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PENROSE SINGULARITY THEOREMS

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Contents

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Abstract

In this work, we will review Penrose singularity theorems. Following the publication of Einstein's General Theory of Relativity in 1916, the possible existence of singularities in relativistic space-times garnered attention, which initiated a significant debate where various ideas were proposed in order to provide meaning to singularities. In 1965, Roger Penrose published the first modern singularity theorems, which generalised the existence of singularities in any relativistic space-time under certain topological conditions. To present the theorems in an accessible and coherent way, we will follow the structure of Penrose's publication (as referenced in [17]). We will first establish the causal and chronological order relations between points in space-time. Subsequently, we will address the physical viability of the space-times. To do so, we will study the Cauchy problem and the necessary conditions that a relativistic space-time must satisfy to avoid containing time loops. In the final section, we will explore geodesic congruences in the space-time by applying general relativity, and we will define the concept of trapped surfaces, which is a key notion in Penrose developments. By following this line of reasoning, we will be able to prove the existence of incomplete geodesics defined within space-time. Therefore, we will observe the existence of singularities at the end and the beginning of geodesics on space-times.

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Resum

En aquest treball, revisarem els teoremes de singularitat de Penrose. Després de la publicació de la Teoria de la Relativitat General d'Einstein el 1916, la possible existència de singularitats en espai-temps relativistes va rebre atenció, la qual cosa inicià un debat sobre les possibles definicions de singularitats. Roger Penrose publicà el 1965 els primers teoremes de singularitat moderns, els quals generalitzaven l'existència de singularitats en qualsevol espai-temps relativista sota certes condicions topològiques. Per tal de presentar els teoremes d'una manera accessible i coherent, seguirem l'estructura proposada per Penrose (vegeu [17]). En primer lloc, establirem les relacions d'ordre causal i cronològic entre punts de l'espai-temps. Seguidament, n'abordarem la viabilitat física a través de l'estudi del problema de Cauchy i les condicions necessàries que han de satisfer per evitar bucles temporals. En la darrera secció, explorarem congruències de geodèsiques en espai-temps gràcies a la relativitat general. Alhora, definirem la noció de superfícies atrapades, la qual serà clau en els desenvolupaments de Penrose. Seguint aquest raonament, demostrarem l'existència de geodèsiques incompletes a l'espai-temps, i doncs, de singularitats al final i a l'inici de geodèsiques.

Introduction

The main goal of this work is to present a comprehensive proof to the Penrose singularity theorems, which are essential to our understanding of space-time singularities in general relativity. We will establish the theoretical foundations upon which the Theorems will be stated. This framework is commonly referred to as *causal structure*. The primary purpose of this mathematical framework is to set relations between points in space-time. This is the crucial innovation that Penrose will bring to the debate on singularity theorems, which will allow his results to be independent of assumptions regarding the metric of space-time. This will make Penrose's singularity theorems applicable to any physically relevant space-time.
This new approach to singularities in space-time succeeded in convincing the general relativity community that singularities were a generic characteristic of the solutions to the Einstein's field equations. However, they did not close the debate about the correct definition of singularities. It became clear that there is not a single way to define them. Instead, when we specify the term 'space-time singularity' it will refer to a family of interrelated behaviours on a relativistic space-time.
The history of singularity theorems in general relativity theory is far from a simple analysis on the existence of singularities in solutions to the field equations. Soon after the final publications of Einstein's general relativity theory in 1916, the first solutions to the Einstein field equation were provided by Schwarzschild. One year after the Schwarzschild's solution was published, David Hilbert pointed out the existence of two *apparently* singular points. He suggested, with good intuition, that one of the singularities could not be evaded by just a coordinate transformation. Since then, the concept of singularities began to gain attention. This would mark the beginning of a long journey to profoundly understand the meaning and true significance of singularities. Indeed, such a journey would lead to crucial advances in the true comprehension of space-time. Prior to Penrose's results, the development of singularity theorems would go through many stages. In the 1930s, the Tolman-Lemaître model was proposed as a general solution to the field equations for dust with spherical symmetry. Some important results of this model include the instability of Einstein static universe and an ubiquitous initial singularity of Friedman creation-time type for expanding models capable of explaining the observed cosmological redshifts. Note that this singular behaviour emerged from symmetry assumptions, which were not truly realistic. Therefore, Tolman and Lemaître dropped the symmetry assumptions and studied spatially homogeneous but anisotropic models. Again, the singularity remained, indicating that the anisotropy of space could not prevent the existence of singularities.
The results of critical mass obtained by Chandrasekhar in 1931, along with the debate on the notion of singularity, motivated Oppenheimer and Snyder to con-

sider solutions to the field equations that described gravitational collapse processes. Thus, they proved that, under spherical symmetry, the light emitted from a collapsing star would be asymptotically redshifted towards an external observer. Therefore, Oppenheimer and Snyder proved that singularities could exist not only as an initial singularity, but also a final singularity after gravitational collapse.
In 1955, further research showed that in a cosmological context, singularities in general relativistic space-times are not artifacts of symmetry assumptions. Raychaudhuri produced a geometrical result, later known by his own name, which proved how geodesics of a congruence tend to a caustic. This was called *focusing* or *Raychaudhuri effect*, which does not necessary lead to singularities. Later, in collaboration with Komar, they produced a singularity theorem for irrotational dust. They assumed that the universe was made of dust moving along geodesics and supposed the existence of matter singularities.
Finally, in 1965 Penrose produces the singularity theorems presented in this work. He wanted to prove that deviations from spherical symmetry were not able to prevent the formation of singularities within general relativity theory. To achieve this goal, Penrose presented the idea of *incompleteness* to describe the singular space-times, along with the idea of *closed trapped surface*. These topics will be addressed in the final section of this work
This work will follow a similar structure as the one presented by Penrose in his publication in 1965, as referenced in [17]. We will divide this project into three main parts. In the first chapter we present the initial approach to the causal structure. That is, we set a basic framework of definitions on differential geometry. Afterwards, we present the relations in the causal structure and their properties, alongside with the kind of paths the points follow in space-time to follow. We will finish this chapter zooming-out from just one-to-one relations between points in space-time to focus into the set of all chronologically and causally related points. This will lead us to the intuitive concept of *light cones*. In the second chapter we will continue developing the causal structure. These notions will help us to incorporate some physical relevance to the problem, as for example, the notion of physically acceptable space-time, where time-travel to the past is not allowed or the Cauchy problem. At the end of this chapter, we will introduce some notions on the causal curves topology. The proofs for the core results of this work will be based on topological arguments. The last chapter is strongly oriented towards the theorems. To this end, we present a first section in which we continue developing necessary properties of causal curves together with the introduction of the required results obtained by Raychaudhuri. To end this section, we will present some final theorems about conjugate points. In the final section, we will introduce the notion of closed trapped surfaces to rapidly attain the singularity theorems.

Notation: We will use the Einstein notation: Given a tensorial expression, we obtain the same expression in Einstein notation by removing the summation sign so that repeated indexes in the resultant expression indicate a sum over all the possible values of the index $X^iY_i = \sum_i X^iY_i$ or $g^{ij}g_{ij} = \sum_{i,j} g^{ij}g_{ij}$
We will use n as a natural number $n \in \mathbb{N}$ where $0 \in \mathbb{N}$.
Additionally, we will conveniently write $\partial_i \equiv \partial/\partial x^i$.

Chapter 1

Causality and chronology

In this chapter we will give the first intuitions for causal structure. We begin by defining basic concepts of differential geometry to rapidly address the definition of space-time. Afterwards, we will define the kind of trajectories observers follow along this structure as well as their properties, as will be discussed in section 1.2. Finally, we will zoom out from a single trajectory towards the bundle of all possible trajectories, that is, the past and future of a present point in space-time.

1.1 Basic definitions

Let M be a topological space.

Definition 1.1. An ($n-$dimensional) *chart* at $p \in M$ is a map $\varphi : U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$ where $U \subseteq M$ is an open set containing p and φ is an homeomorphism onto an open subset of $\mathbb{R}^n$. The *coordinate functions* of the chart are the real-valued functions on U given by the coordinates of φ, that is, they are the functions $x^i = u^i \circ \varphi : U \longrightarrow \mathbb{R}$ where $u^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are the standard coordinates on $\mathbb{R}^n$ with $u^i(a^1, \ldots, a^n) = a^i$. Thus, for every $q \in U$, $\varphi q = (x^1 q, \ldots, x^n q)$ so we write $\varphi = (x^1, \ldots, x^n)$. We call φ a *coordinate map*, U the *coordinate neighbourhood* and the collection $(x^1, \ldots, x^n)$ *coordinates* or a *coordinate system* at p.

Definition 1.2. An *atlas* $\mathcal{A}$ for M is a family of charts on M $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ which covers M. An atlas will be *smooth* if for every chart (U, ϕ) and (V, ψ), with $U \cap V \neq \emptyset$, the maps $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ and $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ are C^∞.

Definition 1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two atlases for M. We say that two atlases are *equivalent* if for all $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$, the maps $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are C^∞. A *smooth structure* Φ on M is an equivalence class $[\mathcal{A}]$ of smooth atlases on M.

Definition 1.4. An $n-$dimensional *smooth manifold* is a pair (M, Φ) of a second-countable Hausdorff topological space M and a differentiable structure Φ on M such that every chart in Φ takes values in $\mathbb{R}^n$.

Definition 1.5. Let M be a smooth manifold. A function $f : M \longrightarrow \mathbb{R}$ is *smooth* if $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}$ is C^∞ for all chart (U, φ) on M. We will denote the set of all smooth functions on M by $\mathcal{F}(M)$. It is a vector space (see [4]).

Definition 1.6. A *tangent vector* at p is a map $T : \mathcal{F}(M) \longrightarrow \mathbb{R}$ such that, for any coordinate system $\varphi : U \longrightarrow \mathbb{R}^n$ with $p \in U$, there exists a $n-$tuple $(a^1, \ldots, a^n)$ of real numbers with the following property.

$$T(f) = \Sigma_{i=1}^n a^i \frac{\partial}{\partial u^i}(f \circ \varphi^{-1})|_{\varphi(p)} \quad \text{for each } f \in \mathcal{F}(M):$$

Remark 1.7. Given a coordinate system φ about p, let $x^i = u^i \circ \varphi$ denote the ith coordinate function of φ. By $\partial/\partial x^i$ for all $i = 1, \ldots, n$ is meant the tangent vector at p defined by:

$$\frac{\partial}{\partial x^i}(f) = \frac{\partial}{\partial u^i}(f \circ \varphi^{-1})|_{\varphi(p)} \quad \text{for each } f \in \mathcal{F}(M)$$

Definition 1.8. The *tangent space to M at p* is the vector space of all tangent vectors at a point $p \in M$. It is denoted as T_pM.

Definition 1.9. Let M be a smooth manifold. A *vector field* on M is an assignment of a tangent vector $X_p \in T_pM$ for all $p \in M$. That is, a vector field is a map $X : M \longrightarrow \bigsqcup_{p \in M} T_pM$. The set of all smooth vector field over M will be denoted as $\mathcal{X}(M)$. It is a vector space (see [4]).

Theorem 1.10. *Let M be a smooth $n-$dimensional manifold. Let (U, φ) be a chart at $p \in U$ with coordinate functions $x^1, \ldots, x^n$. Let T_pM be the tangent space of M at p. The set $\{\partial/\partial x^i\}_{i=1}^n$ is a basis of T_pM. For all vector $T \in T_pM$ can be written as:*

$$T = T^i \frac{\partial}{\partial x^i}|_p$$

Hence, $\dim(T_pM) = n$.

Proof. See [16, Section 1.2] □

Definition 1.11. A *Lorentzian tensor field g* on an $n-$dimensional smooth manifold M is a symmetric non-degenerate smooth $(0, 2)$-tensor field with signature $(1, n - 1)$.

Let g be a Lorentzian tensor field. That is, g maps each point $p \in M$ onto a scalar product on the tangent space on p as $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$. Notice that since g_p varies differentially with p, the tensor g, will be smooth. In other words, for all open set $U \subseteq M$ and for all $X, Y \in \mathcal{X}(U)$, where $\mathcal{X}(U)$ is the set of all vector fields defined at U, $g(X,Y) : M \longrightarrow \mathbb{R}$ with $g(X,Y)(p) = g_p(X_p, Y_p) \in \mathbb{R}$ is C^∞.

Definition 1.12. A *space-time* is a pair (M, g), where M is a connected real $4-$dimensional C^∞ Hausdorff manifold and g is a Lorentzian tensor field on M.

From now on, let us consider M as a space-time.

Remark 1.13. For any $p \in M$, we can find a suitable basis on T_pM where g_p takes the form $\mathrm{diag}(1, -1, -1, -1)$.

Remark 1.14. Let (U, φ) be a chart on a space-time (M, g). Let $x^1, \ldots, x^4$ be the coordinate functions for φ. Let $X, Y \in \mathcal{X}(U)$. We can write:

$$g(X,Y) = g(X^\mu \partial_\mu, Y^\nu \partial_\nu) = X^\mu Y^\nu g(\partial_\mu, \partial_\nu) = g_{\mu\nu} X^\mu Y^\nu$$

Where we have considered $g = g_{\mu\nu} dx^\mu \bigotimes dx^\nu$. Since g is symmetric, we obtain $g_{\mu\nu} = g_{\nu\mu}$. From definition, $(g_{\mu\nu})^{-1} = g^{\mu\nu}$.

Let $p \in M$ be a point and let $X_p \in T_pM$ be a tangent vector. If $g_p(X_p, X_p)$ is positive, negative or zero then X_p is said to be timelike, spacelike or null respectively.

Definition 1.15. A space-time M is *time orientable* if it is possible to make a consistent continuous choice all over M of one component of the set of timelike vectors at each point of M. A space-time is *time oriented* when we label the chosen timelike vectors as *future pointing* and the remaining ones as *past pointing*.

Remark 1.16. In case M is not time-orientable, we can always choose a $2-$fold covering M' of M which is time-orientable. In consequence, from now on, we will consider every space-time as time-orientable. More about $2-$fold coverings of space-times can be seen at [17, Section 1].

Definition 1.17. A *path* is a C^∞ map $\mu : \Sigma \longrightarrow M$ where $\Sigma \subseteq \mathbb{R}$ is an interval.

Definition 1.18. Let $\mu : \Sigma \longrightarrow \mathbb{R}$ be a path. Let $a = \inf \Sigma$ and $b = \sup \Sigma$. A point $x \in M$ will be a *future endpoint* (respectively, *past endpoint*) of μ if for all sequence of points $\{u_i\}_i \subseteq \Sigma$, if $u_i \to a$ then $\mu(u_i) \to x$ (respectively if $u_i \to b$ then $\mu(u_i) \to x$).

Remark 1.19. A path is *timelike* if its tangent vector is timelike at every point. We say it is *future-oriented* if the tangent vector is future-pointing at every point. Analogously for causal and null paths. We will call smooth paths as *curves*. For convenience, all timelike or causal curves are required to contain their endpoints. For instance, for a curve with both endpoints, Σ will be a closed interval.

Remark 1.20. Let $\gamma \subseteq M$ be a curve. It is called *past endless* if it does not have a past endpoint and *future endless* if it does not have a future endpoint. In case neither a future endpoint nor a past endpoint exist, γ is called *endless*. The later notion will be of crucial importance when discussing the first singularity theorem.

Definition 1.21. A *connection* ∇ is a map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ with $\nabla(X, Y) = \nabla_X Y$ satisfying for any vector fields $X, Y, Z \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$ these properties:

1. $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$
2. $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
3. $\nabla_{fX} Y = f \nabla_X Y$
4. $\nabla_X (fY) = X(f)Y + f(\nabla_X)Y$

The expression $\nabla_X Y$ will be read as the *covariant derivative* of Y along X for the ∇ connection. In addition, we can define the covariant derivative of a vector field Y at a point p with respect to a tangent vector $T \in T_pM$. In order to do so, we need to choose a vector field $X \in \mathcal{X}(M)$ with $X_p = T$ and write $\nabla_T Y = (\nabla_X Y)_p \in T_pM$. This is well-defined by the properties above.

In this paper, ∇ will be considered to be the *Levi-Civita* connection. That is, the unique torsion-free connection which is compatible with the metric tensor g (see [4]). More precisely, ∇ is characterized by the properties:

1. $\nabla_X Y - \nabla_Y X = [X, Y]$
2. $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

where $[X, Y]$ is the *Lie bracket* for any $X, Y \in \mathcal{X}(M)$.

Definition 1.22. Let (U, φ) be a chart with coordinate functions $x^1, \ldots, x^n$ on an $n-$dimensional Lorentzian manifold. The functions $\Gamma^{\mu}_{\nu\rho} : U \longrightarrow \mathbb{R}$ defined by:

$$\nabla_{\partial_\nu} \partial_\rho = \Gamma^{\mu}_{\nu\rho} \partial_\mu$$

are the *Christoffel symbols* in (U, φ).

Remark 1.23. Let (U, φ) be a chart with coordinate functions $x^1, \ldots, x^n$ on an $n-$dimensional Lorentzian manifold. Consider two vector fields $X = X^\mu \partial_\mu$ and $Y = Y^\nu \partial_\nu$. We can write $\nabla_X Y = \nabla_{X^\mu \partial_\mu} Y^\nu \partial_\nu = X^\mu \nabla_{\partial_\mu} Y^\nu \partial_\nu = X^\mu((\nabla_{\partial_\mu} Y^\nu)\partial_\nu + Y^\nu \nabla_{\partial_\mu} \partial_\nu) = X^\mu((\partial_\mu Y^\nu)\partial_\nu + Y^\nu \Gamma^\rho_{\mu\nu} \partial_\rho)$. Therefore $\nabla_X Y = (X^\mu Y^\nu \Gamma^\rho_{\mu\nu} + X^\mu \partial_\mu Y^\rho)\partial_\rho$. Analogously, $\nabla_{\partial_\mu} Y = \nabla_\mu Y = (Y^\nu \Gamma^\rho_{\mu\nu} + \partial_\mu Y^\rho)\partial_\rho = (\nabla_\mu Y)^\rho \partial_\rho$.
With this final expression, we have $\nabla_X Y = (\nabla_X Y)^\nu \partial_\nu = X^\mu (\nabla_\mu Y)^\nu \partial_\nu$.

Remark 1.24. In physics bibliography and by Penrose himself, there is a certain abuse of notation where $(\nabla_\mu Y)^\nu$ is usually written as $\nabla_\mu Y^\nu$.

Definition 1.25. An *affinely parametrised geodesic* (a.p.geodesic) consists on a path μ, with tangent vector field T, which satisfies the so-called *geodesic equation* $\nabla_T T = 0$ at every point of μ.

Remark 1.26. Let us explain the geodesic equation. Let μ be a path contained in M and $p \in \mu$ a point. Take the tangent vector field $T \in T_p M$ to μ at p. The geodesic equation is the covariant derivative of T at p respect to itself. That is, we choose an extension vector field of T defined as $X \in \mathcal{X}(M)$ with $X|_\mu = T$. We can write $\nabla_T T = (\nabla_X X)|_\mu = 0$.

Remark 1.27. We can rewrite the geodesic equation. Let (U, φ) be a chart with coordinate functions $x^1, \ldots, x^4$. Let μ be a path with tangent vector field T. We can write $0 = \nabla_T T = T^\mu (\nabla_\mu T)^\nu \partial_\nu$ (which by 1.24 could be written as $T^\mu \nabla_\mu T^\nu = 0$).

Definition 1.28. The *exponential map on* $p \in M$ is the map $\exp_p : U \longrightarrow M$, where $U \subseteq T_p M$ is an open subset defined by the condition that $X \in U$, $\exp_p X = \gamma_X(1)$ where γ_X is a unique geodesic such that $\gamma_X(0) = p$ and $\gamma'_X(0) = X$.

Definition 1.29. A space-time M is said to be *geodesically complete* if for every $p \in M$, $\exp_p$ is defined on the whole $T_p M$.

Several regions of $T_p M$ can be mapped to the same point of M or the exponential map can present a pathological behaviour for certain elements of $T_p M$. Let us introduce the following concepts.

Definition 1.30. Let $p \in M$ be a point and let $Q \subseteq T_p M$ be a star-shaped neighbourhood of $0 \in T_p M$ such that $Q \longrightarrow \exp_p(Q)$ is a diffeomorphism. Then, $\exp_p(Q) \ni a$ is said to be a *normal neighbourhood* of p.
Let N be a normal neighbourhood of any $p \in M$, such that N is also a normal neighbourhood of any other point $q \in N$. Then N is said to be *simply convex*.

Definition 1.31. A choice of an orthonormal basis $\{e_i\}_{i=1}^4$ for $T_p M$ is equivalent to an isomorphism $e : T_p M \longrightarrow \mathbb{R}^4$ with $e(X^\mu e_\mu) = (X^1, \ldots, X^4)$. Therefore, on a normal neighbourhood N of a point $p \in M$, we have local coordinates $e \circ \exp_p^{-1} : N \longrightarrow \mathbb{R}^4$. Such coordinates are called *normal coordinates* centred at p.

Proposition 1.32. *Let N be simply convex. Then, for any pair of points $p, q \in N$ there is a unique geodesic from p to q.*

Definition 1.33. A *simple region* N is a simply convex neighbourhood such that $\overline{N}$ is compact.

Proposition 1.34. *Some properties of simple regions include:*

1. *Let N be a simple region. Then, for every $p, q \in \overline{N}$ there exists a unique geodesic from p to q. The geodesics are continuous functions of $(p, q) \in \overline{N} \times \overline{N}$.*
2. *The boundary ∂N is compact. Every closed subset $Q \subseteq N$ is compact.*
3. *The space-time M can be covered by a locally finite number of simple regions.*

The full proof of 1.32 and 1.34 can be seen at [13] and [12].
Now, we are going to explain the *Jacobi fields*. This notion plays a key role in the field of differential geometry when studying manifold curvatures and geodesic behaviour. In order to do so, we will begin by working on geodesic congruences. For the sake of this work's structure, they are explained in Chapter 3.

Consider a geodesic congruence expressed as $\Gamma(s,t)$ so that $\Gamma_s(t) = \Gamma(s,t)$ is a geodesic. Let $T(s,t) = \partial\Gamma(s,t)/\partial t$ and $S(s,t) = \partial\Gamma(s,t)/\partial s$ be the tangent and the deviation vector fields between geodesics respectively. Therefore, Jacobi fields can measure the variation of geodesics given a representative one. With the deviation vector field we can study how small perturbations to the representative geodesic change its behaviour, whereas with the tangent vector field we can comprehend the evolution of initial conditions.Taking into account that T and S are basis vectors adapted to a coordinate system, their commutator becomes $[S, T] = 0$.
Let (U, φ) be a chart with coordinate functions $x^1, \ldots, x^4$. We can write $[S, T]^\mu = S^\nu \nabla_\nu T^\mu - T^\nu \nabla_\nu S^\mu = 0$ and thus $\nabla_S T = \nabla_T S$. Let us denote $D = \nabla_T$ as the *propagation derivative*, so that $DS^\mu = S^\nu \nabla_\nu T^\mu$. The second derivative will be:

$$D^2 S^\mu = D(S^\nu \nabla_\nu T^\mu) = T^\lambda \nabla_\lambda (S^\nu \nabla_\nu T^\mu) = (T^\lambda \nabla_\lambda S^\nu)(\nabla_\nu T^\mu) + T^\lambda S^\nu \nabla_\lambda \nabla_\nu T^\mu =$$

$$= (S^\lambda \nabla_\lambda T^\nu)(\nabla_\nu T^\mu) + T^\lambda S^\nu (\nabla_\nu \nabla_\lambda T^\mu + R^\mu_{\rho\lambda\nu} T^\rho) =$$

$$= (S^\lambda \nabla_\lambda T^\nu)(\nabla_\nu T^\mu) + S^\nu \nabla_\nu (T^\lambda \nabla_\lambda T^\mu) - (S^\nu \nabla_\nu T^\mu) - (S^\nu \nabla_\nu T^\lambda) \nabla_\rho T^\mu + R^\mu_{\rho\lambda\nu} T^\rho T^\lambda S^\nu =$$

$$R^\mu_{\rho\lambda\nu} T^\rho T^\lambda S^\nu$$

We have used the identity $[\nabla_\lambda, \nabla_\nu] T^\mu = (\nabla_\lambda \nabla_\nu - \nabla_\nu \nabla_\lambda) T^\mu = R^\mu_{\rho\lambda\nu} T^\rho$.

Remark 1.35. We denote as $R^{\mu}_{\rho\lambda\nu}$ the Riemann curvature tensor, where $R(X,Y)Z = X^{\lambda}Y^{\nu}Z^{\rho}R^{\mu}_{\rho\lambda\nu}\partial_{\mu}$. We can obtain the Ricci tensor $R_{\rho\nu} = R^{\mu}_{\rho\mu\nu}$ and the Ricci scalar as $R = g^{\rho\nu}R_{\rho\nu}$. We will apply these notions at Chapter 3. The complete definition of them is out of the scope of this work. More can be seen at [4, Section 5.7].

Definition 1.36. Let (U, φ) be a chart with coordinate functions $x^1, \dots, x^4$. Let T and S be the tangent and deviation vector fields of a 1-parameter system of a.p.geodesics. We call *geodesic deviation equation*:

$$D^2 S^{\mu} = R^{\mu}_{\nu\lambda\rho} T^{\nu} T^{\lambda} S^{\rho}$$

Definition 1.37. A *Jacobi field* is a vector field along an a.p.geodesic which satisfies the geodesic deviation equation.

Proposition 1.38. *With the same notation as 1.36. If $T^{\mu}T_{\mu} \equiv$ constant for every geodesic of the system, then $T_{\mu}S^{\mu} \equiv$ constant along the geodesic.*

Proof. $D(T_{\mu}S^{\mu}) = T_{\mu}D(S^{\mu}) = T_{\mu}D(S^{\mu}) = T_{\mu}S^{\nu}\nabla_{\nu}T^{\mu} = \frac{1}{2}S^{\nu}\nabla_{\nu}(T_{\mu}T^{\mu}) = 0$
Using the geodesic equation $D(T_{\mu}) = 0$ □

Definition 1.39. Let S be a Jacobi field over an a.p.geodesic γ. We say that two points $p, q \in \gamma$ are conjugate if $S(p) = S(q) = 0$ and $S \not\equiv 0$.

1.2 Trips, curves and geodesics

Definition 1.40. A *trip* is a curve which is piece-wise a future oriented timelike geodesic. For any $p, q \in M$, we say $p \ll q$ if and only if there exists a trip from p to q. The relation $p \ll q$ states that there exists a sequence of points $\{x_i\}_{i=0}^{n}$, $n \geq 1$, and a timelike geodesic, called *segment*, with past endpoint x_{i-1} and future endpoint x_i for each $i = 1, \dots, n$ while setting $x_0 = p$ and $x_n = q$. We read that p *chronologically precedes* q if and only if $p \ll q$.

Definition 1.41. A *causal trip* is defined the same way as a trip, but allowing geodesics to be null. In this case, for any point $p, q \in M$ we say that p *causally precedes* q if and only if $p \prec q$.

Proposition 1.42. *Let $a, b \in M$ be two different points. If:*

1. $a \ll b \implies a \prec b$
2. $a \ll b$ *and* $b \ll c \implies a \ll c$

3. *$a \prec b$ and $b \prec c \implies a \prec c$*

Proof. At 1), by definition it is easy to see that if a chronologically precedes b, it must also causally precede b. At 2), let γ be a trip from a to b and μ a trip from b to c. By joining γ and μ by b we can show that there is a trip from a to c so that $a \ll c$. At 3), by applying both 1) and 2), we see that if $a \ll b$ and $b \ll c$, which implies $a \prec b$ and $b \prec c$, then $a \ll c$ so that $a \prec c$. □

Definition 1.43. Let us define the *chronological future* of p as $I^+(p) = \{q \in M | p \ll q\}$ and the *chronological past* as $I^-(p) = \{q \in M | q \ll p\}$. Analogously, let us define the *causal future* $J^+(p) = \{q \in M | p \prec q\}$ and the *causal past* as $J^-(p) = \{q \in M | q \prec p\}$. Given a set $S \subseteq M$, we define:

$$I^{\pm}(S) = \bigcup_{p \in S} I^+(p) \quad \text{and} \quad J^{\pm}(S) = \bigcup_{p \in S} J^{\pm}(p)$$

Proposition 1.44. *$I^+(p)$ is open for any $p \in M$.*

Proof. Let $x \in I^+(p)$ be a point in the chronological future of p, so that there exists a trip γ from p to x. Consider a point y in the last segment of γ with a tangent vector given by $\exp_y^{-1}(x) \in T_yM$. Since γ is future timelike, we have $g(\exp_y^{-1}(x), \exp_y^{-1}(x)) > 0$. Let $N \ni x$ be a simple region which also contains y. Let $Q \subseteq \exp_y^{-1}(N)$ be an open subset of all timelike future-pointing vectors. Since $Q \longrightarrow \exp_y(Q)$ is a diffeomorphism, then $\exp_y(Q) \ni x$ will be open. We know $\exp_y(Q) \subseteq I^+(y) \subseteq I^+(p)$ which implies $I^+(p)$ is open.

□

Proposition 1.45. *For any $x, y \in M$ and any $S \subseteq M$. The following statements are true:*

1. *$x \in I^+(y) \iff y \in I^-(x)$ (Analogously with $J^{\pm}$).*
2. *$I^+(S) = I^+(\overline{S})$*
3. *$I^+(S) = I^+(I^+(S)) \subseteq J^+(S) = J^+(J^+(S))$*

Proof. 1. Derived by applying proposition 1.42

2. We must show that the future of S can be reached from $\overline{S}$. Consider two points $x \in \overline{S}$ and $y \in I^+(x)$. There exists some $z \in I^-(y) \cap S$. Therefore, any point in the future of a point in $\overline{S}$, lies on the future of a point in S. See that $I^+(S) \subseteq I^+(\overline{S})$ is also true since $I^+(S)$ is open (see proposition 1.44).

3. Let us write $I^+(I^+(S)) = \bigcup_{x\in I^+(S)}\{y \in M | x \ll y\}$. Consider a point $y \in I^+(I^+(S))$. There exists some $x \in I^+(S)$ which chronologically precedes y ($x \ll y$). Since $x \in I^+(S)$, there exists some $z \in S$ satisfying $z \ll x \ll y \implies y \in I^+(S)$ and thus $I^+(I^+(S)) \subseteq I^+(S)$.
Conversely, if $x \in I^+(S)$, there exists some $z \in S$ satisfying $z \ll x$, so that there is a trip from z to x. Choose another point y in this trip. Then $y \in I^+(S)$ and $x \in I^+(y)$ such that $x \in I^+(I^+(S))$. Therefore $I^+(S) \subseteq I^+(I^+(S))$.

□

Definition 1.46. Let N be a simple region. Let us define the *world function* as the map $\Phi : N \times N \longrightarrow \mathbb{R}$ with $\Phi(p,q) = g(\exp_p^{-1}(q), \exp_p^{-1}(q))$. Note that $\Phi(p,q)$ is a continuous function of $(p,q) \in \overline{N} \times \overline{N}$. See $\Phi(p,q) = \Phi(q,p)$ and positive, negative or zero whether xy is timelike, spacelike or null.

Definition 1.47. Let N be a simple region. Consider a fixed point $p \in N$. The *hypersurfaces* defined as $H_{p,K} = \{x \in M | \Phi(p,x) = K\}$ for a constant value $K \in \mathbb{R}$ are smooth in $N \setminus \{p\}$ and spacelike, timelike or null whether K is positive, negative or zero respectively. Additionally, the geodesic px is orthogonal to $H_{p,K}$ at x.

Proof. Let us prove firstly the smoothness of $H_{p,K}$ over $N \setminus \{p\}$. Consider a simple region $N \ni p$. From $\exp_p^{-1}(N) \subseteq T_pM$, we can choose normal coordinates (t,x,y,z) over T_pM, so that $H_{p,K} = \{t^2 - x^2 - y^2 - z^2 = K\}$. Since $\exp_p^{-1}|_N$ is well behaved and the equation of $H_{p,K}$ in normal coordinates is smooth for all $K \in \mathbb{R}$ (except at the origin when $K = 0$), we can conclude that $H_{p,K}$ is smooth.
Secondly, let $q \in H_{p,K}$ vary along some curve with tangent vector field T so that pq describes a $1-$parameter system of a.p.geodesics of squared length K. Then T is a Jacobi field where $T_p = 0$. From 1.38 we know that if $g(T,T)$ is constant along the curve, then $g(T,S)$ too. Since $T_p = 0$, then $g(T,S) = 0$ such that T must be orthogonal, at q, to the direction of pq.

□

Proposition 1.48. *Let N be a simple region. Suppose a, b, $c \in \overline{N}$ are such that ab and bc are future causal with a joint on b (if both of them are null), or suppose that there is a timelike curve $\gamma \subseteq \overline{N}$ from a to c. Then, ac is a future timelike geodesic.*

Proof. Let us write the geodesic $\beta = ab \cup bc$ or take γ as a timelike curve. Let Φ be the world function and consider the function $\Phi_a(x) = \Phi(a,x)$, where x varies along β or γ into the future causal direction defined by the vector field T. Since both of them are future causal, then $\Phi_a(x) \geq 0$, being null only if ax is null and T tangent to ax . Consequently, $\Phi_a(c) = \Phi(a, c) > 0$ and thus ac is future timelike. □

Proposition 1.49. *If* $a \ll b$, $b \prec c \implies a \ll c$. *Same for* $a \prec b$, $b \ll c \implies a \ll c$

Proof. Let α be a trip from a to b and γ the causal trip from b to c. Since γ is compact, cover it by a finite sequence of simple regions. Set $x_0 = b \in N_{i_0}$. Let $x_1 \in \gamma \cap \overline{N}_{i_0}$ be the future endpoint of such connected component. Choose $y_1 \in N_{i_0}$ on the final segment of $\alpha \setminus \{x_0\}$. By 1.42, the geodesic y_1x_1 is future timelike. If $x_1 = c$ the proof would end. Otherwise, $x_1 \in N_{i_1} \setminus N_{i_0}$, so that let $x_2 \in \gamma \cap \overline{N}_{i_2}$ be the future endpoint of such connected component and choose $y_2 \in y_1x_1 \cap N_{i_1} \setminus \{x_1\}$. Then y_2x_2 is future timelike. Either $x_2 = c$ or we may continue. Since we have a finite number of connected components $\gamma \cap \overline{N}_i$, we will eventually finish by following this process. □

Proposition 1.50. *Let* α *be a null geodesic from* a *to* b *and* β *a null geodesic from* b *to* c. *Then* $a \ll c$ *or else* $\alpha \cup \beta$ *is a unique null geodesic from* a *to* c.

Proof. If $\alpha \cup \beta$ is not a unique null geodesic , then α and β must have different "directions". Otherwise, let $x \in ab$ and $y \in bc$ be in their respective segments, then $a \prec x \ll y \prec c \implies a \ll c$. □

Proposition 1.51. *If* $a \not\ll b$ *and* $a \prec b$ *then there is a null geodesic from* a *to* b

Proof. Let γ be a causal trip from a to b. If γ has a timelike segment, we can apply 1.49 to obtain $a \ll b$, which leads to a contradiction. Instead, let γ contain null segments with joints. By applying 1.50 repeatedly, we would obtain $a \ll b$ unless γ is null. □

The following proposition shows the equivalence between trips and curves. To do so, we need to smooth out the trips and conversely, we need to stagger the curves.

Proposition 1.52. $a \ll b \iff$ *there is a timelike curve from* a *to* b

Proof. Let γ be a timelike curve from a to b. Cover γ with a finite number of simple regions N_i. Set $x_0 = a$ and $x_1 \in \gamma \cap \overline{N}_{i_0}$ future endpoint from x_0, so that x_0x_1 is future timelike. Let $x_1 \in N_{i_1} \setminus N_{i_0}$ and let $x_2 \in \gamma \cap \overline{N}_{i_1}$ be the future endpoint from x_1 such that x_1x_2 is future timelike too. Since there is a finite number of connected

components $\gamma \cap \overline{N}_i$, we will eventually reach b by applying this process.
Let us prove the converse. Consider two consecutive segments of the trip between a and b, denoted as λ and μ, jointed at q. Take the normal coordinates at T_qM so that $\exp_q^{-1}(\lambda) = (-\tau, \tau \tan\chi, 0, 0)$ and $\exp_q^{-1}(\mu) = (\tau, \tau \tan\chi, 0, 0)$ where τ varies over non-negative values and $0 \le \chi \le \pi/4$ is fixed. Choose a particular $\tau_0 > 0$ so that $(-\tau_0, \tau_0 \tan\chi, 0, 0)$ and $(\tau_0, \tau_0 \tan\chi, 0, 0)$ can be connected by a C^∞ curve $\eta \subseteq T_qM$. By choosing a small enough τ_0, we assure $\exp_q(\eta)$ to be timelike in M. With this, we smooth $\lambda \cup \mu$ into a curve. Repeating this process for every pair of segments of the trip, we will reach the equivalent curve. □

Definition 1.53. γ is a causal curve $\iff$ for any point a, $b \in \gamma$ and for every open set Q containing the segment $ab \in \gamma$, there is a causal trip from a to b in Q

In Penrose's work, trips will be more usual than smooth curves for the sake of simplicity and easiness of proofs. If curves were used, we should constantly apply smoothening arguments as mentioned in 1.52.

1.3 Pasts and Futures

In this section, we will define the past and future sets along with their properties. Informally, in a Minkowski metric, the past or future sets can be regarded as the past and future cones.

Definition 1.54. A set $F \subseteq M$ is called *future set* if there exists some $S \subseteq M$ such that $F = I^+(S)$. We say F is future set if and only if $F = I^+(F)$. Note that F is an open set by proposition 1.44.

Definition 1.55. A subset $P \subseteq M$ a called a *past set* if there exists some $S \subseteq M$ such that $P = I^-(S)$. Note that P is a past set if and only if $P = I^-(P)$.

Proposition 1.56. *Let $F \subseteq M$ be a future set.* $\overline{F} = \{x \in M | I^+(x) \subseteq F\}$

Proof. If $p \in \{x \in M | I^+(x) \subseteq F\}$, then $I^+(p) \subseteq F$ such that every trip $\gamma \subseteq I^+(p)$ contains points arbitrarily near to p in F. Thus, $p \in \overline{F}$. Conversely, if $p \in \overline{F}$, consider a point $q \in I^+(p)$. Since $I^-(q)$ is open, there exists some $r \in F$ so that if $q \in I^+(r) \implies q \in F \implies I^+(p) \subseteq F \implies p \in \{x \in M | I^+(x) \subseteq F\}$. □

In the following proposition, we denote $\neg F = M \setminus F$.

Proposition 1.57. *Let $F \subseteq M$ be a future set.*

a) $\overline{F} = \neg I^-(\neg F)$

b) $\partial F = (\neg F) \cap (\overline{F})$

c) $F = I^+(\overline{F})$

Proof. a) Let $S \subseteq M$ be a subset and take $F = I^+(S)$. Since there are no closed trips, assume $p \not\ll p$ for all $p \in M$. The complementary is $\neg F = \{x \in M | x \notin F\}$. If $p \in \neg F$, then $p \in \partial F$, $p \in \overline{P} = \overline{I^-(S)}$ or $p \in \neg(\overline{F} \cup \overline{P})$.
For all $p \in \neg F$, consider a point $q \in I^-(\neg F)$. Let us study the following cases. If:

- $p \in \partial F$, then $q \in \neg(\overline{F} \cup \overline{P})$ or $q \in \overline{P}$
- $p \in \neg(\overline{F} \cup \overline{P})$, then $q \in \neg(\overline{F} \cup \overline{P})$ or $q \in \overline{P}$
- $p \in \overline{P}$, then $q \in P$

Consider a point $r \in \neg I^-(\neg F)$. Then $r \notin \overline{P}$ and $r \notin \neg(\overline{F} \cup \overline{P})$. This means $r \in \partial F$ or $r \in F$, so that $I^+(r) \subseteq F$. Therefore, $r \in \overline{F}$. Conversely, if $r \in \overline{F}$, then $I^+(r) \subseteq F$, that is, $r \notin \overline{P}$ and $r \notin \neg(\overline{F} \cup \overline{P})$, which implies $r \in \neg I^-(\neg F)$.

b) The proof is stems from set theory.

c) Recalling the second statement from proposition 1.45, if $F \subseteq M$ then $F = I^+(F) = I^+(\overline{F})$.

□

Remark 1.58. Note that in the proof of the last statement, $I^+(\overline{F}) \neq \overline{F}$ as $\overline{F}$ is neither a future set nor open, only its closure.

Proposition 1.59. *Let $Q \subseteq M$. The following statements are equivalent:*

1. $I^+(Q) \subseteq Q$
2. $I^-(\neg Q) \subseteq \neg Q$
3. $I^+(Q) \cap I^-(\neg Q) = \emptyset$
4. $\overset{\circ}{Q} = I^+(Q)$
5. $\partial Q = [\neg I^+(Q)] \cap [\neg I^-(\neg Q)]$

Proof. For the sake of brevity, we refer to [17, Section 3] □

Proposition 1.60. *If $Q = \overset{\circ}{Q}$ and $I^+(Q) \subseteq Q$ then Q is a future set.*

Proof. Since we can write $Q = \overset{\circ}{Q} = I^+(Q)$, it becomes a future set. □

Proposition 1.61. *Let $S_i \subseteq M$, $Q = I^+(Q)$ and $R = I^+(R)$. Then:*

a) $\bigcup_i I^+(S_i) = I^+(\bigcup_i S_i)$

b) $Q \cap R = I^+(Q \cap R)$

Proof. a) If $p \in \bigcup_i I^+(S_i)$, there exists a point $q \in S_k$, for some index k, such that $p \in I^+(q) \subseteq I^+(S_k) \subseteq \bigcup_i I^+(S_i)$. Then, $q \in S_k \subseteq \bigcup_i S_i$ and since $p \in I^+(q)$ we obtain $p \in I^+(\bigcup_i S_i)$. Conversely, if $p \in I^+(\bigcup_i S_i)$, there exists a point q for some index k such that $q \in S_k$ satisfies $p \in I^+(q)$. Then, $p \in \bigcup_i I^+(S_i)$.

b) $Q \cap R = I^+(Q) \cap I^+(R) \supseteq I^+(Q \cap R)$. Since Q and R are open, then $Q \cap R$ is open, and we can apply the proposition 1.60 to show that $Q \cap R = I^+(Q \cap R)$. □

Proposition 1.62. *If $p \prec q \implies I^+(p) \supseteq I^+(q)$*

Proof. Let $r \in I^+(q)$, so that $q \ll r$. Since $p \prec q \ll r$ we have $p \ll r$. Then, r lies to the future of p ($r \in I^+(p)$). □

Proposition 1.63. *Let $S \subseteq M$. Then $J^+(S) \subseteq \overline{I^+(S)}$*

Proof. If $x \in S$, there exists some $y \in J^+(S)$ satisfying $x \prec y \implies I^+(x) \supseteq I^+(y)$, and since $I^+(x) \subseteq I^+(S) \implies x \in \{x \in M | I^+(x) \subseteq I^+(S)\} = \overline{I^+(S)}$. □

Definition 1.64. A subset $S \subseteq M$ is *achronal* if no pair of points $x, y \in S$ is chronologically related.

Definition 1.65. A subset $B \subseteq M$ is an *achronal boundary* if $B = \partial F$ for a future set $F \subseteq M$.

Remark 1.66. From definitions, every achronal boundary is an achronal set, but the converse is not true.

Proposition 1.67. *$B \subseteq M$ is an achronal boundary $\iff$ there exists some $T \subseteq M$ such that $B = \partial I^-(T)$*

Proof. If B is an achronal boundary, there must be a future set $F \subseteq M$ such that $B = \partial F$. Choose $T = \neg F$ so that $B = \partial I^-(\neg F)$ because $I^-(x) \subseteq I^-(\neg F)$ if and only if $x \notin F$, and $x \notin I^-(\neg F)$ if and only if $I^+(x) \subseteq F$. Conversely, if there is a $T \subseteq M$ such that $B = \partial I^-(T)$, we can take the time inverse and obtain that B is an achronal boundary. □

The following proposition will allow us to dissect every space-time M into "three disjoint pieces" .

Proposition 1.68. *If $B \neq \emptyset$ is an achronal boundary, there exists a unique pair of future and past sets F, P such that $M = F \sqcup P \sqcup B$, then $B = \partial F = \partial P$. In addition, for any trip or timelike curve γ from P to F, there exists a unique point $p \in B$ such that $\gamma \cap B = \{p\}$.*

Proof. Let us prove the existence of P and F sets. From the definition of achronal boundaries, there exists a future set F satisfying $B = \partial F$. By proposition 1.67, there exists a past set $P = I^-(\neg F)$ with $B = \partial P$. Thus, we get $B = \partial F = \partial P$.
Before proving the uniqueness of such sets, let us show that for any trip or timelike curve γ from past to future, there exists a unique point $p \in B$ satisfying $\gamma \cap B = \{p\}$. Let us write $M = F \sqcup P \sqcup B$ as a decomposition of space-time with $P = I^-(P)$ and $F = I^+(F)$, where γ goes from a point $a \in P$ to a point $b \in F$. Note that $\gamma \cap (\neg F)$ and $\gamma \cap (\neg P)$ are closed such that $[\gamma \cap (\neg F)] \cup [\gamma \cap (\neg P)] = \gamma \cap (\neg F \cup \neg P) = \gamma \cap M = \gamma$ and thus $[\gamma \cap (\neg F)] \cap [\gamma \cap (\neg P)] = \gamma \cap B = \{p\}$ which must be unique since B is achronal.
Finally, let us prove the uniqueness of such sets. Suppose there exists another pair of past and future sets P' and F' which also satisfy $M = F' \sqcup P' \sqcup B$. Thus, either $P \cap F' \neq \emptyset$ or $P' \cap F \neq \emptyset$. Without loss of generality, consider the first case, so that there exists a point $x \in P \cap F'$ and another point $y \in B$ connected by a curve ζ. From the second point of this proof, we see that ζ cannot exit F' without crossing B in the past direction, thus into the P, and ζ cannot exit P without crossing B towards F' into the future direction. This implies $\zeta \subseteq P \cap F'$ but $y \notin P \cap F'$, leading us to a contradiction. □

Proposition 1.69. *Any achronal boundary is a topological $3-$manifold.*

Proof. Broad lines of this proof will be given in favour of brevity of this paper. We have to show that any achronal boundary B is locally homeomorphic to E^3. In order to do so, choose P and F as 1.68 and a point $a \in B$, contained in a simple region N where we choose normal coordinates. Then, we take a region $Q \subseteq N$ and establish a one-to-one mapping between $B \cap Q$ and the interior of a sphere in $\mathbb{R}^3$. A more extensive proof can be seen at [17, Section 3]. □

Having defined most of the properties of the past and future sets, we are going to prove the existence of future endless trips or geodesics given a past endpoint (analogously for past endless trips and future endpoints). This set of arguments will be of great use when proving the singularity theorems.

Proposition 1.70. *Let F be a future set and let $B = \partial F$ be an achronal boundary. Let $x \in B$ be a point and suppose there exists an open neighbourhood $Q \ni x$ satisfying:*

a) $\forall y \in Q \cap F \quad \exists \gamma$ *trip from* $z \in F \setminus Q$ *to* y*; or equivalently*

b) $F = I^+(F \setminus Q)$

Then, there exists a null geodesic $\eta \subseteq B$ with future endpoint x.

Proof. Firstly, let us prove the equivalence between $a)$ and $b)$. Let us write $F = I^+(F \setminus Q) = \bigcup_{y \in F \setminus Q} I^+(y) = \bigcup_{y \in F \setminus Q} \{z \in M | y \ll z\}$. For any $y \in Q \cap F = Q \cap \bigcup_{x \in F \setminus Q} I^+(x)$ there exists some $z \in F \setminus Q$ such that $z \ll y$.
Conversely, we must show that if $y \in F$, there exists a $z \in F \setminus Q$ satisfying $z \ll y$. Let F be a future set, so that there exists a point $w \in F$ with $w \ll y$. If $w \in F \cap Q$, we use the hypothesis to obtain $z \ll w \ll y$. If $w \in F \setminus Q$, then we can take $z = w$.
Now, let us prove the main implication. Let $N \subseteq Q$ be a simple region containing $x \in \overline{N \cap F}$ since $x \in \overline{F}$ and $N \ni x$ is open, and let $\{y_i\}_{i=1}^n \subseteq N \cap F$, with $n \geq 1$, be a sequence of points converging towards x. Let $z_i \in F \setminus Q \subseteq F \setminus N$ be such that $z_i \ll y_i$ along a trip γ_i. Let $v_i \in F \cap \partial N$ be the past endpoint of $\gamma_i \cap \overline{N}$ which terminates at y_i. Note that $v_i y_i$ is timelike and since ∂N is compact, the sequence of v_i accumulates to $v \in \overline{F} \cap \partial N$. Since $v_i y_i$ is timelike, if $y_i \to x$ then vx will be timelike or null, so that $\Phi(v_i, x_i) \to \Phi(v, x) \geq 0$. Note that vx cannot be timelike since $v \in \overline{F}$ and $x \not\ll F$, then vx is a null geodesic. Let us denote it as η.
Finally, we see that $\eta \not\subseteq F$ because if there existed some $w \in F$ with $w \prec x$, there would exist some $u \in F$ satisfying $u \ll w \prec x \implies u \ll x$, where $x \in F$, which is a contradiction, whereas $v_i y_i \subseteq F$. Therefore, $\eta \subseteq B$.

□

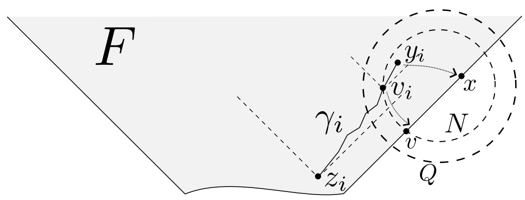

Figure 1.1: Diagram for 1.70. $F \subseteq M$ is a future set in Minkowski space. The dotted arrows show where sequence of points tend to accumulate.

Theorem 1.71. *Let $S \subseteq M$ be a subset and let $B = \partial I^+(S)$ be an achronal boundary. If $x \in B \setminus \overline{S}$, there exists a null geodesic $\eta \subseteq B$ with future endpoint x, which is either past endless or has a past endpoint on $\overline{S}$*

Proof. Note $\overline{S}$ is closed and $x \notin \overline{S}$, so that we can choose an open neighbourhood $Q \ni x$ where $Q \cap \overline{S} = \emptyset$. With this, the first hypothesis of proposition 1.70 is satisfied. Therefore, there exists a null geodesic $\eta \subseteq B$ with future endpoint x. We can extend η into the past on B such that either η is past endless or not. If η is not past endless, since B is closed, there exists a past endpoint $y \in B$. If $y \notin \overline{S}$, we can apply again proposition 1.70 to obtain a null geodesic $\zeta \subseteq B$ with future endpoint y. However, if ζ and η are both null in different directions, by proposition 1.50 some points in ζ would chronologically precede x. This leads us to a contradiction since B is achronal. Thus, ζ must continue η. □

Proposition 1.72. *Let $S \subseteq M$ be a subset, $B = \partial I^+(S)$ an achronal boundary. Let η, $\zeta \subseteq B$ be null geodesics with endpoint $x \in B \setminus \overline{S}$.*

a) If x is the past endpoint of one or both of the geodesics, then $\eta \cup \zeta \subseteq B$ is a null geodesic.

b) If x is the future endpoint of both geodesics and neither one of them is contained into the other one, then every extension of one of the geodesics towards the future beyond x leaves B and enters $I^+(S)$.

Proof. a) Let x be the past endpoint of one of the geodesics and the future endpoint of the other one. For instance, x will be the future endpoint of η and the past endpoint of ζ. Since $\eta, \zeta \subseteq B$ are both null, and B is achronal, then by 1.50 $\eta \cup \zeta$ is a null geodesic. If x is the past endpoint of both of them, then by Theorem 1.71, there exists another null geodesic $\xi \subseteq B$ with x as the future endpoint such that $\xi \cup \eta \cup \zeta \subseteq B$ is null.

b) Let us denote as $\xi \subseteq B$ the extension of η into the future beyond x. By a), $\xi \cup \eta \cup \zeta$ must be a single null geodesic, which is impossible unless $\eta \subseteq \zeta$ or $\zeta \subseteq \eta$. Let us exclude that situation, so that $\xi \nsubseteq B$. Since $\xi \subseteq J^+(x)$, then $\xi \setminus \{x\} \subseteq I^+(S)$ by proposition 1.63.

□

Chapter 2

Causal conditions and space-time topologies

In this chapter we will continue developing notions on causal structure. Firstly, we will state the definition of strong causality and its importance regarding the singularity theorems. Afterwards, we will show the implications of strong causality failure at one point. Finally, we will focus on particular regions of space-time determined by its initial data and the topology of causal curves.

2.1 Causality

Having developed core concepts of Lorentzian geometry and before attaining results from general relativity, it is important to note the *causal hierarchy of space-time*, which comprises several levels of causal properties applied to space-time. At the top level lies the strongest causal condition, known as *global hiperbolicity*, which will be discussed in detail later in this work. Conversely, at the bottom lies the weakest condition, referred to as *non-totally viciousity*, which requires that at least one point in space-time not to chronologically precede itself. As we can observe, a space-time becomes more physical as stronger the causality condition is. In this work, we will focus on "physical" space-times. This is the reason why we will be discussing the notions of *strong causality condition* and even stronger ones.

Definition 2.1. A space-time M is said to be *future distinguishing at* $p \in M$ if and only if for any other point $q \in M$, $I^+(p) \neq I^+(q)$ is satisfied. It is analogously defined for *past distinguishing* space-times.

Definition 2.2. An open subset $Q \subseteq M$ is *causally convex* if and only if for all trip γ, the intersection $Q \cap \gamma$ is connected.

Definition 2.3. A space-time M is *strongly causal* at a point $p \in M$ if and only if there exists arbitrarily small causally convex neighbourhoods Q of p.

Therefore, a strongly causal space-time will not contain quasi-closed chronological trips at any neighbourhood. Note that for any point in our space-time, we could always choose an arbitrarily small neighbourhood where strong causality fails. If it was the case, it should be in order to preserve its global structure. However, we are going to introduce a series of definitions to justify the existence of strongly causal space-times.
The following definition will denote the sets that contain all the possible trips between two points in the space-time.

Definition 2.4. Let $Q \subseteq M$ be an open subset. For all x, y, we write $x \ll_Q y$ if and only if there exists a trip $\gamma \subseteq Q$ from x to y. Analogously, $x \prec_Q y$ but with causal trips. Let us denote:

$$\langle x, y\rangle_Q = \{z \in Q | x \ll_Q z \ll_Q y\}; \quad \langle x, y\rangle_M = \langle x, y\rangle = I^+(x) \cap I^-(x)$$

Proposition 2.5. *$\langle x, y\rangle$ is open. $\langle x, y\rangle_Q$ is open if $Q \subseteq M$ is open in M.*

Proof. Since past and future sets are open, $I^+(x) \cap I^-(x) = \langle x, y\rangle_M$ is open. Thus, $\langle x, y\rangle_Q$ is open in Q as a space-time. If $Q \subseteq M$ is open, then $\langle x, y\rangle_Q$ will also be open in M. □

Proposition 2.6. *Let $N \subseteq M$ be a simple region. For all $x, y \in N$, and for every trip $\gamma \subseteq N$, then $\gamma \cap \langle x, y\rangle_N$ is connected.*

Proof. Given any $x, y \in N$, let us consider $\langle x, y\rangle_N \neq \emptyset$. Let $\eta \subseteq N$ be a trip, and consider some points $u, v \in \eta \cap \langle x, y\rangle_N$ with $u \ll v$ along η. Since N is a simple region, there are unique geodesics xu and vy such that $xu \cup \eta \cup vy$ is a connected trip contained in $\langle x, y\rangle_N$ (by definition of $\langle x, y\rangle_N$). Since this applies to any pair of points $u, v \in \eta \cap \langle x, y\rangle_N$, we see that $\eta \cap \langle x, y\rangle_N$ is connected. □

Proposition 2.7. *Let $N \subseteq M$ be a simple region, and let $Q \subseteq N$ be an open subset containing a point p. There exists a pair of points $u, v \in Q$ such that $p \in \langle u, v \rangle_N \subseteq Q$*

Proof. Let us choose Minkowski normal coordinates for N with origin at p. Let us define a ball $B = \{t^2 + x^2 + y^2 + z^2 \leq \epsilon^2\} \subseteq Q$ for an $\epsilon > 0$, and choose the coordinates $u = (-\epsilon/2, 0, 0, 0)$ and $v = (\epsilon/2, 0, 0, 0)$. For any timelike geodesic γ which crosses the hemisphere $\{t^2 + x^2 + y^2 + z^2; t < 0\}$, there exists a point $q \in \gamma \cap I^-(v)$ such that $vq \cup qu$ is a trip. Let $w \in \langle u, v \rangle_N$ be a point so that uw and wv are future timelike, and $uw \cup vw$ will be a trip which cannot be extended beyond q. Then $\langle u, v \rangle \subseteq B \subseteq Q$ □

Proposition 2.8. *Any simple region $N \subseteq M$, regarded as a space-time, is also strongly causal*

Proof. It follows from the definitions of simply convex neighbourhoods and propositions 2.7 and 2.6. □

With the above proposition, we can always work with strongly causal space-times by dealing with simple regions.

Definition 2.9. Let $N \subseteq M$ be a simple region. An open subset $L \subseteq M$ is a *local causality neighbourhood* if it is causally convex and $\overline{L} \subseteq N \subseteq M$.

Proposition 2.10. *A space-time M is strongly causal at p if and only if p is contained in a local causality neighbourhood L.*

Proof. If M is strongly causal at p, there exists an open causally convex neighbourhood $Q \ni p$. By choosing a simple region $N \subseteq M$ such that $Q = \overline{Q} \subseteq N$ we ensure local causality.
Conversely, let $L = \overset{\circ}{L} \ni p$ be a local causality neighbourhood, and let $N \subseteq M$ be a simple region where $\overline{L} \subseteq N$. By 2.7, consider any pair of points $u, v \in L$ such that $p \in \langle u, v \rangle_N \subseteq L$. With this, $\langle u, v \rangle_N$ is open and for any trip γ, we must have $\gamma \cap \langle u, v \rangle_N$ connected. Then $\langle u, v \rangle_N$ is causally convex and thus, M is strongly causal at p. □

Proposition 2.11. *The set of strongly causal points at M is open.*

Proof. It follows from 2.10. □

Proposition 2.12. *Let $A \subseteq M$ be strongly causal, then A can be covered by a locally finite set of local causality neighbourhoods. If A is compact, then with a finite number of local causality neighbourhoods will be sufficient.*

Proof. It follows from 2.10 and [13]. □

Proposition 2.13. *Let L be a local causality neighbourhood. There are no future or past endless causal trips γ contained in L.*

Proof. Let us prove it by contradiction. Suppose that there is a future endless causal trip $\gamma \subseteq L$. Let $\{p_i\}_i$ be an infinite sequence of points along γ. Since $\overline{L}$ is a closed subset of a simple region N, then $\overline{L}$ is compact, so that the sequence of points $\{p_i\}_i$ must accumulate towards a point $p \in \overline{L}$. Therefore, for every neighbourhood $Q \ni p$ there are infinitely many i such that $p_i \in Q$. Note that p cannot be a future endpoint of γ since it is future endless. Consequently, there are points along γ into the future and into the past beyond Q.
We can choose a pair of points $u, v \in Q$ such that $p \in \langle u, v \rangle_N \subseteq Q \subseteq N$. Then, $\langle u, v \rangle_N$ must contain infinitely many $p_i \in \gamma$ but does not contain the infinitely many points between each p_i. This means that γ must leave and re-enter $\langle u, v \rangle_N$, such that each $p_i \in \langle u, v \rangle_N$ while the intercourse between each point is outside of it. We have found a contradiction with 2.6.

□

Proposition 2.14. *M is not strongly causal at $p \in M \iff \exists q \neq p, q \prec p$ such that $\exists x, y$ with $x \ll p$, $q \ll y$ then $x \ll y$*

Proof. Suppose M is not strongly causal at p. Let $N \subseteq M$ be a simple region containing p. Consider a sequence of nested neighbourhoods $Q_i = \langle u_i, v_i \rangle_N \ni p$ ($Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n$) converging on p, that is $\bigcap_i Q_i = \{p\}$. Take $\overline{Q}_i \subseteq N$.
Since strong causality fails at p, each Q_i fails to be a local causality neighbourhood. This means that trips re-enter Q_i. Consider a trip γ_i with past endpoint $a_i \in Q_i$ and future endpoint $d_i \in Q_i$. Let $\gamma_i \cap \partial N = b_i$ and $\gamma_i \cap \partial N = c_i$ be two points where γ_i exits and re-enters N respectively. Since ∂N is compact, take $b, c \in \partial N$ to be the accumulation points of $\{b_i\}_i$ and $\{c_i\}_i$ respectively. Since $c_i d_i$ is timelike, then cp must be causal. Suppose a point $q \in cp$ and let us choose a point y with $q \ll y$. Consider also a point x in the past of p, $x \ll p$.
With all, $x \ll p \ll a_i \ll b_i \ll c_i \ll y \implies x \ll y$.

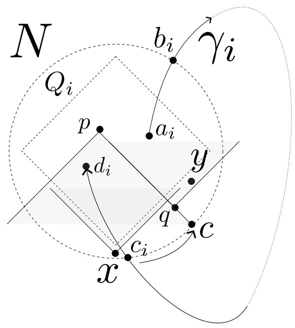

Figure 2.1: Diagram for the implication considering strong causality failure at p. It is shown $Q_i \subseteq N$ and a trip γ_i which intersects Q_i at a disconnected set.

Conversely, assume there exists two different points p, q, with $q \prec p$ such that there exists x, y satisfying $x \ll p$ and $q \ll y$, then $x \ll y$. Let $P \ni p$ and $Q \ni q$ be two disjoint open neighbourhoods. We must show that P is not causally convex independently of how small it is chosen. Let $z \in I^+(p) \cap P$ and take $x \in I^-(p) \cap P$ and $y \in I^-(z) \cap I^+(q) \cap Q$. Note $P \cap I^+(p) \neq \emptyset$ and we have $q \prec p \ll z$ so that $q \in I^-(z)$. Since $I^-(z)$ is open, we will have $Q \cap I^+(q) \cap I^-(z) \neq \emptyset$. We can choose a trip from $x \ll p$, $q \prec p$, $q \ll y$, $y \ll z$ which re-enters P. $\square$

Proposition 2.15. *If M is strongly causal at p, then M is future-distinguishing at p.*

Proof. Let us prove it by contraposition. Let p, q be two different points at M such that $I^+(p) = I^+(q)$. Let $P \ni p$ and $Q \ni q$ be two disjoint open neighbourhoods. Let $x \in P \cap I^+(p) = P \cap I^+(q)$, then $q \ll x$. Let $y \in Q$ be a point with $q \ll y \ll x$. Since $I^+(p) = I^+(q)$, we obtain $p \ll y$. Thus, there is a trip from p to x passing by $y \in Q$ (and $y \notin P$). Since this is valid for every P, we obtain that M is not strongly causal at p. $\square$

Definition 2.16. A space-time M is *stably causal* if it cannot be made to contain closed trips by arbitrarily small perturbations of the metric.

The condition of *causal stability* on a space-time lays higher on the causal hierarchy. Here it is mentioned to note the existence of a maximally restrictive causality condition. This concept was developed by Hawking by introducing the *global time function*, as it can be seen at [8].

Proposition 2.17. *If $p \ll q$ and $p \ll r$, there exists a point w such that $w \ll q$, $w \ll r$ and $p \ll w$.*

Proof. Since $p \in I^-(q) \cap I^-(r)$ is open, there is a point $w \in I^-(q) \cap I^-(r)$ with $p \ll w$. □

Proposition 2.18. *Let $x, p, q, r, s \in M$ be such that $x \in \langle p, q \rangle \cap \langle r, s \rangle$. There exists $u, v \in M$ such that $x \in \langle u, v \rangle \subseteq \langle p, q \rangle \cap \langle r, s \rangle$*

Proof. If $x \in \langle p, q \rangle \cap \langle r, s \rangle \iff p \ll x \ll q$ and $r \ll x \ll s$. If $p \ll q$, $p \ll s$, there exists a point v with $x \ll v$, $v \ll q$, $v \ll s$ and $p \ll v$. Analogously for u into the past by using 2.17. □

Definition 2.19. A topology (M, τ) is an *Alexandrov topology* if for any open set $U \subseteq \tau$ and for any point $x \in U$, there is an open set $B \subseteq \mathcal{B}$ such that $x \in B \subseteq U$. Note $\mathcal{B}$ is a base of a topology whose open sets are $\langle u, v \rangle$ for any $u, v \in M$

In all the previous work in this paper, we have defined causal and chronological relations between points on a manifold. When doing this, we have been dealing with the *manifold topology*, which does not contain any physical meaning. However, we can construct a topology with higher physical meaning based on the causal and chronological relations. By doing so, we can admit structures different from the usual smooth manifolds. Therefore, as it will be seen further in this document, singularity theorems may depend on space-times being strongly causal. Since such theorems must be applicable to all space-times satisfying the required conditions, we must take a step forward and separate ourselves from manifold structure. Alexandrov topologies and the following theorem allows a global understanding of space-times and the causality relations in itself. More can be seen at [14], but it will be no longer discussed for the sake of brevity.

Theorem 2.20. *The following statements are equivalent:*

a) M is strongly causal

b) The Alexandrov topology agrees with the manifold topology

c) The Alexandrov topology is Hausdorff

Proof. Let us begin by proving $a) \implies b)$. Suppose M is strongly causal. We must show that any open set in the manifold topology is an open set in the Alexandrov topology. Let p be a point where M is strongly causal and let $P \ni p$ be an open set of the manifold topology. We have to show that an Alexandrov neighbourhood of p exists in P. Let $N \subseteq M$ be a simple region with $p \in N \subseteq P$. Let Q be a causally convex neighbourhood with $p \in Q \subseteq N$. Then, there are two points $u, v \in Q$ such

that $p \in \langle u,v\rangle_N \subseteq Q$. In order to facilitate the proof, let us show, $\langle u,v\rangle_N = \langle u,v\rangle_M$ by contradiction: Suppose $\{w \in N | u \ll_N w \ll_N v\} \neq \{w \in M | u \ll w \ll v\}$. The set $\langle u,v\rangle_N$ exhausts all possible trips in N between u and v, and since $N \subseteq M$, it exhausts all possible trips on M. Thus, if $\langle u,v\rangle_N \neq \langle u,v\rangle$ there would be a trip leaving and re-entering $\langle u,v\rangle_N$, which is a contradiction. Therefore, $p \in \langle u,v\rangle_N \subseteq Q \subseteq P$
From $b) \implies c)$ the proof is direct, since M is already Hausdorff in the manifold topology.
From $c) \implies a)$, let us prove it by contraposition. Suppose M is not strongly causal at $p \iff \exists q \neq p, q \prec p$ such that $\exists x \ll p, q \ll y$, then $x \ll y$. Let $p \in \langle x,u\rangle$, $q \in \langle v,w\rangle$ and choose $y \in I^-(u) \cap \langle v,w\rangle$. Since $x \ll y$, then $y \in \langle x,u\rangle \implies \langle x,u\rangle \cap \langle v,w\rangle \neq \emptyset$ and thus, we have proven that any Alexandrov set intersects any other Alexandrov set, showing that M is not Hausdorff. This leads us to the required contradiction.

□

Definition 2.21. A point $p \in M$ is a *vicious point* if closed trips pass through it. The set of all vicious points of M and the boundary are defined as:

$$V = \bigcup_{x \in M} \langle x,x\rangle; \quad \partial V = \bigcup_{x \in M} \partial\langle x,x\rangle$$

Definition 2.22. Let $S \subseteq M$. If $U \subseteq M$ such that for any two different points $x, y \in S$, if $x \ll_U y$ and $y \ll_U x$ then S will be said to be *vicious with respect to U*.

Proposition 2.23. *If $\langle x,x\rangle \cap \langle y,y\rangle \neq \emptyset$, then $\langle x,x\rangle = \langle y,y\rangle$*

Proof. Let $z \in \langle x,x\rangle \cap \langle y,y\rangle$ be a point, then $x \ll z \ll x$ and $y \ll z \ll y \implies x \ll z \ll y$ and $y \ll z \ll x$. Therefore, $\langle x,x\rangle \subseteq \langle y,y\rangle$ and $\langle y,y\rangle \subseteq \langle x,x\rangle$. □

Note that this means that V is the union of disjoint sets.

Proposition 2.24. *For all $p \in \partial V$, M is not strongly causal at p.*

Proof. The proof is derived from the definition of strong causality, since given any point $p \in \partial V$, trips would re-enter any open neighbourhood of p. □

Proposition 2.25. *If future distinction fails at $p \notin V$, then $p \in \gamma \subseteq \neg V$, where γ is a null geodesic along which future distinction fails.*

Proof. Let p be the point where M is not future distinguishing. Note that $p \notin V \iff p \notin I^+(p)$, thus $p \in \partial I^+(p)$. Let $q \neq p$ be another point with $I^+(p) = I^+(q)$. Consider the achronal boundary $B = \partial I^+(p) = \partial I^+(q)$. By applying Theorem 1.72, we can obtain a past-endless null geodesic $\eta \subseteq \partial I^+(p)$ with future endpoint p. Now, for all points $r \in \eta \subseteq \partial I^+(p)$, if $r \prec p$ then $I^+(r) \supseteq I^+(p)$. If $p \in \partial I^+(p)$, we obtain $I^+(r) = I^+(p)$ for all $r \in \eta$. Finally, for all $r \in \eta$, $r \notin I^+(p) = I^+(r)$, thus $r \notin V \implies \eta \subseteq \neg V$

□

Theorem 2.26. *Let strong causality fail at $p \in M$. Then, at least one of the following statements is true:*

a) $p \in V$

b) There exists a past endless null geodesic $\eta \subseteq \partial V$ along which future distinction fails and $p \in \eta$.

c) There exists a future endless null geodesic $\eta \subseteq \partial V$ along which past distinction fails and $p \in \eta$.

d) There exists a past endless null geodesic $\eta \subseteq \partial V$ along which future distinction fails, and there exists a future endless null geodesic $\zeta \subseteq \partial V$ along which past distinction fails, where $p \in \zeta$ and $p \in \eta$.

e) There exists an endless null geodesic $\eta \subseteq \partial V$ along which strong causality fails and $p \in \eta$.

Proof. Let $N \subseteq M$ be a simple region with $p \in N$, and let $Q_i = \langle u_i, v_i \rangle_N$ be a nested sequence of subsets such that $\{p\} = \bigcap_i Q_i$. By 2.14, let γ_i be a trip such that $\gamma_i \cap Q_i$ is a disconnected set. Let $a_i \in Q_i$ be the past endpoint and $d_i \in Q_i$ the future endpoint. Denote $b_i = \gamma_i \cap \partial N$ as the first point where γ_i leaves N and $c_i \in \gamma_i \cap \partial N$ the point where γ_i last re-enters N. Since ∂N is a compact set, the sequence of points accumulates at $b, c \in \partial N$. Since pb_i and $c_i p$ are future timelike, then pb and cp are both future causal. With this situation, there are several cases to be discussed:

I If pb, cp are both timelike, there is an index i such that $b_i \in I^+(p)$ and $c_i \in I^-(p) \implies p \in V$

II If pb is timelike and cp is null, there exists a point $x \in \langle p, b \rangle_N$ with $c \ll x$ and an index i so that $c_i \ll x \ll b_i$. Taking into account $b_i \ll c_i$, then $x \in V$. Therefore, $\langle p, b \rangle_N \subseteq V$ and $p \in \overline{V}$. There are two possible cases:

i If $p \in V = \overset{\circ}{V}$, then it is the same case as I.

ii If $p \notin V$, then $p \in \partial V$, for all $y \in \langle c, x \rangle$ we get $p \ll x \ll b_i \ll c_i \ll y$ for some i big enough. Then $I^+(c) \subseteq I^+(p)$, and since $c \prec p$ we can obtain $I^+(c) \supseteq I^+(p)$. We can state $I^+(c) = I^+(p)$ so that future distinction fails at p. By proposition 2.25, p lies on a past endless null geodesic $\gamma \subseteq \neg V$ along which future distinction fails. Let γ be the maximal extension of pc into the past such that for all $q \in \gamma$, $I^+(q) = I^+(p)$. (Note that this means that the same accessible points from p are accessible from any point $q \in \gamma$, and thus $\langle p, b \rangle_N$ and $\langle q, x \rangle$ must have some sort of overlapping). Since future distinction fails at q, any point $z \in \langle q, x \rangle$ lies on a trip from p. This means $\langle p, b \rangle_N \cap \langle x, x \rangle \neq \emptyset$, so that $\langle p, b \rangle_N = \langle x, x \rangle$. For instance, let $x' \in \langle p, b \rangle_N = \langle x, x \rangle$, and since $z \ll x$ ($z \in \langle q, x \rangle$) we have $x' \ll z$. With this, $z \ll x \ll x' \ll z$ so that $z \in V$ and $\langle q, x \rangle \subseteq V$. Then, for all $q \in \gamma$ we have $q \in \partial V$.

III If pb is null and cp is timelike, we have the same situation as before, only time-reversed.

IV If pb and cp are both null with different directions, so that $c \ll b$. For some i, we get for any $x \in \langle c, b \rangle_N$, $x \ll b_i \ll c_i \ll x \implies \langle c, b \rangle_N \subseteq V$. With this, $p \in \overline{\langle c, b \rangle}_N \subseteq \overline{V}$. Thus, we see that $\langle c, b \rangle_N \cap \langle x, x \rangle \neq \emptyset$ so that $\langle c, b \rangle_N = \langle x, x \rangle$. Again, suppose $p \notin V = \overset{\circ}{V}$ (else we would go back to the first case), so that $p \in \partial V$. Choose $r \in pb$, different from p and b, and consider $r \in V$. This means, there is a closed trip from r to itself, which means there must be a point $w \in I^+(r)$ at which such closed trip leaves N. Thus, $w \in I^+(p)$. Then, the proof follows the same lines as previously for $IIii$) changing b by w, proving again the b) statement. Analogously, if we take a point along cp, we can prove c) statement following the same reasoning. Consider then a point along pb and cp so that both segments are contained at ∂V. Take $p' \in pb$ and regain the argument for b) and $p'' \in cp$ to regain the argument for c). With this, we can prove d) statement for p.

V If pb and cp are both null in the same direction, say it along a geodesic γ, we obtain future timelike geodesics η_i by maximally extending $c_i d_i$, where $\eta_i \cap \partial N = e_i$. Same for $a_i b_i$ denoted as ζ_i, with $\zeta_i \cap \partial N = f_i$, where $c_i e_i$, $f_i b_i \subseteq \overline{N}$. With this, we obtain $f_i \ll b_i \ll c_i \ll e_i$ which implies that strong causality fails at b and c and along the segment bc. By repeating this argument for p instead of b and c, we take b' as the original b. If we suppose $b' \notin \gamma$, we will have $p \ll b'$ and a point $b'' \in \partial N$ on the trip from p to b' instead of b, leading to b) again. So, let us take $b' \in \gamma$. We can repeat this argument for each point along γ, leading to γ violating strong causality. By this construction, η_i and ζ_i provide the required points in the neighbourhoods of points of γ for a large enough i. Taking two different points $u, v \in \gamma$ with $u \prec v$, and considering two neighbourhoods U and W for u and v respectively, then we can use η_i and ζ_i so that they have $m_i \in U$ and $n_i \in V$ respectively with $n_i \ll m_i$. If $u \ll x$ and $y \ll v$, choose $U \subseteq I^-(x)$ and $W \subseteq I^+(y)$. Then $y \ll n_i \ll m_i \ll x$.

□

Proposition 2.27. *If M is compact, then it contains closed trips.*

Proof. We are going to show that compactness in the Alexandrov topology implies the existence of closed trips, since we have seen the equivalence between manifold and Alexandrov topologies in 2.20. Therefore, let M be covered by a finite number of sets $\langle x_i, y_i \rangle$. Then, for each y_i there is a j such that $y_i \in \langle x_j, y_j \rangle$. We obtain $y_{i_1} \ll y_{i_2} \ll y_{i_3} \ll \ldots$ Since there is a finite number of y_i, at each subsequence there must be repetitions, leading to closed trips. □

With this final proposition, we prove that physical space-times, for instance the one we live in, are not compact.

2.2 Domains of dependence

The concept of domains of dependence will be of importance in singularity theorems and initial value problems. That is, if a point belongs to a domain of dependence of a $S \subseteq M$, then initial data given at S, completely and uniquely determines the state of any system at this point.

Definition 2.28. Let $S \subseteq M$ be an achronal subset. Let us define the future, past and total domains of dependence as:

$$D^+(S) = \{x \in M | \forall \gamma \ni x \text{ past endless trip}, \gamma \cap S \neq \emptyset\}$$

$$D^-(S) = \{x \in M | \forall \gamma \ni x \text{ future endless trip}, \gamma \cap S \neq \emptyset\}$$

$$D(S) = \{x \in M | \forall \gamma \ni x \text{ endless trip}, \gamma \cap S \neq \emptyset\} = D^+(S) \cup D^-(S)$$

A domain of dependence is the largest region in M in which the physics can be predicted from the knowledge of the initial data at S.

Definition 2.29. Let $S \subseteq M$ be an achronal closed set. We define the future, past and total *Cauchy horizon* as:

$$H^+(S) = \{x \in M | x \in D^+(S) \text{ and } I^+(x) \cap D^+(S) = \emptyset\}$$

$$H^-(S) = \{x \in M | x \in D^-(S) \text{ and } I^-(x) \cap D^-(S) = \emptyset\}$$

$$H(S) = H^+(S) \cup H^-(S)$$

Remark 2.30. We can restate $H^{\pm}(S) = D^{\pm}(S) \setminus I^{\mp}(D^{\pm}(S))$

A set of identities for domains of dependence and Cauchy horizons when S is closed can be seen at [17, Section 5]. They will not be stated in this document for the sake of brevity.

Definition 2.31. Let $S \subseteq M$ be achronal. Let us define the *edge* as:

$$\text{edge}(S) = \{x \in M | \forall Q \ni x \ \exists y, z \in Q \ \exists \alpha, \beta \text{ trips connecting } y \text{ and } z$$

$$\text{such that } (\alpha \veebar \beta) \cap S \neq \emptyset\}$$

The set edge(S) can be regarded as the set of accumulation points of S which are not in S and whose neighbourhood fails to be a 3-manifold.

Remark 2.32. Note $\overline{S} \setminus S \subseteq \text{edge}(S) \subseteq \overline{S}$. If edge$(S) = \emptyset$, then S is *edgeless* and S must be closed.

Proposition 2.33. *Let $S \subseteq M$ be achronal. A point $p \notin$ edge(S) $\iff$ there exists an open connected set $Q \ni p$ such that $S \cap Q$ is an achronal boundary in Q as a space-time. ($\emptyset$ is regarded as an achronal boundary in Q)*

Proof. If $Q \ni p$ is connected such that $S \cap Q$ is an achronal boundary in Q as a space-time, there exists a future set $F \subseteq M$ with $S \cap Q = \partial F$ and all trip from P to F intersects $S \cap Q$ in a unique point by proposition 1.68. Note that:

- If $S \cap Q = \emptyset$, for all $y, z \in Q$ we obtain $\langle y, z\rangle \cap S = \emptyset$
- If $S \cap Q \neq \emptyset$, either all trip between a pair of points in Q intersects S or none of them.

Conversely, suppose $p \notin$ edge(S). Then, there exists an open neighbourhood $P \ni p$ which contains a trip γ from $y \in P$ to $z \in P$, and every other trip between both points intersects S if and only if γ does. Choose a simple region $N \subseteq P$. Let $p \in N$ and choose $y, z \in N$ so that $p \in \langle y, z\rangle_N$. Set $Q = \langle y, z\rangle_N$. Depending on whether every trip contained in Q intersects S or whether each trip in Q does not intersect $S \cap Q$, we have the following cases:

- For all trip $\gamma \in Q$, we have $\gamma \cap (S \cap Q) = \emptyset \implies S \cap Q = \emptyset$ which is an achronal boundary on Q.
- For all trip $\gamma \in Q$, we have $\gamma \cap S \neq \emptyset$. Then we can consider $F_Q = \bigcup_{q \in S \cap Q} \langle q, z \rangle_N$, which is a future set on Q. For all $x \in Q$, we have $x \in \partial F_Q \subseteq Q \iff x \in S$. Then $S \cap Q$ is an achronal boundary in Q as a space-time.

□

Corollary 2.34. *Every achronal boundary in M is edgeless.*

Proof. Derived from 2.33. □

Proposition 2.35. *Let $S \subseteq M$ be achronal. Then:*

a) $\text{edge(S)} = \overline{\text{edge(S)}}$

b) $I^+(\text{edge(S)}) \cap D^+(S) = \emptyset$

c) $\text{edge(S)} = \text{edge}(H^+(S))$.

Proof. We will provide general lines of proofs for the sake of giving context, but not extending too much.

a) Derived by applying the above proposition.

b) Let us prove it by contradiction. Suppose there is a point $x \in I^+(\text{edge}(S))$ and let $y \in \text{edge}(S)$ be such that $y \ll x$; and $x \in D^+(S)$, then $I^-(x) \cap J^+(S) \subseteq D^+(S)$. Since $\partial D^+(S) = H^+(S) \cup S$, the only possibility for a point $y \in \text{edge}(S)$ to precede $x \in D^+(S)$ would be $y \prec x$ and $y \not\ll x$, which contradicts $x \in I^+(\text{edge}(S))$.

c) Let $p \in \text{edge}(S)$ be a point. All neighbourhood of p contains two points and two trips connecting them, one of them crosses S. Take a segment between points in S and $H^+(S)$. Choose a neighbourhood for $edge(H^+(S))$, where we may be able to show $\text{edge}(S) \subseteq edge(H^+(S))$. By the same kind of argument, we can prove the converse.

□

Theorem 2.36. *Let $S \subseteq M$ be achronal. For every point $p \in H^+(S) \setminus \mathrm{edge}(S)$ there exists a null geodesic $\eta \subseteq H^+(S)$ with future endpoint p and either past endless or has a past endpoint in* $\mathrm{edge}(S)$

Proof. Let $W = I^+(H(S))$. Let $x \in W$ be the future endpoint of two trips trips α, β. Take α as the past endless trip and β with a past endpoint at S. Since $I^+(W) \subseteq W$ and $W = \mathring{W}$, then $I^+(W) = W$, so that W is a future set. Thus, $H^+(S)$ is part of the achronal boundary ∂W (the remaining part of ∂W will be $\partial I^+(S) \setminus S$). Let us write $H^+(S) = \partial W \cap D^+(S)$. There are several cases:

- If $x \in H^+(S) \setminus S$, then $\exists \beta \nexists \alpha$ trips.
- If $x \in \partial I^+(S) \setminus S$, then $\exists \alpha \nexists \beta$ trips.
- If $x \in S$, then β degenerates and $\nexists \alpha$

Let $p \in H^+(S) \setminus \mathrm{edge}(S)$ be a point and consider a simple region $Q \ni p$ such that $\partial I^+(S) \cap \overline{Q} = S \cap \overline{Q}$. We can do this in two ways:

- If $p \in H^+(S) \setminus S$, we can choose Q so that $\overline{Q} \subseteq I^+(\beta)$
- If $p \in S \setminus \mathrm{edge}(S)$. Take the set $\langle y, z \rangle_N$ so that every trip from y to z intersects S. Then, choose Q so that $\overline{Q} \subseteq \langle y, z \rangle_N$, where $\langle y, z \rangle_N$ is taken so that every trip from y to z intersects S.

Therefore, any point $y \in \overline{Q} \cap \overline{I^+(S)}$ is the future endpoint of a β trip (possibly degenerate). Since $I^+(p) \subseteq W$, let $x \in I^+(p)$ be the future endpoint of an α trip which meets $\partial Q \cap \overline{I^+(S)}$ at a point q. Now, q will also be the future endpoint of a β trip, so that $q \in W \cap \partial Q \subseteq W \setminus Q$. Thus, the conditions for proposition 1.70 are satisfied, and from the achronalicity of $H^+(S)$, there exists a null geodesic $\eta \subseteq H^+(S)$ with future endpoint p which can be endlessly extended into the past along $\partial H^+(S)$ or, until it meets $\mathrm{edge}(S)$.

□

Definition 2.37. A *Caucy hypersurface* for M is an achronal set $S \subseteq M$ for which $D(S) = M$.

Proposition 2.38. *Let $S \subseteq M$ be an achronal set. For all endless null geodesic $\eta \subseteq M$, if $S \cap \eta \neq \emptyset$ is compact, then $D(S) = M$.*

Proof. Let us prove it by contradiction. Suppose $D(S) \neq M$, so that either $H^+(S)$ or $H^-(S)$ is not empty. Without loss of generality, let us suppose $H^+(S) \neq \emptyset$. Then, there is a null geodesic $\eta \subseteq H^+(S)$ whose maximal extension intersects S,

where $\eta \cap S$ is compact by hypothesis. Therefore, we can continue $\eta \cap S$ along η until it reaches edge(S). We need to show that edge$(S) = \emptyset$ to find the required contradiction. In order to do so, recall corollary 2.34, so that we just need to show that S is an achronal boundary, that is $S = \partial I^+(S)$. Since S is achronal, then $S \subseteq \partial I^+(S)$.
Conversely, let us prove it by contradiction. Suppose $p \notin S$. Since p belongs to an achronal boundary, there exists an endless null geodesic $\tilde{\eta} \subseteq \partial I^+(S)$ which contains p. By hypothesis, since $\tilde{\eta} \cap S$ is compact we can find another point $q \in \partial I^+(S)$ between p and $\tilde{\eta} \cap S$. Again, for any null geodesic $\zeta \ni q$ with different direction from $\tilde{\eta}$, we will have $\zeta \cap \partial I^+(S) = \{q\}$. By applying the hypothesis, $\zeta \cap S \neq \emptyset$ is compact. Summarizing, ζ will be a null geodesic, $\zeta \cap \partial I^+(S) = \{q\}$ and $\zeta \cap S = \emptyset$ only if $q \in S$, which contradicts $q \notin S$. Thus, $S = \partial I^+(S)$ leads us to stating that S is an edgeless achronal boundary, which also contradicts the hypothesis. □

Proposition 2.39. *Let $S \subseteq M$ be achronal and $x \in D^+(S) \setminus H^+(S)$. Then, for every past endless causal trip γ with future endpoint x we have $\gamma \cap (S \setminus (H^+(S) \cup \mathrm{edge}(S))) \neq \emptyset$, and $\gamma \cap I^-(S)$ is a point.*

Proof. Assume $x \notin S$ to ensure the proof is not trivial. Suppose $x \in \operatorname{int} D^+(S) = D^+(S) \setminus (H^+(S) \cup S)$ and let $y_1 \in I^+(x) \cap D^+(S)$ be a point. Let γ be a past endless causal trip with future endpoint x. Since γ is compact, we can cover it by a finite number of simple regions. Take a locally finite system of simple regions N_i. Let $x = x_1 \in N_{i_1}$ and choose $y_1 \in N_{i_1}$. Let $x_2 \in N_{i_2}$ for some $i_2 \neq i_1$ be the past endpoint of the connected component $\overline{N}_{i_1} \cap \gamma$, so that $x_2 \in \partial N_{i_1}$. We have $x_2 \prec x_1$ and $x_2 \ll y_1$. Choose $y_2 \in N_{i_2}$ with $x_2 \ll y_2 \ll y_1$, and let $x_3 \in \partial N_{i_2}$ be the past endpoint of $\overline{N}_{i_2} \cap \gamma$. By following this process, we get $\cdots \ll y_3 \ll y_2 \ll y_1 \in D^+(S)$, where $y_{r+1}y_r$ is future timelike in N_r, for $r = 1, 2 \ldots$ No segment of γ can re-enter N_i, and since $\{N_i\}_i$ is locally finite, there exists a sequence of points $\{x_i\}_i \subseteq \gamma$ towards an endless past. Then $\bigcup_{i \in \mathbb{N}} y_{i+1}y_{i+2} \equiv \eta$ is a past endless trip which gets closer to γ as more into the past it goes. Since $y_1 \in D^+(S)$, there exists a point z and some k such that $\eta \cap S = \{z\} \in y_k y_{k-1}$. We have $x_k \ll z$ so that $x_k \notin D^+(S)$. Thus, some point $w \in \gamma$ lies on $\partial D^+(S)$. Note $w \notin H^+(S)$ and $w \notin \mathrm{edge}(S)$ since $w \prec x$ would imply $w \in H^+(S) \cap (\neg D^+(S))$ by 2.36 b). Therefore, $w \in S \setminus (H^+(S) \cup \mathrm{edge}(S))$ and also $x_k \in I^-(S)$.

□

Proposition 2.40. *Let $S \subseteq M$ be achronal. If $y \in \operatorname{int} D^+(S)$, then $J^-(y) \cap I^+(S) = J^-(y) \cap \operatorname{int} D^+(S)$ and $J^-(y) \cap J^+(S) = J^-(y) \cap D^+(S)$*

Proof. The proof to this proposition stems from set theory. □

Proposition 2.41. *Let $S \subseteq M$ achronal and let $p \in \operatorname{int} D^+(S)$ be a point. The space-time M is strongly causal at p.*

Proof. Let us prove it by contradiction. Let $p \in \operatorname{int} D^+(S)$ and let M not to be strongly causal at p. There is a closed trip through p, denoted as η, such that there exists a point w satisfying $\eta \cap S = w \in S$. This leads us to the contradiction given by $w \ll w$, since S is achronal. Thus, $D^+(S) \cap V = \emptyset$ and $\operatorname{int}(D^+(S)) \cap \overline{V} = \emptyset$. Recall Theorem 2.26, the first four statements of which require $p \in \overline{V}$, so $e)$ applies on this case. Take $p \in \operatorname{int} D^+(S)$ and let γ be and endless null geodesic containing p along which strong causality fails. Since $I^-(S)$ and $D^+(S)$ are both open and γ is endless, we can find some $y \in I^+(q) \cap I^-(S)$ and $x \in I^-(p) \cap \operatorname{int} D^+(S)$ with $x \ll y$. Then, there exists trips connecting each x, y towards S. This means, there is a trip between any two points of S, which contradicts the achronalicity of S. □

Proposition 2.42. *Let $S \subseteq M$ be achronal and $x \in \operatorname{int} D^+(S)$, then $J^-(x) \cap J^+(S)$ is compact.*

Proof. Let us prove it by contradiction. Denote $A = J^-(x) \cap J^+(S)$. Suppose A is not compact, so that there is a sequence of points $\{a_i\}_i \subseteq A$ without an accumulation point in A. We will use this sequence to construct a past endless trip γ with future endpoint in $D^+(S)$ such that $\gamma \cap S = \emptyset$ which will lead us to the required contradiction. Cover A by a locally finite system of simple regions $\{N_i\}_i$.

1. Let $x = x_0 \in N_{i_0}$ and choose $y_0 \in I^+(x) \cap D^+(S) \cap N_{i_0}$. From our initial suppositions, all $a_i \in A \subseteq J^-(x)$, so that future causal trips can be traced from each a_i to x_0. Since there is not an infinite number of a_i in N_{i_0}, from the compacity of the boundary of a simple region, these causal trips must meet in a set of points which accumulate at $z_0 \in \partial N_{i_0}$. With this, z_0x_0 is future causal and z_0y_0 is future timelike. Since $z_{i_0} \notin N_{i_0}$, for some $i_1 \neq i_0$ we will have $z_0 \in N_{i_1}$.

2. Take $x_1, y_1 \in z_0y_0 \cap N_{i_1}$ with $z_0 \ll x_1 \ll y_1 \ll y_0$. Repeating the same argument as before with $a_i \in I^-(x_1)$, we obtain an accumulation point $z_1 \in \partial N_{i_1}$ of final intersections of causal trips from a_i to x_1 with z_1x_1 future causal and z_1y_1 future timelike.

By repeating this process, we obtain $y_{i+1} \ll y_i$ for all $i \in \mathbb{N}$. We see that $\bigcup_{i \in \mathbb{N}} y_{i+1}y_i \equiv \gamma$ constitutes a past endless trip since the points y_i do not accumulate in a unique simple region N_i (if not, γ would constitute a bad trip). Lastly, let us prove by contradiction that $\gamma \cap S = \emptyset$. If $\gamma \cap S \neq \emptyset$, there would be some point $y_i \in I^-(S)$, and since each $a_j \ll y_i$, it would contradict the fact that $a_j \in J^+(S)$ since S is achronal.

□

Proposition 2.43. *Let $S \subseteq M$ be achronal and let $y \in \operatorname{int} D^+(S)$ be a point. Then* $\overline{I^-(y) \cap D^+(S)} = J^-(y) \cap D^+(S)$

Proof. Note $I^-(y) \cap D^+(S) \subseteq J^-(y) \cap D^+(S) \subseteq \overline{I^-(y) \cap D^+(S)}$ and $J^-(y) \cap D^+(S) = J^-(y) \cap J^+(S)$ is compact, which proves the statement. □

Proposition 2.44. *If $S \subseteq M$ is achronal, then* $\operatorname{int} D(S)$ *is strongly causal.*

Proof. From proposition 2.42, we saw that M is strongly causal at $\operatorname{int} D^+(S)$. By the same argument, we see that $\operatorname{int} D^-(S)$ is strongly causal. Therefore, $D(S) = D^+(S) \cup D^-(S)$ will be strongly causal too by a similar argument. □

Proposition 2.45. *Let $S \subseteq M$ be achronal and consider two points $u, v \in \operatorname{int} D(S)$. Then $J^+(u) \cap J^-(v)$ is compact.*

Proof. From proposition 2.43, given a point $x \in \operatorname{int} D^+(S)$, $J^-(x) \cap J^+(S)$ is compact. We can argue the same for $D^-(S)$. By choosing two points $u, v \in D(S)$, $J^+(u) \cap J^-(v)$ will be a closed subset of a compact set for a given $x \in D(S)$. Therefore, $J^+(u) \cap J^-(v)$ will be compact. □

Definition 2.46. A space-time M is *globally hyperbolic* if and only if M is strongly causal such that for any $u, v \in M$, the set $J^+(u) \cap J^-(v)$ is compact.

Theorem 2.47. *A space-time M is globally hyperbolic if and only if there exists a Cauchy hypersurface for M.*

Proof. If $M = D(S)$, then $D(S) = \operatorname{int} D(S)$ which is strongly causal. For any $u, v \in M$, we obtain that $J^+(u) \cap J^-(v)$ is compact, thus M is globally hyperbolic. The converse requires a more extensive approach and working with the time functions, as it can be seen at [3]. Further details will not be provided for the sake of brevity.

□

2.3 Causal curves topology

The notion of global hiperbolicity was developed such that given any two points on a space time, the set of causal curves connecting them is compact. Intuitively, this requires the space-time not to have "holes" or "singularities" between both points. In this section we will prove this other view to hiperbolicity by introducing a topology on collections of curves between two points.

Definition 2.48. Let $K \subseteq M$ be the open set of strongly causal points at M. Call $\mathcal{C} \subseteq K$ the set of causal curves lying in K, $\mathcal{K} \subseteq K$ the set of causal trips in K and $\mathcal{T} \subseteq K$ the set of all trips in K. We have the following inclusions:

$$\mathcal{T} \subseteq \mathcal{K} \subseteq \mathcal{C}$$

Let $C \subseteq K$ be a subset and $A, B \subseteq C$. Let us define:

$$\mathcal{C}_C(A, B) = \{\gamma \in C | \gamma \text{ is a causal curve from } A \text{ to } B\}$$

We are interested when C is compact, and A, B are closed. We want to topologize $\mathcal{C}$ such that any $\mathcal{C}_C(A, B)$ becomes compact. Then, we could define:

$$l : \mathcal{C}_C(A, B) \longrightarrow \mathbb{R}$$

As upper-semicontinuous (as proven in the following section), which with compacity implies the existence of $\max(l) \in \mathcal{C}_C(A, B)$. In the following section, we will see that under certain circumstances, $\max(l)$ is reached by geodesics without conjugate points.

Definition 2.49. Let us topologize $\mathcal{C}$ with a base of open sets of the form $\mathcal{C}_R(P, Q)$ where P, Q, R are open on N and $P, Q \subseteq R$. Note that if $\gamma \in \mathcal{C}_R(P, Q)$ and $\gamma \in \mathcal{C}_{R'}(P', Q')$, then $\gamma \in \mathcal{C}_{R''}(P'', Q'') \subseteq \mathcal{C}_R(P, Q) \cap \mathcal{C}_{R'}(P', Q')$ where $R'' = R \cap R'$, $P'' = P \cap P''$ and $Q'' = Q \cap Q'$.

This kind of topology allows obtaining smooth curves from trips and also, the notion of a sequence of curves γ_i approaching a limiting curve γ when γ_i are contained in any open set of M containing γ.

Proposition 2.50. *$\mathcal{K}$ is dense in $\mathcal{C}$ and $\mathcal{T}$ is dense in $\mathcal{K}$*

Proof. Let $\gamma \in \mathcal{C}$ be a causal curve and $\mathcal{R} = \mathcal{C}_R(P, Q) \ni \gamma$ a neighbourhood. We can cover γ by simple regions contained in R, with which we can obtain a causal trip $\eta \subseteq \mathcal{R}$ by a similar construction as in 1.49. Let $\gamma' \in \mathcal{K}$ be a causal trip and suppose a neighbourhood to it $\mathcal{R}' = \mathcal{C}_{R'}(P', Q')$. Analogously to the above case, we obtain $\eta' \in \mathcal{R}'$. Let p, q be respectively the past and future endpoints of γ'. Let $r \in Q'$ be such that $q \ll_{R'} r$, we have $p \prec_{R'} q$ where $p \ll_{R'} r$ and choose η' from p to r in Q'. $\square$

Theorem 2.51. *Let $C \subseteq K$ be compact and $A, B \subseteq C$ closed, then $\mathcal{C}_C(A, B)$ is compact.*

Proof. For any sequence of causal curves $\{\gamma_i\}_i \subseteq \mathcal{C}_C(A, B)$ there is an accumulation curve $\gamma \in \mathcal{C}_C(A, B)$. More details to this proof are quite tedious and would be too extensive for this work. These can be seen at [17, Section 6]. □

Corollary 2.52. *Let $S \subseteq M$ be achronal and strongly causal. Let $x \in \operatorname{int} D^+(S)$ and $y, z \in \operatorname{int} D(S)$, then $\mathcal{C}(S, \{x\})$ and $\mathcal{C}(\{y\}, \{z\})$ are both compact.*

Proof. Derived by applying 2.42, 2.43, 2.45 and 2.46. □

Remark 2.53. For all globally hyperbolic space-time, $\mathcal{C}(\{x\}, \{y\})$ is compact for any $x, y \in M$. With this we could recover the original notion of global hiperbolicity initially introduced to deal with the solutions of hyperbolic differential equations defined on manifolds. Thus, a space-time will be said to be hyperbolic if for any given pair of points, the collection of causal curves between them is compact; that is, there are no "holes" in between the points.

Chapter 3

Singularity Theorems

3.1 Hypersurfaces

Definition 3.1. The *length* of a causal trip γ is defined as:

$$l(\gamma) = \sum_{i=1}^{k} \sqrt{\Phi(p_{i-1}, p_i)}$$

Where $p_{i-1}p_i$ are successive segments of γ lying in a simple region N_i, where Φ is the world function. Thus, $l(\gamma) \geq 0$.

Remark 3.2. The length as defined in 3.1 is called *proper time*, which is measured by an observable moving along the trip or curve we are considering. Intuitively, we measure "distances" with a clock.

Remark 3.3. We can talk about the triangle inequality when focusing on special relativity. It will become of great use when discussing the first singularity theorem. Since Minkowski metric is not positive-definite, the reverse triangle inequality is satisfied $l(p_0p_2) \geq l(p_0p_1) + l(p_1p_2)$ for any $p_0, p_1, p_2 \in M$. This has to do with the so called *Twin Paradox*, but the particularities of this will not be discussed so as not to digress.

Proposition 3.4. Let N be a simple region and let $p, q \in N$ be such that pq is future-causal. If $\eta \subseteq N$ is the causal trip pq and $\eta' \subseteq N$ is any other causal trip from p to q, then $l(\eta) > l(\eta')$.

Proof. Let pq be timelike, otherwise the proof would be trivial. Choose the Minkowski normal coordinates (t, x, y, z) for N with origin at some point at the extension to the past of pq. Consider a new region $\hat{N}$ given by $t > (x^2 + y^2 + z^2)^{1/2}$ and choose there new coordinates, given by $T = (t^2 - x^2 - y^2 - z^2)^{1/2}$ and $X^a = x^a/t$, for $a = 1, 2, 3$. These become a synchronous coordinate system for N whose line element can be written as $ds^2 = dT^2 - \gamma_{\alpha\beta} dX^\alpha dX^\beta$ with $\alpha, \beta = 1, 2, 3$. Note that if X is constant, we have timelike geodesics and if T is constant, we obtain orthogonal hypersurfaces to the geodesics. If $l(\eta') = \int_{T_0}^{T_1} [1 - \gamma_{\alpha\beta} \frac{dX^\alpha}{dT} \frac{dX^\beta}{dT}]^{1/2} dT$ where T_0, T_1 are respectively the coordinates for p, q, we will reach the maximum value when $dX^a/dT = 0$ for any $a = 1, 2, 3$, thus giving the geodesic η. □

Definition 3.5. Let $p \prec q$ be two points along a causal curve γ. Let us define $\xi = \{x_i\}_{i=0}^k \subseteq \gamma$, and $n \geq 1$, as a finite sequence of points along γ between $x_0 = p$ and $x_k = q$. Suppose a simple region N_i containing both points x_i, x_{i+1} together with the portion of γ from x_i to x_{i+1}. Since $x_i \prec x_{i+1}$, the portion $x_i x_{i+1}$ will be future casual.
Denote the causal trip as $\gamma_\xi = \bigcup_{i=0}^{k-1} x_i x_{i+1}$. Let Ξ be the set of allowed sequences ξ. We have $l(\gamma_{\xi'}) \leq l(\gamma_\xi) \iff \xi \subseteq \xi'$ by 3.4 and $l(\gamma_{\xi''}) \leq \min\{l(\gamma_\xi), l(\gamma'_\xi)\} \iff \xi' \cup \xi = \xi''$. We can finally define:

$$l : \mathcal{C}(\{p\}, \{q\}) \longrightarrow \mathbb{R}; \quad \gamma \to \inf_{\xi \in \Xi} \{l(\gamma_\xi)\}$$

Which provides sense to the notion of length of causal curves with endpoints. Note the infimum exists since $l(\gamma) \geq 0$. We can extend this notion to all $\mathcal{C}$ if we allow l to reach infinity for endless causal curves. However, we will work with causal curves having both endpoints. Therefore, the length map can be expressed as $l : \mathcal{C}(A, B) \longrightarrow \mathbb{R}$ for any $A \subseteq C$, $B \subseteq C$ and $C \subseteq K$.

Theorem 3.6. *$l : \mathcal{C}_C(A, B) \longrightarrow \mathbb{R}$ is upper semi-continuous*

Proof. Recall that upper semi-continuity for l will be given if $l^{-1}((-\infty, a))$ is open in $\mathcal{C}_C(A, B)$ for any a. In order to do so, we must show that for any causal curve $\gamma \in \mathcal{C}$ from $p \in A$ to $q \in B$ with $l(\gamma) \subseteq (-\infty, a)$, there is a neighbourhood $\mathcal{R} \ni \gamma$ in $\mathcal{C}$ such that for any $\gamma' \in \mathcal{R}$, $l(\gamma') < a$ is satisfied.
Suppose $l(\gamma) = b < a$ and choose a $\xi \in \Xi$ such that $l(\gamma_\xi) < b + (a - b)/2$. Consider a sequence of points $\{x_i\}_{i=0}^k \subseteq \gamma$ and choose them close enough so that each pair x_i, x_{i+1} belongs to a local causality neighbourhood denoted as L_i, with $\overline{L}_i \subseteq N_i$ and which satisfies $L_i \cap L_j \neq \emptyset$ if $j = i \pm 1$. Since the length of a geodesic in N_i is a continuous function of its endpoints, we can choose another local causality neighbourhood U_i for each x_i with $U_0 \subseteq L_0$ and $U_j \subseteq L_j \cap L_{j-1}$ for all $j = 1, \ldots, k$. Choose the neighbourhoods U_i small enough so that for any two points $x \in U_i$ and $y \in U_{i+1}$ the length of the geodesics xy satisfy $l(xy) - l(x_i x_{i+1}) \leq |a - b|/2k$. Define $V_i = \bigcup_{y \in U_i; z \in U_{i+1}} \langle y, z \rangle$ so that $V_i \subseteq L_i$ by the causal convexity of L_i. Note

$V_i \cap V_j \neq \emptyset$ if $j = i \pm 1$. Define $R = \bigcup_i V_i$, $P = V_0$ and $Q = V_{k-1}$ and suppose $\gamma' \in \mathcal{C}_R(P, Q)$. Since each U_i is causally convex, from the definition of V_i, the curve γ' meets $V_{i-1} \cap V_i$ in a point between two points of U_i. Then, $x'_0, \ldots, x'_k \in \gamma'$ with $x'_i \in U_i$ so that the length of the causal trip $\eta = \bigcup_{i \in \mathbb{N}}^{k-1} x'_i x'_{i+1}$ is $l(\eta) < b + \frac{1}{2}(a - b) + kx\frac{a-b}{2k} = a$, therefore $l(\gamma') < a$ as required. □

Corollary 3.7. *If A and B are closed subsets of a compact and strongly causal set C, there is a causal curve $\gamma \subseteq C$ from A to B which maximizes the length of such curves.*

Proof. Derived by applying Theorem 2.51 and 3.6. □

Proposition 3.8. *Let A and B be closed subsets of a compact and strongly causal set C. Let $\gamma \in \mathcal{C}_C(A, B)$ be a causal curve which maximizes $l(\gamma)$. If $\gamma \subseteq \overset{\circ}{C}$, then γ is a causal geodesic.*

Proof. Cover $\gamma \subseteq \overset{\circ}{C}$ by a finite system of simple regions contained in C. By 3.4 and 3.5, the intersection of each simple region with γ is a geodesic, then γ is a geodesic too. □

Definition 3.9. Let γ be a timelike geodesic and Σ a smooth spacelike hypersurface orthogonal to γ at p. We say *q is conjugate to Σ at p* if and only if there exists a non-trivial Jacobi field S on γ with $S(p) = 0$ and $S \not\equiv 0$, which comes from a 1-parameter system of a.p.geodesics orthogonal to Σ at their intersection.
In case γ was null, Σ would also be null and γ orthogonal to it.

Consider a family of hypersurfaces Σ defined by a scalar field $f \equiv$ constant. Let (U, φ) be a chart with coordinate functions $x^1, \ldots, x^4$. The normal vector field to Σ at a point $p \in \Sigma$ will be defined as $\zeta^\mu = g^{\mu\nu} \nabla_\mu f$, which will be orthogonal to all vectors in $T_p\Sigma \subseteq T_pM$. If ζ^μ is timelike, the hypersurface Σ will be spacelike and viceversa. If ζ^μ is null, Σ will also be null. In the later case, ζ^μ will also be tangent to the hypersurface Σ since $\zeta^\mu \nabla_\mu f = \zeta^\mu \zeta_\mu = 0$, which means that they are orthogonal between each other. Thus, the vectors ζ^μ are orthogonal to Σ and to themselves. Let us prove that ζ^μ are in fact tangent vectors to null geodesics contained in the null hypersurface Σ by checking that they satisfy the geodesic equation. Consider a curve $\gamma \subseteq \Sigma$ with tangent vector field ζ^μ. Then:

$$\zeta^\mu \nabla_\mu \zeta_\nu = \zeta^\mu \nabla_\mu \nabla_\nu f = \zeta^\mu \nabla_\nu \nabla_\mu f = \zeta^\mu \nabla_\nu \zeta_\mu = \frac{1}{2} \nabla_\nu (\zeta^\mu \zeta_\mu) = 0$$

By applying 1.21 d), we can see that $\nabla f(\partial_\nu) = \partial_\nu f$, for a $f \in \mathcal{F}(M)$. Then it applies $\nabla_\mu \nabla_\nu f = \partial_\mu \partial_\nu f - \Gamma^\rho_{\mu\nu} \nabla_\rho f = \partial_\nu \partial_\mu f - \Gamma^\rho_{\nu\mu} \nabla_\rho f = \nabla_\nu \nabla_\mu f$. Consequently,

since null geodesics orthogonal to a null hypersurface are contained in it, we may say that null hypersurfaces are *generated* by null geodesics.
Because of this situation, the concept of conjugate points to a null hypersurface lose meaning. Thus we may define the notion of *cross-section* and conjugate points to them, as it may be discussed below:

Definition 3.10. Let Σ be a null hypersurface of a 4-dimensional space-time M. The *cross-section* Λ of Σ is a $2-$dimensional subset such that the normal vector ζ to Σ is nowhere tangent to Λ and each null geodesic generator intersects Λ at most once. For a n-dimensional space-time, the cross-section will be $(n-2)-$dimensional.

Definition 3.11. Let γ be a null geodesic and Λ a cross-section such that γ intersects Λ orthogonally at p. A point $q \in \gamma$ is *conjugate* to Λ at γ if and only if there exists a non-trivial Jacobi field S on γ such that $S(p) = 0$ and $S \not\equiv 0$, from a 1-parameter system of null a.p.geodesics orthogonal to Λ at their intersections.

In some contexts, it is useful to think about conjugate points in an alternative way, which involves the below statements:

Definition 3.12. A *congruence of geodesics* is a set of curves in a region of M where each point of the region lies on a unique curve.

Remark 3.13. Suppose a timelike geodesic γ_0 orthogonal to a spacelike hypersurface Σ. Let us consider the *congruence of timelike geodesics* (γ) *orthogonal to* Σ and let us focus on those geodesics of the congruence lying in some neighbourhood of γ_0. Choose a neighbourhood $Q \ni \gamma_0$ small enough so that the future-pointing tangent vectors to (γ) constitute a smooth tangent vector field T. Consider a chart (U, φ) with coordinate functions $x^1, \ldots, x^4$. Let $T^\mu T_\mu = 1$ and the geodesic equation is satisfied $T^\mu \nabla_\mu T^\nu = 0$. Consider the deviation vector S between geodesics. Let us consider the propagation derivative of S along the tangent vector field T, that is $\nabla_T S$. Let us write it as:

$$DS^\mu = T^\nu \nabla_\nu S^\mu = B^\mu_\nu S^\nu$$

By using $\nabla_T S = \nabla_S T$, proven in the first section. It has been denoted $B_{\mu\nu} = \nabla_\nu T_\mu$, which measures the deviation of neighbouring geodesics in relation to being perfectly parallelly transported.
In Q, consider a set of particles whose time evolution is described by γ_0. Since $T^\mu B_{\mu\nu} = T^\mu \nabla_\nu T_\mu = 0$ and $T^\nu B_{\mu\nu} = T^\nu \nabla_\nu T_\mu = 0$ by the fact that $T^\mu T_\mu = 1$ and the geodesic equation respectively, then we note that $B_{\mu\nu}$ belongs to the subspace of $T_p M$ of normal vectors to T. Since the $B_{\mu\nu}$ tensor is a $(0,2)-$tensor, it can be separated into the sum of a antisymmetric and symmetric tensors, which can also

be separated into the trace and crossed terms. Thus, we obtain $B_{\mu\nu} = \frac{1}{3}\theta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$, where $P_{\mu\nu} = g_{\mu\nu} - T_\mu T_\nu$ is the projection tensor onto a vector subspace of the tangent space. With this, we present the following definition.

Definition 3.14. With the same notation as specified in 3.13, let us define the *divergence* $\theta = P^{\mu\nu}B_{\mu\nu} = \nabla_\mu T^\mu$, the *shear* $\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3}\theta P_{\mu\nu} = \frac{1}{2}(\nabla_\mu T_\nu + \nabla_\nu T_\mu) - \frac{1}{3}\theta P_{\mu\nu}$ and the *rotation* $\omega_{\mu\nu} = B_{[\mu\nu]} = \frac{1}{2}(\nabla_\mu T_\nu - \nabla_\nu T_\mu)$. Respectively, they measure the change of volume, the distortion of the shape and the rotation of Q. Since T are parallelly transported along the geodesics, the rotation $\omega_{\mu\nu} = 0$.

We can consider a scalar field t which measures the distance along γ from Σ, satisfying $T_\mu = \nabla_\mu t$. The t can be used as the time coordinate in a *synchronous coordinate system* with line element $ds^2 = dt^2 - q_{\mu\nu}dx^\mu dx^\nu$ for $\mu, \nu = 1, 2, 3$, where $q_{\mu\nu}$ is a positive 3 by 3 matrix whose components are functions of t, x^1, x^2, x^3.
For null geodesic congruences, consider a null geodesic γ_0 orthogonal to a 2-surface Δ_0. Let Ω_0 be a hypersurface generated by null geodesics orthogonal to a cross-section Λ_0. Consider a neighbourhood $Q \ni \gamma_0$. Let γ be a null geodesic which generates a null hypersurface. The tangent vector to it satisfies $g(T, T) = 0$ and $g(T, S) = 0$, where S is the vector which indicates the variation between geodesics of the congruence, with $g(S, S) = 0$. Thus, a normal vector to T and S will also be null.
Consider a family of cross-sections Λ parametrised by u which varies on a direction not contained at Ω_0. For $u = 0$ we denote the cross-section we obtain as Λ_0. With this, we obtain a 1-parameter family of null hypersurfaces Ω parametrised by u. Let $u \equiv$ constant be a null hypersurface. We obtain an analogous coordinate system to that for timelike geodesics, denoted as *null coordinate system* (u, v, x^1, x^2), with line element: $ds^2 = 2du(dv + \frac{1}{2}adu + b_\lambda dx^\lambda) - r_{\lambda\mu}dx^\lambda dx^\mu$ for $\lambda, \mu = 2, 3$. The coordinate v is chosen as an affine parameter for γ such that $(u, v) \equiv (\text{constant}, 0)$ gives us a cross-section Λ and particularly, $(u, v) = (0, 0)$ provides us Λ_0. Note that each geodesic will be defined by a constant value of (u, x^1, x^2). The b_λ and $r_{\lambda\mu}$ are functions of u, v, x^1, x^2 and $r_{\lambda\mu}$ is a symmetric positive 2 by 2 matrix.

Proposition 3.15. *With the same notation as 3.13. Let the divergence be* $\theta = \nabla_\mu T^\mu$ *and* $D = \nabla_T = T^\mu \nabla_\mu$ *for a curve* γ *with tangent vector field* T*. They satisfy* $D\Delta = \theta\Delta$*, where* Δ *is proportional to the volume of* Σ *if* γ *is timelike or to an element of surface of* Λ *if* γ *is null.*

Proof. If γ is timelike, take a synchronous coordinate system with $X_0 = T = \partial/\partial t$ and $X_a = \partial/\partial x^a$ for $a = 1, 2, 3$. If γ is null, take a null coordinate system $X_0 = T = \partial/\partial v$, $X_1 = \partial/\partial u$ and $X_\lambda = \partial/\partial x^\lambda$ for $\lambda = 2, 3$. For these coordinates, the 4-volume is defined by $\Delta_4 = \sqrt{|\det g_{\lambda\tau}|}$. The $g_{\lambda\tau}$ are the components of the metric tensor, and can take different forms depending on the coordinate system we are working on. Respectively, we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & -q_{\alpha\beta} & \\ 0 & & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & a & b_2 & b_3 \\ 0 & b_2 & & \\ 0 & b_3 & & -r_{\lambda\mu} \end{bmatrix}$$

Thus, $\Delta = \sqrt{-\det q_{\alpha\beta}}$ or $\Delta = \sqrt{-\det r_{\lambda\mu}}$. Now, compute:

$$\theta\Delta = \nabla_\mu T^\mu \Delta = (\partial_\mu T^\mu)\sqrt{-g} + T^\mu \partial_\mu(\sqrt{-g}) = (\partial_\mu T^\mu)\sqrt{-g} + T^\mu \nabla_\mu \sqrt{-g} - \Gamma^\mu_{\mu\lambda}\sqrt{-g}T^\mu =$$

$$= D\Delta + \sqrt{-g}(\partial_\mu T^\mu + \Gamma^\mu_{\mu\lambda} T^\mu) = D\Delta$$

Where we denoted $\sqrt{|\det g_{\kappa\tau}|} = \sqrt{-g}$ and $\nabla_\mu \sqrt{-g} = \partial_\mu(\sqrt{-g}) - \sqrt{-g}\Gamma^\mu_{\mu\nu}$ and the geodesic equation with the tangent vector to the geodesics, as $0 = T^\mu \nabla_\mu T^\nu = T^\mu(\partial_\mu T^\nu + \Gamma^\nu_{\mu\lambda} T^\lambda)$.

□

Proposition 3.16. *With the same notation as 3.13 and 3.15. A point w is conjugate to Σ or Λ_0 at γ_0 if and only if $\Delta = 0$ at w. In addition, the divergence θ exists and is continuous at any point of γ_0 at which $\Delta \neq 0$, and θ is unbounded near any point w at which $\Delta = 0$, such that θ becomes infinitely positive at the future of w and infinitely negative at its past on γ_0.*

Proof. A conjugate point w to Σ_0 or Λ_0 is given by a non-trivial Jacobi field S on γ_0 with $S(w) = 0$ and $S \not\equiv 0$, which connects to neighbouring geodesics also orthogonal to Σ_0 or Λ_0. Given a coordinate system, the Jacobi field must be a linear combination of $X_0, \ldots, X_3$ such that it becomes linearly dependent at w, so that $\Delta = 0$ since it depends on the coordinates. For the second part of the proposition, note that if $D\Delta = \theta\Delta$ is equivalent to $D \log \Delta = \theta$. Then take limits of Δ approaching the null value by both sides. □

Remark 3.17. Note that one solution we can always find from the deviation geodesic equation is the Jacobi field defined by the assignation $t \to tS(t)$, which vanishes if and only if $t = 0$. From this, we can infer that $\Delta \propto (t - t_1)^r$ where r is the number of possible independent lineal combinations of X_μ, given by a chosen coordinate system, which vanish at the conjugate point. This is denoted as the *conjugate degree* such that if $\dim M = n$, then $r \leq n - 1$. This degree denotes the multiplicity of the conjugate points, so that conjugate points cannot accumulate. The only similar situation would be r conjugate points lying on the same point.

Proposition 3.18. *With the same notation as remark 1.35 and 3.15. Take $\Delta > 0$ for simplicity. If γ_0 is timelike, then $D^2\Delta^{1/3} \leq \frac{1}{3}R_{\mu\nu}T^\mu T^\nu \Delta^{1/3}$. If γ_0 is null, then $D^2\Delta^{1/2} \leq \frac{1}{2}R_{\mu\nu}T^\mu T^\nu \Delta^{1/2}$.*

Proof. Let us write

$$DB_{\mu\nu} = T^\sigma \nabla_\sigma B_{\mu\nu} = T^\sigma \nabla_\sigma \nabla_\nu T_\mu$$

By applying $g^{\mu\nu}B_{\mu\nu} = g^{\mu\nu}\nabla_\nu T_\mu = \nabla^\mu T_\mu = \nabla_\mu T^\mu = \theta$ to the above expression, we achieve:

$$D\theta = T^\sigma \nabla_\sigma \nabla_\nu T^\nu = T^\sigma \nabla_\nu \nabla_\sigma T^\nu + T^\sigma R^\nu_{\mu\nu\sigma} T^\mu =$$
$$= \nabla_\nu (T^\sigma \nabla_\sigma T^\nu) - (\nabla_\nu T^\sigma)(\nabla_\sigma T^\nu) + R_{\mu\sigma} T^\sigma T^\mu = -B^\sigma_\nu B^\nu_\sigma + R_{\mu\sigma} T^\sigma T^\mu =$$
$$= -g^{\sigma\sigma} B_{\sigma\nu} B^\nu_\sigma + R_{\mu\sigma} T^\sigma T^\mu = -B_{\sigma\nu} B^{\sigma\nu} + R_{\mu\sigma} T^\sigma T^\mu$$

Where we have used $\nabla_\nu (T^\sigma \nabla_\sigma T^\nu) = \nabla_\nu T^\sigma \nabla_\sigma T^\nu + T^\sigma \nabla_\nu \nabla_\sigma T^\nu$ and the geodesic equation. Reorganizing the indexes we obtain:

$$D\theta = -B_{\sigma\nu} B^{\sigma\nu} + R_{\sigma\nu} T^\sigma T^\nu$$

Since $\omega_{\sigma\nu} = 0$ (see 3.14), we have $\nabla_\sigma T_\nu = \nabla_\nu T_\sigma$. Developing the previous expression:

$$D\theta + \frac{1}{3}\theta^2 = R_{\mu\nu} T^\mu T^\nu - S_{\mu\nu} S^{\mu\nu}$$

Where $S_{\mu\nu} = \nabla_\mu T^\nu - \frac{1}{3}\theta(g_{\mu\nu} - T_\mu T_\nu)$ is the shear tensor and $S_{\mu\nu}S^{\mu\nu} = \nabla_\mu T_\nu \nabla^\mu T^\nu + \frac{1}{3}\theta^2 \geq 0$. Since $S_{\mu\nu}T^\mu = 0$, the shear tensor is orthogonal to the tangent vector, so that they lie on a spacelike hypersurface with $S_{\mu\nu}S^{\mu\nu} \geq 0$. See that:

$$D^2\Delta^{1/3} = D(\frac{1}{3}\Delta^{-2/3}\theta\Delta) = \frac{1}{3}D(\theta\Delta^{1/3}) = \frac{1}{3}[(D\theta)\Delta^{1/3} + \theta D\Delta^{1/3}] =$$
$$= \frac{1}{3}\Delta^{1/3}[D\theta + \frac{1}{3}\theta^2] = \frac{1}{3}\Delta^{1/3}[R_{\mu\nu}T^\mu T^\nu - S_{\mu\nu}S^{\mu\nu}] \leq \frac{1}{3}R_{\mu\nu}T^\mu T^\nu \Delta^{1/3}$$

In case T was considered to be null, we could rewrite:

$$D\theta + \frac{1}{2}\theta^2 = R_{\mu\nu} T^\mu T^\nu - \sigma_{\mu\nu}\sigma^{\mu\nu}$$

In order to provide this expression, we have considered another null vector N satisfying $T^\mu N_\mu = 1$. Let Λ_0 be the cross-section, that is, the space of orthogonal vectors to N and T. The 2-metric of Λ_0 will take the form $\gamma_{\mu\nu} = g_{\mu\nu} - T_\mu N_\nu - N_\mu T_\nu$ with $\gamma^{\mu\nu}\nabla_\mu T_\nu = \theta$. Since $\sigma_{\mu\nu} = \sigma_{\nu\mu} = \nabla_\mu T_\nu - \frac{1}{2}\theta\gamma_{\mu\nu}$, we can get $\sigma_{\mu\nu}\sigma^{\mu\nu} = (\nabla_\mu T_\nu)(\nabla^\mu T^\nu) = (\nabla_\mu T^\mu)\gamma_{\mu\nu}\gamma^{\mu\nu}(\nabla_\nu T^\nu) = 2\theta^2 \geq 0$. More details about this procedure can be seen at [1, Appendix F]. Therefore, we can write:

$$D^2\Delta^{1/2} = D(\frac{1}{2}\Delta^{-1/2}\theta\Delta) = \frac{1}{2}D(\theta\Delta^{1/2}) = \frac{1}{2}[(D\theta)\Delta^{1/2} + \theta D\Delta^{1/2}] =$$

$$= \frac{1}{2}\Delta^{1/2}[D\theta + \frac{1}{2}\theta^2] = \frac{1}{2}\Delta^{1/2}[R_{\mu\nu}T^\mu T^\nu - \sigma_{\mu\nu}\sigma^{\mu\nu}] \leq \frac{1}{2}R_{\mu\nu}T^\mu T^\nu \Delta^{1/2}$$

□

Remark 3.19. We can strengthen both of the statements in 3.18 by applying the so-called *energy conditions* of general relativity. In this case, we want to understand the properties of Einstein's field equations that hold for a variety of different sources of energy. It is useful to think about field equations without specifying the theory of matter from which we derive the *energy-momentum tensor* $T_{\mu\nu}$. The energy conditions limit $T_{\mu\nu}$ and are coordinate-invariant restrictions to it. Therefore, we construct scalars from the energy-momentum tensor. We are interested in the so-called *strong energy condition* which implies that gravitation is attractive. More about energy conditions can be seen at [1, Section 4].
With the same notation as 3.18, consider any vector field X^μ as the tangent vector to the path followed by a particle. The quantity $T_{\mu\nu}X^\mu X^\nu$ defines the total energy measured at each point along the path. Using Einstein's field equations:

$$\frac{8\pi G}{c^4}T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

Where $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar respectively [4, Section 5.14]. Let us conveniently re-write it as $\frac{c^4}{8\pi G}R_{\mu\nu}T^\mu T^\nu = (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)T^\mu T^\nu$. The strong energy condition will imply $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)T^\mu T^\nu \leq 0$ so that we can make a stronger statement where $D^2\Delta^{1/3} \leq 0$ and $D^2\Delta^{1/2} \leq 0$. This inequality means that when geodesics of the congruence (γ) start to converge, they must within a finite number of parameters converge to points with $\Delta = 0$, whose locus becomes a 2-surface called caustic of Σ (where we assume γ_0 to be a complete geodesic). This phenomena is called as *Raychaudhuri effect*, and will be of great importance in the first singularity theorem. Note that in the absence of an energy condition, a similar effect still persists, as it will be proven in the following proposition.

Proposition 3.20. *With the same notation as in 3.13 and 3.14. Let $\alpha = 1/3$ or $\alpha = 1/2$ for γ_0 timelike or null respectively. Let $\Delta^\alpha = A$ and $D\Delta^\alpha = -B$ at some point $a \in \gamma_0$. Suppose $\alpha R_{\mu\nu}T^\mu T^\nu \leq k^2$ along the segment $ab \subseteq \gamma_0$ where a parameter t (or v) at $b \in \gamma_0$ is greater than itself at a by at least $\frac{1}{k}\tanh^{-1}(\frac{Ak}{B})$. Then $\Delta = 0$ at some point in ab.*

Proof. From 3.18, we have $D^2\Delta^\alpha \leq k^2\Delta^\alpha$. Let us solve the differential equation $D^2x = k^2x$ for some x. Applying initial conditions at a, that is t (or v) $= 0$, $x(0) = A$ and $DX(0) = -B$, we obtain $x(t) = A\cosh(kt) - \frac{1}{k}B\sinh(kt)$. Let us find the value t^* for the parameter so that $\Delta = 0$: $x(t^*) = A\cosh(kt^*) - \frac{1}{k}B\sinh(kt^*) = 0$ and thus $t^* = \frac{1}{k}\tanh^{-1}(\frac{Ak}{B})$. □

Note that if we assure $B > Ak$ taking $A, k \geq 0$, then $\Delta = 0$ at the future of a.

Proposition 3.21. *Let γ be a causal geodesic from p to q. Let Σ be a smooth spacelike hypersurface if γ is timelike, or a $2-$surface if Σ is null.*

a) Let q be a conjugate point to p at γ, or else;

b) Let γ be orthogonal to Σ at p and q conjugate to Σ on γ.

Then, there exists a first point $q' \in I^+(p) \cap \gamma$ satisfying a) or b), which varies continuously with p and γ, maintaining Σ fixed in b) for simplicity.

Proof. Let (U, φ) be a chart with coordinate functions $x^1, \ldots, x^4$. The isolation of conjugate points implies the existence of a first conjugate point q'. Recall the geodesic deviation equation, whose solutions depend on the deviation vector S^μ and its derivative DS^μ at a point of γ. If the Riemann tensor $R^\mu_{\nu\rho\lambda}$ varies, then the solutions of the deviation equation must vary. We obtain the same result if instead of varying the Riemann tensor we vary γ. We must see now that this implies the continuous variance of conjugate points.
Let $r \in \gamma = \gamma_0$ be a point in a caustic of the congruence (γ) where ($\Delta = 0$). If γ_0 is timelike, we have $\Delta^{1/3} \propto (t - t_1)^{1/3}$ or $(t - t_1)^{2/3}$ or $(t - t_1)$, else if γ_0 is null, we have $\Delta^{1/2} \propto (v - v_1)^{1/2}$ or $(v - v_1)$. Then, recall 3.20, for a sufficiently small interval of γ_0 about r, we can choose two points a and b within this interval for which any congruence $(\tilde{\gamma})$ near the congruence (γ) and for any Riemann tensor $\tilde{R}^\mu_{\nu\rho\lambda}$ that differs slightly from $R^\mu_{\nu\rho\lambda}$, the condition $\tilde{\Delta} = 0$ holds. □

Proposition 3.22. *With the same notation as 3.13. If there are no conjugate points on γ_0 to Σ_0, Λ_0 or to a point p, there is a synchronous (γ_0 timelike) or null (γ_0 null) coordinate system valid at a neighbourhood of γ_0. For geodesics through p, at a neighbourhood for the portion of γ_0 to the future of p.*

Proof. Some general lines for the proof will be given. By the proposition 3.21, there exists a neighbourhood $Q \ni \gamma_0$ which does not intersect the set of points of the congruence (γ) where $\Delta = 0$ (except at p if geodesics go through p). At Q the congruence is well-defined. Taking a construction similar to that of 3.14 would lead to the required coordinate systems. □

Theorem 3.23. *Let γ be a causal geodesic from p to q.*

a) If there exists a point $r \in \gamma$ conjugate to p or q, there is a causal trip η from p to q with $l(\eta) > l(pq)$. In case η is null, then $p \ll q$.

b) *Let Σ be a spacelike hypersurface if γ is timelike, or a 2-surface if γ is null. Let $p \in \Sigma$ be such that either γ is not orthogonal to Σ at p or else, γ is orthogonal to Σ and there is a conjugate point $r \in \gamma$ at Σ between p and q. Then, there is a causal trip η from Σ to q and $l(\eta) > l(\gamma)$. If γ is null, then $q \in I^+(\Sigma)$.*

Proof. Let γ be timelike and orthogonal to Σ (in the case b)). Let r be the first conjugate point to Σ or p. Choose a synchronous coordinate system $\mathcal{S}$ on a neighbourhood of the portion of γ from p to r; and also at p in case b). At $\mathcal{S}$, choose a scalar field t which measures the distance from p (case a)) or Σ (case b)) along timelike geodesics through p or which intersect Σ orthogonally. Take a point $w \in \gamma \cap I^+(r)$ near enough to r so that there are no conjugate points in the segment rw. Suppose a point $r' \in \gamma \cap I^-(r)$ close enough to r so that r' is not conjugate to w either. Then, at the segment $r'w$ we can consider another synchronous coordinate system $\tilde{\mathcal{S}}$ with $\tilde{t}$ scalar field which measures *minus the distance to* w. Let (U, φ) be a chart with coordinate functions $x^1, \dots, x^4$. Note that $\tilde{T}^\mu = \nabla^\mu \tilde{t}$ are future pointing timelike vectors so that $\tilde{T} = T$ along $\gamma = \gamma_0$.
Since r is conjugate to p or Σ, there exists a non-trivial Jacobi field, denoted as X on $\gamma = \gamma_0$, from a 1-parameter subfamily of the congruence (γ) of time-lines (timelike geodesics orthogonal to a hypersurface given by $t \equiv$ constant) of $\mathcal{S}$, which contains γ_0, satisfying $X(r) = 0$ and $DX(r) \neq 0$. Let t_0 be the value of t at r ($l(pr) = t_0$) so that it can be written $X = (t_0 - t)Y$, where Y is a smooth vector field defined along γ_0 orthogonal to γ_0 with $Y(r) \neq 0$. Thus, Y is space-like at r with $Y^\mu Y_\mu < 0$.
Consider now the geodesic η given by $Y^\mu \nabla_\mu Y^\nu = 0$. Let us discuss the variation of the length $t - \tilde{t}$ along this geodesic to construct a geodesic with greater length: Suppose $\eta : I \subseteq \mathbb{R} \longrightarrow M$ is parametrized by s, with an interval I. Suppose, without loss of generality, that $I = [0, a)$ (for some real $a > 0$) so that $\eta(0) = r'$ and there will be some $s_0 \in I$ satisfying $\eta(s_0) = r'' \in \eta$ near enough to r. Consider ϕ as a function defined over η. Let us write $\phi = f(\eta(s))$ for a well-behaved function $f \in \mathcal{F}(M)$. With this, we can compute the Taylor series expansion for ϕ such as:

$$f(\eta(s)) = f(\eta(0)) + s\nabla_Y \phi + s^2 \nabla_Y^2 \phi + \mathcal{O}(Y^3(\phi))$$

Write $\phi = t - \tilde{t}$, at r' we will have $f(\eta(0)) = f(r') = (t - \tilde{t})(r')$. We see:

$$f(r'') = f(r') + s_0 \nabla_Y \phi + s_0^2 \nabla_Y^2 \phi + \mathcal{O}(Y^3(\phi))$$

Let us combine both expression as $f(r'') - f(r') = s_0 \nabla_Y \phi(r'') + s_0^2 \nabla_Y^2 \phi(r'')$.
Since Y is orthogonal to γ_0, we have $\nabla_Y \phi = Y^\mu \nabla_\mu (t - \tilde{t}) = Y^\mu (T_\mu - \tilde{T}_\nu) = 0$, and $\nabla_Y^2 \phi = Y^\mu Y^\nu \nabla_\mu \nabla_\nu (t - \tilde{t}) > 0$ since $Y^\mu Y^\nu \nabla_\mu \nabla_\nu t = \frac{1}{t_0 - t} Y^\nu X^\mu \nabla_\mu T_\nu = \frac{1}{t_0 - t} Y^\nu D X_\nu = \frac{1}{t_0 - t} Y^\nu D[(t_0 - t) Y_\nu] = -\frac{1}{t_0 - t} Y^\nu Y_\nu + Y^\nu D Y_\nu$. We obtain $\nabla_Y^2 \phi = (\frac{\tilde{t} - t}{(t_0 - \tilde{t})(t_0 - t)}) Y_\nu > 0$

in the near past of r on γ_0 since $Y_\nu Y^\nu < 0$ and $t_0 - \tilde{t} > 0$, $t_0 - t > 0$, $\tilde{t} - t < 0$. Finally, we see that $(t - \tilde{t})(r'') - (t - \tilde{t})(r') > 0$, with which we construct a trip ν with relevant geodesics of (γ) up to r'' together with the segment $r''w$, so that the length of ν is greater than the length up to w along γ_0. Thus, $l(\nu \cup wq) > l(\gamma_0)$.
In case b), but now γ is not orthogonal to Σ at p. Suppose there exists a point $w \in I^+(p) \cap \gamma$. Choose a synchronous coordinate system $\tilde{\mathcal{S}}$ with $-\tilde{t}$ as before. Take Z as a tangent vector to Σ not orthogonal to γ so that $Z_\mu T^\mu = Z_\mu \nabla^\mu \tilde{t} < 0$ at p. Choose $p' \in \Sigma$ near enough to p on some curve $\zeta \subseteq \Sigma$ with tangent vector Z. We can apply a similar reasoning as before. Let ψ to be a function over ζ so that $f(\zeta(s)) = f(\zeta(0)) + s\nabla_Z\psi(0) + \mathcal{O}(Z^2(\psi))$. Then $f(p') - f(p) = s_0\nabla_Z\psi(p')$ and since $s_0 > 0$ and $\nabla_Z\psi = Z^\mu\nabla_\mu(-\tilde{t}) > 0$ at p'. Finally, $l(p'w \cup wq) > l(\gamma)$.
Consider the case when γ is null (and orthogonal to Σ in the case b)). The proof follows similarly as before. Instead of $t_0 - t$ and t we have $v_0 - v$ and u respectively. We must consider $\mathcal{S}$ and $\tilde{\mathcal{S}}$ as null coordinate systems with $\nabla_\mu u = T_\mu = \tilde{T}_\mu = \nabla\tilde{u}$ along $\gamma = \gamma_0$ where $u = \tilde{u} = 0$ along γ_0, and take $X = (v_0 - v)Y$. Analogously, we have $Y^\mu Y^\nu \nabla_\mu\nabla_\mu(u - \tilde{u}) > 0$ in the near past of r on γ_0, call it r', where $Y^\mu T_\mu = 0$ and $Y^\mu Y_\mu < 0$. Set $U = \partial/\partial u$ at r'. Let π be the plane between U and Y. By applying $\exp_{r'}$ to a point at π we obtain $xU + yY$, and consider $u = u(x, y)$ and $\tilde{u} = \tilde{u}(x, y)$. At $r' = (0, 0)$ we have $u = \tilde{u} = 0$ and $\partial u/\partial x(r') = \partial\tilde{u}/\partial x(r') = 1$ since $U(u) = U^\mu\nabla_\mu u = \partial u/\partial u$, and $U(\tilde{u}) = U^\mu\nabla_\mu\tilde{u} = U^\mu\tilde{T}_\mu = U^\mu T_\mu = U(u) = 1$. In addition, $\partial u/\partial y(r') = \partial\tilde{u}/\partial y(r') = 0$ since $Y^\mu\nabla_\mu u = Y^\mu T_\mu = 0$ and $Y^\mu\nabla_\mu\tilde{u} = Y^\mu\tilde{T}_\mu = 0$. Consider the curve given by $4x + (A + \tilde{A})y^2 = 0$ where $A = \partial^2 u/\partial y^2(r')$ and $\tilde{A} = \partial^2\tilde{u}/\partial y^2(r')$, satisfying $A > \tilde{A}$ as it has been proven before. Let us write a Taylor's expansion near r': $(u - \tilde{u})(x, y) = \frac{1}{2}y^2(A - \tilde{A}) > 0$ so that with $\tilde{u} < 0$ we have $\tilde{u} < 0 < u$ for a small enough $y > 0$. Let us take the point (x, y) which satisfies the previous expression so that $\exp_{r'}(x, y) = r''$. Since $\tilde{u} < 0$ at this point, then $r'' \ll w$ and since $u > 0$ at the same point, $p \ll r''$. Now, $p \ll r'' \ll w \prec q \implies p \ll q$ as we wanted to see for a) when γ null and with $q \in I^+(\Sigma)$ for case b).
If γ is null and not orthogonal to Σ, we can repeat the same proof as when γ is timelike but taking $\tilde{\mathcal{S}}$ as a null synchronous coordinate system. Since $\tilde{u} < 0$, we have a point $p' \in \Sigma$ which, as previously for timelike, maximizes the length towards w so that $p' \ll w \ll q \implies q \in I^+(\Sigma)$. □

Proposition 3.24. *Let γ be a causal geodesic from p to q. If one of the following statements is true:*

a) $\nexists x, y \in \gamma$ such that x and y are conjugate; or else, *b)* *Σ is a spacelike hypersurface (or a $2-$surface) orthogonal to a timelike (or null) γ at p and there are no conjugate points along γ to Σ.*

Then, there is a neighbourhood $Q \ni \gamma$ such that for any causal curve $\eta \in Q$ from p to q (a)) or from Σ to q (b)), $l(\eta) \leq l(\gamma)$ is satisfied and $l(\eta) = l(\gamma)$ if and only if $\eta = \gamma$

Proof. From proposition 3.22, the existence of a coordinate system is ensured for some neighbourhood Q. Then, the proof follows a similar argument as in 3.4. □

3.2 Singularity Theorems

In the following theorem, we will prove the existence of incomplete geodesics in a space-time M that satisfies certain conditions. That is, the existence of geodesics of finite length without past endpoint, leading us to consider the existence of some kind of singularity.

Theorem 3.25. *Let M be a space-time which satisfies:*

a) *There is a closed spacelike hypersurface $\Sigma \subseteq M$ whose normals diverge at every point of Σ.*

b) *The energy condition (remark 3.13) is satisfied at every point of M.*

Then, there is a past endless geodesic $\gamma \subseteq M$ with $l(\gamma) < \infty$.

Proof. Let us prove it by contradiction, so that we will consider every past endless geodesics to be of infinite length. It will be proven considering first Σ to be a Cauchy hypersurface and secondly as not.
Suppose Σ is a Cauchy hypersurface. By the Raychaudhuri effect, every past endless geodesic with infinite length orthogonal to Σ, denoted as γ, will find a point where $\Delta = 0$. Denote q as such point, which must be conjugate to Σ since $\Delta = 0$ at it. Given that Σ is a closed hypersurface and, by proposition 3.21, such conjugate points move continuously, there exists an upper bound B to the distance from Σ to q along γ. Considering that every past endless geodesic is of infinite length, they must extend infinitely into the past beyond B. Let w be a point whose distance to Σ is greater than B. Thus, q lies between Σ and w. By proposition 3.23, γ is not maximal at the portion from w to Σ. This statement must be true for every past endless geodesic orthogonal to Σ through w having a conjugate point in between. However, since Σ is a Cauchy hypersurface, M must be globally hyperbolic, and thus strongly causal everywhere. If we have $w \in D^-(\Sigma)$, since Σ is achronal, $\mathcal{C}(\Sigma, w)$ must be compact so that by corollary 3.7, there is a causal geodesic from Σ to q of maximal length. This leads us to a contradiction since

portions from Σ to w on any γ are not maximal.
Suppose Σ is not a Cauchy hypersurface. If Σ is not an achronal hypersurface at M, we can find a covering M^* of M which contains a discrete set of isomorphic achronal copies of Σ. Therefore, assume Σ is an achronal hypersurface. Let γ be a past endless geodesic with infinite length. The above argument shows that if each γ has infinite length into the past, then each γ must have points in $I^-(\Sigma) \setminus D^-(\Sigma)$ so that they intersect $H = H^-(\Sigma)$. Note that $D^-(\Sigma)$ will lay on the area covered by portions of these curves with future endpoints in Σ and length B. Since Σ is closed, then H will also be closed, so that it is compact. Given that Σ is spacelike and edgeless, we have $H \cap \Sigma = \emptyset$ showing that H must be a compact C^0-manifold without boundary. Set $f = \gamma \cap H$ and $p(f)$ as the maximum of the lengths of the segments of γ from a fixed f to Σ. We can find some $f_0 \in H$ with $p(f_0)$ minimum. By 2.36, there exists a future null geodesic $\eta \subseteq H$ with past endpoint f_0. Take $f_1 \in \eta$ at the near future of f_0. Note that $p(f_1) > p(f_0)$. If a point f_2 is taken to the future of f_1 on the maximal curve from f_1 to Σ, we must have $l(f_0 f_2) > l(f_0 f_1) + l(f_1 f_2)$. Then, we obtain a trip ζ from f_0 to Σ of length $l(\zeta) = k > p(f_0)$. Choose a point $b \in \zeta$ at a greater distance from Σ than $p(f_0)$ and let it approach f_0 until we reach a limiting curve $\gamma \ni b$, by the compactness of Σ. We obtain $p(f_0) \geq k > p(f_0)$, which leads to another contradiction. □

Remark 3.26. This shows that if we reversed time for a closed region in a space which is expanding outwards, we will find a singular point from which all geodesics emerge; due to the existence of incomplete past endless timelike geodesics.

Lemma 3.27. *Let $S \subseteq M$ be an achronal set. If there is a null geodesic $\gamma \subseteq I^+(S)$ or $H^+(S)$, then γ does not contain conjugate points, except possibly at its endpoints.*

Proof. By contradiction, suppose γ contains a point conjugate to one of its endpoints. Since $I^+(S)$ and $H^+(S)$ are achronal sets and γ is a null geodesic lying in any of both, from 3.23, its endpoints should be chronologically related, which contradicts the achronalicity. □

Lemma 3.28. *If there are no closed trips in M and for every endless null geodesic in M there are two conjugate points, then M is strongly causal.*

Proof. Let us prove it by contradiction. Suppose M is not strongly causal, so that there is a point $p \in M$ where strong causality fails. By applying Theorem 2.26 e), there is an endless null geodesic $\eta \ni p$ along which strong causality fails. That is, for any pair of dfferent points $u \neq v$ satisfying $u \prec v$ such that there are $x \neq y$ with

$u \ll x$ and $y \ll v$ then $y \ll x$. According to the hypothesis, along η there is a pair of conjugate points. Thus, taking into account 3.23, we get $u \ll v$ so that we can find some $y \in I^-(v) \cap I^+(u)$ where $y \ll x$ and $x \ll y$ for some $x \in I^-(v) \cap I^+(u)$. This leads us to the existence of a closed trip, which is a contradiction. □

Definition 3.29. Let $S \subseteq M$ be an achronal non-empty closed set for which $E^+(S) = J^+(S) \setminus I^+(S)$ is compact. Then S is said to be a *future-trapped set*. Analogously for *past-trapped* sets. Any future-trapped set is compact since $S \subseteq E^+(S)$.

Remark 3.30. The sets $E^+(S)$ for any $S \subseteq M$ are called *horismotic sets*. This notion is difficult to convey if we restrict ourselves to Minkowski metrics. However, it becomes much easier to think of $E^+(S)$ as compact sets when thinking in non-planar metric, such as the Schwarzschild one.

Lemma 3.31. *If S is future-trapped and $\overline{I^+(S)}$ is strongly causal, there exists a future endless timelike curve or trip $\gamma \subseteq \operatorname{int}[D^+(E^+(S))]$*

Proof. We need to show first that $H = H^+(E^+(S))$ is non-compact or empty. Any trip which leaves $\operatorname{int}[D^+(E^+(S))]$ must cross H. Note that if $H = \emptyset$, the proof is trivial. Suppose then that $H \neq \emptyset$ and cover it by a finite set of local causality neighbourhoods $L_1, \ldots, L_k$. Consider a point $p \in I^+(S) \setminus D^+(E^+(S))$, and let it be in some $L_i \ni p$. Thus, there is another point $q_i \in I^+(S) \setminus D^+(E^+(S))$ such that $q_{i_1} \ll p$ and $q_i \in L_{i_1} \setminus L_i$ for some index i_1. By repeating this process k times, we obtain $\cdots \ll q_{i_k} \ll q_{i_{k-1}} \ll \ldots q_{i_2} \ll q_{i_1} \ll p$ so that there are at least two of them lying on the same L_{i_j}. This means, there would be a trip leaving and re-entering L_{i_j}, which contradicts the strong causality of $\overline{I^+(S)}$. Thus, H is non-compact.
Consider a nowhere vanishing vector field $\xi \in \mathcal{X}(M)$, over which we define a congruence of curves from $E^+(S)$. If one of such curves leaves $\operatorname{int}[D^+(E^+(S))]$, it must intersect H, and thus establishing and homeomorphism between $E^+(S)$ and H. However, this leads to a contradiction since $E^+(S)$ is compact but H has been proven otherwise. More details can be seen at [9, Section 8.2] □

The proof for the following theorem is based on [11] and [9, Section 8.2].

Theorem 3.32. *No space-time can satisfy simultaneously the following statements:*

a) M does not contain closed trips.

b) Every endless causal geodesic in M contains a pair of conjugate points.

c) There is a future-trapped set $S \subseteq M$.

Proof. By contradiction, suppose the three statements are satisfied simultaneously. Since we are considering that M contains no closed trips and every endless causal geodesic contains a pair of conjugate points, 3.29 is satisfied so that M must be strongly causal. Hence, since there is a future-trapped surface $S \subseteq M$, 3.31 holds, so that there exists a future endless timelike curve $\gamma \subseteq \text{int}\,[D^+(E^+(S))]$. Consider the set $T = \overline{I^-(\gamma)} \cap E^+(S)$ and let us show that T is past-trapped by proving that $E^-(T)$ is compact (Note that T is closed and achronal since $\overline{I^-(\gamma)}$ is closed and $E^+(S)$ is closed and achronal):
Since $\gamma \subseteq \text{int}\,[D^+(E^+(S))]$, any past endless curve with future endpoint on γ intersects $T \subseteq E^+(S)$. Taking into account that $I^-(\gamma)$ sectioned by T gives $I^-(T)$ since $\overline{I^-(T)} \subseteq I^-(\gamma)$, we see that $\partial I^-(T) \subseteq T \cup \partial I^-(\gamma)$. In addition, note that $E^-(T) \setminus T$ must be generated by the null geodesics $\eta \subseteq \partial I^-(\gamma)$ with future endpoint on edge$(T) \subseteq T$. The null geodesics $\eta \subseteq \partial I^-(\gamma)$ extend infinitely into the future by proposition 2.36. By applying $b)$ we see that every maximal extension of η must contain a pair of conjugate points p and q, such that $p \prec q$. However, by lemma 3.27, we see that $p \in I^-(\gamma)$, which implies that η must have a past endpoint at p or at $I^+(p)$. From 3.21, for each η chose p and q so that the segments between p and q along the extension of η sweep out a compact region. Analogously, the segments of the extension of η from p to edge(T) sweep out another compact region $C \subseteq M$. With all of this, $E^-(T) = \partial I^-(T) \cap (C \cup T) \subseteq C \cup T$, and since $C \cup T$ is compact, $E^-(T)$ must be compact too. We have shown that T is past-trapped.
By the time reverse of 3.31, there exists a past endless curve $\alpha \subseteq \text{int}\,[D^-(E^-(T))]$. Choose a point $a_0 \in \alpha$ so that we can find a point $c_0 \in \gamma$ with $a_0 \ll c_0$. Choose an infinite sequence of points into the past $\{a_i\}_i \subseteq \gamma$ and into the future $\{c_i\}_i \subseteq \gamma$ so that $a_i \ll c_i$ for all i. Since $a_i \in \text{int}\,[D^-(E^-(T))]$ and $c_i \in \text{int}\,[D^+(E^+(S))]$, we see that $J^+(a_i) \cap J^-(T)$ and $J^-(c_i) \cap J^+(S)$ are compact and strongly causal, so that we obtain the compact and strongly causal set $J^+(a_i) \cap J^-(c_i)$. By applying 3.7 we find a maximal causal geodesic μ_i from a_i to c_i which intersects T at a point q_i.Since T is compact, as $i \to \infty$, then $q_i \to q \in T$, providing an endless causal geodesic μ which crosses T at the point q as a limiting curve from μ_i. Again, by applying $b)$, there is a pair of conjugate points $u, v \in \mu$ where $u \prec v$, and since conjugate points vary continuously, u will be the limit point of some sequence $\{u_j\}_j$ and so v for $\{v_j\}_j$ where u_j and v_j are conjugate points on the maximal extension of μ_j which converges to μ. However, the sequences $\{a_i\}_i$ and $\{c_i\}_i$ cannot accumulate at any point of the segment uv of μ, so that if $j \to \infty$, then $a_j \ll u_j$ and $v_j \ll c_j$ along μ_j. Thus, we obtain a maximal geodesic with a pair of conjugate points, contradicting the fact that a maximal geodesic "evades" a pair of conjugate points along a curve, as seen in proposition 3.23.

□

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