UNIVERSITAT DE BARCELONA

Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

Chern-Simons Theory

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Barcelona, 15 de gener de 2025

Contents

Abstract

The aim of this thesis is to prove that the set of critical points of the Chern-Simons classical action for a closed, three-dimensional spacetime manifold M and a compact, simply connected Lie group G is the set of flat G-connections over M.

To establish this result, we first develop the foundational theory of Lie groups, Lie algebras and principal bundles – fibre bundles with a Lie group as their fibre.

2020 Mathematics Subject Classification. 57R22, 57R20, 58J28.

Resum

Títol del TFG: Teoria de Chern-Simons.

L'objectiu d'aquest TFG és demostrar que el conjunt de punts crítics de l'acció clàssica de Chern-Simons per a una varietat espai-temps tridimensional tancada M i un grup de Lie compacte i simplement connex G és el conjunt de connexions planes de G sobre M.

Per establir aquest resultat, primer desenvolupem la teoria fonamental dels grups de Lie, les àlgebres de Lie i els fibrats principals, que són fibrats amb un grup de Lie com a fibra.

Acknowledgements

I would like to begin by wholeheartedly thanking my thesis supervisor, Prof. Ricardo García, for his skilled, continued and understanding support thoughout my whole thesis. Even though my interest in geometry and mathematical physics goes way back, this thesis has convinced me of the beauty of the field and will certainly shape my academic journey moving forward.

Next, I would like to thank my friends for ultimately being my ballast through this long five and a half years of double degree. Special thanks to Laura and Marc-Jamel, for being the best pair of friends one could wish for.

Lastly, I would like to thank my family above all. To my brother for bearing with me all these years and for being that wonderful counterbalance in my life. And to my parents, who, of course, need no explanation.

Introduction

It would only be a slight exaggeration to say that the language of modern theoretical physics is that of *Lagrangian theories* – remarkably so in fields as important as quantum mechanics or general relativity.

In order to further understand what these theories entail, we must first talk about *fields*. Restricting ourselves to the classical case, we can simply define a field as a physical entity that is defined at every point of time and space, or more generally at every point on a *smooth manifold*. Fields can be classified into scalar, vectorial or tensorial, depending on the type of value they output at each point. For instance, a function assigning to each point of a solid the temperature at that point is a scalar field, while the electric field produced by a charge is of vectorial type. Mathematically, we typically think of fields as smooth *sections* of smooth *fibre bundles*, that is, spaces that locally take the form of a product space but could globally have a different topology. A well-known example of a fibre bundle are *vector bundles*, including, for instance, the tangent bundle $TM \to M$ of a smooth manifold M. We will denote the set of fields for a particular fibre bundle by $\mathcal{F}$.

Of course, one of the main goals of physics is to predict and model the evolution of physical entities, which leads us to (classical) field theories: physical frameworks that study the dynamics of fields and how these interact with matter through *field equations*, without considering quantisation effects. A notable formalism within classical field theory is *Lagrangian field theory*, which we now proceed to define. Stepping again into the realm of mathematics, the main object of study is the *Lagrangian*[2], formally defined as a smooth map

$$\mathcal{L} : \mathcal{F} \longrightarrow \Omega^n(M).$$

Here, M denotes an n-dimensional manifold, and $\Omega^n(M)$ represents the vector space of top forms on M. Typically, M is interpreted as *spacetime*.

We can then define the *action* as

$$\mathcal{S} : \mathcal{F} \longrightarrow \mathbb{R}, \quad \phi \longmapsto \int_M \mathcal{L}(\phi), \tag{1}$$

whenever this integral makes sense (i.e. it exists and it is finite). The key piece of insight that motivates this whole construction is the following principle, known as *Maupertuis's principle* or the *principle of least action*[3]:

[2]Or, more precisely, *Lagrangian density*.

[3]Albeit common, this is somewhat of a misnomer: as the principle clearly states, the action need not be minimal but simply extremal.

Maupertuis's Principle: *A field $\phi \in \mathcal{F}$ is a solution of the field equation if and only if it is a* critical point *of the action $\mathcal{S}$.*

Field equations, as noted above, represent the laws governing fields, defining the conditions that fields must satisfy to be deemed physically plausible. Alternatively, they may emphasize specific fields that exhibit desirable mathematical properties. They are expressed as $f(\phi) = 0$, where

$$f : \mathcal{F} \longrightarrow V$$

is a function from the set of fields to a vector space V.

It is then the job of the physicist – or the mathematician – to find out what the lagrangian of the system is, proceeding later to calculate the critical points of the action to unveil what fields are compatible with the field equations.

This is where Chern-Simons theory comes into play: in their seminal paper [11], mathematicians S-S. Chern and J. Simons introduced the *Chern-Simons form*[4]. Reserving the details for Chapter 3 below, these forms – interpretable as the Lagrangians of a classical field theory – are defined over the space of *connections* (our fields) of *principal G-bundles*, a type of fibre bundle that has a Lie group G as its fibre. A Lie group, being a group that is also a smooth manifold with smooth multiplication and inverse operations, perfectly embodies the concept of *continuous symmetry*, which explains its appeal and prominence in physical theories.

Closely tied with Lie groups, it is no surprise that principal G-bundles are also ubiquitous in physics. Furthermore, they are interesting mathematical objects in their own right, exhibiting a rich structure stemming from the interplay between (Lie) groups and manifolds.

Chern-Simons theory is particularly powerful in dimension $3 = 2 + 1$, which reflect spacetimes with two spatial and one temporal dimension. Although this thesis focuses on the classical version of the theory – which already demands a remarkable level of subtlety – its quantum counterpart has had a profound impact in physics, mathematics and even certain aspects of quantum computing.

In particular, the objective of the thesis was to develop the basic notions of Chern-Simons theory for the particularly simple case of a spacetime manifold M that is *closed* – compact and without border – and a Lie group G that is compact and simply connected. The compact condition on M ensures that the action (1) is well-defined, while limiting ourselves to manifolds without border is purely for the sake of brevity. The hypothesis on G allow us to deal only with *trivialisable* principal G-bundles, which are not only locally a product space but also globally. With these hypothesis, we have above all focused on characterising the set of critical points of the Chern-Simons 3-action, which is Proposition 3.28 below.

Structure of this work

In order to arrive to Proposition 3.28, we have structured our work as follows. In Chapter 1, we introduce the fundamental concepts of Lie groups and Lie algebras. Lie algebras,

[4] For manifolds of arbitrary dimension, although we shall focus on (closed) three-manifolds.

defined as the tangent space of a Lie group at the identity element, are particularly significant because they encapsulate much of the structure of the Lie group while being simpler in nature – as they form a vector space.

In the following chapter, we establish the foundations of principal bundle theory, covering the basic definitions as well as explaining how connections and curvature are defined in this case.

Finally, in Chapter 3, we touch on Chern-Simons theory. After defining essential objects for the theory such as the *Maurer-Cartan form* or the group of *gauge transformations*, we define the Chern-Simons 3-form (Definition 3.12) and prove its most important properties in Proposition 3.14. Lastly, after introducing the *category* of all connections over a given spacetime M, we define the Chern-Simons action and finish, as mentioned before, characterising the set of solutions of this classical field theory – which turn out to be the set of *flat connections*, that is, connections for which the associated curvature identically vanishes.

Chapter 1

Lie Groups and Lie Algebras

1.1 Lie Groups

Lie groups, and by extension Lie algebras, lie at the intersection between group theory, differential geometry and linear algebra. They are fundamental in the construction of principal bundles, which in turn are tightly related to Chern–Simons theory.

As such, they are central to our work. Thus, let us begin by giving the definition of a Lie group,

Definition 1.1. *A* Lie Group G *is both a smooth manifold and a group, such that the group operations of multiplication,*

$$\mu : G \times G \to G, \quad (g, h) \mapsto g \cdot h := \mu(g, h)$$

and inverse,

$$\kappa : G \to G, \quad g \mapsto g^{-1}$$

are smooth as maps between smooth manifolds.

We will typically omit the dot when referring to the product of elements in G, indicating $g \cdot h$ simply by juxtaposition gh.

We also assume familiarity with the basics of differential geometry, so we do not elaborate further on standard concepts like *smooth manifolds* or *smooth maps* between manifolds. Unless explicitly stated, the terms 'manifold' and 'map' are to be understood as shorthand for 'smooth manifold' and 'smooth map between smooth manifolds', respectively.

Notation: For the remainder of this thesis, G will denote an arbitrary Lie group and μ will represent the group multiplication, unless explicitly stated otherwise.

We further define the ***dimension*** of a Lie group G as its dimension when viewed as a manifold.

Notation: If $f : N \to M$ is a smooth map between manifolds and $p \in N$, we denote the differential of f at p by

$$f_{*,p} : T_pN \to T_{f(p)}M$$

instead of the more conventional notation df_p to avoid confusion later on with the exterior derivative of k-forms.

A key property of Lie groups is that they are *homogeneous*. Formally, this means the following:

Proposition 1.2. *For any $g \in G$, the map defined by left multiplication by g*

$$l_g : G \to G, \quad x \mapsto gx$$

is a diffeomorphism.

Proof. Denote by $\iota_g : G \hookrightarrow \{g\} \times G$ the inclusion map. By virtue of the *regular level set theorem* (see [1], p. 105), $\{g\} \times G$ is easily seen to be a regular submanifold of $G \times G$. Hence, since the restriction of a smooth map to a regular submanifold is still smooth, $l_g = \mu|_{\{g\}\times G} \circ \iota_g$ is smooth.

Moreover, $l_{g^{-1}}$ is quite clearly l_g's inverse, and is smooth for analogous reasons to that which prove l_g's smoothness. □

Roughly speaking, this seemingly simple result conveys the idea that the group looks the same from the perspective of any of its elements. Thus, we can focus on the surroundings of G's identity, which we will denote as e_G. If no confusion arises regarding the group to which the identity element belongs, we may simply refer to it as e. As we will see, this idea of homogeneity will later motivate the importance of the Lie algebra $\mathfrak{g}$, which will be defined further in the text.

Definition 1.3. *Suppose that G and H are Lie groups. A map $f : G \to H$ is said to be a* Lie group homomorphism *if it is smooth and a group homomorphism.*

Remark 1.4. We note that, since for any $g, h \in G$, a Lie group homomorphism $F : G \to H$ must satisfy

$$F(gh) = F(g)F(h).$$

Consequently, a smooth map $F : G \to H$ between Lie groups is a Lie group homomorphism if, and only if,

$$F \circ l_g = l_{F(g)} \circ F, \quad \forall g \in G.$$

Having covered the basic definitions in Lie group theory, let us now present the first, and notably important, example of a Lie group: the *general linear group* $\mathrm{GL}(n, \mathbb{R})$ of degree n over the reals, for any positive integer n.

Notation: Fix n a positive integer. We define $\mathcal{M}(n, \mathbb{R})$ to be the set of all $n \times n$ matrices with real coefficients. Given $M \in \mathcal{M}(n, \mathbb{R})$ we denote by M_{ij} the entry of M in the i-th row and the j-th column.

Proposition 1.5. *The general linear group,*

$$\mathrm{GL}(n, \mathbb{R}) := \{M \in \mathcal{M}(n, \mathbb{R}) : \det(M) \neq 0\},$$

is a Lie group with the operation of matrix multiplication.

Proof. The map

$$\det : \mathcal{M}(n, \mathbb{R}) \to \mathbb{R}$$

is continuous. Therefore, $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is an open subset of $\mathcal{M}(n, \mathbb{R})$. Now, $\mathcal{M}(n, \mathbb{R})$ can be identified with the manifold $\mathbb{R}^{n^2}$, which allows its open subset $\mathrm{GL}\,(n)$ to naturally inherit a manifold structure.

It remains to be proven that the group operations, multiplication and inverse, are smooth. This follows directly from the fact that, in both cases, the entries of the result matrix are smooth functions of the entries from the input matrices. For matrix multiplication, this is immediately clear, whereas for matrix inversion, we can apply Cramer's rule from linear algebra. According to this result,

$$\left(M^{-1}\right)_{ij} = \frac{\operatorname{adj}(M)_{ij}}{\det(M)},$$

where $\operatorname{adj}(M)$ denotes the *adjugate*[1] of M. Provided that $\det(M) \neq 0$, this shows that $\left(M^{-1}\right)_{ij}$ is a smooth function of the entries of M. □

During the course of the preceeding proof we argued why we can view $\mathrm{GL}(n, \mathbb{R})$ as an open subset of $\mathbb{R}^{n^2} \sim \mathbb{R}^{n\times n}$. Let us fix some notation that will be useful later on:
Notation: We denote the standard coordinates of $\mathbb{R}^{n\times n}$ by $r^{ij}, 1 \leq i, j \leq n$. Taking

$$\phi : \mathrm{GL}(n, \mathbb{R}) \to \phi(\mathrm{GL}(n, \mathbb{R})) \subset \mathbb{R}^{n\times n}$$

to be the global chart identifying $\mathrm{GL}(n, \mathbb{R})$ with an open subset of $\mathbb{R}^{n\times n}$, let $x^{ij} := r^{ij} \circ \phi$. We may express $A \in T_I(\mathrm{GL}(n, \mathbb{R}))$, with I the identity map, as[2]

$$A = a^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_I = a^{ij}\, \partial_{ij}\big|_I\,,$$

for coefficients $a^{ij} \in \mathbb{R}$.
Interestingly, in the case $G = \mathrm{GL}(n, \mathbb{R})$, we can give a closed expression for the differential of l_g at the identity, (where l_g is the map introduced in Proposition 1.2):

Proposition 1.6. *For any $g = [g^{ij}] \in \mathrm{GL}(n, \mathbb{R})$ and any $A = a^{ij}\, \partial_{ij}\big|_I \in T_I(\mathrm{GL}(n, \mathbb{R}))$,*

$$l_{g_{*,I}}(A) = \left(\sum_k g^{ik} a^{kj}\right) \partial_{ij}\big|_g$$

Proof. This directly follows from considering a curve $\gamma(t)$ starting at A, the $\mathbb{R}$-linearity of the derivative $\frac{d}{dt}$ and the definition of l_g. □

1.2 Lie Algebras

1.2.1 Left-invariant Vector Fields

Notation: If M is a (smooth) manifold, we define $\mathfrak{X}(M)$ to be the real vector space of all smooth vector fields – i.e. the vector space of all smooth sections of the tangent bundle TM. As before, 'vector field' will mean 'smooth vector field', and we will reserve the nomenclature 'rough vector field' for a vector field that is not necessarily smooth. Lastly, for $X \in \mathfrak{X}(M)$ and $p \in M$, we denote the image of X at p as $X_p \in T_pM$.
Similarly, we define $C^\infty(M)$ to be the set of all real-valued smooth functions. If $f \in C^\infty(M)$ and X is a rough vector field, then Xf denotes the function on M given by $(Xf)(p) := X_p f, p \in M$.

[1] $\operatorname{adj}(M)_{ij} := (-1)^{i+j}\cdot$ (determinant of the $(n-1)\times(n-1)$ matrix resulting from deleting row i and column j of M).
[2] We will use Einstein summation convention throughout this thesis, whenever possible.

Since the following standard fact about vector fields will be used in several proofs throughout the text, we record it here for convenience:

Proposition 1.7. *Let X be a rough vector field on a manifold M. Then,*

$$X \in \mathfrak{X}(M) \iff \forall f \in C^\infty(M), Xf \in C^\infty(M).$$

Proof. The proof is outlined in [1], p.151. □

Definition 1.8. *Take X a rough vector field in G. We say that X is a* left-invariant vector field *if for all $g, h \in G$,*

$$X_{gh} = (l_g)_{*,h}(X_h).$$

We define $L(G)$ to be the set of all left-invariant vector fields.

By the linearity of the differential, it is straightforward to see that $L(G)$ has a real vector space structure. Furthermore, we do not need to assume that a left-invariant vector field is smooth, as this is automatically satisfied:

Proposition 1.9. *$L(G)$ is a vector subspace of $\mathfrak{X}(G)$.*

Proof. Choose $X \in L(G)$. By proposition 1.7, it suffices to show that, for any $f \in C^\infty(G)$, we have $Xf \in C^\infty(G)$. To demonstrate this, we will express Xf as the composition of smooth functions.

Assume $I \subset \mathbb{R}$ is an open interval containing 0, and that $\gamma : I \to G$ is a smooth curve such that $\gamma(0) = e$ and $\gamma'(0) = X_e$. Define $c(t) = g\gamma(t)$, for $g \in G$. Then, c is also a smooth curve on I, satisfying $c(0) = g$ and

$$c'(0) = (g\gamma)'(0) = (l_g \circ \gamma)'(0) = l_{g_{*,e}}\left(\gamma'(0)\right) = l_{g_{*,e}}\left(X_e\right) = X_g.$$

Where we have used the chain rule and that X is left-invariant. Now, the function

$$\begin{aligned} G \times I &\xrightarrow{\mathrm{Id}_G \times \gamma} G \times G \xrightarrow{\mu} G \xrightarrow{f} \mathbb{R} \\ (g,t) &\mapsto (g, \gamma(t)) \mapsto g\gamma(t) \mapsto f(g\gamma(t)) \end{aligned}$$

is smooth, as all functions involved in the above diagram are smooth. Consequently, its derivative with respect to t

$$\tilde{f} : G \times I \longrightarrow \mathbb{R}, \quad (g,t) \mapsto \left.\frac{d}{ds}\right|_{s=t} f(g\gamma(s))$$

is also smooth. Thus, picking $g \in G$ and defining $\iota_0 : G \hookrightarrow G \times I$, $\iota_0(g) = (g, 0)$ to be the inclusion map, we see that for $g \in G$

$$(Xf)(g) = X_g f = \left(c'(0)\right) f = \left.\frac{d}{dt}\right|_{t=0} f(c(t)) = \left.\frac{d}{dt}\right|_{t=0} f(g\gamma(t)) = \tilde{f} \circ \iota_0(g).$$

Allowing us to conclude that, being the composition of two smooth function, Xf is also smooth. □

Another notable fact about left-invariant vector fields is that they are completely defined by their value at e, in the following sense. Take $A \in T_eG$, and define a rough vector field A^L by setting

$$A^L_g := l_{g_{*,e}}(A), \quad \forall g \in G. \tag{1.1}$$

Since $l_{gh} = l_g \circ l_h$, it is clear that, for all $g, h \in G$,

$$A^L_{gh} = l_{gh_{*,e}}(A) = l_{g_{*,h}}(l_{h*,e}(A)) = l_{g_{*,h}}(A^L_h),$$

so that, in fact, A^L belongs to $L(G)$. Even more is true,

Proposition 1.10. *The map*

$$\mathrm{ev}_e : L(G) \longrightarrow T_eG, \quad X \mapsto X_e$$

is a vector space isomorphism, with inverse map $A \mapsto A^L$.

Proof. By the definition of the sum and product by a scalar of vector fields, the map is linear. It is also apparent that $(\cdot)^L : T_eG \longrightarrow L(G)$ is its inverse. For example, considering $A \in T_eG$,

$$\mathrm{ev}_e(A^L) = l_{g_{*,e}}(A) = \mathrm{Id}_{T_eG}(A) = A.$$

□

1.2.2 The Lie Algebra of a Lie Group

Definition 1.11. *A* Lie algebra *over a field F is a pair $(V, [\ ,\ ])$ where V is a vector space over F and*

$$[\ ,\ ] : V \times V \longrightarrow V$$

is a map, called the Lie bracket*, that satisfies the following properties:*

LB1 *It is F-bilinear.*

LB2 *It is anticommutative: $[Y, X] = -[X, Y]$*

LB3 Jacobi identity:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

If $(V, [\ ,\ ]_V)$ and $(W, [\ ,\ ]_W)$ are two Lie algebras, a ***Lie algebra homomorphism*** is a linear map $f : V \to W$ that preserves brackets – i.e. for all $u, v \in V$, $f([u, v]_V) = [f(u), f(v)]_W$.

Example 1.12. Given a manifold M, we know from standard differential geometry that we can endow $\mathfrak{X}(M)$ with a Lie algebra structure by defining, for $X, Y \in \mathfrak{X}(G)$, a new vector field $[X, Y] \in \mathfrak{X}(M)$ satisfying

$$[X, Y]f = X(Yf) - Y(Xf), \quad \forall f \in C^\infty(M). \tag{1.2}$$

Unless explicitly stated, we always assume that $\mathfrak{X}(M)$ is a Lie algebra, with the Lie algebra defined above.

The isomorphism established in Proposition 1.10 proves to be highly useful, as it allows us to transfer additional structures between the vector spaces involved. For our purposes, the most significant application is using the Lie algebra structure on $\mathfrak{X}(G)$ (cf. Example 1.12) to define a Lie bracket on T_eG.

Indeed, for $A, B \in T_eG$ we define

$$[A, B] := \left([A^L, B^L]\right)_e, \tag{1.3}$$

where $(\cdot)^L : T_eG \longrightarrow L(G)$ is the map from the previous section. In fact, one can easily show that the subspace $L(G)$ is ***closed*** under this Lie bracket. More precisely, if again we take $A, B \in T_eG$, the following identity is an easy consequence of the preceeding definitions:

$$[A^L, B^L] = [A, B]^L \in L(G).$$

It is then routine to check that the axioms of Definition 1.11 hold. When viewing T_eG as a Lie algebra with this induced Lie bracket, we commonly denote it by $\mathfrak{g}$.

Remark 1.13. By the very definition of the Lie bracket on $\mathfrak{g}$ given in Eq. (1.3), the map discussed in Proposition 1.10 is a Lie algebra isomorphism.

In the proof of Proposition 1.5 we identified $G = \mathrm{GL}(n, \mathbb{R})$ as an open subset of $\mathbb{R}^{n\times n}$. Consequently, the vector space component of its Lie algebra, denoted $\mathfrak{gl}(n, \mathbb{R}) := \mathfrak{g}$, can be thought of as $\mathcal{M}(n, \mathbb{R})$. The proposition below provides an explicit expression for the Lie bracket in this Lie algebra, under this identification.

Proposition 1.14. *The induced Lie bracket on $\mathfrak{gl}(n, \mathbb{R})$ (1.3) is the **commutator bracket**,*

$$[A, B] = AB - BA, \quad A, B \in \mathcal{M}(n, \mathbb{R}),$$

where we are identifying $\mathfrak{gl}(n, \mathbb{R})$ with $\mathcal{M}(n, \mathbb{R})$ and thus juxtaposition stands for matrix multiplication.

Proof. Using the notation introduced in the paragraphs following Proposition 1.5, we can express $A, B \in \mathfrak{gl}(n, \mathbb{R})$ as

$$A = a^{ij}\, \partial_{ij}\big|_I \quad \text{and} \quad B = b^{ij}\, \partial_{ij}\big|_I,$$

for real coefficients a^{ij} and b^{ij}. The current identification then sends $A \in \mathfrak{gl}(n, \mathbb{R})$, for instance, to $\widetilde{A} = [a^{ij}] \in \mathcal{M}(n, \mathbb{R})$. We prove the equality by direct computation. Suppose that

$$[A, B] := [A^L, B^L]_I = c^{ij}\, \partial_{ij}\big|_I,$$

for some coefficients $c^{ij} \in \mathbb{R}$. We aim to show that

$$c^{ij} = (AB - BA)^{ij} = \sum_k a^{ik}b^{kj} - b^{ik}a^{kj}.$$

In order to do so, we first solve for c^{ij},

$$c^{ij} = [A^L, B^L]_I x^{ij} = (A^L)_I(B^L x^{ij}) - (B^L)_I(A^L x^{ij}) = A(B^L x^{ij}) - B(A^L x^{ij}), \tag{1.4}$$

where we have used the definition (1.2) of the Lie bracket in $\mathfrak{X}(\mathrm{GL}(n,\mathbb{R}))$ and of left-invariant vector field (1.1). Proposition 1.6 now provides an explicit expression for $(B^L)_g, g = [g^{ij}] \in \mathrm{GL}(n,\mathbb{R})$, from which we deduce that

$$(B^L)_g x^{ij} \overset{1.6}{=} \left[(\sum_k g^{lk} b^{ks})\, \partial_{ls}|_g \right] x^{ij} = \sum_k g^{ik} b^{kj} = \sum_k x^{ik}(g) b^{kj} \Rightarrow B^L x^{ij} = \sum_k b^{kj} x^{ik}.$$

Therefore,

$$A(B^L x^{ij}) = \left[a^{ls}\, \partial_{ls}|_I \right] \left(\sum_k b^{kj} x^{ik} \right) = \sum_k a^{ls} b^{kj}\, \partial_{ls}|_I\, (x^{ik}) = \sum_k a^{ik} b^{kj}.$$

This last equality, jointly with symmetry considerations and Equation (1.4), prove the desired result. □

We conclude this subsection showing that all Lie groups accept global smooth frames, (i.e. they are ***parallelisable***):

Proposition 1.15. *Suppose G is a Lie group of dimension n with Lie algebra $\mathfrak{g}$. If $B_1, \dots B_n$ is a basis for $\mathfrak{g}$, $B_1^L, \dots B_n^L \in L(G)$ is a smooth frame for the tangent bundle $TG \to G$.*

Proof. The vector fields $B_i^L, 1 \le i \le n$, are smooth by Proposition 1.9. Furthermore, the diffeomorphism $l_g : G \to G, g \in G$, induces a linear isomorphism

$$(l_g)_{*,e} : \mathfrak{g} \longrightarrow T_g G.$$

This readily implies that the vectors $(B_i^L)_g = (l_g)_{*,e}(B_i), 1 \le i \le n$, are a basis for $T_g G$, for all $g \in G$. □

1.2.3 Lie Algebra Homomorphisms

Consider a Lie group homomorphism $F : G \to H$, and denote the respective Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$. The aim of this subsection is to show that the differential of F at the identity e_G is a Lie algebra homomorphism,

$$F_{*,e_G} : \mathfrak{g} \longrightarrow \mathfrak{h}.$$

To do so, we introduce the following definition, which will also be useful in later sections:

Definition 1.16. *Let $F : N \to M$ be a map between manifolds. Choose X and $\overline{X}$ rough vector fields on N and M, respectively. We say that X and $\overline{X}$ are F-**related**, and denote it by $X \sim^F \overline{X}$, if, for all $p \in N$,*

$$\overline{X}_{F(p)} = F_{*,p}(X_p). \tag{1.5}$$

An easy to prove, but nevertheless significant, result concerning F-relatedness is the one below:

Proposition 1.17. *Suppose $F : N \to M$ is a map between manifolds. If X and Y are vector fields on N that are F-related to the vector fields $\overline{X}$ and $\overline{Y}$ on M, respectively, then $[X, Y]$ is F-related to $[\overline{X}, \overline{Y}]$.*

A proof of the above proposition can be found in [1], p.160.

Proposition 1.18. *Let $F : G \to H$ be a Lie group homomorphism. Then, $F_{*,e_G} : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. Take $A, B \in \mathfrak{g}$. For $g \in G$,

$$([F_{*,e_G}(A)]^L)_{F(g)} = \left(l_{F(g)}\right)_{*,e_H}(F_{*,e_G}(A)) = (l_{F(g)} \circ F)_{*,e_G}(A).$$

As seen in Remark 1.4, $l_{F(g)} \circ F = F \circ l_g$. Consequently,

$$([F_{*,e_G}(A)]^L)_{F(g)} = (F \circ l_g)_{*,e_G}(A) = F_{*,g}((A^L)_g),$$

demonstrating that the vector fields A^L and $(F_{*,e_G}A)^L$ are F-related. An analogous result holds for B^L and $(F_{*,e_G}B)^L$, hence implying that

$$[A^L, B^L] \sim^F [(F_{*,e_G}A)^L, (F_{*,e_G}B)^L]$$

by Proposition 1.17. Evaluating the F-relatedness relation (1.5) for this two vector fields at $F(e_G) = e_H$ proves the result. □

Remark 1.19. Let $F : N \to M$ be a diffeomorphism, and take $X \in \mathfrak{X}(N)$. We can extend the notion of F-relatedness to that of a ***pushforward*** induced by F, *defining* $F_*(X) \in \mathfrak{X}(M)$ by

$$(F_*X)_{F(p)} := F_{*,p}(X_p), \forall p \in N.$$

Note that we need the surjectivity of F to ensure that $F_*(X)$ is defined for all $q \in M$, and its injectivity in order to assign a unique $p \in N$ for any $F(p) \in M$.

1.3 The Exponential Map of a Lie Group

1.3.1 The Exponential Map

The exponential map is of great importance in Lie group theory as it establishes a deep connection between a Lie group and its Lie algebra. Before defining it, we first introduce several definitions and results concerning vector fields on a Lie group G.

Definition 1.20. *Let X be a rough vector field on a manifold M. Suppose $I \subset \mathbb{R}$ is an open interval and let $\gamma : I \to M$ be a smooth curve on M. We say that γ is an* integral curve *of X if, for all $t \in I$,*

$$\gamma'(t) = X_{\gamma(t)}.$$

An integral curve $\gamma : I \to M$ of X is said to be ***maximal*** if it cannot be extended – i.e. if J is an open interval containing I and $\gamma' : J \to M$ is an integral curve of X satisfying $\gamma'|_I = \gamma$, then $J = I$. If γ is a maximal integral curve, $0 \in I$ and $p = \gamma(0)$, we say that γ is the maximal integral curve of X around p, denoted by $\theta(\cdot\,; p) : I^{(p)} \to M$. Whenever it is necessary to specify the vector field explicitly, we will write the curve as $\theta_X(\cdot\,; p)$.

The ***Fundamental Theorem on Flows*** (see, for instance, [2], p.212) asserts that if X is a (smooth) vector field on M, then there exists a unique maximal solution around $p \in M$, for every p.

Regarding left-invariant vector fields on Lie groups, we have the following result.

Proposition 1.21. *Let $\theta(\cdot\,;g) : I^{(g)} \to G$ be the maximal integral curve of $X \in \mathfrak{X}(G)$ around $g \in G$. Then, $I^{(g)} = \mathbb{R}$.*

Proof. A detailed proof is given in [3], p. 119. □

We can now define the exponential map,

Definition 1.22. *Let G be a Lie group with its corresponding Lie algebra $\mathfrak{g}$. We define*

$$\exp : \mathfrak{g} \longrightarrow G, \quad A \mapsto \theta_{A^L}(1;e),$$

to be the exponential map *of the Lie group G.*

This notion of exponential is well defined because we know that (I) the left-invariant vector field associated to an element of the Lie algebra is smooth by Proposition 1.9, (II) there is a unique maximal integral curve around $e \in G$ by virtue of the Fundamental Theorem on Flows and (III) we can evaluate this curve at 1 since its domain, $I^{(e)}$, is all $\mathbb{R}$ (Proposition 1.21).

Proposition 1.23. *The exponential map satisfies the following properties,*

1. *Fix $A \in \mathfrak{g}$ and $g \in G$. Then, $\theta_{A^L}(t;g) = g\exp(tA)$.*
2. *The map $\exp : \mathfrak{g} \to G$ is smooth.*

Proof. Property 1 can be easily seen to hold (its proof can be found in [3], p. 120). Let us focus, then, on the second property. We define[3] $X \in \mathfrak{X}(G \times \mathfrak{g})$ by

$$X_{(g,A)} := \Big((A^L)_g, 0\Big), \quad \forall g \in G, \forall A \in \mathfrak{g}.$$

Here, 0 represents the zero vector of the vector space $T_A(\mathfrak{g})$, for any $A \in \mathfrak{g}$ under consideration. By property 1, the *flow* of X is

$$\theta_X(t;(g,A)) = (g\exp(tA), A), \quad t \in \mathbb{R}.$$

Now, a standard fact in the study of differential equations is that if a vector field is smooth, then its flow also is. Thus, if $\mathrm{pr}_1 : G \times \mathfrak{g} \to G$ is the projection onto the first component,

$$\exp(A) = \mathrm{pr}_1(\theta_X(1;(e,A))),$$

so that exp is seen to be the composition of smooth functions and, hence, is also smooth. □

Notation: We will also denote $\exp(A)$ by e^A.

[3]Here, and in the rest of the text, we do the following identification: if M and N are manifolds, with $m \in M$ and $n \in N$, $T_{(m,n)}(M \times N) \sim T_m M \times T_n N$.

1.3.2 The Adjoint Representation

Definition 1.24. *Take $g \in G$. We define its* conjugation map *by*

$$c_g : G \longrightarrow G, \quad h \mapsto ghg^{-1}.$$

Now, given $g \in G$, we can consider the differential of its conjugation map, $(c_g)_{*,e} : \mathfrak{g} \to \mathfrak{g}$, thereby defining a map

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g}), \quad g \mapsto (c_g)_{*,e}.$$

Since $(c_{g^{-1}})_{*,e}$ is easily shown to be the inverse of $(c_g)_{*,e}$, this map is well defined (i.e. $\mathrm{Ad}(g) \in \mathrm{GL}(\mathfrak{g})$ for all $g \in G$). It is called the ***adjoint representation of the Lie group*** G.

Similarly to how we expressed the exponential map as a composition of smooth functions, one can see that the adjoint map is also smooth (for example, see [2], p. 534). This proves that the adjoint is an example of a more general concept: that of a ***smooth representation*** of a Lie group.

Definition 1.25. *Suppose V is a finite-dimensional vector space. A* smooth representation*, or simply a representation, of G on V is a smooth map*

$$\rho : G \longrightarrow \mathrm{GL}(V),$$

that is also a group homomorphism.

We give one last interesting result involving the adjoint's differential map at e. Set

$$\mathrm{ad} = (\mathrm{Ad})_{*,e} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}), \tag{1.6}$$

where $\mathfrak{gl}(\mathfrak{g})$ stands for the Lie algebra of the Lie group $\mathrm{GL}(\mathfrak{g})$. This map is usually referred to as the ***adjoint representation of the Lie algebra*** $\mathfrak{g}$.

Proposition 1.26. *For any $A, B \in \mathfrak{g}$, $ad(A)(B) = [A, B]$.*

Proof. A detailed proof can be found in [3], p.124. □

Chapter 2

Principal bundles

In this chapter we introduce the topic of principal bundles. Ubiquitous in many areas of differential geometry and modern theoretical physics, their definition shares some similarities to that of vector bundles[1], but with the vector space structure replaced by a Lie group. This shift in fibre has profound implications on how several fundamental notions, such as connections and curvature, are defined in this more intricate case.

Throughout this chapter, assume that M is a smooth manifold.

2.1 Vector Bundles

In this chapter we collect a series of definitions and results related to vector bundles, that will serve as stepping stones when we develop the theory of principal bundles in later sections.

2.1.1 Vector Bundles and Subbundles

Definition 2.1. *Assume that M, E and F are manifolds. We say that a smooth, surjective map $\pi : E \to M$* trivialises with fibre F *if there exists an open cover $\{U_\alpha\}$ for M together with a collection of diffeomorphisms $\{\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times F\}$ that are* fibre-preserving *– i.e. such that the following diagram is commutative:*

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_\alpha & \end{tikzcd},$$

where $\mathrm{pr}_1 : U_\alpha \times F \to U_\alpha$ is the projection onto the first factor. We shall refer to the elements of the open cover U_α as trivialising open sets *and to the corresponding fibre-preserving diffeomorphisms ϕ_α as* trivialisations.

If $\pi : E \to M$ trivialises with fibre F, we equivalently say that π is a ***fibre bundle with fibre*** F. In this case, we refer to E as the ***total space*** of the bundle, M as the ***base manifold*** and π as the ***projection map***.

[1]Similarly to previous nomenclature, we will always assume a vector bundle to be smooth unless explicitly stated otherwise.

Let r be a positive integer. Recall that a ***vector bundle of rank*** r consists of a smooth, surjective map $\eta : E \to M$ between manifolds that locally trivialises with fibre $\mathbb{R}^r$, such that $\eta^{-1}(\{p\})$ is isomorphic (as a vector space) to $\mathbb{R}^r$ for all $p \in M$.

Typically, we will be given an assignation $M \ni p \to E_p$, where E_p is a real vector space of dimension r, and we will want to prove that the disjoint union

$$E := \bigsqcup_{p \in M} E_p$$

is a vector bundle over M when equipped with the projection

$$\eta : E \longrightarrow M, \quad E_p \ni X_p \longmapsto p. \tag{2.1}$$

The next proposition provides a sufficient condition for this to be the case:
Notation: If $f : A \to B$ is a surjective map between sets and $S \subset B$, we define $A|_S := f^{-1}(S)$. Whenever $S = \{b\}$ consists of a single point $b \in B$, we will denote $f^{-1}(S)$ as $A|_b$ instead of the more correct $A|_{\{b\}}$.

Proposition 2.2. *Continuing with the notation established in the previous paragraph, suppose $p \mapsto E_p$ maps $p \in M$ to a real vector space E_p of fixed dimension r. Assume that we are given*

- *An open cover*[2] $\{U_\alpha\}$ *of M.*
- *A set of bijective maps $\{\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r\}$ such that $\phi_\alpha(E_p) \subset \{p\} \times R^r$, $p \in U_\alpha$, with*

  $$\phi_{\alpha,p} := \phi_\alpha|_{E_p} : E_p \longrightarrow \{p\} \times \mathbb{R}^r$$

  a linear vector space isomorphism.
- *For every pair of open sets U_α and U_β with nontrivial intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, a smooth map*

  $$\tau_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{GL}(r, \mathbb{R}),$$

  called a transition function, *such that the composition*

  $$\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta{}^{-1} : U_{\alpha\beta} \times \mathbb{R}^r \longrightarrow U_{\alpha\beta} \times \mathbb{R}^r$$

  can be written as

  $$\phi_{\alpha\beta}(p, v) \longmapsto (p, \tau_{\alpha\beta}(p)v), \quad \forall p \in U_{\alpha\beta}\ \forall v \in \mathbb{R}^r.$$

In that case, E has a unique topology and smooth structure making it into a smooth manifold and a rank-r vector bundle over M, with η from Eq. (2.1) as projection and $\{(U_\alpha, \phi_\alpha)\}$ as local trivialisations.

Proof. A rigorous proof can be found in [2], p.253, Lemma 10.6. □

[2]We will typically omit the indexing set $A \ni \alpha$ to lighten the notation.

Definition 2.3. *Let $\eta : E \to M$ and $\eta' : E' \to M'$ be two vector bundles of ranks* r *and* r′*, respectively. A* vector bundle map *consists of a pair of maps $f : M \to M'$ and $F : E \to E'$ such that (1) the diagram*

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ M & \xrightarrow{f} & M' \end{array}$$

is commutative and (2) for all $p \in M$,

$$F|_{E_p} : E_p \to E'_{F(p)}$$

is linear. We define the rank of F at $p \in M$ *to be the rank of the linear map $F|_{E_p}$.*

In the case $M = M'$, if $F : E \to E'$ is a vector bundle map with $f = 1_M : M \to M$ the identity over M, we alternatively say that F is a vector bundle map *over* M.

For reasons that will be apparent later on, specially in the chapter on connections on vector bundles, we will be interested in studying ***vector subbundles***: subsets of vector bundles that are vector bundles themselves. More precisely,

Definition 2.4. *Suppose $\eta : E \to M$ and $\lambda : F \to M$ are two vector bundles. We say that λ is a* vector subbundle *of η if F is a regular submanifold of E and the inclusion map $\iota : F \hookrightarrow E$ is a vector bundle map over M.*

We now state, without proof, a standard and powerful sufficient condition for determining when a union of vector subspaces forms a (smooth) vector subbundle,

Lemma 2.5. *Suppose $\eta : E \to M$ is a vector bundle of rank r and take $0 \leq k \leq r$ a nonnegative integer. Consider a disjoint union $W = \bigsqcup_{p \in M} W_p$ of vector subspaces $W_p \subset E_p$ of dimension k, and let $\lambda : W \to M$ be the natural projection map.*

Assume that for every $p \in M$, there exists an open neighbourhood U_p of p such that there exist smooth sections $s_1, \ldots s_m$ of E over U_p, with $m \geq k$, that span $W_q, \forall q \in U_p$. Then, λ is a vector subbundle of η.

Proof. For a complete proof see [3], p.175. □

A common method of finding new vector subbundles is to consider the kernel and image of vector bundle maps with constant rank:

Lemma 2.6. *Let $\eta : E \to M$ and $\eta' : E' \to M$ be vector bundles and suppose $F : E \to E'$ is a vector bundle map over M. Define the sets*

$$\mathrm{Ker}(F) := \bigcup_{p \in M} \mathrm{Ker}\left(F|_{E_p}\right) \text{ and } \mathrm{Im}(F) := \bigcup_{p \in M} \mathrm{Im}\left(F|_{E_p}\right),$$

equipped with the natural projection into M. Then, $\mathrm{Ker}(F)$ *and* $\mathrm{Im}(F)$ *are vector subbundles of η and η', respectively, if, and only if, F has constant rank.*

Proof. The ‘only if’ statement is clear, since the fibre of a vector subbundle must have the same dimension everywhere. Let us focus then in the ‘if’ part of the proof. We will first show that the result holds for $\mathrm{Im}(F)$, and subsequently use this to demonstrate that the same is true for $\mathrm{Ker}(F)$. Hence, assume the rank of F is constant and equal to k and that the dimensions of η and η' are r and r', respectively.

Choose $p \in M$ and a neighbourhood U of p for which a *local smooth frame* $s_1, \dots, s_r$ for η exists. By reordering the sections in the frame if necessary, we can assume that

$$\mathrm{span}(F \circ s_1(p), \dots, F \circ s_k(p)) = \mathrm{Im}\left(F|_{E_p}\right),$$

so that the vectors $(F \circ s_1)(p), \dots, (F \circ s_k)(p)$ are linearly independent. Being linearly independent is an *open* condition, as we may think of it in terms of the determinant of a matrix being nonzero, and therefore we can assume that another, possibly smaller, neighbourhood U_0 of p in M exists such that $(F \circ s_1)(q), \dots, (F \circ s_k)(q)$ is linearly independent in E'_q for all $q \in U_0$. Since F has constant rank, this collection of vectors forms a basis for each fibre $\mathrm{Im}\left(F|_{E_p}\right)$. Given that we have proven this for an arbitrary $p \in M$, Lemma 2.5 ensures that $\mathrm{Im}(F)$ is a vector subbundle of η'.

Continuing with the same notation, define V to be the total space of the vector subbundle of $\eta|_{E|_{U_0}}$ spanned by the sections $s_1, \dots, s_k$. By construction, the restriction $F|_V : V \to \mathrm{Im}(F)|_{U_0}$ is bijective. In fact, this is enough to show that $F|_V$ is a *vector bundle isomorphism*: we know that the inverse of a linear map is linear, and that taking the inverse of a matrix is a smooth operation (see the proof of Proposition 1.5). We define the vector bundle map

$$\psi : E|_{U_0} \longrightarrow E|_{U_0}, \quad \psi(X) := X - (F|_V)^{-1}(F(X)).$$

Following the definitions and using the rank-nullity theorem, the below facts are easy to check:

1. $E|_{U_0} = V \oplus \mathrm{Ker}(F)|_{U_0}$,
2. ψ maps the subbundles V and $\mathrm{Ker}(F)|_{U_0}$ into $\mathrm{Ker}(F)|_{U_0}$,
3. ψ restricts to the identity map on $\mathrm{Ker}(F)|_{U_0}$.

From this we can conclude that $\mathrm{Im}(\psi) = \mathrm{Ker}(F)|_{U_0}$. This simultaneously demonstrates that ψ has constant rank and that, by the previous paragraph, $\mathrm{Ker}(F)|_{U_0}$ is vector subbundle of $E|_{U_0}$. This readily implies that $\mathrm{Ker}(F)$ is a subbundle of E, as desired. □

2.1.2 Connection and Curvature

As in the previous subsection, let $\eta : E \to M$ be a vector bundle of rank r. Similar to our discussion of smooth manifolds and smooth maps in the previous chapter, we assume the reader is familiar with the notion of a connection on a vector bundle. For a rigorous definition, see [3], Definition 10.1.

Let us then consider a vector bundle connection on η,

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E), \quad (X, s) \mapsto \nabla_X s.$$

In this context, we define the ***curvature of the vector bundle*** to be the map

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$
$$(X, Y, s) \mapsto R(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

Definition 2.7. *Let $\eta : E \to M$ and $\lambda : F \to M$ be vector bundles over a manifold M. Suppose $\alpha : \Gamma(E) \to \Gamma(F)$ is an $\mathbb{R}$-linear map. We say that α is* local *if for every open set U on M and every $s \in \Gamma(E)$,*

$$s|_U \equiv 0 \Rightarrow \alpha(s)|_U \equiv 0.$$

It is straighforward to see that ∇ is itself a local operator, meaning that if either s or X vanish on an open set U of M, then $\nabla_X s$ does too. It is a standard result in differential geometry that ensures that any local operator can be restricted to an open set (see [3], Theorem 7.20). Thus, if U is an open set on M, we define

$$\nabla^U : \mathfrak{X}(U) \times \Gamma(U, E) \longrightarrow \Gamma(U, E), \quad \nabla^U_{X|_U}(s|_U) = (\nabla_X s)|_U,$$

for all $X \in \mathfrak{X}(M)$ and all $s \in \Gamma(E)$. We typically slightly abuse notation and denote ∇^U simply by ∇.

Suppose that, in addition, η trivialises over U, with trivialisation $\phi : E|_U \to U \times \mathbb{R}^r$. Let $e_1, \ldots, e_r$ be a local smooth frame of E over U. Since ∇ satisfies *Leibniz's rule*,

$$\nabla_X(fs) = (Xf)s + f\nabla_X s, \quad \forall X \in \mathfrak{X}(U)\ \forall s \in \Gamma(U, E),$$

any section $\nabla_X s$ can be expressed as a combination of the sections $\nabla_X e_j$. We define the ***components of a vector bundle connection*** over U with respect to the frame $e_1, \ldots, e_r$ to be the $r \cdot r = r^2$ set of scalar 1-forms $\omega^i_j \in \Omega^1(U)$ satisfying

$$\nabla_X e_j = \omega^i_j(X)e_i, \quad \forall X \in \mathfrak{X}(U).$$

Similar considerations now apply to the curvature R, for which we can analogously define components $\Omega^i_j \in \Omega^2(U)$ satisfying

$$R(X, Y)e_j = \Omega^i_j(X, Y)e_i.$$

These components, ω^i_j and Ω^i_j, are deeply connected through the ***second structural equation*** (cf. [3], Theorem 11.1):

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j, \tag{2.2}$$

where $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior derivative of scalar forms.

2.1.3 Parametric Families of Differential Forms

Definition 2.8. *Suppose k and n are positive integers with $k \leq n$. If $1 \leq j_1 < \cdots < j_k \leq n$ is a sequence of strictly ascending integers, we introduce the* multi-index

$$J := (j_1, \ldots, j_k).$$

If $\alpha^1, \ldots, \alpha^n \in \Omega^1(M)$ are differential forms on a manifold M, we define

$$\alpha^J := \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \in \Omega^k(M).$$

We conclude this section with a brief reminder of 1-parameter families of differential forms. Assume $I \subset \mathbb{R}$ is an open interval, and let $k \in \mathbb{N}$ and M be a manifold of dimension $n \in \mathbb{N}$. Consider a family of differential forms

$$\{\alpha_t \in \Omega^k(M) : t \in I\},$$

varying with the *parameter* $t \in I$, called a ***1-parameter family of forms***. We say that the family $\{\alpha_t\}_{t \in I}$ ***varies smoothly with*** t if for every $p \in M$ there exists a coordinate neighbourhood U_p of p on which

$$(\alpha_t)_q = a_J(t,q)(dx^J)_q, \quad \forall (t,q) \in I \times U_p,$$

for some smooth functions a_J on $I \times U_p$.

If $\{\alpha_t\}_{t \in I}$ is a family of k-forms varying smoothly with t, we define its ***derivative with respect to*** t at $(t_0, q) \in I \times U_p$ as

$$\left(\frac{d}{dt}\bigg|_{t=t_0} \alpha_t \right)_q := \frac{\partial a_J}{\partial t}(t_0,q)(dx^J)_q.$$

It is easy to check that this definition that does not depend on the particular choice of neighbourhood U_p that we make (a proof is given in [1], p.378, Problem 20.3). Leibniz's rule holds when differentiating wedge products of 1-parameter families,

Proposition 2.9. *Suppose $\{\beta_t\}_{t \in I}$ and $\{\gamma_t\}_{t \in J}$ are smooth families of scalar differential k and l-forms, respectively. Then,*

$$\frac{d}{dt}(\beta_t \wedge \gamma_t) = \frac{d}{dt}(\beta_t) \wedge \gamma_t + \beta_t \wedge \frac{d}{dt}(\gamma_t).$$

Proof. See [1], p.222, Proposition 20.1 for a detailed proof. □

We can analogously integrate a smooth 1-parameter family of k-forms by setting

$$\left(\int_a^b \omega_t dt \right)_p := \left(\int_a^b a_J(t,p) dt \right) (dx^J)_p,$$

for $a, b \in I$.

2.2 Introduction to Principal Bundles

Definition 2.10. *A* smooth right action *of a Lie group G on M is a group-theoretic right action $\lambda : M \times G \to M$ that is also smooth. A manifold M together with a right action λ is called a G-manifold, or we may also alternatively say that G **acts smoothly** on M.*

As we did before with the multiplication map of a group, we will typically denote $\lambda(p,g)$ by $p \cdot g$ or pg. On the other hand, as any left action can be associated with an equivalent right action (using the rule $p \cdot g := g^{-1} \cdot p$) no generality is lost in the definition by focusing on right actions.

Definition 2.11. *A map $f : N \to M$ between G-manifolds is said to be* G-equivariant *if*

$$f(pg) = f(p)g, \quad \forall p \in N \; \forall g \in G.$$

Recall Definition 2.1:

Definition 2.12. *Let $\pi : P \to M$ be a surjective, smooth map that trivialises with fibre G a Lie group. Then, π is a* principal G-bundle *if*

PB1 *G acts smoothly and freely on P.*

PB2 *π is fibre-preserving: for all $p \in P$ and $g \in G$, $\pi(pg) = \pi(p)$.*

PB3 *The fibre-preserving local trivialisations ϕ_α are G-*equivariant*, where the action of G on $U_\alpha \times G$ is defined by right multiplication. In other words,*

$$\phi_\alpha(pg) = \phi_\alpha(p) \cdot g = (\mathrm{pr}_1(\phi_\alpha(p)), \mathrm{pr}_2(\phi_\alpha(p)) \cdot g), \quad \forall p \in P|_{U_\alpha} \ \forall g \in G.$$

We would like to remark that condition *PB3* is properly stated. Specifically, if $p \in P|_{U_\alpha}$ and $g \in G$, by property *PB2*, we have $\pi(pg) = \pi(p) \in U_\alpha$. Thus, pg belongs to the domain of ϕ_α.

Going forward, we may slightly abuse notation and refer to a principal G-bundle $\pi : P \to M$ exclusively by its projection map π.

Having defined the 'objects'[3] in this new category of principal G-bundles, let us define the 'morphisms':

Definition 2.13. *Suppose that $\pi : P \to M$ and $\pi' : P' \to M'$ are principal G-bundles. A* morphism *between π and π' consists of a pair of maps $f : M \to M'$ and $F : P \to P'$, such that F is G-equivariant and the diagram*

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

is commutative.

If $M = M'$ and $F : P \to P'$ is a map such that $(F, 1_M)$ is a principal bundle morphism, we say that F is a principal bundle morphism *over* M.

Definition 2.14. *Two principal G-bundles $\pi : P \to M$, $\pi' : P' \to M$ over the same* base manifold *M are said to be* isomorphic *if there exists principal bundle morphisms $F : P \to P'$ and $G : P' \to P$ over M that are inverses of each other,*

$$F \circ G = 1_{P'}, \quad G \circ F = 1_P.$$

Example 2.15. If M is a manifold and G is a Lie group, it is easy to verify that the projection into the first factor $\mathrm{pr}_1 : M \times G \to M$ constitutes a principal G-bundle, called the ***explicitly trivial principal bundle***. Consequently, a principal G-bundle $\pi : P \to M$ isomorphic to $M \times G \to G$ is called ***trivialisable***.

If $\pi : P \to M$ is a principal G-bundle, we define the ***fibre above*** $x \in M$ to be the set $P_x := P|_{\{x\}}$. The next proposition follows almost by construction,

Proposition 2.16. *Let $\phi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$ be a trivialisation of the principal G-bundle π. Then,*

- *The restriction of ϕ_α to P_x, $x \in U_\alpha$, denoted $\phi_{\alpha,x} : P_x \to \{x\} \times G$, is a diffeomorphism.*

[3]Only the most rudimentary notions of category theory will be sparsely used throughout the thesis. As its use is not extensive and deviates considerably from the task at hand, we will omit giving the basic definitions and instead refer the interested reader to [1], p.110, Section 3.10.

- *ϕ_α acts transitively on each fibre P_x, for any $x \in M$.*

Proof. The first property follows directly from the fact that trivialisations are fibre-preserving diffeomorphisms, while the second property holds due to the first property and the transitivity of the action of G on $\{x\} \times G$. $\square$

Remark 2.17. In Proposition 2.2 we talked about the *transition functions* of a vector bundle $\eta : E \to M$. In a similar fashion, given now two trivialising open sets U_α and U_β with nonempty intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$ for a principal G-bundle π, one may construct analogous smooth functions $\tau_{\alpha\beta} : U_{\alpha\beta} \to G$ satisfying

$$\phi_\alpha \circ \phi_\beta : U_\alpha \cap U_\beta \times G \longrightarrow U_\alpha \cap U_\beta \times G, \quad (x, g) \mapsto (x, \tau_{\alpha\beta}(x)g).$$

The proof that such functions exist is identical, *mutatis mutandis*, to that of the vector bundle scenario, with only one additional fact: a G-equivariant map of G into itself must be a *left translation* (i.e. of the form $g \mapsto hg$ for some fixed $h \in H$). Since this is easily shown, we omit the proof (see, for example, [3], p. 244).

2.2.1 Fundamental Vector Fields

Let P be a G-manifold, and choose $A \in \mathfrak{g}$ along with $p \in P$. Define the curve

$$\gamma : \mathbb{R} \longrightarrow P, \quad t \mapsto p \cdot e^{tA},$$

where e^{tA} refers to the exponential map discussed in Section 1.3. Now, the smoothness of the action of G on P and of the exponential map ensures that γ is smooth. Thus, we can consider the *a priori* rough vector field (called ***fundamental vector field on*** P ***associated to*** A)

$$\underline{A} : p \in P \longmapsto \underline{A}_p = \left.\frac{d}{dt}\right|_{t=0} \left(p \cdot e^{tA}\right) \in T_pP. \tag{2.3}$$

Remark 2.18. It is not hard to see, from the definitions and the properties of the exponential map, that the flow of $\underline{A}$ is $\theta(t; p) = p \cdot e^{tA}$.

Proposition 2.19. *For any $A \in \mathfrak{g}$, its fundamental vector field on P is smooth.*

Proof. Take $f \in C^\infty(P)$. Then, by Proposition 1.7, it suffices to show that $\underline{A}_p f$ is also smooth. By definition,

$$\underline{A}_p f = \left(\left.\frac{d}{dt}\right|_{t=0} p \cdot e^{tA}\right) f = \left.\frac{d}{dt}\right|_{t=0} f\left(p \cdot e^{tA}\right) = \left.\frac{d}{dt}\right|_{t=0} f\left(\mu(p, e^{tA})\right) = \Lambda(t = 0, p),$$

where we have defined

$$\Lambda : \mathbb{R} \times P \longrightarrow \mathbb{R}, \quad (s, p) \mapsto \left.\frac{d}{dt}\right|_{t=s} f\left(\mu(p, e^{tA})\right).$$

Being the composition of smooth functions, Λ is likewise smooth, and so is its restriction $\Lambda|_{\{0\}\times P} = \underline{A}_p f$. $\square$

We have hence constructed a map

$$\sigma : \mathfrak{g} \longrightarrow \mathfrak{X}(P), \quad A \mapsto \underline{A}$$

Moreover,

Proposition 2.20. *σ is a Lie algebra homomorphism.*

In order to prove this, we first offer an alternative description of fundamental vector fields. If $p \in P$, we define the (smooth) map

$$j_p : G \longrightarrow P, \quad g \mapsto p \cdot g. \tag{2.4}$$

Consider $A \in \mathfrak{g}$. From Proposition 1.23, we know e^{tA} to be the maximal integral curve of A^L, beginning at $e_G \in G$. By the properties of the differential,

$$(j_p)_{*,e}(A) = \left.\frac{d}{dt}\right|_{t=0} \left(j_p \circ e^{tA}\right) = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{tA} = \underline{A}_p. \tag{2.5}$$

Where we have used the definition of the map j_p in the second equality and that of $\underline{A}$ in the last.

Proof (of Proposition 2.20). Since Equation (2.5) demonstrates the linearity of σ, it is enough to show that, if A and B belong to $\mathfrak{g}$,

$$[\underline{A}, \underline{B}] = [\sigma(A), \sigma(B)] = \sigma([A, B]) = \underline{[A, B]}.$$

We will prove this equality pointwise. Thus, fix $p \in P$.

We claim that $A^L \sim^{j_p} \underline{A}$. Indeed,

$$\underline{A}_{pg} \overset{(2.5)}{=} (j_{pg})_{*,e}(A) \overset{j_{pg}=j_p \circ l_g}{=} (j_p)_{*,g}\left(l_{g*,e}A\right) = (j_p)_{*,g}\left(A^L_g\right), \forall g \in G,$$

where, as before, $l_g : G \to G$ is left translation by g. Similarly we have that $B^L \sim^{j_p} \underline{B}$, so that by Proposition 1.17,

$$[\underline{A}, \underline{B}]_{pg} = (j_p)_{*,g}\left([A^L, B^L]_g\right).$$

Evaluating at $g = e_G$ and taking into account Remark 1.13, we obtain the desired result. $\square$

To conclude this section, we study more in depth the map defined in (2.4). Firstly, let us remind ourselves that if $\rho : S \times G \to S$ is a right-action of a group G on a set S, we define the ***stabiliser*** of $s \in S$ as

$$\mathrm{Stab}(s) := \{g \in G : s \cdot g = s\}.$$

Proposition 2.21. *Let P be a G-manifold. Then $\mathrm{Stab}(p)$ is itself a Lie group for any $p \in P$.*

Proof. For $h \in G$, define $r_h : P \to P$ and $r'_h : G \to G$ by $r_h(p) := ph$ and $r'_h(g) := gh$. Since the action of G on P is associative, we have the identity

$$j_p \circ r'_h = r_h \circ j_p.$$

Taking differentials at $g \in G$, the previous equation becomes

$$(j_p)_{*,gh} \circ r'_{h*,g} = r_{h*,pg} \circ (j_p)_{*,g}$$

This shows that the map j_p has constant rank, as both maps r'_h and r_h are easily seen to be diffeomorphisms. As a consequence of the ***constant-rank level set theorem*** (e.g. see [1], p.116), $\mathrm{Stab}(p) = j_p^{-1}(\{p\})$ is a regular submanifold of G. Thus, the restriction to $\mathrm{Stab}(p)$ of the product $\mu : G \times G \to G$ and inverse $\kappa : G \to G$ maps of G are again smooth, as desired. ☐

In the proposition below we return to the notation $\theta_X(\cdot\,; p)$ introduced in section 1.3 regarding maximal integral curves of a vector field X on a manifold.

Proposition 2.22. *Choose $A \in \mathfrak{g}$ along with $p \in P$. Then, $\underline{A}_p = 0$ if, and only if, A belongs to $\mathrm{Lie}(\mathrm{Stab}(p))$, the Lie algebra of the Lie group $\mathrm{Stab}(p)$.*

Proof. By a slight abuse of notation, we denote the exponential map for G, $\exp_G : \mathfrak{g} \to G$, and the one for $\mathrm{Stab}(p)$, $\exp_{\mathrm{Stab}(p)} : \mathrm{Lie}(\mathrm{Stab}(p)) \to \mathrm{Stab}(p)$, with the same notation exp. Also, by Remark 2.18, $\theta_{\underline{A}}(t; p) = p \cdot \exp(tA)$.

($\Leftarrow$) Assume that $A \in \mathrm{Lie}(\mathrm{Stab}(p))$, so that e^{tA} lies in $\mathrm{Stab}(p)$ for all $t \in \mathbb{R}$. Accordingly,

$$\underline{A}_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{tA} = \left.\frac{d}{dt}\right|_{t=0} p = 0.$$

($\Rightarrow$) Reciprocally, take A such that $\underline{A}_p = 0$. This implies that the curve $\gamma(t) = p$ is a maximal integral curve of $\underline{A}$ starting at p. In accordance with the fundamental theorem on flows, this maximal solution has to be unique, so that, for all $t \in \mathbb{R}$, $p \cdot \exp(tA) = p$. In other words, $\exp(tA) \in \mathrm{Stab}(p)$ for all t, so that by standard results from Lie group theory (namely, see Proposition 20.9, p.521, in [2]), we may conclude that $A \in \mathrm{Lie}(\mathrm{Stab}(p))$. ☐

2.2.2 Vertical and Horizontal Distributions of the Tangent Bundle TP

In this section, we define the notion of *vertical* and *horizontal* vectors in a principal G-bundle $\pi : P \to M$. As we shall see, the notion of a vertical vector arises naturally from the structure of the bundle, whereas defining horizontal vectors requires additional structure. Accordingly, we begin with vertical vectors.

Let us begin with an observation: the projection map π is a ***submersion***. By this, we mean that its differential $\pi_{*,p}$ is surjective for all $p \in P$.

Indeed, take $\phi_\alpha : P|_U \to U_\alpha \times G$ a trivialisation. From (2.1), we have the equality $\pi|_{P|_U} = \mathrm{pr}_1 \circ \phi_\alpha$, with $\mathrm{pr}_1 : U_\alpha \times G \to U_\alpha$ the projection into the first factor. Since pr_1 is clearly a submersion and ϕ_α is a diffeomorphism, it follows that $\pi|_{P|_U}$ – and thus π – is itself a submersion.

Remark 2.23. Considering a trivialisation $\phi_\alpha : P|_U \to U_\alpha \times G$ just as above, we can also conclude that if $\pi : P \to M$ is a principal G-bundle, then $\dim(P) \geq \dim(G)$.

Definition 2.24. *Let U, V and W be vector spaces over a field F and suppose $i : U \to V$ and $j : V \to W$ are linear maps between them. We say that the sequence*

$$0 \hookrightarrow U \xrightarrow{i} V \xrightarrow{j} W \to 0,$$

where 0 represents the trivial vector space, $0 \hookrightarrow U$ denotes the inclusion map and $W \to 0$ is the trivial map, is a short exact sequence of vector spaces *if i is injective, j is surjective and* $\operatorname{Im} i = \operatorname{Ker} j$*.*

We now define the ***vertical tangent subspace*** at $p \in P$ as $\mathcal{V}_p := \operatorname{Ker} \pi_{*,p} \subset T_pP$. By the previous observation, we have that

$$0 \hookrightarrow \mathcal{V}_p \hookrightarrow T_pP \xrightarrow{\pi_{*,p}} T_{\pi(p)}M \to 0$$

is a short exact sequence. Most interestingly, the vertical tangent subspace is closely connected with the Lie group's Lie algebra $\mathfrak{g}$, as the next proposition shows:

Proposition 2.25. *Let $\pi : P \to M$ be a principal G-bundle, and for $p \in P$, consider the map $j_p : G \to P$ defined in (2.4). Then, the following statements hold:*

1. *$(j_p)_{*,e}(\mathfrak{g}) \subset \mathcal{V}_p$.*
2. *The restriction $j_{p*} := (j_p)_{*,e} : \mathfrak{g} \longrightarrow \mathcal{V}_p$ is a vector space isomorphism.*
3. *The disjoint union $\mathcal{V} = \bigsqcup_{p \in P} \mathcal{V}_p$ is a vector subbundle of the tangent bundle TP.*

Proof. Fix $p \in P$. We observe that the map $\pi \circ j_p$ is constant by the fibre-preserving property ***PB2*** of Definition 2.12. Hence, for all $A \in \mathfrak{g}$,

$$\pi_{*,p}\left(j_{p*}(A)\right) = \left(\pi \circ j_p\right)_{*,e}(A) = 0.$$

Thus proving part (i) of the proposition.

Let us focus now on the second statement. We proceed by firstly showing that j_{p*} is injective and subsequently by proving that $\mathcal{V}_p$ and $\mathfrak{g}$ have the same dimension. Choose $A \in \mathfrak{g}$ such that $0 = j_{p*}(A) = \underline{A}_p$. Now, the action of G on P is by hypothesis free, so that by Propositon 2.22 A must be zero.

On the other hand, we know $\pi_{*,p}$ to be surjective, so that by the *rank-nullity theorem*,

$$\dim(\mathcal{V}_p) = \dim(T_pP) - \dim(T_xM)$$

where $x = \pi(p)$. Consider a trivialisation $\phi_\alpha : P|_U \to U_\alpha \times G$ about x, so that $x \in U_\alpha$. Then, applying the rank-nullity theorem to the vector space isomorphism ${\phi_\alpha}_{*,p}$, we are able to conclude that

$$\dim(T_pP) = \dim(T_x(U_\alpha)) + \dim(T_gG) = \dim(T_xM) + \dim(\mathfrak{g}),$$

where $g = \mathrm{pr}_2(\phi_\alpha(p))$.

For the last statement, assume that $\dim(\mathfrak{g}) = n$ and take $B_1, \dots, B_n$ a basis for $\mathfrak{g}$. By virtue of Proposition 2.19, the vector fields $\underline{B}_1, \dots, \underline{B}_n$ are smooth, so that by Lemma 2.5 $\mathcal{V}$ is indeed a vector subbundle of TP. $\square$

At this point, we can define ***horizontal vectors*** in a natural, intuitive way. We define a ***distribution*** to be a vector subbundle of the tangent bundle TP.

Definition 2.26. *Suppose $\pi : P \to M$ is a principal G-bundle. A distribution $\mathcal{H}$ is said to be* horizontal *if, at any given $p \in P$,*

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p.$$

Unlike the distribution of vertical vectors $\mathcal{V}$, which is inherently defined for all principal bundles, there is no canonical way to assign a horizontal distribution to a principal bundle. Instead, a horizontal distribution is determined by the choice of a *connection*, which we introduce in the next section.

2.2.3 The Pullback of a Principal Bundle

Much as vector bundles, and differential forms for that matter, it is possible to pullback a principal G-bundle $\pi : P \to M$ through a smooth map on base manifolds $f : N \to M$. Since the technical details of this are remarkably similar to the vector bundle case, we limit ourselves to present the construction of the pullback principal G-bundle and relay the proof that this construction is well-defined to [12], p. 216, Subsection 5.1.7.

As a set, we define

$$f^*P := \{(n, p) \in N \times P : f(n) = \pi(p)\}.$$

We can define a (right) action of G on f^*P by $(n, p) \cdot g := (n, pg)$, for all $(n, p) \in f^*P$ and all $g \in G$. We likewise define the maps

$$\begin{aligned} q : f^*P &\longrightarrow N, \\ (n,p) &\longmapsto n \end{aligned} \qquad \begin{aligned} F : f^*P &\longrightarrow P. \\ (n,p) &\longmapsto p \end{aligned}$$

It is thus clear that the diagram

$$\begin{array}{ccc} f^*P & \xrightarrow{F} & P \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array}$$

is commutative, and it is through this diagram that we endow $q : f^*P \to N$ with the principal G-bundle structure. Specifically, if $\phi : P|_U \to U \times G$ is a local trivialisation of π over $U \subset M$, we let $V := f^{-1}(U)$ and define

$$\psi : q^{-1}(V) \longrightarrow V \times G, \quad (n, p) \longmapsto (n, \mathrm{pr}_2(\phi(p))),$$

where $\mathrm{pr}_2 : U \times G \to G$ is the projection onto the second factor. Then, if $\{(U_\alpha, \phi_\alpha)\}$ is a trivialising open cover for π, one can easily deduce that $\{(V_\alpha, \psi_\alpha)\}$ is a trivialising open cover for q.

2.3 Connections on a Principal Bundle

2.3.1 Vector-valued Differential Forms

As we shall shortly see, a connection on a principal bundle serves as an example of a *vector-valued* form, which generalize conventional differential forms by allowing an arbitrary vector space as its codomain. We will reserve the term *scalar form* for $\mathbb{R}$-valued differential forms.

The goal of this subsection is to show how different properties of forms can be adapted to this scenario.

Notation: Suppose V is a (real) vector space. We denote its ***dual vector space*** by V^* and its ***k-th exterior power*** by $\bigwedge^k V$.

Notation: Let $\eta : E \to M$ be a vector bundle. We denote by $\Gamma(E)$ the vector space of sections of E over M, that is, maps $s : M \to E$, satisfying $\eta \circ s = 1_M$. If U is an open set of M, we denote the space of all sections of E over U by $\Gamma(U, E)$.

A possible way of defining the vector space of k-forms on a manifold M, $\Omega^k(M)$, is by setting

$$\Omega^k(M) = \Gamma(\bigwedge^k T^*M).$$

In this expression,

$$(\bigwedge^k T^*M)_x := \bigwedge^k T_x^*M, \quad \text{for all } x \in M.$$

In a similar fashion, if V is now an arbitrary real vector space, we define a V-valued k-form to be an element of

$$\Gamma\left(\bigwedge^k T^*M \otimes V\right) =: \Omega^k(M, V), \tag{2.6}$$

where if $\eta : E \to M$ is a vector bundle, we denote by $E \otimes V$ the tensor product vector bundle of η with the trivial bundle $M \times V \to M$.

Take T and V two real vector spaces. Denote the vector space of ***alternating*** k-multilinear maps from the cartesian product T^k into V by $A_k(T, V)$. Then, the ***universal mapping property for alternating k-linear maps*** (see [3], Theorem 19.6, p.166), establishes the following vector space isomorphism,

$$\left(\bigwedge^k T^*\right) \otimes V \simeq A_k(T, V),$$

that we will implicitly use whenever we are dealing with vector-valued forms. Thus, after fixing a basis $v_1, \ldots v_n$ for V, we can think of $\alpha \in \Omega^k(M, V)$ as

$$\alpha = \alpha^i \otimes v_i,$$

where the $\alpha^i \in \Omega^k(M)$, for all i. By this notation we mean that, if we choose $x \in M$ and $u_1, \ldots u_k \in T_xM$,

$$\alpha_x(u_1, \ldots, u_k) = \alpha_x^i(u_1, \ldots, u_k)v_i \in V.$$

Under this identification, we will say that α is smooth if all *coordinate functions* α^i are smooth; it can be readily seen that this notion is basis-independent. To simplify the notation, we will typically write $\alpha^i \otimes v_i$ simply as $\alpha^i v_i$.

Remark 2.27. Up to now, we have discussed forms mapping into a *fixed* vector space V. More in general, we could consider ***forms with values in vector bundles***, if we allow the vector space to vary from point to point. More precisely, consider $\eta : E \to M$ a vector bundle. Then, following an analogous reasoning to the one that lead us to Eq. (2.6), we define an E-valued k-form on M as an element of

$$\Gamma\left(\bigwedge^k T^*M \otimes E\right) =: \Omega^k(M, E).$$

Next, we would like to define a generalization of the wedge product. Consider T, V and U vector spaces, along with vector-valued forms $\alpha \in A_k(T, U)$ and $\beta \in A_l(T, V)$, for k and l nonnegative integers.
Notation: If n is a positive integer, we denote by S_n the group of ***permutations*** of the set $\{1, \dots, n\}$. We will write $\operatorname{sgn}(\sigma)$ to refer to the sign of a permutation $\sigma \in S_n$.

Definition 2.28. *With the notation as in the previous paragraph, we define $\alpha \wedge \beta \in A_{k+l}(T, U \otimes V)$ to be*

$$(\alpha \wedge \beta)(t_1, \dots, t_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\ \alpha\Big(t_{\sigma(1)}, \dots, t_{\sigma(k)}\Big) \otimes \beta\Big(t_{\sigma(k+1)}, \dots, t_{\sigma(k+l)}\Big),$$

for any $t_1, \dots, t_{k+l} \in T$.

If we are further given a vector space W and a bilinear map $\widetilde{\lambda} : V \times U \to W$ (which we will always equivalently think of as a linear map $\lambda : V \otimes U \to W$), we can easily construct a W-valued form $\alpha \cdot \beta \in A_{k+l}(T, W)$,

$$(\alpha \cdot \beta)(t_1, \dots, t_{k+l}) := \lambda((\alpha \wedge \beta)(t_1, \dots, t_{k+l})), \quad t_1, \dots, t_{k+l} \in T.$$

Analogous proofs to the scalar case show that this is well defined and that the product is $\mathbb{R}$-bilinear. When applied pointwise to differential vector-valued forms on manifolds, this definition extends to a map

$$\Omega^k(M, V) \times \Omega^l(M, U) \longrightarrow \Omega^{k+l}(M, W),$$

as the next proposition shows.

Proposition 2.29. *Let $\alpha \in \Omega^k(M, V)$ and $\beta \in \Omega^l(M, V)$. For vectors v_i in V and u_j in U, suppose that $\alpha = \alpha^i v_i$ and $\beta = \beta^j u_j$, and take $\lambda : V \otimes U \to W$ a bilinear map. Then,*

$$\alpha \cdot \beta = (\alpha^i \wedge \beta^j)\ \lambda(v_i \otimes u_j) \in \Omega^{k+l}(M, W),$$

where $\alpha^i \wedge \beta^j$ denotes the standard wedge product of scalar forms.

Proof. Fix $x \in M$ and $t_1, \dots t_{k+l} \in T_x M$. We prove the equality pointwise:

$$\begin{aligned}
&(\alpha \cdot \beta)_x(t_1, \dots, t_{k+l}) \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\ \lambda\Big(\alpha_x\left(t_{\sigma(1)}, \dots, t_{\sigma(k)}\right) \otimes \beta_x\left(t_{\sigma(k+1)}, \dots, t_{\sigma(k+l)}\right)\Big) \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \sum_{i,j} \operatorname{sgn}(\sigma) \alpha^i_x \left(t_{\sigma(1)}, \dots, t_{\sigma(k)}\right) \beta^j_x \left(t_{\sigma(k+1)}, \dots, t_{\sigma(k+l)}\right)\ \lambda(v_i \otimes u_j) \\
&\quad = \sum_{i,j} \lambda(v_i \otimes u_j) \left[\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \alpha^i_x \left(t_{\sigma(1)}, \dots, t_{\sigma(k)}\right) \beta^j_x \left(t_{\sigma(k+1)}, \dots, t_{\sigma(k+l)}\right)\right] \\
&\quad = \sum_{i,j} \lambda(v_i \otimes u_j) \Big[(\alpha^i \wedge \beta^j)_x(t_1, \dots, t_{k+l})\Big].
\end{aligned}$$

The first equality follows by definiton, while in the second one we have used λ's bilinearity and the expression of α and β in terms of their respective components α^i and β^j. Afterwards, we have simply swapped the order of the two (finite) sums, and used the definition of the scalar wedge product.

Regarding smoothness: we can begin by assuming, without loss of generality, that the vectors v_i and u_j form a basis for the spaces V and U, respectively. Indeed, suppose this was not the case for v_i, for example, and let $\tilde{v}_1, \dots, \tilde{v}_n$ be a basis for V. Further suppose that $v_i = c_i^l \tilde{v}_l$, for $c_i^l \in \mathbb{R}$. We can then write

$$\alpha = \alpha^i v_i = \alpha^i (c_i^l \tilde{v}_l) = (c_i^l \alpha^i) \tilde{v}_l,$$

where the forms $c_i^l \alpha^i$ are smooth by hypothesis ($\alpha \in \Omega^k(M, V)$).

By expressing the vectors $\lambda(v_i \otimes u_j) = a_{ij}^l z_l$ in terms of a basis z_l for W, much in the same way as we have just done for the v_i's, and using that the wedge product of smooth scalar forms is smooth, we see that

$$\alpha \cdot \beta = (\alpha^i \wedge \beta^j)(a_{ij}^l z_l) = (a_{ij}^l \alpha^i \wedge \beta^j) z_l$$

does belong to $\Omega^{k+l}(M, W)$. $\square$

Remark 2.30. We will be particularly interested in the case where $V = U = W = \mathfrak{g}$, the Lie algebra of a Lie group G. Here, the Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ naturally serves as a bilinear map $\widetilde{\lambda}$. In this setting, we will express the product of two $\mathfrak{g}$-valued forms as $[\alpha \wedge \beta]$ instead of the more generic $\alpha \cdot \beta$.

In the next proposition we prove a first result in this direction, generalizing Jacobi identity (cf. Definition 1.11) for vector-valued forms:

Proposition 2.31. *Let $\mathfrak{g}$ be a Lie algebra and M a manifold. Consider $\mathfrak{g}$-valued forms $\alpha \in \Omega^p(M, \mathfrak{g}), \beta \in \Omega^q(M, \mathfrak{g})$ and $\gamma \in \Omega^p r(M, \mathfrak{g})$ for $p, q, r \in \mathbb{N}$. Then,*

$$(-1)^{pr}[\alpha \wedge [\beta \wedge \gamma]] + (-1)^{pq}[\beta \wedge [\gamma \wedge \alpha]] + (-1)^{qr}[\gamma \wedge [\alpha \wedge \beta]] = 0.$$

Proof. This is a direct consequence of Proposition 2.29 and the fact that the wedge product of scalar forms is ***graded commutative***: if $\alpha^i \in \Omega^p(M), \beta \in \Omega^q(M)$,

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha. \tag{2.7}$$

$\square$

We can obtain an analogue to graded commutatitivity for forms with values in a Lie algebra,

Proposition 2.32. *Suppose $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\ ,\ ]$ and let M be a manifold. Take $\alpha \in \Omega^k(M, \mathfrak{g}), \beta \in \Omega^l(M, \mathfrak{g})$. Then,*

$$[\alpha \wedge \beta] = (-1)^{kl+1}[\beta \wedge \alpha].$$

Proof. Let B_i be a basis for $\mathfrak{g}$ and express the $\mathfrak{g}$-valued forms in terms of their components, $\alpha^i \otimes B_i$ and $\beta = \beta^j \otimes B_j$. Then,

$$\begin{aligned}
[\alpha \wedge \beta] &= (\alpha^i \wedge \beta^j)\ [B_i \otimes B_j] && \text{(Prop. 2.29)} \\
&= (-1)^{kl}(\beta^j \wedge \alpha^i)\ [B_i \otimes B_j] && \text{(Eq. (2.7))} \\
&= (-1)^{kl+1}(\beta^j \wedge \alpha^i)\ [B_j \otimes B_i] && \text{(Def. 1.11)} \\
&= (-1)^{kl+1}[\beta \wedge \alpha].
\end{aligned}$$

$\square$

Example 2.33. Allow us to consider the case $V = U = W = \mathfrak{gl}(n, \mathbb{R})$ in further detail. As discussed previously, its Lie bracket, which we saw in Proposition 1.14 to equal the commutator bracket, defines a product of $\mathfrak{gl}(n, \mathbb{R})$-valued forms α and β which we denote $[\alpha, \beta]$. However, another possible choice for a linear map λ is the matrix product,

$$\lambda : \mathfrak{gl}(n, \mathbb{R}) \otimes \mathfrak{gl}(n, \mathbb{R}) \longrightarrow \mathfrak{gl}(n, \mathbb{R}), \quad A \otimes B \longmapsto AB.$$

To distinguish it from the previous case, we will denote the product of $\mathfrak{gl}(n, \mathbb{R})$-valued forms induced by this choice of λ by $\alpha \cdot \beta$. This two types of products are intimately related: suppose $\alpha = \alpha^{ij} e_{ij}$ and $\beta = \beta^{ij} e_{ij}$, where e_{ij} is the standard basis for $\mathfrak{gl}(n, \mathbb{R})$. Then,

$$\begin{aligned}
[\alpha \wedge \beta] &= (\alpha^{ij} \wedge \beta^{ls}) \; [e_{ij}, e_{ls}] \\
&= (\alpha^{ij} \wedge \beta^{ls})(e_{ij} e_{ls} - e_{ls} e_{ij}) \\
&= (\alpha^{ij} \wedge \beta^{ls}) e_{ij} e_{ls} - (\alpha^{ij} \wedge \beta^{ls}) e_{ls} e_{ij} \\
&= (\alpha^{ij} \wedge \beta^{ls}) e_{ij} e_{ls} - (-1)^{\deg(\alpha)} (\beta^{ls} \wedge \alpha^{ij}) e_{ls} e_{ij} \\
&= \alpha \cdot \beta + (-1)^{\deg(\alpha)\deg(\beta)+1} \beta \cdot \alpha.
\end{aligned}$$

The second equality follows by Proposition 1.14, and the second-to-last by Equation (2.7). In the particular case $\alpha = \beta$,

$$[\alpha \wedge \alpha] = \begin{cases} 2\alpha \cdot \alpha, & \deg(\alpha) \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \tag{2.8}$$

Vector-valued differential forms can be pullbacked in a similar manner to scalar forms. Namely, if $F : M \to N$ is a map between manifolds, V is a vector space and $\alpha \in \Omega^k(N, V)$, we define the ***pullback*** of ω through F to be $F^*\omega \in \Omega^k(M, V)$, with

$$(F^*\omega)_x(t_1, \ldots, t_k) := \omega\left(F_{*,x}(t_1), \ldots, F_{*,x}(t_1)\right), \quad \forall x \in M \; \forall t_1, \ldots, t_k \in T_x M.$$

We close off this section defining the exterior derivative of a vector-valued form.

Definition 2.34. *Suppose $\alpha \in \Omega^k(M, V)$ is a V-valued form, such that $\alpha = \alpha^i \otimes v_i$ for a basis v_i for V. We define its* exterior derivative *as the vector-valued $k+1$-form*

$$d\alpha := (d\alpha^i) v_i, \tag{2.9}$$

where $d\alpha^i$ denotes the exterior derivative of the scalar form α^i.

It is straightfoward to see that this definition is basis-independent.

Remark 2.35. Many of the properties displayed by the scalar version of the exterior derivative are also true for the vector-valued scenario. For example, if M is a manifold and

$$\alpha \in \Omega^*(M) := \bigoplus_{k \geq 0} \Omega^k(M)$$

is a scalar differential form on M, we know that $d^2\alpha = 0$. Now consider $\beta \in \Omega^*(M, V)$ for an arbitrary vector space V. Since, following (2.9), d acts on the scalar differential form components of β, it is also true that $d^2\beta = 0$. Applying the definitions in this section, jointly with the analogous properties of the scalar case, one may also show that:

- d commutes with the pullback: if $F : N \to M$ is a map between manifolds,

$$d(F^*(\beta)) = F^*(d\beta).$$

- d is an ***antiderivation***: if $\gamma \in \Omega^k(M, V)$ and $\beta \in \Omega^l(M, V)$,

$$d(\gamma \cdot \beta) = (d\gamma) \cdot \beta + (-1)^k \gamma \cdot (d\beta).$$

2.3.2 Connections on a Principal Bundle

At the end of subsection 2.2.2 we remarked on the fact that there is not a natural way to endow a principal G-bundle with a horizontal distribution. The objective of this subsection is precisely to demonstrate how one can specify a horizontal distribution through a certain family of $\mathfrak{g}$-valued forms.

Notation: If $g \in G$, we denote by $r_g : P \to P$ the map

$$p \mapsto r_g(p) := pg. \tag{2.10}$$

Definition 2.36. *Consider a principal G-bundle $\pi : P \to M$. An* Ehresmann connection *is a $\mathfrak{g}$-valued (smooth) form $\omega \in \Omega^1(P, \mathfrak{g})$, such that:*

EC1 *It is constant on fundamental vector fields: for any $A \in \mathfrak{g}$ and any $p \in P$, $\omega_p(\underline{A}_p) = A$.*

EC2 *It is G-equivariant: for any $g \in G$, $r_g^*(\omega) = \mathrm{Ad}(g^{-1}) \circ \omega$.*

Here, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of G introduced in Subsection 1.3.2. If there is no risk of confusion, we will typically refer to an Ehresmann connection simply as a *connection*.

Notation: Given a principal G-bundle $\pi : P \to M$, we define $\mathcal{A}_P$ to be the set of all connection forms on π.

We now give the main result of this subsection:

Theorem 2.37. *A connection $\omega \in \mathcal{A}_P$ on a principal G-bundle $\pi : P \to M$ defines a right-invariant horizontal distribution $\mathcal{H}$, by assigning $\mathcal{H}_p = \mathrm{Ker}(\omega_p)$.*

Remark 2.38. By a ***right-invariant distribution*** we mean a distribution $\mathcal{K}$ such that, for all $g \in G$ and $p \in P$,

$$r_{g_{*,p}}(\mathcal{K}_p) \subset \mathcal{K}_{pg}.$$

Proof of Theorem 2.37. Fix $p \in P$. We will prove the result in steps:

1. $\mathcal{H}_p \oplus \mathcal{V}_p = T_pP$, for all $p \in P$

Consider $X_p \in \mathcal{H}_p \cap \mathcal{V}_p$. Now, if $X_p \in \mathcal{V}_p$, Proposition 2.25 (ii) ensures that there exists $A \in \mathfrak{g}$ such that $\mathrm{Ker}(\omega_p) = \mathcal{H}_p \ni X_p = j_{p*}(A) = \underline{A}_p$, so that

$$0 = \omega_p\left(\underline{A}_p\right) \overset{EC1}{=} A.$$

Thus showing that $\mathcal{H}_p \cap \mathcal{V}_p = \{0\}$. In addition, and again by axiom *EC1*, we know that ω_p is surjective. Recalling that $\mathfrak{g}$ and $\mathcal{V}_p$ are isomorphic, the rank-nullity theorem therefore implies that

$$\dim(T_pP) = \dim(\mathcal{H}_p) + \dim(\mathfrak{g}) = \dim(\mathcal{H}_p) + \dim(\mathcal{V}_p).$$

In particular, the above equation shows that the dimension of $\mathcal{H}_p$ does not depend on the choice of $p \in P$.

2. $\mathcal{H}$ is right-invariant

Take $g \in G$ and $X_p \in \mathcal{H}_p$. We want to see whether $r_{g_{*,p}}(X_p) \in \mathrm{Ker}(\omega_{pg})$,

$$\omega_{pg}(r_{g_{*,p}}(X_p)) = (r_g^*\omega)_p(X_p) \overset{EC2}{=} \mathrm{Ad}(g^{-1})\left(\omega_p(X_p)\right) \overset{X_p \in \mathcal{H}_p}{=} \mathrm{Ad}(g^{-1})(0) = 0.$$

Where the first equality follows from the definition of the pullback of a form.

3. $\mathcal{H}$ is a smooth vector subbundle of TP

This is a direct consequence of Lemma 2.6. Indeed, consider the trivial vector bundle $P \times \mathfrak{g} \to P$ over P and the vector bundle map

$$\widetilde{\omega} : TP \longrightarrow P \times \mathfrak{g}, \quad X_p \in T_pP \mapsto (p, \omega_p(X_p)).$$

Since ω_p is surjective for all $p \in P$, it has constant rank. Lemma 2.6 then implies that $\mathcal{H} = \mathrm{Ker}(\widetilde{\omega})$ is a smooth subbundle of TP, completing the proof. $\square$

Remark 2.39. Theorem 2.37 shows how can we can associate a connection with a horizontal, right-invariant distribution. A natural question to pose now is whether a horizontal, right-invariant distribution $\mathcal{H}$ defines in turn a connection $\omega_{\mathcal{H}}$. The answer is positive, and we refer the interested reader to [3], Theorem 28.1 for a proof. In conclusion, specifying a horizontal, right-invariant distribution is equivalent to giving a connection.

2.3.3 Vertical and Horizontal Components of Vector Fields

Throughout this section, let $\pi : P \to M$ be a principal G-bundle with connection $\omega \in \Omega^1(P, \mathfrak{g})$. Denote by $\mathcal{H}$ the horizontal distribution defined by ω.

Definition 2.40. *Suppose $p \in P$, and take $X_p \in T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$. Then, we know there exist a unique $v(X_p) \in \mathcal{V}_p$ and $h(X_p) \in \mathcal{H}_p$ such that*

$$X_p = v(X_p) + h(X_p).$$

These as referred to as, respectively, the vertical *and* horizontal components *of the vector X_p. A rough vector field X over P is said to be vertical (resp. horizontal) if, for every $p \in P$, X_p is vertical (resp. horizontal).*

A first property of the vertical and horizontal components is that they commute with the differential of the map r_g, for any $g \in G$:

Proposition 2.41. *For every $g \in G$ and $p \in P$, the differential of the right translation $(r_g)_{*,p}$ commutes with the vertical and horizontal projections.*

Proof. We aim to show that, given $g \in G$, $p \in P$ and $X_p \in T_pP$,

$$(r_g)_{*,p}(v(X_p)) = v((r_g)_{*,p}(X_p)) \quad \text{and} \quad (r_g)_{*,p}(h(X_p)) = h((r_g)_{*,p}(X_p)).$$

We begin by decomposing X_p into its vertical and horizontal components,

$$X_p = v(X_p) + h(X_p) \Rightarrow (r_g)_{*,p}(X_p) = (r_g)_{*,p}(v(X_p)) + (r_g)_{*,p}(h(X_p)).$$

Now, differentiating the identity $\pi \circ r_g = \pi$ we conclude that $(r_g)_{*,p}(v(X_p)) \in \mathcal{V}_{pg}$. Analogously, Theorem 2.37 ensures that the distribution $\mathcal{H}$ is right-invariant, implying that $(r_g)_{*,p}(h(X_p)) \in \mathcal{H}_{pg}$. The fact that the descomposition into vertical and horizontal components is unique implies the result we set out to prove. $\square$

This notion of vertical and horizontal can be extended to vector fields, where if $X \in \mathfrak{X}(P)$, we define $v(X)$ (and analogously $h(X)$) to be the rough vector field $v(X)_p = v(X_p)$, for all $p \in P$.

Proposition 2.42. *If $X \in \mathfrak{X}(P)$, then the vector fields $v(X)$ and $h(X)$ are smooth.*

Proof. As in Proposition 2.25, define $j_{p*} := (j_p)_{*,e} : \mathfrak{g} \longrightarrow \mathcal{V}_p$. Observe that, if $X_p \in T_pP$ for $p \in P$, then $v(X_p) = (j_{p*} \circ \omega_p)(X_p)$. Indeed, from Proposition 2.25 (ii) itself it is clear that $(j_{p*} \circ \omega_p(X_p)) \in \mathcal{V}_p$. Furthermore,

$$\omega_p\left(X_p - j_{p*} \circ \omega_p(X_p)\right) = \omega_p(X_p) - \omega_p\left(\underline{\omega_p(X_p)}_p\right) \overset{EC1}{=} \omega_p(X_p) - \omega_p(X_p) = 0,$$

proving the claim.

Thus, if we write $\omega = \omega^i B_i$ for a basis B_i of $\mathfrak{g}$ and scalar 1-forms ω^i, we have an explicit expression for the vector field $v(X)$,

$$v(X)_p = (j_{p*} \circ \omega_p)(X_p) = j_{p*}\left(\omega^i_p(X_p)B_i\right) = \omega^i_p(X_p)\underline{B_i}_p, \quad p \in P.$$

Allowing p to vary freely, the previous equation implies that

$$v(X) = \omega^i(X)\underline{B_i}.$$

From Proposition 2.19, it follows that $v(X)$ is smooth, and, consequently, $h(X) = X - v(X)$ is also smooth. $\square$

In many instances, given $X \in \mathfrak{X}(M)$, we will be interested in *lifting* X to P. This translates into a (smooth) assignation $\widetilde{X}_p$, for every $p \in P$, in such a way that $\pi_{*,p}\left(\widetilde{X}_p\right) = X_x$, where $x := \pi(p)$. A connection provides such a smooth assignation, that additionally ensures that the vector field $\widetilde{X} \in \mathfrak{X}(P)$ is horizontal.

Consider $p \in P$ and write $x = \pi(p) \in M$. Since $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$, we have the vector space isomorphism

$$\mathcal{H}_p \cong \frac{T_pP}{\mathcal{V}_p}.$$

At the same time, the linear map $\pi_{*,p} : T_pP \to T_xM$ induces a corresponding vector space isomorphism

$$\frac{T_pP}{\mathcal{V}_p} = \frac{T_pP}{\mathrm{Ker}(\pi_{*,p})} \cong T_xM.$$

Composing this two isomorphisms we obtain a third vector space isomorphism,

$$\psi_p : \mathcal{H}_p \longrightarrow T_xM, \quad h \mapsto \pi_{*,p}(h).$$

from which we deduce that if $X_x \in T_xM$ and $p \in P_x$, there exists a unique $\widetilde{X}_p \in \mathcal{H}_p$ such that $\pi_{*,p}(\widetilde{X}_p) = X_x$. In a pointwise fashion way we can now define a rough vector field $\widetilde{X} \in \mathfrak{X}(P)$ on P for a given $X \in \mathfrak{X}(M)$, called its ***horizontal lift***.

Definition 2.43. *A smooth vector field X on a G-manifold P is said to be **right-invariant** if X is r_g-related to itself for all $g \in G$.*

Proposition 2.44. *Let $X \in \mathfrak{X}(M)$. Then, $\widetilde{X}$ is a smooth, right-invariant vector field on P.*

Proof. Choose $p \in P$ and $g \in G$, and, like before, denote $x = \pi(p)$. Recall that since π is fibre-preserving, $\pi \circ r_g = \pi$. Thus,

$$\pi_{*,pg}\left((r_g)_{*,p}(\widetilde{X}_p)\right) = (\pi \circ r_g)_{*,p}(\widetilde{X}_p) = \pi_{*,p}(\widetilde{X}_p) = X_x = \pi_{*,pg}\left(\widetilde{X}_{pg}\right).$$

By the uniqueness of the horizontal lift, we conclude that $\widetilde{X}_{pg} = (r_g)_{*,p}(\widetilde{X}_p)$, proving that $\widetilde{X}$ is right-invariant.

Secondly, $\widetilde{X}$ is smooth by the local triviality condition of a principal bundle. Indeed, suppose that $\phi : P|_U \to U \times G$ is a trivialisation of $\pi : P \to M$. Define the smooth vector field

$$Z \in \mathfrak{X}(U \times G), \quad Z_{(x,g)} := (X_x, 0),\ x \in U \text{ and } g \in G.$$

Here, 0 represents the zero vector of $T_g G$. If $\mathrm{pr}_1 : U \times G \to U$ is the projection into the first factor, a quick computation shows that

$$(\mathrm{pr}_1)_{*,(x,g)}(Z_{(x,g)}) = X_x,$$

implying that Z lifts X over U. Define Y to be the pushforward of Z induced by ϕ^{-1} (see Remark 1.19),

$$Y := (\phi^{-1})_*(Z) \in \mathfrak{X}(P|_U).$$

Proposition 2.42 then assures that $h(Y)$ is a (horizontal) smooth vector field over $P|_U$. What is more, if $U \times G \ni (x,g) = \phi(q), q \in P|_U$,

$$\begin{aligned}
\pi_{*,q}\left(Y_q\right) &= \pi_{*,q}\left(\left[(\phi^{-1})_*(Z)\right]_q\right) \\
&= \pi_{*qp}\left((\phi^{-1})_{*,(x,g)}(Z_{(x,g)})\right) \\
&= \left(\pi \circ \phi^{-1}\right)_{*,(x,g)}(Z_{(x,g)}) \\
&= (\mathrm{pr}_1)_{*,(x,g)}(Z_{(x,g)}) \\
&= X_x.
\end{aligned}$$

The first equality follows from the definition of pushforward and the third one by the definition of a trivialisation of a principal bundle. Thus,

$$\pi_{*,q}\left(h(Y_q)\right) = \pi_{*,q}\left(Y_q\right) - \pi_{*,q}\left(v(Y_q)\right) = X_x + 0.$$

Again by the uniqueness of the horizontal lift, $\widetilde{X}\big|_U = h(Y) \in \mathfrak{X}(P|_U)$, proving the desired result. □

We end this section with a result that will prove useful in upcoming proofs.

Notation: If X and Y are smooth vector fields on P, we denote the ***Lie derivative of*** Y ***along*** X by $\mathcal{L}_X Y \in \mathfrak{X}(P)$.

Proposition 2.45. *Let $\pi : P \to M$ be a principal G-bundle with connection ω. Consider the fundamental vector field $\underline{A}$, for $A \in \mathfrak{g}$ and $X \in \mathfrak{X}(P)$. Then,*

(i) If X is horizontal, then $[\underline{A}, X]$ also is horizontal.

(ii) If X is right-invariant, then $[\underline{A}, X] = 0$.

Proof. Fix $p \in P$.
(i) From Remark 2.18 we know that the flow of $\underline{A}$ is $p \cdot e^{tA}$. Therefore,

$$[\underline{A}, X]_p = \mathcal{L}_{\underline{A}}(X)_p = \lim_{t \to 0} \frac{[(r_{e^{-tA}})_*(X)]_p - X_p}{t}. \tag{2.11}$$

The first equality follows from the identification of the Lie derivative with the Lie bracket (as shown in [1], p.225, Theorem 20.4), while the following equality is by definition of Lie derivative. As stated in Theorem 2.37, the horizontal distribution is right-invariant. Hence, if X_p is horizontal, $[(r_{e^{-tA}})_*(X)]_p \in \mathcal{H}_p$. Since this is true for all t for which the expression makes sense, the difference between both vectors, and thus also the limit $[\underline{A}, X]_p$, is likewise horizontal.
(ii) By definition of right-invariance, $[(r_{e^{-tA}})_*(X)] = X$, so that (2.11) is identically zero. □

2.3.4 Existence of Connections on Principal Bundles

As one might expect, connections are a fundamental concept in the study of principal G-bundles, as well as in their applications – such as in modern theoretical physics. It is hence worthwhile to study when are connections guaranteed to exist given a principal G-bundle $\pi : P \to M$.

In this subsection, we will circumscribe ourselves to explain the line of reasoning by which, under our current definitions, connections on π always exist, leaving the proofs and the heavy weight-lifting to [4]. First, a definition:

Definition 2.46. *A topological space M is said to be* paracompact *if every open cover $\{U_\alpha\}_{\alpha \in A}$ of M has an open refinement that is locally finite. In other words, if there exists another open cover $\{V_\beta\}_{\beta \in B}$ of M such that*

1. *(Open refinement) For every $\alpha \in A$, there exists $\beta \in B$ such that $V_\beta \subset U_\alpha$.*

2. *(Local finiteness) For every $x \in M$ there exists a neighbourhood W of x in M such that the set*

$$\{\beta \in B : W \cap V_\beta \neq \emptyset\}$$

is finite.

Even though we have not explicitly stated it until now, in this thesis we assume that a manifold is by definition Hausdorff and second countable. It is then a consequence of the ***Urysohn metrisation lemma*** (see [13], p. 215, Theorem 34.1) that any manifold M is necessarily paracompact. Therefore, the hypothesis of the following theorem always hold:

Theorem 2.47. *Let $\pi : P \to M$ be a principal G-bundle and C a, possibly empty, closed subset of M. If M is paracompact, every connection defined over C can be extended to a connection in P. In particular, P admits a connection if M is paracompact.*

Proof. Consult [4], p. 67, Theorem 2.1, for a detailed proof. □

2.4 Curvature on a Principal Bundle

The goal of this section is to generalize curvature to principal G-bundles. For this purpose, our guiding principle will be the definition of curvature in the vector bundle case.

As in Section 2.1, let $\eta : E \to M$ be a vector bundle of rank r, equipped with a vector bundle connection ∇ and its corresponding curvature map R. Relative to a smooth local frame over an open set U of M, the components of the connection and the curvature are denoted by ω^i_j and Ω^i_j, respectively. Our starting point is then the second structural equation (2.2):

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j.$$

The results from Subsection 2.3.1 now allow us to adopt a new perspective: we can regard ω^i_j and Ω^i_j as components of $\mathcal{M}(n,\mathbb{R})$-valued forms $\omega \in \Omega^1(U, \mathcal{M}(n,\mathbb{R}))$ and $\Omega \in \Omega^2(U, \mathcal{M}(n,\mathbb{R}))$, respectively, with respect to the standard basis of $\mathcal{M}(n,\mathbb{R})$.

Indeed, define the matrices of forms $\Omega = [\Omega^i_j]$ and $\omega = [\omega^i_j]$. Recalling the notation introduced in Example 2.33, we can express the second structural equation in matricial form,

$$\Omega = d\omega + \omega \cdot \omega \overset{(2.8)}{=} d\omega + \frac{1}{2}[\omega \wedge \omega]. \tag{2.12}$$

Of course, the notation until now has been purposefully picked to be reminiscent of (Ehresmann) connections on principal G-bundles. Equation (2.12) then justifies the definition of curvature on a principal G-bundle:

Definition 2.48. *Let $\pi : P \to M$ be a principal G-bundle with Ehresmann connection $\omega \in \mathcal{A}_P$. We define its* curvature *as the $\mathfrak{g}$-valued form $\Omega \in \Omega^2(P, \mathfrak{g})$,*

$$\Omega := d\omega + \frac{1}{2}[\omega \wedge \omega].$$

Notation: Whenever we wish to indicate that Ω is the curvature form associated with the connection $\omega \in \mathcal{A}_P$, we will denote it by Ω^ω.

Notice how we have transitioned from a particular Lie bracket for the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$ in Eq. (2.12) to an arbitrary Lie algebra in Definition 2.48. The next theorem summarizes the most important properties of the curvature form,

Theorem 2.49. *Suppose $\pi : P \to M$ is a principal G-bundle, where G is Lie group with Lie algebra $\mathfrak{g}$. Assume $\omega \in \Omega^1(P, \mathfrak{g})$ is a connection on π and denote by Ω its associated curvature form. Then, Ω satisfies:*

(i) Horizontality*: for all $p \in P$ and $X_p, Y_p \in T_pP$,*

$$\Omega(X_p, Y_p) = (d\omega)_p(h(X_p), h(Y_p)). \tag{2.13}$$

(ii) G-equivariance*: for all $g \in G$,*

$$r_g^*\Omega = (\mathrm{Ad}(g^{-1})) \circ \Omega.$$

(iii) Second Bianchi identity*: $d\Omega = [\Omega \wedge \omega]$.*

We will need the following lemma for the proof of Theorem 2.49:

Lemma 2.50. *Take $\pi : P \to M$ a principal G-bundle. For $p \in P$, consider $X_p \in T_pP$. Then,*

(i) If $X_p \in \mathcal{V}_p$, there exists a fundamental vector field $\underline{A}$ for $A \in \mathfrak{g}$ such that $\underline{A}_p = X_p$.

(ii) If $X_p \in \mathcal{H}_p$, there exists a horizontal vector field $Z \in \mathfrak{X}(M)$ such that its horizontal lift satisfies $\widetilde{Z}_p = X_p$.

Proof. *(i)* In Proposition 2.25 we showed that $j_{p*} : \mathfrak{g} \to \mathcal{V}_p, j_{p*}(B) = \underline{B}_p$, is a vector space isomorphism. Thus, we need only consider the fundamental vector field $\underline{A}$, for $A := (j_{p*})^{-1}(X_p)$.

(ii) From the existence of *bump functions* (see, for instance, [2], Prop. 2.25) and the local triviality of the vector bundle TM, it is straightforward to see that, given any $x \in M$ and $X_x \in T_xM$, there exists $Y \in \mathfrak{X}(M)$ such that $Y_x = X_x$. Let Z then be the extension of $\pi_{*,p}(X_p) \in T_{\pi(p)}M$. By the uniqueness of the horizontal lift, $\widetilde{Z}_p = X_p$. □

We can now prove the previous theorem,

Proof of Theorem 2.49. *(i)* Since $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$ and Equation (2.13) is linear, we may assume without loss of generality that X_p and Y_p are either vertical or horizontal and prove the equality by cases. By the alternating property of Ω, there are only three cases to study: (1) both vectors are horizontal, (2) both are vertical and (3) X_p is vertical and Y_p is horizontal. All three are similarly proved, so we focus in the last of them, for example, and refer the interested reader to [3], Theorem 30.4, for the full proof.

Suppose then that $X_p \in \mathcal{V}_p$ and that $Y_p \in \mathcal{H}_p$. By Lemma 2.50, there exists $A \in \mathfrak{g}$ and $Z \in \mathfrak{X}(M)$ such that $\underline{A}$ and $\widetilde{Z}$ extend X_p and Y_p, respectively. Now, the *global formula for the exterior derivative* ([1], Theorem 20.14) implies that

$$d\omega(\underline{A}, \widetilde{Z}) = \underline{A}\omega(\widetilde{Z}) - \widetilde{Z}\omega(\underline{A}) - \omega([\underline{A}, \widetilde{Z}]).$$

Each of the terms on the right-hand side is zero. Indeed, $\omega(\widetilde{Z}) = 0$ because $\widetilde{Z}$ is horizontal. Following the properties of the connection, the second term can be rewritten as $\widetilde{Z}(A)$, which is also zero since its a vector field acting on a constant. Finally, Proposition 2.45 ensures that $[\underline{A}, \widetilde{Z}]$ is horizontal, implying that $\omega([\underline{A}, \widetilde{Z}]) = 0$.

(ii) Choose $g \in G$. Remark 2.35 implies that the pullback and the exterior derivative commute, while it is also easy to see that the same is true for the pullback and the product of forms. Hence,

$$\begin{aligned} r_g^*\Omega = r_g^* d\omega + \frac{1}{2} r_g^*[\omega \wedge \omega] &= d(r_g^*\omega) + \frac{1}{2}[r_g^*\omega \wedge r_g^*\omega] \\ &\overset{EC2}{=} d(\mathrm{Ad}(g^{-1}) \circ \omega) + \frac{1}{2}[\mathrm{Ad}(g^{-1}) \circ \omega \wedge \mathrm{Ad}(g^{-1}) \circ \omega]. \end{aligned}$$

Now, if we express $\omega = \omega^i \otimes B_i$ in terms of its components with respect to a basis B_i of the Lie algebra $\mathfrak{g}$, we observe that while d acts on the scalar forms ω^i, $\mathrm{Ad}(g^{-1})$ acts on the B_i's, so that both maps commute. On the other hand, Proposition 1.18 ensures that $\mathrm{Ad}(g^{-1})$ is a Lie algebra homomorphism, implying that

$$r_g^*\Omega = \mathrm{Ad}(g^{-1}) \left(d\omega + \frac{1}{2}[\omega \wedge \omega] \right) = \mathrm{Ad}(g^{-1}) \circ \Omega.$$

(iii) We proceed by direct computation:

$$\begin{aligned} d\Omega &= d(d\omega + \frac{1}{2}[\omega \wedge \omega]) \\ &= 0 + \frac{1}{2}([d\omega \wedge \omega] - [\omega \wedge d\omega]) && \text{(Remark 2.35)} \\ &= [d\omega \wedge \omega] && \text{(Proposition 2.32)} \\ &= [\Omega \wedge \omega] - \frac{1}{2}[[\omega \wedge \omega] \wedge \omega]. && \text{(Definition of } \Omega) \end{aligned}$$

Lastly, Proposition 2.31 implies that the triple product $[[\omega \wedge \omega] \wedge \omega]$ vanishes. Thus, $d\Omega = [\Omega \wedge \omega]$, as desired. □

Chapter 3

Chern-Simons Theory

In this last chapter we give a glimpse of Chern-Simons (classical) theory, focusing on closed three manifolds and compact and simply connected Lie groups.

3.1 Maurer-Cartan Form

Let G be a Lie group with Lie algebra $\mathfrak{g}$.

Definition 3.1. *The Maurer-Cartan form $\theta \in \Omega^1(G, \mathfrak{g})$ is the $\mathfrak{g}$-valued 1-form on G satisfying*

$$\theta_g(X_g) := (l_{g^{-1}})_{*,g}(X_g), \quad \forall g \in G \ \forall X_g \in T_gG.$$

Remark 3.2. If $X \in L(G)$ is a left-invariant vector field on G,

$$\theta_g(X_g) = (l_{g^{-1}})_{*,g}(X_g) = (l_{g^{-1}})_{*,g}\left((l_g)_{*,e}(X_e)\right) = X_e,$$

for any $g \in G$. Thus, $\theta(X) = X_e$ is a constant $\mathfrak{g}$-valued function.

Let us verify that θ is indeed smooth. Consider $f \in C^\infty(G)$ and $X \in \mathfrak{X}(G)$. Then, by the linearity of $\theta_g, g \in G$:

$$\theta_g(f(g)X_g) = f(g)\theta_g(X_g) \Rightarrow \theta(fX) = f\theta(X). \tag{3.1}$$

Applying Proposition 1.15 and the equation above, we see that it is only necessary to prove smoothness for left-invariant vector fields, but this trivially holds by Remark 3.2.

Proposition 3.3. *The Maurer-Cartan form satisfies the* Maurer-Cartan equation,

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0.$$

Proof. Similarly to how we argued that θ is smooth, it suffices to show the above equality for left-invariant vector fields $X, Y \in L(G)$. On the one hand, the global formula for the exterior derivative implies that

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) = 0 - 0 - [X, Y]_e.$$

In the last equality we have used Eq. (3.1), taking into account that a vector field on a constant function is zero. On the other hand, by Definition 2.28 and Remark 1.13,

$$[\theta \wedge \theta](X, Y) = [\theta(X), \theta(Y)] - [\theta(Y), \theta(X)] = 2[X_e, Y_e] = 2[X, Y]_e.$$

□

3.2 Gauge Transformations

Proposition 3.4. *Let $\pi : P \to M$ and $\pi' : P' \to M$ be principal G-bundles. Then, any principal bundle morphism $F : P \to P'$ over M is necessarily a principal bundle isomorphism.*

Proof. By the definition of principal bundle morphism, we have that F is G-equivariant and that $\pi' \circ F = \pi$.

We first prove that F is a bijection. Regarding injectivity, consider $p, q \in P$ such that $F(p) = F(q)$. Then,

$$\pi(p) = \pi'(F(p)) = \pi'(F(q)) = \pi(q) \overset{\text{Prop. 2.16}}{\Longrightarrow} q = pg,$$

for some $g \in G$. Thus, $F(p) = F(q) = F(pg) = F(p)g$. But the action of G on P' is free, implying that $g = e$ and therefore that $p = q$. Surjectivity can be proven with similar arguments, and one can easily see that the inverse of F, $H : P' \to P$, is G-equivariant and satisfies the commutativity condition $\pi \circ H = \pi'$.

We now turn to show that H is continuous. Consider the set

$$\mathfrak{U} = \{\, P'|_U \subset P' : U \subset M \text{ is a trivialising open set for } \pi' \}.$$

Since $\mathfrak{U}$ is an open cover of P', the *local formulation of continuity* (see [13], p. 107, Theorem 18.2(f)), implies that it suffices to verify the continuity of

$$H_U := H|_{P'|_U} : P'|_U \longrightarrow P$$

for all $U \in \mathfrak{U}$.

From the identity $\pi' \circ F = \pi$, it follows that $F^{-1}(P'|_U) = P|_U$. Since this set is open, we may equivalently check the continuity of H_U by restricting the codomain. Reusing the symbol H_U, we rewrite it as

$$H_U : P'|_U \longrightarrow P|_U.$$

By choosing a smaller $U \subset M$ if necessary, we may assume that U is also a trivialising open set for π. In this case, we can regard H_U as

$$H_U : U \times G \longrightarrow U \times G, \quad (x, g) \longmapsto (x, \Lambda(x, g)),$$

where $\Lambda : U \times G \to G$ is a map still to be determined. As noted in Remark 2.17, the G-equivariance of H_U implies that the restriction

$$\Lambda_x : G \longrightarrow G, \quad g \longmapsto \Lambda(x, g)$$

must be a left translation. Consequently, there exists a function $h : U \longrightarrow G$ such that

$$\Lambda(x, g) = h(x)g, \qquad \text{for all } x \in U \text{ and all } g \in G.$$

The inverse of H_U, which is of course the restriction of F to $P|_U$ as thought as a map on $U \times G$, can then be expressed as

$$F_U := H_U^{-1} : U \times G \longrightarrow U \times G, \quad (x, g) \longmapsto (x, h(x)^{-1}g).$$

Since F is continuous (being smooth), F_U also is, ensuring that the map $(x, g) \mapsto h(x)^{-1}g$ is continuous. Finally, as the group operations (multiplication and inversion) in G are smooth and thus in particular continuous, the continuity of $x \mapsto h(x)$ follows. This implies the continuity of H_U.

By a similar argument, changing "continuous" by "smooth" in the above reasoning, we conclude that H is smooth, completing the proof. □

This proposition allows us to define the ***group of gauge transformations*** $\mathcal{G}_P$ of a principal G-bundle $\pi : P \to M$,

Corollary 3.5. *The set of principal G-bundle automorphisms,*

$$\mathcal{G}_P := \{F : P \to P \mid F \text{ is a principal } G\text{-bundle morphism over } M\},$$

is a group under the operation of map composition.

There is a useful alternative way to view a gauge transformation $F \in \mathcal{G}_P$. Observe that for every $p \in P$, there exists a unique $g_p \in G$ such that

$$F(p) = p \cdot g_p.$$

Indeed, while existence follows again from Proposition 2.16, the freedom of the action of G on P ensures uniqueness. We can thus consider a map $u_F = u : P \to G, p \mapsto g_p$. Moreover, the G-equivariance of F imposes an extra condition on u, namely

$$(p \cdot u(p))g = F(p)g = F(pg) = pg \cdot u(pg) \implies u(pg) = g^{-1}u(p)g, \quad \forall p \in P \; \forall g \in G.$$

This motivates the definition of the following set,

$$C^\infty(P,G)^G := \{u \in C^\infty(P,G) \mid u(pg) = g^{-1}u(p)g, \text{ for all } p \in P \text{ and } g \in G\}.$$

Note that this set has a natural group structure induced by the product in G. If $u, v \in C^\infty(P,G)^G$, we define $u \cdot v \in C^\infty(P,G)^G$ to be the function

$$(u \cdot v)(p) := u(p) \cdot v(p), \quad \text{for all } p \in P.$$

We denote the (group) inverse of $u \in C^\infty(P,G)^G$ by u^{-1}. That is, $u^{-1}(p) := (u(p))^{-1}$, for all $p \in P$.

Proposition 3.6. *The map*

$$\zeta : \mathcal{G}_P \longrightarrow C^\infty(P,G)^G, \quad F \longmapsto u_F$$

defined above is a group isomorphism.

Proof. We begin by showing that ζ is well-defined, i.e. that $u_F = u$ is smooth. Take a trivialisation $\phi : P|_U \to U \times G$ of π. We can then give an explicit expression for $u|_{P|_U}$ in terms of the projection onto the second factor $\mathrm{pr}_2 : U \times G \to G$,

$$u(p) = (\mathrm{pr}_2(\phi(p)))^{-1} \cdot \mathrm{pr}_2(\phi(F(p))),$$

for any $p \in P|_U$. This expression is manifestly smooth by the definition of Lie group and the smoothness of F.

Next, we demonstrate that ζ is a group homomorphism. If $F, G \in \mathcal{G}_P$,

$$p \cdot u_{F \circ G}(p) = (F \circ G)(p) = F(G(p)) = \\ = F(p \cdot u_G(p)) = F(p) \cdot u_G(p) = p \cdot [(u_F \cdot u_G)(p)].$$

Thus, if we can show that the map sending $u \in C^\infty(P, G)^G$ to $\mathcal{G}_P \ni F_u(p) = p \cdot u(p)$ is well-defined and serves as the inverse of ζ, the proof will be complete. Since the latter is clear, we show only the former: the smoothness of both u and the action of G on P ensure that F_u is likewise smooth. Furthermore, the definition of $C^\infty(P, G)^G \ni u$ implies that F_u is a G-equivariant map satisfying that $\pi \circ F = \pi$, this is, that $F_u \in \mathcal{G}_P$. □

As justified by the preceding proposition, in what follows we will view the group of gauge transformations either as $\mathcal{G}_P$ or $C^\infty(P, G)^G$. Perhaps the most interesting aspect of $\mathcal{G}_P$ is that it defines a natural action on $\mathcal{A}_P$, the set of connections on $\pi : P \to M$:

Proposition 3.7. *The pullback defines a (right) group-theoretic action of $\mathcal{G}_P$ on $\mathcal{A}_P$,*

$$\mathcal{A}_P \times \mathcal{G}_P \longrightarrow \mathcal{A}_P, \quad (\omega, F) \longmapsto F^*\omega.$$

In terms of the identification $\mathcal{G}_P \cong C^\infty(P, G)^G$, this action translates into

$$\mathcal{A}_P \times C^\infty(P, G)^G \longrightarrow \mathcal{A}_P, \quad (\omega, u) \longmapsto \omega \cdot u := \mathrm{Ad}(u^{-1})(\omega) + u^*(\theta),$$

where $\theta \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form introduced in Section 3.1. Under this last action, the curvature form $\Omega^\omega \in \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g})$ associated to ω transforms as a tensor. That is,

$$\Omega^{\omega \cdot u} = \mathrm{Ad}(u^{-1}) \circ \Omega^\omega.$$

For the proof, we will need the lemma below concerning the differential of the action of G on P. Note that, since $l_g : G \to G, g \in G$, is a diffeomorphism, we may identify $T_g G$ with $l_{g*,e}(\mathfrak{g})$.

Lemma 3.8. *Denote by $\lambda : P \times G \to P$ the smooth and free right-action of G on P. Taking $g \in G$ and identifying $T_g G$ with $l_{g*,e}(\mathfrak{g})$, we can write the differential of λ at $(p, g) \in P \times G$ as*

$$\lambda_{*,(p,g)}(X_p, (l_g)_{*,e}(A)) = (r_g)_{*,p}(X_p) + \underline{A}_{pg}, \quad X_p \in T_pP, A \in \mathfrak{g}.$$

Proof. This is a direct consequence of the linearity of the differential,

$$\lambda_{*,(p,g)}(X_p, (l_g)_{*,e}(A)) = \lambda_{*,(p,g)}(X_p, 0) + \lambda_{*,(p,g)}(0, (l_g)_{*,e}(A)),$$

and the definitions of r_g and of fundamental vector fields (see (2.10) and (2.3), respectively). □

Proof of Proposition 3.7. Fix $\omega \in \mathcal{A}_P$ and $F \in \mathcal{G}_P$ with its associated $u \in C^\infty(P, G)^G$. To begin, we show that

1. $F^*\omega \in \mathcal{A}_P$: Consider $A \in G$ and $p \in P$. Then, if $F_* := F_{*,p}$ and $(F \circ j_p)_* := (F \circ j_p)_{*,e}$,

$$(F^*(\omega))_p(\underline{A}_p) = \omega_{F(p)}(F_*(\underline{A}_p)) = \omega_{F(p)}\left((F \circ j_p)_*(\underline{A}_p)\right).$$

Since F is G-equivariant, $F \circ j_p = j_{F(p)}$, allowing us to conclude that $(F^*(\omega))_p(\underline{A}_p) = A$. That $F^*\omega$ is right-equivariant of type Ad is also easy to verify and mainly uses that ω is itself right-equivariant of type Ad and that r_g ($g \in G$), commutes with F.

2. $\omega \cdot u := \mathrm{Ad}(u^{-1})(\omega) + u^*(\theta)$: Differentiating the equality

$$F(p) = \lambda(p, u(p)), \quad p \in P,$$

we obtain that

$$F_{*,p} = \lambda_{*,(p,u(p))}(X_p, u_{*,p}(X_p)), \quad X_p \in T_P.$$

We can write $u_{*,p}(X_p)$ more suggestively as

$$u_{*,p}(X_p) = (l_{u(p)})_{*,e}\left[(l_{u(p)^{-1}})_{*,u(p)} \circ u_{*,p}(X_p)\right] = (l_{u(p)})_{*,e}\left[(u^*\theta)_p(X_p)\right].$$

Lemma 3.8 then implies that

$$F_{*,p} = (r_{u(p)})_{*,p}(X_p) + \underline{u^*(\theta)_p(X_p)}_{p\cdot u(p)}. \tag{3.2}$$

Therefore,

$$\begin{aligned}(F^*\omega)_p(X_p) = \omega_{p\cdot u(p)}(F_{*,p}(X_p)) &= ((r_{u(p)})^*\omega)_p(X_p) + u^*(\theta)_p(X_p)\\ &= (\mathrm{Ad}(u(p)^{-1}) \circ \omega)_p(X_p) + u^*(\theta)_p(X_p).\end{aligned}$$

3. The curvature transforms as a tensor: Given that

$$\begin{aligned}\Omega^{\omega\cdot u} &= d(\omega \cdot u) + \frac{1}{2}[\omega \cdot u \wedge \omega \cdot u]\\ &= d(F^*\omega) + \frac{1}{2}[F^*\omega \wedge F^*\omega]\\ &= F^*(\Omega^\omega),\end{aligned}$$

we can compute for $p \in P$ and $X_p, Y_p \in T_pP$:

$$(\Omega^{\omega\cdot u})_p(X_p, Y_p) = (\Omega^\omega)_{F(p)}(F_{*,p}(X_p), F_{*,p}(Y_p)).$$

Using Eq. (3.2) and the fact that Ω^ω is horizontal and right-equivariant of type Ad, we obtained the desired result. □

Even though we have up to now only considered the set of connections $\mathcal{A}_P$ relative to a specified principal G-bundle $\pi : P \to M$, it is useful to consider the set of all G-connections at once. To do this, we define the ***category of G-connections over M***, $\mathbf{Conn}_M^G$, as follows:

- An object in $\mathbf{Conn}_M^G$ is a connection $\omega \in \mathcal{A}_P$ on any principal G-bundle $\pi : P \to M$.
- If $\omega, \omega' \in \mathrm{Ob}(\mathbf{Conn}_M^G)$ are two connections on $\pi : P \to M$ and $\pi' : P' \to M$, respectively, a morphism $\Lambda \in \mathrm{Mor}(\omega, \omega')$ is any principal G-bundle morphism $\Lambda : P \to P'$ such that $\omega = \Lambda^*\omega'$.

It is not difficult to show that $\mathbf{Conn}_M^G$ does indeed satisfy the axioms of a category. Note that, by Proposition 3.4, all morphisms in the category are invertible, so that in fact $\mathbf{Conn}_M^G$ is what is called a ***groupoid***.

Remark 3.9. Recalling the definition of a connection 2.36, we see that $\mathcal{A}_P$ is actually an ***affine subspace*** of the vector space $\Omega^1(P, \mathfrak{g})$; this holds because the difference $\omega_2 - \omega_1$ of any two connections $\omega_1, \omega_2 \in \mathcal{A}_P$ belongs to the vector space $\Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})$. This observation enables us to view $\mathcal{A}_P$ as an example of a ***Fréchet manifold***[1], a type of infinite-dimensional manifold. Since

$$\mathrm{Ob}(\mathbf{Conn}^G_M) = \bigsqcup_{P \in \mathcal{P}} \mathcal{A}_P,$$

where $\mathcal{P}$ is the collection of all principal G-bundles over M, the differential structure of $\mathcal{A}_P$ carries on to the class of objects of $\mathbf{Conn}^G_M$. In the interest of brevity, we will touch upon such matters in a fleeting and naive manner, simply pointing out when functions defined over $\mathrm{Ob}(\mathbf{Conn}^G_M)$, (or $\overline{\mathbf{Conn}^G_M}$, see the definition below), are smooth.

There is a natural equivalence relation stemming from Proposition 3.4,

Definition 3.10. *We say that $\omega, \omega' \in \mathrm{Ob}(\mathbf{Conn}^G_M)$ are* equivalent *if there exists a morphism $\Lambda \in Mor(\omega, \omega')$. We denote the set of equivalence classes by $\overline{\mathbf{Conn}^G_M}$.*

3.3 Chern-Simons Theory

3.3.1 Chern-Simons Form

From this point onward we denote by

$$\langle \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$$

a linear, symmetric and Ad-invariant real-valued map on $\mathfrak{g} \otimes \mathfrak{g}$. In other words, $\langle \cdot \rangle$ is a linear map satisfying

- *Symmetry:* $\langle A \otimes B \rangle = \langle B \otimes A \rangle$
- Ad-*invariance:* $\langle \mathrm{Ad}(g) A \otimes \mathrm{Ad}(g) B \rangle = \langle A \otimes B \rangle$,

for any $A, B \in \mathfrak{g}$ and any $g \in G$.

Proposition 3.11. Ad-*invariance implies* ad-invariance*: for any $A, B, C \in \mathfrak{g}$,*

$$\langle [A, B] \otimes C \rangle + \langle B \otimes [A, C] \rangle = 0.$$

Proof. Consider the real-valued function

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \langle \mathrm{Ad}(e^{tA}) B \otimes \mathrm{Ad}(e^{tA}) C \rangle.$$

Ad-invariance implies that γ is constant and equal to $\langle B \otimes C \rangle$, so that its derivative with respect to t must vanish. The result now follows from this observation and from computing this derivative,

$$\left. \frac{d}{dt} \right|_{t=0} \langle \mathrm{Ad}(e^{tA}) B \otimes \mathrm{Ad}(e^{tA}) C \rangle, \tag{3.3}$$

where we recall the identities (cf. Equation (1.6) and Proposition 1.26, respectively)

$$\mathrm{ad} = \mathrm{Ad}_{*,e}, \quad \mathrm{ad}(A)(B) = [A, B],$$

and note that *Leibniz's rule* applies to (3.3) given that $\langle \cdot \rangle$ is linear. $\square$

[1] See [9], p.56, Subsection 4.3.3, for details.

Definition 3.12. *Let $\pi : P \to M$ be a principal G-bundle, and let $\omega \in \mathcal{A}_P$ be a connection with associated curvature form $\Omega \in \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g})$. We define the* Chern-Simons 3-form *of the connection ω as*

$$\alpha(\omega) := \langle \omega \wedge \Omega \rangle - \frac{1}{6} \langle \omega \wedge [\omega \wedge \omega] \rangle \in \Omega^3(P).$$

Remark 3.13. As we will see in the following proposition, the motivation behind the definition of the Chern-Simons 3-form essentially lies in the fact that $\alpha(\omega)$ is an ***antiderivative*** of the Chern-Weil[2] 4-form $\langle \Omega \wedge \Omega \rangle$ (see Definition A.6), meaning that

$$d(\alpha(\omega)) = \langle \Omega \wedge \Omega \rangle.$$

Section 8 of [10] shows that $\alpha(\omega)$ is but a particular example of a more general construction to build antiderivatives (in the sense given above) of other Chern-Weil forms, for other invariant, symmetric mappings belonging to $I^k(G), k \in \mathbb{N}$.

Proposition 3.14. *Mantaining the notation of Definition 3.12, $\alpha(\omega)$ satisfies*

(i) $d(\alpha(\omega)) = \langle \Omega \wedge \Omega \rangle$

(ii) If $F \in \mathcal{G}_P$ is a gauge transformation with associated map $u \in C^\infty(P, G)^G$,

$$F^*(\alpha(\omega)) = \alpha(\omega) + d\langle \mathrm{Ad}(u^{-1}) \circ \omega \wedge u^*\theta \rangle - \frac{1}{6} u^* \langle \theta \wedge [\theta \wedge \theta] \rangle.$$

Before proving both results by direct computation, we list two facts that will be useful in our derivation:

Lemma 3.15. *Suppose $\alpha \in \Omega^p(P, \mathfrak{g})$, $\beta \in \Omega^q(P, \mathfrak{g})$ and $\gamma \in \Omega^r(P, \mathfrak{g})$. Then,*

(i) $\langle \alpha \wedge \beta \rangle = (-1)^{pq} \langle \beta \wedge \alpha \rangle$

(ii) $\langle [\alpha \wedge \beta] \wedge \gamma \rangle = (-1)^{pq+1} \langle \beta \wedge [\alpha \wedge \gamma] \rangle$

Proof. Both properties are easy to deduce. The first one follows from the graded commutativity property for scalar forms (2.7) and the symmetry of $\langle \cdot \rangle$; the second one holds by the ad-invariance of $\langle \cdot \rangle$ (Proposition 3.11) and again by (2.7). □

Proof of Proposition 3.14. (i) We study both terms in

$$d(\alpha(\omega)) = d\langle \omega \wedge \Omega \rangle - \frac{1}{6} d\langle \omega \wedge [\omega \wedge \omega] \rangle$$

independently. By the antiderivation property of d (Remark 2.35), we can express the first term in the right-hand side as

$$d\langle \omega \wedge \Omega \rangle = \langle d\omega \wedge \Omega \rangle - \langle \omega \wedge d\Omega \rangle. \tag{3.4}$$

[2] Although not indispensable for the development of Chern-Simons theory, characteristic classes provide important motivation for the theory. For this reason, a short appendix on the Chern-Weil homomorphism and characteristic classes is included at the end of this thesis.

However,

$$\begin{aligned}\langle \omega \wedge d\Omega \rangle &= \langle \omega \wedge [\Omega \wedge \omega] \rangle && \text{(Theorem 2.49)}\\ &= -\langle \omega \wedge [\omega \wedge \Omega] \rangle && \text{(Proposition 2.32)}\\ &= -\langle [\omega \wedge \omega] \wedge \Omega \rangle && \text{(Lemma 3.15).}\end{aligned}$$

Therefore, Eq. (3.4) can be written as

$$d\langle \omega \wedge \Omega \rangle = \langle (d\omega + [\omega \wedge \omega]) \wedge \Omega \rangle \overset{\text{Def. 2.48}}{=} \langle \Omega \wedge \Omega \rangle + \frac{1}{2}\langle [\omega \wedge \omega] \wedge \Omega \rangle.$$

Thus, its sufficient to show that

$$-\frac{1}{6} d\langle \omega \wedge [\omega \wedge \omega] \rangle = -\frac{1}{2}\langle [\omega \wedge \omega] \wedge \Omega \rangle \overset{\text{Lemma 3.15}}{\iff} d\langle \omega \wedge [\omega \wedge \omega] \rangle = 3\langle \Omega \wedge [\omega \wedge \omega] \rangle. \tag{3.5}$$

Observe that

$$\langle \Omega \wedge [\omega \wedge \omega] \rangle = \langle d\omega \wedge [\omega \wedge \omega] \rangle + \frac{1}{2}\langle [\omega \wedge \omega] \wedge [\omega \wedge \omega] \rangle = \langle d\omega \wedge [\omega \wedge \omega] \rangle,$$

where the second term in the right-hand side vanishes because it equals

$$\langle [\omega \wedge \omega] \wedge [\omega \wedge \omega] \rangle = \langle \omega \wedge [\omega \wedge [\omega \wedge \omega]] \rangle = 0.$$

In the first equation we have once again used Lemma 3.15 and in the second one Proposition 2.31. Let us now examine the left-hand side of (3.5):

$$d\langle \omega \wedge [\omega \wedge \omega] \rangle = \langle d\omega \wedge [\omega \wedge \omega] \rangle - \langle \omega \wedge d[\omega \wedge \omega] \rangle$$

Reasoning analogously to how we have so far, we deduce that the last term in the right-hand side is equivalent to

$$\begin{gathered} d[\omega \wedge \omega] = [d\omega \wedge \omega] - [\omega \wedge d\omega] = -2[\omega \wedge d\omega] \Longrightarrow \\ \langle \omega \wedge d[\omega \wedge \omega] \rangle = -2\langle \omega \wedge [\omega \wedge d\omega] \rangle = -2\langle [\omega \wedge \omega] \wedge d\omega \rangle = -2\langle d\omega \wedge [\omega \wedge \omega] \rangle. \end{gathered}$$

Summarizing,

$$d\langle \omega \wedge [\omega \wedge \omega] \rangle = 3\langle d\omega \wedge [\omega \wedge \omega] \rangle = 3\langle \Omega \wedge [\omega \wedge \omega] \rangle,$$

as desired.

(ii) The proof of this property follows in a similar fashion to that of (i), so that we will only outline the initial steps of it here. The interested reader can refer to [10], p.10, Proposition 3.2, for a complete development of the argument.

By the properties of the pullback and the definition of $\alpha(\omega)$, one can easily verify that $F^*(\alpha(\omega)) = \alpha(F^*\omega)$. In that case, we can expand $\alpha(F^*\omega) = \alpha(\omega \cdot u)$ according to Proposition 3.7, and then play around with the resulting terms to obtain the desired expression, using as before the properties of the curvature form stated in Theorem 2.49, Lemma 3.15 and the Maurer-Cartan equation proven in Proposition 3.3. □

3.3.2 Chern-Simons Action

We will begin with a simplification, motivated by the following result:

Proposition 3.16. *Let G be a simply connected Lie group. Then, any principal G-bundle $\pi : P \to M$, with M of dimension at most three, is trivialisable.*

The proof of this fact (which can be found in [9], p.53, Lemma 4.1.1), is out of the scope of this thesis. Nevertheless, it justifies the following hypothesis for a first approximation to Chern-Simons theory:

Hypothesis 3.17. *From now on, we will assume G to be a simply connected and compact Lie group.*

Notation: In keeping with the conventions of the field, we will denote by X (instead of M) the base manifold of a principal G-bundle $\pi : P \to X$. We will assume X to be a ***closed*** (compact and without boundary) and oriented 3-manifold.
The following characterisation of trivialisable principal G-bundles will prove instrumental in our definition of the Chern-Simons action:

Proposition 3.18. *Suppose $\pi : P \to X$ is a principal G-bundle. Then,*

$$\pi \text{ is trivialisable } \iff \exists\, s : X \to P \text{ global smooth section.}$$

Proof. Assume first that π is trivialisable, with principal bundle isomorphism $F : P \to M \times G$. In that case, it is clear that

$$s : X \longrightarrow P, \quad x \longmapsto F^{-1}(x, e),$$

is a global section for π. Reciprocally, suppose we are given a global section $s : X \to P$ for π. Then, it can readily be shown that

$$H : X \times G \longrightarrow P, \quad (x, g) \longmapsto s(x) \cdot g,$$

is a principal bundle isomorphism, with inverse

$$F : P \longrightarrow X \times G, \quad p \longmapsto (\pi(p), g_p), \text{ where } g_p \in G : p = s(\pi(p)) \cdot g_p.$$

□

As in the vector bundle case, we denote by $\Gamma(P)$ the vector space of all sections of P over X.

Corollary 3.19. *Assume $s, s' \in \Gamma(P)$ are two global sections of $\pi : P \to X$. Then, there exists $F \in \mathcal{G}_P$ such that*

$$F \circ s(x) = s'(x), \quad \forall x \in X.$$

Proof. We can simply compose the isomorphisms from the previous proposition for the different sections,

$$\begin{aligned} F : & P \xrightarrow{\cong} X \times G \xrightarrow{\cong} P \\ & p \longmapsto (\pi(p), g_p^s) \longmapsto s'(\pi(p)) g_p^s, \end{aligned}$$

where $g_p^s \in G$ is such that $p = s(\pi(p)) g_p^s$, to obtain the principal bundle automorphism we are looking for. □

We now give the map that will serve as a precursor of what we will finally name *Chern-Simons action*,

Definition 3.20. *Suppose $\pi : P \to X$ is a trivialisable principal G-bundle. We define the map*

$$\begin{aligned} S_X^P : \Gamma(P) \times \mathcal{A}_P &\longmapsto \mathbb{R} \\ (s, \omega) &\longmapsto \int_X s^*(\alpha(\omega)), \end{aligned}$$

where $\alpha(\omega)$ is the Chern-Simons 3-form discussed in the previous subsection.

The above definition of S_X^P is obviously dependent on the section $s : X \to P$ that we choose, which is unfortunate because we would our action to act solely on $\mathcal{A}_P$. As Corollary 3.19 shows, if we aim to study how S_X^P changes with different sections $s \in \Gamma(P)$, we can restrict our attention to composing sections with gauge transformations:

Proposition 3.21. *Consider $F \in \mathcal{G}_P$ an automorphism of the principal G-bundle $\pi : P \to X$, with associated map $u \in C^\infty(P, G)^G$. Let $\omega \in \mathcal{A}_P$ be a connection on π and $s \in \Gamma(P)$ a global section. Then,*

$$S_X^P(F \circ s, \omega) = S_X^P(s, \omega) - \frac{1}{6} \int_X \langle \varphi_u \wedge [\varphi_u \wedge \varphi_u] \rangle, \tag{3.6}$$

where we have defined $\varphi_u := (u \circ s)^* \theta$.

Proof. One may easily verify through direct computation that

$$(F \circ s)^* = s^* \circ F^*.$$

Hence,

$$\begin{aligned} S_X^P(F \circ s, \omega) &= \int_X (F \circ s)^*(\alpha(\omega)) = \int_X s^* \circ (F^*(\alpha(\omega))) = \\ \int_X s^* \circ &\left[\alpha(\omega) + d\langle \mathrm{Ad}(u^{-1}) \circ \omega \wedge u^*\theta \rangle - \frac{1}{6} u^* \langle \theta \wedge [\theta \wedge \theta] \rangle \right], \end{aligned} \tag{3.7}$$

with the last equality following from Proposition 3.14. Applying ***Stokes's theorem*** (see [2], p.411, Theorem 16.11), we deduce that the integral of the second term in the second line above vanishes,

$$\int_X d\langle \mathrm{Ad}(u^{-1}) \circ s^*\omega \wedge \varphi_u \rangle = \int_{\partial X} \langle \mathrm{Ad}(u^{-1}) \circ s^*\omega \wedge \varphi_u \rangle = 0.$$

Here, $\partial X = \emptyset$ denotes the (manifold) boundary of X, which is empty by hypothesis since X is closed. We see that the two remaning terms in (3.7) exactly coincide with the statement we wanted to prove. □

The preceeding proposition lights a possible way to make S_X^P section-independent by adding an extra hypothesis on the form $\langle \varphi_u \wedge [\varphi_u \wedge \varphi_u] \rangle$, or equivalently on $\langle \theta \wedge [\theta \wedge \theta] \rangle$. To do so, we need the next proposition:

Proposition 3.22. *The form $\langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G)$ is closed. That is,*

$$d\langle \theta \wedge [\theta \wedge \theta] \rangle = 0.$$

Proof. The proof is similar to the one given in Proposition 3.14. Therefore, we proceed with a direct computation,

$$d\langle \theta \wedge [\theta \wedge \theta] \rangle = \langle d\theta \wedge [\theta \wedge \theta] \rangle - \langle \theta \wedge d[\theta \wedge \theta] \rangle. \tag{3.8}$$

Both of these terms on the right-hand side are independently zero. For the first one, the Maurer-Cartan equation (Proposition 3.3) implies that

$$\langle d\theta \wedge [\theta \wedge \theta] \rangle = -\frac{1}{2}\langle [\theta \wedge \theta] \wedge [\theta \wedge \theta] \rangle = 0,$$

the last equality following for the same reasons $\langle [\omega \wedge \omega] \wedge [\omega \wedge \omega] \rangle$ vanished in Proposition 3.14. With respect to the second term in (3.8),

$$d[\theta \wedge \theta] = -2\, d^2\theta = 0,$$

where we have once again Maurer-Cartan's equation. □

We make the following *normalization* hypothesis,

Hypothesis 3.23. *Assume that the closed form $\langle \theta \wedge [\theta \wedge \theta] \rangle$ represents an* integral class[3] *in $H^3(G)$.*

Without getting into unnecessary detail, the main consequence of this assumption is that the integral of $\langle \varphi_u \wedge [\varphi_u \wedge \varphi_u] \rangle$ appearing in Equation (3.6) is an integer. Therefore, if this hypothesis holds, we can consider the map

$$\widetilde{S_X^P} : \mathcal{A}_P \longrightarrow \mathbb{R}/\mathbb{Z}, \quad \omega \longmapsto S_X^P(s, \omega),$$

for an arbitrary section $s \in \Gamma(P)$. Here, $\mathbb{R}/\mathbb{Z}$ stands for the quotient of $\mathbb{R}$ by the equivalence relation

$$a \sim b \iff a - b \in \mathbb{Z}, \quad \text{for all } a, b \in \mathbb{R}.$$

We typically abuse notation and we again write S_X^P for $\widetilde{S_X^P}$.

Notice that we could even define $\widetilde{S_X^P}$ over $\mathrm{Ob}(\mathbf{Conn}_X^G)$, simply by choosing a global section s for whichever principal G-bundle over X we are considering. We will denote this map by

$$S_X : \mathrm{Ob}(\mathbf{Conn}_X^G) \longrightarrow \mathbb{R}/\mathbb{Z},$$

called the ***Chern-Simons action on closed manifolds.***

Theorem 3.24 (Properties of the Chern-Simons action)**.** *The Chern-Simons action is smooth and satisfies*

(i) (Functoriality) *Consider two principal G-bundles $\pi : P \to X$ and $\pi' : P' \to X'$. If the pair of maps $(F : P' \to P, f : X' \to X)$ consitute a principal bundle morphism, with f an orientation-preserving diffeomorphism, and $\omega \in \mathcal{A}_P$,*

$$S_{X'}(F^*\omega) = S_X(\omega).$$

[3] See section 1 of Appendix A for a formal definition of cohomology classes.

(ii) (Orientation) *Let $-X$ represent the same manifold as X but with opposite orientation. Then, for any G-connection ω,*

$$S_{-X}(\omega) = -S_X(\omega).$$

(iii) (Additivity) *If $X = X_1 \sqcup X_2$ is a disjoint union, with respective connections $\omega_i, i = 1, 2$, over X_i,*

$$S_{X_1 \sqcup X_2}(\omega_1 \sqcup \omega_2) = S_{X_1}(\omega_1) + S_{X_2}(\omega_2).$$

Proof. The smoothness of S_X follows from the smoothness of the Chern-Simons 3-form $\alpha(\omega)$ (Definition 3.12) with respect to the G-connection ω. Regarding functoriality, fix $s' : X' \to P'$ a global section for $\pi' : P' \to X'$. Then,

$$s := F \circ s' \circ f^{-1} : X \to P$$

is a global section for π. In that case,

$$\begin{aligned} S_{X'}(F^*\omega) &= \int_{X'} (s')^*(\alpha(F^*(\omega))) \\ &= \int_{X'} (F \circ s')^*(\alpha(\omega)) \\ &= \int_X (f^{-1})^*(F \circ s')^*(\alpha(\omega)) \qquad (3.9) \\ &= \int_X s^*(\alpha(\omega)) \\ &= S_X(\omega). \end{aligned}$$

Where in Equation (3.9) we have used that f is an orientation-preserving diffeomorphism. On the other hand, properties (ii) and (iii) are direct consequences of the standard properties of integrals of forms over manifolds. □

Remark 3.25. We can deduce from the functoriality property above that two equivalent connections $\omega, \omega' \in \mathrm{Ob}(\mathbf{Conn}_X^G)$, in the sense of Definition 3.10, yield the same value for the Chern-Simons action:

$$S_X(\omega) = S_X(\omega').$$

As a consequence, we may regard[4] S_X as defined on *fields* (the connections) modulo *symmetries* (the principal G-bundle morphisms over X),

$$S_X : \overline{\mathbf{Conn}_X^G} \longrightarrow \mathbb{R}/\mathbb{Z}.$$

3.3.3 Classical Solutions

In common physics parlance, the term ***classical solutions*** refers to the critical points of the action of the theory under consideration. In this subsection, we aim to find these classical solutions for the Chern-Simons action. For the sake of simplicity, we will focus our attention on a particular principal G-bundle $\pi : P \to X$. Therefore, our objective is to study the critical points of S_X^P.

[4]Notice that we are using notation somewhat loosely, indistinctly writing S_X for the action both when it is defined over $\mathrm{Ob}(\mathbf{Conn}_X^G)$ or $\overline{\mathbf{Conn}_X^G}$. It should be clear by the context to which one we are referring to.

Definition 3.26. *Let $\pi : P \to X$ be a principal G-bundle. Consider an interval $I \subset \mathbb{R}$ along with a family of connections $\{\omega(t)\}_{t\in I} \subset \mathcal{A}_P$. We say that $\omega(t)$* varies smoothly with *t if, after fixing a basis B_i for $\mathfrak{g}$, the component functions $\omega^i(t), t \in I$,*

$$\omega(t) = \omega^i(t)\ B_i,$$

vary smoothly with t in the sense of Subsection 2.1.3.

Lemma 3.27. *Set $I = [0, 1]$ and suppose $\{\omega(t)\}_{t\in I} \subset \mathcal{A}_P$ is a smoothly varying family of connections for the principal G-bundle $\pi : P \to X$. Then, $\omega(t)$ pastes to a unique connection on the principal G-bundle*

$$1_I \times \pi : I \times P \to I \times X, \quad (t, p) \longmapsto (t, \pi(p)). \tag{3.10}$$

Proof. Let $f : I \times X \to X$ be the projection into the second factor. We consider the *pullback* principal G-bundle f^*P (cf. Subsection 2.2.3),

$$\begin{array}{ccc} f^*P & \xrightarrow{F} & P \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ I \times X & \xrightarrow{f} & X \end{array}.$$

Observe that since

$$f^*P := \{((t, x), p) \in (I \times X) \times P : x = \pi(p)\} = \{((t, \pi(p)), p) : t \in I, p \in P\},$$

q is evidently isomorphic to the product bundle $1_I \times \pi : I \times P \to I \times X$, defined in Equation (3.10). Now, by Remark 2.39, we can specify a connection on $Q := I \times P$, by giving a horizontal distribution on TQ. This is straightforward to do by the definition of the principal bundle $1_I \times \pi$ and the fact that, by Theorem 2.37, $\omega(t)$ itself defines a horizontal distribution $\mathcal{H}^t$ in TP, for all $t \in I$.

To see why this is so, let us study what do vertical vectors look like in $1_I \times \pi$. Consider $(t, p) \in Q$ and $(X_t, X_p) \in T_{(t,p)}Q$. Then,

$$(1_I \times \pi)_{*,(t,p)}(X_t, X_p) = (1_I \times \pi)_{*,(t,p)}(X_t, 0) + (1_I \times \pi)_{*,(t,p)}(0, X_p) = X_t + \pi_{*,p}(X_p),$$

Therefore, (X_t, X_p) is vertical if, and only if, $X_t = 0$ and $X_p \in \mathcal{V}_p$, where $\mathcal{V}_p \subset T_pP$ is the space of vertical vectors defined by $\pi_{*,p}$. In other words, $\mathcal{V}_{(t,p)} = \mathcal{V}_p$. Consequently,

$$T_{(t,p)}Q = T_tI \oplus T_pP = T_tI \oplus (\mathcal{H}^t_p \oplus \mathcal{V}_p) \cong (T_tI \oplus \mathcal{H}^t_p) \oplus \mathcal{V}_p = (T_tI \oplus \mathcal{H}^t_p) \oplus \mathcal{V}_{(t,p)}.$$

We can thus define the horizontal distribution on Q

$$\mathcal{H}_{(t,p)} := T_tI \oplus \mathcal{H}^t_p.$$

In terms of a $\mathfrak{g}$-valued form $\nu \in \mathcal{A}_Q$, we can write this assignation as

$$\nu_{(t,p)}(X_t, X_p) := \omega(t)(X_p).$$

$\square$

Proposition 3.28. *Let $\pi : P \to X$ be a principal G-bundle and $\{\omega(t)\}_{t\in I} \subset \mathcal{A}_P$, $I := [0,1]$, a smoothly varying family of G-connections. Denoting $\omega := \omega(0)$, $\dot{\omega} := \left.\frac{d}{dt}\right|_{t=0} \omega(t)$ and $\Omega := \Omega^\omega$,*

$$\left.\frac{d}{dt}\right|_{t=0} S_X^P(\omega_t) = 2\int_X \sigma^*\langle \Omega \wedge \dot{\omega}\rangle,$$

where $\sigma \in \Gamma(P)$ is an arbitrary global section of π.

Proof. Using Lemma 3.27, we paste $\omega(t)$ into a single connection ν on the pullback principal bundle $1_{[0,t]} \times \pi : [0,t] \times P \to [0,t] \times X$. If $s \in [0,t]$ and $p \in P$, the curvature form of ν equals

$$\begin{aligned}\Omega^\nu_{(s,p)} &= (d\nu)_{(s,p)} + \frac{1}{2}[\nu_{(s,p)}, \nu_{(s,p)}] \\ &= (d\omega(s))_p + (ds \wedge \frac{d}{ds}\omega(s))_{(s,p)} + \frac{1}{2}[\omega(s)_p, \omega(s)_p] \\ &= \Omega_p^{\omega(s)} + (ds \wedge \frac{d}{ds}\omega(s))_{(s,p)}.\end{aligned}$$

We can write this equality more succinctly as

$$\Omega^\nu_s = \Omega^{\omega(s)} + ds \wedge \frac{d}{ds}\omega(s). \tag{3.11}$$

On the other hand, if ∂ represents the boundary operator, $\partial X = \emptyset$ implies that

$$\partial([0,t] \times X) = \partial([0,t]) \times X = \{0,t\} \times X. \tag{3.12}$$

Setting $\widetilde{\sigma} := 1_{[0,1]} \times \sigma : [0,1] \times X \to Q$, this characterization of $\partial([0,t] \times X)$ allows us to write

$$\begin{aligned} S_X^P(\omega(t)) - S_X^P(\omega(0)) &= \int_X \sigma^*(\alpha(\omega(t)) - \alpha(\omega(0))) \\ &= \int_{\partial([0,t]\times X)} (\widetilde{\sigma})^*(\alpha(\nu)) && \text{(Eq. (3.12))} \\ &= \int_{[0,t]\times X} d\left((\widetilde{\sigma})^*(\alpha(\nu))\right) && \text{(Stokes' Thm.)} \\ &= \int_{[0,t]\times X} (\widetilde{\sigma})^*\langle \Omega^\nu \wedge \Omega^\nu\rangle && \text{(Prop. 3.14)} \\ &= 2\int_{[0,t]\times X} \sigma^*\langle \Omega^{\omega(s)} \wedge (ds \wedge \frac{d}{ds}\omega(s))\rangle && \text{(Eq. (3.11))} \\ &= 2\int_0^t \int_X \sigma^*\langle \Omega^{\omega(s)} \wedge \frac{d}{ds}\omega(s)\rangle. \end{aligned}$$

The result now follows by differentiating with respect to t and evaluating at $t = 0$. □

Remark 3.29. What this proposition is telling us, most interestingly, is that the critical points of the action are ***flat connections***, i.e. connections $\omega \in \mathcal{A}_P$ with vanishing curvature,

$$\Omega^\omega \equiv 0.$$

Conclusions

In this thesis we have aimed to introduce the most fundamental notions of Chern-Simons theory, focusing on the specific case of closed three-manifolds and compact, simply connected Lie groups. To conclude our work, I would like to highlight different topics that seem like a natural continuation of the work presented here.

For starters, it would be an understatement to say that we have barely scratched the surface of a topic as rich as Lie groups and Lie algebras, developing the subject only to the extent that we needed it for in subsequent chapters. Thus, it would be most interesting to expand on subtopics like Lie subgroups or the *closed-subgroup theorem* on the one hand or the structure theory and classification of Lie algebras on the other.

While in the second chapter we have given an arguably robust introduction to principal bundles, we have not had the opportunity to discuss important concepts such as the frame bundle or the generalisation of the covariant derivative to this case. Intimately related to this last topic is the idea of the *associated* (and in particular, adjoint) bundle for a given smooth representation $\rho : G \to \mathrm{GL}(V)$ of the Lie group G on the vector space V, or the vector space of *tensorial forms of type* ρ, essential in the study of principal bundles.

Lastly, there are a myriad of ways we could have further developed Chern-Simons theory or explored adjacent topics. Firstly, the jump in complexity when considering manifolds with boundary is noteworthy, as it requires the introduction of the *Chern-Simons line bundle*. Another possible generalisation could be to consider non-trivialisable principal bundles, as seen in [8].

We could have even considered other gauge theories altogether. A notable example could be *Yang-Mills theory*, a gauge theory based on a special unitary group $\mathrm{SU}(n)$.

Appendix A

Characteristic Classes

While not essential for the development of Chern-Simons theory, characteristic classes provide a key source of motivation. In particular, Chern-Simons theory introduces a secondary class of characteristic invariants that remain nontrivial for ***flat bundles*** – principal G-bundles equipped with a flat connection. This appendix offers a brief overview of the most important aspects of characteristic classes, with an emphasis on the Chern-Weil Theorem.

Throughout this appendix, let M be a smooth manifold. We begin by recalling the basics of *de Rahm Cohomology*:

A.1 De Rahm Cohomology

Definition A.1. *Suppose $k \in \mathbb{N}$. We define the vector spaces of closed and exact k-forms, respectively, as*

$$\begin{aligned} Z^k(M) &:= \{\alpha \in \Omega^k(M) : d\alpha = 0\}, \\ B^k(M) &:= \{\alpha \in \Omega^k(M) : \alpha = d\beta, \beta \in \Omega^{k-1}(M)\}. \end{aligned}$$

By definition, $B^0(M) := \{0\}$.

The main objective of de Rham cohomology is precisely to quantify in what measure closed forms fail to be exact. To do so, one defines the quotient vector space,

$$H^k(M) := {Z^k(M)}\big/{B^k(M)}, \quad \text{for all } k \in \mathbb{N},$$

called ***de Rahm cohomology class in degree*** k. The equivalence class of $\alpha \in Z^k(M)$, that we will denote as $[\alpha]$, is called its ***cohomology class***. Two forms with the same cohomology class are said to be ***cohomologous***. Crucially, de Rahm cohomology is *invariant under diffeomorphism* (see, for example, [1], p. 278, Remark 24.7), meaning that if M and N are diffeomorphic, $H^k(M)$ and $H^k(N)$ are isomorphic as vector spaces for all natural k.

Set

$$H^*(M) := \bigoplus_{k\in\mathbb{N}} H^k(M).$$

Using the next proposition, we can then define a product in $H^*(M)$ that gives the vector space the structure of an *algebra*,

Proposition A.2. *Fix $k, l \in \mathbb{N}$ and a manifold M. Take $\alpha \in Z^k(M)$ and $\beta \in Z^l(M)$. Then,*

(i) The wedge product is also closed, $\alpha \wedge \beta \in Z^{k+l}(M)$.

(ii) If $\alpha' \in Z^k(M)$ and $\beta' \in Z^l(M)$ are cohomologous with α and β, respectively,

$$[\alpha' \wedge \beta'] = [\alpha \wedge \beta].$$

For a proof of this result, we refer the interested reader to [1], p. 279. Hence, $H^*(M)$ becomes a graded algebra with the product

$$[\alpha] \wedge [\beta] := [\alpha \wedge \beta], \quad \text{for all } [\alpha] \in H^k(M), [\beta] \in H^l(M).$$

A.2 Chern-Weil Theorem

Let V be a finite-dimensional real vector space and fix k a positive integer. We say that a map

$$f : \bigotimes^k V \longrightarrow \mathbb{R}$$

is ***symmetric*** if

$$f(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}) = f(v_1 \otimes \cdots \otimes v_k)$$

for any $v_1, \ldots, v_k \in V$ and any $\sigma \in S_k$.

Definition A.3. *Let V be a finite-dimensional real vector space and k a positive integer. We define $S^k(V^*)$ to be the set of linear, symmetric maps*

$$f : \bigotimes^k V \longrightarrow \mathbb{R}.$$

We are interested in endowing $S^k(V^*)$ with a product: for any given k, l positive integers, we define

$$\Box : S^k(V^*) \times S^l(V^*) \longrightarrow S^{k+l}(V^*)$$

by

$$(P \Box Q)(v_1, \ldots, v_{k+l}) := \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} P(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) Q(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}),$$

for all $P \in S^k(V^*)$, $Q \in S^l(V^*)$ and $v_1, \ldots, v_{k+l} \in V$. Defining $S^0(V^*) := \mathbb{R}$, we set

$$S^*(V^*) := \bigoplus_{k \in \mathbb{N}} S^k(V^*).$$

Suppose that $V = \mathfrak{g}$, the Lie algebra of a Lie group G. In this case, we are interested in a particular subalgebra of $S^*(V^*)$:

Definition A.4. *Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. We define $I^k(G) \subset S^k(\mathfrak{g}^*)$, $k \in \mathbb{N}$, to be subset of* $\mathrm{Ad}(G)$-invariant *maps,*

$$f\left(\mathrm{Ad}(g)(B_1), \ldots, \mathrm{Ad}(g)(B_k)\right) = f(B_1, \ldots, B_k), \quad \forall g \in \ \forall B_1, \ldots, B_k \in \mathfrak{g}.$$

It is straightforward to verify that $I^*(\mathfrak{g}) := \bigoplus_{k\in\mathbb{N}} I^k(\mathfrak{g})$ is a subalgebra of $S^*(\mathfrak{g}*)$. We now have all the ingredients to state the main result of this appendix,
Notation: Let V be a vector space, M a manifold and $k \in \mathbb{N}$; consider $\alpha \in \Omega^k(M, V)$. For n a nonnegative integer, we denote

$$\alpha^n := \alpha \wedge \overset{(n)}{\cdots} \wedge \alpha \in \Omega^{kn}(M, \otimes^n V).$$

Theorem A.5 (Chern-Weil Homomorphism)**.** *Suppose $\pi : P \to M$ is a principal G-bundle. For a positive integer k, choose $f \in I^k(G)$ and consider the form $f(\Omega^k) \in \Omega^{2k}(P)$. Then,*

(i) $f(\Omega^k)$ is basic: there exists a unique $\Lambda \in \Omega^{2k}(M)$ such that

$$f(\Omega^k) = \pi^*\Lambda.$$

(ii) Λ is closed

(iii) The cohomology class $[\Lambda] \in H^{2k}(M)$ is independent of the connection ω.

Furthermore, the map

$$w : I^*(G) \longrightarrow H^*(M), \quad f \longmapsto [\Lambda],\ (\pi^*(\Lambda) = f(\Omega^k)).$$

is an $\mathbb{R}$-algebra homomorphism, called the Chern-Weil homomorphism.

Proof. A detailed proof is given in [5], p. 294, Theorem 1.1. □

Finally, some definitions:

Definition A.6. *A* Chern-Weil *form is a differential form that can be expressed as $f(\Omega)$ for $f \in I^*(G)$. Its cohomology class $[\Lambda] = w(f)$ is called the* characteristic class of f.

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