Effects of noise in opinion dynamics

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Abstract: This work analyzes the non-equilibrium properties of the Voter Model and its noisy extensions (Noisy Voter Model and Voter Model with Global Noise). For the Voter Model on regular lattices, the known conservation of average magnetization $\langle m \rangle$ is confirmed, and the interface density $\langle \rho \rangle$ decays toward consensus following a dimension-dependent scaling. In the noisy models on all-to-all networks, the magnetization distribution from simulations matches the stationary Fokker-Planck solution. The Noisy Voter Model exhibits a bias toward $m = 0.5$, causing a bimodal-to-unimodal transition as noise a increases. Global noise flattens the distribution, becoming uniform at $a = 1$. The average consensus time $\langle t \rangle$ peaks at the central value of initial magnetization. Increasing a delays consensus in the Noisy Voter Model, but accelerates it under global noise by disrupting metastable clusters. These results highlight the distinct roles of local and global noise in collective dynamics.
Keywords: Non-equilibrium physics, Statistical mechanics, Stochastic processes, Spin models, Collective phenomena, Monte Carlo simulations.
SDGs: 4. Quality education.

I. INTRODUCTION

One important field in physics is non-equilibrium physics, which studies how systems evolve over time outside thermal equilibrium. Interestingly, this has numerous applications ranging from opinions spreading in social groups [1], to the dynamics of ecosystems [2], and traffic flow [3]. A key approach to describing such systems is through the lens of stochastic processes.

Stochastic processes are collections of random variables that describe the temporal evolution of systems under randomness. In non-equilibrium systems, the latter is not just a perturbation but instead drives the dynamics. Monte Carlo simulations use random numbers to mimic this inherent stochasticity, generating sequences of events with prescribed probabilities and reproducing fluctuations and emergent behaviors typical of these systems.

An example of such a stochastic, out-of-equilibrium process is the Voter Model. It drives the evolution of a d-dimensional, hypercubic lattice of particles (agents) initially assigned a spin (opinion) $s = 0, 1$, for which periodic boundary conditions are assumed. At each time step, a randomly selected particle adopts the spin of a uniformly randomly chosen neighbor. The system eventually reaches consensus: an absorbing state with all spins aligned and from which the dynamics can no longer escape, breaking ergodicity. Despite its simplicity, the model captures essential non-equilibrium features such as lack of detailed balance, domain coarsening (growth and merge of aligned regions), and metastable states, as the system can spend long periods in the same configuration due to the clusters.

When simulating these models, unless otherwise stated, the initial state is assumed to approach $m = 0.5$ for large enough systems, since the probabilities of an agent holding the opinion 0 and 1 are equal. Thus, the system evolves from a disordered, symmetric state to an ordered state characterized by consensus, spontaneously breaking spin symmetry.

To account for the possibility of spontaneous change of opinion, two noisy variants are proposed. In the Noisy Voter Model, with probability a, a randomly chosen particle flips its spin, and with probability $1 - a$, copies a neighbor. In practice, this means that the particle is chosen first, followed by the type of update, and, in case of the noisy event, forced to change its spin. This biases the system toward magnetization $m = 0.5$: an imbalance in spin populations increases the likelihood of flipping to the minority spin in a noise event. Thus, consensus states are not absorbing, so the system becomes ergodic.

In the Voter Model with Global Noise [4], noise acts independently of the system's configuration. During a noisy update, selected with probability a, the direction of the spin flip, either $0 \to 1$ or $1 \to 0$ (both equally likely), is chosen at random before selecting the agent, removing the bias in the Noisy Voter Model.

Beyond regular lattices in which neighbors are just the adjacent sites, alternative interaction structures can provide valuable insights. In this context, the complete graph or all-to-all network is introduced. In it, every agent interacts with the others, becoming their neighbor.

The aim of this work is to simulate the Voter Model and their noisy modifications, analyzing their dynamic behavior and long-term properties, and contrasting the results with theoretical predictions. In particular, the study is organized around three main objectives:

1. To computationally prove magnetization conservation and show that the interface density evolves with dimension-dependent scaling laws in the Voter Model on regular, low-dimensioned lattices. Both variables are described in Methodology;

2. To analyze the distribution of magnetization values in both the Noisy Voter Model and the Voter Model with Global Noise defined on a complete graph;

3. To study the average time it takes to firstly achieve consensus for the three models in the Voter Model family, defined on a complete graph.

II. METHODOLOGY

A mathematical framework must first be defined to proceed with the study of the introduced models.

To determine the state of a system at a given time, two variables are considered: magnetization m and interface density (or fraction of active links) ρ. The first one indicates the global state of the system and its proximity to consensus ($m = 0, 1$), as it is expressed as it follows:

$$m = \frac{1}{N} \sum_{i}^{N} s_i \tag{1}$$

The interface density reflects the local state by counting links between adjacent, different spins, quantifying the amount of boundaries. It reads [5]:

$$\rho = \frac{1}{\sum_i^N k_i} \sum_{i}^{N} \sum_{j \in \eta(i)} (s_i - s_j)^2 \tag{2}$$

where k_i is the number of neighbors of node i, and η denotes its neighborhood.

It is convenient to introduce diffusion D and drift v. The first represents the randomness of individual decisions and the intrinsic noise of the system. For a complete graph, it is expressed as [6]:

$$D(m) = \frac{1}{2} \frac{\delta m^2}{\delta t} [\mathbf{R}(m) + \mathbf{L}(m)] \tag{3}$$

$\mathbf{R}$ and $\mathbf{L}$ are the raising and lowering operators, respectively. They denote the probabilities of an increase or decrease in m by one agent's change: $\mathbf{R}(m) \equiv \mathbf{P}(m \to m + \delta m)$, $\mathbf{L}(m) \equiv \mathbf{P}(m \to m - \delta m)$. They can be intuitively deduced for each model.

Because $m \in [0, 1]$, the smallest change from a single update is $\delta m = \frac{1}{N}$. Since each update changes the state of one agent, and a Monte Carlo step is defined as the time required for an average of one update per agent, the time increase due to an individual update is $\delta t = \frac{1}{N}$.

The drift represents a systematic bias or trend in the dynamics. When the model prefers one state over another, this introduces a directional change towards that state in the overall state distribution. However, drift can also occur without a preference, for example, as a restoring tendency towards a balanced state, as happens in the Noisy Voter Model. It is given by [6]:

$$v(m) = \frac{\delta m}{\delta t} [\mathbf{R}(m) - \mathbf{L}(m)] \tag{4}$$

	Voter Model	Noisy Voter Model	Voter Model with Global Noise
$\mathbf{R}(m)$	$(1-m)m$	$(1-a)(1-m)m + a(1-m)$	$(1-a)(1-m)m + \frac{a}{2}$
$\mathbf{L}(m)$	$m(1-m)$	$(1-a)m(1-m) + am$	$(1-a)m(1-m) + \frac{a}{2}$
$D(m)$	$\frac{1}{N}m(1-m)$	$\frac{1}{N}[(1-a)(m-m^2) + \frac{a}{2}]$	$\frac{1}{N}[(1-a)(m-m^2) + \frac{a}{2}]$
$v(m)$	0	$a(1-2m)$	0

TABLE I: Expressions for $\mathbf{R}(m)$, $\mathbf{L}(m)$, D and v for the Voter Model, Noisy Voter Model, Voter Model with Global Noise.

$\mathbf{R}(m)$, $\mathbf{L}(m)$, $D(m)$ and $v(m)$ for each model are shown in Table I.

All the analytical expressions and definitions above apply to the theoretical framework. To validate and further explore these dynamics, simulations are implemented by coding the program in `Fortran` and visualizing the data in `Python` with the `NumPy` and `Matplotlib` libraries. Note that the resultant stochastic process evolves in magnetization space, not in real space.

III. RESULTS AND DISCUSSION

A. Temporal evolution of m and ρ in the Voter Model on a regular, low-dimensioned lattice

The Voter Model is unbiased and symmetric, so neither consensus state is preferred. Thus, if a sufficiently large number of simulations are considered, exploring many of all possible configurations in the ensemble, the average magnetization $\langle m \rangle$ is expected to be conserved over time. This is computationally proved and shown in Figure 1.

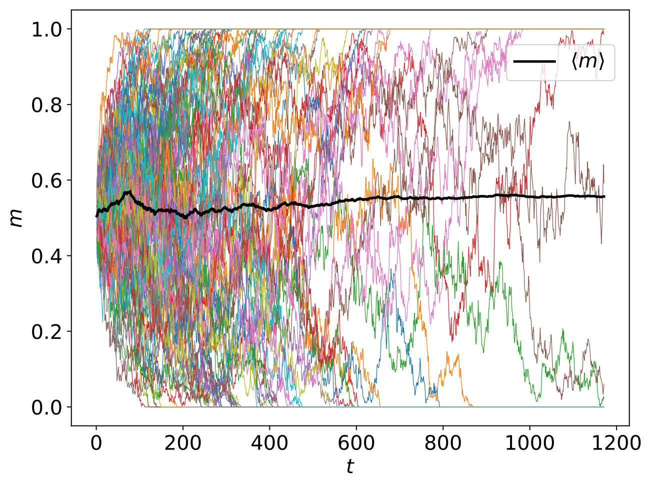

FIG. 1: Evolution of magnetization m over time t, measured in Monte Carlo time steps, for 100 simulations (in different colors) of the Voter Model. Ensemble average of value m is plotted in black, showing an approximately constant behaviour. The lattice considered is $N = 512$-sized and $d = 3$-dimensioned.

In Figure 2, the time evolution of the average value of ρ is shown, just until consensus is reached ($\rho = 0$). In $d = 1$, it follows the potential law $\langle\rho\rangle \sim t^{-1/2}$ and in $d = 2$, the logarithmic decay $\langle\rho\rangle \sim \frac{1}{\ln(t)}$ [7]. Increasing N delays consensus due to more possible disagreements. For $d \geq 3$, the system can maintain persistent interfaces and opinion coexistence, making consensus times very long or effectively infinite [8].

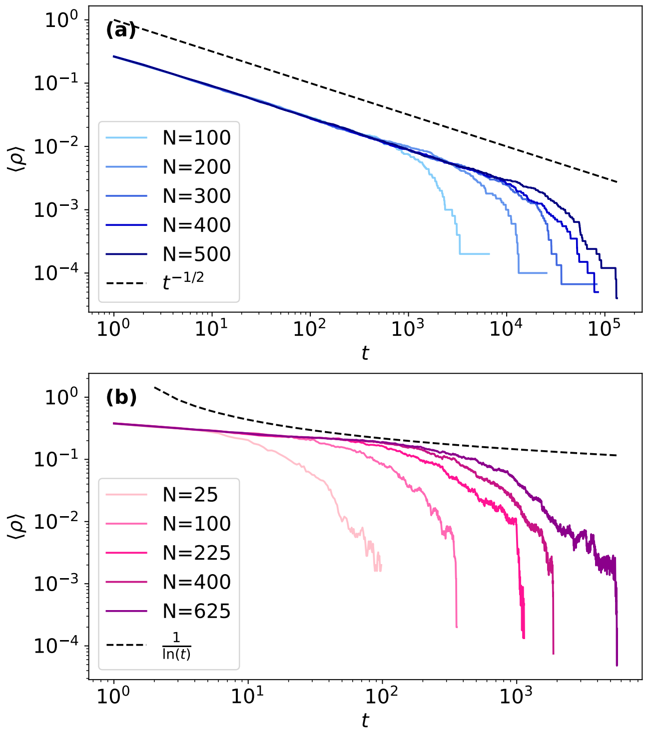

FIG. 2: Evolution of ρ over time t, measured in Monte Carlo time steps, for increasing values of N for the Voter Model defined on a (a) one- and (b) two-dimensioned lattice. Dimension-dependent, decay laws are included in each panel.

B. Magnetization distribution in voter models with noise on a fully connected graph

Some equally spaced magnetization values of a long-time simulation on a complete graph are recorded. The distribution is expected to converge to the stationary probability density $c_{\text{stat}}(m)$, solution of the stationary Fokker-Planck equation. The Fokker-Planck equation governs the time evolution of the probability density $c(m, t)$ of observing magnetization m at time t, and can be deduced from the evolution in one time step of c [6]:

$$\begin{aligned} c(m, t + \delta t) = &\mathbf{R}(m{-}\delta m)\, c(m{-}\delta m, t) \\ &+ \mathbf{L}(m{+}\delta m)\, c(m{+}\delta m, t) \\ &+ \left[1{-}\mathbf{R}(m){-}\mathbf{L}(m)\right] c(m, t) \end{aligned} \quad (5)$$

Expanding it to second order in δm and first order in δt (see Appendix A for more details), it results in the Fokker-Planck or Forward Kolmogorov equation:

$$\frac{\partial c(m, t)}{\partial t} = \frac{\partial}{\partial m}[v(m)c(m, t)] + \frac{\partial^2}{\partial m^2}[D(m)c(m, t)] \quad (6)$$

The stationary state ($\frac{\partial}{\partial t} = 0$) leads to a second-order partial differential equation for the stationary probability $c_{stat}(m)$, derived using the drift and diffusion in Table I.

For the Noisy Voter Model,

$$\begin{aligned} &\frac{\partial}{\partial m}\left[a(1 - 2m)\, c(m, t)\right] \\ &\quad + \frac{1}{N}\frac{\partial^2}{\partial m^2}\left(\left[(1 - a)(m - m^2) + \frac{a}{2}\right] c(m, t)\right) = 0 \end{aligned} \quad (7)$$

For the Voter Model with Global Noise,

$$\frac{1}{N}\frac{\partial^2}{\partial m^2}\left[(1 - a)(m - m^2) + \frac{a}{2}\right] = 0 \quad (8)$$

Equation (7) is solved as in [9]:

$$c_{stat} = \frac{1}{Z}\exp\left[-N\int^m \frac{-v(m') + \frac{D'(m')}{N}}{D(m')}dm'\right] \quad (9)$$

being Z the normalization constant.

After solving (9), the simulation results can be compared to the the theoretical prediction, as done in Figure 3. As noticeable, as a gets larger, the expected value

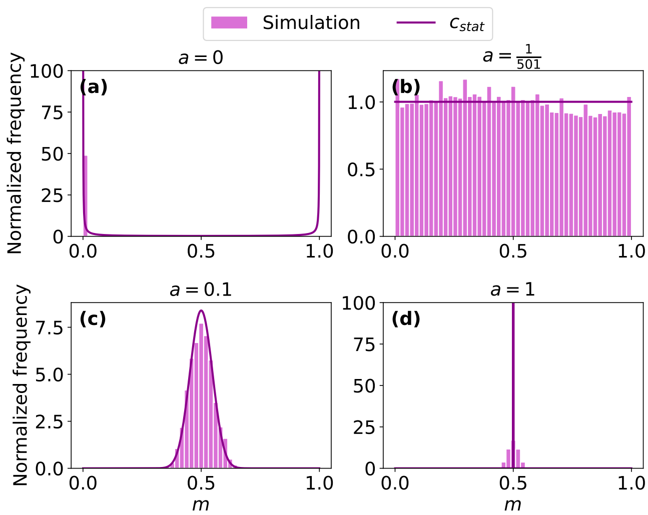

FIG. 3: Histogram of magnetization of a single long-term Noisy Voter Model for (a) $a = 0$, (b) $a = \frac{1}{501}$, (c) $a = 0.1$ (d) $a = 1$ defined on an $N = 500$-sized complete graph. Solution to stationary Fokker-Planck is also plotted.

of m approaches 0.5. A bimodal-to-unimodal transition occurs at $a_c = \frac{1}{N+1}$ [10], where the second derivative of the stationary distribution at $m = 0.5$ changes sign, signaling the shift in dominance from diffusion to drift. At $a = 1$, there is no copying mechanism, but just a random flipping that breaks local correlations, introducing a restoring force towards the central magnetization. In a mean-field picture, this is seen as the drift term pulling the system toward the central value, dominating over the diffusive dynamics of the Voter Model.

To solve Equation (8) `Mathematica`'s `NDSolve` is used. It computes approximate solutions when analytical ones are not feasible, by returning an interpolating numerical function that can be evaluated at any point within the domain [11]. Reflecting boundary conditions are imposed to ensure that total probability remains inside the domain. They mathematically read:

$$\frac{d}{dm}[D(m)c(m)]\bigg|_{m=0} = 0 \qquad \frac{d}{dm}[D(m)c(m)]\bigg|_{m=1} = 0$$

Figure 4 shows solution to (8) and simulation results,

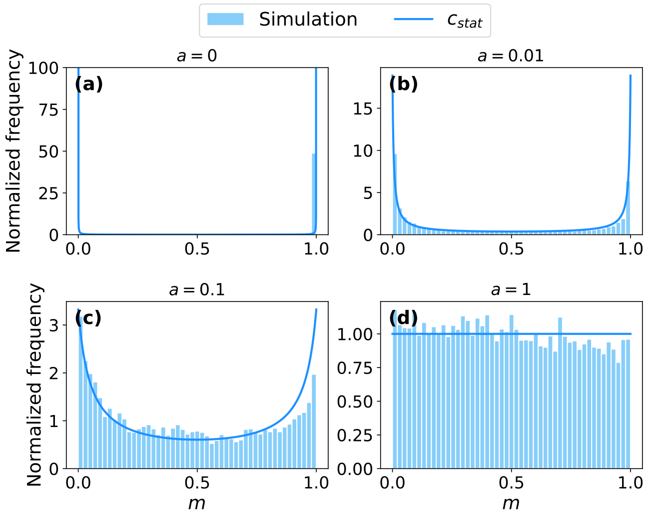

FIG. 4: Histogram of magnetization of a long-term Voter Model with Global Noise for (a) $a = 0$, (b) $a = 0.01$, (c) $a = 0.1$, (d) $a = 1$, on an $N = 500$-sized complete graph.

illustrating that as a increases, the distribution is flattened, without favoring any particular state. At $a = 1$, all configurations become equally likely, the central peak is gone and the model reduces to a Random Walk.

C. First passage processes on a complete graph

First passage processes study the time it takes for a stochastic variable to reach a predefined boundary for the first time, starting from an initial condition [12]. Here, it is the average time to achieve consensus given a predefined initial magnetization m_0.

Let $t(m_0)$ be the average time to consensus as a function of the initial magnetization:

$$\begin{aligned} t(m_0) = \delta t + \mathbf{R}(m_0)t(m_0+\delta m_0) \\ + \mathbf{L}(m_0)t(m_0-\delta m_0) \\ + [1-\mathbf{R}(m_0)-\mathbf{L}(m_0)]t(m_0) \end{aligned} \qquad (10)$$

It yields the average consensus time as the time for a single step plus the average time to reach consensus after taking this step. The last terms account for the transitions $m_0 \to m_0 \pm \delta m$ and $m_0 \to m_0$, respectively [6].

The Backward Kolmogorov equation, a second-order partial differential equation describing the dependence of t on m_0, is derived by expanding (10) to second order in δm_0 (detailed in Appendix B), enabling approximation of the discrete dynamics by a continuous diffusion process:

$$v(m_0)\frac{dt}{dm_0} + D(m_0)\frac{d^2t}{dm_0^2} = -1 \qquad (11)$$

Recovering D and v for each model in Table I, it is easy to get to their Backward Kolmogorov equation.

For the Voter Model, the Noisy Voter Model and the Voter Model with Global Noise, respectively:

$$\frac{m_0(1-m_0)}{N}\frac{d^2t}{dm_0^2} = -1 \qquad (12)$$

$$\begin{aligned} &[-a(2m_0-1)]\frac{dt}{dm_0} \\ &+ \left[\frac{1}{N}\left(\frac{a}{2}+(1-a)(m_0-m_0^2)\right)\right]\frac{d^2t}{dm_0^2} = -1 \end{aligned} \qquad (13)$$

$$\frac{1}{N}\left(\frac{a}{2}+(1-a)(m_0-m_0^2)\right)\frac{d^2t}{dm_0^2} = -1 \qquad (14)$$

Solving the Backward Kolmogorov equation yields the theoretical time to consensus. The boundary conditions $t(0) = 0$, $t(1) = 0$ are considered, since $m_0 = 0, 1$ correspond to zero and full probability of initially assigning to each agent an opinion 1, which implies that both already start in consensus. Equation (12) is solved analytically using `Mathematica`'s `DSolve`, which attempts to find an exact symbolic solution to differential equations [11]. The expression provided is:

$$T(m_0) = N[(m-1)\ln(1-m) - m\ln m] \qquad (15)$$

Equations (13) and (14) are complex, so their solution is carried out using `NDSolve`. These resolutions enable a comparison between the theoretical time to consensus and the one obtained in simulation, as done in Figure 5.

The maximum t value occurs at $m_0 = 0.5$, the point farthest from the boundaries. In the Noisy Voter Model, increasing a rapidly raises t. This is explained as noise pushes the system away from consensus by strengthening the drift opposing it and stabilizing the biased, non-consensus states. Only low a values are considered, since for $a \gtrsim 0.003$, t becomes so large that it exceeds `Python`'s floating-point limits, resulting in overflow.

For the Voter Model with Global Noise, increasing a decreases $\langle t \rangle$. As noise increases, more fluctuations are introduced, allowing the system to explore configurations faster and escape from locally stable states. Paradoxically, full randomness $a = 1$ leads to the fastest consensus, as fluctuations dominate and prevent the formation

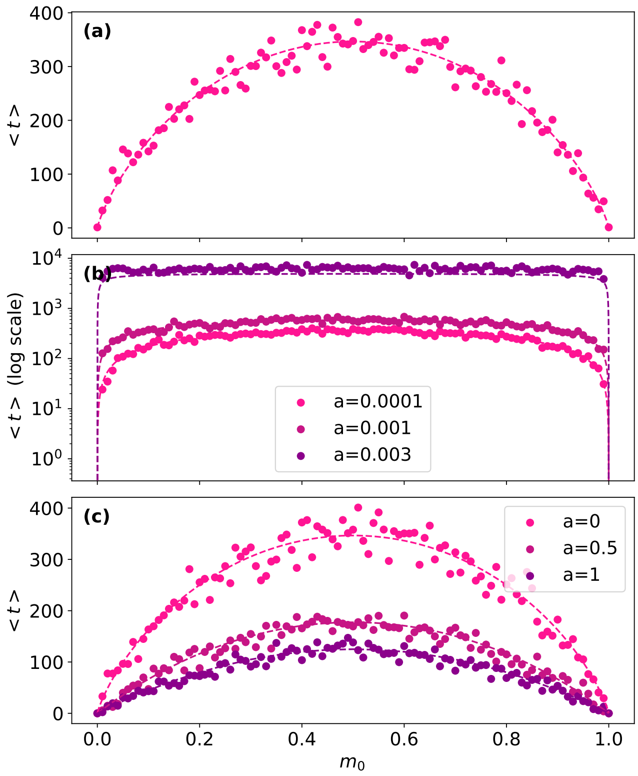

FIG. 5: Average time to consensus, in Monte Carlo steps, against initial magnetization on an $N = 500$ complete graph for the (a) Voter Model, (b) Noisy Voter Model, (c) Voter Model with Global Noise. 100 simulations per m_0, $\Delta m_0 = 0.01$. Dashed lines are solution to Backward Kolmogorov.

and persistence of local opinion clusters, responsible for the slowness in the Voter Model, since the only changing mechanism is copying neighbors' opinions. Noise accelerates the mixing of opinions throughout the population, speeding up the diffusion process and making consensus more likely to occur sooner.

IV. CONCLUSIONS

It is graphically proved that $\langle m \rangle$ remains constant for the Voter Model defined on large, regular systems by performing 100 simulations and recording m at each time step. For one- and two-dimensional N-sized systems, varying N, ρ values are kept, averaged over all simulations and plotted against t. As expected, the evolutions are well approximated by the adjustments $\langle \rho \rangle_{1D} \sim t^{-1/2}$ and $\langle \rho \rangle_{2D} \sim \frac{1}{\ln(t)}$, verifying the model's correctness. The distribution of m in a simulation of the noisy models on all-to-all networks is studied, showing convergence to the stationary Fokker-Planck solution, which gives the probability of each value of m. In the representation of the Noisy Voter Model, a change in modality is observed while increasing the noise probability a, reflecting how the noise-induced drift counteracts diffusion and dominates for $a > \frac{1}{N+1}$. Lastly, the average consensus time is measured as a function of m_0 for all models and found to adjust to their respective solution of the Backward Kolmogorov equation, which relates consensus time to m_0.

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[1] Castellano, C. et al. (2009). Statistical physics of social dynamics. *Reviews of Modern Physics*. 81(2),591-646. https://doi.org/10.1103/RevModPhys.81.591

[2] McKane, A. J. & Newman, T. J. (2005). Predator-Prey Cycles from Resonant Amplification of Demographic Stochasticity. *Physical Review Letters*. 94(21), 218102. https://doi.org/10.1103/PhysRevLett.94.218102

[3] Helbing, D. (2001). Traffic and related self-driven many-particle systems.*Reviews of Modern Physics*. 73(4), 1067–1141. https://doi.org/10.1103/RevModPhys.73.1067

[4] Artime, O. et al. (2018). First-passage distributions for the one-dimensional Fokker-Planck equation. *Physical Review E*. 98(4). https://doi.org/10.1103/PhysRevE.98.042143

[5] Artime, O. (2020). Temporal Activity Patterns in Social Dynamics. (PHD Thesis, Universitat de les Illes Balears)

[6] Sood, J. et al. (2008). Voter Models on Heterogeneous Networks. *Physical Review E*, 77(4), 041121. https://doi.org/10.1103/PhysRevE.77.041121

[7] Mobilia, M. (2003). Does a Single Zealot Affect an Infinite Group of Voters? *Physical Review Letters*. 91(2). https://doi.org/10.48550/arXiv.cond-mat/0304670

[8] Ben-Naim, E. et al., (1996). Coarsening and persistence in the voter model. *Physical Review E*, 53(4), 3078–3087. https://doi.org/10.1103/PhysRevE.53.3078

[9] Peralta, A.F. et al. (2018). Analytical and numerical study of the non-linear noisy voter model on complex networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science*. 28(7). https://doi.org/10.48550/arXiv.1803.06861

[10] Khalil, N., & Toral, R. (2019). The noisy voter model under the influence of contrarians. *Physica A: Statistical Mechanics and its Applications*, 515, 81-92. https://doi.org/10.1016/j.physa.2018.09.178

[11] Wolfram Research, Inc. (n.d.). NDSolve - Numerical Differential Equation Solver. Retrieved from https://reference.wolfram.com/language/ref/NDSolve.html

[12] Redner, S. (2001). *A guide to first-passage processes*. Cambridge University Press. https://doi.org/10.1017/CBO9780511606014

Efectes del soroll en dinàmiques d'opinió

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Resum: S'analitza les propietats fora de l'equilibri del Model del Votant i les seves variants sorolloses (Model del Votant Sorollós i Model del Votant amb Soroll Global). Pel Model del Votant en xarxes regulars, la conservació de la magnetització mitjana $\langle m \rangle$ es confirma, i la densitat d'interfície $\langle \rho \rangle$ decau cap al consens segons una llei d'escala depenent de la dimensió. En xarxes completament connectades, la distribució de magnetització de les simulacions coincideix amb la solució estacionària de Fokker-Planck. El Model del Votant Sorollós mostra una preferència per $m = 0.5$, donant lloc a una transició bimodal-unimodal en augmentar el soroll a. El soroll global aplana la distribució, que esdevé uniforme per $a = 1$. El temps mitjà de consens $\langle t \rangle$ presenta un màxim pel valor central de magnetització inicial. L'increment d'a retarda el consens en el Model del Votant Sorollós, però l'accelera en el Model del Votant amb Soroll Global en trencar agrupacions metastables. Els resultats contrasten els rols del soroll local i global en dinàmica col·lectiva.
Paraules clau: Física fora de l'equilibri, Mecànica estadística, Processos estocàstics, Models d'espí, Fenòmens col·lectius, Simulacions Monte Carlo.

ODSs: Aquest TFG esta relacionat amb l'Objectiu de Desenvolupament Sostenible (ODS) 4. Educació de qualitat.

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

El contingut d'aquest TFG, elaborat en el marc del grau de Física, es relaciona amb l'ODS 4 (Educació de qualitat), concretament amb la fita 4.4, que promou l'adquisició de competències tècniques i científiques en àmbits com la programació, la modelització matemàtica i la física estadística. El treball fomenta la formació universitària avançada i el desenvolupament d'habilitats per a la recerca i l'anàlisi de fenòmens complexos.

Appendix A: TRANSITION FROM DISCRETE TO CONTINUOUS TO DERIVE FOKKER-PLANCK EQUATION

The starting point is

$$\begin{aligned} c(m, t+\delta t) = &\mathbf{R}(m-\delta m) c(m-\delta m, t) \\ &+\mathbf{L}(m+\delta m) c(m+\delta m, t) \\ &+[1-\mathbf{R}(m)-\mathbf{L}(m)] c(m, t) \end{aligned} \tag{A1}$$

The expansions in Taylor to second order in δm are:

$$\begin{aligned} c(m \pm \delta m, t) &\approx c(m, t) \pm \delta m \frac{\partial c}{\partial m}+\frac{\delta m^2}{2} \frac{\partial^2 c}{\partial m^2} \\ \mathbf{R}(m-\delta m) &\approx \mathbf{R}(m)-\delta m \frac{d \mathbf{R}}{d m}+\frac{\delta m^2}{2} \frac{d^2 \mathbf{R}}{d m^2} \\ \mathbf{L}(m+\delta m) &\approx \mathbf{L}(m)+\delta m \frac{d \mathbf{L}}{d m}+\frac{\delta m^2}{2} \frac{d^2 \mathbf{L}}{d m^2} \end{aligned}$$

The products $\mathbf{R}(m-\delta m) c(m-\delta m, t)$ and $\mathbf{L}(m+\delta m) c(m+\delta m, t)$ can be expressed for expanded expressions in δm:

$$\begin{aligned} \mathbf{R}(m-\delta m)\, c(m-\delta m, t) \approx &\left(\mathbf{R}-\delta m \mathbf{R}^{\prime}+\frac{\delta m^2}{2} \mathbf{R}^{\prime \prime}\right) \\ &\left(c-\delta m c^{\prime}+\frac{\delta m^2}{2} c^{\prime \prime}\right) \\ \approx &\mathbf{R} c-\delta m\left(\mathbf{R} c^{\prime}+\mathbf{R}^{\prime} c\right) \\ &+\delta m^2\left(\frac{1}{2} \mathbf{R} c^{\prime \prime}+\frac{1}{2} \mathbf{R}^{\prime \prime} c+\mathbf{R}^{\prime} c^{\prime}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{L}(m+\delta m)\, c(m+\delta m, t) \approx &\left(\mathbf{L}+\delta m \mathbf{L}^{\prime}+\frac{\delta m^2}{2} \mathbf{L}^{\prime \prime}\right) \\ &\left(c+\delta m c^{\prime}+\frac{\delta m^2}{2} c^{\prime \prime}\right) \\ \approx &\mathbf{L} c+\delta m\left(\mathbf{L} c^{\prime}+\mathbf{L}^{\prime} c\right) \\ &+\delta m^2\left(\frac{1}{2} \mathbf{L} c^{\prime \prime}+\frac{1}{2} \mathbf{L}^{\prime \prime} c+\mathbf{L}^{\prime} c^{\prime}\right) \end{aligned}$$

Notice that it has been defined $\mathbf{R} \equiv \mathbf{R}(m)$, $\mathbf{L} \equiv \mathbf{L}(m)$ and $c(m,t) \equiv c$. Furthermore, $\mathbf{R}'$ and $\mathbf{R}''$ respectively denote the first and second derivative of $\mathbf{R}$ with respect to m. The same works for $\mathbf{L}$. c' and c'' respectively denote the partial first and second derivative with respect to m.

With this, Equation (A1) can be expressed as:

$$\begin{aligned} c(m, t+\delta t) \approx &\mathbf{R} c-\delta m\left(\mathbf{R} c^{\prime}+\mathbf{R}^{\prime} c\right) \\ &+\delta m^2\left(\frac{1}{2} \mathbf{R} c^{\prime \prime}+\frac{1}{2} \mathbf{R}^{\prime \prime} c+\mathbf{R}^{\prime} c^{\prime}\right) \\ &+\mathbf{L} c+\delta m\left(\mathbf{L} c^{\prime}+\mathbf{L}^{\prime} c\right) \\ &+\delta m^2\left(\frac{1}{2} \mathbf{L} c^{\prime \prime}+\frac{1}{2} \mathbf{L}^{\prime \prime} c+\mathbf{L}^{\prime} c^{\prime}\right) \\ &+[1-\mathbf{R}-\mathbf{L}]\, c \end{aligned}$$

Grouping non-proportional to δm, proportional to δm and proportional to δm^2 terms, and simplifying:

$$\begin{aligned} c(m, t+\delta t) \approx &\, c(m, t)+\delta m\left([\mathbf{L}-\mathbf{R}] c^{\prime}+\left[\mathbf{L}^{\prime}-\mathbf{R}^{\prime}\right] c\right)+ \\ &+\delta m^2\left(\frac{1}{2}\left([\mathbf{R}+\mathbf{L}] c^{\prime \prime}+\left[\mathbf{R}^{\prime \prime}+\mathbf{L}^{\prime \prime}\right] c\right)+\left[\mathbf{R}^{\prime}+\mathbf{L}^{\prime}\right] c^{\prime}\right) \end{aligned}$$

Expanding $c(m, t+\delta t)$ to first order in δt, that is, $c(m,t+\delta t) \approx c(m,t) + \frac{\partial c}{\partial t}\delta t$, dividing by δt and remembering $\delta t = \delta m$:

$$\begin{aligned} \frac{\partial c}{\partial t} \approx &\left([\mathbf{L}-\mathbf{R}] c^{\prime}+\left[\mathbf{L}^{\prime}-\mathbf{R}^{\prime}\right] c\right) \\ &+\delta m\left(\frac{1}{2}\left([\mathbf{R}+\mathbf{L}] c^{\prime \prime}+\left[\mathbf{R}^{\prime \prime}+\mathbf{L}^{\prime \prime}\right] c\right)+\left[\mathbf{R}^{\prime}+\mathbf{L}^{\prime}\right] c^{\prime}\right) \end{aligned} \tag{A2}$$

Recovering the diffusion D and drift v expressions, as functions of $\mathbf{R}(m)$ and $\mathbf{L}(m)$:

$$\begin{aligned} D(m) &= \frac{1}{2} \frac{\delta m^2}{\delta t}[\mathbf{R}(m)+\mathbf{L}(m)] \\ v(m) &= \frac{\delta m}{\delta t}[\mathbf{R}(m)-\mathbf{L}(m)] \end{aligned}$$

It is possible to express $\mathbf{R}(m)$ and $\mathbf{L}(m)$ in terms of $D(m)$ and $v(m)$:

$$\begin{aligned} \mathbf{R}(m) &= D(m) \frac{\delta t}{\delta m^2}+v(m) \frac{1}{2} \frac{\delta t}{\delta m} \\ \mathbf{L}(m) &= D(m) \frac{\delta t}{\delta m^2}-v(m) \frac{1}{2} \frac{\delta t}{\delta m} \end{aligned}$$

Their first and second derivatives, are:

$$\begin{aligned} \mathbf{R}^{\prime}(m) &= D^{\prime}(m) \frac{\delta t}{\delta m^2}+v^{\prime}(m) \frac{1}{2} \frac{\delta t}{\delta m} \\ \mathbf{L}^{\prime}(m) &= D^{\prime}(m) \frac{\delta t}{\delta m^2}-v^{\prime}(m) \frac{1}{2} \frac{\delta t}{\delta m} \\ \mathbf{R}^{\prime \prime}(m) &= D^{\prime \prime}(m) \frac{\delta t}{\delta m^2}+v^{\prime \prime}(m) \frac{1}{2} \frac{\delta t}{\delta m} \\ \mathbf{L}^{\prime \prime}(m) &= D^{\prime \prime}(m) \frac{\delta t}{\delta m^2}-v^{\prime \prime}(m) \frac{1}{2} \frac{\delta t}{\delta m} \end{aligned}$$

Therefore, substituting $\mathbf{R}$ and $\mathbf{L}$ and their derivatives in (A2) and canceling δt with δm:

$$\begin{aligned} \frac{\partial c}{\partial t} \approx &\, v(m) \frac{\partial c}{\partial m}+\frac{\partial v}{\partial m} c \\ &+D(m) \frac{\partial^2 c}{\partial m^2}+\frac{\partial D}{\partial m} \frac{\partial c}{\partial m}+\frac{1}{2} \frac{\partial^2 D}{\partial m^2} c \end{aligned}$$

Finally, grouping terms yields the Fokker-Planck equation:

$$\frac{\partial c(m, t)}{\partial t} \approx \frac{\partial}{\partial m}[v(m)\, c(m, t)]+\frac{\partial^2}{\partial m^2}[D(m)\, c(m, t)] \tag{A3}$$

Appendix B: TRANSITION FROM DISCRETE TO CONTINUOUS TO DERIVE BACKWARD KOLMOGOROV EQUATION

The starting point is the time to consensus, as a function of initial magnetization:

$$\begin{aligned} t(m_0) = \delta t &+ \mathbf{R}(m_0) t(m_0 + \delta m_0) \\ &+ \mathbf{L}(m_0) t(m_0 - \delta m_0) \\ &+ [1 - \mathbf{R}(m_0) - \mathbf{L}(m_0)] t(m_0) \end{aligned} \tag{B1}$$

Expanding $t(m_0 \pm \delta m_0)$ to second order in δm_0:

$$\begin{aligned} t(m_0) \approx \delta t &+ \mathbf{R}(m_0) \left[t(m_0) + \frac{dt}{dm_0}\delta m_0 + \frac{1}{2}\frac{d^2 t}{dm_0^2}\delta m_0^2 \right] \\ &+ \mathbf{L}(m_0) \left[t(m_0) - \frac{dt}{dm_0}\delta m_0 + \frac{1}{2}\frac{d^2 t}{dm_0^2}\delta m_0^2 \right] \\ &+ [1 - \mathbf{R}(m_0) - \mathbf{L}(m_0)] t(m_0) \end{aligned}$$

Recovering the diffusion D and drift v expressions evaluated at $m = m_0$, as functions of $\mathbf{R}(m_0)$ and $\mathbf{L}(m_0)$:

$$\begin{aligned} D(m_0) &= \frac{1}{2}\frac{\delta m_0^2}{\delta t}[\mathbf{R}(m_0) + \mathbf{L}(m_0)] \\ v(m_0) &= \frac{\delta m_0}{\delta t}[\mathbf{R}(m_0) - \mathbf{L}(m_0)] \end{aligned}$$

It is possible to express $\mathbf{R}(m_0)$ and $\mathbf{L}(m_0)$ in terms of $D(m_0)$ and $v(m_0)$:

$$\begin{aligned} \mathbf{R}(m_0) &= D(m_0)\frac{\delta t}{\delta m_0^2} + v(m_0)\frac{1}{2}\frac{\delta t}{\delta m_0} \\ \mathbf{L}(m_0) &= D(m_0)\frac{\delta t}{\delta m_0^2} - v(m_0)\frac{1}{2}\frac{\delta t}{\delta m_0} \end{aligned}$$

Substituting $\mathbf{R}(m)$ and $\mathbf{L}(m)$ in Equation (B1):

$$\begin{aligned} t(m_0) = \delta t &+ \left[D(m_0)\frac{\delta t}{\delta m_0^2} + \frac{v}{2}\frac{\delta t}{\delta m_0} \right] \left(t(m_0) + \frac{dt}{dm_0}\delta m_0 \right. \\ &\qquad \left. + \frac{1}{2}\frac{d^2 t}{dm_0^2}\delta m_0^2 \right) \\ &+ \left[D(m_0)\frac{\delta t}{\delta m_0^2} - \frac{v}{2}\frac{\delta t}{\delta m_0} \right] \left(t(m_0) - \frac{dt}{dm_0}\delta m_0 \right. \\ &\qquad \left. + \frac{1}{2}\frac{d^2 t}{dm_0^2}\delta m_0^2 \right) \\ &+ \left[1 - 2D(m_0)\frac{\delta t}{\delta m_0^2} \right] t(m_0) \end{aligned}$$

Expanding it, the expression gets simplified to:

$$t \approx \delta t + v(m_0)\frac{dt}{dm_0}\delta t + D(m_0)\frac{d^2 t}{dm_0^2}\delta t + t$$

And finally, substituting $\delta t = \delta m = \frac{1}{N}$:

$$v(m_0)\frac{dt}{dm_0} + D(m_0)\frac{d^2 t}{dm_0^2} \approx -1 \tag{B2}$$

which is the Backward Kolmogorov equation, a second-order differential equation expressing the time to consensus as a function of m_0.