The Dirac Equation in the Schwarzschild Metric

Author: Sami Calvo Muñoz, scalvomu7@alumnes.ub.edu
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Cristiano Germani, germani@ub.edu

Abstract: This thesis studies the possibility of a primordial black hole (PBH) trapping an electron in a stable orbit, potentially allowing detection through the emitted electromagnetic radiation from the accelerated charge. Firstly, the Dirac equation is analysed in the Minkowski metric, generalising afterwards the formulation from flat to curved backgrounds. Using the separation of variables method, it is demonstrated that no non-trivial stationary solutions exist outside the horizon, in accordance with the no-hair theorem. Meanwhile, non-stationary solutions do exist and are unique for smooth initial conditions. Lastly, the spinor norm's time evolution is analysed to identify bound states, revealing two quantised circular orbits dependent on the fermion's mass and angular momentum. The expression for the quantised radius, velocity, and energy has also been derived. With this information, one can calculate the energy spectrum for a trapped electron around a PBH, giving a way of experimentally proving the existence of primordial black holes.

Keywords: Dirac equation, Schwarzschild metric, existence, bound orbits, PBH detection

SDGs: Quality education (number 4)

I. INTRODUCTION

I-1. Relativistic equations and the Dirac equation

After the appearance and success of the Schrödinger equation in 1926, physicists quickly sought a relativistic version of it. The first successful one was the Klein-Gordon equation (KG equation, from now on):

$$(\hbar^2 \partial_a \partial^a + m^2 c^2)\phi = 0, \tag{1}$$

which is manifestly covariant. Nevertheless, this equation has two major problems: It yields negative energy solutions and negative probability densities, as is now shown.

Plane-wave solutions are a solution to the KG equation:

$$\phi = N e^{\frac{-ip_a x^a}{\hbar}} \quad ; \quad p^a = (E/c, \vec{p}) \to p_a x^a = \frac{E}{c}t - \vec{p}\cdot\vec{x}.$$

Plugging this into (1):

$$0 = ((-ip^a)(-ip_a) + m^2c^2)\psi \to E = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}.$$

So a negative-energy solution appears. Defining the conserved charged current as:

$$j^a = i\hbar(\phi^* \partial^a \phi - (\partial^a \phi^*)\phi),$$

and applying (1) to it, one can see that $\partial_a j^a = 0$, the conservation law associated to the current. Then, for the plane-wave solutions, one gets $j^0 = \pm 2N^2|E|/c$, so the probability current can indeed be negative [1].

To solve this, Dirac attempted to find an alternative relativistic equation, one with first-order derivatives instead of second-order, which, Dirac argued, would eliminate the negative probability. Therefore, he made the following *Ansatz*:

$$(i\hbar\gamma^a \partial_a - mc)\psi = 0, \tag{2}$$

where the γ^a are yet to be determined.

First, equivalence to the KG equation is required by applying $(-i\hbar\gamma^a\partial_a - mc)$ to (2) and comparing with the KG equation:

$$(-i\hbar\gamma^a\partial_a - mc)(i\hbar\gamma^b\partial_b - mc) =$$
$$\hbar^2\gamma^a\gamma^b\partial_a\partial_b + i\hbar mc(\gamma^a\partial_a - \gamma^b\partial_b) + m^2c^2.$$

The second term immediately vanishes (as seen by relabelling the dummy indices $a \leftrightarrow b$), and the first term, due to the interchangeability of partial derivatives, can be rewritten as $\frac{1}{2}(\gamma^a\gamma^b + \gamma^b\gamma^a)$. Therefore, comparing with (1), the following constraint is found:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \tag{3}$$

where the curly brackets indicate anti-commutation. This and some other properties force the γ's to be matrices with an even number of dimensions greater than two. One could choose any specific representation, but the easiest one is the 4x4 matrices. The specific form of the matrices is arbitrary, as long as they fulfil (3), but the one used in this work will be:

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix}, \tag{4}$$

where the σ^j are the three Pauli matrices and each entry is a diagonal 2x2 block [1][2]. Note that the signature $(-+++)$ is used.

I-2. General Covariance

Now, the problem of a formulation of the Dirac equation in a general coordinate system in a covariant manner is addressed. The easiest way to do it is to use the so-called Vierbein formalism in order to define a covariant derivative.

In this formalism, instead of working directly with a general system of coordinates, one considers a normal coordinate system at each space-time point. Then, the metric expressed in those coordinates will be simply the Minkowski metric η_{ab}. In terms of a general coordinate system, the metric tensor is related to η_{ab} as:

$$g_{\mu\nu}(x) = V_\mu^a(x) V_\nu^b(x) \eta_{ab}, \tag{5}$$

where the dependence on x is explicitly shown to recall that the Vierbein V_μ^a depends on the spacetime point, as does the general metric.

Henceforth, the convention is used that Latin alphabet letters denote local coordinates, whereas Greek letters refer to the general coordinate system. The Vierbein transforms as a contravariant vector, and any tensor can be expressed in either of the two coordinate systems as follows:

$$A_a = V_a^\mu A_\mu.$$

Now, the following definition for the spinor covariant derivative is made [3][4]:

$$\nabla_a = V_a^\mu(\partial_\mu + \Gamma_\mu) \quad ; \quad \Gamma_\mu = \frac{1}{2}[\gamma^a, \gamma^b] V_a^\nu \nabla_\mu V_{b\nu}.$$

With this prescription, the Dirac equation in a general metric may be obtained by replacing all ∂_a with ∇_a and contracting all Latin indices with the Vierbein V_μ^a:

$$(iV_a^\mu \gamma^a V_\mu^b \nabla_b - m)\psi \coloneqq (i\gamma^\mu \nabla_\mu - m)\psi = 0,$$

where $\hbar$ and c have been set to 1. The γ^a fulfil the same anticommutation relations (3) as before, whereas the γ^μ satisfy (3) but replacing η^{ab} by $g^{\mu\nu}$.

II. THEORETICAL STUDY AND RESULTS

II-1. Methodology and objectives

In this section, the existence and non-existence of solutions are studied, followed by an analysis of the norm outside the horizon.

This is done by first formulating the Dirac equation in the Schwarzschild metric. The resulting equation is then reduced to a system of coupled PDEs using separation of angular variables. Existence and uniqueness results are obtained via analytical theorems from PDE theory. Subsequently, the system is analysed to identify bound orbits, comparing the results to those of GR and applying them to the interesting case of primordial blackholes.

II-2. Existence and non-existence theorems

The Schwarzschild metric describes a spherically symmetric space-time around a non-charged and non-spinning point mass. The expression of the metric in spherical coordinates (for an asymptotic observer) is:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where the event horizon is at $r = 2GM$. The Dirac equation in this metric is $i\partial_t \bar{\psi} = \mathcal{H}\bar{\psi}$ with:

$$\mathcal{H} = i\gamma^0\gamma^1 f(r)^2 \hat{\mathcal{L}} + f(r)\left(\frac{\gamma^1}{r}\hat{K} - \gamma^0 m\right), \tag{6}$$

where $f(r) = \sqrt{1 - \frac{2GM}{r}}$, $\hat{\mathcal{L}} = \partial_r + \frac{1}{r}$, $\hat{K}$ is an angular operator and $\bar{\psi} = \frac{1}{f(r)^{1/2}}\psi$ [5] .

Then, $\hat{K}$ commutes with the Hamiltonian $\mathcal{H}$, so $\bar{\psi}$ can simultaneously be an eigenvalue of both $\hat{K}$ and $\mathcal{H}$. Nevertheless, it will firstly be assumed that $\bar{\psi}$ is only an eigenfunction of $\hat{K}$ with eigenvalue k.

Because the only gamma matrices that appear in (6) are antidiagonal (and their product is diagonal) in representation (4), the spinor $\bar{\psi}$ can be decomposed into two components, and one gets the following coupled PDE system:

$$\begin{cases} \left[-\partial_t + f(r)^2\hat{\mathcal{L}}\right] F + f(r)\left(\frac{k}{r} - m\right) G = 0 \\ \left[\partial_t + f(r)^2\hat{\mathcal{L}}\right] G + f(r)\left(\frac{k}{r} + m\right) F = 0 \end{cases}, \tag{7}$$

where $\bar{\psi} = \begin{pmatrix} F(r)\phi^{\pm}_{j\tilde{m}} \\ G(r)\phi^{\mp}_{j\tilde{m}} \end{pmatrix}$ and the functions $\phi^{\pm}_{j\tilde{m}}$ are the eigenfunctions of $\hat{K}$, the angular spinors [11].

Let j be the *total* angular momentum. Then, if $j = l + \frac{1}{2}$ (parallel coupling), $k = -\left(j + \frac{1}{2}\right) = -(l+1)$ and if $j = l - \frac{1}{2}$ (antiparallel), $k = j + \frac{1}{2} = l$, where the coupling of spin and angular momentum is for the *upper* component of $\bar{\psi}$, and opposite for the lower one. Therefore, since $l \in \mathbb{N}$, $k \in \mathbb{Z}$ can only take non-zero *discrete* values.

From now on, the theorems in Appendix A (which are cited from the literature) are used, and their header starts with an "A".

By doing the change of variable $x = \sqrt{1-\frac{1}{r}}$ [12] in (7) one gets:

$$\begin{cases} \left(\hat{\mathcal{L}} - \partial_t\right) F = x(m-(1-x^2)k)G \\ \left(\hat{\mathcal{L}} + \partial_t\right) G = -x(m+(1-x^2)k)F \end{cases}, \tag{8}$$

$$\text{where} \quad \hat{\mathcal{L}} = (1-x^2)\left(\frac{1}{2}x(1-x^2)\partial_x + x^2\right).$$

Now, there is no longer a singularity at $r = 1$ (corresponding to $x = 0$) for the coefficients, which happened for (7) because of $f(r)$. With this, one gets the following theorem:

Theorem 1. *There is no non-trivial bounded solution to (8) in the stationary case.*

Proof. The stationary version of (8) is achieved by substituting ∂_t by $-i\varepsilon$, being ε the energy (and the spinor now has a global phase of $e^{-i\varepsilon t}$). With this change, equations (8) still have analytic coefficients, so Theorem A.1, asserting the existence and uniqueness of solutions for an ODE system with analytic coefficients such as the stationary version of (8), may be applied. Finally, for the solution to remain bounded everywhere, F and G must vanish on the horizon (otherwise, the term $f^{-\frac{1}{2}}$ would make the solution diverge there). Since the ODE system and the boundary conditions are homogeneous, by uniqueness, $\bar{\psi} = 0$ is the only solution. □

This result is one of the consequences of the No-hair Theorem, which asserts that a black hole is fully characterised by its mass, spin, and charge [7].

Now, the attention is turned to the non-stationary case. Since the equations are now *partial* differential equations, there are no results as nice as Theorem A.1:

Theorem 2. *A unique C^∞ solution to (8) exists for every C^∞ initial condition. The solution is local outside the horizon, but may be extended arbitrarily close to the horizon and to infinity. The solution is also normalisable outside the horizon.*

Proof. If the domain is restricted to a region arbitrarily close to $x = 0$ and $x = 1$, A (the matrix in Theorem A.2) has *distinct* eigenvalues in the whole domain. Consequently, Theorem A.2, which guarantees the existence and uniqueness of a PDE system with C^∞ initial conditions such as (8), may be applied. For the normalisation condition, note that the change of unknown $\tilde{\psi} = r\bar{\psi}$ can be made and the PDE coefficients are still C^∞. Since the solution is C^∞, its spatial integral in a bounded set (in this case, $x \in (\delta_0, \delta_1)$) is finite. Therefore:

$$\int_{0+\delta_0}^{1-\delta_1} \tilde{\psi}^2 dx = \int_{r_m}^{r_M} \frac{\tilde{\psi}^2}{r^2\sqrt{1-\frac{1}{r}}} dr = \int_{r_m}^{r_M} \frac{\bar{\psi}^2}{\sqrt{1-\frac{1}{r}}} dr < \infty,$$

where r_M and r_m are the maximum and minimum r, respectively, the domain reaches, as discussed above. □

Theorem A.2 also yields existence, but not uniqueness, in the case of a C^m initial condition, for $m \in \mathbb{N}^+$.

II-3. Circular orbits

To proceed with the analysis of bound orbits, it is necessary to uncouple equations (8). This can be achieved by solving for one component, such as G, and substituting it into the equation for F, leading to a second-order *ordinary* differential equation. After some lengthy calculations, one gets:

$$\partial_t F = A(x)\partial_x^2 F + B(x)\partial_x F + C(x)F + D(x)\partial_t^2 F, \tag{9}$$

where the exact form of the coefficients is shown in Appendix B. For a Dirac particle, the norm (without the angular dependence) is given by:

$$P(t) := \left\|\bar{\psi}(t)\right\|^2 = \int \frac{r^2|\bar{\psi}(t)|^2}{f(r)} dr = \int \frac{|\bar{\psi}(t)|^2}{x(1-x^2)^4} dx,$$

since the radial volume element for the Schwarzschild metric is $dV_r = \frac{r^2 dr}{f(r)}$. For simplicity, F will also be assumed to be real (which it is if and only if the initial condition is real, since the coefficients of its PDE are real). Using the PDE for F (and recalling that $\bar{\psi} = f(r)^{-1/2}F$):

$$\frac{dP}{dt} = \langle \partial_t \tilde{F} | \tilde{F} \rangle + \langle \tilde{F} | \partial_t \tilde{F} \rangle = \langle \tilde{F} | (L + L^\dagger) \tilde{F} \rangle,$$

where $\tilde{F} = \frac{F}{x(1-x^2)^4}$ and, since $F \in \mathbb{R}$, $L = L^\dagger$. Therefore $L + L^\dagger = 2A(x)\partial_x^2 + B(x)\partial_x + 2C(x) + 2D(x)\partial_t^2$ and thus:

$$\frac{dP}{dt} = 2\int \left[\tilde{C}F^2 + F\left(\tilde{A}\partial_x^2 + \tilde{B}\partial_x + \tilde{D}\partial_t^2\right)F\right] dx,$$

where the tilded coefficients are the untilded ones divided by $x(1-x^2)^4$. It will also be assumed that the second derivative (in time or space) of F is on the order of or, at most, a few orders of magnitude larger than F (which will be the case for wavefunctions that are smooth enough). Considering this and that $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$ are many orders of magnitude less than $2\tilde{C}$ [13], except around $x = 0$, it can be approximated that

$$\frac{dP}{dt} \approx \int 2\tilde{C}(x)F^2 dx. \tag{10}$$

While this simplification may lose accuracy in scenarios where the wavefunction is sharply peaked or rapidly oscillating, it suffices for capturing the general behaviour of bound states, as the existence of bound orbits.

Since this integral in (10) may be computed for any interval in x (between 0 and 1), regions where $2\tilde{C}$ is positive correspond to regions where the wavefunction increases with time. In particular, if there is a maximum or a divergence, the wavefunction will clump there, corresponding to a *circular orbit*.

Firstly, the behaviour of the norm outside the horizon is analysed for the two cases $|k| > m$ and $|k| < m$.

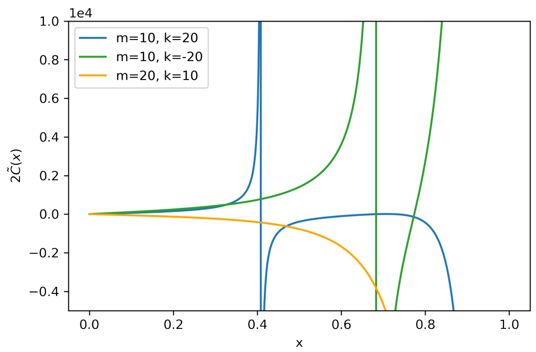

FIG. 1: Plot of the coefficient $2\tilde{C}(x)$ in equation (10) for three representative sets of parameters m and k.

- $\mathbf{|k| > m}$
  For this case, $2\tilde{C}$ has a divergence at $x = \sqrt{\frac{1}{3}(1-\frac{m}{k})}$, which corresponds to $r_1 = \frac{3r_s}{2+\frac{m}{k}}$. This divergence is positive, so it corresponds to a bound orbit, as discussed above. Also note that for a massless particle, this orbit is at $r = \frac{3}{2}r_s$, which is exactly the result predicted by general relativity alone.

  In addition to this, for $k > 0$, $2\tilde{C}$ has a maximum, and therefore an orbit, at $x = \sqrt{1-\frac{m}{k}}$, corresponding to $r_2 = \frac{k}{m}r_s > r_1$.

- $\mathbf{m \leq |k|}$
  In this case, there is no divergence and therefore no bound orbit.

In conclusion, for $|k| > m$ there two orbits, an inner one at r_1 and an outer one at r_2, where the second only appears for positive k. On the other hand, for $|k| < m$, there are no orbits. This contrasts with the result of GR, where the last stable orbit is at $r = 3r_s$ for massive particles, regardless of their mass and angular momentum.

In both cases, when one includes $\tilde{D}$ into (10), there is a positive divergence at $x = 0$ (the horizon), so once the function falls into the black hole, it stays at the horizon forever. This can be interpreted considering that the time coordinate used corresponds to that of an asymptotic observer. For this observer, time stops at the horizon so that anything that falls into the black hole seems to become frozen there.

All previous analysis has been done for the first component of the spinor, F. For the second component, G, the results are very similar, but the behaviour of the wavefunction swaps with respect to the value of k: the orbits for F for $k > 0$ are the ones for G for $k < 0$, and vice versa for negative k. This swapping is absolutely expected due to F and G having opposite couplings.

In the semiclassical limit, where the fermion is highly energetic and its Compton wavelength is small, it is reasonable to interpret the eigenvalue k as corresponding to the classical angular momentum, since the fermion may be treated as a point particle. For a circular orbit in the Schwarzschild metric, $L = mvr\sqrt{\frac{1+\frac{r_s}{2r-3r_s}}{1-\frac{r_s}{r}}}$ [8]. For $k > 0$, $k = l$ and for $k < 0$, $k = -(l+1)$. With this and (after recovering $\hbar$, $2GM$ and c, $m \to r_s mc/\hbar$) $r_1 = \frac{3r_s}{2+\frac{r_s mc}{k\hbar}}$ one can find the corresponding quantised velocity after substituting the expression for r_1 into L:

$$v_1 = c\left(2+\frac{r_s mc}{k\hbar}\right)\frac{(|k|-k_0)\hbar}{r_s mc}\sqrt{\frac{1-\frac{2+\frac{r_s mc}{k\hbar}}{3}}{1+\frac{1}{\frac{6}{2+\frac{r_s mc}{k\hbar}}-3}}},$$

where $k_0 = 0$ for $k > 0$ and $k_0 = 1$ for $k < 0$ (that comes from the fact that $|k| = l$ for the first case but $|k| = l+1$ for the second). For $k > 0$, v_1 turns out to be negative for every value of k. That is because, in GR, circular orbits are always above $\frac{3}{2}r_s$, whereas for $k > 0$ the opposite is the case, and $L \notin \mathbb{R}$ for orbits lower than that.

Doing a similar computation for $r_2 = \frac{k\hbar}{mc}$ $(k > 0)$ yields:

$$v_2 = c\sqrt{\frac{1-\frac{r_s mc}{k\hbar}}{1+\frac{1}{\frac{2k\hbar}{r_s mc}-3}}}.$$

Now, one can calculate the quantised energy. In the Schwarzschild metric, the energy *per unit mass* for an orbit is $E^2 = \left(1+\frac{L^2}{m^2r^2c^2}\right)\left(1-\frac{r_s}{r}\right)c^4$ [8]. Using the expression for the radius r_1:

$$\frac{E_1^2}{c^4} = \left[1+\frac{(|k|-k_0)^2\hbar^2}{9m^2c^2r_s^2}\left(2+\frac{r_s mc}{k\hbar}\right)^2\right]\left[1-\frac{1}{3}\left(2+\frac{r_s mc}{k\hbar}\right)\right].$$

E_1 is a function that essentially grows linearly with $|k|$ for sufficiently large $|k|$. For the radius r_2:

$$E_2^2 = 2\left(1-\frac{mr_s c}{k\hbar}\right)c^4.$$

For all the expressions, there is also the condition $|k|\hbar > r_s mc$.

With this information, one could detect primordial black holes, if they exist, by their characteristic emission spectrum, given that they trap an electron in a bound orbit, similar to how it is done with atoms.

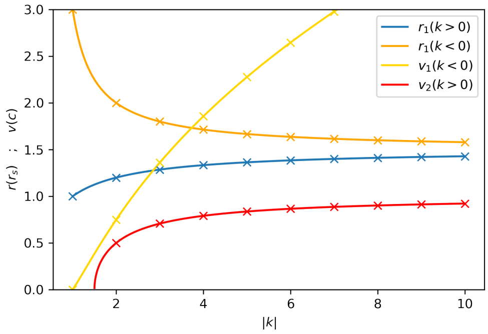

FIG. 2: Quantised radius and velocity in terms of the absolute value of the quantum number k, where it has been chosen that $r_s = c = \hbar = m = 1$ for simplicity. The crosses mark the integer values of $|k|$.

III. CONCLUSIONS AND FURTHER WORK

This study, under the assumption of separation of angular variables, has established the non-existence of stationary solutions, aligning with one consequence of the No-hair Theorem [7], and the existence of non-stationary solutions to the Dirac equation in Schwarzschild spacetime.

After that, the behaviour of the solution has been analysed, finding that, for $|k|\hbar > r_s mc$, bound circular orbits can exist at $r_1 = \frac{3r_s}{2+\frac{r_s mc}{k\hbar}}$ and $r_2 = \frac{k\hbar}{r_s mc} r_s > r_1$, whereas for negative k only the inner orbit appears. Moreover, since k is a natural number, the orbits are quantised.

For $|k|\hbar < r_s mc$, as expected [14], there is no orbit and the particle just falls into the BH. Moreover, the result that the infalling particles become frozen at the horizon, because of the slowing of coordinate time there, has also been found to be correct in this framework.

After that, the physical interpretation of identifying the angular momentum eigenvalue with the classical one, valid for small Compton wavelengths, has also been discussed. With this, the expressions for the quantized velocity and energy have been derived.

With this, the characteristic energy spectrum of a trapped electron orbiting around a PBH can be obtained, giving hope for spectroscopic detection of primordial black holes, especially in the early universe when energy and free electrons were abundant.

Finally, since separation in the angular variables has been done, only circular orbits have been considered, and therefore other, more general orbits can, in principle, exist for various radii. Lastly, since the Dirac equation implies the KG equation, all positive results discussed in this work also apply to bosons.

Regarding possible future work, no quantisation with creation/annihilation operators has been done. Therefore, it is expected that all the predictions of GR concerning the stability and possibility of circular orbits are recovered when doing the classical limit and introducing quantisation. In particular, it is expected that the orbits below $r = 3r_s$ become unstable, since that is the limit for stable circular orbits predicted by GR for massive particles. It would also be interesting to study why, physically speaking, the possible circular orbits are different for opposite spin-orbit coupling.

Acknowledgments

I want to thank my advisor, Cristiano Germani, and his PhD student, Laia Montellà, for their constant help and advice throughout the project.

Bibliography

[1] Ll. Garrido, *Física d'Altes Energies* (Univ. de Barcelona, Barcelona, 2015).
[2] E. Peskin and V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, Reading, MA, 1995).
[3] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, (Camb. Univ. Press, Cambridge, 1982).
[4] L. Shapiro. "Covariant derivative of fermions and all that". Universe **2022**, 8(11), 586 (2022).
[5] D. R. Brill and J. A. Wheeler. "Interaction of Neutrinos and Gravitational Fields". Rev. Mod. Phys. **29**, 465 (1957).
[6] R. Sini and V.C. Kuriakose. "Absorption cross-section and Emission Spectra of Schwarzschild Black Hole in Dirac Field". Mod. Phys. Letters **23**, 33 (2008).
[7] S. D. Mathur, *Lecture notes 4: The No-hair Theorem* (The Ohio State Univ., Ohio, 2023).
[8] L. Ryder, *Introduction to General Relativity* (Camb. Univ. Press, Cambridge, 2009)
[9] G. Teschl, *Ordinary Differential Equations and Dynamical Systems* (American Math. Society, Rhode Isl., 2012).
[10] P. D. Lax, *Hyperbolic Partial Differential Equations*, (American Math. Society, Rhode Isl., 2006).

Footnotes

[11] See [6] for the exact expression of the angular spinors
[12] Henceforth, units are chosen so that $2GM$=1
[13] For example, for $m = 10$ and $k = 20$, at x=0.25 $2\tilde{C} \sim 520$ while $2\tilde{A} \sim -0.4$, $2\tilde{B} \sim 0.45$ and $2\tilde{D} \sim 33$. The difference is even larger for more disparate values of m and k
[14] At least, what *is* expected is that for sufficiently large masses the gravitational pull is enough to fully absorb it

Appendix A

In this Appendix, all the theorems from the literature that have been used throughout the text are presented.

Theorem 4.7, section 4.3, of [9]:

Theorem A.1. *Let $A(z)$ be a matrix- and $b(z)$ a singlevariable-functions that are analytic in the simply connected domain $\Omega \subset \mathbb{C}$. Then, a unique solution $\boldsymbol{w}$ exists, in the whole of Ω, to the coupled ODE system*

$$\boldsymbol{w}' = A(z)\boldsymbol{w} + b(z)$$

with initial conditions $\boldsymbol{w}(z_0) = \boldsymbol{w}_0$.

Theorem 4.1, section 4.1, of [10]:

Theorem A.2. *Let A and B be C^∞ coefficient matrices in the open $\Omega \subset \mathbb{R}^2$. Let A have distinct real eigenvalues. Then the PDE system*

$$\partial_t \boldsymbol{u} + A\partial_x \boldsymbol{u} + B\boldsymbol{u} = 0$$

with initial condition

$$\boldsymbol{u}(x, t = 0) = \boldsymbol{u}_0(x)$$

has a unique C^∞ solution for every C^∞ u_0 in Ω. Moreover, if the initial condition is C^k, there exists a C^k solution.

Appendix B

The expressions for the coefficients appearing in (9) are:

$$A(x) = -\frac{1}{2}x^2(1-x^2)^2 g(x) \quad ; \quad D(x) = \frac{2g(x)}{(1-x^2)^2} \quad ; \quad B(x) = -x^3\frac{f(x)}{m + k(3x^2-1)} \quad ; \quad f(x) = k(-x^4 + 2x^2 - 1)$$
$$C(x) = -D(x)x^2\left\{(1-x^2)^2\left(\frac{1}{2}\frac{m(1-3x^2) + f(x)}{m - k(1-x^2)} + x^2\right) + m^2 - (1-x^2)^2k^2\right\} \quad ; \quad g(x) = \frac{m - k(1-x^2)}{m + k(3x^2-1)}$$

L'equació de Dirac en la mètrica de Schwarzschild

Author: Sami Calvo Muñoz, scalvomu7@alumnes.ub.edu
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Cristiano Germani, germani@ub.edu

Resum: Aquest TFG estudia la possibilitat que un forat negre primordial (PBH) pugui atrapar un electró en una òrbita estable, la qual cosa permetria la seva detecció mitjançant la radiació electromagnètica emesa per la càrrega accelerada. En primer lloc, s'analitza l'equació de Dirac en la mètrica de Minkowski i, posteriorment, es generalitza la formulació dels espai-temps plans als corbats. Mitjançant el mètode de separació de variables, es demostra que no existeixen solucions estacionàries no trivials fora de l'horitzó, en acord amb el teorema del *no-hair*. Tanmateix, sí que existeixen solucions no estacionàries, les quals són úniques per a condicions inicials suaus. Finalment, s'analitza l'evolució temporal de la norma de l'espinor per identificar estats lligats, i es demostra la existència de dues òrbites circulars quantitzades que depenen de la massa i del moment angular del fermió. També s'ha derivat l'expressió per al radi, la velocitat i l'energia quantitzats. Amb aquesta informació, es pot calcular l'espectre energètic d'un electró atrapat al voltant d'un PBH, cosa que oferiria una via experimental per demostrar l'existència dels forats negres primordials.

Paraules clau: Equació de Dirac, existència, òrbites lligades, detecció de PBHs

ODSs: Educació de qualitat: número 4