Facultat de Matemàtiques
i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

On the sheaf theoretic de Rham theorem and the Witten Laplacian

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Realitzat a: Departament de Matemàtiques i Informàtica
Barcelona, January 15, 2025

Abstract

The aim of this work is to establish a dialogue between topology, differential geometry, and certain modern developments in theoretical physics involving supersymmetry. First, the construction of the de Rham theorem is presented, followed by its proof through the elegant theory of sheaves, bringing forth algebraic invariants of the manifold derived from the properties of differential objects. Next, harmonic differential forms are studied using Hodge theory, demonstrating the main decomposition theorem as well as the existence and uniqueness of harmonic representatives in the de Rham cohomology groups. Finally, Witten's ideas concerning supersymmetry preservation are discussed, and a proof of Morse inequalities is presented using Witten's deformed Laplacian.

La intenció d'aquest treball és establir un diàleg entre la topologia, la geometria diferencial i alguns desenvolupaments moderns de la física teòrica que involucren supersimetria. Es presenta, primerament, la construcció del teorema de de Rham seguida de la corresponent demostració mitjançant l'elegant teoria de feixos, tot fent emergir invariants algebraics en la varietat a partir de propietats d'objectes diferencials. A continuació, s'estudien les formes diferencials harmòniques mitjançant la teoria de Hodge, demostrant-se el principal teorema de descomposició, així com l'existència i unicitat de representants harmònics als grups de cohomologia de de Rham. Finalment, es discuteixen les idees de Witten involucrant la preservació de la supersimetria, i es presenta una demostració de les desigualtats de Morse mitjançant el Laplacià deformat de Witten.

Acknowledgements

The following pages are the result of the unwavering support of my parents, sister, and partner, as well as the counsel and inspiration of my wise and centenarian grandfather. None of this would have been possible without the guidance and advice of Ricardo, nor without the companionship of my friends throughout this long journey.

2020 Mathematics Subject Classification. 55S35, 55U15, 57N65, 14F40, 46M20, 55N30

Contents

1 Introduction **1**

2 Cohomology and the de Rham theorem **4**
2.1 Tensor and exterior algebras . . . 4
2.2 The exterior derivative . . . 8
2.3 Singular cohomology . . . 11
2.4 De Rham cohomology . . . 12
2.5 Integration and the de Rham theorem . . . 14

3 Sheaf theory **18**
3.1 Presheaves and sheaves . . . 18
3.2 The sheafification functor . . . 20
3.3 Resolutions of sheaves . . . 21
3.4 Sheaf cohomology . . . 23
3.5 Proof of the de Rham theorem . . . 27

4 Harmonic forms **29**
4.1 The Hodge star operator . . . 29
4.2 The Laplace-Beltrami operator . . . 31
4.3 Functional analysis and elliptic operators . . . 33
4.4 Hodge decomposition . . . 36

5 Morse theory and supersymmetry **40**
5.1 The Morse inequalities . . . 40
5.2 Supersymmetric quantum mechanics . . . 42
5.3 Witten's deformation . . . 43
5.4 Proof of the Morse inequalities . . . 48

References **52**

A Basics of differential geometry on manifolds **53**

Chapter 1

Introduction

Humanity's interest in the geometry of the surrounding world dates back thousands of years. From the Greek, $\gamma\varepsilon\omega\mu\varepsilon\tau\rho\acute{\iota}\alpha$ literally means "measurement of the land", which is why it is no surprise that one of the earliest known milestones was, around 200 BC, the calculus of the Earth's circumference by the Greek Eratosthenes. Similarly, in 150 AC, Ptolemy introduced the stereographic projection for measuring its shape. Over the following centuries, Euclid's *Elements* continued to dominate human technical knowledge, leaving a lasting impact on art, architecture, and many other disciplines. However, applying the calculus developed by Newton and Leibniz to curved forms, such as the Earth, continued to present significant challenges to geometers.

In 1827, Gauss proved the *Theorema Egregium*, which states that the Gaussian curvature of a surface is an intrinsic property and does not depend on how the surface is embedded in Euclidean space. This discovery propelled the study of spaces in their own right. Over time, topological spaces gained increasing importance, with metric properties losing relevance compared to the preservation of an object's characteristics under continuous deformations. Through this process, the primary setting for geometry came to be known as the manifold, which will also serve as the stage for the pages that follow. Differentiable manifolds are, in brief, topological spaces that are locally flat, and a structure of compatible local charts allows us to apply the laws of calculus within them locally.

At the end of the 19th century, Hilbert and his colleagues in Göttingen initiated a movement demanding greater rigor in the development of mathematics. In an effort to systematize the study of topology, Henri Poincaré published *Analysis Situs* (1895), followed by five supplementary articles written between 1899 and 1904. In this work, Poincaré used algebraic structures to distinguish between non-homeomorphic topological spaces, laying the foundations of algebraic topology. This introduced the notions of the "fundamental group" and "simplicial homology" [11]. Later simplifications of the theory appeared thanks to the contributions of Whitney, Cartan, Weyl, and Noether.

Under the Eilenberg-Steenrod axioms, the homology theories of a nice enough topological space determine the same homology modules as the singular homology of the space in question. Prior to Poincaré's formalization, homological algebra had already been addressed in the works of Riemann (1857) and Betti (1871), emerging abstractly from chain complexes. Similarly, the abstract treatment of the algebra of cochains gives rise to the cohomology modules, seemingly disconnected from any topological meaning. Among the various cohomology theories, one of the most common is known as de Rham cohomology, which arises in the context of differential geometry as a measure of the extent to which closed differential forms fail to be exact. Back in 1928, Élie Cartan had published the idea that differential forms and topology should be linked. In his 1931 thesis, Georges de Rham developed, via Stokes' Theorem, an isomorphism between the de Rham cohomology modules and the singular cohomology modules.

$$\tilde{\Psi} : H^k_{dR}(M) \to H^k_{\text{sing}}(M, \mathbb{R}).$$

But, why differential forms? This is a hard question to answer without some more work, but the basic idea is that forms can be both differentiated and integrated without the help of any additional geometric structure. This allows us, throughout Chapter 2, to develop the framework of differential geometry on a manifold necessary to state de Rham's theorem, following primarily [19] while also referencing [2] for a more computational treatment of cohomology. Along the way, we will see that it is highly useful to define an operator on differential forms called the "exterior derivative". Indeed, unlike other operators such as the partial derivative, the exterior derivative preserves the tensorial nature of an object. Proving the de Rham isomorphism, however, is no easy task. Although it could be done within the framework of topology as developed in [4], in this text we could not resist turning to the elegant theory of sheaves to present the proof in Chapter 3. To do so, we will momentarily diverge from [19] and primarily follow [3], [18], [20] and [21].

Sheaf theory has the remarkable beauty of connecting local information with the global behaviour of a space. This process frequently takes center stage in many real-world situations; for example, weather forecasting does not rely on knowing the pressure and temperature at every point in the atmosphere but rather on the proper extension of a local set of data [16]. It all began when Jean Leray, a French mathematician and artillery officer, was imprisoned by the Germans in 1940. Fearing that his captors would make them devote his efforts to warlike purposes if they discovered his true area of expertise —hydrodynamics— he claimed to be a harmless topologist. During the next five years of imprisonment, Leray dedicated himself to research in topology, culminating in the development of sheaf theory, a revolutionary concept that Henri Cartan and Alexander Grothendieck would further develop throughout the 1950s.

Throughout Chapter 3, we will construct the sheaves of differential forms and smooth singular cochains and observe that both allow for the articulation of a resolution of the constant sheaf $\mathbb{R}$, suggesting the existence of a map between them. Subsequently, we will see that, despite having an exact sequence of sheaves, the sequence induced by global sections may exhibit failures in exactness, providing an initial insight into cohomology. By introducing the canonical Godement resolution, we will demonstrate that the exactness defects of the global section sequences induced by the resolutions of the sheaves of differential forms and smooth singular cochains are, in fact, isomorphic, culminating in a proof of de Rham's theorem. Finally, we will have succeeded in relating the topology of the manifold to the differential geometry constructed on it.

Greatly interested in de Rham's thesis, the British mathematician W.V.D. Hodge introduced a new operator, called the Hodge star, which generalized the duality between the real and imaginary parts of a holomorphic 1-form. Within this framework, in Chapter 4, we will once again follow [19] and require the differentiable manifold to be compact and equipped with a Riemannian structure. This metric will induce an inner product on differential forms, allowing us to define a dual operator to the exterior derivative. During the 1930s, Hodge conjectured and proved that the de Rham cohomology classes admit a distinguished representative with the property that both the exterior derivative and its adjoint, that is, d and δ, annihilate it. These representatives are called harmonic forms, and we will introduce them within the formalism of the Laplace-Beltrami operator, $\Delta = d\delta + \delta d$, a generalization of the Laplacian to manifolds.

The Hodge decomposition theorem, presented in Chapter 4, proved to be a significant challenge to demonstrate. Just as Hodge relied on Weyl's assistance to prove it, in this text we will draw heavily on various results from the theory of elliptic operators. While some of these results are explained and proven within the text, the focus has been on providing an intuitive understanding of their role without overburdening the work. For detailed proofs, we will frequently refer to [5] and [8]. Finally, we will prove that the de Rham cohomology modules are isomorphic to the kernels of the Laplace-Beltrami operator when restricted to k-forms. Thus, the Betti numbers, defined as the dimension of the singular homology modules and equal to the dimension of the de Rham cohomology modules thanks to the de Rham theorem, will also correspond to the dimension of the space of forms annihilated by the Laplacian.

$$\beta_k = \dim H^k_{\text{sing}}(M, \mathbb{R}) = \dim H^k_{dR}(M) = \dim \ker \Delta|_{\Omega^k(M)}.$$

Now let us travel to the 1980s, when Edward Witten was working within the framework of supersymmetric theories [22], [23]. Supersymmetric quantum mechanics transforms the Hilbert space of a system into a $\mathbb{Z}_2$-graded Hilbert space [10], separating bosonic variables from fermionic ones. The symmetry that exchanges these types of variables is encoded in the number of ground states of a given system, i.e., forms annihilated by a certain Hamiltonian. When the Hamiltonian is Δ, we are essentially studying the Betti numbers of the manifold. However, one can see that the task of finding the spectrum of the Hamiltonian can become quite complex, which is why Witten simplified it by introducing a deformation of the Laplacian via

$$\Delta_T = e^{-Tf}\Delta\, e^{Tf},$$

where f is a Morse function and $T > 0$. This new operator is called the Witten Laplacian.

This final Chapter 5 allows us to bring topology, differential geometry, and theoretical physics into dialogue. Indeed, as T grows large, the spectrum of the Witten Laplacian restricted to k-forms captures a finite number of eigenvalues that become asymptotically small, while the rest grow, splitting the spectrum into two very distinct parts. The number of small eigenvalues is exactly the number of critical points of the Morse function f with Morse index equal to k, denoted m_k. Throughout the chapter, following [12], [23] and [24], we will see that this provides an alternative way to prove the Morse inequalities, which relate the number of critical points of a Morse function with a given Morse index k to the k-th Betti numbers.

$$\sum_{j=0}^{k}(-1)^{k-j}\beta_j \leq \sum_{j=0}^{k}(-1)^{k-j}m_j.$$

Chapter 2

Cohomology and the de Rham theorem

Differentiable manifolds provide the setting in which many real systems take place. However, much of their information is hidden within their complex topology. In this first chapter, we explore differential geometry on manifolds, extracting the de Rham cohomology modules and connecting them to the manifold's topology by stating the de Rham theorem. The fundamental tools needed to follow the text can be found in appendix A. Throughout the chapter, our main reference will be [19], although we will often also refer to [2] and [4].

2.1 Tensor and exterior algebras

Given a differentiable manifold M, the tangent and cotangent bundles provide the framework for defining fundamental objects known as vector fields and 1-forms (Appendix A). To generalize these concepts, we will explore how more sophisticated constructions can be developed on our manifold. Let V, W, and Z denote finite-dimensional real vector spaces. As usual, V^* will denote the dual space of V consisting of all real-valued linear functions on V.

Definition 2.1.1. Let $F(V,W)$ be the vector space over $\mathbb{R}$ generated by the points of $(v,w) \in V \times W$, where $v \in V$ and $w \in W$. Let $R(V,W)$ be the subspace of $F(V,W)$ generated by the elements of the following forms:

$$(v_1 + v_2, w) - (v_1, w) - (v_2, w)$$

$$(v, w_1 + w_2) - (v, w_1) - (v, w_2)$$

$$(av, w) - a(v, w)$$

$$(v, aw) - a(v, w)$$

where $a \in \mathbb{R}$, $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$. The quotient space $F(V,W)/R(V,W)$ is called the *tensor product of V and W* and is denoted by $V \otimes W$. Its cosets are denoted by $v \otimes w$.

Let $\varphi : V \times W \to V \otimes W$ denote the bilinear map $(v,w) \mapsto v \otimes w$. Then, whenever Z is a vector space and $l : V \times W \to Z$ is a bilinear map, there exists a unique linear map $\tilde{l} : V \otimes W \to Z$ such that $\tilde{l} \circ \varphi = l$. This is known as the *universal mapping property.* The pair $V \otimes W$ and φ are unique with this property in the sense that if Y is a vector space and $\tilde{\varphi} : V \times W \to Y$ is a bilinear map satisfying the universal mapping property, then there exists an isomorphism $\alpha : V \otimes W \to Y$ such that $\alpha \circ \varphi = \tilde{\varphi}$.

It can be proven that $V\otimes W$ is canonically isomorphic to $W\otimes V$, $V\otimes(W\otimes U)$ to $(V\otimes W)\otimes U$ and $(V\otimes W)^*$ to $V^*\otimes W^*$. Furthermore, because of the universal mapping property, the bilinear map

$$V^*\times W\to \mathrm{Hom}(V,W), \qquad (f,w)(v)=f(v)\cdot w$$

for each $f\in V^*$ and $w\in W$ determines uniquely a linear map $\alpha : V^*\otimes W\to \mathrm{Hom}(V,W)$ which can be shown to be an isomorphism. As a consequence,

$$\dim V\otimes W = (\dim V)\cdot(\dim W).$$

Let $\{v_i : i=1,\dots,n\}$ and $\{w_j : j=1,\dots,m\}$ be bases for V and W respectively. Then $\{v_i\otimes w_j : i=1,\dots,n \text{ and } j=1,\dots,m\}$ is a basis of $V\otimes W$.

Definition 2.1.2. The *vector space* $V_{r,s}$ *of tensors of type* (r,s) associated with V is the vector space

$$V\otimes \overset{r)}{\dots} \otimes V\otimes V^*\otimes \overset{s)}{\dots}\otimes V^*$$

and the direct sum

$$T(V)=\bigoplus_{r,s\geq 0} V_{r,s}$$

is called the *tensor algebra of* V, where $V_{0,0}=\mathbb{R}$. The elements of $T(V)$ are called *tensors*. The tensor algebra is a non-commutative, associative, graded algebra under the tensor product: given $u=u_1\otimes\ldots\otimes u_{r_1}\otimes u_1^*\otimes\ldots\otimes u_{s_1}^*\in V_{r_1,s_1}$ and $v=v_1\otimes\ldots\otimes v_{r_2}\otimes v_1^*\otimes\ldots\otimes v_{s_2}^*\in V_{r_2,s_2}$, then, using $(V^*)^*\cong V$,

$$u\otimes v = u_1\otimes\ldots\otimes u_{r_1}\otimes v_1\otimes\ldots\otimes v_{r_2}\otimes u_1^*\otimes\ldots\otimes u_{s_1}^*\otimes v_1^*\otimes\ldots\otimes v_{s_2}^*\in V_{r_1+r_2,s_1+s_2}.$$

Definition 2.1.3. Let us define the subalgebra of $T(V)$ given by $C(V)=\bigoplus_{k=0}^{\infty} V_{k,0}$. Let $I(V)$ be the two-sided ideal in $C(V)$ generated by the set of elements of the form $v\otimes v$ for $v\in V$, and set

$$I_k(V)=I(V)\cap V_{k,0}.$$

It follows that

$$I(V)=\bigoplus_{k=0}^{\infty} I_k(V)$$

is a graded ideal in $C(V)$. The *exterior algebra* $\Lambda(V)$ is the graded algebra $C(V)/I(V)$. If we set

$$\Lambda^k(V)=V_{k,0}/I_k(V)\quad (k\geq 2),\quad \Lambda^0(V)=\mathbb{R},\quad \Lambda^1(V)=V$$

then

$$\Lambda(V)=\bigoplus_{k=0}^{\infty}\Lambda^k(V)$$

has an algebra structure under the *exterior product*, being the image of the tensor product under the projection map $\pi : T(V)\to\Lambda(V)$. That is, $\alpha\wedge\beta=\pi(\alpha\otimes\beta)$, and will also be referred to as the *wedge product*.

Lemma 2.1.4. Let us show some of the main properties of the exterior algebra.

(i) If $\alpha\in\Lambda^k(V)$ and $\beta\in\Lambda^l(V)$, then $\alpha\wedge\beta=(-1)^{kl}\beta\wedge\alpha\in\Lambda^{k+l}(V)$.

(ii) If $\dim V=n$, then $\dim\Lambda^0(V)=1$, $\dim\Lambda^n(V)=1$ and $\Lambda^k(V)=\{0\}$ for $k>n$.

(iii) The element corresponding to the multilinear map $\det : V\times \overset{n)}{\dots} \times V \to \mathbb{R}$ spans $\Lambda^n(V)$.

(iv) If $v_1, \dots, v_n$ is a basis for V, then $\{v_{i_1} \wedge \dots \wedge v_{i_k} : i_1 < \dots < i_k\}$ is a basis for $\Lambda^k(V)$.

(v) $v_1, \dots, v_k \in V$ are linearly dependent if and only if $v_1 \wedge \dots \wedge v_k = 0$. This is why Grassmann called it "exterior": for the elements to be non-zero, each must be "exterior" to the space spanned by the others.

Corollary 2.1.5. From (iv), $\dim \Lambda(V) = 2^n$ and $\dim \Lambda^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$, for $0 \le k \le n$.

Definition 2.1.6. A multilinear map $h : V\times \overset{r)}{\dots} \times V \to W$ is called an *alternating map* if

$$h(v_{\sigma(1)}, \dots, v_{\sigma(r)}) = (\mathrm{sgn}\ \sigma) \cdot h(v_1, \dots, v_r)$$

for all permutations on r letters $\sigma \in S_r$. The vector space of alternating multilinear functions $V\times \overset{r)}{\dots} \times V \to \mathbb{R}$ is denoted by $A^r(V)$, and we set $A^0(V) = \mathbb{R}$.

As seen for the tensor product, we can also state the *universal mapping property* for the exterior algebra. Let $\varphi : V\times \overset{k)}{\dots} \times V \to \Lambda^k(V)$ be defined by $(v_1, \dots, v_n) \mapsto v_1 \wedge \dots \wedge v_n$. Then φ is an alternating multilinear map. To each alternating multilinear map into a vector space $h : V\times \overset{k)}{\dots} \times V \to W$, there corresponds uniquely a linear map $\tilde{h} : \Lambda^k(V) \to W$ such that the following diagram commutes.

$$\begin{tikzcd}
\Lambda^k(V) \arrow[drr, dashed, "\tilde{h}"] & & \\
V \times \dots^{k)} \times V \arrow[u, "\varphi"] \arrow[rr, "h"] & & W
\end{tikzcd}$$

In the special case in which $W = \mathbb{R}$, the diagram above establishes a natural isomorphism

$$\Lambda^k(V)^* \cong A^k(V)$$

We shall now consider various dual pairings between the spaces $V_{r,s}$, $\Lambda^k(V)$, $\Lambda(V)$ and their corresponding dual spaces built on the dual V^* of V.

Definition 2.1.7. Let V and W be real finite dimensional vector spaces. A *pairing* of V and W is a bilinear map $(\ ,\ ) : V \times W \to \mathbb{R}$. A pairing is called *non-singular* if whenever $w \neq 0$ in W, there exists an element $v \in V$ such that $(v, w) \neq 0$, and whenever $v \neq 0$ in V, there exists an element $w \in W$ such that $(v, w) \neq 0$.

Let V and W be non-singularly paired by $(\ ,\ )$, and define $\varphi : V \to W^*$ by $\varphi(v)(w) = (v, w)$ for $v \in V$ and $w \in W$, which is clearly bijective. Similarly, there is a bijective map $W \to V^*$. Therefore, V and W have the same dimension and hence φ is an isomorphism of V with W^*. Followingly, this idea will be applied to several spaces.

Definition 2.1.8 (A non-singular pairing of $(V^*)_{r,s}$ with $V_{r,s}$)**.** This pairing is defined as the bilinear map $(V^*)_{r,s} \times V_{r,s} \to \mathbb{R}$ which on the elements

$$v^* = v_1^* \otimes \dots \otimes v_r^* \otimes u_{r+1} \otimes \dots \otimes u_{r+s} \in (V^*)_{r,s}$$

$$u = u_1 \otimes \dots \otimes u_r \otimes v_{r+1}^* \otimes \dots \otimes v_{r+s}^* \in V_{r,s}$$

yields $(v^*, u) = v_1^*(u_1) \dots v_{r+s}^*(u_{r+s})$. This pairing establishes an isomorphism

$$(V^*)_{r,s} \cong (V_{r,s})^*.$$

The obvious extension of the universal mapping property shows that there is a natural isomorphism

$$(V_{r,s})^* \cong M_{r,s}(V)$$

where $M_{r,s}(V)$ is the vector space of all multilinear functions

$$V \times \overset{r)}{\dots} \times V \times V^* \times \overset{s)}{\dots} \times V^* \to \mathbb{R}.$$

And this yields $(V^*)_{r,s} \cong M_{r,s}(V)$.

Definition 2.1.9 (A non-singular pairing of $\Lambda^k(V^*)$ with $\Lambda^k(V)$)**.** This pairing is defined as the bilinear map $\Lambda^k(V^*) \times \Lambda^k(V) \to \mathbb{R}$ which on the elements $v^* = v_1^* \wedge \dots \wedge v_k^* \in \Lambda^k(V^*)$ and $u = u_1 \wedge \dots \wedge u_k \in \Lambda^k(V)$ yields

$$(v^*, u) = \frac{1}{k!}\det(v_i^*(u_j)).$$

This pairing establishes an isomorphism

$$\Lambda^k(V^*) \cong \Lambda^k(V)^*$$

which composed with the natural isomorphism found above yields

$$\Lambda^k(V^*) \cong A^k(V).$$

Since the dual space of a finite direct sum is canonically isomorphic to the direct sum of dual spaces, we obtain isomorphisms

$$\Lambda(V^*) = \bigoplus_{k=0}^{\infty} \Lambda^k(V^*) \cong \bigoplus_{k=0}^{\infty} \Lambda^k(V)^* = (\Lambda(V))^*$$

and thus, we obtain and isomorphism

$$\Lambda(V^*) \cong A(V) = \bigoplus_{k=0}^{\infty} A^k(V)$$

We shall make use of the identifications via the pairings showed without further comment. Recall that $\Lambda(V^*)$ is an algebra under the wedge product. The pairing in Definition 2.1.9 establishes an isomorphism $\alpha : \Lambda(V^*) \to \Lambda(V)^*$. From this, we also obtain an algebra structure $\wedge$ on $A(V)$. If $f \in A^k(V)$ and $g \in A^l(V)$, then this induced algebra structure takes the form

$$(f \wedge g)(v_1, \dots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\pi \in S_{k+l}} (sgn\ \pi) f(v_{\pi(1)}, \dots, v_{\pi(k)}) g(v_{\pi(k+1)}, \dots, v_{\pi(k+l)}).$$

Definition 2.1.10. An endomorphism $\psi : \Lambda(V) \to \Lambda(V)$ is

(i) a *derivation* if $\psi(u \wedge v) = \psi(u) \wedge v + u \wedge \psi(v)$ for $u, v \in \Lambda(V)$,

(ii) an *anti-derivation* if $\psi(u \wedge v) = \psi(u) \wedge v + (-1)^k u \wedge \psi(v)$ for $u \in \Lambda^k(V), v \in \Lambda(V)$,

(iii) *of degree m* if $\psi : \Lambda^k(V) \to \Lambda^{k+m}(V)$ for all k, assuming that $\Lambda^k(V) = \{0\}$ if $k < 0$.

2.2 The exterior derivative

Definition 2.2.1. Let M be a differentiable manifold. We define

(i) $T_{r,s}(M) = \bigcup_{p\in M}(T_pM)_{r,s}$ the *tensor bundle of type* (r,s) *over* M.

(ii) $\Lambda(M) = \bigcup_{p\in M}\Lambda(T_p^*M)$ the *exterior algebra bundle over* M.

(iii) $\Lambda^k(M) = \bigcup_{p\in M}\Lambda^k(T_p^*M)$ the *exterior* k*-bundle over* M.

The three bundles above have natural manifold structures such that the canonical projection maps to M are C^∞.

Definition 2.2.2. We call *(smooth) tensor field of type* (r,s) *on* M to a C^∞ mapping $M \to T_{r,s}(M)$.

Definition 2.2.3. We call *(differential) form on* M to a C^∞ mapping $M \to \Lambda(M)$. The set of all differential forms on M is denoted by $\Omega^*(M)$.

Definition 2.2.4. We call *(differential)* k*-form on* M to a C^∞ mapping $M \to \Lambda^k(M)$. The set of all differential k-forms on M is denoted by $\Omega^k(M)$.

From now on, unless needed for emphasis, we will drop the adjectives of "smooth" and "differential". Note that an element $\omega \in \Omega^k(M)$, $\omega : M \to \Lambda^k(M)$, assigns to each point $p \in M$ an alternating k-tensor $\omega_p \in A^k(T_pM)$. In particular, if U is an open subset of M, then $\omega \in \Omega^k(U)$ if $\omega_p \in A^k(T_pM)$ for all $p \in U$.

Definition 2.2.5. A *differential* 0*-form on* M is a real valued function on M, that is, $\Omega^0(M) = C^\infty(M)$. This happens because $\Lambda^0(M) = M \times \mathbb{R}$, and smooth liftings of M into $M \times \mathbb{R}$ are simply graphs of C^∞ functions on M.

Forms can be added, given a product and multiplied by scalars. This allows us to state the following properties.

Definition 2.2.6. The wedge product extends pointwise to differential forms on a manifold. If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then $\omega \wedge \eta \in \Omega^{k+l}(M)$ such that

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$$

at all $p \in M$.

Despite being defined on the exterior algebra, this latter property allows us to speak directly of the wedge product of differential forms.

Definition 2.2.7. The wedge product of a 0-form $f \in C^\infty(M)$ and a k-form $\omega \in \Omega^k(M)$ is defined as the k-form $f\omega$ where

$$(\omega \wedge f)_p = (f \wedge \omega)_p = f(p)\ \omega_p.$$

Remark 2.2.8. A map $\alpha : M \to T_{r,s}(M)$ is a smooth tensor field of type (r,s) if and only if for each coordinate system $(U, x_1, \ldots, x_n)$ on M,

$$\alpha|_U = \sum a_{i_1,\ldots,i_r;j_1,\ldots,j_s} \frac{\partial}{\partial x_{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \ldots \otimes dx_{j_s},$$

where $a_{i_1,\ldots,i_r;j_1,\ldots,j_s} \in C^\infty(U)$. Recall that we are skipping the classical tensor notation of lower and upper indices.

Remark 2.2.9. A map $\omega : M \to \Lambda^k(M)$ is a differential k-form if and only if for each coordinate system $(U, x_1, \ldots, x_n)$ on M,

$$\omega|_U = \sum_{i_1 < \cdots < i_k} b_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

where $b_{i_1,\ldots,i_k} \in C^\infty(U)$. In general, we will omit this cumbersome notation and disregard the summation.

Since $\Lambda^1(T_p^*M) = T_p^*M$, we can refer to Definition A.19 and observe that $df_p \in \Lambda^1(T_p^*M)$. This makes the differential of a smooth function on M (a 0-form) into a 1-form, $df : M \to \Lambda^1(M)$. The differential of a function is also known as the *exterior derivative of the* 0*-form* f, an operator that extends to $\Omega^*(M)$ as described below.

Definition 2.2.10. The $\mathbb{R}$-linear map $d_U : \Omega^k(U) \to \Omega^{k+1}(U)$ defined as

$$d_U\omega = \sum_{i_1 < \cdots < i_k} db_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

is called the *exterior derivative of* ω *on* U.

Let $p \in U$, then $d_U\omega$ is independent of the chart. Since

$$db_{i_1,\ldots,i_k} = \sum_i \frac{\partial b_{i_1,\ldots,i_k}}{\partial x_i} dx_i,$$

one can see that applying twice we get

$$\begin{aligned} d_U^2\omega &= \sum_{i,j} \sum_{i_1 < \cdots < i_k} \frac{\partial^2 b_{i_1\cdots i_k}}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i<j} \sum_{i_1 < \cdots < i_k} \left(\frac{\partial^2 b_{i_1\cdots i_k}}{\partial x_i \partial x_j} - \frac{\partial^2 b_{i_1\cdots i_k}}{\partial x_j \partial x_i} \right) dx_i \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0. \end{aligned}$$

Definition 2.2.11 (Exterior derivative)**.** The *exterior derivative of a differential k-form* is the unique linear operator

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

such that for $k \geq 0$ and $\omega \in \Omega^k(M)$ we have $(d\omega)|_p = (d_U\omega)_p$ for all $p \in M$, and that satisfies the following properties for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$:

(i) d is an anti-derivation. That is,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(ii) $d \circ d = 0$.

Let $\psi : M \to N$ be a smooth map and $p \in M$. As shown in Appendix A, we can consider the pushforward $\psi_* : T_pM \to T_{\psi(p)}N$ as well as its dual $\psi^* : T^*_{\psi(p)}N \to T_p^*M$. The latter induces a bundle homomorphism $\hat{\psi}^* : \Lambda(N) \to \Lambda(M)$ which can be extended to act on differential forms. If ω is a form on N, it can be pulled back to a form on M by setting

$$\hat{\psi}^*(\omega)|_p = \psi^*(\omega|_{\psi(p)}).$$

We will no longer make any difference between ψ^* and $\hat{\psi}^*$. This feature, called the *pullback of a k-form*, actually makes a big difference between differential forms and vectors. While forms can be pulled back under a smooth mapping from its range to its domain, vector fields do not display such pleasing behaviour.

Proposition 2.2.12. Let $\psi : M \to N$ and $\phi : N \to L$ be smooth maps, then the pullback of the composition yields $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ such that the following diagram commutes.

$$\begin{tikzcd} \Omega^k(L) \arrow[r, "\phi^*"] \arrow[dr, "(\phi \circ \psi)^*"'] & \Omega^k(N) \arrow[d, "\psi^*"] \\ & \Omega^k(M) \end{tikzcd}$$

Proposition 2.2.13. Let $\psi : M \to N$ be a smooth map. If ω and η are differential forms on N, such that ω and η have the same order, then

1. (preservation of the vector space structure) $\psi^*(a\omega + b\eta) = a(\psi^*\omega) + b(\psi^*\eta)$ for all $a, b \in \mathbb{R}$.
2. (preservation of the wedge product) $\psi^*(\omega \wedge \eta) = \psi^*\omega \wedge \psi^*\eta$.
3. (commutation with the differential) $\psi^*(d\omega) = d(\psi^*\omega)$, i.e., the following diagram commutes:

$$\begin{tikzcd} \Omega^k(N) \arrow[r, "d"] \arrow[d, "\psi^*"'] & \Omega^{k+1}(N) \arrow[d, "\psi^*"] \\ \Omega^k(M) \arrow[r, "d"'] & \Omega^{k+1}(M) \end{tikzcd}$$

Remark 2.2.14. The pullback of the identity map is an identity map. That is,

$$(\mathrm{id}_M)^* = \mathrm{id}_{\Omega^k(M)}$$

Example 2.2.15. On $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ are both 1-dimensional, while $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are both free of rank three over the algebra of C^∞-functions, so the following identifications are possible

$$\begin{tikzcd} \{\text{functions}\} \arrow[r, dotted, no head, "\cong"] & \{\text{0-forms}\} \arrow[r, dotted, no head, "\cong"] & \{\text{3-forms}\} \\ f \arrow[r, leftrightarrow] & f \arrow[r, leftrightarrow] & f\ dx \wedge dy \wedge dz \end{tikzcd}$$

and

$$\begin{tikzcd} \left\{ \begin{matrix} \text{vector} \\ \text{fields} \end{matrix} \right\} \arrow[r, dashed, no head, "\cong"] & \{\text{1-forms}\} \arrow[r, dotted, no head, "\cong"] & \{\text{2-forms}\} \\ X = (f_1, f_2, f_3) \arrow[r, leftrightarrow] & \begin{matrix} f_1\ dx + f_2\ dy \\ +f_3\ dz \end{matrix} \arrow[r, leftrightarrow] & \begin{matrix} f_1\ dy \wedge dz - f_2\ dx \wedge dz \\ +f_3\ dx \wedge dy \end{matrix} \end{tikzcd}$$

On 1-forms we have

$$d(f_1\ dx + f_2\ dy + f_3\ dz) = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy \wedge dz$$
$$-\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dx \wedge dz + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy.$$

On 2-forms,

$$d(f_1 dy \wedge dz - f_2 dx \wedge dz + f_3 dx \wedge dy) = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx \wedge dy \wedge dz.$$

In summary, d(0-forms) = gradient, d(1-forms) = curl and d(2-forms) = divergence. This shows that the exterior derivative is the ultimate abstract extension of vector calculus on $\mathbb{R}^3$. In its action on differential forms, we will see that it has the remarkable ability to connect the differential geometry tools described so far with the topological properties of the manifold.

In any course on algebraic topology, one learns how homology arises as an algebraic invariant within topological spaces. Cohomology, often defined as the dualization of homology, plays a similar role. However, being a contravariant theory, it has better properties than homology.

2.3 Singular cohomology

Singular cohomology is the contravariant version of singular homology. To introduce it, we will briefly review the latter, omitting many of the proofs typically covered in an introductory course on algebraic topology.

Definition 2.3.1. For each $k \geq 1$, the *standard k-simplex in $\mathbb{R}^k$* is defined to be

$$\Delta^k = \Big\{(a_1, \ldots, a_k) \in \mathbb{R}^k : \sum_{i=1}^{k} a_i \leq 1, \ a_i \geq 0\Big\}.$$

and for $k = 0$ we set $\Delta^0 = \{0\}$. Now, let M be a smooth manifold. A *differentiable singular k-simplex σ in $U \subset M$* is a map $\Delta^k \to U$ which extends to a C^∞ map of a neighbourhood of Δ^k into U. We let $S_k(U)$ denote the free abelian group generated by the singular k-simplices in U. Its elements are called *smooth singular k-chains with real coefficients.*

For each $k \geq 0$, we define the collection of maps $g_i^k : \Delta^k \to \Delta^{k+1}$ for $0 \leq i \leq k+1$ as follows. For $k = 0$, $g_0^0(0) = 1$ and $g_1^0(0) = 0$, and for $k \geq 1$,

$$g_i^k(a_1, \ldots, a_k)(= \begin{cases} \left(1 - \sum_{j=1}^{k} a_j, a_1, \ldots, a_k\right) & i = 0 \\ (a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_k) & 1 \leq i \leq k+1 \end{cases}$$

Definition 2.3.2. We define the *i-th face of a differentiable singular k-simplex σ* to be the $(k-1)$-simplex $\sigma^i = \sigma \circ g_i^{k-1}$ and the *boundary of σ* to be the $(k-1)$-chain

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i \sigma^i.$$

A quick calculation shows that $g_i^{k+1} \circ g_j^k = g_{j+1}^{k+1} \circ g_i^k$ for $k \geq 0$ and $i \leq j$. It follows that

$$\partial \circ \partial = 0.$$

The boundary operator then induces linear transformations for all $k \geq 1$:

$$\partial : S_k(U) \to S_{k-1}(U).$$

We can set $U = M$. The elements of $\ker(\partial : S_k(M) \to S_{k-1}(M))$ are called *differential k-cycles*, the elements of $\operatorname{im}(\partial : S_{k+1}(M) \to S_k(M))$ are called *differential k-boundaries* and their quotient space is called the *k-th differential singular homology group of M with real coefficients*, denoted by

$$H_k^{\text{sing}}(M) = \frac{\ker(\partial : S_k(M) \to S_{k-1}(M))}{\operatorname{im}(\partial : S_{k+1}(M) \to S_k(M))}.$$

Let $S^k(U, \mathbb{R})$ denote the $\mathbb{R}$-module of homomorphisms $\operatorname{Hom}(S_k(U), \mathbb{R})$. Elements of $S^k(U, \mathbb{R})$ are called *smooth singular k-cochains on U* and are functions which assign to each singular k-simplex in U an element of $\mathbb{R}$. The $\mathbb{R}$-module operations are defined by

$$(\lambda f)(\sigma) = \lambda(f(\sigma)), \qquad (f+g)(\sigma) = f(\sigma) + g(\sigma), \qquad \forall f, g \in S^k(U, \mathbb{R}), \lambda \in \mathbb{R},$$

and extend into homomorphisms of $S_k(U)$ by linearity. Furthermore, the *coboundary homomorphism* is defined for all $k \geq 0$ by

$$\delta : S^k(U, \mathbb{R}) \to S^{k+1}(U, \mathbb{R}), \qquad \delta f(\sigma) = f(\partial \sigma)$$

for $f \in S^k(U, \mathbb{R})$ and $\sigma \in S_{k+1}(U)$. It follows that $\delta \circ \delta = 0$, and therefore it makes sense to talk about *singular cohomology*:

$$H^k_{\text{sing}}(M, \mathbb{R}) = \frac{\ker(\delta : S^k(M, \mathbb{R}) \to S^{k+1}(M, \mathbb{R}))}{\operatorname{im}(\delta : S^{k-1}(M, \mathbb{R}) \to S^k(M, \mathbb{R}))}$$

where, as one could imagine, the elements of $\ker(\delta : S^k(M, \mathbb{R}) \to S^{k+1}(M, \mathbb{R}))$ are called *k-cocycles* and the elements of $\operatorname{im}(\delta : S^{k-1}(M, \mathbb{R}) \to S^k(M, \mathbb{R}))$ are called *k-coboundaries.* In all definitions regarding singular cohomology, one could replace $\mathbb{R}$ with a general field K, and the results would hold at a more general level. However, for our purposes, we restrict ourselves to the case of real coefficients.

Remark 2.3.3. Singular (co)homology is typically defined on any topological space by taking continuous chains (cochains). On a smooth manifold, the resulting modules are isomorphic to those defined above.

2.4 De Rham cohomology

Let us reopen the toolbox of differential geometry on manifolds. Recall that the exterior derivative is a local operator; as shown in [14], this follows from its property of being an antiderivation. This means that for all $k \geq 0$, whenever a k-form $\omega \in \Omega^k(M)$ is such that $\omega_p = 0$ for all $p \in U \subset M$, then $d\omega \equiv 0$ on U. Equivalently, for all $k \geq 0$, if two k-forms $\omega, \eta \in \Omega^k(M)$ agree on an open subset U, then $d\omega \equiv d\eta$ on U.

Definition 2.4.1. A differential k-form $\omega \in \Omega^k(M)$, $k \geq 0$, is said to be *closed* if $d\omega = 0$. The set of closed differential k-forms on M is denoted by $\mathcal{Z}^k(M)$.

Definition 2.4.2. A differential k-form $\omega \in \Omega^k(M)$, $k \geq 0$, is said to be *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. The set of exact differential k-forms on M is denoted by $\mathcal{B}^k(M)$.

Recall that the sum of two closed (exact) k-forms is also a closed (exact) k-form, and so is its product by a scalar. Since $d^2 = 0$, one can see that every exact form is closed.

Proposition 2.4.3. Let $\psi : M \to N$ be a smooth map of manifolds. Then the pullback map ψ^* sends closed forms to closed forms and exact forms to exact forms.

Proof. Since the pullback commutes with d, then $d(\psi^*(\omega)) = \psi^*(d\omega) = 0$ and $\psi^*(\omega)$ is closed. Now, if $\omega = d\eta$ is exact, then $\psi^*(\omega) = \psi^*(d\eta) = d(\psi^*(\eta))$, meaning $\psi^*(\omega)$ is exact. □

Definition 2.4.4. We define a *cochain complex of modules* to be a sequence of modules $C^0, C^1, C^2 \dots$ and homomorphisms $f^k : C^k \to C^{k+1}$ such that $f^{k+1} \circ f^k = 0$.

It can be seen that the exterior derivative, along with the modules of differential k-forms, forms a cochain complex known as the *de Rham complex*:

$$0 \to C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \to 0$$

Definition 2.4.5. The *k-th de Rham cohomology group of M* is the quotient

$$H^k_{dR}(M) = \frac{\mathcal{Z}^k(M)}{\mathcal{B}^k(M)} = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1} \to \Omega^k(M))}$$

Hence, the de Rham cohomology of a smooth manifold measures the extent to which closed forms fail to be exact. It is clear that the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ induces a map $d : H^k_{dR}(M) \to H^{k+1}_{dR}(M)$ by sending $[\omega] \to [d\omega]$.

Proposition 2.4.6. We define the *wedge product of cohomology classes* represented by $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ by

$$[\omega] \wedge [\eta] = [\omega \wedge \eta] \in H^{k+l}_{dR}(M).$$

One can see that the wedge product of two closed forms is a closed form, and that the result of the previous proposition does not depend on representatives.

Proposition 2.4.7. If a smooth manifolds M has m connected components, then its de Rham cohomology in degree zero is $H^0_{dR}(M) = \mathbb{R}^m$.

Proof. Since there are no non-zero exact 0-forms, we have $H^0_{dR}(M) = \mathcal{Z}^0(M)$. Suppose that $f \in C^\infty(M)$ is a closed 0-form on M. Because of Remark A.21 (ii), we have

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i = 0$$

and this happens if and only if for all i we have $\partial f / \partial x_i = 0$ in U. That means that f is locally constant in U. Such a function must be constant in each connected component of M, and hence can be specified by a set of m real numbers. □

Proposition 2.4.8. In a smooth n-manifold M, $H^k_{dR}(M) = 0$ for each $k > n$.

Proof. At any point p, the tangent space T_pM is a vector space of dimension n. If $\omega \in \Omega^k(M)$, then $\omega_p \in \Lambda^k(T^*_pM)$, and because of Lemma 2.1.4(ii), $\Lambda^k(T^*_pM) = \{0\}$ for $k > n$. □

Definition 2.4.9. Let $\psi : M \to N$ be a smooth map of manifolds. Its pullback ψ^* induces a linear map of quotient spaces called the *pullback map in cohomology*,

$$\psi^\# : H^k_{dR}(N) \to H^k_{dR}(M), \qquad [\omega] \mapsto [\psi^*(\omega)]$$

Remark 2.4.10. From Proposition 2.2.12 and Remark 2.2.14 it follows that

(i) The identity map id_M induces an identity map $\mathrm{id}_M^{\#} : H^k_{dR}(M) \to H^k_{dR}(M)$.

(ii) Let $\psi : M \to N$ and $\phi : N \to L$ be smooth maps, then $(\phi \circ \psi)^{\#} = \psi^{\#} \circ \phi^{\#}$.

Theorem 2.4.11. Let U be the unit ball in Euclidean space $\mathbb{R}^n$. For each $k \geq 1$ there is a linear transformation $h_k : \Omega^k(U) \to \Omega^{k-1}(U)$ such that

$$h_{k+1} \circ d + d \circ h_k = \mathrm{id}_{\Omega^k(U)}.$$

This result is essential for proving the homotopy invariance of cohomology. Specifically, if two manifolds M and N are smoothly homotopy equivalent, their k-th cohomology groups are isomorphic for all k. This also plays a crucial role in the proof of the following lemma.

Lemma 2.4.12. [Poincaré lemma in Euclidean spaces] Let U be a star-convex open set in $\mathbb{R}^n$. For $k \geq 1$, every closed k-form in U is exact. Therefore, $H^k_{dR}(\mathbb{R}^n) = 0$ for each $k \geq 1$.

Proposition 2.4.13. Let $\psi : M \to N$ be a diffeomorphism of manifolds. Then the pullback map in cohomology $\psi^{\#} : H^k_{dR}(N) \to H^k_{dR}(M)$ is an isomorphism.

Lemma 2.4.14. [Poincaré lemma on manifolds] Let M be a smooth n-manifold. Then for all $p \in M$ there exists an open neighbourhood U such that every closed k-form on U is exact for $k \geq 1$.

Proof. Let (U, φ) be a coordinate system on a smooth n-manifold M such that $p \in U$. We know that the coordinate map $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a diffeomorphism. We choose U such that $\varphi(U)$ is an open ball in $\mathbb{R}^n$. The Poincaré lemma in Euclidean spaces tells us that every closed k-form on $\varphi(U)$ is exact for $k \geq 1$, meaning that $H^k_{dR}(\varphi(U)) = 0$ for $k \geq 1$. Now, Proposition 2.4.12 ensures that $H^k_{dR}(U) = 0$ for $k \geq 1$, concluding the proof. □

For more complex manifolds, there are various strategies to compute their de Rham cohomology, including the Mayer-Vietoris technique. All of these methods are thoroughly explained and developed in [2].

2.5 Integration and the de Rham theorem

Definition 2.5.1. Due to Lemma 2.1.4 (ii), $\Lambda^n(V)$ is one-dimensional for any n-dimensional vector space V, and therefore $\Lambda^n(V) - \{0\}$ has two disjoint connected components. An *orientation* is a choice of one of these components. A manifold M is said to be *orientable* if there is a consistent choice of orientation for T_p^*M at each point $p \in M$.

Definition 2.5.2. Let us define

$$O = \bigcup_{p \in M} \{0_p \in \Lambda^n(T_p^*M)\}.$$

Since each $\Lambda^n(T_p^*M) - \{0_p\}$ has two connected components, the previous definition states that M is orientable if $\Lambda^n(M) - O$ has two components. A non-connected manifold is said to be orientable if each component is orientable.

Definition 2.5.3. Let M be oriented and $v_1, \ldots, v_n$ a basis of T_pM with dual basis $v_1^*, \ldots, v_n^*$. The former is said to be a *(ordered) oriented basis* if $v_1^* \wedge \ldots \wedge v_n^*$ belongs to the orientation.

Definition 2.5.4. Let $\psi : M \to N$ be a differentiable map between orientable n-manifolds. It is said to be *orientation preserving* if the induced map $\psi^* : \Lambda^n(N) \to \Lambda^n(M)$ maps the component of $\Lambda^n(N) - O'$ determining the orientation on N into the component of $\Lambda^n(M) - O$ determining the orientation on M.

Proposition 2.5.5. Let M be a differentiable n-manifold. Then the following are equivalent:

(i) M is orientable.

(ii) There is a collection $\Phi = \{(V, \varphi)\}$ of coordinate systems on M such that $M = \bigcup_{(V,\varphi)\in\Phi} V$ and $\det\left(\frac{\partial x_i}{\partial y_j}\right) > 0$ on $U_1 \cap U_2$, whenever $(U_1, x_1, \ldots, x_n)$, $(U_2, y_1, \ldots, y_n) \in \Phi$.

(iii) There is a nowhere vanishing n-form on M.

Definition 2.5.6. Let ψ be a diffeomorphism of a bounded open set D in $\mathbb{R}^n$ with a bounded open set $\psi(D) \subset \mathbb{R}^n$. Let $J\psi$ denote the determinant of the Jacobian matrix of ψ:

$$J\psi = \det\left(\frac{\partial \psi_i}{\partial r_j}\right)$$

Let f be a bounded continuous function on $\psi(D)$ and A a subset of D. Then

$$\int_{\psi(A)} f = \int_A f \circ \psi \; |J\psi|$$

Definition 2.5.7. Let the standard orientation of $\mathbb{R}^n$ be determined by the n-form $dr_1 \wedge \ldots \wedge dr_n$ and ω be an n-form on an open set $D \subset \mathbb{R}^n$. Then there is a uniquely determined function f on D such that $\omega = f dr_1 \wedge \ldots \wedge dr_n$. If $A \subset D$, the integration of the n-form ω in $\mathbb{R}^n$ is defined to be

$$\int_A \omega = \int_A f$$

and the previous change of variable formula can be re-stated as

$$\int_{\psi(A)} \omega = \pm \int_A \psi^*(\omega)$$

where $\pm$ expresses the preservation of orientation.

Definition 2.5.8. Since a 0-form is just a function, the integral of a 0-form ω over the 0-simplex σ is just $\omega(\sigma(0))$. For $k \geq 0$, the k-form ω can be pulled back via σ to a k-form $\sigma^*(\omega)$ on a neighbourhood of Δ^k. In this case, the *integral of the k-form ω over the k-simplex σ* is

$$\int_\sigma \omega = \int_{\Delta^k} \sigma^*(\omega)$$

and extends linearly to k-chains. We shall present two versions of Stokes' theorem, for the proofs of which we refer to [19, p. 144-148].

Theorem 2.5.9 (Stokes' theorem I)**.** Let c be a k-chain, ($k \geq 1$), in a differentiable manifold M, and let ω be a smooth $(k-1)$-form defined on a neighbourhood of the image of c. Then

$$\int_{\partial c} \omega = \int_c d\omega.$$

Example 2.5.10. Let M be the unit circle S^1. Since there are no non-zero k-forms for $k > \dim(S^1) = 1$, it follows that for $k > 1$, we have $H^k_{dR}(S^1) = 0$. Additionally, because there are no exact 0-forms and a closed 0-form on a connected manifold is just a constant function, we have $H^0_{dR}(S^1) \cong \mathbb{R}$.

Although the polar coordinate function θ is not globally defined on S^1, its differential $d\theta$ is a globally defined, nowhere vanishing 1-form. It is not exact, because if it were, its integral over S^1 would be 0 instead of 2π. Since $H^2_{dR}(S^1) = 0$, all 1-forms on S^1 are closed. We claim that if α is a 1-form on S^1, then there is a constant c such that $\alpha - c \cdot d\theta$ is exact. Recall that all 1-forms sin S^1 are $f(\theta)d\theta$, $\theta \in (0, 2\pi)$. For $\alpha = f(\theta)d\theta$, we define

$$c = \frac{1}{2\pi}\int_{S^1} \alpha, \qquad g(\theta) = \int_0^\theta (f(\theta) - c)d\theta.$$

g is a well-defined function on S^1, and $dg = (f(\theta) - c)d\theta = \alpha - cd\theta$. Therefore, every 1-form on S^1 differs from a real multiple of $d\theta$ by an exact form. This yields

$$H^1_{dR}(S^1) \cong \mathbb{R}.$$

Definition 2.5.11. Let M be now an oriented n-manifold. A subset $D \subset M$ is said to be a regular domain if for each $p \in M$, one of the following holds:

(i) There is an open neighbourhood of p contained in $M \backslash D$.

(ii) There is an open neighbourhood of p contained in D.

(iii) There is a centered coordinate system (U, φ) about p such that $\varphi(U \cap D) = \varphi(U) \cap H^n$, where H^n is the half-space of $\mathbb{R}^n$ defined by $r_n \geq 0$.

A second version of the Stokes' theorem enables us to integrate $(n-1)$-forms with compact support over regular domains.

Theorem 2.5.12 (Stokes' theorem II)**.** Let D be a regular domain in an oriented n-manifold M and $\omega \in \Omega^{n-1}(M)$ with compact support. Then

$$\int_D d\omega = \pm \int_{\partial D} \omega$$

where the sign corresponds to the choice of orientation of ∂D.

Corollary 2.5.13. Let ω be a smooth $(n-1)$-form on a compact oriented n-manifold M. Since $\partial D = \varnothing$, then

$$\int_M d\omega = 0.$$

We have seen that on a smooth manifold M, a smooth k-form can be integrated over a continuous k-chain to yield a real number. Thus,

$$\int_{()} \omega : S_k(M) \to \mathbb{R}, \qquad \sigma \mapsto \int_\sigma \omega$$

is a k-cochain on M. Because of Theorem 2.5.9, the map

$$\Psi : \Omega^k(M) \to S^k(M, \mathbb{R}), \qquad \omega \mapsto \int_{()} \omega$$

satisfies

$$\Psi(d\omega) = \int_{()} d\omega = \int_{\partial()} \omega = \delta \int_{()} \omega = \delta(\Psi(\omega)).$$

where we have used that $\delta f(\sigma) = f(\partial\sigma)$ for $f \in S^k(M, \mathbb{R})$ and $\sigma \in S_{k+1}(M)$. This relationship between the exterior derivative in forms and the coboundary operator in cochains shows that the image of a closed k-form is a k-cocycle, since if $\omega \in \Omega^k(M)$ such that $d\omega = 0$, then

$$\delta(\Psi(\omega)) = \Psi(d\omega) = \Psi(0) = 0.$$

Moreover, one can also check that the image of an exact k-form is a k-coboundary. If $\omega \in \Omega^k(M)$ such that $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$, then

$$\Psi(\omega) = \int_{()} \omega = \int_{()} d\eta = \int_{\partial()} \eta = \delta \int_{()} \eta.$$

This implies that the homomorphism Ψ induces a well-defined map between the respective cohomology groups. Moreover, de Rham theorem takes this even further by asserting the strength of this map.

Theorem 2.5.14 (de Rham Theorem)**.** The induced map

$$\tilde{\Psi} : H^k_{dR}(M) \to H^k_{\text{sing}}(M, \mathbb{R}), \qquad [\omega] \mapsto \left[\int_{()} \omega\right]$$

in cohomology is an isomorphism.

Let us gain some intuition about this result. We know that homology essentially counts the number of k-cycles that fail to be k-boundaries. De Rham cohomology, on the other hand, can be thought of as the failure of local solutions to glue together into a global solution. Indeed, since we are dealing with a manifold, any closed form ω is "locally trivial" in the sense that the manifold can be covered by contractible charts, and over each of these charts, a solution to $d\alpha = \omega$ exists, as guaranteed by the Poincaré lemma. The cohomology class $[\omega]$ measures the obstruction to the existence of a global solution to this equation.

The de Rham theorem connects these two ideas by telling us that the dualization of the first is equivalent to the second. There are various proofs of this theorem, some of which are framed within topology itself, as developed in [4]. However, for us, this provides the perfect opportunity to dive into the world of sheaf theory, which we will explore in detail in the next chapter.

Chapter 3

Sheaf theory

"Think about it like the mathematical object is a plot of land and a sheaf is like a garden on top of it."

Mark Agrios

Sheaf theory is a fundamental tool in both algebraic topology and algebraic geometry. Its strength lies in its ability to connect local information with the global behaviour of a space. As one might expect, this feature is especially useful for manifolds, since they are locally Euclidean. This allows us to tackle problems locally using familiar tools from analysis, and then "glue" these local solutions together to uncover global invariants, which will take shape through the introduction of sheaf cohomology.

This approach will enable us to present a particularly elegant—though not unique—proof of the de Rham theorem. The results outlined in this chapter are derived under the assumption that the manifold M is Hausdorff and paracompact. Generally, K will be a principal ideal domain; when K is $\mathbb{Z}$, the K-modules will be abelian groups, and when K is $\mathbb{R}$, the K-modules will be real vector spaces. We will mainly follow [3], [18], [20] and [21].

3.1 Presheaves and sheaves

Definition 3.1.1. A *presheaf $\mathcal{F}$ of K-modules on M* is a contravariant functor from the category of open sets and inclusions to the category of K-modules and homomorphisms of K-modules. That is, it assigns to every open set $U \subset M$ a K-module $\mathcal{F}(U)$ and to every inclusion of open sets $i_U^V : V \to U$ a K-module homomorphism $\mathcal{F}(i_U^V) := \rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ called the *restriction from U to V*, satisfying that $\rho_U^U = id_{\mathcal{F}(U)}$ for all $U \subset M$ and $\rho_W^V \circ \rho_V^U = \rho_W^U$ if $W \subset V \subset U \subset M$.

The elements of each K-module $\mathcal{F}(U)$, also denoted by $\Gamma(U, \mathcal{F})$, are called *sections* of $\mathcal{F}$ over U. When $U = M$, we refer to elements of $\mathcal{F}(M)$ as *global sections of $\mathcal{F}$*.

Definition 3.1.2. Let $\mathcal{F}$ be $\mathcal{G}$ be presheaves on M, a *morphism of presheaves* $f : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $\{f_U : \mathcal{F}(U) \to \mathcal{G}(U)\}$ such that the diagram commutes.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho^U_V = \mathcal{F}(i^V_U)"'] & \mathcal{G}(U) \arrow[d, "(\rho^U_V)' = \mathcal{G}(i^V_U)"] \\ \mathcal{F}(V) \arrow[r, "f_V"] & \mathcal{G}(V) \end{tikzcd}$$

If for all $U \subset M$, f_U are isomorphisms, then f is an isomorphism of presheaves.

Example 3.1.3. The functor that assigns to every open set $U \subset M$ the $\mathbb{R}$-module $C^\infty(U)$ and to every inclusion of open sets the usual restriction of C^∞ functions is a presheaf.

In Appendix A we define germs of functions, which encode their local behaviour. The corresponding notion for a presheaf is the *stalk* of the presheaf at a point.

Definition 3.1.4. A *directed set* is a set I with a binary relation $\leq$ satisfying reflexivity, transitivity and the existence of an upper bound.

Definition 3.1.5. A *directed system of K-modules* is a collection of K-modules $\{G_i\}_{i\in I}$ indexed by a directed set I and a collection of morphisms $f^a_b : G_a \to G_b$ indexed by pairs $a, b \in I$ such that $f^a_a = \mathrm{id}_{G_a}$ and $f^a_c = f^b_c \circ f^a_b$ for $a \leq b \leq c$ in I.

We introduce an equivalence relation $\sim$ on $G = \bigsqcup_i G_i$ for which $g_a \in G_a$ and $g_b \in G_b$ are equivalent if there exists an upper bound c of a and b such that $f^a_c(g_a) = f^b_c(g_b)$ in G_c. We call *direct limit* of the direct system, denoted by $\lim_{\to i\in I} G_i$, to the quotient of the disjoint union G by the equivalence relation $\sim$.

Definition 3.1.6. Given $p \in M$, the set of all neighbourhoods of p with reverse inclusion form a directed set. Thus, if $\mathcal{F}$ is a presheaf of K-modules, $\{\mathcal{F}(U)\}_{U\ni p}$, where U ranges over all open neighbourhoods of p, is a directed system of K-modules and its direct limit

$$\mathcal{F}_p = \lim_{\to p\in U} \mathcal{F}(U)$$

is called the *stalk of $\mathcal{F}$ at p*. An element of $\mathcal{F}_p$ is called a *germ of sections at p*.

It is easy to check that a morphism of presheaves of K-modules on M, $f : \mathcal{F} \to \mathcal{G}$, induces a morphism of stalks $f_p : \mathcal{F}_p \to \mathcal{G}_p$ by sending the germ at p of a section $s \in \mathcal{F}(U)$ to the germ at p of the section $f(s) \in \mathcal{G}(U)$.

The stalk of a presheaf embodies in it the local character of the presheaf about the point. However, there are no clear instructions on how the stalks of a presheaf extend to global sections. A *sheaf* is a presheaf with two additional properties that establish a connection between its local and global behaviour.

Definition 3.1.7. A *sheaf $\mathcal{S}$ of K-modules on M* is a presheaf satisfying the following conditions for any open set $U \subset M$ and any open cover $\{U_i\}$ of U.

(i) *(Locality)* If $s, t \in \mathcal{S}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.

(ii) *(Gluing)* If $\{s_i \in \mathcal{S}(U_i)\}$ is a collection of sections such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all i, j, then there is a section $s \in \mathcal{S}(U)$ such that $s|_{U_i} = s_i$ for each i.

Example 3.1.8. The functor associating to every open set U the K-module of constant real-valued functions on U and to every inclusion the restriction of functions is a presheaf which satisfies the locality axiom, but not the gluing one. Indeed, if U_1 and U_2 are disjoint open sets in M and $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ have different values, then there is no constant function s on $U_1 \cup U_2$ that restricts to s_1 in U_1 and to s_2 in U_2.

Example 3.1.9. The functor associating to every open set U the $\mathbb{R}$-module of locally constant real-valued functions on U and to every inclusion the restriction of functions is a presheaf which is also a sheaf. If G is a $\mathbb{R}$-module, the sheaf of locally constant functions with values in G is called the *constant sheaf with values in G*.

Example 3.1.10. The presheaf Ω^k on M that assigns to each open set U the $\mathbb{R}$-module of differential k-forms on U is a sheaf. Equivalently we can define the sheaf of differential forms Ω^* on M assigning to each U the $\mathbb{R}$-module of differential forms on U. They respectively assign to the inclusion $i : U \to V$ the restrictions $r_k : \Omega^k(V) \to \Omega^k(U)$ and $r_* : \Omega^*(V) \to \Omega^*(U)$.

3.2 The sheafification functor

Let us now refer to [7] to introduce the following tool. Associated to a presheaf $\mathcal{F}$ on a manifold M there is another topological space $\mathcal{E}_\mathcal{F}$ called *étalé space of $\mathcal{F}$*. As a set, this is just the disjoint union of all the stalks of $\mathcal{F}$, that is,

$$\mathcal{E}_\mathcal{F} = \bigsqcup_{p \in M} \mathcal{F}_p,$$

and we also define a projection map

$$\pi : \mathcal{E}_\mathcal{F} \to M, \qquad \mathcal{F}_p \mapsto p.$$

Definition 3.2.1. A *section of the étalé space $\mathcal{E}_\mathcal{F}$ over $U \subset M$* is a map $t : U \to \mathcal{E}_\mathcal{F}$ such that $\pi \circ t = \mathrm{id}_U$. Recall that for $U \subset M$, $s \in \mathcal{F}(U)$ and $s_p \in \mathcal{F}_p$ being the germ of s at p, then the element $s \in \mathcal{F}(U)$ defines a section of the étalé space over U by

$$\tilde{s} : U \to \mathcal{E}_\mathcal{F}, \qquad p \mapsto s_p \in \mathcal{F}_p.$$

The collection $\{\tilde{s}(U) : U \subset M \text{ open}, s \in \mathcal{F}\}$ satisfies the conditions to be a basis for a topology on $\mathcal{E}_\mathcal{F}$, making $\mathcal{E}_\mathcal{F}$ into a topological space which is locally homeomorphic to M. Let $\tilde{F}$ be the presheaf that associates to each open subset $U \subset M$ the module of continuous sections of $\mathcal{E}_\mathcal{F}$ over U, denoted by $\Gamma(U, \mathcal{E}_\mathcal{F})$. Under pointwise addition of sections, this is easily seen to be a sheaf called the *sheafification* or the *associated sheaf* of the presheaf $\mathcal{F}$. There is an obvious presheaf morphism $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ sending a section $s \in \mathcal{F}(U)$ to the section $\tilde{s} \in \tilde{\mathcal{F}}(U)$.

Example 3.2.2. Let us now bring together Examples 3.1.8 and 3.1.9. The first presheaf associated to each open subset $U \subset M$ the module of constant real-values functions. At each point $p \in M$, the stalk $\mathcal{F}_p$ is $\mathbb{R}$. The étalé space is thus $M \times \mathbb{R}$ with the product topology of the given topology on M and the discrete topology on $\mathbb{R}$. The sheafification $\tilde{\mathcal{F}}$ is the sheaf of locally constant real-valued functions, thus, the latter example.

One can observe that if $\mathcal{S}$ is already a sheaf, then the sheaf of sections of the associated space $\mathcal{E}_{\mathcal{S}}$ is isomorphic to the original sheaf, $\mathcal{S} \cong \tilde{\mathcal{S}}$. In that case, we will not distinguish between the notations $\mathcal{S}(U)$ and $\Gamma(U, \mathcal{E}_{\mathcal{S}})$. This is precisely what occurs with the sheaf of differentiable forms.

Definition 3.2.3. A *sheaf morphism* $\varphi : \mathcal{S} \to \mathcal{T}$ is by definition a morphism of presheaves. The *presheaf kernel*, $U \mapsto \ker(\varphi_U : \mathcal{S}(U) \to \mathcal{T}(U))$ is a sheaf called *kernel of* φ and denoted $\ker \varphi$. However, the *presheaf image*, $U \mapsto \operatorname{im}(\varphi_U : \mathcal{S}(U) \to \mathcal{T}(U))$, is not always a sheaf. The *image of* φ, denoted $\operatorname{im}\varphi$, is then the sheaf associated to the presheaf image of φ. The sheaf morphism is said to be *injective* if $\ker \varphi = 0$ and *surjective* if $\operatorname{im}\varphi = \mathcal{T}$.

Definition 3.2.4. A sheaf $\mathcal{S}$ over M is a subsheaf of the sheaf $\mathcal{T}$ if for every open set $U \subset M$, $\mathcal{S}(U)$ is a K-submodule of $\mathcal{T}(U)$ and the inclusion $i : \mathcal{S} \to \mathcal{T}$ is a presheaf morphism. In that case, the *quotient sheaf* is the sheaf associated to the presheaf $U \mapsto \mathcal{T}(U)/\mathcal{S}(U)$.

3.3 Resolutions of sheaves

Definition 3.3.1. If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are sheaves of K-modules over M and $\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C}$ is a sequence of sheaf morphisms, then it is *exact at* $\mathcal{B}$ if, for each p, the induced sequence on stalks

$$\mathcal{A}_p \xrightarrow{g_p} \mathcal{B}_p \xrightarrow{h_p} \mathcal{C}_p$$

is exact. That is, for each $p \in M$, $\ker(h_p) = \operatorname{im}(g_p)$. A *short exact sequence* is a sequence $0 \to \mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C} \to 0$ which is exact at $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ where 0 denotes the (constant) zero sheaf.

Thus, exactness is a local property. However, even if local exactness holds (that is, exactness of the sequence of stalks at each point), the associated presheaf-level sequence

$$0 \to \Gamma(U, \mathcal{A}) \to \Gamma(U, \mathcal{B}) \to \Gamma(U, \mathcal{C}) \to 0$$

for each open set $U \subset M$ might not be exact. Consequently, globally exact sequences may fail to be constructed. We have already seen that such obstructions are measured by cohomology. Now, using sheaf theory, we will develop similar tools to study these impediments.

Example 3.3.2. Let $\mathcal{A}$ be a subsheaf of $\mathcal{B}$. Then

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{q} \mathcal{B}/\mathcal{A} \to 0$$

is exact where i is the natural inclusion and q the natural quotient mapping.

Definition 3.3.3. A *graded sheaf* is a family of sheaves indexed by integers. A *sequence of sheaves* is a graded sheaf $\mathcal{S}^* = \{\mathcal{S}_i\}_{i \in \mathbb{Z}}$ connected by sheaf mappings

$$\dots \xrightarrow{\varphi_{i-1}} \mathcal{S}_i \xrightarrow{\varphi_i} \mathcal{S}_{i+1} \xrightarrow{\varphi_{i+1}} \mathcal{S}_{i+2} \xrightarrow{\varphi_{i+2}} \dots$$

Definition 3.3.4. A *differential sheaf* is a sequence of sheaves such that the composition of two consecutive maps is zero. That is, for each $i \in \mathbb{Z}$, $\varphi_{i+1} \circ \varphi_i = 0$.

Definition 3.3.5. A *resolution of a sheaf* $\mathcal{S}$ is an exact sequence of sheaves of the form

$$0 \to \mathcal{S} \to \mathcal{S}_0 \xrightarrow{\varphi_0} \mathcal{S}_1 \xrightarrow{\varphi_1} \mathcal{S}_2 \xrightarrow{\varphi_2} \mathcal{S}_3 \xrightarrow{\varphi_3} \dots$$

denoted by $0 \to \mathcal{S} \to \mathcal{S}_*$.

Example 3.3.6. Let M be a differential n-manifold and Ω^k the sheaf of real-valued differential k-forms on M. There is a resolution of the constant sheaf with values in $\mathbb{R}$ given by

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \longrightarrow 0$$

where i is the inclusion and d is the exterior derivative. Since $d^2 = 0$ it is clear that it is a differential sheaf. Because of Poincaré lemma (2.4.14), if $U \subset \mathbb{R}^n$ is a star-shaped domain and $f \in \Omega^k(U)$ such that $df = 0$, then f is exact. Therefore, since we can find representatives in local coordinates in star-shaped domains, the induced mapping d_p on stalks at $p \in M$ is exact. The exactness of the first term follows from the fact that if $f \in \Omega^0(M)$ and $df = 0$, then f is locally constant. We denote the resolution by $0 \to \mathbb{R} \to \Omega^*$.

It is of vital importance to observe here that if we disregard the constant sheaf $\mathbb{R}$ and take global sections of the previous resolution, we obtain precisely the de Rham complex. Therefore, it becomes evident that the study of the defect in the exactness of the global section sequence induced by the resolution just presented is of particular interest to us.

Example 3.3.7. Let M be a differential n-manifold, and let $S^k(U, \mathbb{R})$ represent the vector space of differential singular cochains on U with coefficients in $\mathbb{R}$. Denote the coboundary operator by $\delta : S^k(U, \mathbb{R}) \to S^{k+1}(U, \mathbb{R})$, and let $\mathcal{S}^k(\mathbb{R})$ be the sheaf on M generated by the presheaf $U \mapsto S^k(U, \mathbb{R})$, equipped with the induced differential map δ. If U is the unit ball in Euclidean space, then the sequence

$$\dots \to S^{k-1}(U, \mathbb{R}) \xrightarrow{\delta} S^k(U, \mathbb{R}) \xrightarrow{\delta} S^{k+1}(U, \mathbb{R}) \xrightarrow{\delta} \dots$$

is exact since $\ker \delta / \mathrm{im} \delta$ is the classical singular cohomology, which is well-known to be zero for $k > 0$. Furthermore, since $\ker(\delta : S^0(U, \mathbb{R}) \to S^1(U, \mathbb{R})) \cong \mathbb{R}$, we have the following resolution by differential cochains with coefficients in $\mathbb{R}$:

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} \mathcal{S}^0(\mathbb{R}) \xrightarrow{\delta} \mathcal{S}^1(\mathbb{R}) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{S}^n(\mathbb{R}) \longrightarrow \dots$$

which we abbreviate by $0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{S}^*(\mathbb{R})$.

As before, if we omit the constant sheaf $\mathbb{R}$ and take global sections, we obtain the complex of smooth singular cochains. Recall that the homomorphism introduced at the end of the previous chapter induces a natural homomorphism of differential sheaves, $\Psi : \Omega^* \to \mathcal{S}^*(\mathbb{R})$, defined by integration over chains:

$$\Psi_U : \Omega^*(U) \to \mathcal{S}^*(U, \mathbb{R}), \qquad \Psi_U(\omega)(c) = \int_c \omega,$$

which at its turn induces a homomorphism of resolutions:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "i"] \arrow[d, "id_{\mathbb{R}}"] & \Omega^0 \arrow[r, "d"] \arrow[d, "\Psi"] & \Omega^1 \arrow[r, "d"] \arrow[d, "\Psi"] & \dots \arrow[r, "d"] & \Omega^n \arrow[r, "d"] \arrow[d, "\Psi"] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & \mathcal{S}^0(\mathbb{R}) \arrow[r, "\delta"] & \mathcal{S}^1(\mathbb{R}) \arrow[r, "\delta"] & \dots \arrow[r, "\delta"] & \mathcal{S}^n(\mathbb{R}) \arrow[r, "\delta"] & 0
\end{tikzcd}$$

As seen before, Stokes' theorem ensures that the mapping Ψ commutes with the differentials, making the diagram commutative. We will now see how resolutions can be used to represent the cohomology groups of a manifold.

3.4 Sheaf cohomology

Since all sheaves are presheaves, we will denote them by $\mathcal{F}$ and save $\mathcal{S}$ for a special type of sheaves. Given a sheaf $\mathcal{F}$, there is a natural functor Γ of global sections, which to $\mathcal{F}$ associates $\Gamma(M, \mathcal{F}) = \mathcal{F}(M)$. This functor has values in the category of K-modules. As we have anticipated earlier, it is left-exact but not right-exact, i.e. a surjective morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves does not necessarily induce a surjective morphism at the level of global sections. Sheaf cohomology is a theory which is used to compute and understand this defect in exactness of induced sequences of global sections via the use of invariants, namely the images under the functor $H^p(M,)$, of the sheaves $\ker \varphi$, $\mathcal{F}$ and $\mathcal{G}$. We need to introduce a class of sheaves for which the inexactness of global section sequences is solved.

Definition 3.4.1. Let $\mathcal{F}$ be a sheaf on M and S be a closed subset of M. Let $\Gamma(S, \mathcal{F}) = \mathcal{F}(S) := \varinjlim_{U \supset S} \mathcal{F}(U)$ where the direct limit runs over all open sets U containing S.

Definition 3.4.2. A sheaf $\mathcal{F}$ over M is *soft* if for any closed subset $S \subset M$, the restriction $\mathcal{F}(M) \to \mathcal{F}(S)$ is surjective. That is, any section of $\mathcal{F}$ over S can be extended to a global section.

Remark 3.4.3. If we were to drop the paracompactness assumption, it would be necessary to use *flabby sheaves* instead of soft sheaves. A sheaf $\mathcal{S}$ is said to be flabby if the restriction map $\mathcal{S}(M) \to \mathcal{S}(U)$ is surjective for every open subset $U \subset M$. Moreover, it can be proven that every flabby sheaf is also soft.

Theorem 3.4.4. If $\mathcal{A}$ is a soft sheaf and

$$0 \to \mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C} \to 0$$

is a short exact sequence of sheaves, then the following induced sequence of global sections,

$$0 \to \Gamma(M, \mathcal{A}) \xrightarrow{g} \Gamma(M, \mathcal{B}) \xrightarrow{h} \Gamma(M, \mathcal{C}) \to 0,$$

is also exact. For the proof, we refer to [21, p. 52], from which we also deduce the following results.

Corollary 3.4.5. If $\mathcal{A}$ and $\mathcal{B}$ are soft sheaves, and $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact, then $\mathcal{C}$ is a soft sheaf.

Corollary 3.4.6. If $0 \to \mathcal{S}_0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \dots$ is an exact sequence of soft sheaves, then the induced sequence of global sections is also exact.

Proof. Let $\mathcal{K}_i = \ker(\mathcal{S}_i \to \mathcal{S}_{i+1})$. By definition, the short sequence below is exact:

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{S}_i \longrightarrow \mathcal{K}_{i+1} \longrightarrow 0$$

Since $\mathcal{K}_0 = 0$, the previous sequence for $i = 0$ yields $\mathcal{K}_1 = \mathcal{S}_0$. Since $\mathcal{S}_0$ is soft, Theorem 3.4.4 ensures that the following is a short exact sequence:

$$0 \longrightarrow \Gamma(M, \mathcal{K}_1) \longrightarrow \Gamma(M, \mathcal{S}_1) \longrightarrow \Gamma(M, \mathcal{K}_2) \longrightarrow 0$$

On the other hand, since $\mathcal{K}_1$ and $\mathcal{S}_1$ are soft, the first short exact sequence together with Corollary 3.4.5 yield that $\mathcal{K}_2$ is also a soft sheaf. This argument holds recursively, and one deduces that $\mathcal{K}_i$ is soft for all i. We then obtain short exact sequences

$$0 \longrightarrow \Gamma(M, \mathcal{K}_i) \longrightarrow \Gamma(M, \mathcal{S}_i) \longrightarrow \Gamma(M, \mathcal{K}_{i+1}) \longrightarrow 0.$$

We can use them to patch together the following sequence:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(M,\mathcal{S}_0) \arrow[rr] \arrow[dr] & & \Gamma(M,\mathcal{S}_1) \arrow[rr] \arrow[dr] & & \dots \\
& & \Gamma(M,\mathcal{K}_1) \arrow[ur] \arrow[dr] & & \Gamma(M,\mathcal{K}_2) \arrow[ur] \arrow[dr] & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

resulting in the desired long exact sequence. This process of combining short exact sequences will be used in some upcoming proof.

□

Definition 3.4.7. A sheaf $\mathcal{F}$ on M is *fine* if for each locally finite open cover $\{U_i\}$ of M there exists, for each i, an endomorphism l_i of $\mathcal{F}$ such that supp $l_i \subset U_i$ and $\sum_i l_i = \mathrm{id}_{\mathcal{F}}$.

We will proceed as follows. Note that one same sheaf can admit different resolutions (as in Examples 3.3.6 and 3.3.7). Next, for a given sheaf, we will construct a canonical resolution. Since the examples mentioned are resolutions of the same constant sheaf $\mathbb{R}$, we will relate the defects in the exactness of the global section sequences induced by each of the example resolutions to the defect in the exactness of the global section sequence induced by the canonical resolution.

First, then, let us construct the well-known *Godement resolution*. Given a sheaf $\mathcal{S}$, recall the construction of the space $\mathcal{E}_{\mathcal{S}}$. Let $\mathcal{C}^0(\mathcal{S})$ denote the presheaf defined by

$$\mathcal{C}^0(\mathcal{S}) = \{f : U \to \mathcal{E}_{\mathcal{S}} : \pi \circ f = \mathrm{id}_U\}.$$

This presheaf is actually a soft sheaf, referred to as the *sheaf of discontinuous sections of $\mathcal{S}$ over M* (as opposed to the sections, which were defined to be continuous). Consequently, there is an evident injection

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{C}^0(\mathcal{S})$$

And thus we can consider the quotient sheaf $\mathcal{F}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{S})/\mathcal{S}$ and define $\mathcal{C}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^1(\mathcal{S}))$. By induction, we define

$$\mathcal{F}^i(\mathcal{S}) = \mathcal{C}^{i-1}(\mathcal{S})/\mathcal{F}^{i-1}(\mathcal{S}), \qquad \mathcal{C}^i(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^i(\mathcal{S}))$$

We now have the following short exact sequences of sheaves

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{C}^0(\mathcal{S}) \longrightarrow \mathcal{F}^1(\mathcal{S}) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{F}^i(\mathcal{S}) \longrightarrow \mathcal{C}^i(\mathcal{S}) \longrightarrow \mathcal{F}^{i+1}(\mathcal{S}) \longrightarrow 0$$

By splicing them together similarly to the proof of Corollary 3.4.6, we obtain the following long exact sequence called the *Godement canonical resolution of* $\mathcal{S}$

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{C}^0(\mathcal{S}) \longrightarrow \mathcal{C}^1(\mathcal{S}) \longrightarrow \mathcal{C}^2(\mathcal{S}) \longrightarrow \dots$$

abbreviated by $0 \to \mathcal{S} \to \mathcal{C}^*(\mathcal{S})$. We can now give a definition of cohomology groups of a space with coefficients in a given sheaf. By taking global sections to the previous long sequence, we obtain

$$0 \longrightarrow \Gamma(M, \mathcal{S}) \longrightarrow \Gamma(M, \mathcal{C}^0(\mathcal{S})) \longrightarrow \Gamma(M, \mathcal{C}^1(\mathcal{S})) \longrightarrow \Gamma(M, \mathcal{C}^2(\mathcal{S})) \longrightarrow \dots$$

Recall that, for the construction of quotients, this sequence forms a *cochain complex of modules*, meaning that the composition of two consecutive maps is zero. Moreover, it is exact at $\Gamma(M, \mathcal{C}^0(\mathcal{S}))$ and remains exact everywhere if $\mathcal{S}$ is soft, due to Corollary 3.4.6. Let $C^*(M, \mathcal{S}) = \Gamma(M, \mathcal{C}^*(\mathcal{S}))$ and we can rewrite the previous sequence as $0 \to \Gamma(M, \mathcal{S}) \to C^*(M, \mathcal{S})$. Now, as covered in any introductory course on homological algebra and described in the previous chapter, cochain complexes provide the framework for defining the following algebraic invariants.

Definition 3.4.8. Let $\mathcal{S}$ be a sheaf over M. For $q \geq 0$, the *sheaf cohomology groups of M of degree q with coefficients in $\mathcal{S}$* are defined to be

$$H^q(M, \mathcal{S}) = H^q(C^*(M, \mathcal{S})) = \frac{\ker(C^q(M, \mathcal{S}) \to C^{q+1}(M, \mathcal{S}))}{\operatorname{im}(C^{q-1}(M, \mathcal{S}) \to C^q(M, \mathcal{S}))}, \qquad C^{-1}(M, \mathcal{S}) = 0$$

The functorial properties of cohomology groups are summarized in the following list.

(I) $H^q(M, \mathcal{S}) = 0$ for $q < 0$ and there is an isomorphism $H^0(M, \mathcal{S}) \cong \Gamma(M, \mathcal{S})$ such that for each morphism $\mathcal{S} \to \mathcal{S}'$, the following diagram commutes

$$\begin{tikzcd} H^0(M, \mathcal{S}) \arrow[r, "\cong"] \arrow[d] & \Gamma(M, \mathcal{S}) \arrow[d] \\ H^0(M, \mathcal{S}') \arrow[r, "\cong"] & \Gamma(M, \mathcal{S}') \end{tikzcd}$$

(II) $H^q(M, \mathcal{S}) = 0$ for all $q > 0$ if $\mathcal{S}$ is a soft sheaf.

(III) For any sheaf morphism $h : \mathcal{S} \to \mathcal{T}$ there is, for each $q \geq 0$, a group homomorphism

$$h_q : H^q(M, \mathcal{S}) \to H^q(M, \mathcal{T})$$

such that (i) $h_0 = h_M : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{T})$, (ii) h_q is the identity map if h is the identity map for $q \geq 0$ and (iii) $g_q \circ h_q = (g \circ h)_q$ for all $q \geq 0$ for $g : \mathcal{T} \to \mathcal{R}$ being a second sheaf morphism.

(IV) For each short exact sequence of sheaves $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$, there are group homomorphisms $H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}')$ for all $q \geq 0$ such that the following induced long sequence is exact.

$$\dots \to H^{q-1}(M, \mathcal{S}'') \to H^q(M, \mathcal{S}') \to H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}') \to \dots$$

(V) For each morphism of short exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}' & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{S}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}' & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}'' & \longrightarrow & 0 \end{array}$$

the following diagram commutes.

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & H^q(M,\mathcal{S}'') & \longrightarrow & H^{q+1}(M,\mathcal{S}') & \longrightarrow & H^{q+1}(M,\mathcal{S}) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & H^q(M,\mathcal{T}'') & \longrightarrow & H^{q+1}(M,\mathcal{T}') & \longrightarrow & H^{q+1}(M,\mathcal{T}) & \longrightarrow & \ldots \end{array}$$

Let us now sketch the proof for (III), (IV) and (V) and refer to [21, p. 57-58] for the rest. Given the map $h : \mathcal{S} \to \mathcal{T}$, we define first the map $h^0 : \mathcal{C}^0(\mathcal{S}) \to \mathcal{C}^0(\mathcal{T})$ by letting $h^0(s_p) = (h \circ s)_p$, where s is a discontinuous section of $\mathcal{S}$. This induces a quotient map

$$\tilde{h}^0 : \mathcal{F}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{S})/\mathcal{S} \to \mathcal{C}^0(\mathcal{T})/\mathcal{T}$$

which, at its turn, induces

$$h^1 : \mathcal{C}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^1(\mathcal{S})) \to \mathcal{C}^1(\mathcal{T}) = \mathcal{C}^0(\mathcal{F}^1(\mathcal{T})).$$

Repeating the procedure above, we obtain, for each $q \geq 0$, maps $h^q : \mathcal{C}^q(\mathcal{S}) \to \mathcal{C}^q(\mathcal{T})$, and the induced section maps induce a complex map $h^* : C^*(M,\mathcal{S}) \to C^*(M,\mathcal{T})$. Moreover, if $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ is an exact sequence of sheaves, then this implies that

$$0 \to \mathcal{C}^*(\mathcal{S}') \to \mathcal{C}^*(\mathcal{S}) \to \mathcal{C}^*(\mathcal{S}'') \to 0$$

is an exact sequence of complexes of sheaves. However, the sheaves in this sequence are all soft, and hence it follows that

$$0 \to C^*(M,\mathcal{S}') \to C^*(M,\mathcal{S}) \to C^*(M,\mathcal{S}'') \to 0$$

is an exact sequence of cochain complexes of modules. It now follows from elementary homological algebra that there is a long exact sequence for the derived cohomology groups

$$\ldots \to H^q(C^*(M,\mathcal{S}')) \to H^q(C^*(M,\mathcal{S})) \to H^q(C^*(M,\mathcal{S}'')) \to H^{q+1}(C^*(M,\mathcal{S}')) \to \ldots$$

where the maps $H^q(C^*(M,\mathcal{S}'')) \to H^{q+1}(C^*(M,\mathcal{S}'))$ are defined through the snake lemma.

The sheaf cohomology groups, then, account for the defect in the exactness of the global section sequence induced by the Godement resolution. In the next section, we will see how this relates to the defect in the exactness of the sequences of greater interest to us. This relationship will be particularly useful when the sheaves in the resolutions satisfy the following property.

Definition 3.4.9. A resolution of a sheaf $\mathcal{S}$ over M, $0 \to \mathcal{S} \to \mathcal{A}^*$, is called *acyclic* if $H^q(M,\mathcal{A}^p) = 0$ for all $q > 0$ and $p \geq 0$. Note that fine or soft resolutions are necessarily acyclic.

3.5 Proof of the de Rham theorem

Theorem 3.5.1. Let $\mathcal{S}$ be a sheaf over M and let $0 \to \mathcal{S} \to \mathcal{A}^*$ be a resolution of $\mathcal{S}$. Then, there is a natural homomorphism

$$\gamma^p : H^p(\Gamma(M, \mathcal{A}^*)) \longrightarrow H^p(M, \mathcal{S})$$

where

$$H^p(\Gamma(M, \mathcal{A}^*)) = \frac{\ker(\Gamma(M, \mathcal{A}_p) \to \Gamma(M, \mathcal{A}_{p+1})}{\mathrm{im}(\Gamma(M, \mathcal{A}_{p-1}) \to \Gamma(M, \mathcal{A}_p))} \quad \text{if } p \geq 1$$
$$H^0(\Gamma(M, \mathcal{A}^*)) = \Gamma(M, \mathcal{S}).$$

Moreover, if the resolution is acyclic, then γ^p is an isomorphism.

Proof. Let $\mathcal{K}_p = \ker(\mathcal{A}_p \to \mathcal{A}_{p+1}) = \mathrm{im}(\mathcal{A}_{p-1} \to \mathcal{A}_p)$ so that $\mathcal{K}_0 = \mathcal{S}$. Similarly to the proof of Corollary 3.4.6, we have short exact sequences

$$0 \longrightarrow \mathcal{K}_{p-1} \longrightarrow \mathcal{A}_{p-1} \longrightarrow \mathcal{K}_p \longrightarrow 0$$

and because of property (III), this yields an exact sequence

$$\begin{aligned} 0 \to \Gamma(M, \mathcal{K}_{p-1}) &\longrightarrow \Gamma(M, \mathcal{A}_{p-1}) \longrightarrow \Gamma(M, \mathcal{K}_p) \longrightarrow H^1(M, \mathcal{K}_{p-1}) \longrightarrow H^1(M, \mathcal{A}_{p-1}) \\ &\longrightarrow H^1(M, \mathcal{K}_p) \longrightarrow H^2(M, \mathcal{K}_{p-1}) \longrightarrow \dots \end{aligned}$$

Moreover, we notice that $\ker(\Gamma(M, \mathcal{A}_p) \to \Gamma(M, \mathcal{A}_{p+1})) \cong \Gamma(M, \mathcal{K}_p)$ so that

$$H^p(\Gamma(M, \mathcal{A}^*)) \cong \frac{\Gamma(M, \mathcal{K}_p)}{\mathrm{im}(\Gamma(M, \mathcal{A}_{p-1}) \to \Gamma(M, \mathcal{K}_p))}$$

And therefore, from the exact sequence above, we have defined

$$\gamma_1^p : H^p(\Gamma(M, \mathcal{A}^*)) \longrightarrow H^1(M, \mathcal{K}_{p-1})$$

and γ_1^p is injective. If the resolution is acyclic, then $H^1(M, \mathcal{A}_{p-1}) = 0$ and the long exact sequence above ensures that the previous map is also surjective, therefore making γ_1^p an isomorphism. Similarly we can consider the following short exact sequences for $2 \leq r \leq p$

$$0 \longrightarrow \mathcal{K}_{p-r} \longrightarrow \mathcal{A}_{p-r} \longrightarrow \mathcal{K}_{p-r+1} \longrightarrow 0$$

and from the induced long exact sequences we obtain

$$\gamma_r^p : H^{r-1}(M, \mathcal{K}_{p-r+1}) \longrightarrow H^r(M, \mathcal{K}_{p-r})$$

where again γ_r^p are isomorphisms if the resolution is acyclic. We define now

$$\gamma_p = \gamma_p^p \circ \gamma_{p-1}^p \circ \dots \circ \gamma_2^p \circ \gamma_1^p$$

and thus

$$H^p(\Gamma(M, \mathcal{A}^*)) \xrightarrow{\gamma_1^p} H^1(M, \mathcal{K}_{p-1}) \xrightarrow{\gamma_2^p} H^2(M, \mathcal{K}_{p-2}) \xrightarrow{\gamma_3^p} \dots \xrightarrow{\gamma_p^p} H^p(M, \mathcal{K}_0) = H^p(M, \mathcal{S})$$

being an isomorphism if the resolution is acyclic, since it would be a composition of isomorphisms. The assertion that γ_p is natural means that if

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d, "f"] & \mathcal{A}^* \arrow[d, "g"] \\ 0 \arrow[r] & \mathcal{T} \arrow[r] & \mathcal{B}^* \end{tikzcd}$$

is a homomorphism of resolutions, then

$$\begin{tikzcd} H^p(\Gamma(M,\mathcal{A}^*)) \arrow[r, "\gamma^p"] \arrow[d, "g_p"'] & H^p(M,\mathcal{S}) \arrow[d, "f_p"] \\ H^p(\Gamma(M,\mathcal{B}^*)) \arrow[r, "\gamma^p"] & H^p(M,\mathcal{T}) \end{tikzcd}$$

is also commutative, where g_p is the induced map on the cohomology of the complexes. This follows from the listed properties above. □

Corollary 3.5.2. Let the following be a homomorphism between resolutions of sheaves

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d, "f"] & \mathcal{A}^* \arrow[d, "g"] \\ 0 \arrow[r] & \mathcal{T} \arrow[r] & \mathcal{B}^* \end{tikzcd}$$

Then there is an induced homomorphism $H^p(\Gamma(M,\mathcal{A}^*)) \stackrel{g_p}{\to} H^p(\Gamma(M,\mathcal{B}^*))$ which is an isomorphism if f is an isomorphism of sheaves and the resolutions are both acyclic.

As a consequence, we obtain the abstract de Rham theorem.

Theorem 3.5.3 (Abstract de Rham theorem)**.** Let M be a differentiable n-manifold. Then the natural mapping

$$\tilde{\Psi} : H^k(\Omega^*(M)) \to H^k(\mathcal{S}^*(M,\mathbb{R}))$$

induced by integration of differential forms over C^∞ singular chains with real coefficients is an isomorphism.

Proof. As seen before, consider the homomorphism between resolutions of $\mathbb{R}$ given by

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] \arrow[d, "id_{\mathbb{R}}"] & \Omega^0 \arrow[r, "d"] \arrow[d, "\Psi"] & \Omega^1 \arrow[r, "d"] \arrow[d, "\Psi"] & \dots \arrow[r, "d"] & \Omega^n \arrow[r, "d"] \arrow[d, "\Psi"] & 0 \\ 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & \mathcal{S}^0(\mathbb{R}) \arrow[r, "\delta"] & \mathcal{S}^1(\mathbb{R}) \arrow[r, "\delta"] & \dots \arrow[r, "\delta"] & \mathcal{S}^n(\mathbb{R}) \arrow[r, "\delta"] & 0 \end{tikzcd}$$

The sheaves $\Omega^*(M)$ and $\mathcal{S}^*(\mathbb{R})$ are both soft. Moreover, for all $k \geq 0$, the sheaves $\Omega^k(M)$ are fine and the sheaves $\mathcal{S}^k(\mathbb{R})$ are soft for an argument involving cup-product structure (for which we refer to [19]). In view of the previous corollary, this concludes the proof. □

Chapter 4

Harmonic forms

We have established that the de Rham cohomology groups are topological invariants of a differentiable manifold M. Now, we will see that if M is compact and is equipped with a Riemannian structure, we can select certain closed differential forms as representatives for the de Rham cohomology classes. These representatives, known as harmonic forms, are not only closed but also co-closed, meaning they vanish under the adjoint of the exterior derivative. Consequently, the Hodge decomposition theorem enables us to directly relate the de Rham cohomology groups $H^k_{dR}(M)$ to harmonic k-forms. Many of the results presented in this chapter involve the analysis of elliptic operators. We will primarily reference [8], [18], and [19] for detailed proofs, as our focus will be on the underlying algebraic and topological arguments.

4.1 The Hodge star operator

We have introduced two fundamental operations on differential forms: the exterior product and the exterior derivative. We now turn to the third and final key operation, the Hodge star. Let V be a n-dimensional vector space over $\mathbb{R}$ and let B be a bilinear form on V. Then B induces a bilinear form on $\Lambda^k(V)$, also denoted by B, determined by its value on decomposable elements as

$$B(\alpha, \beta) = \det(B(\alpha_i, \beta_j)), \qquad \alpha = \alpha_1 \wedge \ldots \wedge \alpha_k,\ \beta = \beta_1 \wedge \ldots \wedge \beta_k.$$

Suppose we have a fixed element $\omega \in \Lambda^n(V)$ which identifies the one-dimensional exterior algebra $\Lambda^n(V)$ with $\mathbb{R}$. Given $k \geq 0$, the wedge product induces a map

$$\varphi_\wedge : \Lambda^k(V) \times \Lambda^{n-k}(V) \to \Lambda^n(V) \cong \mathbb{R}, \qquad (\alpha, \beta) \mapsto \alpha \wedge \beta$$

for $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^{n-k}(V)$. This can be viewed as a pairing as in Definition 2.1.7, therefore identifying $\Lambda^{n-k}(V)$ with $\Lambda^k(V^*) \cong (\Lambda^k(V))^*$. On the other hand, B induces a map $\Lambda^k(V) \to (\Lambda^k(V))^*$ by sending $\alpha \mapsto B(\alpha, \cdot)$. Therefore, their composition yields a map

$$\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$$

called the *Hodge star operator*, characterized by

$$\alpha \wedge \star\beta = B(\alpha, \beta)\omega.$$

Proposition 4.1.1. If one changes the fixed element ω for $\hat{\omega} = \lambda\omega$, $\lambda \in \mathbb{R}$, then $\hat{\star} = \lambda\star$.

Proposition 4.1.2. Given an endomorphism $J : V \to V$, let $\hat{B}(u, v) = B(u, Jv)$. We can extend J to a map $J : \Lambda^k(V) \to \Lambda^k(V)$ by

$$J(v_1 \wedge \ldots \wedge v_k) = Jv_1 \wedge \ldots \wedge Jv_k$$

so that the extended bilinear forms are also related by $\hat{B}(\alpha, \beta) = B(\alpha, J\beta)$. This means that

$$\hat{\star} = \star \circ J.$$

Since we will be working in a Riemannian manifold, the bilinear form that we will use is the *metric tensor*, which gives an inner product between elements of T_pM, for all $p \in M$. Since the metric also gives an isomorphism between T_pM and T_p^*M, it furthermore provides an inner product there, denoted by $\langle \cdot, \cdot \rangle_g$.

Definition 4.1.3. Given a Riemannian n-manifold M with local coordinate functions being $(x_1, \ldots, x_n)$, let $|g|$ denote the determinant of the matrix representation of the metric tensor. Our fixed element $\omega \in \Lambda^n(M)$ will be given by *volume form of* (M, g), defined by

$$d\nu_M = \sqrt{|g|} dx_1 \wedge \ldots \wedge dx_n.$$

On the other hand, given $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_k$ and $\beta = \beta_1 \wedge \ldots \wedge \beta_k$, the extension of the inner product to the exterior algebra is defined by

$$\langle \alpha, \beta \rangle_g = \det(\langle \alpha_i, \beta_j \rangle_g).$$

We are now in conditions to extend the previous definitions to our Riemannian manifold, and therefore we can build an abstract version of the Hodge star operator by requiring

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle_g d\nu_M.$$

Recall that, although the wedge product is defined within the exterior algebra, it remains well-defined when extended to differential forms, as shown in Definition 2.2.6. The previous expression of the Hodge star via the wedge product guarantees that this operator also extends smoothly to differential forms. Since we have learned to integrate differential forms, let us abuse notation and transform the last expression into the definition of the inner product in $\Omega^k(M)$,

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle_g d\nu_M = \int_M \alpha \wedge \star\beta$$

which is symmetric since the metric is symmetric too. We will denote $|\alpha|^2 = \langle \alpha, \alpha \rangle_g$ and $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$. If we choose an orthonormal basis for a vector space T_pM, $\{e_i\}_{i=1,\ldots,n}$, we can rewrite the Hodge star operator more explicitly following the next proposition. One can discern that the intuition underlying the operator is that of orthogonality.

Proposition 4.1.4. Given an orthonormal basis $\{e_i\}_{i=1,\ldots,n}$, the Hodge star operator satisfies

$$\star(e_1 \wedge \ldots \wedge e_k) = \pm e_{k+1} \wedge \ldots \wedge e_n$$

where the sign depends on orientation. We set $\star(1) = \pm e_1 \wedge \ldots \wedge e_n$ and $\star(e_1 \wedge \ldots \wedge e_n) = \pm 1$.

Proposition 4.1.5. The Hodge star operator for (M, g) satisfies the identity

$$\star\star = (-1)^{k(n-k)}$$

Proof. Let $\alpha, \beta \in \Omega^k(M)$. Note that

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta = (-1)^{k(n-k)} \int_M \star\beta \wedge \alpha.$$

On the other hand, since the inner product is symmetric, as it is induced by the metric, which is also symmetric, and since the Hodge star preserves the metric, we can write

$$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \langle \star\beta, \star\alpha \rangle = \int_M \star\beta \wedge \star\star\alpha.$$

Comparing the last two expressions we see that $\star\star\alpha = (-1)^{k(n-k)}\alpha$ and the proof is concluded. $\square$

4.2 The Laplace-Beltrami operator

Proposition 4.2.1. The adjoint operator of the exterior derivative defined through the inner product induced by the metric becomes

$$\delta = (-1)^{n(k+1)+1} \star d \star .$$

Proof. Let $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$. Because of Stokes's theorem, we know that

$$0 = \int_M d(\alpha \wedge \star\beta) = \int_M d\alpha \wedge \star\beta - (-1)^k \int_M \alpha \wedge d \star \beta = \langle d\alpha, \beta \rangle - \int_M \alpha \wedge (-1)^k d \star \beta.$$

Then we need an operator δ such that

$$\langle \alpha, \delta\beta \rangle = \int_M \alpha \wedge \star\delta\beta = \int_M \alpha \wedge (-1)^k d \star \beta.$$

This last equality allows us to write $\star\delta\beta = (-1)^k d \star \beta$, enabling a direct comparison. By applying again the Hodge operator to $\delta\beta \in \Omega^{k-1}(M)$ we get

$$(-1)^{(k-1)(n-k+1)}\delta\beta = (-1)^k \star d \star \beta.$$

Expanding and noting that changing the sign of exponents does not affect their parity, along with the observation that k^2 and k always share the same parity, we conclude the proof. $\square$

For any twice-differentiable real-valued function f defined on Euclidean space $\mathbb{R}^n$, the *Laplace operator* maps f to the divergence of its gradient vector field. This operator can be generalized to differential forms as follows.

Definition 4.2.2. We define the *Laplace-Beltrami operator*, also called the *Laplacian*, by

$$\Delta = \delta d + d\delta$$

A sanity check would be to verify that this operator reduces to the known Laplacian when restricted to 0-forms in $\mathbb{R}^n$, specifically $-\sum_i \frac{\partial^2}{\partial x_i^2}$. In this context, 0-forms are simply smooth functions $f \in C^\infty$. Since δ acts as the zero map on 0-forms, we obtain:

$$\begin{aligned}\Delta f = \delta df = (-1)^{2n+1} \star d \star df &= - \star d \star \left(\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \right) \\ &= - \star d \left(\frac{\partial f}{\partial x_1} dx_2 \wedge \ldots \wedge dx_n + \cdots + \frac{\partial f}{\partial x_n} dx_1 \wedge \ldots \wedge dx_{n-1} \right)\end{aligned}$$

$$= - \star \left(\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \right) dx_1 \wedge \ldots \wedge dx_n = - \left(\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \right).$$

Theorem 4.2.3. The Laplace-Beltrami operator is self-adjoint and positive definite.

Proof. Let $\alpha, \beta \in \Omega^k(M)$ for $k > 0$. Since δ is the adjoint of d by the induced inner product, we see that

$$\langle \Delta\alpha, \beta \rangle = \langle d\delta\alpha, \beta \rangle + \langle \delta d\alpha, \beta \rangle = \langle \delta\alpha, \delta\beta \rangle + \langle d\alpha, d\beta \rangle = \langle \alpha, d\delta\beta \rangle + \langle \alpha, \delta d\beta \rangle = \langle \alpha, \Delta\beta \rangle.$$

This proves self-adjointness. On the other hand, one sees that

$$\langle \Delta\alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle = \|d\alpha\|^2 + \|\delta\alpha\|^2 \geq 0,$$

and it is zero only when $d\alpha = 0$ and $\delta\alpha = 0$, a case that we will be of main interest for the rest of the text.

□

Definition 4.2.4. The k-forms in the kernel of the Laplacian are called *k-harmonic forms* and are denoted by

$$\mathcal{H}^k = \{\omega \in \Omega^k(M) : \Delta\omega = 0\}.$$

Proposition 4.2.5. $\Delta\alpha = 0$ if and only if $d\alpha = 0$ and $\delta\alpha = 0$.

Proof. While one implication is clear, the other follows from the fact that if $\Delta\alpha = 0$ then

$$0 = \langle \Delta\alpha, \alpha \rangle = \langle (d\delta + \delta d)\alpha, \alpha \rangle = \langle \delta\alpha, \delta\alpha \rangle + \langle d\alpha, d\alpha \rangle$$

meaning $d\alpha = \delta\alpha = 0$. □

Corollary 4.2.6. The harmonic 0-forms are the constant functions.

Finding an harmonic form is related to solving $\Delta\omega = 0$. More generally, let us focus in solving $\Delta\omega = \alpha$. Let ω be a solution of the latter scenario, then we can build a *bounded linear functional* $l : \Omega^k(M) \to \mathbb{R}$ by

$$l(\beta) = \langle \omega, \beta \rangle.$$

Now, given any $\gamma \in \Omega^k(M)$,

$$l(\Delta\gamma) = \langle \omega, \Delta\gamma \rangle = \langle \Delta\omega, \gamma \rangle = \langle \alpha, \gamma \rangle.$$

Definition 4.2.7. A *weak solution* of $\Delta\omega = \alpha$ is a bounded linear functional

$$l : \Omega^k(M) \to \mathbb{R}$$

such that $l(\Delta\gamma) = \langle \alpha, \gamma \rangle$ for all $\gamma \in \Omega^k(M)$.

We have seen that an ordinary solution $\omega \in \Omega^k(M)$ of $\Delta\omega = \alpha$ determines a weak solution. The regularity theorem will tell us that weak solutions also determine ordinary solutions. Before stating it, we will have to develop some basic tools of functional analysis.

4.3 Functional analysis and elliptic operators

Definition 4.3.1. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $g : V \to \mathbb{R}$ satisfying $g(\lambda p) = \lambda g(p)$ for all $\lambda > 0$ and $p \in V$, and $g(p+q) \leq g(p) + g(q)$ for $p, q \in V$ is called a *sublinear functional.*

Theorem 4.3.2 (Hahn-Banach)**.** Let $W \subset V$ be a vectorial subspace, g a sublinear functional and $h : W \to \mathbb{R}$ a linear functional satisfying $h(p) \leq g(p)$ for all $p \in W$. Then we can extend h to a functional $f : V \to \mathbb{R}$ on all V such that $f(p) \leq g(p)$ for all $p \in V$.

For the proof we refer to [5, p. 1]. We will now define some norms that will be useful in what follows. Let $\mathcal{P}$ denote the complex vector space of C^∞ functions defined on $\mathbb{R}^n$ which have values in $\mathbb{C}^m$ and are 2π-periodic in each variable.

Definition 4.3.3. Let $Q = \{p \in \mathbb{R}^n : 0 < x_i(p) < 2\pi, i = 1, \ldots, n\}$ be the open cube. For $\psi, \varphi \in \mathcal{P}$, we define the L^2-inner product and its norm by

$$\langle \psi, \varphi \rangle_{L^2} = \frac{1}{(2\pi)^n} \int_Q \psi \cdot \overline{\varphi} \qquad \text{and} \qquad \|\psi\|_{L^2}^2 = \langle \psi, \psi \rangle_{L^2}.$$

Now, let $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α_i are integers. Let us now adopt the Schwarz notation and write

$$D^\alpha = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

In the context of partial differential equations (PDEs), solutions are often not smooth or differentiable everywhere. However, we do not wish to disregard them entirely, as we might in other contexts [5, p. 201]. To address this, we shift our perspective from describing functions based on their pointwise values, $u(x)$, to a functional viewpoint. That is, we characterize them by their action on a set of test functions through integration.

Definition 4.3.4. Let ψ be a function on $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ an open subset. We say that g is the *α-th weak derivative of ψ* if it satisfies

$$\int_U \psi D^\alpha \phi = (-1)^{|\alpha|} \int_U g \ \phi, \qquad \forall \phi \in C^\infty(\mathbb{R}^n).$$

We will set $g = D^\alpha \psi$.

Definition 4.3.5. We define the *Sobolev space* $W^{j,p}(\mathbb{R}^n)$ to be

$$W^{j,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n), \ \forall |\alpha| \leq j\}.$$

For $p = 2$, the Sobolev space $W^{j,2}(\mathbb{R}^n)$ is actually a *Hilbert space* and therefore denoted by $H_j(\mathbb{R}^n)$. This space is equipped with the following norm:

$$\|u\|_{H_j} = \left(\sum_{|\alpha| \leq j} \|D^\alpha u\|_{L^2}^2 \right)^{1/2}.$$

Thanks to norm equivalence, this can be viewed in the following more practical manner. Let $\xi = (\xi_1, \ldots, \xi_n)$ be an integer n-tuple, then any function $u \in H_j(\mathbb{R}^n)$ can be written in terms of Fourier series as

$$u(x) = \sum_\xi u_\xi e^{ix\xi}, \qquad \text{where} \qquad u_\xi = \int_{\mathbb{R}^n} u(x) e^{-i\xi x} dx.$$

One can think of weak derivatives as formal derivatives of Fourier series, and also define an inner product and a norm as follows. Further details about the following few pages can be found in [19, p. 229-235].

$$D^\alpha u = \sum_\xi (i\xi)^\alpha u_\xi e^{ix\xi},$$

$$\langle u, v\rangle_j = \sum_\xi (1+|\xi|)^j |u_\xi||v_\xi| = \sum_{|\alpha|=0}^{j} \langle D^\alpha u, D^\alpha v\rangle_{L^2},$$

$$|u|_j^2 = \langle u, u\rangle_j = \sum_\xi (1+|\xi|)^j |u_\xi|^2.$$

Definition 4.3.6. Equivalent to the previous definition, is the presentation of the Sobolev space as

$$H_j(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : |u|_j < \infty\},$$

which we just write as H_j. Since $j > j'$ implies $H_j \subset H_{j'}$ we denote by H_∞ the union of all H_j. This definition is standardized and can be found, for example, in [19, p. 231].

We can see that $\mathcal{P}$ is a subspace of H_j for all k. In fact, it is dense as it contains the sequences with only finitely many terms non-zero. Moreover, $H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

Proposition 4.3.7. If $u \in H_{j+[\alpha]}$, then $|D^\alpha u|_j \leq |u|_{j+[\alpha]}$. Therefore, D^α is a bounded operator from $H_{j+[\alpha]}$ to H_j.

Proposition 4.3.8. If $u \in H_{j+l}$, then $|u|_{j+l} = \sup_{v\in H_{j+l}} \frac{|\langle u,v\rangle_j|}{|v|_{j-l}}$.

For $l = 0$, the previous proposition establishes that $|u|_j = 0$ if and only if $|\langle u, v\rangle_j| = 0$ for all $v \in H_j$. This naturally leads us to another important question: given $u \in H_j$, represented as a formal Fourier series, under what conditions does u correspond to an actual function, i.e., when does the Fourier series converge? According to the Sobolev embedding theorem, this occurs when $j \geq \left\lceil \frac{n}{2} \right\rceil + 1 + m$, then $u \in C^m(\mathbb{R}^n)$, meaning that u belongs to the space of functions with continuous derivatives up to order m. Thus, for sufficiently large j, the functions in the Sobolev space H_j have the required smoothness and convergence properties. For proofs and further study of Sobolev spaces we refer to [5, p. 201-307].

Definition 4.3.9. Given a non negative integer l, a *differential operator* L *of order* l on complex valued smooth functions $\mathbb{R}^n \to \mathbb{C}$ is a map that can be written as

$$L = \sum_{|\alpha|=0}^{l} a_\alpha(x) D^\alpha,$$

where the coefficients $a_\alpha(x)$ are complex valued $C^\infty(\mathbb{R}^n)$ functions and for some α with $|\alpha| = l$, $a_\alpha(x) \not\equiv 0$. A differential operator L is *periodic* if $a_\alpha(x)$ are periodic. Note that a differential operator L of order l on $C^m(\mathbb{R}^n)$ functions is a $m \times m$ matrix where the entries L_{ij} are differential operators on complex-valued $C^\infty(\mathbb{R}^n)$ functions.

Definition 4.3.10. The polynomial p obtained by replacing the partials $\partial/\partial x_i$ by their Fourier dual variables ξ_i, typically interpreted as momentum or frequency, is called the *total symbol of* L, that is

$$p(x, \xi) = \sum_{|\alpha|\leq l} a_\alpha(x) \xi^\alpha.$$

The highest homogeneous component of the total symbol, namely

$$\sigma(x,\xi) = \sum_{|\alpha|=l} a_\alpha(x)\xi^\alpha,$$

is called the *principal symbol of L*.

Definition 4.3.11. A differential operator L is *elliptic* if for all $\xi \neq 0$ we have $\sigma(x,\xi) \neq 0$. This ensures that L is invertible in the Fourier domain for high frequencies, which is key to the rich theory of elliptic operators.

Definition 4.3.12. If L is a periodic differential operator with entries

$$L_{ij} = \sum_{|\alpha|=0}^{l} a_\alpha^{ij} D^\alpha,$$

we define its adjoint L^* to be the differential operator with entries

$$L_{ij}^* = \sum_{|\alpha|=0}^{l} D^\alpha \overline{a_\alpha^{ij}}$$

where $\overline{z}$ denotes the complex conjugate of z. The adjoint L^* satisfies the adjoint property for the L^2-norm on $\mathcal{P}$:

$$\langle L\varphi, \psi\rangle_{L^2} = \langle \varphi, L^*\psi\rangle_{L^2}, \qquad \psi, \varphi \in \mathcal{P}.$$

This follows from integration by parts. We restrict ourselves to periodic functions since the boundary term vanishes. One can readily verify that the Laplacian in $\mathbb{R}^n$ is an elliptic operator of principal symbol 2. However, the generalized Laplace-Beltrami operator Δ differs from the standard differential operators as we have defined them, since it acts on k-forms rather than scalar functions. Despite this, due to the underlying manifold structure, the Laplace-Beltrami operator locally induces a corresponding differential operator, which from now on we will denote by L.

Theorem 4.3.13. The induced operator L is an order two elliptic differential operator. Furthermore, it is invariant under coordinate changes, [19, p. 250-251].

Why is ellipticity so important? The following inequality shows that elliptic operators provide a way to understand higher-order derivatives through lower-order ones. Given an equation $Lu = f$ and a weak solution, which means finding some $u \in H_j$ that satisfies the equation, ellipticity allows us to infer higher-order derivatives of u. This is followed by applying the Sobolev embedding lemma to determine the regularity class to which the solution belongs. The proofs for the following two theorems can be found in detail in [19, p. 240-243].

Theorem 4.3.14 (Fundamental inequality)**.** Let L be an elliptic operator on $\mathcal{P}$ of order l. For all $u \in H_{j+l}$, there exists a constant $c \in \mathbb{R}$ such that $|u|_{j+l} \leq c(|Lu|_j + |u|_j)$.

We are now prepared to elevate weak solutions to actual solutions of our equation through the following main theorem.

Theorem 4.3.15 (Regularity theorem)**.** Let η be a differentiable k-form and $l : \Omega^k(M) \to \mathbb{R}$ a bounded linear functional satisfying

$$l(\Delta\varphi) = \langle \eta, \varphi \rangle$$

for all $\varphi \in \Omega^k(M)$. Then there exists a differentiable k-form ζ such that $l(\beta) = \langle \zeta, \beta \rangle$ for all $\beta \in \Omega^k(M)$.

Corollary 4.3.16. Let $\alpha \in \Omega^k(M)$ and l be a weak solution of the equation $\Delta\omega = \alpha$. Then there exists $\zeta \in \Omega^k(M)$ such that $\Delta\zeta = \alpha$.

Proof. Using the regularity theorem, there exists a k-form ζ such that $l(\beta) = \langle \zeta, \beta \rangle$ for all k-forms β. By definition of weak solution, we have

$$l(\Delta\beta) = \langle \alpha, \beta \rangle.$$

On the other hand

$$l(\Delta\beta) = \langle \zeta, \Delta\beta \rangle = \langle \Delta\zeta, \beta \rangle,$$

and therefore for all k-forms β we have

$$\langle \Delta\zeta, \beta \rangle = \langle \alpha, \beta \rangle,$$

meaning $\Delta\zeta = \alpha$, since $\langle \cdot, \cdot \rangle$ is non-degenerate. $\square$

4.4 Hodge decomposition

Lemma 4.4.1. Let $\{\alpha_n\}_n$ be a sequence of k-forms on M such that for a constant $c \in \mathbb{R}$ we have $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$. Then there exists a Cauchy subsequence of $\{\alpha_n\}_n$.

See the proof in [19, p. 248-249].

Corollary 4.4.2. The space of harmonic k-forms, $\mathcal{H}^k$, is finite dimensional.

Proof. Suppose it is not. Then we can find an orthonormal basis of infinite length $\{\alpha_n\}_n$. Since $\|\alpha_n\| = 1$ and $\|\Delta\alpha_n\| = 0$, the previous lemma ensures that there is a Cauchy subsequence. However, this is impossible since the distance between any two elements of the basis is 1. $\square$

Corollary 4.4.3. Let $\beta \in (\mathcal{H}^k)^\perp$. Then there exists a constant $c \in \mathbb{R}$ such that $\|\beta\| \leq c\|\Delta\beta\|$.

Proof. Suppose the contrary. Then we can find a sequence $\{\beta_n\}_n$ in $(\mathcal{H}^k)^\perp$ such that $\|\beta_n\| = 1$ and $\|\Delta\beta_n\| \to 0$. It can be supposed to be Cauchy because of the last lemma. We define a functional l by

$$l(\phi) = \lim_{n\to\infty} \langle \beta_n, \phi \rangle,$$

where the limit exists since the sequence is Cauchy. Now

$$l(\Delta\alpha) = \lim_{n\to\infty} \langle \beta_n, \Delta\alpha \rangle = \lim_{n\to\infty} \langle \Delta\beta_n, \alpha \rangle = 0,$$

for all $\alpha \in \Omega^k(M)$, so l is a weak solution of $\Delta\omega = 0$. The regularity theorem ensures that there is an actual solution β such that $\Delta\beta = 0$, so $\beta \in \mathcal{H}^k$. On the other hand, we know that $\lim_{n\to\infty}\langle \beta_n, \Delta\alpha \rangle = \langle \beta, \Delta\alpha \rangle$ for all α, so $\beta = \lim_{n\to\infty} \beta_n$. Since $\|\beta_n\| = 1$, we have $\|\beta\| = 1$, and since $\beta_n \in (\mathcal{H}^k)^\perp$ we have $\beta \in (\mathcal{H}^k)^\perp$, which is a contradiction with $\beta \in \mathcal{H}^k$. $\square$

Theorem 4.4.4 (Hodge decomposition theorem)**.** Let M be a compact Riemannian n-manifold and $0 \leq k \leq n$. Then we have an orthogonal direct sum decomposition

$$\Omega^k(M) = \Delta(\Omega^k) \oplus \mathcal{H}^k = \text{im } d \oplus \text{im } \delta \oplus \ker \Delta$$

Proof. Since $\mathcal{H}^k$ is finite dimensional, we have an orthogonal decomposition

$$\Omega^k(M) = \mathcal{H}^k \oplus (\mathcal{H}^k)^\perp,$$

therefore it is enough to show that $(\mathcal{H}^k)^\perp = \Delta(\Omega^k)$. One inclusion is easy; if $\omega \in \Omega^k(M)$ and $\alpha \in \mathcal{H}^k$, then

$$\langle \Delta\omega, \alpha \rangle = \langle \omega, \Delta\alpha \rangle = 0,$$

and therefore $\Delta\omega \in (\mathcal{H}^k)^\perp$ meaning $\Delta(\Omega^k) \subset (\mathcal{H}^k)^\perp$.

For the other inclusion, we take $\alpha \in (\mathcal{H}^k)^\perp$ and define the linear functional l on $\Delta(\Omega^k)$ by

$$l(\Delta\phi) = \langle \alpha, \phi \rangle, \qquad \forall \phi \in \Omega^k(M).$$

Let H denote the projection operator to the space of harmonic forms. Take $\psi = \phi - H(\phi) \in (\mathcal{H}^k)^\perp$ and therefore $\Delta\psi = \Delta\phi$. Then

$$|l(\Delta\psi)| = |\langle \alpha, \psi \rangle| \leq \|\alpha\| \cdot \|\psi\|.$$

Since $\psi \in (\mathcal{H}^k)^\perp$, Corollary 4.4.3 states that there exists a constant c such that $\|\psi\| \leq c\|\Delta\psi\|$. Now

$$|l(\Delta\phi)| = |l(\Delta\psi)| \leq \|\alpha\| \cdot \|\psi\| \leq c\|\alpha\| \cdot \|\Delta\psi\| \leq c\|\alpha\| \cdot \|\Delta\phi\|.$$

Applying the Hahn-Banach theorem with the sublinear functional $p(\phi) = c\|\alpha\| \cdot \|\phi\|$ we extend l to all $\Omega^k(M)$. Then l is a weak solution of $\Delta\zeta = \alpha$. Because of the regularity theorem, there exists a k-form ω such that $\Delta\omega = \alpha$. Therefore, $\alpha \in \Delta(\Omega^k)$ concluding the proof. $\square$

This theorem has a significant application in the context of de Rham cohomology classes, which we recently examined. Let $\alpha \in \Omega^k(M)$. Because of Hodge decomposition theorem, we can express $\alpha = \Delta\beta + H(\alpha)$. We denote β by $G(\alpha)$ where the operator $G : \Omega^k(M) \to (\mathcal{H}^k)^\perp$ is called *Green operator*.

Proposition 4.4.5. The Green operator G commutes with d, δ and Δ. In fact, it commutes with any linear operator which commutes with the Laplace-Beltrami operator.

Proof. Given $T : \Omega^k(M) \to \Omega^q(M)$ such that $T\Delta = \Delta T$, let $\pi_{(\mathcal{H}^k)^\perp}$ denote the projection mapping from $\Omega^k(M)$ onto $(\mathcal{H}^k)^\perp$. By definition we have

$$G = (\Delta|_{(\mathcal{H}^k)^\perp})^{-1} \circ \pi_{(\mathcal{H}^k)^\perp}.$$

If $\eta \in \mathcal{H}^k$, then $\Delta T(\eta) = T(\Delta\eta) = 0$, thus $T(\mathcal{H}^k) \subset \mathcal{H}^k$. Similarly, if $\alpha \in \Delta(\Omega^k)$ then there exists $\omega \in \Omega^k$ such that $\alpha = \Delta\omega$. Now $T(\alpha) = T(\Delta\omega) = \Delta T(\omega) \in \Delta(\Omega^q)$, meaning $T((\mathcal{H}^k)^\perp) \subset (\mathcal{H}^k)^\perp$. It follows that

$$T \circ \pi_{(\mathcal{H}^k)^\perp} = \pi_{(\mathcal{H}^k)^\perp} \circ T$$

and that means that

$$T \circ \Delta|_{(\mathcal{H}^k)^\perp} = \Delta|_{(\mathcal{H}^k)^\perp} \circ T$$

and therefore

$$T \circ (\Delta|_{(\mathcal{H}^k)^\perp})^{-1} = (\Delta|_{(\mathcal{H}^k)^\perp})^{-1} \circ T.$$

It follows that T commutes with G. $\square$

Theorem 4.4.6. Every de Rham cohomology class has a unique harmonic representative.

Proof. Let ω be a closed k-form. Then we can write

$$\omega = \Delta G(\omega)+H(\omega) = d\delta G(\omega)+\delta dG(\omega)+H(\omega) = d\delta G(\omega)+\delta G(d\omega)+H(\omega) = d\delta G(\omega)+H(\omega).$$

and since they differ by an exact form, ω and $H(\omega)$ belong to the same cohomology class. We have then an harmonic representative. We now claim that this representative is unique. Given α, β two harmonic k-forms in the same cohomology class, they differ by some exact form: $\alpha = \beta + d\eta$. Since α, β are harmonic, they are co-closed and therefore

$$\langle d\eta, \beta - \alpha\rangle = \langle \eta, \delta\beta - \delta\alpha\rangle = \langle \eta, 0\rangle = 0,$$

so $\beta - \alpha$ is orthogonal to $d\eta$. Since $d\eta = \alpha - \beta$, then $\|\alpha - \beta\|^2 = -\langle \alpha - \beta, \beta - \alpha\rangle = -\langle d\eta, \beta - \alpha\rangle = 0$ and because of the non-degeneracy of the inner product, $\alpha = \beta$. ☐

We know that all harmonic forms are closed. On the other hand, any closed $\omega \in \Omega^k(M)$ form decomposes in $\omega = \alpha + d\eta$ with $\alpha \in \ker \Delta|_{\Omega^k}$, thus having also associated an harmonic form, yielding

$$H^k_{dR}(M) \cong \ker \Delta|_{\Omega^k} \equiv \mathcal{H}^k.$$

Example 4.4.7. In Example 2.5.10 we saw that $H^1_{dR}(S^1)$ was generated by the 1-form $d\theta$. The Laplacian in S^1 has de form $d^2/d\theta^2$. One can see that $\frac{d^2}{d\theta^2}(d\theta) = 0$, that is, $d\theta$ is an harmonic form in accordance with the last expression.

The power of harmonic forms goes beyond merely assigning a simple representative to the vast classes in the de Rham cohomology groups, which are determined by closed but non-exact forms. In fact, the following lemma shows that choosing representatives that are also co-closed ensures that they minimize the norm induced by the metric, $\|\alpha\|^2 = \langle \alpha, \alpha\rangle$.

Lemma 4.4.8. Given a closed k-form $\omega \in \Omega^k(M)$, then $\delta\omega = 0$ if and only if ω is the unique form in its de Rham cohomology class with minimum norm.

Proof. Suppose $\delta\omega = 0$, and denote by $[\omega]$ the de Rham cohomology class of ω. Then for another element of $[\omega]$, $\omega + d\eta$, we have

$$\langle \omega + d\eta, \omega + d\eta\rangle = \langle \omega, \omega\rangle + 2\langle \omega, d\eta\rangle + \langle d\eta, d\eta\rangle$$

$$= \|\omega\|^2 + 2\langle \delta\omega, \eta\rangle + \|d\eta\|^2 = \|\omega\|^2 + \|d\eta\|^2 > \|\omega\|^2,$$

and ω is the unique element with minimum norm.
Now assume ω is the element of its de Rham cohomology class with minimum norm, but $\delta\omega \neq 0$. Using the fact that $\delta\omega \neq 0$, we will show it is possible to shift ω slightly to get an element of smaller norm. Define

$$f(t) = \|\omega + d(\delta t)\|^2.$$

Then

$$f'(0) = \lim_{t\to 0} \frac{1}{t}\left(\langle \omega + d(\delta t), \omega + d(\delta t)\rangle - \|\omega\|^2\right) = \lim_{t\to 0} \frac{1}{t}\left(2\langle \omega, d\delta t\omega\rangle + \langle d\delta t\omega, d\delta t\omega\rangle\right)$$

$$= \lim_{t \to 0} (2\langle \delta\omega, \delta\omega \rangle + t\langle d\delta\omega, d\delta\omega \rangle) = 2\langle \delta\omega, \delta\omega \rangle \neq 0,$$

so f cannot assume a minimum at 0 and ω cannot be the minimum. This shows that for a closed k-form ω with $\delta\omega \neq 0$ and for a small $t > 0$, then $\omega - t\, d\delta\omega$ will be an element of the same cohomology class but with smaller norm, concluding the proof.

□

Since any differentiable manifold can be equipped with a Riemannian metric, the de Rham cohomology groups of a compact, oriented, differentiable n-manifold are all finite dimensional. Under these circumstances, we define the following bilinear function

$$H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}, \qquad ([\varphi], [\psi]) \mapsto \int_M \varphi \wedge \psi,$$

where φ and ψ are the representatives of the corresponding cohomology classes. To check that the map is well defined, we take an other representative $\varphi' = \varphi + d\eta$. Because of the second version of Stokes' theorem, we obtain

$$\int_M \varphi' \wedge \psi = \int_M \varphi \wedge \psi + \int_M d\eta \wedge \psi = \int_M \varphi \wedge \psi + \int_M d(\eta \wedge \psi) = \int_M \varphi \wedge \psi.$$

Theorem 4.4.9 (Poincaré duality for the de Rham cohomology of a compact, oriented, n-manifold M)**.** The previous bilinear function is a non-singular pairing, and therefore determines isomorphisms yielding

$$H^k_{dR}(M) \cong (H^{n-k}_{dR}(M))^*.$$

Proof. Given a non-zero cohomology class representative $[\varphi] \in H^k_{dR}(M)$, we must find a non-zero cohomology class representative $[\psi] \in H^{n-k}_{dR}(M)$. Choose a Riemannian structure on M. We know that we can assume φ to be an harmonic representative of $[\varphi]$. Since $[\varphi]$ is not zero, then φ is not identically zero. Since $\star\Delta = \Delta\star$, it follows that $\star\varphi$ is also an harmonic form and therefore is closed. Note that $\star\varphi$ represents a cohomology class $[\star\varphi] \in H^{n-k}_{dR}(M)$. The proof is completed by noting that the defined inner product is non-degenerate,

$$([\varphi], [\star\varphi]) \mapsto \int_M \varphi \wedge \star\varphi = \langle \varphi, \varphi \rangle \neq 0.$$

□

We have seen how the choice of representatives in the de Rham cohomology modules, using harmonic forms, allows us to uncover new properties of these spaces. This privileged perspective, in turn, reveals new insights into the topology of the manifold. Let us define the *k-th Betti number of M* by

$$\beta_k = \dim H^k_{dR}(M).$$

Theorem 4.4.9 asserts then that $\beta_k = \beta_{n-k}$. We define the *Euler-Poincaré characteristic* by

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k_{dR}(M) = \sum_{k=0}^{n} (-1)^k \beta_k$$

and therefore we obtain the following consequence.

Corollary 4.4.10. If $n = \dim M$ is odd, then $\chi(M) = 0$.

Chapter 5

Morse theory and supersymmetry

"Now I finally understand Morse theory!"

Edward Witten to Raoul Bott

Throughout history, there is no doubt that physicists have been deeply indebted to mathematicians. The logical framework of mathematics has been indispensable for shaping empirical sciences like physics, which have never fully repaid that support. However, the development of supersymmetry within physical systems, primarily driven by Edward Witten, represents one of the rare instances where intuition from the actual world has simplified mathematical proofs, specifically those in Morse theory. In the present chapter, basic understanding of quantum mechanics will be assumed, though it will not be essential for grasping the mathematical foundations presented. Based on the framework of supersymmetric quantum mechanics, we will present Witten's analytical proof of the Morse inequalities. Although this is not the most topological approach—developed in [13]—it fits seamlessly with the concepts we have learnt in the previous chapters.

5.1 The Morse inequalities

Morse theory provides a powerful framework for analyzing the topology of smooth manifolds by examining the behaviour of smooth functions defined on them. A classic example, as discussed in [13], involves a torus $M = T^2$ positioned tangentially to a plane V. In this setup, a function $f : M \to \mathbb{R}$ is defined to represent the height of each point on the torus relative to V. Let M^a denote the sublevel set $\{p \in M \mid f(p) \leq a\}$. Milnor observed that the homotopy type of M^a changes precisely at the critical points of f. Combined with the inequalities established in this chapter, this demonstrates the profound link between the critical points of smooth functions and the underlying topology of the manifold.

Definition 5.1.1. Let M be a n-dimensional, compact, oriented manifold. Let $f \in C^\infty(M)$. A point $p \in M$ is called a *critical point of f* if $df(p) = 0$, that is, given a coordinate system (U, φ) and coordinate functions $\{x_1, \ldots, x_n\}$,

$$\left.\frac{\partial f}{\partial x_1}\right|_p = \cdots = \left.\frac{\partial f}{\partial x_n}\right|_p = 0.$$

This definition is independent of the choice of coordinate system about p. Indeed, given a different coordinate system (V, ϕ) with coordinate functions $\{y_1, \dots, y_n\}$ we have

$$\left.\frac{\partial f}{\partial y_i}\right|_p = \left.\frac{\partial f}{\partial x_i}\right|_p \cdot \left.\frac{\partial(\phi \circ \varphi^{-1})}{\partial y_i}\right|_p = 0, \qquad \forall i = 1, \dots, n.$$

We denote the set of critical points of f by $\mathrm{Crit}(f)$.

Definition 5.1.2. Let $p \in \mathrm{Crit}(f)$. We define the *Hessian of f at p on T_pM* to be a symmetric bilinear map given by

$$\mathrm{Hess}_f(\ \cdot\ ,\ \cdot\ )(p) \equiv d^2f|_p : T_pM \times T_pM \to \mathbb{R}, \qquad d^2f|_p(X_p, Y_p) = X(Y(f))(p).$$

By setting $X_p = \sum_i a_i \frac{\partial}{\partial x_i}|_p$ and $Y_p = \sum_j b_j \frac{\partial}{\partial x_j}|_p$, one can see that

$$d^2f|_p(X_p, Y_p) = X\left(\sum_j b_j \frac{\partial f}{\partial x_j}\right)(p) = \sum_{i,j} a_i b_j \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_p.$$

Definition 5.1.3. Again, let $p \in \mathrm{Crit}(f)$. We say it is *non-degenerate* if the Hessian of f at p is non-singular, i.e.,

$$\det\left(\left.\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}\right|_p\right) \neq 0.$$

Again, one can see that this does not depend on the coordinate system.

Definition 5.1.4. A $C^\infty(M)$ function on M is said to be a *Morse function* if all the critical points of this function are non-degenerate.

For any bilinear form B defined on a vector space V, the *index of B* is defined to be the maximal dimension of any subspace W on which B is negative definite. We call *index of f at p* to the index of $d^2f|_p$ on $T_pM \times T_pM$. If f is a Morse function, then the index of f at p is called the *Morse index of f at p*. In the next lemma, we will see that it characterizes the local behaviour of f near p. From now on, we will omit the subscripts $|_p$, as we will primarily focus on local arguments.

Since non-degenerate critical points are isolated, the requirement of compactness for M implies that a Morse function has a finite amount of critical points. It is also known that there always exists a Morse function on M [13, p. 32]. This provides the following result.

Lemma 5.1.5 (Morse lemma)**.** For any critical point $p \in M$ of a Morse function f, there is a coordinate system (U_p, φ) and coordinate functions $y = (y_1, \dots, y_n)$ such that $\varphi(p) = 0$ and

$$f \circ \varphi^{-1}(y) = \frac{1}{2}y_1^2 - \dots - \frac{1}{2}y_{n_f(p)}^2 + \frac{1}{2}y_{n_f(p)+1}^2 + \dots + \frac{1}{2}y_n^2$$

where $n_f(p)$ denotes the Morse index of f at p. Let m_k denote the number of critical points $p \in M$ of f such that $n_f(p) = k$. The *Morse inequalities*, for which an analytic proof will be given in this chapter, can be stated as follows.

Theorem 5.1.6 (Morse inequalities)**.** For any integer k such that $0 \leq k \leq n$, one has

(i) (Weak Morse inequalities).

$$\beta_k \leq m_k.$$

(ii) (Strong Morse inequalities).

$$\beta_k - \beta_{k-1} + \cdots + (-1)^k \beta_0 \leq m_k - m_{i-k} + \cdots + (-1)^k m_0.$$

Moreover,

$$\beta_n - \beta_{n-1} + \cdots + (-1)^n \beta_0 = m_n - m_{n-1} + \cdots + (-1)^n m_0.$$

It is easy to see that the weak inequalities follow from the strong inequalities. Morse inequalities have the power to extract information about the topology of a manifold from the critical points of a Morse function. For example, one can notice that the last expression gives a way to derive $\chi(M)$ by calculating the alternating sum of numbers of critical points up to index n. Next, we will introduce a framework within theoretical physics where we aim to study the eigenspaces of a new Laplacian, very similar to the Laplace-Beltrami operator. However, this time, the eigenspaces will be localized around the critical points of a Morse function, allowing us to relate them to the topology of the manifold.

5.2 Supersymmetric quantum mechanics

Quantum mechanics has been one of the most significant revolutions in the history of physics. Its development in the past century has triggered a cascade of remarkable predictions and advancements, culminating in groundbreaking theories such as quantum chromodynamics and the Standard Model of particle physics. However, modern theoretical physics faces the monumental challenge of unifying the Standard Model, formulated through quantum field theory, with Einstein's theory of general relativity.

A key insight into this challenge comes from the *Coleman-Mandula theorem*, which states that space-time symmetries cannot be combined with internal symmetries in a nontrivial way without violating fundamental physical principles. *Supersymmetry* escapes this limitation because it adds *fermionic symmetries*, which are qualitatively different from the usual ones. These symmetries change bosons into fermions and vice versa, extending the possibilities while staying consistent with the Coleman-Mandula theorem. For our purposes, it suffices to know that bosonic fields commute while fermionic fields anti-commute. This distinction naturally aligns bosonic fields with symmetric tensors and fermionic fields with antisymmetric tensors. However, the supersymmetric formalism offers a more elegant framework by treating bosonic fields as differential forms of even degree and fermionic fields as differential forms of odd degree. Indeed, if $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,

$$\omega \wedge \eta = \begin{cases} +\eta \wedge \omega & \text{if } \ k, l \text{ even,} \\ -\eta \wedge \omega & \text{if } \ k, l \text{ odd.} \end{cases}$$

Definition 5.2.1. A *supersymmetric quantum mechanics theory with two supercharges* is a quantum mechanics theory with a positive definite $\mathbb{Z}_2$-graded Hilbert space $\mathbb{H}$, an even operator $H : \mathbb{H} \to \mathbb{H}$ as the *hamiltonian* and odd self-adjoint operators $Q_1, Q_2 : \mathbb{H} \to \mathbb{H}$ called *supercharges*. These operators obey the following commutation rules:

$$Q_1^2 = Q_2^2 = 2H \quad \text{and} \quad \{Q, Q^\dagger\} = 0,$$

where the second relation invokes the Poisson bracket. As a consequence, the supercharges are conserved:

$$[H, Q_1] = [H, Q_2] = 0.$$

The operator responsible for defining the $\mathbb{Z}_2$-grading is denoted by $(-1)^F$. We define the even subspace of the Hilbert space $\mathbb{H}$, where $(-1)^F = 1$, as $\mathbb{H}^B$, and the odd subspace, where $(-1)^F = -1$, as $\mathbb{H}^F$. These subspaces are referred to as the *Hilbert space of bosonic states* and the *Hilbert space of fermionic states*, respectively. The hamiltonian preserves the decomposition

$$\mathbb{H} = \mathbb{H}^B \oplus \mathbb{H}^F$$

while the supercharges map one subspace to the other

$$Q_1, Q_2 : \mathbb{H}^B \to \mathbb{H}^F, \qquad Q_1, Q_2 : \mathbb{H}^F \to \mathbb{H}^B.$$

That is, the supercharges are operators that exchange bosonic states for fermionic states and vice versa. In line with our discussion and following Witten's procedure in [23], we take

$$\mathbb{H}^B = \Omega^{\text{even}}(M) = \bigoplus_{k \text{ even}} \Omega^k(M) \quad \text{and} \quad \mathbb{H}^F = \Omega^{\text{odd}}(M) = \bigoplus_{k \text{ odd}} \Omega^k(M),$$

and define the supercharges and the hamiltonian by

$$Q_1 = d + \delta, \quad Q_2 = i(d + \delta) \quad \text{and} \quad H = d\delta + \delta d = \Delta.$$

Accordingly, Witten argues that supersymmetry is, in fact, a symmetry of the system only when the vacuum has zero energy, meaning that there exists an eigenvalue of the Hamiltonian equal to zero. This led Witten to study the existence of harmonic forms and, as a consequence of the theory developed in the previous chapter, also the Betti numbers of the manifold. However, finding the spectrum of an operator like the Laplacian is far from trivial, compelling Witten to resort to the technique developed below.

5.3 Witten's deformation

For physicists, the Hilbert space formed by functions on M is, in some sense, more fundamental than the points of M themselves [1]. This perspective drove Witten to investigate the Betti numbers through the properties of functions on M; given a Morse function f, Witten deformed the exterior derivative and its adjoint by

$$d_T = e^{-Tf} \cdot d \cdot e^{Tf} \quad \text{and} \quad \delta_T = e^{Tf} \cdot \delta \cdot e^{-Tf}$$

for $T \in \mathbb{R}^+$. Since the Hodge star commutes with 0-forms, one can check that for $\alpha, \beta \in \Omega^k(M)$

$$\begin{aligned}\langle d_T\alpha, \beta\rangle &= \int_M e^{-Tf}\, d(e^{Tf}\alpha) \wedge \star\beta = \int_M d(e^{Tf}\alpha) \wedge \star(e^{-Tf}\beta) = \langle d(e^{Tf}\alpha), e^{-Tf}\beta\rangle \\ &= \langle e^{Tf}\alpha, \delta e^{-Tf}\beta\rangle = \int_M \alpha \wedge \star(e^{Tf}\delta\; e^{-Tf}\beta) = \langle \alpha, \delta_T\beta\rangle.\end{aligned}$$

When we focus on $T \to +\infty$, one might think about $1/T$ as being the Planck constant. It is clear that $d_T^2 = e^{-Tf}d^2e^{Tf} = 0$, and therefore we can consider the *deformed de Rham complex*, given by

$$0 \to \Omega^0(M) \xrightarrow{d_T} \Omega^1(M) \xrightarrow{d_T} \Omega^2(M) \xrightarrow{d_T} \dots \xrightarrow{d_T} \Omega^n(M) \to 0,$$

and define its cohomology groups by

$$H^k_{T,dR}(M) = \frac{\ker d_T|_{\Omega^k(M)}}{\text{im } d_T|_{\Omega^{k-1}(M)}}.$$

Proposition 5.3.1. For any $T \geq 0$ and $0 \leq k \leq n$, we have

$$H^k_{T,dR}(M) \cong H^k_{dR}(M)$$

and therefore

$$\dim H^k_{T,dR}(M) = \dim H^k_{dR}(M) = \beta_k.$$

Proof. For all $k > 0$, $T \geq 0$ let us consider the isomorphisms $\phi_T^k : \Omega^k(M) \to \Omega^k(M)$ sending $\omega \mapsto e^{-Tf}\omega$. If we can proof that all the squares of the diagram below commute, then the previous isomorphisms induce isomorphisms between the respective cohomology modules, since they will send closed forms to closed forms and exact forms to exact forms.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(M) \arrow[r, "d"] \arrow[d, "\phi_T^0"'] & \Omega^1(M) \arrow[r, "d"] \arrow[d, "\phi_T^1"'] & \dots \arrow[r, "d"] & \Omega^{n-1}(M) \arrow[r, "d"] \arrow[d, "\phi_T^{n-1}"'] & \Omega^n(M) \arrow[r, "d"] \arrow[d, "\phi_T^n"'] & 0 \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "d_T"] & \Omega^1(M) \arrow[r, "d_T"] & \dots \arrow[r, "d_T"] & \Omega^{n-1}(M) \arrow[r, "d_T"] & \Omega^n(M) \arrow[r, "d_T"] & 0
\end{tikzcd}$$

Now, recall that for $\omega \in \Omega^{k-1}(M)$,

$$d_T(\phi_T^{k-1}(\omega)) = d_T(e^{-Tf}\omega) = e^{-Tf} \cdot d \cdot e^{Tf}(e^{-Tf}\omega) = e^{-Tf}d\omega = \phi_T^k(d\omega).$$

$\square$

At its turn, the deformed operators define a deformed Laplacian, commonly referred to as the *Witten Laplacian*

$$\Delta_T = d_T\delta_T + \delta_T d_T$$

satisfying both $\langle \Delta_T\alpha, \beta\rangle = \langle \alpha, \Delta_T\beta\rangle$ (self-adjointness) and $\langle \Delta_T\alpha, \alpha\rangle \geq 0$ (positive definite) $\forall \alpha, \beta \in \Omega^k(M)$, $k \geq 0$. The motivation for defining this new operator will become evident as we proceed.

We now introduce two new operators on forms that will help us express the Witten Laplacian in a more comprehensive way. Recall that the metric tensor induces a bundle isomorphism $\flat : TM \to T^*M$ sending $u \mapsto \langle u, \cdot\ \rangle_g$ and its inverse is denoted by $\sharp : T^*M \to TM$.

Definition 5.3.2. Given an $w_p \in T_p^*M$, we define the *exterior product by* ω_p to be the map $\omega_p\wedge : \Lambda^k(T_p^*M) \to \Lambda^{k+1}(T_p^*M)$. This trivially extends to forms. Notice that this is a particular case of an object already mentioned at the beginning of the previous chapter.

Definition 5.3.3. Given an $w_p \in T_p^*M$, we define the *interior product by* ω_p to be the map $i_{w_p} : \Lambda^k(T_p^*M) \to \Lambda^{k-1}(T_p^*M)$ such that $i_{w_p^\sharp}$ is the adjoint of $w_p\wedge$. Again, this trivially extends to forms, and $\langle \omega \wedge \alpha, \beta\rangle = \langle \alpha, i_{\omega^\sharp}\beta\rangle$ for $\alpha \in \Omega^k(M)$, $\beta \in \Omega^{k+1}(M)$.

With this new formalism, the deformed operators can be expressed as follows. For $\alpha \in \Omega^k(M)$,

$$d_T\alpha = e^{-Tf}d(e^{Tf}\alpha) = e^{-Tf}(d(e^{Tf})\wedge\alpha + e^{Tf}d\alpha) = e^{-Tf}(Te^{Tf}df\wedge\alpha + e^{Tf}d\alpha) = (d + Tdf\wedge)\alpha.$$

and also

$$\langle d_T\alpha, \beta\rangle = \langle d\alpha, \beta\rangle + T\langle df\wedge\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle + T\langle\alpha, i_{df^\sharp}\beta\rangle = \langle\alpha, (\delta + Ti_{df^\sharp})\beta\rangle$$

and therefore $d_T = d + Tdf\wedge$ and $\delta_T = \delta + Ti_{df^\sharp}$.

Proposition 5.3.4. Δ_T is an elliptic operator with the same symbol as Δ.

Proof. From the previous expressions, it is evident that d_T is the sum of d and a zero-order differential operator, and therefore shares the same symbol as d. Similarly, δ_T retains the same symbol as δ. According to the formal rules for calculating symbols, Δ_T and Δ also share the same symbol and Δ_T is elliptic. $\square$

Equipped with the inner product induced by the metric, the space of differential k-forms can be extended to the Hilbert space of square integrable k-forms, $L^2(M, \Lambda^k(M))$, which we will denote by $L^2_k(M)$. Furthermore, stronger results on spectral decompositions of elliptic operators, as detailed in [9, Chapter 8], establish that the spectrum of a self-adjoint operator L, denoted by $\mathrm{Spec}(L)$, consists of a sequence of eigenvalues $0 \le \lambda_0(T) \le \lambda_1(T) \le \ldots \to \infty$, each with finite multiplicity. Let us denote the eigenspaces of the Witten Laplacian by

$$F^k_{\mu,T} = \{\omega \in \Omega^k(M) : \Delta_T\omega = \mu\omega\},$$

for $\mu \in \mathbb{R}$. We refer to [17] for the proof that, for elliptic operators such as Δ_T, the following decomposition holds:

$$\Omega^k(M) = \bigoplus_{\lambda\in\mathrm{Spec}(\Delta_T)} F^k_{\lambda,T}.$$

We now define

$$\mathcal{E}^k(\lambda, \Delta_T) = \bigoplus_{\mu\le\lambda} F^k_{\mu,T}$$

and

$$N^k(\lambda, \Delta_T) := \dim \mathcal{E}^k(\lambda, \Delta_T) = \#\{j : \lambda_j(T) \le \lambda\}.$$

On the other hand, since

$$d_T(\mathcal{E}^k(\lambda, \Delta_T)) \subset \mathcal{E}^{k+1}(\lambda, \Delta_T),$$

we can consider the cochain complex given by

$$0 \to \mathcal{E}^0(\lambda, \Delta_T) \xrightarrow{d_T} \mathcal{E}^1(\lambda, \Delta_T) \xrightarrow{d_T} \ldots \xrightarrow{d_T} \mathcal{E}^n(\lambda, \Delta_T) \xrightarrow{d_T} 0.$$

Proposition 5.3.5. The previous complex $(\mathcal{E}^k(\lambda, \Delta_T), d_T)$ has the same cohomology as the complex $(\Omega^k(M), d_T)$. That is,

$$H^k(\mathcal{E}^*(\lambda, \Delta_T)) \cong H^k_{dR}(M)$$

and therefore $\dim H^k(\mathcal{E}^*(\lambda, \Delta_T)) = \beta_k$.

Proof. Let us fix $\mu > 0$. Since $d_T(F^k_{\mu,T}) \subset F^{k+1}_{\mu,T}$, we can consider the following cochain complex

$$0 \to F^0_{\mu,T} \xrightarrow{d_T} F^1_{\mu,T} \xrightarrow{d_T} \dots \xrightarrow{d_T} F^n_{\mu,T} \xrightarrow{d_T} 0.$$

Consider $\omega \in F^k_{\mu,T}$ such that $d_T\omega = 0$. Since $\mu\omega = \Delta_T\omega = d_T\delta_T\omega$, then $\omega = d_T(\delta_T\omega/\mu)$ for $\delta_T\omega/\mu \in F^{k-1}_{\mu,T}$. This shows that

$$\ker(d_T : F^k_{\mu,T} \to F^{k+1}_{\mu,T}) = \text{im } (d_T : F^{k-1}_{\mu,T} \to F^k_{\mu,T}),$$

which proves the exactness of the complex. This means that the cohomology modules $H^k(\mathcal{E}^*(\lambda, \Delta_T))$ are exactly the cohomology modules $H^k(F^*_{0,T})$, where $F^k_{0,T} = \ker \Delta_T|_{\Omega^k}$.

Since Δ_T is an elliptic operator, its Hodge decomposition follows in the same way as for the Laplace-Beltrami operator. A more general formulation of this result can be found in [21, p. 147]. Thus, for $\omega \in \Omega^k(M)$, we have

$$\omega = \alpha + d_T\beta$$

for some $\alpha \in \ker \Delta_T|_{\Omega^k} = F^k_{0,T}$, $\beta \in \Omega^{k-1}(M)$. Therefore, the cohomology classes coincide, $[\omega] = [\alpha]$. Together with Theorem 5.3.1, this concludes the proof. □

For a concise proof of the previous result based on homotopy theory, we refer to [12, p. 26]. Furthermore, straightforward manipulations involving the connections of the manifold M—which we omit here, as they are unnecessary for our purposes—allow us to express the Witten Laplacian in the form of a Schrödinger operator with potential $T^2|df|^2$.

Proposition 5.3.6 (Bochner formula)**.** The Witten Laplacian can be written in terms of the Laplace-Beltrami operator as

$$\Delta_T = \Delta + T^2|df^2| + T\sum_{i,j} \text{Hess}_f\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big)[dx_i\wedge, i_{dx_j}],$$

as seen in [12, p. 27-28]. Note that when $df \neq 0$, the potential gets huge as $T \to +\infty$. In exploring whether supersymmetry is preserved, Witten was particularly interested in the ground states. Consequently, the behaviour of the Witten Laplacian is especially important near critical points, where $df = 0$. We will now study what happens to Δ_T in the neighbourhood of $\text{Crit}(f)$.

Following [12], let us denote by $\text{Crit}(f; r)$ the set of critical points of f with Morse index r. Let $p \in \text{Crit}(f; r)$, and $(U_p, x_1, ..., x_n)$ be a coordinate system in a neighbourhood of p. Locally we have $|df|^2 = |x|^2$. Let us denote $dx_J = dx_{i_1} \wedge ... \wedge dx_{i_k}$, $J = \{i_1, ..., i_k\}$. Now, using the expression of the Laplacian in $\mathbb{R}^n$ and the fact that Hess_f is diagonal with eigenvalues -1 for $j \leq r$ and $+1$ for $j > r$, the form of the Witten Laplacian given by Proposition 5.3.6 for $\alpha dx_J \in \Omega^k(U_p)$ yields

$$\Delta_T(\alpha \cdot dx_J) = \sum_{j=1}^n \Big[-\Big(\frac{\partial^2}{\partial x_j^2}\Big)\alpha + T^2|x_j|^2\alpha\Big] dx_J + \alpha T \sum_{j=1}^n \varepsilon_j [dx_j\wedge, i_{dx_j}] dx_J$$

where $\varepsilon_j = -1$ for $j \leq r$ and $\varepsilon_j = 1$ for $j > r$. Let us now define the following useful operator called the *model operator*:

$$\Delta'_{T,r} = \sum_{j=1}^n H_j + \sum_{j=1}^n \varepsilon_j K_j$$

where

$$H_j = -\Big(\frac{\partial^2}{\partial x_j^2}\Big) + T^2|x_j|^2 \quad \text{ and } \quad K_j = T[dx_j\wedge, i_{dx_j}].$$

It is known - [9, p. 233] - that the spectrum of the harmonic oscillator $-(\frac{\partial^2}{\partial y^2}) + y^2$ on $L^2(\mathbb{R})$ consists of the eigenvalues $\{2N+1\}_{N\in\mathbb{N}}$ with multiplicity one, and the eigenfunctions can be chosen to be of the form

$$\phi_N(y) = H_N(y)e^{-y^2/2}$$

where $H_N(y)$ denotes the N'th Hermite polynomial. Let us now follow [6, p. 10] and change variables by $y_j = \sqrt{T}x_j$. One can see that now $(\partial^2/\partial x_j^2) = T(\partial^2/\partial y_j^2)$ and the previously defined operator becomes

$$H_j = T\Big[-\Big(\frac{\partial^2}{\partial y_j^2}\Big) + |y_j|^2\Big],$$

which has the same spectrum as the harmonic oscillator but scaled by T, that is, $\{T(2N+1)\}_{N\in\mathbb{N}}$. Similarly, the eigenfunctions become $\phi_N(\sqrt{T}y_j)$. On the other hand, $K_j(dx_J) = T\varepsilon_j^J dx_J$ where $\varepsilon_j^J = 1$ if $j \in J$ and $\varepsilon_j^J = -1$ otherwise. Since the operators H_{j_1} and H_{j_2} commute for any $1 \le j_1 < j_2 \le n$, we find that $L_k^2(\mathbb{R}^n)$ has de following orthonormal basis of eigenforms of $\Delta'_{T,r}$:

$$\{\phi_{N_1}(\sqrt{T}x_1)\dots\phi_{N_n}(\sqrt{T}x_n)dx_J : N_1,\dots,N_n \in \mathbb{N}\}$$

with corresponding eigenvalues

$$\{T\sum_{j=1}^n(2N_j + 1 + \varepsilon_j\varepsilon_j^J) : N_1,\dots,N_n \in \mathbb{N}\}.$$

Theorem 5.3.7. The spectrum of $\Delta'_{T,r}$ on $L_k^2(\mathbb{R}^n)$ is given by the previous set of eigenvalues. Moreover,

$$\ker\Big(\Delta'_{T,r}|_{L_k^2(\mathbb{R}^n)}\Big) = \begin{cases} 0 & \text{if } r \neq k, \\ \mathbb{R}e^{-\frac{T|x|^2}{2}}dx_1\wedge\ldots\wedge dx_r & \text{if } r = k. \end{cases}$$

As T grows, all other eigenvalues are of the form $C\cdot T$ for some $C > 0$.

Proof. An eigenvalue

$$T\sum_{j=1}^n(2N_j + 1 + \varepsilon_j\varepsilon_j^J)$$

vanishes if and only if all parentheses vanish (since they are all positive). This is the case if and only if $N_j = 0$ and $\varepsilon_j\varepsilon_j^J = -1$ for all $j = 1,\dots,n$. This means precisely $J = \{1,\dots,r\}$, where r is the Morse index, after the definitions of ε_j and ε_j^J. The corresponding eigenvalue is

$$\phi_0(\sqrt{T}x_1)\cdots\phi_0(\sqrt{T}x_n)dx_1\wedge\ldots\wedge dx_r = e^{-\frac{T|x|^2}{2}}dx_1\wedge\ldots\wedge dx_r.$$

□

When physicists discover that something can be described in terms of harmonic oscillators, it can only mean good news. The proof of the following theorem will allow us, through a simple algebraic manipulation, to conclude Witten's proof of the Morse inequalities. This theorem can be proved either using functional analysis -[24, p. 82-89]- or by resorting to the min-max principle for self-adjoint operators -[15]-. In this text, we will follow [12, p. 30-32] and focus on the latter approach.

5.4 Proof of the Morse inequalities

One of the key remarks of Witten in [23] is that the eigenforms of the Witten Laplacian concentrate near $\mathrm{Crit}(f)$ as $T \to +\infty$. We now study the spectrum of Δ_T comparing it with the spectrum of Δ'_T by means of the min-max principle.

The idea will be to show that there exists a spectral gap that grows asymptotically as T increases, which precisely separates m_k eigenvalues of $\Delta_T|_{\Omega^k(M)}$ that are asymptotically small (ground states) from the rest, which grow significantly. On a topological level, this will imply that for a good choice of λ, $\dim \mathcal{E}^k(\lambda, \Delta_T) = m_k$. Since this complex has the same cohomology as the de Rham complex, we will be able to compare the numbers m_k with the Betti numbers. On a physical level, the number of ground states in a system is fundamental within the framework of supersymmetry, as it is where the correspondence between bosons and fermions, as asserted by the theory, is determined.

Theorem 5.4.1. There exist constants $C_1, C_2 > 0$ such that

$$\mathrm{Spec}(\Delta_T) \subset [0, e^{-C_1 T}] \cup [C_2 T, +\infty], \qquad T \gg 1.$$

Also, $\Delta_T|_{L^2_k(M)}$ has exactly m_k eigenvalues (counted with multiplicity) in $[0, e^{-C_1T}]$, that is,

$$N^k(e^{-C_1T}, \Delta_T) = m_k.$$

Proof. The proof consists of two distinct parts. In the first part, it is shown that there are at least m_k eigenvalues within the interval $[0, e^{-C_1T}]$. In the second part, it is demonstrated that the (m_k+1)-th eigenvalue grows linearly as $T \to +\infty$. This reveals that, for large T, the spectrum of the deformed Laplacian corresponds to the spectrum of the sum of m_k model operators $\Delta'_{T,k}$. Additionally, we can observe how each critical value contributes exactly to one ground state to the spectrum.

Consider a C^∞ bump function $\eta : \mathbb{R} \to \mathbb{R}$ with compact support $\mathrm{supp}\, \eta = [-2, 2]$ such that $\eta = 1$ on $[-1, 1]$. Set $\eta_\varepsilon(t) = \eta(\varepsilon t)$, $\varepsilon > 0$. Fix a Morse index r and an small enough $\varepsilon > 0$. Consider, for every $p \in \mathrm{Crit}(f; r)$, the coordinate system $(U_p, x_1, ..., x_n)$ and the map $\psi_{p,\varepsilon} : U_p \to \mathbb{R}$ given by $\psi_{p,\varepsilon}(x) = \eta_\varepsilon(x_1) \dots \eta_\varepsilon(x_n)$. Note that $\mathrm{supp}\, \psi_{p,\varepsilon} = \{x \in U_p : |x_j| \leq 2\varepsilon, j = 1, \dots, n\}$. We now transport the forms from $\ker \Delta'_{T,r}|_{L^2_r(\mathbb{R}^n)}$ to M defining

$$\omega_{p,T} = \begin{cases} \frac{1}{\sqrt{a_T^n}} e^{-\frac{T|x|^2}{2}} \psi_{p,\varepsilon}(x) dx_1 \wedge \dots \wedge dx_r & \text{on } U_p, \\ 0 & \text{on } M \setminus \mathrm{supp}\, \psi_{p,\varepsilon}, \end{cases}$$

where $a_T = \int_{\mathbb{R}} e^{-Ty^2} \eta_\varepsilon^2(y)\, dy$. Note that we can compute the following integral as a product for each coordinate

$$\langle \omega_{p,T}, \omega_{p,T} \rangle = \int_M \omega_{p,T} \wedge \star \omega_{p,T} = \frac{1}{a_T^n} \left[\int_{\mathbb{R}} e^{-Tx^2} \eta_\varepsilon^2(x) dx \right]^n = 1.$$

Now, since $\mathrm{supp}\, \omega_{p,T} \subset U_p$, the r-forms $\omega_{p,T}$ are linearly independent when p runs in $\mathrm{Crit}(f; r)$. We define

$$J_T^r = \bigoplus_{p \in \mathrm{Crit}(f;r)} \langle \omega_{p,T} \rangle.$$

In order to use the min-max principle, we define the *quadratic form associated to* Δ_T by

$$Q_T(u) = \langle \Delta_T u, u\rangle, \qquad u \in \Omega^k(M)$$

Again, separating coordinates, we get for large T

$$\begin{aligned} Q_T(\omega_{p,T}) &= \langle \Delta_T \omega_{p,T}, \omega_{p,T}\rangle = \langle \Delta'_{T,r}\omega_{p,T}, \omega_{p,T}\rangle = \int_M \Delta'_{T,r}\omega_{p,T} \wedge \star\omega_{p,T} \\ &= \frac{1}{a_T^n}\left[\int_{\mathbb{R}} \Big(-\frac{\partial^2}{\partial x^2} + T^2x^2\Big)(\eta_\varepsilon(x)e^{-\frac{-Tx^2}{2}})\eta_\varepsilon(x)e^{-\frac{-Tx^2}{2}}dx\right]^n \\ &= \frac{1}{a_T^n}\left[\int_{\mathbb{R}} \Big(-(\eta''_\varepsilon - 2\eta'_\varepsilon xT + \eta_\varepsilon T^2x^2)e^{-Tx^2/2} + T^2x^2\eta_\varepsilon e^{-Tx^2/2}\Big)\eta_\varepsilon(x)e^{-\frac{-Tx^2}{2}}dx\right]^n \\ &= \frac{1}{a_T^n}\left[\int_{\mathbb{R}} (-\eta''_\varepsilon\eta_\varepsilon + 2\eta'_\varepsilon\eta_\varepsilon xT)e^{-Tx^2}dx\right]^n \le Ce^{-Tn\varepsilon^2/2}. \end{aligned}$$

where we have used that the function in parenthesis has support $[\varepsilon, 2\varepsilon]$ and that $a_T \ge \frac{1}{2}\sqrt{\pi/T}$ for $T \gg 1$. As T grows, there is always a constant $C_1 > 0$ for which we can write $Ce^{-Tn\varepsilon^2/2} = e^{-C_1T}$. Let $\lambda_1(T) \le \lambda_2(T) \le \dots$ be the spectrum of Δ_T in $L^2_r(M)$. According to [12, A. 43], the min-max principle tells us that

$$\lambda_j(T) = \inf_{F\subset \mathrm{Dom}(Q)} \sup_{u\in F, \|u\|=1} Q(u)$$

where F runs through the j-dimensional subspaces of the domain of Q, $\mathrm{Dom}(Q)$. Since $\dim J^r_T = m_r$, we have

$$\lambda_{m_r}(T) \le \sup_{u\in J^r_T, \|u\|=1} Q_T(u) \le e^{-C_1T}.$$

This proves the first part. For the second part, an alternative form of the min-max principle is needed -[12, A. 44]-, given by

$$\lambda_j(T) = \sup_{F\subset \mathrm{Dom}(Q)} \inf_{u\in F, \|u\|=1} Q(u)$$

where F now runs over the $(j-1)$-codimensional subspaces of $\mathrm{Dom}(Q)$. This means that all we need is to prove that

$$Q_T(u) \ge C_2T\|u\|^2 \quad \forall u \in (J^r_T)^\perp,$$

since then it follows that

$$\lambda_{m_r+1}(T) \ge \inf_{u\in (J^r_T)^\perp, \|u\|=1} Q_T(u) \ge C_2T$$

for large T. To do so, let us construct a cover $\mathscr{U}$ of M. Let $U_0 = M \setminus \bigcup_{p\in \mathrm{Crit}(f)} \overline{U_p}$ and set $\mathscr{U} = \{U_0\} \cup \{U_p : p \in \mathrm{Crit}(f)\}$. We consider now a partition of unity $\{\varphi_U : U \in \mathscr{U}\}$ subordinated to $\mathscr{U}$ with

$$\sum_{U\in\mathscr{U}} \varphi_U^2 = 1 \quad \text{and} \quad \varphi_{U_p} = 1 \text{ on } \mathrm{supp}\ \omega_{p,T}.$$

Recall that for any $u \in \Omega^k(M)$ we have $u = \sum_{U\in\mathscr{U}} \varphi_U u$. Moreover, since $\sum_{U\in\mathscr{U}} \varphi_U d\varphi_U = 0$,

$$\sum_{U\in\mathscr{U}} |d(\varphi_U u)|^2 = \sum_{U\in\mathscr{U}} (\varphi_U^2|du|^2 + |d\varphi_U \wedge u|^2) = |du|^2 + \sum_{U\in\mathscr{U}} |d\varphi_U \wedge u|^2$$

and
$$\sum_{U\in\mathscr{U}}|\delta(\varphi_U u)|^2=\sum_{U\in\mathscr{U}}(\varphi_U^2|\delta u|^2+|i_{d\varphi_U}u|^2)=|\delta u|^2+\sum_{U\in\mathscr{U}}|i_{d\varphi_U}u|^2.$$
For $u\in L^2_r(M)$, let us calculate the quadratic form making use of the inner product induced by the metric

$$\begin{aligned}\sum_{U\in\mathscr{U}}Q_T(\varphi_U u)&=\sum_{U\in\mathscr{U}}\int_M\Big(|d(\varphi_U u)|^2+|\delta(\varphi_U u)|^2+T^2|df|^2|\varphi_U u|^2\\&\quad+T\sum_{i,j}\mathrm{Hess}_f\Big(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\Big)\langle[dx_i\wedge,i_{dx_j}]\varphi_U u,\varphi_U u\rangle_g\Big)d\nu_M\\&=\int_M\Big(|du|^2+|\delta u|^2+T^2|df|^2|u|^2+T\sum_{i,j}\mathrm{Hess}_f\Big(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\Big)\langle[dx_i\wedge,i_{dx_j}]u,u\rangle_g\Big)\\&\quad+\sum_{U\in\mathscr{U}}\int_M\Big(|d\varphi_U\wedge u|^2+|i_{d\varphi_U}u|^2\Big)=Q_T(u)+\sum_{U\in\mathscr{U}}\int_M\Big(|d\varphi_U\wedge u|^2+|i_{d\varphi_U}u|^2\Big).\end{aligned}$$

and therefore there is a constant $C>0$ such that
$$Q_T(u)\geq\sum_{U\in\mathscr{U}}Q_T(\varphi_U u)-C\|u\|^2.$$

We will revisit this expression later. Next, we will see that the desired inequality, $Q_T(u)\geq C_2T\|u\|^2$, holds for (i) r-forms restricted to U_0; (ii) r-forms restricted to U_p, where $p\in\mathrm{Crit}(f)\setminus\mathrm{Crit}(f;r)$; and finally, (iii) r-forms restricted to U_p, where $p\in\mathrm{Crit}(f;r)$, all of them in $(J_T^r)^\perp$.

To prove (i), take $u\in L^2_r(U_0)$, since $U_0\cap\mathrm{Crit}(f)=\varnothing$ there exists $c>0$ such that $0<c\leq|df|^2$. Now, the dominant term in the Bochner formula is the one with $|df|^2$ and therefore
$$\langle\Delta_T u,u\rangle\geq\frac{c}{2}T^2\|u\|^2\geq\frac{cT_0}{2}T\|u\|^2=K_1T\|u\|^2,\quad\text{for }T\geq T_0.$$
The proof of (ii) follows from Theorem 5.3.7, as if $p\in\mathrm{Crit}(f;l)$, $l\neq r$, and $u\in L^2_r(U_p)$, then
$$\langle\Delta_T u,u\rangle=\langle\Delta'_{T,r}u,u\rangle\geq K_2T\langle u,u\rangle=K_2T\|u\|^2,\quad K_2>0.$$
The proof of (iii) is somewhat more intricate. Given $\tilde u\in L^2_r(U_p)\cong L^2_r(\mathbb{R}^n)$ with $p\in\mathrm{Crit}(f;r)$, extend this form to the entirety of M by setting it to zero outside U_p, an extension we denote by $u\in L^2_r(M)$. Then $\|u\|=\|\tilde u\|$, and on U_p we have $\Delta_T=\Delta'_{T,r}$ and also $\langle\Delta_T u,u\rangle=\langle\Delta'_{T,r}\tilde u,\tilde u\rangle$. By orthogonal decomposition, we can write $\tilde u=\tilde u_1+\tilde u_2$ with $\tilde u_1\in\ker\Delta'_{T,r}$ and $\tilde u_2\perp\ker\Delta'_{T,r}$. Let us denote $dx_J=dx_1\wedge\ldots\wedge dx_r$. We can then express $\tilde u_1$ as $\tilde u_1=\langle\tilde u,e^{-Tx^2/2}dx_J\rangle e^{-Tx^2/2}dx_J$. Note that
$$\begin{aligned}\|\tilde u_1\|=|\langle\tilde u,e^{-Tx^2/2}dx_J\rangle|&=|\langle u,[\psi_{p,\varepsilon}+(1-\psi_{p,\varepsilon})]e^{-Tx^2/2}dx_J\rangle|=|\langle u,(1-\psi_{p,\varepsilon})e^{-Tx^2/2}dx_J\rangle|\\&\leq\|u\|\,\langle(1-\psi_{p,\varepsilon})e^{-Tx^2/2}dx_J,(1-\psi_{p,\varepsilon})e^{-Tx^2/2}dx_J\rangle=K_3e^{-T\varepsilon^2/4}\|\tilde u\|,\end{aligned}$$
for $K_3>0$, where we have used that $u\perp\omega_{p,T}$, the Cauchy-Schwarz inequality and the fact that $(1-\psi_{p,\varepsilon})$ vanishes on $[-\varepsilon,\varepsilon]^n$. On the other hand, from Theorem 5.3.7 we have

$\langle \Delta'_{T,r}\tilde{u}_2, \tilde{u}_2\rangle \geq K_4 T\|\tilde{u}_2\|$ for $K_4 > 0$. Therefore,

$$\begin{aligned} Q_T(u) = \langle \Delta'_{T,r}\tilde{u}, \tilde{u}\rangle = \langle \Delta'_{T,r}\tilde{u}_2, \tilde{u}_2\rangle \geq K_4 T\|\tilde{u}_2\| = K_4 T\|\tilde{u}_2\| = K_4 T(\|\tilde{u}\| - \|\tilde{u}_1\|) \\ \geq K_4 T(1 - K_3 e^{-T\varepsilon^2/4})\|\tilde{u}\| \geq K_5 T\|\tilde{u}\| \end{aligned}$$

for large T. This proves (iii). With the help of the partition of unity presented above and since $Q_T(u) \geq \sum_{U\in\mathscr{U}} Q_T(\varphi_U u) - C\|u\|^2$ for a general $u \in L^2_r(M)$, $u \perp J^r_T$, the proof is concluded. □

To complete the proof of the Morse inequalities, we return to the known complex,

$$0 \to \mathcal{E}^0(\lambda, \Delta_T) \xrightarrow{d_T} \mathcal{E}^1(\lambda, \Delta_T) \xrightarrow{d_T} \dots \xrightarrow{d_T} \mathcal{E}^n(\lambda, \Delta_T) \xrightarrow{d_T} 0.$$

Now knowing that, for a good choice of λ,

$$\dim H^k(\mathcal{E}^*(\lambda, \Delta_T)) = \beta_k \quad \text{and} \quad \dim \mathcal{E}^k(\lambda, \Delta_T) = N^k(\lambda, \Delta_T) = m_k.$$

Proof of the Morse inequalities. Let us denote

$$z_k = \dim\ker d_T|_{\mathcal{E}^k(\lambda,\Delta_T)} \quad \text{and} \quad r_k = \dim\operatorname{im} d_T|_{\dim\mathcal{E}^k(\lambda,\Delta_T)}.$$

By definition, $m_k = \dim \mathcal{E}^k(\lambda, \Delta_T) = z_k + r_k$ and $\beta_k = \dim H^k(\mathcal{E}^*(\lambda, \Delta_T)) = z_k - r_{k-1}$. Therefore

$$\sum_{j=0}^{k}(-1)^{k-j} m_j = r_k + \sum_{j=0}^{k}(-1)^{k-j}\beta_j.$$

Since $r_k \geq 0$ for all k, and $r_{-1} = r_n = 0$, we obtain the strong Morse inequalities. For the weak inequalities, one just notices that $m_k = \beta_k + r_k + r_{k-1} \geq \beta_k$. □

With this, we conclude the proof of the Morse inequalities using Witten's deformed Laplacian, as well as the present text. Before finishing, it is worth looking back and reviewing everything we have learned. We began by studying the differential geometry of a manifold and learning that the extent to which closed differential forms on the manifold failed to be exact provided a measure of its topology, through de Rham's theorem. With Hodge's theorem, we learned that the Laplacian allows us to select a harmonic form as a representative of the classes of these topological invariants, the de Rham cohomology groups. With Witten's work, we discovered that harmonic forms are, in fact, ground states of a physical system very relevant in the context of supersymmetry, and that to study them, we can use a deformation of the Laplacian, whose harmonic forms retain information about the topology of the manifold due to the properties of elliptic operators. Finally, the introduction of Morse functions f has allowed us to separate the spectrum of this new operator and count the number of forms that vanish in terms of the Morse index of the critical points of f.

The work developed has an analogous counterpart in terms of complex manifolds, known as Kähler manifolds. Instead of the de Rham complex, topological invariants are obtained from the Dolbeault complex. Dolbeault cohomology is also closely related to Hodge theory and harmonic forms, as well as playing an important role in the paper where Witten develops his ideas on supersymmetry.

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Appendix A

Basics of differential geometry on manifolds

Let $n \geq 1$ be an integer and let

$$\mathbb{R}^n = \{a = (a_1, \dots, a_n) : a_i \in \mathbb{R}\}$$

be the *n-dimensional Euclidean space.*

Definition A.1. For $i = 1, \dots, n$, the function $r_i : \mathbb{R}^n \to \mathbb{R}$ defined by $r_i(a) = a_i$ is called the ith *canonical coordinate function* on $\mathbb{R}^n$.

Given $\alpha = (\alpha_1, \dots, \alpha_n)$ a n-tuple of non-negative integers, we set $[\alpha] = \sum \alpha_i$ and $\alpha! = \alpha_1! \dots \alpha_n!$ so that:

$$\frac{\partial^\alpha}{\partial r^\alpha} = \frac{\partial^{[\alpha]}}{\partial r_1^{\alpha_1} \dots \partial r_n^{\alpha_n}}$$

Definition A.2. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. We say that f is *differentiable of class C^k on U*, denoted by $f \in C^k(U)$, if for each $[\alpha] \leq k$ the partial derivatives $\frac{\partial^\alpha f}{\partial r^\alpha}$ exist and are continuous on U.

Definition A.3. A *locally Euclidean space M of dimension n* is a Hausdorff n-dimensional topological space for which each point has a neighbourhood homeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$. If φ is a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^n$, then φ is called a *coordinate map*, the functions $x_i = r_i \circ \varphi : U \to \mathbb{R}$ are called the *coordinate functions* and the pair (U, φ) is called a *coordinate system* or simply a *chart.*

Definition A.4. A *differentiable structure $\mathscr{F}$ of class C^k* on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha : \alpha \in A\}$ satisfying:

(i) $\bigcup_{\alpha \in A} U_\alpha = M$.

(ii) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^k for all $\alpha, \beta \in A$.

(iii) If (U, φ) is a coordinate system such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are C^k for all $\alpha \in A$, then $(U, \varphi) \in \mathscr{F}$. That is, $\mathscr{F}$ is a *smooth atlas* which is maximal with respect to (ii).

Definition A.5. A *n-dimensional differentiable manifold of class C^k* is a pair $(M, \mathscr{F})$ consisting of a n-dimensional, second countable, locally Euclidean space M and a differentiable structure $\mathscr{F}$ of class C^k. Generally, we will write only M and focus our attention on C^∞-differential structures, which give rise to the so-called *smooth manifolds.*

Definition A.6. Let $U \subset M$ be open. Then $f : U \to \mathbb{R}$ is a C^∞ *function on* U, $f \in C^\infty(U)$, if $f \circ \varphi^{-1}$ is C^∞ for every coordinate map. The set of all $C^\infty(M)$ functions has $\mathbb{R}$-algebra structure with the operations

$$(f+g)(p) := f(p) + g(p), \quad (fg)(p) := f(p)g(p), \quad (\lambda f)(p) := \lambda f(p), \quad \forall p \in M.$$

Definition A.7. A continuous map $\psi : M \to N$ is *differentiable of class* C^∞, $\psi \in C^\infty(M, N)$ or simply $\psi \in C^\infty$, if and only if $\varphi \circ \psi \circ \tau$ for each coordinate map φ on M and τ on N. This is in fact a local property.

Definition A.8. Given a neighbourhood U of $p \in M$ and a C^∞ function $f : U \to \mathbb{R}$, then (f, U) is said to be equivalent to (g, V) if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. This equivalence class of (f, U) is called *germ of f at p.*

A germ is, locally, the heart of the function, as it is the cereal germ for a grain. Let us denote by $C_p^\infty(M)$ the set of all germs of smooth functions on M at p.

The requirement of the second axiom of countability in manifolds brings forth many useful properties. Due to this, manifolds are normal, metrizable, and paracompact. Paracompactness implies the existence of partitions of unity, a tool that will allow the emergence of global properties from the addition of local ones [19].

A collection $\{U_\alpha\}$ of open subsets of M is a *cover* of a set $W \subset M$ if $W \subset \bigcup_\alpha U_\alpha$. A collection $\{A_\alpha\}$ of subsets of M is *locally finite* if whenever $p \in M$ there exists a neighbourhood W_p of p such that $W_p \cap A_\alpha \neq \varnothing$ for only finitely many α. Given a function φ on a topological space X, the *support of* φ is the subset of X defined by supp $\varphi = \overline{\varphi^{-1}(\mathbb{R} - \{0\})}$.

Definition A.9. A *partition of unity on M* is a collection $\{\varphi_i : i \in I\}$ of C^∞ functions on M such that

(i) The collection $\{\text{supp } \varphi_i : i \in I\}$ is locally finite.

(ii) $\sum_{i \in I} \varphi_i(p) = 1$ for all $p \in M$, and $\varphi_i(p) \geq 0$ for all $p \in M$ and $i \in I$.

It is *subordinate* to the cover $\{U_\alpha\}_{\alpha \in A}$ if for each i there exists an α such that supp $\varphi_i \subset U_\alpha$.

Theorem A.10 (Existence of partitions of unity)**.** Let M be a differentiable manifold and $\{U_\alpha : \alpha \in A\}$ an open cover of M. Then there exists a countable partition of unity $\{\varphi_i : i = 1, 2, 3, \dots\}$ subordinate to the cover $\{U_\alpha\}$ with supp φ compact for each i. If one does not require compact supports, then there is a partition of unity φ_α subordinate to the cover $\{U_\alpha\}$ with at most countably many of the φ_α not identically zero.

The concept of linear approximation to a manifold is embodied in the so-called tangent space of the manifold. We will identify the property of geometric tangent vectors by their action on smooth functions as directional derivatives and establish a bijection between the tangent vectors on the manifold and the derivations of $C^\infty(M)$. Other ways of introducing the tangent space as a linear approximation involve quotient spaces of the previously defined germs. However, this approach is more characteristic of algebraic geometry.

Definition A.11. Given $p \in \mathbb{R}^n$, a *tangent vector to* $\mathbb{R}^n$ at p is a pair $(p; v)$ where $v = (v_1, \dots, v_n) \in \mathbb{R}^n$.

Definition A.12. The set of all tangent vectors to $\mathbb{R}^n$ at p forms a vector space called *tangent space of* $\mathbb{R}^n$ *at* p, denoted by $T_p(\mathbb{R}^n)$ and defined by

$$(p; v) + (p; w) = (p; v + w), \qquad c(p; v) = (p; cv)$$

Definition A.13. A linear map $X_p : C_p^\infty(\mathbb{R}^n) \to \mathbb{R}^n$ satisfying the Leibniz rule, that is, $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)\ \forall f, g \in C_p^\infty(\mathbb{R}^n)$, is called a *derivation at* $p \in \mathbb{R}^n$. The set of all derivations forms a vector space at p denoted by $\mathcal{D}_p(\mathbb{R}^n)$.

Theorem A.14. The linear map

$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n) \qquad\qquad (p; v) \mapsto \mathcal{D}_v = \sum_{i=1}^n v_i \frac{\partial}{\partial r_i}\bigg|_p$$

where $(p; v) = (p; v_1, \dots, v_n)$ and $\mathcal{D}_v$ is the directional derivative in the direction of v, is an isomorphism. Thus, the base of the vector space $T_p(\mathbb{R}^n)$, $\{e_i\}_{i=1,\dots,n}$, corresponds to the base $\{\partial/\partial r_j|_p\}_{j=1,\dots,n}$.

Now, the previous results can be easily extended to manifolds as follows. Definitions 2.2.1, 2.2.2 and 2.2.3 hold when $\mathbb{R}^n$ is replaced by a general smooth manifold M.

Definition A.15. Given a chart $\varphi : U \to \mathbb{R}^n$ with coordinate functions $x_1, \dots, x_n$ and a smooth function $f : M \to \mathbb{R}$ we define the *partial derivative* $\partial f/\partial x_i$ by

$$\left(\frac{\partial}{\partial x_i}\bigg|_p\right)(f) = \frac{\partial (f \circ \varphi^{-1})}{\partial r_i}\bigg|_{\varphi(p)} = \frac{\partial}{\partial r_i}\bigg|_{\varphi(p)} (f \circ \varphi^{-1})$$

Definition A.16. Let $\psi : M \to N$ be a smooth map of smooth manifolds. At each $p \in M$, it induces a linear map of tangent spaces called the *pushforward of a vector*,

$$\psi_* : T_pM \to T_{\psi(p)}N,$$

such that for each $X_p \in T_pM$ we have $\psi_*(X_p)(f) = X_p(f \circ \psi) \in \mathbb{R}$ for $f \in C^\infty_{\psi(p)}(N)$.

Theorem A.17. Let (U, φ) be a coordinate system of M with coordinate functions $\{x_i\}_{i=1,..,n}$. Then $\varphi_* : T_pM \to T_{\varphi(p)}\mathbb{R}^n = \mathbb{R}^n$ is a vector space isomorphism and thus the basis of T_pM is $\{\partial/\partial x_i|_p\}_{i=1,\dots,n}$ such that $x_i = r_i \circ \varphi$.

Hence, if M is a n-dimensional manifold, then T_pM is a vector space of dimension n.

Definition A.18. Let M be a smooth manifold and $p \in M$. The dual space of T_pM is called the *cotangent space of* M *at* p and is denoted by T_p^*M. Its elements are called *co-vectors.*

Definition A.19. Let $f : M \to \mathbb{R}$ be a smooth function. Its *differential* is defined to be the object such that for any $p \in M$ and $X_p \in T_pM$ yields $(df)_p(X_p) = X_p(f)$. Therefore we have $df_p \in T_p^*M$.

Remark A.20. Let (U, φ) be a coordinate system such that $\{x_i\}_{i=1,\dots,n}$ are coordinate functions and $\{\partial/\partial x_i|_p\}_{i=1,\dots,n}$ form a basis of T_pM. Since $x_i : M \to \mathbb{R}$, it makes sense to define their differentials as those objects $dx_i|_p$ such that

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

Therefore the basis of T_p^*M is $\{dx_i|_p\}_{i=1,\dots,n}$.

Remark A.21. One can easily check that the following properties are satisfied.

(i) Any tangent vector $X_p \in T_pM$ can be written uniquely as a linear combination

$$X_p = \sum_{i=1}^{n} X_p^i \frac{\partial}{\partial x_i}\bigg|_p.$$

Suppose that (U, φ) and (V, ψ) are coordinate systems about p with coordinate functions $x_1, \dots, x_n$ and $y_1, \dots, y_n$ respectively. Then

$$\frac{\partial}{\partial y_i}\bigg|_p = \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_j}\bigg|_p \frac{\partial}{\partial x_i}\bigg|_p.$$

(ii) If $(U, x_1, \dots, x_d)$ is a coordinate system about $p \in M$, then $\{dx_i|_p\}$ is the basis of T_p^*M dual to $\{\partial/\partial x_i|_p\}$. If $f : M \to \mathbb{R}$ is a C^∞ function, then

$$df_p = df(p) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\bigg|_p dx_i|_p.$$

Definition A.22. $\psi : M \to N$ be C^∞ and $p \in M$. The *pullback of co-vector* is the linear map $\psi^* : T_{\psi(p)}^*N \to T_p^*M$ such that

$$\psi^*(\omega_p)(X_p) = \omega_p(\psi_*(X_p))$$

whenever $\omega_p \in T_{\psi(p)}^*N$ and $X_p \in T_pM$.

Now, let us observe how, in a natural way, the collection of tangent vectors on a differentiable manifold itself forms another differentiable manifold. Similarly, this holds for its dual object, formed by the linear functions on the tangent spaces. Let M be a smooth manifold with differential structure $\mathscr{F}$. We define

$$TM = \bigcup_{p \in M} T_pM$$

$$T^*M = \bigcup_{p \in M} T_p^*M$$

and consider the natural projections

$$\pi : TM \to M, \qquad \pi(X_p) = p \;\text{ if }\; X_p \in T_pM$$

$$\pi^* : T^*M \to M, \qquad \pi^*(\omega_p) = p \;\text{ if }\; \omega_p \in T_p^*M$$

Let $(U, \varphi) \in \mathscr{F}$ with coordinate functions $x_1, \ldots, x_n$. Now, define one-to-one maps onto open subsets of $\mathbb{R}^{2n}$, $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ and $\tilde{\varphi^*} : (\pi^*)^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\tilde{\varphi}(v) = (x_1(\pi(v)), \ldots, x_n(\pi(v)), dx_1(v), \ldots, dx_n(v))$$

$$\tilde{\varphi^*}(\tau) = (x_1(\pi^*(\tau)), \ldots, x_n(\pi^*(\tau)), \tau\left(\frac{\partial}{\partial x_1}\right), \ldots, \tau\left(\frac{\partial}{\partial x_n}\right))$$

for $v \in \pi^{-1}(U)$ and $\tau \in (\pi^*)^{-1}(U)$. From this maps, the construction of the topology and differential structure in both TM and T^*M is developed in [19, p. 19]. The sets TM and T^*M with these differentiable structures are called respectively *tangent bundle* and *cotangent bundle*. Their points can be conveniently written as pairs (p, X_p) where $p \in M$ and $X_p \in T_pM$ for the former and (p, ω_p) where $p \in M$ and $\omega_p \in T_p^*M$ for the latter. In most of the sections, however, we omit the specification of the point unless it is particularly needed. The set $M_p = \pi^{-1}(p)$ is called the *fiber of TM at p* and is canonically identified with T_pM by mapping each (p, X_p) to X_p.

If $\psi : M \to N$ is a C^∞ map, then the pushforward map defines a C^∞ mapping of the tangent bundles by

$$\hat{\psi}_* : TM \to TN, \qquad \hat{\psi}_*(p, X_p) = (\psi(p), \psi_*(X_p)).$$

Definition A.23. A *vector field X along a curve* $\sigma : [a, b] \to M$ is a mapping $X : [a, b] \to TM$ which *lifts* σ, that is, $\pi \circ X = \sigma$. The vector field is *smooth along* σ if $X : [a, b] \to TM$ is smooth.

Definition A.24. A *vector field X on an open set $U \subset M$* is a *lifting* of U into TM, that is, a map $X : U \to TM$ such that $\pi \circ X = id_U$. Again, the vector field is smooth if $X \in C^\infty(U, TM)$.

The set of smooth vector fields over U acquires structure of vector space over $\mathbb{R}$ and a module over the ring $C^\infty(U)$. If $p \in U$, then $X(p) = X_p$ is an element of T_pM. Vector fields are then defined by mapping each point to point of the fiber of p. Furthermore, if $f \in C^\infty(U)$, then $X(f)$ is the function on U defined by $X(f)(p) = X_p(f)$.

Definition A.25. A *differential 1-form on an open set $U \subset M$* is a lifting of U into T^*M, that is, a map $\omega : U \to T^*M$ such that $\pi^* \circ \omega = id_U$.

Just as in vector fields, the set of smooth 1-forms over U acquires vector space structure over $\mathbb{R}$. If $p \in U$, then $\omega(p) = \omega_p$ is an element of T_p^*M.