UNIVERSITAT DE BARCELONA

Facultat de Matemàtiques
i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

An Introduction to Complex Analysis in Several Variables: Riemann Mapping and Bergman Spaces

Autor: Guillem Vilaseca Vinadé

Director: Dr. Jordi Marzo Sànchez
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 15 de gener de 2025

Abstract

The main goal of this work is to give an introduction of the fundamental concepts in complex analysis in several variables.

It starts by introducing holomorphic functions of several complex variables, their representation via power series, and fundamental results like the Cauchy integral formula.

Then it follows by the Riemann mapping theorem, a cornerstone result that guarantees the existence of conformal mappings between simply connected domains and the unit disc in $\mathbb{C}$. We show also that the Riemann mapping theorem cannot be extended to $\mathbb{C}^n$.

Finally, the last part of the report delves into Bergman spaces, studying their kernels and their connection to the Riemann Mapping Theorem.

2020 Mathematics Subject Classification. 30E20, 32A36, 32A05, 30C35

Agraïments

En primer lloc, vull agrair al meu tutor, el Jordi Marzo, per guiar-me i donar-me suport durant tot aquest treball, sense ell res d'això s'hauria dut a terme.

A l'Aleix i el Pol, companys durant la carrera i amics fora d'ella. Els anys a la facultat han estat molt més entretinguts gràcies a ells.

A la Noelia, que m'ha fet revifar la flama de les matemàtiques durant aquests últims mesos amb la seva passió per l'acadèmia.

Per últim, als meus pares i l'àvia Montserrat. Sempre han posat de la seva part per simplificar-me la vida. Gràcies.

Contents

Introduction

This report is dedicated to exploring key concepts and results in complex analysis, with a particular focus on their extension to several variables. The study begins by introducing the notion of *homomorphic function* in an open subset of

$$\mathbb{C}^n = \{z = (z_1, \dots, z_n) \mid z_i \in \mathbb{C}, i = 1, \dots, n\},$$

which is the Cartesian product of n copies of $\mathbb{C}$. It is followed by fundamental definitions and elementary theorems such as the extended version in several variables of results like the Cauchy integral formula or Cauchy-Riemann equations. In particular, we present, and prove from scratch, Montel's theorem, which can be seen as a complex analog of the Arzelà-Ascoli theorem. Montel's theorem states that *every sequence $f_1, f_2, \dots$ of holomorphic functions in an open set $\Omega \subset \mathbb{C}$ that is locally bounded in Ω has a subsequence that converges compactly in Ω.*

In the second chapter, we define the concept of biholomorphic invariance between domains, laying the groundwork for understanding equivalence in the context of holomorphic mappings. This leads to the statement and proof of the Riemann mapping theorem, which states that *if $D \subset \mathbb{C}$ is a simply connected open set which is not all of $\mathbb{C}$, then D is conformally equivalent to the unit disc*. The chapter is followed by a Poincare's theorem which proves that *there exists no biholomorphic map between the polydisc and the ball in $\mathbb{C}^n$, if $n > 1$*, showing that the Riemann mapping theorem cannot be extended to $\mathbb{C}^n$.

The final chapter introduces, given $\Omega \subset \mathbb{C}^n$, the notion of Bergman spaces

$$A^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} : \|f\|^2 = \int_\Omega |f(z)|^2 d\lambda(z) < \infty \right\},$$

where λ is the Lebesgue measure. Bergman spaces are a type of Hilbert space of functions and have a reproducing kernel. This means that *given any $z \in \Omega$ fixed there exists a function $K(z,w) \in A^2(\Omega)$, named Bergman kernel of Ω, such that for all $f \in A^2(\Omega)$*

$$f(z) = \int_\Omega K(z,w) f(w) d\lambda(w).$$

The chapter is followed by several examples of Bergman kernels, such as the Bergman kernel of the unit disc or unit polydisc in $\mathbb{C}^n$. Finally, we demonstrate the relationship between the Bergman kernel and the Riemann mapping: *let* $\Omega \subset \mathbb{C}$ *be a simply connected domain and let* $K(z, w)$ *be the Bergman kernel of* Ω. *Let* $F : \Omega \to \mathbb{D}$ *be the Riemann mapping with the uniqueness properties* $F(a) = 0$ *and* $F'(a) > 0$ *for some* $a \in \Omega$. *Then*

$$F'(z) = \sqrt{\frac{\pi}{K(a,a)}} K(z,a), \quad z \in \Omega.$$

The study of these concepts has been conducted mainly following the guidance provided by *Holomorphic Functions and Integral Representations in Several Complex Variables*, Chapter 1 by R. Michael Range, *Theory of Complex Functions*, Chapters 2, 3 by Reinhold Remmert, *Comlpex Made Simple*, Chapter 8, 9 by David C. Ullrich, *Complex Analysis*, Chapter 8 by Elias M. Stein & Rami Shakarchi and *Complex Analysis*, Chapter 7 by Friedrich Haslinger.

Chapter 1

Holomorphic functions

In this chapter, we will extend classical complex analysis results from one variable to several variables. We will begin by reviewing key concepts like the complex Euclidean space and the Cauchy-Riemann equations. By drawing analogies with one-variable theory, we will explore the Cauchy integral formula, holomorphic maps, and power series in higher dimensions.

In this chapter we follow the references [Ran86, Chap. 1], [Rem91, Chap. 3] and [Ort97, Chap. 4].

1.1 Complex Euclidean Space

For $n \in \mathbb{N}$, we define the n-dimensional complex number space as

$$\mathbb{C}^n = \{z = (z_1, \dots, z_n) \mid z_i \in \mathbb{C},\ i = 1, \dots, n\}.$$

$\mathbb{C}^n$ is the Cartesian product of n copies of $\mathbb{C}$ and carries the structure of an n-dimensional complex vector space.

Definition 1.1. (Hermitian Inner Product) *Let V be a complex vector space. A Hermitian inner product on V is a function*

$$\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C},$$

which is, for every $u, v \in V$:

- ***Conjugate-symmetric**: $\langle u, v\rangle = \overline{\langle v, u\rangle}$.*
- ***Linear on the first factor**: $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$, for all scalars λ and $\langle u_1, u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$, for all vectors u_1, u_2.*
- ***Positive**: $\langle u, v\rangle \geq 0$.*

- ***Non-degenerate***: *if* $\langle u, v \rangle = 0$ *for every* v *then* $u = 0$.

We say that V *is a complex inner product space.*

The classic example of a Hermitian inner product space is the standard one on $\mathbb{C}^n$,

$$\langle a, b \rangle = \sum a_i \bar{b}_i,$$

for $a, b \in \mathbb{C}^n$. One can define the associated norm $|a| = \langle a, a \rangle^{1/2}$ that induces the Euclidean metric in the usual way: $\text{dist}(a, b) = |a - b|$.

Definition 1.2. (Open ball) *The open ball of radius* $r > 0$ *and center* $a \in \mathbb{C}^n$ *is defined by*

$$B(a, r) = \{z \in \mathbb{C}^n : \ |z - a| < r\}.$$

Given $z = (z_1, \ldots, z) \in \mathbb{C}$, each coordinate z_j can be written as $z_j = x_j + iy_j$, with $x_j, y_j \in \mathbb{C}$, where i is the imaginary unit $\sqrt{-1}$.
The mapping

$$z \to (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$$

establishes an $\mathbb{R}$-linear isomorphism between $\mathbb{C}$ and $\mathbb{R}$. A ball $B(a, r)$ in $\mathbb{C}^n$ is identified with an Euclidean ball in $\mathbb{R}^{2n}$ of equal radius r. Due to this identification, all the usual concepts from topology and analysis on real Euclidean spaces $\mathbb{R}^{2n}$ carry over immediately to $\mathbb{C}^n$.

Definition 1.3. *A set* $\Omega \subset \mathbb{C}^n$ *is said to be* ***open*** *if for every* $a \in \Omega$ *there exists a ball* $B(a, r) \subset \Omega$ *with* $r > 0$.

From now on, unless specified otherwise, Ω will denote an open set in $\mathbb{C}^n$; such Ω will also be called a **domain**, or a **region**.

Definition 1.4. (Open polydisc) *The open polydisc of center* $a \in \mathbb{C}^n$ *and multiradius* $r = (r_1, \ldots, r_n), r_j > 0$ *is defined as the Cartesian product of* n *open discs in* $\mathbb{C}$:

$$P(a, r) = \{z \in \mathbb{C}^n : \ |z_j - a_j| < r_j, \ 1 \leq j \leq n\}.$$

For the sake of simplicity, sometimes we will refer to $P(a, r)$ as P.

1.2 The Cauchy-Riemann equations

Notation 1.5. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^N$. *We call* α *a multiindex and define*

$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$
$$\alpha + 1 := (\alpha_1 + 1, \ldots, \alpha_n + 1),$$
$$\alpha! := \alpha_1! \cdots \alpha_n!.$$

For $z \in \mathbb{C}^n$ and a multiindex a we write

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

and we define the partial derivative operators

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$

For $\Omega \in \mathbb{C}^n$, open, and $k \in \mathbb{N} \cup \{\infty\}$, we define $C^k(\Omega)$ as the space of k times continuously differentiable complex values functions on Ω. (We write $C(\Omega)$ instead of $C^0(\Omega)$).
Turning to $\mathbb{C}^n$, we define

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$
$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

for $j = 1, \ldots, n$, which are named Wirtinger derivatives.

Definition 1.6. *Let $\Omega \subset \mathbb{C}^n$ be open. A function $f : \Omega \to \mathbb{C}$ is called **holomorphic** at a point $a \in \Omega$ if $f \in \mathcal{C}^1(\Omega)$ and satisfies the Cauchy-Riemann equations at a:*

$$\frac{\partial f(a)}{\partial \bar{z}_j} = 0 \quad \text{for } 1 \leq j \leq n.$$

A function f is called holomorphic on Ω if $\forall a \in \Omega, f$ is holomorphic at a. The space of holomorphic functions is denoted as $H(\Omega)$.

1.3 The Cauchy integral formula on polydiscs

A function $f : \Omega \to \mathbb{C}^n$ is said to be holomorphic in each variable separately if for every $z \in \Omega$ and $1 \leq j \leq n$, the function $f_{z_j}(\lambda) = f(z_1, \ldots, z_{j-1}, \lambda, z_{j+1}, \ldots, z_n)$ is holomorphic on Ω.

Definition 1.7. (Distinguished boundary) *The distinguished boundary of a polydisc $P(a,r) \subset \mathbb{C}^n$ is defined as $b_oP = \{w \in \mathbb{C}^n : |w_j - a_j| = r_j,\ 1 \leq j \leq n\}$*

Notice that b_oP is strictly smaller than the topological boundary ∂P of P in case that $n > 1$.
For any function in several complex variables $g \in \mathcal{C}(b_oP)$, in terms of the standard parametrization

$$w_z = a_j + r_j e^{i\theta}$$

of $b_o P(a,r)$, one has

$$\int_{b_o P(a,r)} g(w) dw_1 \dots dw_n = i^n r_1 \dots r_n \int_{[0,2\pi]^n} g(w(\theta)) e^{i\theta_1} \dots e^{i\theta_n} d\theta_1 \dots d\theta_n. \tag{1.1}$$

This property will be useful to prove the following theorem.

Theorem 1.8. (Cauchy Integral Formula) *Let $P(a,r) = P$ be a polydisc in $\mathbb{C}^n$ with multiradius $r = (r_1, \dots, r_n)$. Suppose $f \in \mathcal{C}(\overline{P})$, and f is holomorphic in each variable separately on $\{\lambda \in \mathbb{C} : |\lambda - a_j| < r_j\}$. Then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{b_o P} \frac{f(w) dw_1 \dots dw_n}{(w_1 - z_1) \dots (w_n - z_n)} \quad \text{for } z \in P, \tag{1.2}$$

where $b_o P = \{w \in \mathbb{C}^n : |w_j - a_j| = r_j,\ 1 \le j \le n\}$.

Proof. We will use induction over the number of variables n in f. The case $n = 1$ is the classical Cauchy integral formula, which we assume as known. Suppose that for $n > 1$ the theorem has been proved for $n - 1$ variables. Let us pick $z = (z_1, \dots, z_n) \in P$ fixed and the function $f(z_1, \dots, z_n)$. Applying the inductive hypothesis with respect to $(z_2, \dots, z_n)$ one obtains:

$$f(z_1, z_2, \dots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{b_o P'(a',r')} \frac{f(z_1, w_2, \dots, w_n) dw_2 \dots dw_n}{(w_2 - z_2) \dots (w_n - z_n)} \tag{1.3}$$

where $a' = (a_2, \dots, a_n)$ and $r' = (r_2, \dots, r_n)$.
Fixing $w_2, \dots, w_n$, the classic Cauchy Integral Formula gives us:

$$f(z_1, w_2, \dots, w_n) = \frac{1}{2\pi i} \int_{|w_1 - a_1| = r_1} \frac{f(w_1, \dots, w_n) dw_1}{(w_1 - z_1)}, \tag{1.4}$$

and substituting 1.4 into 1.3 we obtain:

$$f(z_1, \dots, z_n) = \frac{1}{(2\pi i)^n} \int_{b_o P'(a',r')} \left(\int_{|w_1 - a_1| = r_1} \frac{f(w_1, \dots, w_n) dw_1}{(w_1 - z_1) \dots (w_n - z_n)} \right) dw_2 \dots dw_n.$$

Let $g(w) = \frac{f(w_1, \dots, w_n)}{(w_1 - z_1) \dots (w_n - z_n)}$, using the parametrization seen above (1.1) along with Fubini's theorem one gets

$$\begin{aligned} \int_{b_o P'(a',r')} \left(\int_{|w_1 - a_1| = r_1} g(w) dw_1 \right) dw_2 \dots dw_n &= i^n r_1 \dots r_n \int_{[0,2\pi]^n} g(w(\theta)) e^{i\theta_1} \dots e^{i\theta_n} d\theta \\ &= \int_{b_o P(a,r)} g(w) dw \\ &= \int_{b_o P(a,r)} \frac{f(w_1, \dots, w_n) dw_1 \dots dw_n}{(w_1 - z_1) \dots (w_n - z_n)}, \end{aligned}$$

and the result has been proven. □

As in the case of one complex variable, there exists the Cauchy integral formula for derivatives

$$D^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n}\int_{b_oP}\frac{f(w)dw_1\dots dw_n}{(w_1-z_1)^{\alpha_1+1}\dots(w_n-z_n)^{\alpha_n+1}}, \tag{1.5}$$

it can be deduced from applying the Cauchy integral formula (1.2) and differentiating under the integral symbol but we will not write it down in these notes.

Theorem 1.9. (Cauchy estimate) *Let $f \in H(P(a,r))$, then for all $\alpha \in \mathbb{N}^n$,*

$$|D^\alpha f(a)| \leq \frac{\alpha!}{r^\alpha}|f|_{P(a,r)}. \tag{1.6}$$

Proof. Fix $0 < p < r$, by Cauchy Integral theorem by derivatives (1.5), it is know that

$$D^\alpha f(a) = \frac{\alpha!}{(2\pi i)^n}\int_{b_oP(a,p)}\frac{f(w)dw_1\dots dw_n}{(w-a)^{\alpha+1}} \tag{1.7}$$

where $P(a,p) \subset P(a,r)$. Applying $|\cdot|$ on both sides of the equation

$$\begin{aligned}|D^\alpha f(a)| &= \left|\frac{\alpha!}{(2\pi i)^n}\int_{b_oP(a,p)}\frac{f(w)dw_1\dots dw_n}{(w-a)^{\alpha+1}}\right| \\ &\leq \frac{\alpha!}{(2\pi)^n}\int_{b_oP(a,p)}\frac{|f(w)|}{|(w-a)^{\alpha+1}}|dw| \\ &\leq \frac{\alpha!}{(2\pi)^n}\cdot\frac{|f|_{P(a,p)}}{p^{\alpha+1}}\cdot\int_{b_oP(a,p)}|dw| \\ &= \frac{\alpha!|f|_{P(a,p)}(2\pi)^n p}{(2\pi)^n p^{\alpha+1}} = \frac{\alpha!}{p^\alpha}|f|_{P(a,p)}\end{aligned}$$

It has been seen that, for $0 < p < r$, $|D^\alpha f(a)| \leq \frac{\alpha!}{p^\alpha}|f|_{P(a,p)}$. Now, making $p \to r$ the result is proven. □

1.4 Holomorphic maps

Let $\Omega \subset \mathbb{C}^n$ be open and consider a map $F : \Omega \to \mathbb{C}^m$. By writing the map as $F = (f_1,\dots,f_n)$ and $f_j = u_j + iv_j$, for $j = 1,\dots,n$, where u_j, v_j are real valued functions on Ω, we can view $F = (u_1, v_1,\dots,u_m, v_m)$ as a map from $\Omega \subset \mathbb{R}^2 n$ into R^{2m}. If F is differentiable at $a \in \Omega$, its differential $dF(a) : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is a linear transformation with matrix representation given by the (real) Jacobian matrix

$$J_{\mathbb{R}}(F) = \begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial v_m}{\partial x_1} & \cdots & \cdots & \frac{\partial v_m}{\partial y_n}\end{bmatrix}.$$

Definition 1.10. *The map $F : \Omega \to \mathbb{C}^m$ is called* ***holomorphic*** *if its components $f_1, \dots, f_n$ are holomorphic functions on Ω. If F is holomorphic, its differential $dF(a)$ at $a \in \Omega$ is a complex linear map $\mathbb{C}^n \to \mathbb{C}^m$, with the following matrix representation*

$$F'(a) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1}(a) & \cdots & \frac{\partial f_1}{\partial z_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial z_1}(a) & \cdots & \frac{\partial f_m}{\partial z_n}(a) \end{bmatrix}.$$

We call $F'(a)$ the derivative (or complex Jacobian matrix) of the holomorphic map F at a.

Lemma 1.11. *If $\Omega \subset \mathbb{C}^n$ and $F : \Omega \to \mathbb{C}^n$ is holomorphic, then*

$$\det J_{\mathbb{R}} F(z) = |\det F'(z)|^2 \geq 0$$

for $z \in \Omega$.

Proof. After permuting the rows and columns one can write

$$\det J_{\mathbb{R}} F = \det \begin{bmatrix} \left(\frac{\partial u_k}{\partial x_j}\right) & \cdots & \left(\frac{\partial u_k}{\partial y_j}\right) \\ \vdots & & \vdots \\ \left(\frac{\partial v_k}{\partial x_j}\right) & \cdots & \left(\frac{\partial v_k}{\partial y_j}\right) \end{bmatrix},$$

Now subtract i times the left blocks from the right side; it follows that

$$\det J_{\mathbb{R}} F = \det \begin{bmatrix} \left(\frac{\partial f_k}{\partial x_j}\right) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \left(\frac{\partial \overline{f_k}}{\partial x_j}\right) \end{bmatrix} = \det F' \cdot \overline{\det F'},$$

where we have used that $\partial f / \partial z_j = \partial f / \partial x_j$ for holomorphic f. □

1.5 Sequences in spaces of holomorphic functions

In this section we plan to discuss the concepts of locally uniform, compact, and normal convergence in $\mathbb{C}$, these concepts will be used lately to study the Riemann Mapping theorem.

Let $X \subset \mathbb{C}$ be any non-empty set and metric space. A sequence f_n of complex-valued functions $f_n : X \to \mathbb{C}$ is said to be convergent at the point $a \in X$ if the sequence $f_n(a)$ of complex numbers converges in $\mathbb{C}$.

Definition 1.12. *A convergent sequence of functions f_n is called* ***pointwise convergent*** *in a subset $A \subset X$ if it converges at every point of A: then the limit function $f : A \to \mathbb{C}$ is defined via*

$$f(x) := \lim f_n(x), \quad x \in A.$$

Along real-valued functions simple examples show how pointwise convergent sequences can have bad properties: the continuous functions x^n on the interval $[0,1]$ converge pointwise there to a limit function which is discontinuous at the point 1. Such pathologies are eliminated by the introduction of the idea of locally uniform convergence.

Definition 1.13. *A sequence of functions f_n is said to be* ***uniformly convergent*** *in $A \subset X$ to $f : A \to \mathbb{C}$ if every $\epsilon > 0$ has an $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that*

$$|f_n(x) - f(x)| < \epsilon \text{ for all } n \geq n_0 \text{ and all } x \in A;$$

when this occurs the limit function f is uniquely determined.

Definition 1.14. *A sequence of functions $f_n : X \to \mathbb{C}$ is said to be* ***continuously convergent*** *in X, if for every convergent sequence $\{x_n\} \subset X$, the limit $\lim_{n\to\infty} f_n(x_n)$ exists in $\mathbb{C}$.*

Definition 1.15. *A sequence of functions $f_n : X \to \mathbb{C}$ is said to* ***converge compactly*** *in X if it converges uniformly on every compact subset of X.*

Lemma 1.16. *If the sequence f_n converges continuously on X to f, then f is continuous on X (even if the f_n are not themselves continuous).*

Proof. Consider any $x \in X$, any sequence $\{x_n\} \subset X$ convergent to x and any $\epsilon > 0$. There is a strictly increasing sequence $n_k \in \mathbb{N}$ such that $|f_{n_k}(x_k) - f(x_k)| < \epsilon/2$. Since, $\lim_k f_{n_k}(x_k) = f(x)$, there exists a k_ϵ such tat $|f_{n_k}(x_k) - f(x)| < \epsilon/2$ for all $k \geq k_\epsilon$. The continuity of f at x follows:

$$|f(x_k) - f(x)| \leq |f(x_k) - f_{n_k}(x_k)| + |f_{n_k}(x_k) - f(x)| < \epsilon \text{ for all } k \geq k_\epsilon.$$

□

Theorem 1.17. *If the sequence f_n is continuously convergent in X then f_n converges compactly in X to a function $f \in \mathcal{C}(X)$*

Proof. Let f be the limit function. We have seen in 1.16 that $f \in \mathcal{C}(X)$. Suppose there is a compact $K \subset X$ such that $|f - f_n|_K = \sup\{|f(x) - f_n(x)| : x \in K\}$ is not a null sequence. This means that there is an $\epsilon > 0$ and a subsequence n' of indices such that $|f - f_{n'}|_K > \epsilon$ for all n'. In turn the latter means that there are points $x_{n'} \in K$ such that

$$|f(x_{n'}) - f_{n'}(x_{n'})| > \epsilon \text{ for all } n'. \tag{1.8}$$

Because K is compact, we may assume, by passing to a further subsequence if necessary, that the sequence $x_{n'}$ converges, say to x. But then $\lim f(x_{n'}) = f(x)$ because of the continuity of f at x and $\lim f_{n'}(x_{n'}) = f(x)$ by hypothesis. Subtraction gives that $\lim[f(x_{n'}) - f_{n'}(x_{n'})] = 0$, contradicting 1.8. □

To prove Montel's theorem, we will begin by examining Weierstrass's theorem, which explores whether the limit of a holomorphic function remains holomorphic.

Theorem 1.18. (Weierstrass theorem) *Let $\Omega \subset \mathbb{C}$ be an open set and $f_n \in \mathcal{H}(\Omega)$ be a sequence of holomorphic functions in Ω, and suppose that for any compact $K \subset \Omega$, $f_n|_K$ converges to $f|_K$ uniformly in K. Then f is holomorphic in Ω and $f_n^{(k)}$ converges to $f^{(k)}$ uniformly in every compact set of Ω.*

Proof. Let R be a rectangle contained in Ω. As ∂R is compact in Ω, by the statement it is known that $f_n|_{\partial R} \to f|_{\partial R}$, then by the Cauchy-Goursat theorem

$$0 = \int_{\partial R} f_n(z)dz \to \int_{\partial R} f(z)dz$$

when $n \to \infty$.
We have seen that the integral of f on any border of any rectangle in Ω equals 0. Thus, by Morera's theorem[1] f is holomorphic in Ω. Having seen that f is holomorphic it remains to see that $f_n^{(k)} \to f^{(k)}$ uniformly on compact sets. Initially we will prove it on closed discs $\overline{D} \subset \Omega$. In fact, if $D = (a, r)$ and $\overline{D} \subset \Omega$, then there exists $R > r$ such that $\overline{D(a, R)} \subset \Omega$. In this case we can apply Cauchy integral formula in $(f_n^{(k)} - f^k)$ and obtain that for any $z \in \overline{D(a, r)}$,

$$(f_n^{(k)}(z) - f^k(z)) = \frac{k!}{2\pi i} \int_{|z-a|=R} \frac{f_n(w) - f(w)}{(w - z)^{k+1}} dw$$

Then for every $z \in \overline{D(a, r)}$

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k! M_n R}{(R - r)^{k+1}},$$

[1]Morera's theorem states that a continuous function f defined on an open set D in the complex plane that satisfies $\int_\gamma f(z)dz = 0$ for every closed curve γ in D must be holomorphic on D.

where $M_n = \sup\{|f_n(z) - f(z)|, |z-a| = R\}$. By the hypothesis $M_n \to 0$ so $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on $\overline{D(a,r)}$. Finally, if K is not a disc and it is an arbitrary compact in Ω, there exists a sequence of discs $\{D_i\}_{i=1}^n$ such that $K \subset D_1 \cup \cdots \cup D_n$. As $f_n^{(k)} \to f^k$ uniformly on every D_i, then $f_n^{(k)} \to f^k$ uniformly in K. □

Lemma 1.19. *Let $f_n : \Omega \to \mathbb{C}, n \in \mathbb{N}$, be a sequence of functions that is bounded at every point of Ω. Then for every countable subset A of Ω there exists a subsequence g_n of the sequence f_n that converges pointwise in A.*

Proof. Let $a_0, a_1, a_2, ...$ be an enumeration of A. For every $l \in \mathbb{N}$, there exists a subsequence $f_{l0}, f_{l1}, f_{l2}, ...$ of the sequence $f_0, f_1, f_2, ...$ such that:

a) The sequence $(f_{ln})_{n\geq 0}$ converges at a_l.

b) The sequence $(f_{ln})_{n\geq 0}, l \geq 1$, is a subsequence of $(f_{l-1,n})_{n\geq 0}$.

Given the sequences $(f_{kn})_{n\geq 0},\ k < l$, choose a subsequence $(f_{ln})_{n\geq 0}$ of the sequence $(f_{l-1,n})_{n\geq 0}$ which converges at a_l. Then a) and b) are satisfied for all sequences $(f_{kn})_{n\geq 0}, k < l$. From the sequences $f_{l0}, f_{l1}, f_{l2}, \ldots$, construct the diagonal sequence $g_0, g_1, g_2, \ldots$, where $g_n := f_{nn}, n \in \mathbb{N}$. It converges at every point $a_m \in A$ since, by b), from the term g_m on it is a subsequence of the sequence $f_{m0}, f_{m1}, f_{m2}, \ldots$, which converges at a_m by a). □

A family of functions $\mathcal{F} \subset \mathcal{H}(\Omega)$ is called bounded in a subset $A \subset \Omega$ if there exists $M > 0$ such that $|f|_A \leq M$ for any $f \in \mathcal{F}$.

Definition 1.20. *The family $\mathcal{F}$ is called locally bounded in Ω if every point $z \in \Omega$ has a neighborhood $U \subset \Omega$ such that $\mathcal{F}$ is bounded in U.*

Observe that this occurs if and only if $\mathcal{F}$ is bounded on every compact set in Ω. In particular, $\mathcal{F} \subset \mathcal{H}(B)$, where $B = B_r(0)$ is a disc of radius $r > 0$, is locally bounded in B if and only if it is bounded in every disc $B_p(0), p < r$.

Lemma 1.21. *Let $\mathcal{F} \subset \mathcal{H}(\Omega)$ be a locally bounded family in Ω. Then for every point $c \in \Omega$ and every $\epsilon > 0$, there exists a disc $D \subset \Omega$ around c such that*

$$|f(w) - f(z)| \leq \epsilon \quad \text{for all } f \in \mathcal{F} \text{ and all } w, z \in D.$$

Proof. We choose $r > 0$ such that $D_{2r}(c) \subset \Omega$. From the Cauchy integral formula it follows

$$f(w) - f(z) = \frac{1}{2\pi i} \int_{\partial D_{2r}(c)} f(\zeta) \left[\frac{1}{\zeta - w} - \frac{1}{\zeta - z} \right] d\zeta$$
$$= \frac{w - z}{2\pi i} \int_{\partial D_{2r}(c)} \frac{f(\zeta)}{(\zeta - w)(\zeta - z)} d\zeta$$

For any $w, z \in D_r(c)$ and $\zeta \in \partial D_{2r}(c)$, then $r^2 \leq |(\zeta - w)(\zeta - z)|$. Given that, we can find a bound for $|f(w) - f(z)|$ as seen below

$$|f(w) - f(z)| \leq \frac{|w - z|}{2\pi i} \int_{\partial D_{2r}(c)} \left| \frac{f(\zeta)}{(\zeta - w)(\zeta - z)} \right| d|\zeta|$$
$$\leq |w - z| \frac{2}{r} \cdot \sup\{|f|_{D_{2r}(c)}\},$$

for all $w, z \in D_r(c)$ and all $f \in \mathcal{F}$. Since $\mathcal{F}$ is locally bounded, we call $M := \frac{2}{r} \cdot \sup\{|f|_{D_{2r}(c)} : f \in \mathcal{F}\} < \infty$, it is enough to set $D := D_\rho(c)$ with

$$\rho := \min \left\{ \frac{\epsilon}{(2M)}, r \right\}$$

to prove that $|f(w) - f(z)| \leq \epsilon$ for all $f \in \mathcal{F}$ and all $w, z \in D$. Given the defined disc it is obvious that $|w - z| \leq 2\rho$, suppose that $\rho = \frac{\epsilon}{2M}$ then:

$$|f(w) - f(z)| \leq |w - z| \cdot M \leq \epsilon$$

□

Theorem 1.22. (Montel's theorem) *Every sequence $f_0, f_1, f_2...$ of holomorphic functions in Ω that is locally bounded in Ω has a subsequence that converges compactly in Ω.*

Proof. Let $A \subset \Omega$ be a countable dense set, for instance the set of all rational complex numbers in Ω. By Lemma 1.19, there exists a subsequence g_n of the sequence f_n that converges pointwise in A. We want to prove that g_n converges compactly in Ω. To prove this, it is only necessary to prove that it converges continuously in Ω, i.e that:

$$\lim g_n(z_n) \text{ exists for every sequence } z_n \in \Omega \text{ with } \lim_{n\to\infty} z_n = z^* \in \Omega$$

Let $\epsilon > 0$. By Lemma (1.21), there exists a disc $D \in \Omega$ around $z*$ such that $|g_n(w) - g_n(z)| \leq \epsilon \ \forall n$ if $w, z \in D$. Since A is dense in Ω, there exists a point

$a \in A \cap D$. Since $z_n \to z*$, there exists $n_1 \in \mathbb{N}$ such that $z_n \in D$ for all $n \geq n_1$. The following inequality

$$|g_m(z_m) - g_n(z_n)| \leq |g_m(z_m) - g_m(a)| + |g_m(a) - g_n(a)| + |g_n(z_n) - g_n(a)|$$

always holds; hence $|g_m(z_m) - g_n(z_n)| \leq 2\epsilon + |g_m(a) - g_n(a)|$ for all $m, n \geq n_1$. Since $\lim g_n(a)$ exists, there is an n_2 such that $|g_m(a) - g_n(a)| \leq \epsilon$ for all $m, n \geq n_2$. We have proved that $|g_m(z_m) - g_n(z_n)| \leq 3\epsilon$ for all $m, n \geq \max(n1, n2)$; thus the sequence $g_n(z_n)$ is a Cauchy sequence and therefore convergent. □

Remark 1.23. The assertion of the theorem is false for a sequence of real-analytic functions. For example, the sequence

$$f_n(x) = sin(nx),$$

$n \in \mathbb{N}$ is bounded in $\mathbb{R}$ but does not even have pointwise convergent subsequences.

Given $\Omega \subset \mathbb{C}$ open set, we have already seen Weirestrass's theorem 1.18. This result can be extended to $\Omega \subset \mathbb{C}^n$, $n \in \mathbb{N}$, as follows:

Theorem 1.24. *Let $\Omega \in \mathbb{C}^n$, given a sequence of functions of several complex variables $\{f_i : i = 1, 2, \dots\} \subset H(\Omega)$, suppose it converges complactly in Ω to the function $f : \Omega \to \mathbb{C}$. Then $f \in H(\Omega)$, and for each $\alpha \in \mathbb{N}^n$,*

$$\lim_{i \to \infty} D^\alpha f_i = D^\alpha f$$

compactly in D.

The proof of this theorem is the same as in the classical case $n = 1$ and will be omitted.

1.6 Power Series

In this section we are planning to prove that every holomorphic function can be represented locally by a convergent power series.

Definition 1.25. *By a **power series of n variables centered at 0** we mean a series of the form*

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

where $c_\alpha \in \mathbb{C}$ for each $\alpha \in \mathbb{N}^n$.

Remark 1.26. On the index set $\mathbb{N}^n$ there is no canonical order. Therefore by convergence of the above power series, we mean the absolute convergence.

Definition 1.27. *The power series defined above* ***converges*** *at a point z if and only if*

$$\sup \left\{ \sum_{\alpha \in F} |c_\alpha z^\alpha| : F \subset \mathbb{N}^n \text{ finite} \right\} < +\infty.$$

The sum of this series is then the limit of partial sums for any ordering of the elements of the series.

Definition 1.28. *The* ***domain of convergence*** $A = A(\{c_\alpha\})$ *of a power series is the interior of the set of points* $z \in \mathbb{C}^n$ *for which the series converges.*

Now we will see that the well-known result for power series in $\mathbb{C}$ extends to $\mathbb{C}^n$.

Theorem 1.29. *Let* $f \in H(P(a,r))$*. Then the Taylor series of f at a converges to f on* $P(a,r)$*, that is,*

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha f(a)}{\alpha!}(z-a)^\alpha \quad \text{for } z \in P(a,r).$$

Proof. Given $z \in P(a,p) \subset P(a,r)$, the Cauchy integral formula states that

$$f(z) = \frac{1}{(2\pi i)^n} \int_{b_o P(a,p)} \frac{f(w) dw_1 \dots dw_n}{(w_1 - z_1) \dots (w_n - z_n)}. \tag{1.9}$$

The expression $(w-z)^{-1} = (w_1 - z_1)^{-1} \dots (w_n - z_n)^{-1}$ can be written as a geometric series

$$\begin{aligned}
\frac{1}{\prod_{i=1}^n (w_i - z_i)} &= \frac{1}{\prod_{i=1}^n (w_i - a_i + a_i - z_i)} \\
&= \frac{1}{\prod_{i=1}^n (w_i - a_i)} \cdot \frac{1}{1 - \frac{(z_1 - a_i)}{(w_1 - a)}} \cdots \frac{1}{1 - \frac{(z_n - a_i)}{(w_n - a_i)}} \\
&= \frac{1}{\prod_{i=1}^n w_i} \sum_{\alpha \in \mathbb{N}^n} \frac{(z-a)^\alpha}{(w-a)^\alpha} \\
&= \sum_{\alpha \in \mathbb{N}^n} \frac{(z-a)^\alpha}{(w-a)^{\alpha+1}}
\end{aligned}$$

Which converges uniformly for $w \in b_o P(a,p)$ since $\frac{|z_i - a_i|}{|w_i - a_i|} \leq \frac{|z_i - a_i|}{p_i} < 1$, for $i = 1, \dots, n$. Now, substituting this series into (1.9), we get

$$f(z) = \frac{1}{(2\pi i)^n} \int_{b_o P(a,p)} f(w) \sum_{\alpha \in \mathbb{N}^n} \frac{(z-a)^\alpha}{(w-a)^{\alpha+1}} dw_1 \cdots dw_n,$$

it is legitimate to interchange summation and integration, and multiplying and dividing by $\alpha!$, leading to

$$f(z) = \frac{1}{\alpha!} \sum_{\alpha \in \mathbb{N}^n} \left[\frac{\alpha!}{(2\pi i)^n} \int_{b_o P(a,p)} \frac{f(w) dw_1 \cdots dw_n}{(w-a)^{\alpha+1}} \right] (z-a)^\alpha. \tag{1.10}$$

It is clear via Cauchy integral formula for derivatives, that the coefficient of $(z - a)^{\alpha+1}$ in (1.10) is equal to $D^\alpha f(a)$, thus:

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha f(a)}{\alpha!} (z-a)^\alpha \quad \text{for } z \in P(a,p).$$

$\square$

Chapter 2

Riemann Mapping Theorem

The purpose of this character is to understand the Riemann mapping theorem in $\mathbb{C}$ and see that this result no longer true in higher dimensions. In order to study this we will go through some previous definitions and lemmas.

In this chapter, we follow the references [Ull00, Chap. 8-9], [Ran86, Chap. 1] and [Sha03, Chap. 8]. Before we get into the Riemann mapping theorem, I will mention three theorems that will not be proven here as they are basic results of functions of one complex variable.

Theorem 2.1. (Open Mapping Theorem) *Let U be a domain in and $f : U \to \mathbb{C}$ is a non-constant holomorphic function, then f is an open map (sends open subsets of U to open subsets of $\mathbb{C}$).*

Theorem 2.2. (Maximum Modulus Principle) *Let f be a holomorphic function on some connected open subset U of $\mathbb{C}$. If there exists $z_0 \in U$ such that*

$$|f(z_0)| \geq |f(z)|$$

for all z in some neighborhood of z_0, then f is constant on U.

Theorem 2.3. (Rouche's Theorem) *Let $f, g : U \to \mathbb{C}$ be two holomorphic functions on $U \subset \mathbb{C}$ open, and $\gamma : I \to U$ a simple closed curve. Assume f and g have no zeros on $\gamma(I)$ and*

$$|f(z) - g(z)| \leq |g(z)|, \quad \forall z \in \gamma(I).$$

Then f and g have the same number of zeros inside γ, counting multiplicities.

2.1 Conformal maps

In this section $\mathbb{D} = D(0,1)$ will be the unit disc.

Definition 2.4. *Let $U \subset \mathbb{C}^n, V \subset \mathbb{C}^m$ be two domains (open, path connected sets), then a map $f = (f_1, \dots, f_n)$ is called holomorphic if all its components are holomorphic, i.e, $f_i \in H(U), \forall i = 1, \dots, m$.*

Definition 2.5. *Let $U \subset \mathbb{C}^n, V \subset \mathbb{C}^m$ be two domains, if there exists a one-to-one holomorphic map $f : U \to V$ such that the inverse f^{-1} is also holomorphic, then we say that U and V are* ***biholomorphically equivalent*** *or that they are biholomorphic. The map f is called a biholomorphic map.*

Example 2.6.

1. Given the upper half plane $H = \{z \in \mathbb{C} : Im(z) > 0\}$ and the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we can build a biholomorphic map between them. To find this, we will take the Mobius transformation $\phi(z) = \frac{az+b}{z+c}$ which maps the points $\{0, 1, \infty\}$ to $\{-1, 1, -i\}$. Solving the system of equations it is seen that $a = 1, b = -i, c = i$, hence
$$\phi(z) = \frac{z-i}{z+i}.$$
It is clear that $\phi(H) = \mathbb{D}$, ϕ is holomorphic and it has a holomorphic inverse $\phi^{-1}(w) = i\left(\frac{1+w}{1-w}\right)$, hence ϕ is a biholomorphic map between H and $\mathbb{D}$.

2. The set $S := \{(z', z_n) \in \mathbb{C}^N : \text{Im } z_n > |z'|^2\}$ is called the Siegel upper half-space and biholomorphically equivalent to the unit ball B^n in $\mathbb{C}^n$. The biholomorphic map is given by the Cayley transform $\phi : B^n \to S$ of the form
$$\phi(z', z_n) \mapsto \frac{1}{1+z_n}(z', i(1-z_n)),$$
and inverse $\phi^{-1} : S \to B^n$:
$$(w', w_n) \mapsto \frac{2i}{1+w_n}\left(w', -\frac{i}{2}(i-w_n)\right).$$

Definition 2.7. *A map $f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is said to be conformal if it is holomorphic and f' has no zeros in U.*

In Stein's book [Sha03, Chap8] one can see the following proposition that gives an equivalent definition of a conformal map, in these notes we will not get into details with the proof, but the result is worth mentioning.

Theorem 2.8. *If $f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal maps is also holomorphic.*

Hence, if $f \in H(U)$ is injective, we say that f is a conformal equivalence, and that $f(U)$ is conformally equivalent to U.

2.2 Schwarz Lemmas

Definition 2.9. *Let $U \subset \mathbb{C}$ be an open subset, we denote $Aut(U)$ as the group of invertible holomorphic mappings from U to itself.*

Theorem 2.10. (Schwarz Lemma) *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. Furthermore, if $|f'(0)| = 1$ then f is a rotation: $f(z) = \beta z$ for some constant β with $|\beta| = 1$.*

Proof. Since $f(0) = 0$, we can define a function $g \in H(\mathbb{D})$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & (z \neq 0) \\ f'(0) & (z = 0) \end{cases}$$

Suppose $r \in (0,1)$, it is clear that $|g| \leq 1/r$ on $\partial D(0,r)$, and so the Maximum Modulus Theorem shows that $|g| \leq 1/r$ in $D(0,r)$. Since this holds for all $r \in (0,1)$, it follows that $|g| \leq 1$ in $\mathbb{D}$ and hence that $|f(z)| \leq |z|$ and $f'(0)| \leq 1$.

If we have $|f'(0)| = 1$ then we have $|g(z)| = 1$ for some $z \in \mathbb{D}$ and hence the Maximum Modulus Theorem shows that g is constant. □

Theorem 2.11. *Suppose that $\phi \in Aut(\mathbb{D})$ and $\phi(0) = 0$. Then ϕ is a rotation:*

$$\phi(z) = \beta z$$

for some $\beta \in \mathbb{C}$ with $|\beta| = 1$.

Proof. By the Schwarz Lemma, we know that $|\phi'(0)| \leq 1$. Let $\psi = \phi^{-1}$, then $|\psi'(0)| \leq 1$ for the same reason. The chain rule shows that $\psi'(0) = 1/\phi'(0)$, and so we must have $|\phi'(0)| = 1$, hence the Schwarz Lemma shows that ϕ is a rotation. □

Definition 2.12. *If $a \in \mathbb{D}$ then*

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

These objects that we have just defined will come up many times in the rest of the chapter. We will collect a few of their important properties

Lemma 2.13. *i) $\phi_a(a) = 0$ and $\phi_a(0) = a$.*

ii) Each ϕ_a is its own inverse.

iii) $\phi_a \in Aut(\mathbb{D})$ for any $a \in \mathbb{D}$.

Proof. The first and the second parts are immediate by doing the calculations. We will prove the third part. Suppose that $|z| = 1$, then $z\bar{z} = 1$ and

$$|\phi_a(z)| = \frac{|a - z|}{|1 - \bar{a}z|} = \frac{|a - z|}{|z(1 - a\bar{z})|} = \frac{|a - z|}{|z - a|} = 1.$$

Hence, ϕ_a maps the unit circle to itself. Since ϕ_a is a homeomorphism of $\mathbb{C}$, it must map components of the complement of the unit circle in $\mathbb{C}$ to components of the complement of the unit circle, which says that the image of $\mathbb{D}$ must be either $\mathbb{D}$ or the "exterior" of the circle, that is $\{z \in \mathbb{C} : z > 1\}$. But the second case can not be true since $\phi_a(0) = a \in \mathbb{D}$. So ϕ_a maps $\mathbb{D}$ to itself, hence is an automorphism of $\mathbb{D}$. □

Given these properties, it follows immediately that Aut($\mathbb{D}$) is generated by the subgroup of rotations and the subset $\{\phi_a : a \in \mathbb{D}\}$:

Proposition 2.14. *For any $\psi \in Aut(\mathbb{D})$ there exist a unique $a \in \mathbb{D}$ and $\beta \in \mathbb{C}$ with $|\beta| = 1$ such that*

$$\psi(z) = \beta\phi_a(z)$$

for all $z \in \mathbb{D}$.

Proof. Let $a = \psi^{-1}(0)$ and set $\psi_1 = \psi \circ \phi_a$. Now $\psi_1 \in \text{Aut}(\mathbb{D})$ and

$$\psi_1(0) = \psi(a) = 0.$$

hence, Theorem 2.11 shows that ψ_1 is a rotation, which means that exists β with $|\beta| = 1$ such that

$$\psi_1(z) = \beta z$$

for all z. Since ϕ_a is its own inverse, we have $\psi = \psi_1 \circ \phi_a$, and hence

$$\psi(z) = \psi_1(\phi_a(z)) = \beta\phi_a(z)$$

and the result has been proven. □

Now that we know the automorphisms of the disc we can study a more sophisticated version of the Schwarz Lemma. The origin is a special point in the Schwarz Lemma (Theorem 2.10). Inside $\mathbb{D}$, $Aut(\mathbb{D})$ acts transitively, this means that for any $z, w \in \mathbb{D}$ there exists $\omega \in Aut(\mathbb{D})$ with $\omega(z) = w$. We are going to study an "invariant" form of the Schwarz Lemma that reflects this symmetry.
From Theorem 2.10 it is immediate to see that

$$|f(z) - f(0)| \leq |z - 0|, \tag{2.1}$$

which means that the distance from $f(z)$ to $f(0)$ is smaller than the distance from z to 0. The invariant version of the theorem says that if f is any holomorphic map from the disc to itself and $z, w \in \mathbb{D}$ then the distance from $f(z)$ to $f(w)$ is no larger than the distance from z to w. This is not true for the Euclidean metric but it is for the *pseudo-hyperbolic metric*, whose distance is defined as

$$d(z, w) = |\phi_z(w)| \quad \text{for all } z, w \in \mathbb{D}.$$

Before getting into the invariant form of the theorem we will prove an equality that will be useful for the next result.

Lemma 2.15. *For any $a, z \in \mathbb{D}$ we have*

$$1 - |\phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2}.$$

Proof. We simply do the math:

$$\begin{aligned}
1 - |\phi_a(z)|^2 &= 1 - \frac{(a - z)(\bar{a} - \bar{z})}{(1 - \bar{a}z)(1 - a\bar{z})} \\
&= \frac{1 - \bar{a}z - a\bar{z} + |a|^2|z|^2 - (|a|^2 - a\bar{z} - \bar{a}z + |z|^2)}{|1 - \bar{a}z|^2} \\
&= \frac{1 + |a|^2|z|^2 - |a|^2 - |z|^2}{|1 - \bar{a}z|^2} \\
&= \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2}.
\end{aligned}$$

□

Theorem 2.16. (Invariant Schwarz Lemma) *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. If $f \notin Aut(\mathbb{D})$ then*

$$d(f(z), f(w)) < d(z, w)$$

for all $z, w \in \mathbb{D}$ with $z \neq w$ and

$$\frac{|f'(z)|}{1-|f(z)|^2} < \frac{1}{1-|z|^2} \tag{2.2}$$

for all $z \in \mathbb{D}$.

Proof. Lets fix $z_0 \in \mathbb{D}$ and let $f(z_0) = a$. Define

$$g = \phi_a \circ f \circ \phi_{z_0},$$

and it is easy to see that $g(0) = 0$ as

$$g(0) = \phi_a(f(\phi_{z_0}(0))) = \phi_a(f(z_0)) = \phi_a(a) = 0.$$

Theorem 2.10 shows that $|g(z) \leq |z|$ for any $z \in \mathbb{D}$. This shows that

$$|\phi_a(f(\phi_{z_0}(z)))| \leq |z| \quad \text{for any} z \in \mathbb{D}; \tag{2.3}$$

replacing z by $\phi_{z_0}(z)$ this shows that

$$|\phi_a(f(z))| \leq |\phi_{z_0}(z)| \tag{2.4}$$

for all z. In other words $|\phi_{f(z_0)}(f(z))| \leq |\phi_{z_0}(z)|$, or:

$$d(f(z), f(w)) \leq d(z, w).$$

Schwarz theorem also shows that $|g'(0)| \leq 1$, so by the chain rule we have that

$$|\phi_a'(a)||f'(z)||\phi_{z_0}'(0)| \leq 1.$$

Finally, to prove (2.2) it is enough to prove that

$$\frac{|f'(z_0)|(1-|z_0|^2)}{1-|a|^2} < 1.$$

which is proven by the Lemma 2.15 and the fact that $f \neq Aut(\mathbb{D})$. □

In order to prove the Riemann mapping theorem we will need an infinitesimal version of the inequality we have just proved.
If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic we define

$$H_f(z) = \frac{|f'(z)|}{1-|f(z)|^2}.$$

Lemma 2.17. *Let $\Omega \subset \mathbb{C}$ be a simply connected open set, suppose that $f : \Omega \to \mathbb{D}$ and $\omega : \mathbb{D} \to \mathbb{D}$ are holomorphic. If $\omega \in Aut(\mathbb{D})$ then*

$$H_{\omega \circ f}(z) = H_f(z)$$

for all $z \in \Omega$, while if $\omega \notin Aut(\mathbb{D})$ then

$$H_{\omega \circ f}(z) < H_f(z)$$

for all $z \in \Omega$ with $f'(z) \neq 0$. (And in particular, $H_{f^2}(z) < H_f(z)$).

Proof. Invariant Schwarz Lemma (2.16) shows that

$$\frac{|\omega'(z)|}{1 - |\omega(z)|^2} < \frac{1}{1 - |z|^2}$$

for all $z \in \mathbb{D}$ if $\omega \notin Aut(\mathbb{D})$. As the image of $f(z)$ is inside $\mathbb{D}$, in particular we have

$$\frac{|\omega'(f(z))|}{1 - |\omega(f(z))|^2} < \frac{1}{1 - |f(z)|^2},$$

and so

$$H_{\omega \circ f}(z) = \frac{|\omega'(f(z))||f'(z)|}{1 - |\omega(f(z))|^2} < \frac{|f'(z)|}{1 - |f(z)|^2} = H_f(z).$$

□

2.3 Riemann Mapping Theorem

We recall that any non-vanishing holomorphic function in a simply connected set has a holomorphic logarithm and hence a holomorphic square root. This property will be used for the proof of the Riemann mapping theorem and finally we have the necessary tools to prove the following result.

Theorem 2.18. (Riemann mapping theorem) *If $D \subset \mathbb{C}$ is a simply connected open set and $D \neq \mathbb{C}$ then D is conformally equivalent to the unit disc $\mathbb{D}$.*

Proof. Fix $z_0 \in D$, let $\mathcal{F}$ be a family of holomorphic functions $f : D \to \mathbb{D}$ such that f is one-to-one and $f(z_0) = 0$.
Let us suppose that $p \in \mathbb{C} \backslash D$, then the function $z - p$ won't have any zeros in D and hence, by the hypothesis, it has a holomorphic square root. Let $g(z)^2 = z - p$ for $z \in D$, note that it is one-to-one:

if $z, w \in D$ and $g(z) = g(w)$ (or even $g(z) = -g(w)$), then $z = w$ since $z - p = g(z)^2 = g(w)^2 = w - p$.

Also, let us see that $g(D) \cap (-g(D)) = \emptyset$; suppose $\alpha \in g(D) \cap (-g(D))$, then $\alpha = g(z)$ and $\alpha = -g(w)$ for some $z, w \in D$ but as seen before this implies that $z = w$ hence $\alpha = -\alpha$ and this would mean that $\alpha = 0$ but it is a contradiction since g has no zeros in D.
The Open mapping theorem (2.1) shows that $g(D)$ is open, now let us see that $g(D)$ is also not dense in the plane. Suppose there exists $q \in \mathbb{C}, r > 0$ such that $D(q,r) \subset g(D)$, then:

$$D(-q,r) \cap g(D) = (-D(q,r)) \cap g(D) \subset (-g(D)) \cap g(D) = \emptyset,$$

which is a contradiction, hence $g(D)$ is not dense in $\mathbb{C}$.
That is, $|g(z) + q| \geq r$ for all $z \in D$, and if we define h as

$$h(z) = \frac{r/3}{g(z) + q} - \frac{r/3}{g(z_0) + q},$$

then $h \in \mathcal{F}$, so $\mathcal{F}$ is nonempty.
If $r > 0$ and $D(z_o, r) \subset D$ then Cauchy's Estimates shows that

$$|f'(z_o)| \leq \frac{1}{r}, \quad \text{since } |f| < 1.$$

Let $m = \sup\{|f'(z_0)| : f \in \mathcal{F}\}$ and choose a sequence (f_n) in $\mathcal{F}$ such that $|f_n'(z_0)| \to m$. Montel's Theorem (1.22) states that (f_n) has a subsequence converging in $H(D)$, replacing (f_n) by this subsequence we may assume that $F \in H(D)$ and

$$f_n \to F$$

uniformly on every compact subset of D.

It follows that $f_n' \to F'$ in $H(D)$, and hence $|F'(z_0)| = m$. Since $m > 0$, this shows that F is not constant. It is clear that $F(D) \subset \mathbb{D}$, if $|F(z)| = 1$ for some $z \in D$ then the Maximum Modulus Principle (2.2) shows that F is constant, which is a contradiction, hence $F(D) \subset \mathbb{D}$.

The next step is to see that F is one-to-one. Suppose the contrary, hence there exists $z, w \in D$ such that $F(z) = F(w) = p$. If we apply Rouche's Theorem in a small disc around z and a small disc around w, it shows that if n is large enough then $f_n - p$ has at least two zeroes, one near z and one near w, contradicting the fact that f_n is one-to-one.

At this point, we know that $F \in \mathcal{F}$. Now, also by contradiction, let us prove that $F(D) = \mathbb{D}$. Suppose there exists $a \in \mathbb{D} \backslash F(D)$, then $\phi_a \circ F \in H(D)$ has no zero in D. It follows that $\phi_a \circ F$ has a holomorphic square root, which we will denote as $(\phi_a \circ F)^{1/2}$. As seen in the start of the proof, $(\phi_a \circ F)^{1/2}$ is one-to-one and hence

$$G = \phi_b \circ (\phi_a \circ F)^{1/2} \in \mathcal{F},$$

if $b = (\phi_a \circ F)^{1/2}(z_0)$. Since $(\phi_a \circ F)^{1/2}$ is one-to-on, its derivative never vanishes, and so it follows from the Lemma 2.17 that

$$H_{\phi_a \circ F} < H_{(\phi_a \circ F)^{1/2}}$$

in D. And again, by the same Lemma

$$H_G = H_{\phi_b \circ (\phi_a \circ F)^{1/2}} = H_{(\phi_a \circ F)^{1/2}} > H_{\phi_a \circ F} = H_F$$

in D. And in particular, since $F(z_0) = G(z_0) = 0$, we have

$$|G'(z_0)| = H_G(z_0) > H_F(z_0) = |F'(z_0)|,$$

contradicting the maximality of $|F'(z_0)|$. So it has been shown that $F(D) = \mathbb{D}$ and the theorem has been proven. □

Corollary 2.19. *Let D be a simply connected open set, $D \neq \mathbb{C}$, and $z_0 \in D$. Then there exists a unique conformal equivalence $F : D \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$*

Proof. We have seen in the proof of the theorem that this function F exists. Now suppose that there also exists another conformal equivalence $\tilde{F} : D \to \mathbb{D}$ satisfying all the statement conditions, then $\phi = \tilde{F} \circ F^{-1} \in Aut(D)$ satisfies $\phi(0) = 0$ and $\phi'(0) > 0$. By Schwarz Lemma (2.10), $|\phi'(0)| \leq 1$ and $|(\phi^{-1})'(0)| \leq 1$. But the chain rule says that

$$(\phi^{-1})'(0) = \frac{1}{\phi'(0)}$$

so we must have that $|\phi'(0)| = 1$ hence, also by Schwarz lemma ϕ is a rotation then it must be the identity. □

On the other hand, it is impossible to find a higher dimensional analog of Riemann's Theorem. This fact was discovered by H.Poincaré in 1907 ("Les fonctions analytiques de deux variables et la représentation conforme", Rend. Circ. Mat. Palermo 23(1907), 185-220).

Theorem 2.20. *There exists no biholomorphic map between the polydisc and the ball in $\mathbb{C}^n$ if $n > 1$.*

Proof. For simplicity, we will consider the case $n = 2$. Let $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, be the open unit disc in $\mathbb{C}$. Suppose there exists

$$F = (f_1, f_2) : \mathbb{D} \times \mathbb{D} \to B = B(0,1) \subset \mathbb{C}^2$$

biholomorphic. For every fixed $w \in \mathbb{D}$, we define the holomorphic map $F_w : \mathbb{D} \to B$ as

$$F_w(z) = \left(\frac{\partial f_1}{\partial w}(z,w), \frac{\partial f_2}{\partial w}(z,w)\right)$$

Indeed, we will show that this satisfies $\lim_{z\to b\mathbb{D}} F_w(z) = 0$ and this gives the result. To prove this it is enough to show that every sequence $\{z_v\} \subset \mathbb{D}$, with $|z_v| \to 1$ has a subsequence $\{z_{v_j}\}$ with $\lim_{j\to\infty} F_w(z_{v_j}) = 0$.

Given such a sequence $\{z_v\}$, an application of Montel's Theorem 1.22 to the bounded sequence of holomorphic maps $F(z_v, \cdot) : \mathbb{D} \to B$ for $v = 1, 2, \ldots$ gives a subsequence $\{z_{v_j}\}_j$ such that $\{F(z_{v_j}, \cdot)\}_j$ converges uniformly on compacts in $\mathbb{D}$ to a holomorphic map

$$\phi : \mathbb{D} \to \overline{B}.$$

Since F is an homeomorphism and $(z_v, w) \to \partial(\mathbb{D}\times\mathbb{D})$ then $F(z_v, w) \to bB$ for every $w \in \mathbb{D}$, as $z_v \to b\mathbb{D}$, hence $\phi(\mathbb{D}) \subset bB$. If $\phi = (\phi_1, \phi_2)$, then

$$|\phi_1|^2 + |\phi_2|^2 = 1, \tag{2.5}$$

for all $w \in \mathbb{D}$, and notice that for $i = 1, 2$

$$\begin{aligned} \frac{\partial^2}{\partial w \partial \overline{w}}(|\phi_i(w)|^2) &= \frac{\partial^2}{\partial w \partial \overline{w}}(\phi_i(w)\overline{\phi_i(w)}) \\ &= \frac{\partial}{\partial w}(\phi_i(w)\overline{\phi_i'(w)}) \\ &= |\phi_i'(w)|^2. \end{aligned}$$

Hence, by applying $\frac{\partial^2}{\partial \overline{w} \partial w}$ to (2.5) one obtains $|\phi_1'(w)|^2 + |\phi_2'(w)|^2 = 0$, so $\phi' \equiv 0$ on $\mathbb{D}$. Since

$$F_w(z_{v_j}, w) \to \phi'(w) \quad \text{as } j \to \infty,$$

it has been proven that $\lim_{z\to b\mathbb{D}} F_w(z) = 0$, let us show that this is enough. This implies that F_w extends continuously to $\overline{\mathbb{D}}$, with boundary values 0. Since F_w is holomorphic on $\mathbb{D}$, by the Maximum Modulus Principle one obtains

$$\sup_{z\in\mathbb{D}} |F_w(z)| = \sup_{\partial\mathbb{D}} |F_w(z)| = 0.$$

Hence, it follows that $F_w \equiv 0$ on $\mathbb{D}$, i.e, $F(z, w)$ is independent of w and F could not be one-to-one. This gives a contradiction, hence we have seen that F does not exist. □

Chapter 3

Bergman spaces

In the following section, we will study a type of Hilbert space of function named Bergman space, and the so called Bergman kernel. After some examples, later in this chapter, we will discuss its relationship to the Riemann mapping theorem. In this chapter we follow the references [Has10, Chap. 7].

3.1 Elementary properties and the Bergman kernel

Definition 3.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain, the Bergman space $A^2(\Omega)$ is defined by*

$$A^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} \text{ holomorphic} : \|f\|^2 = \int_\Omega |f(z)|^2 d\lambda(z) < \infty \right\}$$

where λ is the Lebesgue measure of $\mathbb{C}^n$. The inner product is given by

$$(f, g) = \int_\Omega f(z)\overline{g(z)} d\lambda(z)$$

for $f, g \in A^2(\Omega)$.

For simplicity, we will start by restricting ourselves to the domains $\Omega \subset \mathbb{C}$. Let $f \in A^2(\Omega)$ and fix $z \in \Omega$. By Cauchy's integral theorem, we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $\gamma_s(t) = z + se^{it},\ t \in [0, 2\pi],\ 0 < s \leq r$ and $D_r(z) = \{w : |w - z| < r\} \subset \Omega$. This integral can be rewritten using polar coordinates, doing the change of variables $\zeta = z + se^{it}$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + se^{it})}{z + se^{i} - z} ise^{it} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + se^{it}) dt, \end{aligned}$$

integrating the equality with respect to s between 0 and r and doing the change $w = z + se^{it}$, we get

$$f(z) = \frac{1}{\pi r^2} \int_{D_r(z)} f(w) d\lambda(w). \tag{3.1}$$

Then, by Cauchy-Schwarz inequality,

$$\begin{aligned} |f(z)| &\leq \frac{1}{\pi r^2} \int_{D_r(z)} 1 \cdot |f(w)| d\lambda(w) \\ &\leq \frac{1}{\pi r^2} \left(\int_{D_r(z)} 1^2 d\lambda(w) \right)^{1/2} \left(\int_{D_r(z)} |f(w)|^2 d\lambda(w) \right)^{1/2} \\ &= \frac{1}{\sqrt{\pi} r} \left(\int_{\Omega} |f(w)|^2 d\lambda(w) \right)^{1/2} \\ &\leq \frac{1}{\sqrt{\pi} r} \|f\|. \end{aligned}$$

Theorem 3.2. *Let $K \subset \Omega \subset \mathbb{C}^n$ be a compact set. Then there exists a constant $C(K)$, only depending on K, such that*

$$\sup_{z \in K} |f(z)| \leq C(K) \|f\|, \tag{3.2}$$

for any $f \in A^2(\Omega)$.

Proof. We have seen above that if $\Omega \subset \mathbb{C}$ then $|f(z)| \leq \frac{1}{\sqrt{\pi} r} \|f\|$. If K is a compact subset of Ω, there is an $r(K) > 0$, only depending on K, such that for any $z \in K$ we have $D_{r(K)}(z) \subset \Omega$ and then get

$$\sup_{z \in K} |f(z)| \leq \frac{1}{\sqrt{\pi} r(K)} \|f\|.$$

If $K \subset \Omega \subset \mathbb{C}^n$, we can find a polydisc

$$P(z, r(K)) = \{w \in \mathbb{C}^n : |w_j - z_j| < r(K), j = 1, \ldots, n\}$$

such that for any $z \in K$ we have $P(z, r(K)) \subset \Omega$. Hence, as $r(K)$ only depends on K the result has been proven. □

As mentioned above, Bergman spaces are indeed Hilbert spaces, so it is a complex inner product space that is a complete metric space with respect to the metric given by the inner product. To show this, we will see that any Bergman space is a closed subspace of $L^2(\Omega)$, which is a Hilbert space[1].

[1]The fact that L^2 is a Hilbert space will not be proven here, it can be found in "The Lebesgue-Stieltjes integral, a practical introduction", M. Carter & B. van Bunt, chapter 9.

Theorem 3.3. *$A^2(\Omega)$ is a Hilbert space.*

Proof. Let $(f_k)_k$ be a Cauchy sequence in $A^2(\Omega)$, by theorem 3.2, it is also a Cauchy sequence with respect to uniform convergence on compact subsets of Ω. Hence the sequence $(f_k)_k$ has a holomorphic limit f with respect to uniform convergence on compact subsets of Ω. On the other hand, the original L^2-Cauchy sequence has a subsequence, which converges pointwise almost everywhere to the L^2-limit of the original L^2-Cauchy sequence, and so the $L^2 - limit$ coincides with the holomorphic function f. Therefore $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and itself a Hilbert space. □

Next we are going to present some basic facts about Hilbert spaces. Their demonstrations are out of our scope so for the moment we will only mention them. (See "Complex Analysis", Friedrich Haslinger, chapter 7).

Theorem 3.4. *Let M be a closed subspace of Hilbert space H. Then there exist uniquely determined mappings*

$$P : H \to M, \quad Q : H \to M^{\perp}$$

such that

- *$x = Px + Qx \forall x \in H$.*
- *For $x \in M$ we have $Px = x$, hence $P^2 = P$ and $Qx = 0$; for $x \in M^{\perp}$ we have $Px = 0$, $Qx = x$, and $Q^2 = Q$.*
- *The distance of $x \in M$ is given by*

$$\inf\{\|x - y\| : y \in M\} = \|x - Px\|.$$

- *For each $x \in H$, we have*

$$\|x\|^2 = \|Px\|^2 + \|Qx\|^2.$$

- *P and Q are continuous, linear, self-adjoint operators.*

P and Q are the orthogonal projections fo H onto M and $M^{\perp}$.

Theorem 3.5. *Let L be a continuous linear function on the Hilbert space H. Then there exists a uniquely determined element $y \in H$ such that $Lx = (x, y) \forall x \in H$.*

For fixed $z \in \Omega$, (3.2) implies that the point evaluation $f \mapsto f(z)$ is a continuous linear function on $A^2(\Omega)$, hence by the Riesz representation theorem 3.5, there exists a uniquely determined function $k_z \in A^2(\Omega)$ such that

$$f(z) = (f, k_z) = \int_\Omega f(w)\overline{k_z(w)}d\lambda(w).$$

For notation purposes, we will set $K(z,w) = \overline{k_z(w)}$. Then we have

$$f(z) = \int_\Omega K(z,w)f(w)d\lambda(w), \quad f \in A^2(\Omega).$$

Definition 3.6. *Let H be a Hilbert space of functions from a set X. Given any $z \in X$, if there exists a function $K_z \in H$ such that for all $f \in H$, $(f, K_z) = f(z)$, then the function K_z is called the* ***reproducing kernel*** *of the Hilbert space H.*

The function of two complex variables $(z,w) \mapsto K(z,w)$ is called **Bergman kernel** of $A^2(\Omega)$ and the above identity (3.1) represents the reproducing property of a Bergman kernel. Next, we are going to use the reproducing property of the kernel to see some properties.

Given the holomorphic function $z \mapsto K_u(z)$, where $u \in \Omega$ is fixed:

$$\begin{aligned}
k_u(z) &= \int_\Omega K(z,w)k_u(w)d\lambda(w) \\
&= \int_\Omega \overline{k_z(w)K(u,w)}d\lambda(w) \\
&= \overline{\left(\int_\Omega K(u,w)k_z(w)d\lambda(w)\right)} \\
&= \overline{k_z(u)},
\end{aligned}$$

we have just seen that $k_u(z) = \overline{k_z(u)}$, or $K(z,u) = \overline{K(u,z)}$.

Theorem 3.7. *The Bergman kernel is uniquely determined by the properties that it is an element of $A^2(\Omega)$ in z and that it is conjugate symmetric and reproduces $A^2(\Omega)$.*

Proof. Let us suppose that there exists another kernel $K'(z,w)$ with these properties. Using its reproducing property we have

$$\begin{aligned}
K(z,w) &= \int_\Omega K'(z,u)K(u,w)d\lambda(u) \\
&= \overline{\left(\int_\Omega K(w,u)K'(u,z)d\lambda(u)\right)} \\
&= \overline{K'(w,z)}.
\end{aligned}$$

Hence $K(z,w) = \overline{K'(w,z)} = K'(z,w)$ and the result has been proven. □

Let $\phi \in L^2(\Omega)$. As seen in Theorem 3.4, since $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ there exists a uniquely determined orthogonal projection $P : L^2(\Omega) \to A^2(\Omega)$. Now we define the function $P\phi \in A^2(\Omega)$, using the reproducing property of the Bergman kernel one obtains

$$P\phi(z) = \int_\Omega K(z,w)P\phi(w)d\lambda(w) = (P\phi, k_z) = (\phi, PK_z) = (\phi, k_z);$$

where we have used that P is a self-adjoint operator and that $PK_z = k_z$. Hence, P is called as the Bergman projection and $P\phi(z)$ can be expressed as

$$P\phi(z) = \int_\Omega K(z,w)\phi(w)d\lambda(w). \tag{3.3}$$

We will see that it is possible to compute the Bergman kernel using a complete orthonormal basis.

Definition 3.8. *Let $A \subset \mathbb{N}$, a subset $\{u_\alpha : \alpha \in A\}$ of a Hilbert space is called orthonormal if $(u_\alpha, u_\beta) = \delta_{\alpha\beta}$ for each $\alpha, \beta \in A$ where $\delta_{\alpha\beta}$ is the Kronecker delta.*

If $(x_k)_k$ is a linearly independent sequence in H, there is a standard procedure, called the Gram-Schmidth process, for converting $(x_k)_k$ into an orthonormal sequence $(u_k)_k$ such that the linear span of $(u_k)_{k=1}^N$ equals the linear span of $(x_k)_{k=1}^N$ for all $N \in \mathbb{N}$. We start by defining $u_1 = x_1/\|x_1\|$. Having defined $u_1, \ldots, u_{N-1}$, we set

$$v_N = x_N - \sum_{j=1}^{N-1} (x_N, u_j)u_j.$$

The element v_N is nonzero because x_N is not in the linear span of $x_1, \ldots, x_{N-1}$ and hence of $u_1, \ldots, u_{N-1}$. So we can set $u_N = v_N/\|v_N\|$ and $(u_k)_{k=1}^N$ has the desired properties.

Theorem 3.9. (Bessel's inequality) *If $\{u_\alpha : \alpha \in A\}$ is an orthonormal set in the Hilbert space H, then for any $u \in H$*

$$\sum_{\alpha \in A} |(u, u_\alpha)|^2 \leq \|u\|^2.$$

Theorem 3.10. *If $\{u_\alpha : \alpha \in A\}$ is an orthonormal set in the Hilbert space H, then the following conditions are equivalent:*

- *If $(u, u_\alpha) = 0$ for all $\alpha \in A$, then $u = 0$.*
- *(Parseval's equation) $\|u\|^2 = \sum_{\alpha \in A} |(u, u_\alpha)|^2$ for all $u \in H$.*

- $u = \sum_{\alpha \in A}(u, u_\alpha)u_\alpha$ *for each* $u \in H$*, where the sum has only countably many nonzero terms and converges in norm to* u *no matter how these terms are ordered.*

An orthonormal set having the properties of the last theorem is called an orthonormal basis of H.

Theorem 3.11. *Let* $K \subset \Omega$ *be a compact subset and* $\{\phi_j\}$ *be an orthonormal basis of* $A^2(\Omega)$*. Then the series*

$$\sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}$$

converges uniformly on $K \times K$ *to the Bergman kernel* $K(z, w)$*.*

Proof. Cauchy-Schwarz inequality gives

$$\sum_{j=1}^{\infty} |\overline{\phi_j(z)}\phi_j(w)| \leq \left(\sum_{j=1}^{\infty} |\phi_j(z)|^2\right)^{1/2} \left(\sum_{j=1}^{\infty} |\phi_j(w)|^2\right)^{1/2}, \tag{3.4}$$

taking a look at the factors in the right side of the inequality, one notices that

$$\begin{aligned} \sup_{z\in K} \left(\sum_{j=1}^{\infty} |\phi_j(z)|^2\right)^{1/2} &= \sup\left\{\left|\sum_{j=1}^{\infty} a_j\phi_j(z)\right| : \sum_{j=1}^{\infty} |a_j|^2 = 1, z \in K\right\} \\ &= \sup\{|f(z)| : \|f\| = 1, z \in K\} \\ &\leq C(K), \end{aligned}$$

where we have used Theorem 3.2 in the last inequality. Hence, 3.4 converges uniformly at $z, w \in K$. Furthermore, the function

$$w \mapsto \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}$$

belongs to $\overline{A^2(\Omega)}$. We denote the sum of the series by $K'(z, w)$, which is conjugate symmetric and that for any $f \in A^2(\Omega)$ we obtain

$$\int_\Omega K'(z, w)f(w)d\lambda(w) = \sum_{j=1}^{\infty} \int_\Omega f(w)\overline{\phi_j(w)}d\lambda(w)\phi_j(z) = f(z)$$

with convergence in the Bergman space $A^2(\Omega)$. From **??**, we obtain uniform convergence on compact subsets of Ω, hence

$$f(z) = \int_\Omega K'(z, w)f(w)d\lambda(w),$$

for all $f \in A^2(\Omega)$, so $K'(z, w)$ is a reproducing kernel. Hence, by the uniqueness of the Bergman kernel (3.7), $K'(z, w) = K(z, w)$. □

3.2 Bergman kernel of the unit disc and other examples

We want to compute an explicit formula for the Bergman kernel $K(z,w)$ of $\mathbb{D}$. In order to use Theorem 3.11 we need an orthonormal basis in $A^2(\mathbb{D})$. The functions

$$\phi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, n = 0,1,2,\dots$$

constitute one. In order to prove this, we will see that for any $f \in A^2(\mathbb{D})$, if $(f,\phi_n) = 0$ for all n, then $f \equiv 0$.
For each $f \in A^2(\mathbb{D})$ with Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we got

$$\begin{aligned}(f,z^n) &= \int_{\mathbb{D}} f(z)\overline{z^n} d\lambda(z) = \int_0^1 \int_0^{2\pi} f(re^{i\theta}) r^n e^{-in\theta} r dr d\theta \\ &= \int_0^1 \int_0^{2\pi} \frac{f(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}} re^{i\theta} d\theta r^{2n+1} dr\end{aligned}$$

Using the fact that

$$a_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z^{n+1}} dz, \quad \text{where } \gamma_r(\theta) = re^{i\theta},$$

we can write (f,z^n) as:

$$(f,z^n) = 2\pi a_n \int_0^1 r^{2n+1} dr = \pi \frac{a_n}{n+1}.$$

By the uniqueness of the Taylor series expansion, we obtain that if $(f,\phi_n) = 0$ for any n, then $f \equiv 0$. This implies that $(\phi_n)_{n=0}^{\infty}$ is an orthonormal basis for $A^2(\mathbb{D})$. By Parseval's equation in Theorem 3.10, we also get

$$\|f\|^2 = \sum_{n=0}^{\infty} |(f,\phi_n)|^2,$$

which is equivalent to $\|f\|^2 = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$.
Theorem 3.11 gives us the following series that sums uniformly in z on all compact subsets of $\mathbb{D}$ to the Bergman kernel $K(z,w)$ as follows

$$K(z,w) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)},$$

where $\phi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$ is the orthonormal basis we have just shown above. This gives us

$$
\begin{aligned}
K(z,w) &= \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{\pi}} z^n \cdot \sqrt{\frac{n+1}{\pi}} \overline{w}^n \\
&= \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)(z\overline{w})^n \\
&= \frac{1}{\pi(1-z\overline{w})^2}.
\end{aligned}
$$

Having computed explicitly the Bergman kernel, its reproducing properties give us that for each $f \in A^2(\mathbb{D})$ we have

$$
f(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{1}{(1-z\overline{w})^2} f(w) d\lambda(w).
$$

Having seen the Bergman kernel formula in $A^2(\mathbb{D})$, we are going to prove a result that will help us to compute the Bergman kernel in higher dimensions.

Theorem 3.12. *Let $\Omega_j \subset \mathbb{C}^n, j = 1,2$ be two bounded domains with Bergman kernels K_{Ω_1} and K_{Ω_2}. Then the Bergman kernel K_Ω of the product domain $\Omega = \Omega_1 \times \Omega_2$ is given by*

$$
K_\Omega((z_1,z_2),(w_1,w_2)) = K_{\Omega_1}(z_1,w_1) K_{\Omega_2}(z_2,w_2), \tag{3.5}
$$

for $(z_1,z_2),(w_1,w_2) \in \Omega_1 \times \Omega_2$.

Proof. Let F be the function on the right-hand side of (3.5). It is clear that $(z_1,z_2) \mapsto F((z_1,z_2),(w_1,w_2))$ belongs to $A^2(\Omega)$ for each fixed $(w_1,w_2) \in \Omega$. The reproducing property of K_{Ω_1} and K_{Ω_2} gives us

$$
f_1(z_1) = \int_{\Omega_1} K_{\Omega_1}(z_1,w_1) f(w_1) d\lambda(w_1),
$$

and

$$
f_2(z_2) = \int_{\Omega_2} K_{\Omega_2}(z_2,w_2) f(w_2) d\lambda(w_2).
$$

Using Fubini's theorem along with the reproducing properties of K_{Ω_1} and K_{Ω_2} one gets

$$
f(z_1,z_2) = \int_{\Omega_1 \times \Omega_2} F((z_1,z_2),(w_1,w_2)) f(w_1,w_2) d\lambda(w_1,w_2).
$$

Hence, by the uniqueness property of the Bergman kernel (Theorem 3.7), $F = K_\Omega$. □

Example 3.13. (Unit polydisc in $\mathbb{C}^n$)

Using Theorem 3.12, it is easy to compute the Bergman kernel of the polydisc $P(a, r)$, where $a = [0, \ldots, 0]$ and $r = [1, \ldots, 1]$. This polydisc is, in fact, n times the Cartesian product of the unit disc, hence it can be expressed as $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$. The Bergman kernel will then be given by

$$K_{\mathbb{D}^n}(z, w) = \prod_{j=1}^{n} K_{\mathbb{D}}(z_j, w_j) = \frac{1}{\pi} \sum_{j=1}^{n} \frac{1}{(1 - z_j \overline{w}_j)^2}.$$

Example 3.14. (Unit ball in $\mathbb{C}^n$) For the computation of the Bergman kernel $K_{\mathbb{B}^n}$ of the unit ball in $\mathbb{C}^n$, we use the Beta function

$$B(k+1, m+1) = \int_0^1 x^k (1-x)^m dx = \frac{\Gamma(k+1)\Gamma(m+1)}{\Gamma(k+m+2)},$$

where $\Gamma(n)$ is the Gamma function. Notice that for $0 \leq a < 1$,

$$\begin{aligned} \int_0^{\sqrt{1-a^2}} x^{2k+1} \left(1 - \frac{x^2}{1-a^2}\right)^{m+1} dx &= \frac{1}{2}(1-a^2)^{k+1} \int_0^1 y^k (1-y)^{m+1} dy \\ &= \frac{1}{2}(1-a^2)^{k+1} B(k+1, m+2) \\ &= \frac{1}{2}(1-a^2)^{k+1} \frac{\Gamma(k+1)\Gamma(m+2)}{\Gamma(k+m+3)}. \end{aligned} \tag{3.6}$$

Given the orthogonal basis $\{z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}\}$ in $A^2(\mathbb{B}^n)$, we need to find coefficients C_α to normalize it. We are going to compute the coefficients by calculating $\|z^\alpha\|^2$:

$$\begin{aligned} \|z^\alpha\|^2 &= \int_{\mathbb{B}^n} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} d\lambda(z) \\ &= \int_{\mathbb{B}^{n-1}} |z_1|^{2\alpha_1} \cdots |z_{n-1}|^{2\alpha_{n-1}} \\ &\quad \cdot \left(\int_{B(0, \sqrt{1-|z_1|^2 - \cdots - |z_{n-1}|^2})} |z_n|^{2\alpha_n} d\lambda(z_n) \right) d\lambda(z_1, \ldots, z_{n-1}). \end{aligned} \tag{3.7}$$

Notice that, by taking polar coordinates in the last integral and applying (3.6) one obtains

$$\begin{aligned} \int_{B(0, \sqrt{1-|z_1|^2 - \cdots - |z_{n-1}|^2})} |z_n|^{2\alpha_n} d\lambda(z_n) &= \int_0^{2\pi} \int_0^{\sqrt{1-|z_1|^2 - \cdots - |z_{n-1}|^2}} \rho^{2\alpha_n + 1} d\rho d\theta \\ &= \frac{\pi}{\alpha_n + 1} (1 - |z_1|^2 - \cdots - |z_{n-1}|^2)^{\alpha_n + 1}, \end{aligned}$$

then we can keep operating (3.7) and get

$$\begin{aligned}\|z^\alpha\|^2 &= \frac{\pi}{\alpha_n+1}\int_{\mathbb{B}^{n-1}} |z_1|^{2\alpha_1}\cdots|z_{n-1}|^{2\alpha_{n-1}}(1-|z_1|^2-\cdots-|z_{n-1}|^2)^{\alpha_n+1}d\lambda(z)\\ &= \frac{\pi}{\alpha_n+1}\int_{\mathbb{B}^{n-1}} |z_1|^{2\alpha_1}\cdots|z_{n-2}|^{2\alpha_{n-2}}(1-|z_1|^2-\cdots-|z_{n-2}|^2)^{\alpha_n+1}\\ &\quad\cdot|z_{n-1}|^{2\alpha_{n-1}}\left(1-\frac{|z_{n-1}|^2}{1-|z_1|^2-\cdots-|z_{n-2}|^2}\right)^{\alpha_n+1}d\lambda(z),\end{aligned}$$

and by using the same argument as above we obtain

$$\begin{aligned}&= \frac{\pi}{\alpha_n+1}\cdot\frac{\pi\Gamma(\alpha_{n-1}+1)\Gamma(\alpha_n+2)}{\Gamma(\alpha_n+\alpha_{n-1}+3)}\\ &\quad\cdot\int_{\mathbb{B}^{n-2}}|z_1|^{2\alpha_1}\cdots|z_{n-2}|^{2\alpha_{n-2}}(1-|z_1|^2-\cdots-|z_{n-2}|^2)^{\alpha_n+\alpha_{n-1}+2}d\lambda(z).\end{aligned}$$

Hence, iterating the same process $n-2$ times we get

$$\begin{aligned}\|z^\alpha\|^2 &= \frac{\pi}{\alpha_n+1}\cdot\frac{\pi\Gamma(\alpha_{n-1}+1)\Gamma(\alpha_n+2)}{\Gamma(\alpha_n+\alpha_{n-1}+3)}\cdots\frac{\pi\Gamma(\alpha_1+1)\Gamma(\alpha_n+\cdots+\alpha_2+n)}{\Gamma(\alpha_n+\cdots+\alpha_1+n+1)}\\ &= \frac{\pi^n\alpha_1!\cdots\alpha_n!}{(\alpha_n+\cdots+\alpha_1+n)!}.\end{aligned}$$

Then, our orthonormal basis for $\mathbb{B}^n$ is $\{\phi_\alpha\}_{\alpha\in\mathbb{N}^n}$, where

$$\phi_\alpha = \sqrt{\frac{(|\alpha|+n)!}{\pi^n\alpha!}}z^\alpha.$$

To compute the Bergman kernel we use Theorem 3.11 and obtain

$$\begin{aligned}K_{\mathbb{B}^n}(z,w) &= \sum_{\alpha\in\mathbb{N}^n}\phi_\alpha(z)\overline{\phi_\alpha(w)}\\ &= \sum_\alpha\frac{(\alpha_n+\cdots+\alpha_1+n)!}{\pi^n\alpha_1!\cdots\alpha_n!}z^\alpha\overline{w}^\alpha\\ &= \frac{1}{\pi^n}\sum_{k=0}\sum_{|\alpha|=k}\frac{(\alpha_n+\cdots+\alpha_1+n)!}{\alpha_1!\cdots\alpha_n!}z^\alpha\overline{w}^\alpha\\ &= \frac{1}{\pi^n}\sum_{k=0}(k+n)(k+n-1)\cdots(k+1)(z_1\overline{w}_1+\cdots+z_n\overline{w}_n)^k\\ &= \frac{n!}{\pi^n(1-(z_1\overline{w}_1+\cdots+z_n\overline{w}_n))^{n+1}}.\end{aligned}$$

3.3 Relationship with the Riemann mapping

In this section we will describe the behavior of the Bergman kernel under biholomorphic maps and study an interesting connection with the Riemann Mapping Theorem.

Theorem 3.15. *Let $F : \Omega_1 \to \Omega_2$ be a biholomorphic map between bounded domains in $\mathbb{C}^n$. Let $f_1, \dots, f_n$ be the components of F and $F'(z) = (\frac{\partial f_j(z)}{\partial z_k})_{j,k=1}^n$. Then*

$$K_{\Omega_1}(z,w) = \det F'(z) K_{\Omega_2}(F(z), F(w)) \overline{\det F'(w)}, \tag{3.8}$$

for all $z, w \in \Omega_1$.

Proof. As seen in Lemma 1.11, the determinant of the Jacobian of the mapping F equals to $|\det F'(z)|^2$. Let $g \in L^2(\Omega_2)$, using the substitution formula for integrals we have

$$\int_{\Omega_2} |g(\zeta)|^2 d\lambda(\zeta) = \int_{\Omega_1} |g(F(z))|^2 |\det F'(z)|^2 d\lambda(z).$$

The map $T_F : g \mapsto (g \circ F) \det F'$ establishes an isometric isomorphism from $L^2(\Omega_2)$ to $L^2(\Omega_1)$, with inverse map $T_{F^{-1}}$, which restricts to an isomorphism between $A^2(\Omega_1)$ and $A^2(\Omega_2)$. Let $f \in A^2(\Omega_1)$ and apply the reproducing property of K_{Ω_2} to the function $T_{F^{-1}} f = (f \circ F^{-1}) \det(F^{-1})'$. Rewriting $F(z)$ as u, we get

$$\int_{\Omega_2} K_{\Omega_2}(u,v) T_{F^{-1}} f(v) d\lambda(v) = T_{F^{-1}} f(u) = f(z)(\det F'(z))^{-1}. \tag{3.9}$$

Since T_F is an isometry,

$$\int_{\Omega_2} T_{F^{-1}} f(v) \overline{K_{\Omega_2}(v,u)} d\lambda(v) = \int_{\Omega_1} f(w) \overline{T_F K_{\Omega_2}(\cdot, u)(w)} d\lambda(w). \tag{3.10}$$

Now, from (3.9) and (3.10), we obtain

$$f(z) = \int_{\Omega_1} \det F'(z) K_{\Omega_2}(F(z), F(w)) \overline{\det F'(w)} f(w) d\lambda(w),$$

hence, the right side of the equation (3.8) has the reproducing property and belongs to $A^2(\Omega_1)$. By the uniqueness theorem it must agree with $K_{\Omega_1}(z,w)$. □

We will now see a useful formula for the orthogonal projections

$$P : j : L^2(\Omega_j) \to A^2(\Omega_j), j = 1, 2.$$

Theorem 3.16. *For all $g \in L^2(\Omega_2)$ one has*

$$P_1(\det F' g \circ F) = \det F'(P_2(g) \circ F).$$

Proof. We rewrite the left-hand side of the equality as $P_1(T_F(g))$ and using Theorem 3.4 it gives us $P_1(T_F(g)) = T_F(g)$. Hence, by (3.3), we obtain

$$P_1(T_F(g))(z) = \int_{\Omega_1} K_{\Omega_1}(z,w) T_F(g)(w) d\lambda(w), \quad z \in \Omega_1.$$

Using (3.8) together with (3.10), it is seen that

$$K_{\Omega_1}(w,z) = [T_F(K_{\Omega_2}(\cdot, F(z)))(W)]\overline{\det F'(z)},$$

and since T_F is isometric, we get

$$\begin{aligned} P_1(T_F(g))(z) &= \det F'(z) \int_{\Omega_1} T_F(g)(w)\overline{T_F(K_{\Omega_2}(\cdot, F(z)))(w)} d\lambda(w) \\ &= \det F'(z) \int_{\Omega_2} g(v)\overline{K_{\Omega_2}(v, F(z))} d\lambda(v) \\ &= \det F'(z)(P_2(g))(F(z)), \end{aligned}$$

and the result has been proven. □

If $\Omega \subset \mathbb{C}$ is a simply connected domain, there is a connection between the Bergman kernel K_Ω of Ω and the Riemann mapping theorem.

Theorem 3.17. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain, and let K_Ω be the Bergman kernel of Ω. Let $F : \Omega \to \mathbb{D}$ be the Riemann mapping with the uniqueness properties $F(a) = 0, F'(a) > 0$ for some $a \in \Omega$. Then*

$$F'(z) = \sqrt{\frac{\pi}{K_\Omega(a,a)}} K_\Omega(z,a), \quad z \in \Omega. \tag{3.11}$$

Proof. The transformation T_F establishes an isometry between $L^2(\Omega)$ and $L^2(\mathbb{D})$ which restricts to be an isometry between $A^2(\Omega)$ and $A^2(\mathbb{D})$. As it is an isometry, it follows that

$$(T_F u, T_F u)_\Omega = (u,u)_\mathbb{D}, \quad u \in L^2(\mathbb{D}),$$

where $(\cdot,\cdot)_\Omega$ is the inner product in $L^2(\Omega)$ and $(\cdot,\cdot)_\mathbb{D}$ is the inner product of $L^2(\mathbb{D})$. For $v \in L^2(\Omega)$ and $G = F^{-1}$ we have

$$(T_G v, T_G v)_\mathbb{D} = (v,v)_\Omega.$$

Using the following polarization identity

$$(u_1,u_2) = \frac{1}{4}(\|u_1+u_2\|^2 - \|u_1-u_2\|^2) - \frac{i}{4}(\|u_1+iu_2\|^2 - \|u_1-iu_2\|^2)$$

one obtains

$$(T_F u_1, T_F u_2)_\Omega = (u_1,u_2)_\mathbb{D} \quad \text{and} \quad (T_G v_1, T_G v_2)_\mathbb{D} = (v_1,v_2)_\Omega \tag{3.12}$$

for $u_1, u_2 \in L^2(\mathbb{D})$ and $v_1, v_2 \in L^2(\Omega)$. $T_F T_G$ is the identity, hence from (3.12) we get

$$(T_F u, v)\Omega = (T_F u, T_F(T_G v))_\Omega = (u, T_G v)_\mathbb{D}. \tag{3.13}$$

Let $h \in A^2(\mathbb{D})$, observe that the inner product $(h, 1)_{\mathbb{D}}$ is:

$$\begin{aligned}(h, 1)_{\mathbb{D}} &= \int_{\mathbb{D}} h(z) \cdot \overline{1} d\lambda(z) \\ &= \int_{\mathbb{D}} h(z) d\lambda(z)\end{aligned}$$

and by (3.1) we get that $(h, 1)_{\mathbb{D}} = \pi h(0)$. Using (3.13), we get that for $f \in A^2(\Omega)$

$$(f, F')_{\Omega} = (G'(f \circ G), 1)_{\mathbb{D}} = \pi G'(0) f(G(0)) = \frac{\pi}{F'(a)} f(a).$$

The function $\frac{\overline{F'(a)}}{\pi} F'(z)$ has the reproducing property and belongs to $A^2(\Omega)$, so by the uniqueness theorem it must be the Bergman kernel of $A^2(\Omega)$. We now get

$$F'(z) = \frac{\pi}{\overline{F'(a)}} K_{\Omega}(z, a),$$

and setting $z = a$, one obtains $F'(a)^2 = \pi K_{\Omega}(a, a)$, which proves the result. □

Remark 3.18. The connection between the Bergman kernel and the Riemann mapping seen in Theorem 3.17, is not merely of theoretical interest; it also has practical implications for numerical approximation. The formula

$$F'(z) = \sqrt{\frac{\pi}{K_{\Omega}(a, a)}} K_{\Omega}(z, a), \quad z \in \Omega$$

provides a concrete method to compute the derivative of the Riemann mapping function directly from the Bergman kernel. Since the Bergman kernel can often be approximated numerically, this theorem serves as a valuable tool for approximating the Riemann mapping function itself.

Conclusions

In this work, we have studied how some of the most important theorems in complex analysis of one variable behave when extending to higher dimensions. This analysis has revealed fundamental differences and limitations that emerge when moving from one complex variable to several, offering a deeper understanding of the structure and scope of complex analysis.

One key result we explored is the Riemann mapping theorem, a central theorem in one variable complex analysis. This theorem guarantees the existence of a conformal map that transforms any simply connected domain $\Omega \subset \mathbb{C}$ onto the unit disc. However, we observed that this theorem does not hold when transitioning to higher dimensions, reflecting the increased complexity and constraints of higher dimensional complex spaces.

We also studied Bergman spaces, which consist of square-integrable holomorphic functions, and investigated their properties. A significant finding is that given a Bergman kernel, it is possible to compute Riemann mappings using it. This is relevant because, given certain conditions, the Bergman kernel can be computed using an orthonormal basis, which can be found numerically. This method provides a practical framework for numerical approximation of Riemann mappings, making their computation more feasible in applied settings.

Looking ahead, future research could focus on studying more advanced results of analysis in several complex variables. Regarding functions of one complex variable, we could see if there are any applications of the Riemann mapping theorem where problems can be solved by mapping them to a different space.

References

[Ran86] R. Michael Range. *Holomorphic Functions and Integral Representations in Several Complex Variables*. Springer-Verlag New York Inc, 1986. Chap. 1.

[Rem91] Reinhold Remmert. *Theory of Complex Functions*. Springer, 1991. Chap. 2, 3.

[Ort97] Joaquin Ortega-Cerdà. *Anàlisi complexa*. 1997. Chap. 4.

[Bru00] M.Carter B. van Brunt. *The Lebesgue-Stieltjes integral*. Springer, 2000. Chap. 9.

[Ull00] David C. Ullrich. *Complex Made Simple*. American Mathematical Society, 2000. Chap. 8, 9.

[Sha03] Elias M. Stein Rami Shakarchi. *Complex Analysis*. Princeton University Press, 2003, p. 206.

[Has10] Friedrich Haslinger. *Complex Analysis*. Walter de Gruyter GmbH, 2010. Chap. 7.

[Yu] Xinwei Yu. *Fubini for continuous functions over intervals*. University of Alberta. URL: https://www.math.ualberta.ca/~xinweiyu/217.1.13f/. (accessed: 14.11.2024).