Phase transitions and self organized criticality of forest fires

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In this work, we study the dynamics and critical behavior of forest fire models, with a focus on the Drossel-Schwabl forest fire model (DSFFM). Through computational simulations, we analyze phase transitions in a two-dimensional lattice model, identifying three main regimes: a mixed phase, a spiral wave phase, and a self-organized critical (SOC) phase. We characterize these phases based on spatial configurations and fire-tree density relationships, revealing discontinuous transitions and hysteresis effects. In the SOC regime, we extract critical exponents for the size distribution, radius of gyration, and burning time of tree clusters, comparing our results with those from the literature. While our simulations reproduce known scaling laws, we also discuss the limitations of the DSFFM, particularly its non-universal scaling at large system sizes and time scales. Despite these limitations, the model remains a useful tool for exploring emergent phenomena in excitable systems and complex ecological dynamics.
Keywords: *Forest fire, self organized criticality, scaling behavior, critical exponents, Drossel-Schwabl forest fire model.*

SDGs: SDG 4: Quality Education, SDG 9: Industry, Innovation and Infrastructure, SDG 13: Climate Action, SDG 15: Life on Land.

I. INTRODUCTION

Complex systems such as ecosystems have been a subject of study for several decades, with the objective of mapping intricate behaviors onto fairly simple models that provide key understanding. In a wide variety of ecosystems, certain parameters showcase characteristic dependencies that give insight on the system's dynamics, such as vegetation biomass in changing precipitation [1] or invasive species spread as a function of the fragmented landscape [2]. These changes of the order parameter as the control parameter varies are called phase transitions, which can be continuous or discontinuous. For continuous phase transitions, the correlation length and fluctuations diverge at the critical point. These behaviors can be characterized by critical exponents, which describe the power-law dependencies of certain physical quantities near the transition. In this work, we will study the critical exponents of the forest fire model in ecology.

Forest fire models are a type of cellular automata which implement dynamics of fire propagation on a forest. Generally, a two dimensional grid with three possible states (tree, fire, empty) for each site is used, and the definition of the dynamics that rule tree growth and fire spread are what differs a model from another.
With the introduction of self organized criticality (SOC from now on) by Bak et al. [5], Drossel and Schwabl introduced a forest fire model (DSFFM from now on), which, due to its simplicity, was deeply studied and revisited by many authors [9, 10, 14]. Our objective herein is to study the criticallity of the DSFFM and compute the value of some critical exponents while comparing them to the ones found by others. In order to introduce DSFFM, we will first review the forest fire model presented by Clar, Drossel and Schwabl [3]. After analysing the type of phase transitions this model presents, we shift our focus to the SOC phase, and use the DSFFM to study its criticality.

II. PHASE TRANSITIONS IN A FOREST FIRE MODEL

In this section we computationally study the model presented by Clar et al. [3], which is an odd type of forest fire model mainly because it presents two types of phase transitions. This model exhibits three forest phases, one of which is the SOC mentioned earlier, that will be introduced and discussed further in Section III.
The model's rules are the following:

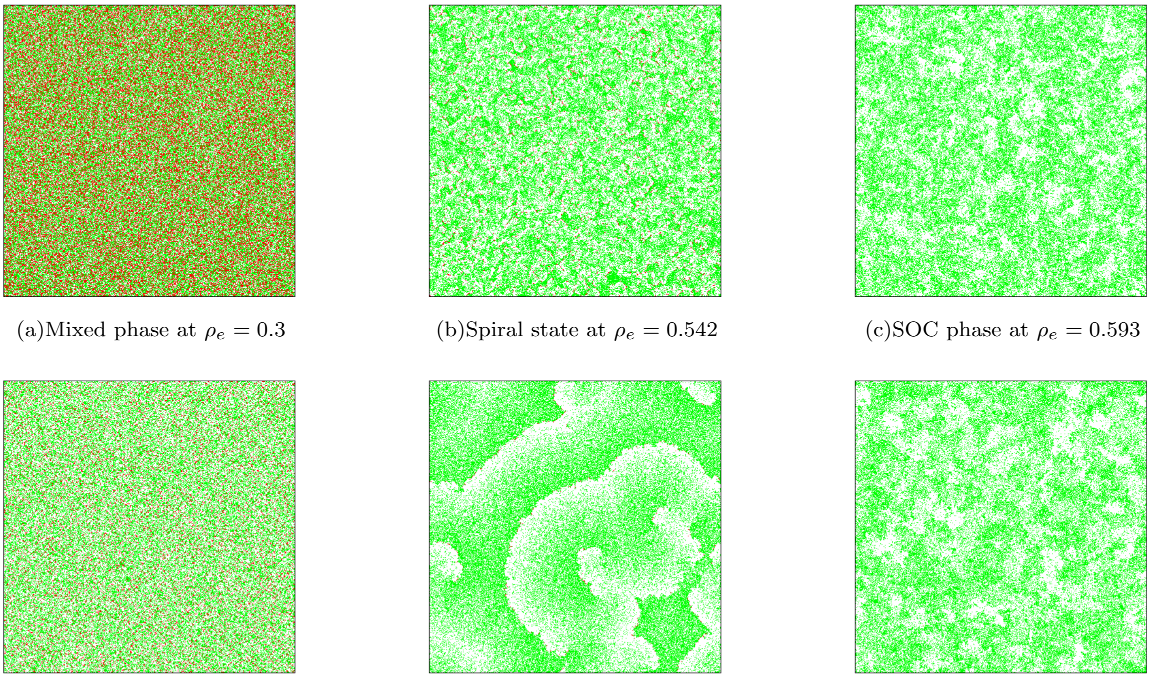
1) All trees on fire will burn down the next time step.

2) The fire on a site will spread on its Von Neumann neighbors in the next time step.

3) After each time step, the same number of trees that have burnt down will grow on randomly chosen empty sites including the ones that just became empty.

4) If the fire dies out, a randomly chosen tree catches fire spontaneously.

The purpose of rule 3 is to set a control parameter that we can vary on the simulations. The implementation of this rule ensures that for each iteration the number of empty sites is constant, therefore our control parameter is the density of empty sites, i.e. the normalized number of empty sites, ρ_e.



(a)Mixed phase at $\rho_e = 0.3$ (b)Spiral state at $\rho_e = 0.542$ (c)SOC phase at $\rho_e = 0.593$

(d)Mixed phase at $\rho_e = 0.542$ (e)Spiral state at $\rho_e = 0.592$ (f)SOC phase at $\rho_e = 0.61$

Figure 1: Lattice configuration for different values of empty site density. Green, red and white dots represent sites with tree, fire or empty states respectively. Figures (b),(d) and (c),(e) showcase distinct configurations for the same or similar values of the control parameter ρ_e (i.e. in hysteresis cycles of Figure 2).

A. Methodology

In order to simulate the previously defined system, a computer program following the four rules was implemented in Fortran. Firstly, a 2-dimensional square lattice of 1000×1000 sites was created. Then, to define the three possible states, a number was assigned to each of them; 0 for empty, 1 for forest and 2 for fire. We imposed a chosen number of empty trees (which sets the control parameter ρ_e value). As initial conditions, we selected a number of fires, leaving the rest of the grid full of trees. To ensure rule 3 was fulfilled, we used the Fisher Yates shuffle [6] over a list of all possible empty sites. This ensures a uniform distribution between the selected empty sites chosen to regrow a burnt tree. The computational implementation was inspired by Durstenfeld's random permutation algorithm [7].
Simulations for different values of the control parameter ρ_e were made with different initial conditions to observe hysteresis behavior, but all measurements with enough steps to ensure convergence to stationary states. The density of fires ρ_f was computed as the average of time-separated values in the stationary state.

B. Results

The results obtained by the implementation of this algorithm are in agreement with those in [3]. For low empty site density, trees and fires coexist at comparable densities. The dynamic of burning and regrowing trees creates a mixed phase seen in Figure 1(a). In this phase, trees and fires blend without an apparent spatial order. This mixed organization remains stable up to a threshold $\rho_e^1 = 0.547$, beyond which the density of fires abruptly falls to small values, signaling a discontinuous phase transition (Figure 2). From this point onward, the abundance of empty sites becomes large enough so that tree clusters lose connectivity and fires can no longer percolate easily from one patch to another, so the system reorganizes in a state with fewer fires that tend to form linear fronts at the edge of tree clusters (Figure 1(b)).

As the control parameter increases, fires form large fronts that span a significant number of sites (Figure 1(e)). Furthermore, the shape of the fronts resembles a spiral wave, propagating in a manner analogous to waves traveling through a medium [3]. The shift from small to large fronts occurs gradually and does not produce any notable changes in the order parameter. This phase is called the spiral state.

While the phase transition from mixed phase to spiral state occurs at ρ_e^1, the reverse phase transition takes place at $\rho_e^2 = 0.542$ where the system discontinuously switches from the spiral to the mixed phase (Figure 2): it "remembers" its history and the phase transition depends on from which side one approaches the region (Figure 2, top right inset). This is called an hysteresis cycle. Figures 1(d) and 1(b), show two

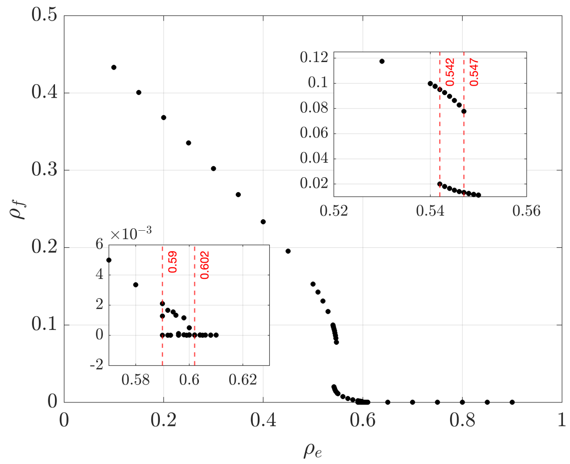


Figure 2: Density of fires as a function of density of empty sites. The insets are magnified windows to show intervals of interest. The right one showcases the first order phase transition hysteresis cicle between the mixed and spiral phases. The left one shows the triangular hysteresis cycle between spiral and SOC phases.

stable phases at ρ_e^2.
At larger values of the control parameter, the density of fires undergoes a continuous phase transition at $\rho_e^3 \approx 0.602$ (Figure 2). The system is reconfigured in a new phase, with characteristics similar to the self organized critical forest, hence it is named SOC phase [3]. In this phase, the empty sites density is high and trees grow in isolated clusters (Figure 1(f)). Since there is no connectivity between clusters, all fires eventually die out. However, thanks to rule (4), fire dynamics remain active: each fire generated by this rule burns down the entire cluster of trees it is connected to and the rule reactivates once the burning is complete. Interestingly, in the stationary state of this phase, certain quantities related to these burnt clusters exhibit critical behavior—hence the term self-organized criticality—which will be examined in Section III.
Again, an hysteresis cycle is observed for the SOC-spiral phase transition (Figure 2 bottom left inset). By approaching the region from the SOC regime, one observes the phase transition at $\rho_e^4 = 0.59$, which differs from ρ_e^3 found earlier. Again, the system remembers its history and behaves accordingly with its past. In Figures 1(e) and 1(c) spatial configurations for spiral and SOC phases in the hysteresis cycle are shown. For larger values of the control parameter, the SOC regime is lost due to lack of trees, and no remarkable dynamics are observed.

III. SELF-ORGANIZED CRITICALITY

Self-organized critical systems are characterized by physical quantities or observables that follow scaling laws, with dependencies described by critical exponents, despite the absence of a tuning parameter or traditional phase transition. This concept was firstly defined by Bak, Tang and Wiesenfeld [5] who based their works on the sandpile model, and was further studied by Drossel and Schwabl [9] as well as Grassberger [8] with their own definition of forest fire model (DSFFM). In this model, each lattice site can either be occupied by a tree or be empty. For each time step, there is a probability p that a tree grows on an empty site and a probability $f \ll p$ that a fire starts spontaneously on a tree (lightning). Trees on fire burn down completely the clusters they are connected to. The time scales of the system are very relevant. It is crucial that the following relation is met to ensure criticality [9],

$$f \ll p \ll T^{-1}(s_{\max}), \tag{1}$$

that is, lightning strikes must occur rarely compared with tree growth, and the time it takes for the cluster of largest size $s_{\max}$ to burn must be infinitely small. Without the first condition, large clusters would not form due to recurrent lightnings. The second condition is necessary for the following reason: when a lightning strikes the largest cluster, it needs some time to burn down, and new trees might grow at the edge of this cluster while it is still burning so that the fire is never extinguished [13]. In order to observe critical behavior, p must be chosen so small that even the largest cluster burns down before a tree is grown.
In the DSFFM, burning happens infinitely fast, so the only relevant parameter is $\theta \equiv p/f$ (i.e. how many trees grow between every lightning) and criticality is observed at the limit $\theta \to \infty$ [9]. Our study focuses on the following quantities: the mean number of tree clusters of size s by unit of volume, $n(s)$, the radius of gyration of tree clusters of size s, $R(s)$, and the time it takes for an entire tree cluster of size s to burn down, $T(s)$. The power law scalings of these quantities are defined as follows [13],

$$n(s) \sim s^{-\tau} \quad R(s) \sim s^{1/\mu} \quad T(s) \sim s^{1/\mu'}. \tag{2}$$

The model's dynamics are described by the following rules: Select a random site in the lattice. If it is a tree, burn the entire cluster where it belongs while measuring its size, radius of gyration and burning time. Otherwise, choose θ random sites and grow a tree in all empty sites of those selected and then proceed with the first step [13].

A. Methodology

The DSFFM rules were implemented in a Fortran program. Each measurement was made for a 1024×1024 lattice. Furthermore, we studied the behavior of the variables for values of θ ranging from 125 to 4000, with the objective of studying the critical limit. Due to finite size efects, raw data had a lot of noise for large cluster sizes, s, and we used logarithmic binning to process it [12] (Figure 3). To compute

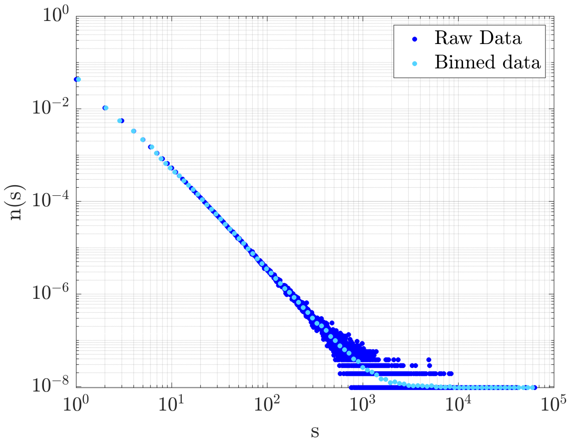


Figure 3: An example of raw and processed data for a measure of $n(s)$.

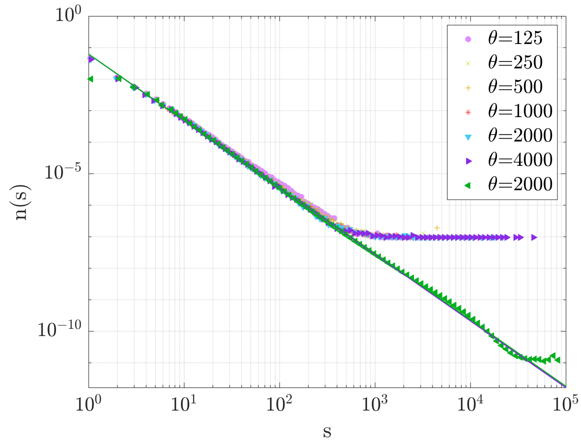


Figure 4: Log-log plot of normalized frequency of forest clusters as a function of cluster size for different values of θ. The potential fit for $\theta = 4000$ is shown. The green line showcases the average over time and its fit for $\theta = 2000$.

critical exponents and its errors we used Microsoft Excel's linear regression tool. To compute $R(s)$ and $n(s)$, we used a recursive algorithm that measured the size of the cluster while simultaneously saving the coordinates of each tree in the cluster to compute $R(s)$ afterwards. For these two measurements, we ran the code for 10 different seeds in order to prevent correlations between consecutive measures. For each seed, we ensured a stationary state by allowing a sufficient relaxation period of 15,000 steps, where each step corresponds to either burning an entire cluster or growing θ trees, and then we measured the size of all tree clusters on the lattice and their respective radius of gyration. To compute the radius of gyration the following definition was used [11]:

$$R(s) = \sqrt{\frac{1}{s}\sum_{i=1}^{s} |r_i - r_{\mathrm{cm}}|^2}; \quad r_{\mathrm{cm}} = \frac{1}{s}\sum_{i=1}^{s} r_i, \tag{3}$$

where r_i is the radial coordinate of the i-th tree of the cluster, defined as $\sqrt{x_i^2 + y_i^2}$ where x and y are the indexes of the discrete matrix.

The measurement of $T(s)$ was made as a time average between multiple measurements on the same lattice, leaving enough time for the system to reach the stationary state and between each mesure to avoid correlations. Statistical data of burning time was taken for $2 \cdot 10^4$ different wildfires. Furthermore, a single computation of $n(s)$ was made using this technique in order to verify that the previous measurements made by averaging over seeds were correct. As we may see in Figure 4, seed averaged values reach a plateau at $s \approx 10^3$ that corresponds to measuring just one cluster of such size, since it is less probable to find a large cluster than a small one. This result is not a property of the system but a lack of statistics. Meanwhile, time averaged evaluation does not show such plateau, but it does deviates from linearity for small s, which seed averaged data do not. Therefore, we found that the two measures complement and corroborate one another.

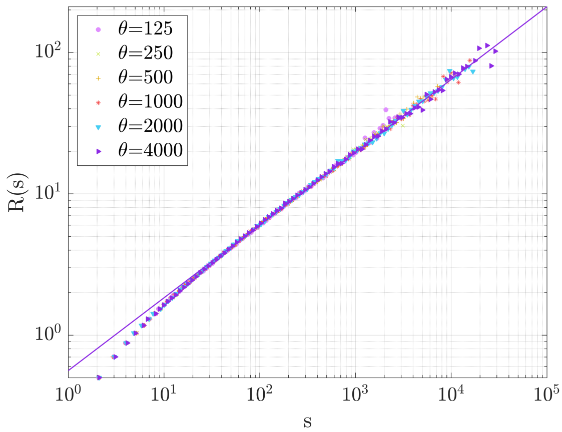


Figure 5: Log-log plot of the radius of gyration as a function of cluster size for different values of θ. The potential fit for $\theta = 4000$ is shown with a line.

B. Results

Figures 4, 5 and 6 show the different results obtained for the previously mentioned variables. We computed every exponent for $\theta = 4000$, except the exponent of time averaged $n(s)$, which was made for $\theta = 2000$. For the exponent of the mean number of tree clusters of size s, we find $\tau = 2.11(1)$ for seed averaged computations and $\tau = 2.10(1)$ for time averaged ones, which are statistically equivalent (Figure 4). Furthermore, both results are statistically compatible with the ones obtained by [10] and [13], $\tau = 2.15(3)$. We obtain the exponent for the radius of gyration $\mu = 1.94(2)$ (Figure 5), which is statistically compatible to the one found by [9] and [10], $\mu = 1.96(1)$ and $\mu = 1.98$ respectively.

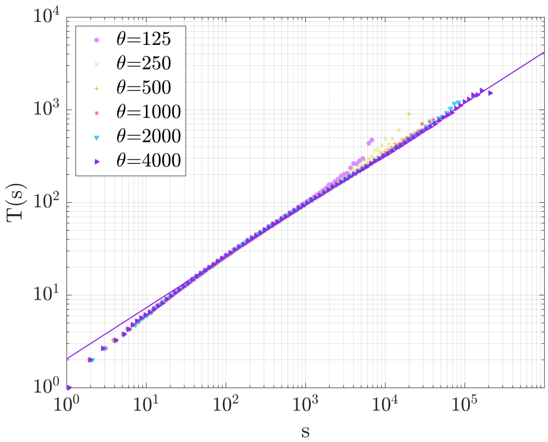


Figure 6: Log-log plot of average burning time for each cluster size. The potential fit for $\theta = 4000$ is shown.

Finally, for the exponent of the average time a fire of size s takes to burn down we obtained $\mu' = 1.81(1)$ (Figure 6), which, albeit close, is not compatible with the one found by [13] $\mu' = 1.89(3)$. We think this discrepancy is due to our interpretation of the variable, since we computed it by measuring how many steps it took the system to have no fires after burning a single tree, whereas its definition is not entirely stated in [13].

Scaling behaviors are clearly seen for all three variables defined, which leads us to think that this SOC forest fire model is critical. However, our simulations are limited by computational power constraints, so we cannot confidently confirm this hypothesis. According to Grassberger [10], these scaling relations are transient; he argues that in the critical limit $\theta \to \infty$, the DSFFM does not exhibit the asymptothic behavior typically associated with critical systems. Instead, it shows effective power-law only within restricted parameter ranges [14]. The DSFFM scaling depends on the scale at which the system is examined; therefore universal scaling laws cannot be assumed. According to [15], the DSFFM appears scale-invariant (i.e. exhibits power-law behavior) for $\theta \lesssim 10^4$, but transitions to a different regime at larger values.

IV. CONCLUSION

We have computationally studied Clar's forest fire model [3], qualitatively arguing each of its phase transitions. Then, we shifted our focus to the SOC system, where we analysed the DSFFM to compute the critical exponents of the radius of gyration, $R(s)$, the normalized number of clusters of size s, $n(s)$ and the average time a cluster of size s takes to burn down, $T(s)$. Results for $R(s)$ and $n(s)$ are statistically compatible with published data [8, 13], but for $T(s)$ not. We argued that our interpretation of $T(s)$ was different from [13]. Furthermore, our results seemed to define scaling behaviors for this SOC model, but we argued that, according to [10, 14, 15], these are transient phases for restricted regions. Even so, the DSFFM is useful to establish a phenomenological understanding of the behavior of real systems [15]. As further work, we suggest studying the efects of an heterogeneous environment such as the presence of rivers or rocks, or defining types of resistant or susceptible trees.

V. ACKNOWLEDGEMENTS

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Transicions de fase i criticitat autoorganitzada en un model d'incendis forestals

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En aquest treball, estudiem la dinàmica i el comportament crític dels models d'incendis forestals, amb un enfocament en el model d'incendi forestal de Drossel-Schwabl (DSFFM). Mitjançant simulacions computacionals, analitzem les transicions de fase en un model de xarxa bidimensional, identificant tres règims principals: una fase mixta, una fase d'ones en espiral i una fase de criticitat autoorganitzada (SOC). Caracteritzem aquestes fases a partir de configuracions espacials i de la relació entre la densitat d'arbres i de focs, revelant transicions discontínues i efectes d'histèresi. En el règim SOC, extraiem els exponents crítics per a la distribució de la mida, el radi de gir i el temps de combustió dels clústers d'arbres, comparant els nostres resultats amb els de la bibliografia. Tot i que les nostres simulacions reprodueixen les lleis d'escala conegudes, també discutim les limitacions del DSFFM, en particular la seva manca d'universalitat en l'escalat a grans dimensions del sistema i en escales temporals llargues. Malgrat aquestes limitacions, el model continua sent una eina útil per explorar fenòmens emergents en sistemes excitables i dinàmiques ecològiques complexes.
Paraules clau: Incendi forestal, criticitat autoorganitzada, escalament, exponents crítics, model d'incendi forestal de Drossel i Schwabl
ODSs: Aquest TFG està relacionat amb els Objectius de Desenvolupament Sostenible (SDGs)

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	X
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	X
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures	X		

El contingut d'aquest TFG part d'un grau de Física contribueix a l'ODS 4 (fita 4.4) promovent l'adquisició de competències tècniques en física computacional i consolidant conceptes de transicions de fase i criticitat, a l'ODS 9 (fita 9.5) mitjançant la recerca científica sobre sistemes complexos, a l'ODS 13 (fita 13.3) millorant la comprensió i sensibilització sobre els incendis forestals en el context del canvi climàtic, i a l'ODS 15 (fites 15.1 i 15.3) aportant coneixement útil per a la conservació i gestió sostenible dels ecosistemes terrestres.

Figure 7: GRAPHICAL ABSTRACT