so quantum teleportation must work for any $|\psi\rangle_C$. This is why Λ can not depend on Charlie's qubit. So, we must write Λ^Φ for specifying that the instructions are associated to the state $|\Phi\rangle_{AB}$.

4.3 Performing Bell measurement on two qubits

In the previous section, one may have noticed that one fundamental step for achieving QT is the Bell measurement performed by Alice on her two qubits. Therefore, we will explain how this process works. However, before exploring this part, it is important to be familiar with the concepts introduced in Section 1.3.

Observables involved

Given an arbitrary 2-qubit state $|\psi\rangle_{\mathcal{O}_1\mathcal{O}_2} = a_0 |00\rangle + a_1 |01\rangle + a_2 |10\rangle + a_3 |11\rangle$, the Bell measurement on this one is the projection of $|\psi\rangle$ onto one of the four Bell states $|\phi_0\rangle, \ldots, |\phi_3\rangle$. They constitute a base of maximally entangled states in $\mathcal{H}^4$.

This measurement is given by the following two observables, which are associated to the parity of states (if the total number of qubits at the state $|1\rangle$ is even or odd) and the relative phase between states. In the standard basis $\{|00\rangle, \ldots, |11\rangle\}$ of $\mathcal{H}^4$, these observables are represented by the matrices:

$$\mathcal{A}_1 := \sigma_1^{\mathcal{O}_1} \otimes \sigma_1^{\mathcal{O}_2} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{A}_2 := \sigma_3^{\mathcal{O}_1} \otimes \sigma_3^{\mathcal{O}_2} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Pauli transformations $\sigma_1^{\mathcal{O}_i}$ and $\sigma_3^{\mathcal{O}_i}$, where $i = 1, 2$, act on the Hilbert space of the observer $\mathcal{O}_i$ (see (1.8)). Indeed, $\mathcal{A}_i$ are observables: both are hermitian operators on a 4-dimensional Hilbert space.

In order to project $|\psi\rangle$ onto one of the four Bell states, we must measure the observables $\mathcal{A}_1$ and $\mathcal{A}_2$. Note that these both commute: $\mathcal{A}_1\mathcal{A}_2 = \mathcal{A}_2\mathcal{A}_1$. For this reason, the measurements can be taken simultaneously. In other words, physical properties do not change if we measure first $\mathcal{A}_1$ or $\mathcal{A}_2$ and later the other one.

Measurements of observables are given by the Born rule. Therefore, first of all is convenient to find the spectral decomposition of both observables:

1) The eigenvalues of $\mathcal{A}_1$ are $\lambda_1^{(1)} = 1$ and $\lambda_2^{(1)} = -1$. The eigenvectors of the former live in a 2-dimensional subspace of $\mathcal{H}^4$ generated by the two states $|1,1\rangle = |\phi_0\rangle$ and $|1,2\rangle = |\phi_1\rangle$. Similarly, for the latter eigenstate its corresponding eigenspace is spanned by $|-1,1\rangle = |\phi_2\rangle$ and $|-1,2\rangle = |\phi_3\rangle$. Then the associated projectors are

$$\Pi_1^{(1)} = |\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$\Pi_2^{(1)} = |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

so $\mathcal{A}_1$ can be written as $\mathcal{A}_1 = \Pi_1^{(1)} - \Pi_2^{(1)}$.

2) With regard to $\mathcal{A}_2$, we have $\lambda_1^{(2)} = 1$ and $\lambda_2^{(2)} = -1$ as well. The eigenspace of the former is generated by $|1,1\rangle = |00\rangle$ and $|1,2\rangle = |11\rangle$, meanwhile for the other is spanned by $|-1,1\rangle = |01\rangle$ and $|-1,2\rangle = |10\rangle$. Therefore, what we have at the end is $\mathcal{A}_2 = \Pi_1^{(2)} - \Pi_2^{(2)}$ with

$$\Pi_1^{(2)} = |00\rangle\langle 00| + |11\rangle\langle 11|,$$
$$\Pi_2^{(2)} = |01\rangle\langle 01| + |10\rangle\langle 10|.$$

Note, indeed, that for both observables their eigenvalues are real numbers and their eigenstates form a basis of $\mathcal{H}^4$.

Measurement of the observables

We will first measure $\mathcal{A}_1$ and then $\mathcal{A}_2$ (the reverse process is performed in the same way). According to the Born rule, after measuring the observable $\mathcal{A}_1$ on

$|\psi\rangle$ we obtain as outcome one of its eigenvalues: $\lambda_1^{(1)} = 1$ with probability P_1 or $\lambda_2^{(1)} = -1$ with P_2. We compute these probabilities using (1.5):

$$\begin{aligned}
P_1 &:= P_{|\psi\rangle}(\mathcal{A}_1 : \lambda_1^{(1)}) = \langle \psi \mid \Pi_1^{(1)} \mid \psi \rangle = \frac{1}{2}\begin{pmatrix}\bar{a}_0 & \cdots & \bar{a}_3\end{pmatrix}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}a_0\\ \vdots\\ \\ a_3\end{pmatrix}\\
&= \frac{1}{2}\left(1+\sum_{i=0}^{3} a_i\bar{a}_{3-i}\right),\\
P_2 &:= P_{|\psi\rangle}(\mathcal{A}_1 : \lambda_2^{(1)}) = \langle \psi \mid \Pi_2^{(1)} \mid \psi \rangle\\
&= \frac{1}{2}\begin{pmatrix}\bar{a}_0 & \cdots & \bar{a}_3\end{pmatrix}\begin{pmatrix}1&0&0&-1\\0&1&-1&0\\0&-1&1&0\\-1&0&0&1\end{pmatrix}\begin{pmatrix}a_0\\ \vdots\\ \\ a_3\end{pmatrix} = \frac{1}{2}\left(1-\sum_{i=0}^{3} a_i\bar{a}_{3-i}\right).
\end{aligned}$$

Note that it is satisfied the fact that $P_1 + P_2 = 1$.

a) If we obtain $\lambda_1^{(1)} = 1$, $|\psi\rangle$ has changed to (or has been projected onto) the state

$$|\psi_1\rangle = \frac{\Pi_1^{(1)}|\psi\rangle}{\left\|\Pi_1^{(1)}|\psi\rangle\right\|} = \frac{1}{2\sqrt{P_1}}\begin{pmatrix}a_0+a_3\\a_1+a_2\\a_1+a_2\\a_0+a_3\end{pmatrix} = b_0|00\rangle + b_1|01\rangle + b_1|10\rangle + b_0|11\rangle,$$

where $b_0 = \dfrac{a_0+a_3}{2\sqrt{P_1}}$ and $b_1 = \dfrac{a_1+a_2}{2\sqrt{P_1}}$. We have used that

$$\left\|\Pi_1^{(1)}|\psi\rangle\right\| = \sqrt{\left\|\Pi_1^{(1)}|\psi\rangle\right\|^2} = \sqrt{\langle \psi \mid \Pi_1^{(1)} \mid \psi\rangle} = \sqrt{P_1}.$$

We can check that, in fact, $\| \, |\psi_1\rangle \, \| = 2|b_0|^2 + 2|b_1|^2 = 1$. One may observe that $|\psi_1\rangle = b_0\sqrt{2}\,|\phi_0\rangle + b_1\sqrt{2}\,|\phi_1\rangle$. Indeed, after this measurement, $|\psi\rangle$ has been projected to the subspace spanned by the eigenstates of $\lambda_1^{(1)}$. We may recall that the probability of this projection, $|\psi\rangle \to |\psi_1\rangle$, is the same as P_1.

Hereafter, we measure $\mathcal{A}_2$ on $|\psi_1\rangle$ and we obtain again only one of its eigenvalues with respective probabilities (both computed using bra-ket notation):

$$\begin{aligned} P_{1,1} :&= P_{|\psi_1\rangle}(\mathcal{A}_2 : \lambda_1^{(2)}) = \langle \psi_1 \mid \Pi_1^{(2)} \mid \psi_1 \rangle \\ &= \langle \psi_1 \mid 00 \rangle \langle 00 \mid \psi_1 \rangle + \langle \psi_1 \mid 11 \rangle \langle 11 \mid \psi_1 \rangle \\ &= | \langle 00 \mid \psi_1 \rangle |^2 + | \langle 11 \mid \psi_1 \rangle |^2 = 2|b_0|^2, \\ P_{1,2} :&= P_{|\psi_1\rangle}(\mathcal{A}_2 : \lambda_2^{(2)}) = \langle \psi_1 \mid \Pi_2^{(2)} \mid \psi_1 \rangle \\ &= \langle \psi_1 \mid 01 \rangle \langle 01 \mid \psi_1 \rangle + \langle \psi_1 \mid 10 \rangle \langle 10 \mid \psi_1 \rangle = 2|b_1|^2; \end{aligned}$$

both together satisfying $P_{1,1} + P_{1,2} = 1$.

a.1) If we get $\lambda_1^{(2)} = 1$ with probability $P_{1,1}$, then the state $|\psi_1\rangle$ has become one generated by $|00\rangle$ and $|11\rangle$:

$$\begin{aligned} |\psi_{1,1}\rangle &= \frac{\Pi_1^{(2)} |\psi_1\rangle}{\left\| \Pi_1^{(2)} |\psi_1\rangle \right\|} = \frac{1}{\sqrt{P_{1,1}}}(|00\rangle \langle 00 \mid \psi_1\rangle + |11\rangle \langle 11 \mid \psi_1\rangle) \\ &= \frac{b_0}{\sqrt{P_{1,1}}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\sqrt{\frac{b_0}{b_0^*}}(|00\rangle + |11\rangle) = \frac{e^{i\theta}}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= |\phi_0\rangle \,, \end{aligned}$$

where at the end we have used the polar form $b_0 = |b_0|e^{i\theta}$ for some $\theta \in \mathbb{R}$.

a.2) In turn, if the outcome is $\lambda_2^{(2)} = -1$, then the resulting state must be one state in a superposition of $|01\rangle$ and $|10\rangle$. Effectively,

$$\begin{aligned} |\psi_{1,2}\rangle &= \frac{\Pi_2^{(2)} |\psi_1\rangle}{\left\| \Pi_2^{(2)} |\psi_1\rangle \right\|} = \frac{1}{\sqrt{P_{1,2}}}(|01\rangle \langle 01 \mid \psi_1\rangle + |10\rangle \langle 10 \mid \psi_1\rangle) \\ &= \frac{b_1}{\sqrt{P_{1,2}}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\sqrt{\frac{b_1}{b_1^*}}(|01\rangle + |10\rangle) = |\phi_1\rangle \,. \end{aligned}$$

b) On the other hand, if we get the other possible outcome after measuring $\mathcal{A}_1$, $\lambda_2^{(1)} = -1$, then $|\psi\rangle$ has been projected onto

$$|\psi_2\rangle = \frac{\Pi_2^{(1)} |\psi\rangle}{\left\| \Pi_2^{(1)} |\psi\rangle \right\|} = \frac{1}{2\sqrt{P_2}} \begin{pmatrix} a_0 - a_3 \\ a_1 - a_2 \\ a_2 - a_1 \\ a_3 - a_0 \end{pmatrix} = c_0 \,|00\rangle + c_1 \,|01\rangle - c_1 \,|10\rangle - c_0 \,|11\rangle \,,$$

where $c_0 = \dfrac{a_0 - a_3}{2\sqrt{P_2}}$ and $c_1 = \dfrac{a_1 - a_2}{2\sqrt{P_2}}$. Again is satisfied that $\| \, |\psi_2\rangle \, \| = 1$. We can see that $|\psi\rangle \to c_0\sqrt{2}\,|\phi_3\rangle + c_1\sqrt{2}\,|\phi_2\rangle$ as we could expect. The outcome probabilities of measuring $\mathcal{A}_2$ on $|\psi_2\rangle$ are

$$\begin{aligned} P_{2,1} :&= P_{|\psi_2\rangle}(\mathcal{A}_2 : \lambda_1^{(2)}) = \langle \psi_2 \mid 00\rangle \langle 00 \mid \psi_2\rangle + \langle \psi_2 \mid 11\rangle \langle 11 \mid \psi_2\rangle = 2|c_0|^2, \\ P_{2,2} :&= P_{|\psi_2\rangle}(\mathcal{A}_2 : \lambda_2^{(2)}) = \langle \psi_2 \mid 01\rangle \langle 01 \mid \psi_2\rangle + \langle \psi_1 \mid 10\rangle \langle 10 \mid \psi_1\rangle = 2|c_1|^2. \end{aligned}$$

It is easy to see that $P_{2,1} + P_{2,2} = 1$.

b.1) If we obtain $\lambda_1^{(2)} = 1$, then we have

$$|\psi_{2,1}\rangle = \frac{\Pi_1^{(2)}\,|\psi_2\rangle}{\left\|\Pi_1^{(2)}\,|\psi_2\rangle\right\|} = \frac{1}{\sqrt{P_{2,1}}}(|00\rangle \langle 00 \mid \psi_2\rangle + |11\rangle \langle 11 \mid \psi_2\rangle) = |\phi_3\rangle\,;$$

b.2) or if it is $\lambda_2^{(2)} = -1$, the state we get is:

$$|\psi_{2,2}\rangle = \frac{\Pi_2^{(2)}\,|\psi_1\rangle}{\left\|\Pi_2^{(2)}\,|\psi_1\rangle\right\|} = \frac{1}{\sqrt{P_{1,2}}}(|01\rangle \langle 01 \mid \psi_2\rangle + |10\rangle \langle 10 \mid \psi_2\rangle) = |\phi_2\rangle\,.$$

In conclusion, after measuring the observables $\mathcal{A}_1$ and $\mathcal{A}_2$ on the state of two qubits $|\psi\rangle$, i.e. performing a Bell measurement, this can end up being one of the four Bell states:

$$\begin{aligned} |\psi\rangle &\to |\phi_0\rangle \quad \text{with probability} \quad P(|\psi\rangle \to |\phi_0\rangle) = P_1P_{1,1} = \frac{|a_0 + a_3|^2}{2}, \\ |\psi\rangle &\to |\phi_1\rangle \quad \text{with probability} \quad P(|\psi\rangle \to |\phi_1\rangle) = P_1P_{1,2} = \frac{|a_1 + a_2|^2}{2}, \\ |\psi\rangle &\to |\phi_2\rangle \quad \text{with probability} \quad P(|\psi\rangle \to |\phi_2\rangle) = P_2P_{2,2} = \frac{|a_1 - a_2|^2}{2}, \\ |\psi\rangle &\to |\phi_3\rangle \quad \text{with probability} \quad P(|\psi\rangle \to |\phi_3\rangle) = P_2P_{2,1} = \frac{|a_0 - a_3|^2}{2}. \end{aligned}$$

Another way of understanding Bell measurement

The process just explained for projecting our original state to one Bell state illustrates how this is achieved experimentally: by obtaining specific measurement values, we can determine with certainty the resulting state of our physical system.

Fortunately, from a theoretical perspective, it is not necessary to follow all those steps. In fact, as Bell states form a base of $\mathcal{H}^4$, then every 2-qubit state can be written in terms of these states. Using the same generic state $|\psi\rangle$ as before and the relations (4.1), this one can be written as

$$|\psi\rangle = \frac{a_0 + a_3}{\sqrt{2}} |\phi_0\rangle + \frac{a_1 + a_2}{\sqrt{2}} |\phi_1\rangle + \frac{a_1 - a_2}{\sqrt{2}} |\phi_2\rangle + \frac{a_0 - a_3}{\sqrt{2}} |\phi_3\rangle .$$

This can be interpreted as $|\psi\rangle$ being in a superposition of the four Bell states. After Bell measurement, our state will be projected onto one of the Bell states $|\phi_i\rangle$ with probability

$$P(|\psi\rangle \to |\phi_i\rangle) = |\langle\phi_i \mid \psi\rangle|^2, \quad \forall i = 0, \ldots, 3.$$

Note that these probabilities align with those computed before. Therefore, using this approach is entirely correct and much easier to compute. In fact, this is the method applied in Section 4.2.

4.4 Using an arbitrary two-qubit state

In this last section, we will consider performing quantum teleportation with an arbitrary two-particle state, which could be both product or entangled then. First of all, we will develop the algebra of Section 4.2 but with a random shared state and we will discuss its consequences. After that, the work we have done in our Physics Final Degree Project [LSC25] will be briefly introduced, as well as a short explanation about the results obtained there.

4.4.1 Generalization of quantum teleportation

Let $|\Phi\rangle_{AB} = a_0 |00\rangle_{AB} + a_1 |01\rangle_{AB} + a_2 |10\rangle_{AB} + a_3 |11\rangle_{AB}$ represent an arbitrary 2-qubit state, and let $|\psi\rangle_C = \alpha |0\rangle_C + \beta |1\rangle_C$ be a random qubit. From now on, we will avoid using the observers' notation, while carefully preserving the order of the basis in $\mathcal{H}^8_{CAB}$. The state of the composite system is:

$$\begin{aligned}
|\Psi\rangle = |\psi\rangle |\Phi\rangle &= \alpha a_0 |000\rangle + \alpha a_1 |001\rangle + \alpha a_2 |010\rangle + \alpha a_3 |011\rangle \\
&\quad + \beta a_0 |100\rangle + \beta a_1 |101\rangle + \beta a_2 |110\rangle + \beta a_3 |111\rangle \\
&\overset{(4.1)}{=} \frac{1}{\sqrt{2}} |\phi_0\rangle \left[(\alpha a_0 + \beta a_2) |0\rangle + (\alpha a_1 + \beta a_3) |1\rangle\right] \\
&\quad + \frac{1}{\sqrt{2}} |\phi_1\rangle \left[(\alpha a_2 + \beta a_0) |0\rangle + (\alpha a_3 + \beta a_1) |1\rangle\right] \\
&\quad + \frac{1}{\sqrt{2}} |\phi_2\rangle \left[(\alpha a_2 - \beta a_0) |0\rangle + (\alpha a_3 - \beta a_1) |1\rangle\right] \\
&\quad + \frac{1}{\sqrt{2}} |\phi_3\rangle \left[(\alpha a_0 - \beta a_2) |0\rangle + (\alpha a_1 - \beta a_3) |1\rangle\right]
\end{aligned} \tag{4.6}$$

After the Bell measurement is performed, the state $|\Psi\rangle$ is projected onto one state of the form $|\widetilde{\Phi}\rangle \otimes |\varphi\rangle$, where $|\widetilde{\Phi}\rangle = |\phi_i\rangle$ and $|\varphi\rangle = |\varphi_i\rangle = \dfrac{\langle \phi_i \mid \Psi\rangle}{\| \langle \phi_i \mid \Psi\rangle \|}$ for some $i = 0, \dots, 3$. Thus, the possible states $|\varphi_i\rangle$ are

$$\begin{aligned}
|\varphi_0\rangle &= \frac{1}{\sqrt{2\mathcal{C}_0}} \begin{pmatrix} \alpha a_0 + \beta a_2 \\ \alpha a_1 + \beta a_3 \end{pmatrix} = \frac{1}{\sqrt{2\mathcal{C}_0}} \begin{pmatrix} a_0 & a_2 \\ a_1 & a_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_0 |\psi\rangle \\
|\varphi_1\rangle &= \frac{1}{\sqrt{2\mathcal{C}_1}} \begin{pmatrix} \alpha a_2 + \beta a_0 \\ \alpha a_3 + \beta a_1 \end{pmatrix} = \frac{1}{\sqrt{2\mathcal{C}_1}} \begin{pmatrix} a_2 & a_0 \\ a_3 & a_1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_1 |\psi\rangle \\
|\varphi_2\rangle &= \frac{1}{\sqrt{2\mathcal{C}_2}} \begin{pmatrix} \alpha a_2 - \beta a_0 \\ \alpha a_3 - \beta a_1 \end{pmatrix} = \frac{1}{\sqrt{2\mathcal{C}_2}} \begin{pmatrix} a_2 & -a_0 \\ a_3 & -a_1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_2 |\psi\rangle \\
|\varphi_3\rangle &= \frac{1}{\sqrt{2\mathcal{C}_3}} \begin{pmatrix} \alpha a_0 - \beta a_2 \\ \alpha a_1 - \beta a_3 \end{pmatrix} = \frac{1}{\sqrt{2\mathcal{C}_3}} \begin{pmatrix} a_0 & -a_2 \\ a_1 & -a_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_3 |\psi\rangle
\end{aligned} \tag{4.7}$$

where $\mathcal{C}_i = \| \langle \phi_i \mid \Psi\rangle \|^2$ and $A_i \in \mathcal{M}_2(\mathbb{C})$, which can not be null. The probability of $|\Psi\rangle \to |\phi_i\rangle |\varphi_i\rangle$ is given by

$$P_i = |(\langle \phi_i| \langle \varphi_i|) |\Psi\rangle|^2 = \left| \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\mathcal{C}_i}} 2\mathcal{C}_i \right|^2 = \mathcal{C}_i.$$

Then, these probabilities are of the form

$$\begin{aligned}
P_0 &= \frac{1}{2} \left(|\alpha a_0 + \beta a_2|^2 + |\alpha a_1 + \beta a_3|^2 \right) & P_1 &= \frac{1}{2} \left(|\alpha a_2 + \beta a_0|^2 + |\alpha a_3 + \beta a_1|^2 \right) \\
P_2 &= \frac{1}{2} \left(|\alpha a_2 - \beta a_0|^2 + |\alpha a_3 - \beta a_1|^2 \right) & P_3 &= \frac{1}{2} \left(|\alpha a_0 - \beta a_2|^2 + |\alpha a_1 - \beta a_3|^2 \right).
\end{aligned} \tag{4.8}$$

Note that at least one of them may be zero and we could not calculate the corresponding state $|\varphi\rangle$. Nevertheless, when $P_i = 0$ the state $|\varphi_i\rangle$ does not exist, so there is no need for considering it. In this context, let us define the set

$$\mathrm{Y} := \{ i \mid P_i \neq 0, \text{ where } 0 \le i \le 3 \}.$$

Afterwards, only one transformation must be applied on the state $|\varphi\rangle$, which depends on $|\widetilde{\Phi}\rangle$, for completing the process. This transformation is a 2×2 unitary matrix $\mathcal{U}$ of the form (1.10). Suppose that $|\varphi\rangle = |\varphi_j\rangle$ for some $j \in \mathrm{Y}$. Then the experiment is successfully finished if and only if there exists a matrix $\mathcal{U}_j \in U(2)$ such that the fidelity is $F = | \langle \varphi_j \mid \mathcal{U}_j^\dagger \mid \psi\rangle |^2 = 1$ (see (4.5)). With this in mind,

$$\begin{aligned}
F = 1 &\Leftrightarrow \mathcal{U}_j |\varphi_j\rangle \propto |\psi\rangle \Leftrightarrow e^{i\omega} \mathcal{U}_j |\varphi_j\rangle = |\psi\rangle \Leftrightarrow e^{i\omega} \mathcal{U}_j A_j |\psi\rangle = |\psi\rangle \\
&\Leftrightarrow \mathcal{B}_j A_j |\psi\rangle = |\psi\rangle \Leftrightarrow \left(\mathcal{B}_j A_j - \mathbb{I} \right) |\psi\rangle = 0,
\end{aligned} \tag{4.9}$$

where ω is some real constant and A_j is one of the matrices from above. Note that $\mathcal{B}_j = e^{i\omega}\mathcal{U}_j$ is a unitary matrix as well, and therefore $\mathcal{B}_j^{-1} = \mathcal{B}_j^{\dagger}$. However, it is a complex problem to deduce from the last expression what conditions must satisfy the matrix A_j or what is the form of $\mathcal{B}_j$. But it is clear that if the latter is known, then we can choose $\mathcal{U}_j = \mathcal{B}_j$, which is determined up to a global phase. Indeed, let be $\mathcal{B}_j \in U(2)$ s.t. $\left(\mathcal{B}_j A_j - \mathbb{I}\right)|\psi\rangle = 0$. If we suppose that Bob performs the transformation $\mathcal{U}_j = e^{i\mu}\mathcal{B}_j$ for some $\mu \in \mathbb{R}$, then we have

$$\left(e^{-i\mu}\mathcal{U}_j A_j - \mathbb{I}\right)|\psi\rangle = 0 \Rightarrow e^{-i\mu}\mathcal{U}_j A_j |\psi\rangle = |\psi\rangle \Rightarrow e^{-i\mu}\mathcal{U}_j |\varphi_j\rangle = |\psi\rangle$$
$$\Rightarrow \mathcal{U}_j |\varphi_j\rangle = e^{i\mu}|\psi\rangle \propto |\psi\rangle \Rightarrow F = |\langle \varphi \mid \mathcal{U}_j^{\dagger} \mid \psi\rangle|^2 = 1.$$

On the other hand, in order to ensure that QT of $|\psi\rangle$ works perfectly, we must contemplate each possible Alice's outcome. Let us consider the fidelity defined as $F_j^{\psi} := |\langle\widetilde{\varphi}_j|\psi\rangle|^2$ for each $j \in \mathrm{Y}$, where $|\widetilde{\varphi}_j\rangle$ is Bob's state at the end of the experiment when Alice's outcome is j. The teleportation of $|\psi\rangle$ would be performed successfully if and only if $F_j^{\psi} = 1$ for each $j \in \mathrm{Y}$. This is equivalent to the existence of $\mathcal{U}_j \in U(2)$ such that (4.9) is satisfied for each $j \in \mathrm{Y}$, where $|\widetilde{\varphi}_j\rangle = \mathcal{U}_j|\varphi_j\rangle$. In conclusion, we can stablish the following criteria:

$$|\Phi\rangle \text{ is } \textit{perfect for teleporting } |\psi\rangle \overset{def}{\iff} \exists \mathcal{U}_j \in U(2) \text{ s.t. } \left(\mathcal{U}_j A_j - \mathbb{I}\right)|\psi\rangle = 0,\ \forall j \in \mathrm{Y}.$$

In this case, the instructions Bob must follow are $\Lambda_j = \mathcal{U}_j$ for each $j \in \mathrm{Y}$, which are defined up to a global phase.

Example 4.1. The following state is perfect for teleporting $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$:

$$|\Phi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Indeed, from (4.6) we get $|\Psi\rangle = \frac{1}{\sqrt{2}}|\phi_0\rangle|\psi\rangle + \frac{1}{\sqrt{2}}|\phi_1\rangle|\psi\rangle$. Therefore, the probability outcomes defined in (4.8) are $P_0 = P_1 = \frac{1}{2}$ and $P_2 = P_3 = 0$. We must find two matrices $\mathcal{U}_0, \mathcal{U}_1 \in U(2)$ such that

$$\left[\mathcal{U}_0 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \mathbb{I}\right] \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \left[\mathcal{U}_1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \mathbb{I}\right] \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which are the same matrix equation. It is easy to see that $\mathcal{U}_0 = \mathcal{U}_1 = \mathbb{I}$ satisfy both relations. Thus, $|\Phi\rangle$ is perfect for teleporting the qubit $|\psi\rangle$ with the instructions $\Lambda_0^{\Phi} = \Lambda_1^{\Phi} = \mathbb{I}$, while the other two can be any. This result makes sense actually. Note that $|\Phi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$, so the shared state is product and Bob posses the state $|\psi\rangle_B$, which is the same state as Charlie's, $|\psi\rangle_C$. Because of $|\Phi\rangle$ being product, then the measurement performed by Alice does not alter in any way Bob's state. Therefore, $|\varphi_0\rangle_B = |\varphi_1\rangle_B = |\psi\rangle_B$, so Bob does not need to apply any transformation on his state as he already have Charlie's state $|\psi\rangle$.

An observation about this example is that it may not be considered a result of quantum teleportation. The reason is because the measurement of Alice and the message of her outcome are irrelevant for the experiment to be successful. In fact, Bob could apply just at the beginning the given instructions on his state and would obtain the same state as Charlie's.

What is required in quantum teleportation is that the shared state $|\Phi\rangle$ is perfect for teleporting any qubit. Moreover, as the qubit to be teleported theoretically is not known, then the instruction Λ^{Φ} that Bob must follow cannot depend on this one, as we already discussed in Section 4.2. In summary, we will say that:

Definition 4.2. A 2-qubit state $|\Phi\rangle$ is *perfect for quantum teleportation (PQT)* if there exists a set of unitary matrices $\Lambda^{\Phi} = \{\Lambda_0^{\Phi}, \Lambda_1^{\Phi}, \Lambda_2^{\Phi}, \Lambda_3^{\Phi}\} \subset U(2)$ (defined up to a global phase) such that for all $|\psi\rangle \in \mathcal{H}^2$ is satisfied $\left(\Lambda_j^{\Phi} A_j(|\psi\rangle) - \mathbb{I}\right)|\psi\rangle = 0$, for each $j \in \Upsilon_{\psi}$. The matrices $A_j(|\psi\rangle)$ are the ones from (4.7).

Nevertheless, the matrices $A_j(|\psi\rangle)$ can depend on $|\psi\rangle$ because of the probabilities P_i. For this reason, if we do not first impose some conditions on the matrices A_j, finding the instructions that satisfy the equations $\left(\Lambda_j^{\Phi} A_j - \mathbb{I}\right)|\psi\rangle = 0$ can be quite challenging.

Some conditions under which a two-qubit state qualifies as PQT

Let us suppose that we desire to teleport a qubit $|\psi\rangle$ with the shared state $|\Phi\rangle$ such that $A_j(|\psi\rangle) \in U(2)$, for each $j \in \Upsilon$. In this case, we can choose $\mathcal{U}_j = A_j^{\dagger}$, which is also unitary, so it is satisfied $\left(A_j^{\dagger} A_j - \mathbb{I}\right) = 0$. Therefore, $|\Phi\rangle$ is perfect for teleporting $|\psi\rangle$ and $\Lambda_j = A_j^{\dagger}$. It can be seen that

$$A_j(|\psi\rangle) \in U(2) \Longleftrightarrow A_j A_j^{\dagger} = \mathbb{I} \Longleftrightarrow \begin{cases} |a_0|^2 + |a_2|^2 = |a_1|^2 + |a_3|^2 = 1/2 \\ a_0\bar{a}_1 = -a_2\bar{a}_3 \quad \text{and} \quad \bar{a}_0 a_1 = -\bar{a}_2 a_3 \\ P_j = 1/4 \end{cases}$$

or, on the other hand,

$$A_j(|\psi\rangle) \in U(2) \Longleftrightarrow A_j^{\dagger} A_j = \mathbb{I} \Longleftrightarrow \begin{cases} |a_0|^2 + |a_1|^2 = |a_2|^2 + |a_3|^2 = 1/2 \\ a_0\bar{a}_2 = -a_1\bar{a}_3 \quad \text{and} \quad \bar{a}_0 a_2 = -\bar{a}_1 a_3 \\ P_j = 1/4 \end{cases}$$

Considering the last relations between the coefficients of $|\Phi\rangle$ and implementing them onto (4.8), we obtain $P_j = 1/4$ which does not depend on $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Thus, we get the following simplified equivalence for each $j \in \mathrm{Y}$:

$$A_j \in U(2) \iff \begin{cases} |a_0|^2 + |a_1|^2 = |a_2|^2 + |a_3|^2 = 1/2 \\ a_0\bar{a}_2 = -a_1\bar{a}_3 \quad \text{and} \quad \bar{a}_0 a_2 = -\bar{a}_1 a_3 \end{cases} \tag{4.10}$$

Let us analyze the consequences of this last result. If the coefficients of the state $|\Phi\rangle$ satisfy these conditions, then $P_j = 1/4$ for all $\alpha, \beta \in \mathbb{C}$ and $A_j \in U(2)$. On the one hand, the sum over all P_j must be one, so $\mathrm{Y} = \{0,1,2,3\}$ for any qubit to be teleported. On the other hand, from (4.7) one may notice that the matrices A_j do not depend on α and β because of the probabilities $P_j \equiv 1/4$. Furthermore, we have observed just above that we can choose $\Lambda_j^\Phi = A_j^\dagger$ and we get $\left(A_j^\dagger A_j - \mathbb{I}\right)|\psi\rangle = 0$ for all $|\psi\rangle \in \mathcal{H}^2$. Therefore, the state $|\Phi\rangle$ is PQT.

It could be appropriate to define from (4.10) the following subset of 2-qubit states in $\mathcal{H}^4$:

$$\mathfrak{A} := \left\{ a_0\,|00\rangle + a_1\,|01\rangle + a_2\,|10\rangle + a_3\,|11\rangle \in \mathcal{H}^4 \;\middle|\; \begin{matrix} |a_0|^2 + |a_1|^2 = |a_2|^2 + |a_3|^2 = \frac{1}{2} \\ a_0\bar{a}_2 = -a_1\bar{a}_3 \wedge \bar{a}_0 a_2 = -\bar{a}_1 a_3 \end{matrix} \right\} \tag{4.11}$$

All in all, we can already confirm the next result:

Proposition 4.3. *Let $|\Phi\rangle = a_0\,|00\rangle + a_1\,|01\rangle + a_2\,|10\rangle + a_3\,|11\rangle$ be a 2-qubit state.*

$$\textit{If} \quad |\Phi\rangle \in \mathfrak{A} \implies |\Phi\rangle \textit{ is PQT.}$$

In this case, the instructions Λ^Φ are of the form (up to a global phase)

$$\Lambda_0^\Phi = \sqrt{2}\begin{pmatrix} \bar{a}_0 & \bar{a}_1 \\ \bar{a}_2 & \bar{a}_3 \end{pmatrix} \qquad \Lambda_1^\Phi = \sqrt{2}\begin{pmatrix} \bar{a}_2 & \bar{a}_3 \\ \bar{a}_0 & \bar{a}_1 \end{pmatrix}$$

$$\Lambda_2^\Phi = \sqrt{2}\begin{pmatrix} \bar{a}_2 & \bar{a}_3 \\ -\bar{a}_0 & -\bar{a}_1 \end{pmatrix} \qquad \Lambda_3^\Phi = \sqrt{2}\begin{pmatrix} \bar{a}_0 & \bar{a}_1 \\ -\bar{a}_2 & -\bar{a}_3 \end{pmatrix}$$

and the outcomes of the Bell measurement are all equiprobable, i.e. $P_j = 1/4$ for all $j = 0,1,2,3$ and for any teleported state $|\psi\rangle \in \mathcal{H}^2$.

Besides that, it is possible to establish a connection between these conditions and the observable $\mathcal{J}_2$ from (3.6):

Proposition 4.4. *Let $|\Phi\rangle = a_0\,|00\rangle + a_1\,|01\rangle + a_2\,|10\rangle + a_3\,|11\rangle$ be a 2-qubit state.*

$$\textit{If} \quad |\Phi\rangle \in \mathfrak{A} \implies \mathcal{J}_2(|\Phi\rangle) = 1.$$

Proof. From (3.4) we have that $\mathcal{J}_2(|\Phi\rangle) = \mathcal{J}_{2,1}(|\Phi\rangle) = 4|a_0a_3 - a_1a_2|^2$. Because of the hypothesis, from (4.10) we know that $A_j \in U(2)$ for all $j = 0,1,2,3$. So these matrices satisfy $|\det A_j| = 1$. Considering the Proposition 4.3, then the outcome probabilities are $P_j = 1/4$ for all j, so from the determinants of the matrices (4.7) we get

$$|\det A_j| = 2|a_0a_3 - a_1a_2| = 1 \Longrightarrow 4|a_0a_3 - a_1a_2|^2 = 1.$$

Therefore, $\mathcal{J}_2(|\Phi\rangle) = 1$, as we wanted to see. □

Example 4.5. The 2-qubit states $|\phi_i\rangle$ and $|\phi_{0,2}(\theta)\rangle = \cos\theta\,|\phi_0\rangle + \sin\theta\,|\phi_2\rangle$ belong to the set $\mathfrak{A}$ defined above. Therefore, according to Propositions 4.3 and 4.4, we already know both states are PQT and maximally entangled. Effectively, we can observe in Table 3.1 that $\mathcal{J}_2(\phi_i) = \mathcal{J}_2(|\phi_{0,2}(\theta)\rangle) = 1$, for all $i = 0,1,2,3$ and for all $\theta \in [0,\pi)$.

For the Bell states $|\phi_i\rangle$, we already know the corresponding instructions (see Table 4.2). Besides, Proposition 4.3 provides as well the matrices of Λ^{ϕ_i}. For instance, for the state $|\phi_0\rangle$ we can see that these only differ from the Pauli matrices (1.9) in a global phase:

$$\Lambda_0^{\phi_0} = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \sigma_0 \qquad \Lambda_1^{\phi_0} = \sqrt{2}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \sigma_1$$

$$\Lambda_2^{\phi_0} = \sqrt{2}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} = e^{i\frac{2\pi}{3}}\sigma_2 \qquad \Lambda_3^{\phi_0} = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \sigma_3$$

For the other states, which are of the form

$$|\phi_{0,2}(\theta)\rangle = \frac{1}{\sqrt{2}}\left(\cos\theta\,|00\rangle + \sin\theta\,|01\rangle - \sin\theta\,|10\rangle + \cos\theta\,|11\rangle\right),$$

the set $\Lambda^{\phi_{0,2}(\theta)}$ is constituted by

$$\Lambda_0^{\phi_{0,2}(\theta)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \Lambda_1^{\phi_{0,2}(\theta)} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

$$\Lambda_2^{\phi_{0,2}(\theta)} = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} \qquad \Lambda_3^{\phi_{0,2}(\theta)} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Physical considerations about the conditions from $\mathfrak{A}$

We have already shown that for any two-qubit state $|\Phi\rangle$ we have the following result:

$$\text{If } \ |\Phi\rangle \in \mathfrak{A} \implies |\Phi\rangle \text{ is PQT and } \mathcal{J}_2(|\Phi\rangle) = 1,$$

where $\mathfrak{A}$ is the subset of states in $\mathcal{H}^4$ defined as (4.11).

One may wonder if there exists a direct connection between PQT states and states with maximum value of the observable $\mathcal{J}_2$ (which theoretically are maximally entangled).

On the one hand, if we manage to find the unitary matrices $\mathcal{U}_j$ such that $\left(\mathcal{U}_j A_j - \mathbb{I}\right)|\psi\rangle = 0$ for all $|\psi\rangle \in \mathcal{H}^2$ and for each $j \in \Upsilon_\psi$, then we might settle the exact conditions —if they exist— for which the state $|\Phi\rangle$ is PQT. Afterwards, it could be possible to proof that if $|\Phi\rangle$ satisfies these hypothetical properties, then $\mathcal{J}_2(|\Phi\rangle) = 1$. This is because, experimentally, it is well known that one of the requirements for a state to be PQT is that it must be maximally entangled: when two particles are maximally entangled then there exists a perfect correlation between the measurements performed on one particle or the other.

On the other hand, it may be a bit fuzzy that $\mathcal{J}_2(|\Phi\rangle) = 1$ implies $|\Phi\rangle$ to be PQT. It is important to emphasize that the Definition 4.2 is based on the fact that Alice always performs a Bell measurement. However, it could be possible that for carrying out QT with a specific 2-particle maximally entangled state, the measurement must be different and related to this state. Therefore, we could have one state such that $\mathcal{J}_2(|\Phi\rangle) = 1$ and not being PQT according to the definition from above.

From the results we have already obtained, we can confirm that 2-qubit states satisfying the conditions of the set $\mathfrak{A}$ are perfect for carrying out quantum teleportation of any qubit. Furthermore, the instructions Bob must follow are perfectly known and they are fully determined, up to a global phase, by the shared state involved. Theoretically, these states must be maximally entangled. Indeed, besides being PQT, we have seen that the observable $\mathcal{J}_2$ acting on these type of states returns a maximum value, in agreement with the theory developed in Section 3.3. This strengthens the fact that the observables $\mathcal{J}_n$ defined in [CST21] are actually a good measurement of entanglement, in particular for 2-qubit states.

Nevertheless, there may exist PQT states which do not belong to the subset $\mathfrak{A} \subset \mathcal{H}^4$. For these ones, it has not been possible to find the general form of the instructions Λ. Even more, there could be maximally entangled states that are not PQT (that is, it is required another measurement different from the Bell one). In addition, 2-qubit states with $\mathcal{J}_2 < 1$ are not from $\mathfrak{A}$, so it is unclear if they are PQT or not. Despite that, the fact that the observable on these states is not maximal suggests that they are not actually maximally entangled states. Therefore, they must not be capable of teleporting any qubit with any kind of measurement.

For all the states just described, which are the ones that do not lie in the set $\mathfrak{A}$ from (4.11), then it is unclear if they are PQT. In addition, if some of them are actually PQT, then it may be complex to find their corresponding instructions. All in all, in these cases it may be interesting to search for the instructions which provide better teleportation results under Bell measurement (i.e. higher fidelity values). In our physics work we have provided a method for achieving this objective.

4.4.2 Optimizing instructions for quantum teleportation

Let us explain briefly the problem statement, the implemented algorithms and the results obtained in our Bachelor's Thesis in Physics [LSC25].

Let $|\Phi\rangle$ represent an arbitrary two-particle state and let us suppose that the instructions for doing QT with it are unknown. The thing we do know is that the general form of Λ_j^Φ is (1.10) for all $j = 0, \ldots, 3$. Therefore, $\Lambda_j^\Phi = \Lambda_j^\Phi(\alpha, \beta, \gamma)$, so they are determined, up to a global phase, by a set of points $\chi_j^\Phi \subset [0, 2\pi)^{\times 3}$.

In this sense, if we try to teleport one qubit $|\psi\rangle$ with that state, we obtain that the fidelities of each possible outcome $j \in \Upsilon_\psi$ can be expressed like

$$F_j^\psi = \left| \langle \varphi_j \mid \left(\Lambda_j^\Phi(\alpha, \beta, \gamma) \right)^\dagger \mid \psi \rangle \right|^2 = F_j^\psi(\alpha, \beta, \gamma),$$

as $|\varphi_j\rangle$ only depends on $|\psi\rangle$ and j. Therefore, in order to find the best Λ_j for teleporting $|\psi\rangle$ we must look for the points (α, β, γ) which give the maximum value of F_j^ψ. However, these ones may depend on $|\psi\rangle$, and so Λ_j. We will try to avoid this dependence by considering a set of m arbitrary qubits $\Omega = \{|\psi_1\rangle, \ldots, |\psi_m\rangle\}$ and defining the following error fidelity:

$$F_j(\alpha, \beta, \gamma) := \prod_{k=1}^{m} F_j^{\psi_k}(\alpha, \beta, \gamma), \quad \forall j = 0, \ldots, 3. \tag{4.12}$$

Given this, the main goal is to find the points χ_j^Φ which maximize F_j for a given state $|\Phi\rangle$, for each $j = 0, \ldots, 3$. Note that we are considering all possible outcomes j but we could have $|\Upsilon_{\psi_{k'}}| < 4$ for some $k' = 1, \ldots, m$. When this happens, for the missing outcome j' in $\Upsilon_{\psi_{k'}}$, we will impose that $F_{j'}^{\psi_{k'}} \equiv 1$ because for $|\psi_{k'}\rangle$ the transformation Λ_j can be any (because it is never used by Bob). If this is another value less than one, it would affect inappropriately on the result of F_j. On the other hand, since (4.12) is an error-type function, it is more convenient to find the probability density function ρ_j of the points (α, β, γ), instead of looking for the maximums of F_j.

Let us fix one arbitrary outcome j. In our physics work, we consider the function $f_j = -\ln F_j \in [0, \infty)$. We know that there exists a density function $\rho_T(x) \propto \exp(-f_j(x)/T) =: g_T(x)$, where $T > 0$ is a scale parameter. Observe that the modes of ρ_T match the minimums of f_j, which are the same as the maximums of F_j. With the purpose of modeling ρ_T, we base our work on *Markov Chain*

Monte Carlo (MCMC) methods [Ser11]. In particular, we use an algorithm that is known as *Multiple Correlated-Try Metropolis* (MCTM) [CL07]. The reader can find in S.A.I the version of this method we implement.

Once this method is written and executed enough times in [Lap], we get perfect results for the cases when the shared state $|\Phi\rangle$ is any of the four Bell states. The interested reader can find in S.A.II the representations of the corresponding 2D marginal densities for each outcome when $|\Phi\rangle = |\phi_0\rangle$ in Figure 4.1.

After the guarantee that the MCTM algorithm works well in our problem, we focus on analyzing the states of the form $|\Phi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$, where $\theta \in [0, \pi)$. They are interesting because they are the result of applying one 4×4 unitary matrix on the Bell state $|\phi_0\rangle$. Moreover, according to (3.4), the amount of their entanglement varies with θ like $4\cos^2\theta\sin^2\theta$.

For all the angles different from $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, the states do not belong to the set $\mathfrak{A}$ defined in (4.11) and also they are not maximally entangled states($\mathcal{J}_2 < 1$), so they must be not PQT. Therefore, it seems useful to implement MCTM for these angles and see how $\Lambda^{\Phi(\theta)}$ varies. After carrying out this method for some values of θ, the conclusions we extract are that for angles in $\left(0, \frac{\pi}{2}\right)$ the instructions Bob must follow are Λ^{ϕ_0}, meanwhile in $\left(\frac{\pi}{2}, \pi\right)$ these are Λ^{ϕ_3}. On the other hand, if $\theta \in \left\{0, \frac{\pi}{2}\right\}$ —the only cases in which $|\Phi(\theta)\rangle$ is product— $\Lambda^{\Phi(\theta)}$ can be both of them. For a illustrative deduction of this conclusions, we suggest to take a look at Figure 4.2 in S.A.II.

We have used these instructions to study how the fidelity results vary when attempting QT using the shared states $|\Phi(\theta)\rangle$, in relation to the values of $\mathcal{J}_2(|\Phi(\theta)\rangle)$. A consistent dependence is observed between the values of this observable and the success of QT (see Figure 4.3 in S.A.II), suggesting that if $|\Phi\rangle$ is PQT, then $\mathcal{J}_2(|\Phi\rangle) = 1$. This aligns with $\mathcal{J}_n$ being a good entanglement measure. Furthermore, as noted in [Pre01], for cases where $\mathcal{J}_2 < 1$, Alice could follow an alternative process to achieve maximum fidelities of $\frac{2}{3}$. Using our method, we improve these results for cases where $\mathcal{J}_2 > 0.125$.

Summary and conclusions

The enigmatic nature of entanglement and its valuable role in technology have captured the interest of researchers worldwide. To better understand and implement this phenomenon, various mathematical frameworks and theories, based on complex geometric algebra, have been developed to provide structure and intuition about quantum entanglement.

In this work, we have seen that the entanglement of pure states can be analyzed using a complex geometrical approach based on Segre embeddings of projective Hilbert spaces. This offers a geometric and intuitive perspective on quantum states and their separability, as well as a practical tool for quantifying entanglement. We refer the interested reader to [CST21, Gat14] for deepen in such topic or to [BBC$^+$19] for a more general understanding of geometry in quantum mechanics.

Among the most significant applications of quantum entanglement is quantum teleportation. We have shown that this phenomenon can be rigorously described through the algebraic principles of quantum mechanics. In the attempt to provide a general description of quantum teleportation, we have proposed a set of conditions useful for determine whether a given two-qubit state is suitable for quantum teleportation. Future research in this area could focus on looking for the exact properties that make a state perfect for teleporting any arbitrary qubit. Additionally, exploring alternative measures on two-qubit states different from the Bell measurement may enhance the efficiency and versatility of quantum teleportation protocols.

Implementing entanglement in order to teleport qubits from one point to another could lay the foundation for a global-scale quantum internet, offering more security and computational power. However, preserving the properties of entanglement unaltered and enabling the transmission of multiple qubits are very challenging problems, so there is still so much to explore and develop in this promising field.

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Supplementary appendix

In this last part, the reader can find some supplementary work that may need in order to fully understand some concepts or steps mentioned before in this work. Each section refers to some specific content related on the context where the reference S.A.# appears in the project.

S.A.I. One version of the Multiple Correlated-Try Metropolis (MCTM)

The main objective is to look for the form of the target density ρ_T, such that $\rho_T(x) \propto \exp(-f_j(x)/T) =: g_T(x)$ We could try to compute $\int_{x \in V} g_T(x)dx$, but this integral is too complex. Fortunately, [CL07] provides an accelerated version of the Metropolis-Hastings method for estimating ρ_T without computing any integral at all. This algorithm is called *Multiple Correlated-Try Metropolis* (MCTM). Being one *Markov Chain Monte Carlo* method, it generates random numbers from the known function g_T, which produces a Markov chain $\{x_t\}_{t=1}^N$ such as its invariant density function matches with our target density $\rho \propto g_T$.

The version of the MCTM method we have used generates a Markov chain $\{x_t\}_1^N$ following these steps:

1. We choose a random point in V as the first member of the chain, x_1, such as $f_j(x_1) \neq \infty$.

2. For $t \geq 1$, let $x := x_t$ be the last point of the chain. From a Gaussian distribution $\mathcal{N}((x^T, \overset{(k)}{\ldots}, x^T)^T, \Sigma_{3k})$, we generate k trial proposals $y_1, \ldots, y_k$. The covariance matrix Σ_{3k} is of the form

$$\Sigma_{3k} = \begin{pmatrix} \Sigma & \Gamma & \cdots & \Gamma \\ \Gamma & \Sigma & \Gamma & \Gamma \\ \cdots & \cdots & \cdots & \cdots \\ \Gamma & \Gamma & \cdots & \Sigma \end{pmatrix}$$

where $\Sigma = \sigma^2 \mathbb{I}_3$ and $\Gamma = \frac{\sigma^2}{1-k}\mathbb{I}_3$ for some σ^2. We will use values of σ^2 around $10-20$. Moreover, the work [CL07] indicates that for that version of the MCTM a number of $k=7$ trial proposals is a good one.

3. We compute $g_T(y_l)$ for each $l = 1, \dots, k$, and then we select one of these points y with probability proportional to $g_T(y)$.

4. We generate $\widetilde{x}_1, \dots, \widetilde{x}_k$ points from the same Gaussian distribution from before, but with y instead of x and conditioning that $\widetilde{x}_k = x$.

5. We compute the *acceptance probability*:

$$\mathcal{A}(y, x_t) = \min\left\{1, \frac{g_T(y_1) + \cdots + g_T(y_k)}{g_T(\widetilde{x}_1) + \cdots + g_T(\widetilde{x}_k)}\right\}.$$

6. The following point of the chain x_{t+1} is y with probability $\mathcal{A}(y,x)$ or x with probability $1 - \mathcal{A}(y,x)$.

7. The steps $2-6$ are repeated with the last point of the chain x_{t+1}, until x_N is reached.

S.A.II. Results from our Physics Final Degree Project

For a clearer understanding of these results, we refer the interested reader to [LSC25].

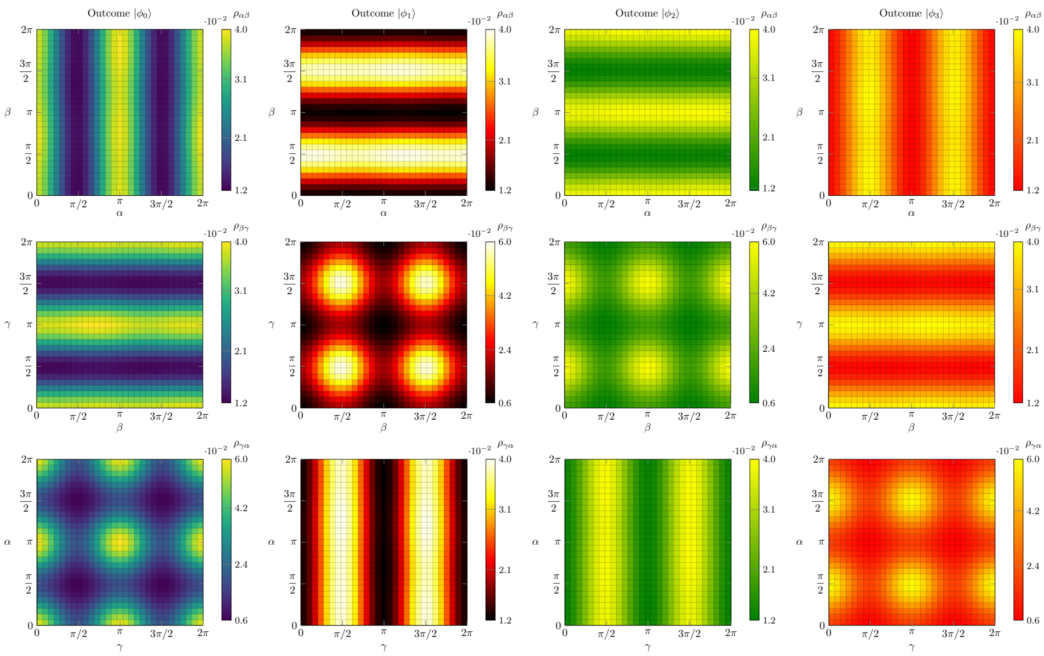

Figure 4.1: For the shared state $|\Phi\rangle = |\phi_0\rangle$, we have represented for each Bell measurement outcome the 2D marginal densities $\rho_{\alpha\beta}$, $\rho_{\beta\gamma}$ and $\rho_{\gamma\alpha}$. These densities have been computed after implementing the MCTM method. Note that, for each outcome, we can get to know the modes of the 3D density $\rho_{\alpha\beta\gamma}$, which effectively agree with the set of points $\chi_j^{\phi_0}$ determining the instructions Λ^{ϕ_0}.

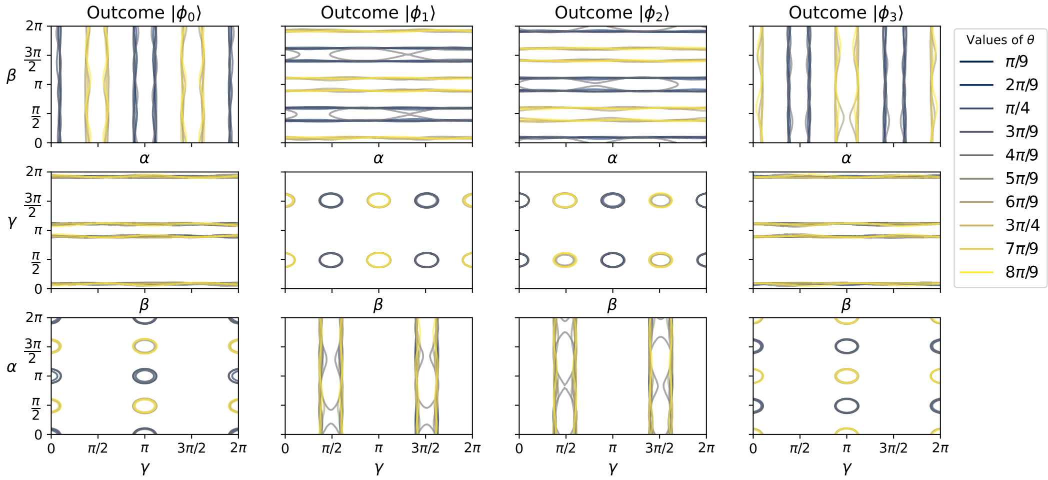

Figure 4.2: Contours of the modes from the densities $\rho_{x,y}$ where $x, y \in \{\alpha, \beta, \gamma\}$, for each outcome. These have been obtained by applying the MCTM algorithm to the states $|\Phi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$ for ten different values of $\theta \in [0, \pi)$.

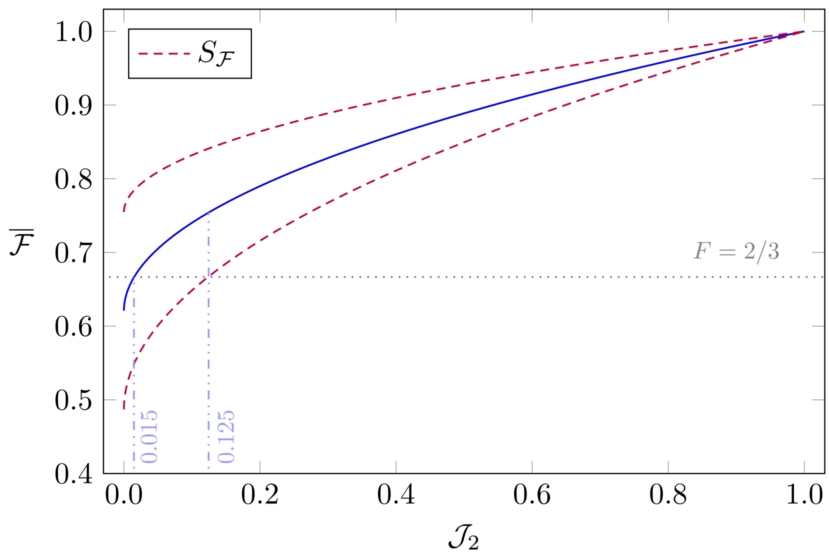

Figure 4.3: Plot of the fidelity results in QT depending on $\mathcal{J}_2$. The data have been obtained from the function $\mathcal{F}_\Phi(\psi) := \sum_{i=0}^{3} p_i F_i^{\psi}$. With this one, we have computed for each $\theta \in [0, \pi)$ the mean value of the set $\{\mathcal{F}_{\Phi(\theta)}(\psi)\}_\psi$ for 10^5 random qubits $|\psi\rangle$. The shared states are of the form $|\Phi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$. The plotted red lines $S_{\mathcal{F}}$ are the standard deviation of the mean value.