Black hole shadows and Brans-Dicke theory

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Abstract: A possible alternative to General Relativity (GR) is Brans-Dicke (BD) theory, a scalar-tensor theory where gravity is mediated both by a symmetric tensor $g_{\mu\nu}$ (the metric) and a scalar field ϕ. In this work, we obtain an expression for a black hole shadow angle valid for static and spherically symmetric configurations of the source in GR and beyond. We derive the BD field equations from the action and find solutions for a static, spherically symmetric spacetime in the weak field limit (analytically) and for a strong field (numerically). We show that for finite values of the coupling constant ω the BD field equations allow for naked singularities, whose existence is controversial. We then conclude that no constraints on the theory can be obtained from black hole shadows observations, since static black holes in BD theory are identical to those in GR.
Keywords: General Relativity, Brans-Dicke theory, black holes.
SDGs: Quality education.

I. INTRODUCTION

General Relativity (GR) is one of the most beautiful, elegant and precisely tested theories in all of physics. Yet we know that it is incomplete. The existence of singularities and the struggle to quantize gravity suggest that there must be a deeper theory from which GR emerges as an effective field theory. The current cosmological tensions may also indicate the presence of new physics beyond GR. As a result, a plethora of alternative theories have been put forward over the years. Some of these fall under the category of scalar-tensor theories, since they propose that gravity is mediated by both a symmetric tensor $g_{\mu\nu}$ (the metric) and a scalar field ϕ. They are all contained within Horndeski's theory, and among them we find Brans-Dicke (BD) theory [1], whose action is

$$S_{BD} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi}\left(R\phi - \frac{\omega}{\phi}g^{\mu\nu}\partial_\nu\phi\partial_\mu\phi\right) + \mathcal{L}_m\right], \quad (1)$$

where g is the metric determinant, R the Ricci scalar, ω the dimensionless BD parameter and $\mathcal{L}_m$ the matter Lagrangian. In fact, this theory can be interpreted as an extension of GR in which the gravitational constant G is no longer constant, but varies in space and time. The function ϕ plays the role of the inverse of the gravitational constant (i.e. $\phi \sim G^{-1}$). It is widely accepted that classical GR is recovered in the limit $\omega \longrightarrow \infty$. Also note that we are working in natural units ($G = c = \hbar = 1$).

ω is a coupling constant that can be constrained with observations and experiments. Solar system and cosmological observations require large values of ω. For instance, data from the Cassini spacecraft constrains $\omega > 10^4$ [2], and CMB data constrains $\omega > 692$ [3]. It is also important, however, to test this theory in strong-field scenarios, where it is known that scalar-tensor theories make different predictions than GR. Our primary goal is to see whether we can constrain ω with observations of the black hole (BH) shadow of Sagittarius A made by the Event Horizon Telescope (EHT) [4].

In the first part of this work we deduce an expression for the angle of a BH shadow, which is the photosphere (the sphere containing unstable, circular orbits of null geodesics or photons) as seen by a distant observer. Secondly, the BD field equations are derived from the action by applying variations with respect to the inverse metric $g^{\mu\nu}$ and the scalar field ϕ. Thirdly, an approximate solution is found for a static and spherically symmetric spacetime in the weak field limit. Then the equations are solved numerically for a strong field. Lastly, we analyse the solutions and discuss why it is not possible to constrain BD using the BH shadow measurements.

II. BLACK HOLE SHADOWS

In this section we will generalize Synge's method [5] for computing a Schwarzschild BH shadow to a general static and spherically symmetric spacetime with metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (2)$$

First of all, let us introduce dimensionless variables: $x \equiv r/2M$, $\tau \equiv t/2M$, where $2M(= 2MG/c^2)$ is the Schwarzschild radius and M the BH's mass. Now the metric reads

$$ds^2 = 4M^2\left[-A(x)d\tau^2 + B(x)dx^2 + x^2d\Omega^2\right], \quad (3)$$

where $d\Omega$ is the differential solid angle. Since the motion of a particle in this metric will be confined to a plane, we can take $\theta = \pi/2$ (equatorial plane) without loss of generality. As photons have no mass, they will follow the null geodesics equation $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$:

$$-A(x)\dot{\tau}^2 + B(x)\dot{x}^2 + x^2\dot{\varphi}^2 = 0, \quad (4)$$

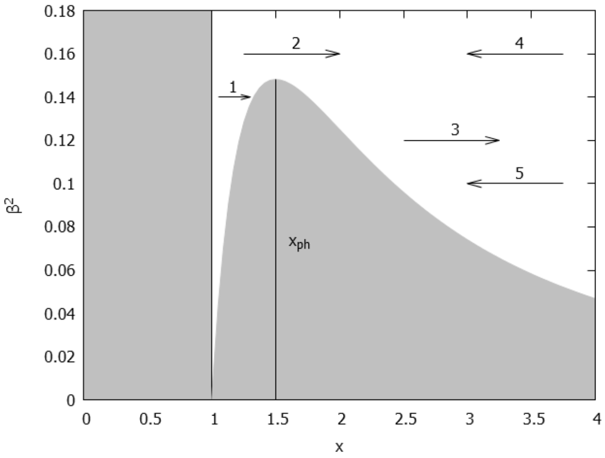

FIG. 1: $\beta^2(x)$ for a Schwarzschild BH, where $A(x) = 1/B(x) = 1 - 1/x$. The forbidden regions are shown in gray. Photon 1 cannot escape the BH, while photons 2 and 3 can. On the other hand, photon 4 will be captured by the BH, while photon 5 will escape.

where a dot indicates differentiation with respect to an affine parameter. Moreover, the metric does not depend on either t or φ. Thus, ∂_t and ∂_φ are Killing vectors and we have the following constants of motion:

$$-(\partial_t)_\mu \dot{x}^\mu = A(x)\dot{\tau} \equiv \alpha\beta,\ (\partial_\varphi)_\mu \dot{x}^\mu = x^2\dot{\varphi} \equiv \alpha, \quad (5)$$

where α and β are constants that depend on the initial conditions. Plugging these expressions into (4) yields

$$\left(\frac{dx}{d\varphi}\right)^2 = \frac{x^4}{B(x)}\left[\frac{\beta^2}{A(x)} - \frac{1}{x^2}\right] \equiv x^4 F(\beta, x). \quad (6)$$

Notice that $F(\beta, x) = 0$ (or $\beta^2 = A(x)/x^2$) defines the curve of turning points. Obviously, the region $F < 0$ is forbidden, as is the region inside the event horizon, which is causally disconnected from the outside Universe. In FIG. 1 these regions are shown for a Schwarzschild BH, where the event horizon and the photosphere are located at $x = 1$ and $x = 3/2$, respectively. A point on the graph represents a point in a photon's trajectory, and the trajectories themselves are horizontal lines. Now let us consider a static observer at x_0 emitting outgoing photons with an angle $0 \leq \psi \leq \pi/2$ with respect to the radial direction. Some photons will be able to escape the black hole's gravitational pull and reach infinity, while others will be captured by the BH.

From FIG. 1 we can see that for $x_0 > x_{ph}$ (outside the photosphere) all rays escape. For $x_0 < x_{ph}$ only those photons inside a cone of semi-angle ψ_c escape. This critical angle corresponds to the maximum value of β, β_c. By setting $d\beta^2/dx = 0$ we get

$$A'(x_{ph}) - 2\frac{A(x_{ph})}{x_{ph}} = 0,\ \beta_c^2 = \frac{A(x_{ph})}{x_{ph}^2}. \quad (7)$$

Now consider an infinitesimal triangle of angle ψ. Using the metric (3) to compute its sides yields

$$\cot\psi = \frac{\sqrt{B(x)}}{x}\frac{dx}{d\varphi}. \quad (8)$$

Plugging (6) and (7) into (8) finally yields,

$$\boxed{\sin^2\psi_c = \frac{A(x_0)}{x_0^2}\frac{x_{ph}^2}{A(x_{ph})}} \quad (9)$$

with x_{ph} given by the first equation in (7). This angle defines the cone of escaping photons. Notice that (9) has two solutions ψ_1, ψ_2 with $\psi_1 + \psi_2 = \pi$, $\psi_2 > \psi_1$. For $x_0 < x_{ph}$ we have to choose ψ_1, and for $x_0 > x_{ph}$ we have to choose ψ_2. However, since the trajectories are reversible, (9) also defines the cone of possible incoming photons. The BH's shadow angular radius (which defines the region from where the observer receives no light) will then be its supplementary: $\alpha_{sh} = \pi - \psi_c$. We can also define the critical impact parameter as $b_c \equiv r_{ph}/\sqrt{A(r_{ph})}$, so that for a distant observer ($x_0 \gg 1$) $\alpha_{sh} \approx b_c/r_0$.

III. BD FIELD EQUATIONS

We derive the BD field equations by varying the action (1) with respect to the metric and the scalar field.

The variation with respect to $g^{\mu\nu}$ ($\frac{\delta S_{BD}}{\delta g^{\mu\nu}} = 0$) yields

$$G_{\mu\nu}\phi + \left[\Box\phi + \frac{\omega}{2\phi}(\nabla\phi)^2\right] g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi = 8\pi T_{\mu\nu}\,, \quad (10)$$

where $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R$ is the Einstein tensor, ∇ the covariant derivative, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ the d'Alembertian, $(\nabla\phi)^2 \equiv g^{\mu\nu}\nabla_\nu\phi\nabla_\mu\phi$ and $T_{\mu\nu}$ the matter energy-momentum tensor.

On the other hand, the variation with respect to ϕ ($\frac{\delta S_{BD}}{\delta\phi} = 0$) gives

$$\Box\phi - \frac{1}{2\phi}(\nabla\phi)^2 + \frac{\phi}{2\omega}R = 0\,. \quad (11)$$

We can simplify this equation by taking the trace of (10) and substituting it into (11). This leads to

$$\Box\phi = \frac{8\pi T}{2\omega + 3}\,, \quad (12)$$

which is a modified Klein-Gordon equation for ϕ with the source given by the trace T of the energy-momentum tensor. More details on the derivation of these equations can be found in the appendix.

IV. WEAK FIELD LIMIT

In order to compute the BH shadow angle we need to determine the metric components near the event horizon, where the field will obviously be strong, cf. equation (9). However, it is illustrative and useful to first solve the equations in the weak field limit, since the exact solution must reduce to the weak field one far enough from the source and, hence, it can be employed to set the boundary conditions in the numerical analysis of Sec. V.

First of all, we consider the field equations (10) and (12) in vacuum ($T_{\mu\nu} = 0$):

$$G_{\mu\nu}\phi + \frac{\omega}{2\phi}(\nabla\phi)^2 g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi = 0, \quad (13)$$

$$\Box\phi = 0. \quad (14)$$

In the weak field limit, we treat the metric as a small perturbation over Minkowski spacetime and expand ϕ to first order. That is,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, |h_{\mu\nu}| \ll 1. \quad (15)$$

$$\phi = \phi_0 + \varepsilon, |\varepsilon| \ll \phi_0. \quad (16)$$

We will only consider first-order terms in the perturbations. Now let us focus on a static, spherically symmetric spacetime. The line element will be given by

$$ds^2 = [-1 + f(r)]\, dt^2 + [1 + g(r)]\, dr^2 + r^2 d\Omega^2, \quad (17)$$

For an asymptotically flat spacetime the conditions $f(r)$, $g(r)$, $\varepsilon(r) \longrightarrow 0$ must be satisfied when $r \longrightarrow \infty$. In this approximation, the equations reduce to

$$R_{\mu\nu}\phi_0 - \nabla_\mu\nabla_\nu\varepsilon(r) = 0, \quad (18)$$

$$\nabla^2\varepsilon(r) = 0. \quad (19)$$

The solution to (19) is straightforward:

$$\varepsilon(r) = \frac{C_1}{r}. \quad (20)$$

The equations for $f(r)$ and $g(r)$ are obtained from the tt and rr components of (18) after computing the Christoffel symbols $\Gamma^\sigma_{\mu\nu} = 1/2 g^{\sigma\rho}(g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho})$ and the Ricci tensor $R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \partial_\nu\Gamma^\sigma_{\mu\sigma} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}$ to first order. We get

$$\frac{1}{2}f''(r) + \frac{f'(r)}{r} = 0 \implies f(r) = \frac{C_2}{r}, \quad (21)$$

$$g'(r) - \left(\frac{2C_1}{\phi_0} + C_2\right)\frac{1}{r^2} = 0 \implies g(r) = -\frac{\frac{2C_1}{\phi_0} + C_2}{r}. \quad (22)$$

Three of the integration constants have already been set to 0 so that spacetime is asymptotically flat. To fix the remaining ones, we need three conditions. In the first place, let us consider a point-like source: $T_{00} = M\delta^{(3)}(\vec{r})$. We can put this into (12) and solve for C_1, taking into account that $\nabla^2(1/r) = -4\pi\delta^{(3)}(\vec{r})$. We find $C_1 = 2M/(2\omega+3)$. We also know that in the Newtonian limit $h_{00} = -2\phi_N$, where $\phi_N = GM/r$ is the Newtonian potential. This fixes $C_2 = -2GM$. Finally, ϕ_0 is related to G by (see [1])

$$\frac{1}{\phi_0} = \frac{2\omega+3}{2\omega+4}G. \quad (23)$$

The solution in the weak field approximation is then

$$\boxed{\varepsilon(r) = \frac{2M}{(2\omega+3)r}}, \quad (24)$$

$$\boxed{ds^2 = -\left[1 - \frac{2M}{r}\right]dt^2 + \left[1 + \frac{1+\omega}{2+\omega}\frac{2M}{r}\right]dr^2 + r^2 d\Omega^2}, \quad (25)$$

which agrees with [1]. Notice that the solution reduces to the Schwarzschild metric when $\omega \longrightarrow \infty$.

V. NUMERICAL SOLUTION

Now we will consider the full equations in vacuum for a static, spherically symmetric spacetime, (13) and (14). With the metric (2) we can compute the Christoffel symbols, the Ricci tensor, the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ and finally the Einstein tensor $G_{\mu\nu}$. We get (see the appendix)

$$G_{tt} = -\frac{A}{r^2 B} + \frac{A}{r^2} + \frac{B'A}{rB^2}, \quad (26)$$

$$G_{rr} = \frac{A'}{Ar} - \frac{B}{r^2} + \frac{1}{r^2}. \quad (27)$$

A prime indicates a derivative with respect to r. Plugging (26) and (27) into (13) gives

$$\frac{-B'}{Br} + \frac{1-B}{r^2} + \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \frac{A'}{2A}\frac{\phi'}{\phi} = 0, \quad (28)$$

$$\frac{A'}{Ar} + \frac{1-B}{r^2} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \frac{B'}{2B}\frac{\phi'}{\phi} = 0. \quad (29)$$

The third equation is obtained after expanding (14):

$$\frac{\phi''}{\phi'} - \frac{B'}{2B} + \frac{A'}{2A} + \frac{2}{r} = 0. \quad (30)$$

Fortunately, (30) can easily be integrated once:

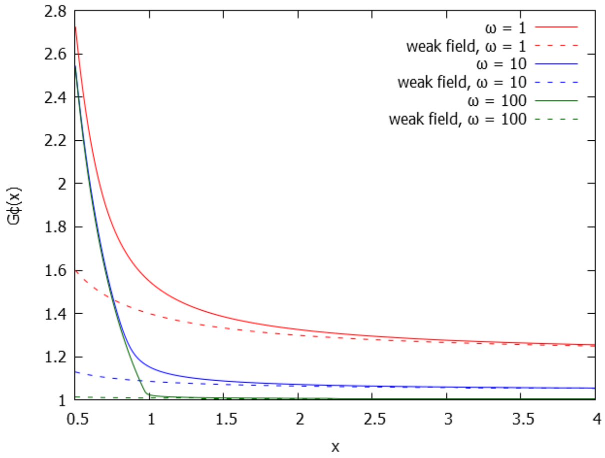

FIG. 2: Numerical and weak field solutions of the scalar field $\phi(x)$ for different values of ω.

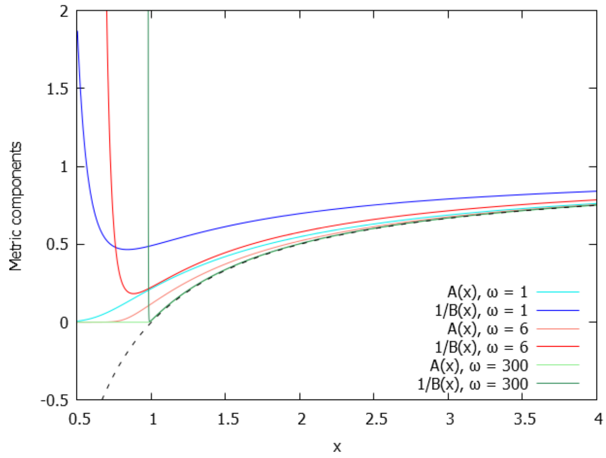

FIG. 3: Numerical solution of the metric components A(x) and 1/B(x) for different values of ω. The dashed line corresponds to the Schwarzschild solution.

$$\frac{d}{dr}\left[\ln\left(\phi' r^2\sqrt{\frac{A}{B}}\right)\right] = 0 \implies \phi' r^2\sqrt{\frac{A}{B}} = K, \tag{31}$$

where K is an integration constant. We can also simplify (29) by plugging ϕ'' from (30). We get

$$\frac{A'}{Ar} + \frac{1-B}{r^2} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 + \frac{A'}{2A}\frac{\phi'}{\phi} + \frac{2}{r}\frac{\phi'}{\phi} = 0. \tag{32}$$

Let us now introduce dimensionless variables: $x \equiv r/2M$, $y \equiv \ln(G\phi)$. With these changes, the dimensionless equations become

$$x^2\sqrt{\frac{A}{B}}y'e^y = C, \tag{33}$$

$$\frac{A'}{Ax} + \frac{1-B}{x^2} - \frac{\omega}{2}(y')^2 + \frac{A'}{2A}y' + \frac{2}{x}y' = 0, \tag{34}$$

$$-\frac{B'}{Bx} + \frac{1-B}{x^2} + \frac{\omega}{2}(y')^2 - \frac{A'}{2A}y' = 0. \tag{35}$$

Now a prime indicates a derivative with respect to x. (33), (34) and (35) form a system of coupled ordinary differential equations with boundary conditions at infinity for the unknown functions $y(x)$, $A(x)$ and $B(x)$, which describe spacetime. No analytical solution has been found without some simplifying assumptions [6]. Instead, we will solve them numerically. To do this, we choose an initial point $x_0 \gg 1$ and make A, B and y match the weak field solution there. The program I have developed, which can be found in [10], solves the equations with the RK4 algorithm for increasing values of x_0 until the variation of the solution near $x = 1$ with respect to the previous iteration falls below a desired precision.

VI. NAKED SINGULARITIES

As we can see in FIG. 2 and FIG. 3 , all functions approach the weak field solution for large x. Moreover, for $x > 1$ and $\omega \gg 1$ we recover the Schwarzschild solution from GR, as expected. Therefore, we can be confident in our derivation and resolution of the equations.

The behavior of the solutions for $x < 1$ is drastically different. As ω increases, A tends to zero and 1/B and ϕ grow more rapidly. What is more, A and 1/B never cross the x axis for any finite ω. This came as a surprise to us, since we expected A and 1/B to resemble the Schwarzschild solution for large ω and thus to have roots near $x = 1$. However, any finite value of the coupling constant (which results in a non-trivial scalar field) prevents this from happening. Only when ω is strictly infinite do we recover classical GR and A and 1/B vanish at $x = 1$. We can actually see this in equation (33). The only way for A and 1/B to vanish is to have $C = 0$, but this is only possible if $y' = 0$ everywhere, which means that the scalar field is constant and we recover GR.

This has a profound implication: there seem to be no event horizons in BD theory (the presence of an event horizon is indicated by A and 1/B crossing zero). Thus, we do not have a BH, but a naked singularity (a singularity causally connected to the rest of the Universe). Indeed, let us prove that if A and 1/B never vanish and remain positive for all r, then there is no event horizon. We will do this by showing that a photon moving radially can always escape to infinity. To do this, we take the metric (2), make θ and φ constant and set $ds^2 = 0$. Thus, the radial null geodesics equation is

$$\frac{dr}{dt} = \pm\sqrt{\frac{A(r)}{B(r)}}. \tag{36}$$

Now it becomes clear that all outgoing photons reach infinity, as there are no turning points in their trajectory.

There is a theorem (Hawking, 1972) which states that stationary BH's in BD theory are indistinguishable from BH's in GR [7]. The gravitational collapse leading to their formation will always result in a constant scalar field. Hawking proves this by assuming the null energy condition

$$R_{ab}l^a l^b \geq 0 \; \forall \text{ null vectors } l^a, \tag{37}$$

which implies that stationary black holes must be static or axially symmetric, and have spherical topology. From this it follows that the scalar field must be constant everywhere. Despite this, Hawking assumes the existence of a BH with a regular event horizon and does not contemplate naked singularities. For that reason our results are compatible with his theorem. It is also widely accepted (but has not been proven) that naked singularities cannot exist (they must be hidden behind a horizon). This is the so-called cosmic censorship conjecture. Singularities are points with infinite curvature where the laws of physics break down. If naked singularities existed, they could influence the outside Universe, causing serious problems to causality and predictability [8]. Nevertheless, the validity of this conjecture is controversial, as it has been seen that regular initial data can sometimes lead to a naked singularity with similar behavior to a BH [9].

With all of this in mind, the question of whether our solutions are unphysical is open and depends on the validity of the cosmic censorship conjecture. Anyway, no constraints on ω can be obtained from observations of BH shadows, since BD and Schwarzschild black holes cannot be told apart.

VII. CONCLUSIONS

In this TFG we have explored black holes in Brans-Dicke theory. We can summarize our results as follows:

- We have presented BD theory as a potential alternative to GR and have highlighted the importance of using strong-field observations, such as BH shadows measurements, to constrain theories beyond GR.

- We have extended Synge's work to find a general expression for a BH shadow's angle in a static and spherically symmetric spacetime in alternative theories of gravity.

- We have derived the field equations in BD theory from its action and have solved them both for a weak and a strong field regimes using analytical and numerical tools, respectively.

- For finite ω we have obtained naked singularities, from which photons can escape. For this solution there is no event horizon. BHs can form in BD from the collapse of massive objects, but they are indistinguishable from those in GR. Therefore, the BH shadow's angle cannot be employed to constrain the BD theory. In passing, we have also discussed the cosmic censorship conjecture.

The main conclusion is then that no constraints on ω can be obtained from static black holes and that other kinds of observations (cosmological, on rotating black holes, etc.) must be used.

Acknowledgments

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Ombres de forats negres i la teoria de Brans-Dicke

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Resum: Una possible alternativa a la Relativitat General (RG) és la teoria de Brans-Dicke (BD), una teoria escalar-tensorial en què la gravetat és mediada tant per un tensor simètric $g_{\mu\nu}$ (la mètrica) com per un camp escalar ϕ. En aquest treball obtenim una expressió per a l'angle de l'ombra d'un forat negre, vàlida per a fonts estàtiques i esfèricament simètriques en RG i més enllà, deduïm les equacions de camp de BD a partir de l'acció i les solucionem per a un espai-temps estàtic i esfèricament simètric en el límit de camp feble (analíticament) i per a un camp fort (numèricament). Mostrem que per a valors finits del paràmetre d'acoblament ω obtenim singularitats nues, l'existència de les quals és controvertida. Finalment, concloem que no es pot obtenir cap restricció sobre la teoria a partir de l'observació d'ombres de forats negres, ja que els forats negres estàtics de BD són idèntics als de RG.
Paraules clau: Relativitat General, teoria de Brans-Dicke, forats negres.
ODSs: Aquest TFG està relacionat amb els Objectius de Desenvolupament Sostenible (SDGs)

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

Aquest TFG de Física té relació amb l'ODS 4, ja que contribueix a l'educació, recerca i comunicació científiques.

APPENDIX

A. Variations of the action

In order to apply variations to the action, we need the following equalities, which can be found in any introductory GR textbook:

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}, \tag{A.1}$$

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{A.2}$$

(A.1) is Jacobi's formula for the variation of the determinant of a matrix and (A.2) the usual definition of the energy-momentum tensor. As for the Ricci scalar R, its variation is

$$\delta R = \delta g^{\rho\sigma} R_{\rho\sigma} + \nabla_\lambda \left[g_{\mu\nu} \nabla^\lambda \delta g^{\mu\nu} - \nabla_\nu \delta g^{\lambda\nu} \right]. \tag{A.3}$$

The second term has to be integrated by parts several times with the condition that variations vanish on the boundary. With all of this and after some tedious calculations, equations (10) and (11) can be obtained.

B. Strong field equations

The only independent, non-vanishing Christoffel symbols $\Gamma^\sigma_{\mu\nu} = 1/2 g^{\sigma\rho}(g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho})$ are

$$\Gamma^t_{tr} = \frac{A'}{2A}, \Gamma^r_{tt} = \frac{A'}{2B}, \Gamma^r_{\theta\theta} = \frac{-r}{B}, \Gamma^r_{rr} = \frac{B'}{2B}, \Gamma^\varphi_{\theta\varphi} = \cot\theta,$$

$$\Gamma^r_{\varphi\varphi} = \frac{-r\sin^2\theta}{B}, \Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \Gamma^\theta_{\varphi\varphi} = -\cos\theta\sin\theta. \tag{B.1}$$

The Ricci tensor is given by $R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}$. The diagonal components are

$$R_{tt} = \frac{A''}{2B} - \frac{A'B'}{4B^2} - \frac{A'A'}{4AB} + \frac{A'}{Br}, \tag{B.2}$$

$$R_{rr} = \frac{-A''}{2A} + \frac{A'A'}{4A^2} + \frac{A'B'}{4AB} + \frac{B'}{Br}, \tag{B.3}$$

$$R_{\theta\theta} = 1 - \frac{1}{B} + \frac{rB'}{2B^2} - \frac{rA'}{2AB}, \tag{B.4}$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta}. \tag{B.5}$$

We can now compute the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$. The result is

$$R = \frac{-A''}{AB} + \frac{A'B'}{2AB^2} + \frac{A'A'}{2A^2B} - \frac{2A'}{ABr} - \frac{2}{r^2B} + \frac{2}{r^2} + \frac{2B'}{rB^2}. \tag{B.6}$$

With the Ricci tensor and scalar we can construct the Einstein tensor $G_{\mu\nu}$. The tt and rr components are given in equations (26) and (27), respectively.