Steady-State Bifurcation in Nonlinear Opinion Dynamics: Analysis on Canonical Networks

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Abstract: This paper explores how collective opinion patterns emerge through steady-state bifurcations in a nonlinear dynamical system. We study a model that describes the time evolution of opinions in a multi-agent system interacting over a social network. We show how this bifurcation emerges, provided that the attention parameter – which quantifies each agent's social susceptibility – exceeds a threshold value. This threshold is solely characterized by other parameters of the system and by the largest eigenvalue of the interaction matrix. Moreover, we see how, near the threshold, the stationary state is approximately proportional to the eigenvector associated with the largest eigenvalue. We apply this framework to canonical networks, including regular, star, Watts-Strogatz, and scale-free graphs. Finally, we investigate the bifurcation unfolding when the agents in the system hold biased opinions.
Keywords: Nonlinear dynamics, bifurcation theory, complex networks, computational physics.
SDGs: Quality Education, Life Below Water, Life on Land, Partnerships for the Goals.

I. INTRODUCTION

An opinion dynamics model is a mathematical model that describes how the opinions of a multi-agent system evolve over time. This multi-agent system consists of a collection of agents who interact socially with one another, as defined by an interaction network. This network specifies how the opinion of one agent influences the opinions of the other agents in the system. These types of models can be used, for instance, to describe collective decision-making problems, which appear in interdisciplinary studies, such as collective animal behaviour or voting pattern models of human social networks.

In this work, we explore opinion formation through a nonlinear dynamical system proposed in [1]. In this model, the opinion update process is fundamentally nonlinear due to the saturation of information of each agent, and exhibits a steady-state bifurcation of opinions when a system parameter is varied. This is a type of dynamical system broadly explored in epidemiology or in neuronal population models.

We restrict our study to the stationary states of the system. This assumption implies that, the results we derive, are applicable provided that the characteristic time to reach the stationary state is shorter than the timescale over which the interaction network undergoes structural changes.

This study seeks to contribute in the following ways:

1. Explores the model for the case of opinion dynamics on two possible options, including reinterpretations and discussions.
2. Presents the mathematical theorem and associated corollaries which describe the bifurcation of opinions in a precise and instructive way (including a mathematical proof of the main theorem made by the author).
3. Applies the theoretical framework to specific interaction networks–regular, star, Watts-Strogatz and scale-free networks–both theoretically and computationally, and illustrates the resulting dynamics.
4. Studies the characteristic time required to reach the stationary opinion states, using both theoretical and computational approaches.
5. Examines the unfolding of bifurcations in the presence of disrupting opinion hubs.

II. NONLINEAR OPINION DYNAMICS MODEL

Consider a network of $N \in \mathbb{N}$ interacting agents, which is described by a static matrix $\mathbf{\Gamma} = (\gamma_{ij}) \in \mathbb{R}^{N\times N}$, with zero entries on the diagonal. Let $x \in \mathbb{R}$ be an agent's *opinion* on a specific idea related to a topic or issue, which indicates whether the agent *favours* (if $x > 0$), *disfavours* (if $x < 0$), or shows *indecision* (if $x = 0$) towards the idea, and with what *intensity* (expressed by $|x|$). We describe the opinions of the N agents in the system using the network opinion vector $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, where x_i is the opinion of the agent i. In this setting, each γ_{ij} describes the influence of the opinion of agent j on the opinion of agent i. We say that the agent i *cooperates* (*competes*) with agent j if $\gamma_{ij} > 0$ ($\gamma_{ij} < 0$).

Suppose that the agent's opinion evolves over time, $x_i(t)$, according to a nonlinear opinion dynamics model described by

$$\frac{dx_i}{dt} = -dx_i + u\tanh\left(\alpha x_i + \sum_{j\neq i}\gamma_{ij}x_j\right) + b_i. \qquad (1)$$

Let us describe the different parameters introduced in this model:

- The parameter $d \geq 0$ is a *damping coefficient*, with the corresponding linear term reflecting the agent's resistance to changing its opinion. Specifically, resistance inhibits opinion formation, and this inhibition increases as d becomes larger.

- The parameter $u \geq 0$ is an *attention coefficient*, and measures the susceptibility of the agent to social influence (which is captured in the hyperbolic tangent term).
- The parameter $\alpha \geq 0$ is a *self-reinforcing coefficient* that captures the agent's tendency to amplify the intensity of its current opinion.
- The parameter $b_i \in \mathbb{R}$ represents the *bias* or *input coefficient* of agent i, and quantifies the agent's preference towards a particular opinion, independently of the opinions of other agents. The collective bias of the system is captured by the network bias vector $\mathbf{b} = (b_1, \ldots, b_N)$.

The nonlinearity of the system appears by consideration of the hyperbolic tangent term. This function is included in the opinion dynamics model as a saturation function, which prevents the unbounded growth of opinion intensities.

III. UNBIASED OPINION DYNAMICS

Throughout this section, we assume that the agents are unbiased, i.e, $\mathbf{b} = \mathbf{0}$. Observe that favouring or disfavouring the discussed idea in the decision-making problem is therefore equivalent, as there is no preference between the two options. Hence, during this section, the opinion states $\mathbf{x}$ and $-\mathbf{x}$ are considered symmetric.

Under this hypothesis, the *neutral state* of the network, $\mathbf{x} = \mathbf{0}$, is a stationary state, as can be readily verified from equation (1). Let us examine the stability of this stationary state, as well as the emergence of other stationary states through a bifurcation.

A. Bifurcation of Stationary States

Here, we present the main result used in our study, which characterizes the stability of the neutral state and the emergence of non-neutral states through a bifurcation, depending on the attention parameter u. This result enables us to describe the system's behaviour based on the structure of the interaction network. I encourage the reader to consult the proof of this theorem in Appendix A.

Theorem 1. (Bifurcation Theorem) Let λ^* denote the eigenvalue of $\mathbf{\Gamma}$ with the largest real part, and let E^* be the associated eigenspace. Suppose:

1. The eigenvalue λ^* is real.
2. If λ is an eigenvalue of $\mathbf{\Gamma}$ and $\lambda \neq \lambda^*$, then $\mathrm{Re}(\lambda) < \lambda^*$.
3. It is satisfied that $\alpha + \lambda^* > 0$.
4. The largest eigenvalue λ^* is simple, that is, the associated eigenspace is one-dimensional: $E^* = \langle \mathbf{v}^* \rangle$ for some normalized vector $\mathbf{v}^* \in \mathbb{R}^N$.

Let

$$u^* \equiv \frac{d}{\alpha + \lambda^*}, \tag{2}$$

which we will refer to as the *threshold parameter*.

Then, the opinion dynamics system described by (1) satisfies:

a) If $u < u^*$, the neutral state $\mathbf{x} = \mathbf{0}$ is a locally stable stationary state.

b) If $u > u^*$, the neutral state $\mathbf{x} = \mathbf{0}$ is a locally unstable stationary state.

c) For $u = u^*$, the system undergoes a steady-state bifurcation at $(\mathbf{x}, u) = (\mathbf{0}, u^*)$, which leads to the emergence of non-zero equilibrium branches $\mathbf{x}^*(u)$ satisfying:

$$\mathbf{x}^*(u \gtrsim u^*) \approx r\mathbf{v}^*, \tag{3}$$

where $r \in \mathbb{R}$ is some proportionality factor.

This is a significant result in the study of the stationary states of the system. It reveals that nonzero opinions can form even without preferences on the opinions (since $\mathbf{b} = \mathbf{0}$), provided that the attention u is greater than u^*.

Let us first analyse the stability of the neutral state. From this result, we deduce that knowing the largest eigenvalue of the network, together with the coefficients of the model, is sufficient to study the stability of the neutral state. The parameter u^* serves as a threshold for the attention parameter u, separating the values for which the neutral state is locally stable from those for which it becomes locally unstable.

Note that the dependence of u^* on the different parameters, as described in (2), is consistent with the interpretations we have assigned to them. Certainly, if the agents exhibit a strong resistance d to changing their opinions, we expect that a larger value of social susceptibility u is required to observe non-neutral stationary states—that is why the threshold u^* increases with d. In contrast, when considering a strong social influence (represented by a large λ^*) or when agents tend to amplify their opinions (parametrized by a large α), we expect that a smaller value of the social susceptibility u is sufficient to observe non-neutral states—hence, u^* decreases as these parameters increase.

Now, we examine the vector $\mathbf{v}^*$ describing the opinion formation at the bifurcation. Notably, if all components of $\mathbf{v}^*$ share the same sign, then for $u \gtrsim u^*$ the system reaches *agreement* equilibrium solutions. This corresponds to the case where all agents hold opinions of the same sign, which implies that the agents unanimously favour or disfavour the discussed idea, although they may differ on the intensity of their opinions. If the opinion intensities also coincide, the agreement becomes a *consensus*. In contrast, if $\mathbf{v}^*$ has components with different signs, then we refer to *disagreement* equilibrium solutions. In this case, at least one pair of agents hold opinions of opposite sign, indicating opposing views. One may observe from (3) that the relations between the components of $\mathbf{v}^*$ are equal to the relations between the agents' stationary opinions when $u \gtrsim u^*$. This implies that the relative stationary opinions between agents depend only on the interaction matrix. This is the only information we can extract from (3) when describing the stationary solutions of the system near the bifurcation, since the proportionality factor r may, in general, depend on the system parameters, and can vary when considering different initial conditions.

B. Perron-Frobenius Matrices and Agreement States

To apply the Theorem 1, the network must be characterized by matrices $\mathbf{\Gamma}$ which satisfy the four hypotheses we have established. A particular case in which these hypotheses are met is when we consider *Perron-Frobenius (PF) matrices*, that is, non-negative irreducible matrices. The following theorem, a proof of which can be found in [7], summarizes the properties of Perron–Frobenius matrices relevant to our study:

Theorem 2. (Perron-Frobenius Theorem) Let $\mathbf{A}$ be a Perron-Frobenius matrix. Then:

a) There exists a positive eigenvalue of $\mathbf{A}$ which is greater than or equal to the module of any other eigenvalue. This eigenvalue is referred to as the *Perron-Frobenius eigenvalue*, denoted by $\lambda_{PF} > 0$.
b) The PF eigenvalue is simple, and its associated normalized eigenvector is the *Perron-Frobenius eigenvector*, denoted by $\mathbf{v}_{PF} \in \mathbb{R}^N$.
c) The PF eigenvector has only strictly positive entries.

Combining the two previous theorems, we obtain the following result:

Corollary 3. (Agreement States) Let $\mathbf{\Gamma} = \gamma \mathbf{A}$, where $\gamma > 0$ is a positive scalar and $\mathbf{A}$ is a Perron-Frobenius matrix with PF eigenvalue λ_{PF} and PF eigenvector $\mathbf{v}_{PF}$.

Then, the hypotheses assumed in Theorem 1 are satisfied. Moreover, the threshold parameter u^* can be expressed as

$$u^* = \frac{d}{\alpha + \gamma \lambda_{PF}}. \tag{4}$$

Additionally, the eigenvector $\mathbf{v}^*$ which determines the bifurcation branches in (3), coincides with $\mathbf{v}_{PF}$, whose entries are all strictly positive, and thereby describing non-trivial agreement solutions.

The described situation can be summarized as follows: in a *cooperative network* (in this setting, characterized by $\gamma > 0$), non-trivial agreement is achieved for attention parameters u slightly above the threshold u^*, regardless of the initial conditions of the system. Note that for $u \gg u^*$, the system generally exhibits disagreement opinion states and becomes highly sensitive to initial conditions due to its intrinsic nonlinearity. This has interesting implications when applying the model to real-world scenarios, as it suggests that, in order for the system to reach agreement, the attention parameter must be above the threshold, but still small enough for the system to remain in the regime where relation (3) holds. The study of this regime of solutions is left for future work.

IV. APPLICATION TO PARTICULAR GRAPHS

In this section, we study undirected graphs with Perron–Frobenius adjacency matrices. Throughout the illustrations in this section (generated with Python) we fix the parameters $\gamma = 0.1$, $\alpha = 1$, and $d = 2$ to compare the opinion dynamics across different networks. The qualitative behaviour of the illustrations remains the same for different choices of parameters, but with a different threshold u^*.

Further details of the studied networks are provided in Appendix C, and the fits to the computationally obtained graphs are included in Appendix E.

A. Regular Graphs

Our study substantially simplifies for regular graphs, since the Perron-Frobenius eigenvalue equals the degree of the graph, $\lambda_{PF} = k$, and the Perron-Frobenius normalized eigenvalue has identical entries (see §3.3 of [4]). Therefore:

$$u^* = \frac{d}{\alpha + \gamma k}, \qquad \mathbf{v}^* = \frac{1}{\sqrt{N}}(1, \ldots, 1). \tag{5}$$

From these expressions, we see how as the network becomes more connected, a lower u is required to reach non-neutral consensus solutions close to the bifurcation. Note that the threshold does not depend on the size of the network N.

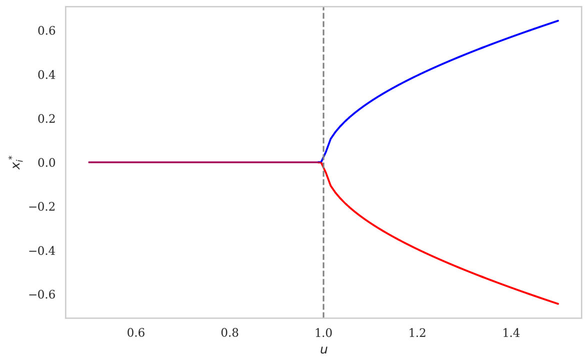

FIG. 1: Bifurcation diagram for the model (1) in the case of a regular graph with $N = 100$ and $k = 10$. We plot the stationary opinion of an arbitrary agent, x_i^*, as a function of u near the threshold $u^* = 1$.

In the case of a regular graph, the stationary opinion of any arbitrary agent is representative of the behaviour of the entire system, in accordance with the agreement solutions described by (5) for $u \gtrsim u^*$. Depending on initial conditions, and near the threshold value, the system will reach a situation where all the agents' opinions are in the positive branch or all in the negative branch, as illustrated in FIG. 1. We can also compute the characteristic time of the system to reach a stationary state, before and after the threshold, as illustrated in FIG. 2.

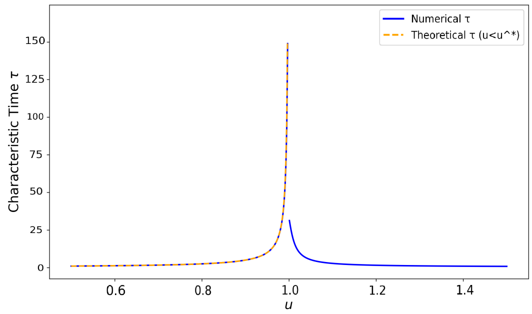

FIG. 2: Characteristic time τ for the system to reach the stationary state with respect to the attention parameter u, for the same network as in FIG. 1. The threshold corresponds to $u^* = 1$. For $u < u^*$, the stationary state is the neutral state, and the characteristic time diverges as $u \to u^*$, according to the theoretical expression $\tau = \frac{1}{d(1-u/u^*)}$ deduced in Appendix B. For $u > u^*$, characteristic time is computed numerically according to the expression (B3). This kind of behaviour is also observed in the rest of the studied networks.

B. Star Graphs

A star graph with N nodes has a single central hub (which we designate as the agent with $i = 0$), and $n = N - 1$ peripheral agents, each connected only to the hub. For this graph, it is satisfied that $\lambda_{PF} = \sqrt{n}$ and (see for example §1.2 in [6]):

$$u^* = \frac{d}{\alpha + \gamma\sqrt{n}}, \quad \mathbf{v}^* = \frac{1}{\sqrt{2n}}(\sqrt{n}, 1, \ldots, 1) \tag{6}$$

Observe how the threshold parameter decreases with n, meaning that for low social susceptibility, non-neutral agreement can be reached for large n. Also, note that for $u \gtrsim u^*$, the ratio between the stationary opinions of the hub and peripheral agents, $x_0^*/x_i^* = \sqrt{n}$, increases with n. This reflects the fact that the hub is socially influenced by a larger number of cooperative agents for increasing n. Observe that this does not imply that the peripheral opinions decrease with n, since such an analysis requires considering the proportionality factor r in (3), which increases with N and u (close to bifurcation) for cooperative networks.

Similar results hold when we consider a wheel graph, as explored in Appendix D.

C. Watts-Strogatz Graphs

For a Watts-Strogatz (WS) graph, we can compute the dependence of the average threshold on the rewiring probability, $\langle u^* \rangle(p)$, for a fixed N, as illustrated in FIG. 3. The average PF eigenvector for a WS graph equals the PF eigenvector of the regular ring lattice (5), which implies that, on average, consensus is reached near the threshold.

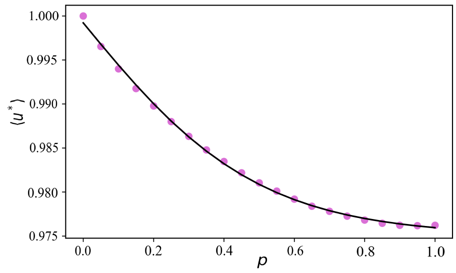

FIG. 3: Average threshold parameter $\langle u^* \rangle$ as a function of the rewiring probability p in a Watts-Strogatz graph with $N = 100$ and underlying ring lattice with $k_0 = 10$. Each average is computed from the largest eigenvalue of the WS graph over 10,000 repetitions.

D. Scale-Free Graphs

Let us consider a scale-free graph with a degree distribution $P(k) \sim k^{-3}$ and a connection $m = 2$. As shown computationally in FIG. 4, the average threshold decreases with N, and the expected Perron–Frobenius eigenvalue exhibits a power-law behaviour $\langle \lambda_{\mathrm{PF}} \rangle \sim N^{0.24}$, which is in agreement with the expected theoretical behaviour $\langle \lambda_{PF} \rangle \sim N^{1/4}$. On the other hand, in the BA graph, the largest components of the average PF eigenvector correspond to the indices of the network's hubs. This implies that, in stationary states near the threshold, the opinions of the hubs in the agreement state are larger than those of the other agents.

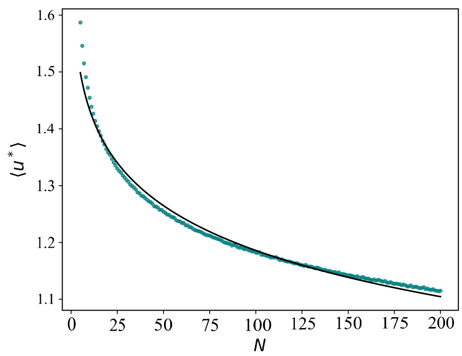

FIG. 4: Average threshold parameter $\langle u^* \rangle$ as a function of N in a scale-free graph. Each average is computed from the largest eigenvalue of the scale-free graph over 1000 repetitions.

V. OPINION UNFOLDING IN THE PRESENCE OF INPUT

In this section, we examine how the introduction of a nonzero bias vector $\mathbf{b}$ in the opinion dynamics can lead the bifurcation of stationary states to evolve into a situation where only one of the previously symmetric equilibrium branches—$\mathbf{x}^*(u)$ or its negative counterpart, $-\mathbf{x}^*(u)$—remains stable. We also see that this selection, which emerges as a response to the distributed inputs, depends only on the network structure.

The fundamental result which describes this situation is a generalization of Corollary 3, which follows from Theorems IV.1 and IV.2 in [2].

Theorem 4. (Bifurcation Unfolding) Let $\mathbf{\Gamma} = \gamma\mathbf{A}$, where $\gamma > 0$ is a positive scalar and $\mathbf{A}$ is a Perron-Frobenius matrix with PF eigenvalue λ_{PF} and PF eigenvector $\mathbf{v}_{PF}$. Let $\mathbf{w}_{PF}$ be the PF eigenvector of the transpose $\mathbf{A^T}$ and let $\mathbf{b}$ be the network bias vector. Then:

a) If $\langle \mathbf{b}, \mathbf{w}_{PF}\rangle = 0$, then model (1) undergoes a supercritical pitchfork bifurcation for $u = u^* = \frac{d}{\alpha+\gamma\lambda_{PF}}$, as described in Corollary 3.

b) If $\langle \mathbf{b}, \mathbf{w}_{PF}\rangle \neq 0$, then the pitchfork unfolds in the direction given by $\langle \mathbf{b}, \mathbf{w}_{PF}\rangle$: if $\langle \mathbf{b}, \mathbf{w}_{PF}\rangle > 0\,(< 0)$, then the only stable equilibrium $\mathbf{x}^*$ which appears at the bifurcation described in Corollary 3 for $u \gtrsim u^*$ satisfies $\langle \mathbf{x}^*, \mathbf{v}_{PF}\rangle > 0\,(< 0)$.

A. Disrupting Opinion Hubs

As an illustrative example of the application of Theorem 4, we study the effect of a hub's input on the stationary opinion of the network. Let us consider the star graph on $n+1$ nodes. In this case, since the graph is undirected, we have that $\mathbf{w}_{PF} = \mathbf{v}_{PF}$ is given by expression (6). Note that the hub's opinion is the most significant in the bifurcation unfolding, since $(\mathbf{v}_{PF})_0$ is the largest entry of the vector.

We suppose that the hub and the peripheral agents have opposing biases: $\mathbf{b} = (b_0, -b, \ldots, -b)$ for some positive real values b_0 and b. It is satisfied that $\langle \mathbf{b}, \mathbf{v}_{PF}\rangle = \frac{1}{\sqrt{2}}(b_0 - b\sqrt{n})$ and, for $u \gtrsim u^*$, $\langle \mathbf{x}^*, \mathbf{v}_{PF}\rangle = r$ according to (3). Then, if $a_0 \equiv \left(\frac{b_0}{b}\right)^2$, the hub's option is adopted in the collective agreement opinion provided that $n < a_0$ (case $r > 0$), whereas the peripheral option is selected for $n > a_0$ (case $r < 0$). The case $n = a_0$ corresponds to the bifurcation analysis, and the resulting opinion depends on initial conditions.

A similar behaviour is observed in the wheel graph, as discussed in Appendix D.

VI. CONCLUSIONS

- Under certain regularity conditions, the model exhibits a steady-state bifurcation of opinions at the neutral state and for a critical value of the attention parameter. This threshold only depends on the model parameters and on the structure of the interaction network. The non-neutral bifurcation branches are proportional to the largest eigenvalue of the network.
- For a Perron-Frobenius interaction network, the bifurcation branches correspond to stationary agreement states. For attention parameters beyond the threshold, the system may exhibit disagreement states.
- The characteristic time to reach a stationary state increases as the attention parameter approaches the threshold. In particular, the time to reach the neutral state diverges at the threshold.
- The average threshold parameter decreases as the rewiring probability increases in a Watts-Strogatz graph. Thus, adding randomness to a regular graph at the neutral state can trigger agreement states for attention parameters slightly below the threshold.
- In the presence of bias, the bifurcation unfolds toward one of the symmetric stationary states. The direction of this unfolding depends solely on the bias vector and the interaction network.

Acknowledgments

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Bifurcacions en Estat Estacionari en Dinàmica No-lineal d'Opinions: Anàlisi en Xarxes Canòniques

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Resum: Aquest article explora com emergeixen patrons col·lectius d'opinió a través de bifurcacions en l'estat estacionari dins d'un sistema dinàmic no lineal. Estudiem un model que descriu l'evolució temporal de les opinions en un sistema multiagent que interacciona a través d'una xarxa social. Mostrem com apareix aquesta bifurcació, sempre que el paràmetre d'atenció – que quantifica la susceptibilitat social de cada agent – superi un valor llindar. Aquest llindar queda determinat exclusivament per altres paràmetres del sistema i pel valor propi més gran de la matriu d'interacció. A més, veiem com, a prop del llindar, l'estat estacionari és aproximadament proporcional al vector propi associat al valor propi més gran. Apliquem aquest marc d'estudi a xarxes canòniques, incloent-hi grafs regulars, estrella, Watts-Strogatz i Barabási-Albert. Finalment, estudiem el desplegament de la bifurcació quan els agents del sistema tenen opinions esbiaixades.
Paraules clau: Dinàmica no lineal, teoria de bifurcacions, xarxes complexes, física computacional.
ODSs: Educació de qualitat, Vida submarina, Vida terrestre, Aliança pels objectius

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	X
6. Aigua neta i sanejament		15. Vida terrestre	X
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	X
9. Indústria, innovació, infraestructures			

Aquest TFG es relaciona amb l'ODS Educació de qualitat, ja que té un caràcter didàctic i promou la investigació científica.

Aquest TFG es relaciona amb els ODS Vida submarina i Vida terrestre, ja que té aplicacions en l'estudi de la dinàmica d'opinió de col·lectius d'animals socials com insectes o bancs de peixos.

Aquest TFG es relaciona amb l'ODS Aliança pels objectius pel seu caràcter col·laboratiu i per les seves aplicacions en l'estudi de la presa de decisions col·lectiva.

Appendix A: Proof of the Bifurcation Theorem

Here, we present a detailed development of the main result of the project.

1. Stability of the Neutral State

Let $\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$ be the autonomous system of ordinary differential equations modelling the opinion dynamics, where $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth function implicitly defined by (1). Note that $\mathbf{F}$ also depends on the system parameters, although this dependence is not made explicit here. The stationary states of the system correspond to the equilibrium points of $\mathbf{F}$, that is, the points $\mathbf{x}^*$ such that $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$.

To examine the stability of the neutral state, we analyse the eigenvalues and eigenvectors of the differential of $\mathbf{F}$ at $\mathbf{x} = \mathbf{0}$, which takes the simple expression:

$$\mathbf{J} \equiv \mathbf{DF}(\mathbf{x} = \mathbf{0}) = (-d + u\alpha)\mathbf{Id} + u\mathbf{\Gamma}, \tag{A1}$$

where $\mathbf{Id} \in \mathbb{R}^{N\times N}$ is the identity matrix. This expression follows directly from (1), noting that the nonlinear terms of $\mathbf{F}$ vanish at $\mathbf{x} = \mathbf{0}$.

The eigenvectors of the interaction matrix $\mathbf{\Gamma}$ are precisely those of $\mathbf{J}$, and if $\mathbf{v}$ denotes an eigenvector of $\mathbf{\Gamma}$ with eigenvalue λ, then the corresponding eigenvalue for $\mathbf{J}$ is $-d + u(\alpha + \lambda)$. Let λ^* denote the eigenvalue of $\mathbf{\Gamma}$ with the largest real part, and let E^* be the associated eigenspace. Let us make some hypotheses:

1. The eigenvalue λ^* is real.
2. If λ is an eigenvalue of $\mathbf{\Gamma}$ and $\lambda \neq \lambda^*$, then $\mathrm{Re}(\lambda) < \lambda^*$.
3. It is satisfied that $\alpha + \lambda^* > 0$.

Let us consider:

$$u^* = \frac{d}{\alpha + \lambda^*}, \tag{A2}$$

which we will refer to as the threshold parameter.

The following discussion provides insight into the origin of the name of u^* by describing the stability of the neutral state depending on the attention parameter:

- If $u < u^*$, then we have $\mathrm{Re}(-d + u(\alpha + \lambda)) \leq -d + u(\alpha + \lambda^*) < 0$ for each eigenvalue λ of $\mathbf{\Gamma}$. This implies that the real part of all the eigenvectors of $\mathbf{J}$ is less than zero. Then, by Hartman–Grobman theorem, the neutral state $\mathbf{x} = \mathbf{0}$ is a local attractor for the dynamics governed by $\mathbf{J}$, and corresponds to a locally stable stationary state.
- If $u > u^*$, then the inequality $-d + u(\alpha + \lambda^*) > 0$ holds. This implies that the eigenspace E^* corresponds to locally repelling directions in the dynamics governed by $\mathbf{J}$. Consequently, the neutral state $\mathbf{x} = \mathbf{0}$ is a locally unstable stationary state, again by Hartman–Grobman theorem.
- If $u = u^*$, then $-d + u(\alpha + \lambda^*) = 0$, which implies that the eigenspace E^* corresponds to the eigenvalue 0 of $\mathbf{J}$. In this case, note that E^* contains all eigenvectors of $\mathbf{J}$ whose associated eigenvalues have zero real part, that is, all eigenvectors associated to eigenvalues on the imaginary axis.

2. Bifurcation of Stationary States

To further analyse the case $u = u^*$, we introduce one last hypothesis:

4. The largest eigenvalue λ^* is simple, that is, the associated eigenspace is one-dimensional: $E^* = \langle \mathbf{v}^* \rangle$ for some normalized vector $\mathbf{v}^* \in \mathbb{R}^N$.

In this case, $\mathbf{v}^*$ is the unique normalized eigenvector of $\mathbf{J}$ with zero eigenvalue, and also the unique one in the imaginary axis. These are precisely the conditions that the system must satisfy for a steady-state bifurcation to occur at $(\mathbf{x}, u) = (\mathbf{0}, u^*)$: for u in a neighbourhood of u^*, non-zero equilibrium branches $\mathbf{x}^*(u)$ emerge, satisfying

$$\mathbf{F}(\mathbf{x}^*(u), u) = \mathbf{0} \tag{A3}$$

throughout this neighbourhood (for more details, see §1.2 in [3]).

Observe that since the neutral solution $\mathbf{x} = \mathbf{0}$ is locally stable for $u < u^*$ and locally unstable for $u > u^*$, the non-zero equilibrium branches exist only for $u > u^*$.

Further analysis using bifurcation theory can be carried out to describe the non-zero equilibrium solutions which arise at the bifurcation point. Specifically, by the Liapunov-Schmidt reduction, there exists a bijective correspondence between the equilibrium branches $\mathbf{x}^*(u)$ which satisfy (A3) in a neighbourhood of u^*, and the solutions near this same neighbourhood of the system

$$\mathbf{f}(\mathbf{x}, u) = \mathbf{0}, \tag{A4}$$

where $\mathbf{f} : \ker \mathbf{J}(u = u^*) \times \mathbb{R} \to \ker \mathbf{J}(u = u^*)$ is a smooth function related to $\mathbf{F}$ (refer to §1.3 in [3] for further discussion).

The key result obtained from this reduction is that since $\ker \mathbf{J}(u = u^*) = \langle \mathbf{v}^* \rangle$, the equilibrium branches are proportional to $\mathbf{v}^*$ near the bifurcation. This completes the proof of Theorem 1.

Appendix B: Characteristic Time

Since we have specified the locally stable stationary states near the threshold in Section III, we may find its characteristic times.

Consider $\mathbf{F}(\mathbf{x})$ for $\mathbf{x} \approx \mathbf{x}^*$ where $\mathbf{x}^*$ is locally stable equilibrium point. Then, we can approximate $\mathbf{F}$ to first order around $\mathbf{x}^*$ as follows:

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{DF}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*). \tag{B1}$$

Suppose that $\mathbf{X} = \mathbf{DF}(\mathbf{x}^*)$ diagonalizes. Then, if $\mathbf{C}$ is a matrix whose columns are the eigenvectors of $\mathbf{X}$, and $\mathbf{D} = \mathbf{diag}(d_1, \dots d_N)$ the diagonal matrix with the corresponding eigenvectors in the diagonal entries, we have that $\mathbf{D} = \mathbf{C}^{-1}\mathbf{XC}$. If we introduce the change of variables $\tilde{\mathbf{x}} = \mathbf{X}^{-1}(\mathbf{x} - \mathbf{x}^*)$, the system $\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \approx \mathbf{X}(\mathbf{x} - \mathbf{x}^*)$ transforms to

$$\frac{d\tilde{\mathbf{x}}}{dt} \approx \mathbf{D}\tilde{\mathbf{x}} \tag{B2}$$

as can be easily verified. This system has the trivial solutions $\tilde{x}_i(t) \approx \tilde{x}_i(0)\exp(d_i t)$, where note $\mathrm{Re}(d_i) < 0$ since $\mathbf{x}^*$ is locally stable. Then, we define the characteristic times $\tau_i \equiv \frac{1}{|d_i|}$, and the dominant characteristic time corresponds to the largest one:

$$\tau = \frac{1}{\min_{1 \le i \le N} |d_i|}. \tag{B3}$$

Particularly, for $u < u^*$, we can express the dominant characteristic time of the neutral state $\mathbf{x} = \mathbf{0}$ as a function of the model parameters. In this case, by $(A1)$ we have that the dominant characteristic time is

$$\tau = \frac{1}{d - u(\alpha + \lambda^*)} = \frac{1}{d(1 - u/u^*)}, \tag{B4}$$

provided that $\mathbf{\Gamma}$ diagonalizes. Observe that τ increases with u, until we reach the bifurcation at $u = u^*$ where $\tau \to \infty$.

Appendix C: Studied Networks

In this appendix, we briefly describe the key features of the matrices studied in Section IV.

Regular Graphs: The first studied network is the one described by a regular graph. In a regular graph, each agent is connected to the same number of neighbours, i.e, has the same degree k. Equivalently, each row sum of $\mathbf{A}$ equals k. Note that if E is the total number of connections between agents in the graph, by the Handshaking lemma we have the relation $kN = 2E$, which implies that for an odd N the degree k must be even.

Some examples of networks defined by regular graphs include the fully connected network, characterized by $k = N - 1$, or a ring lattice, where each agent has links to $k/2$ subsequent neighbours and, by the cyclic structure, also to $k/2$ preceding neighbours.

Star Graph: Other relevant types of non-random graphs are not regular. Instead, some nodes accumulate significantly more connections than others. These highly connected nodes are commonly referred to as hubs. A star graph has one central hub, and the rest of the agents, named peripheral agents, are connected to it. Graphs which present hub structures can be used to describe real-world scenarios in which certain agents attract the attention of the others. For instance, in animal behaviour, this may occur when one individual behaves differently from the rest of its neighbours.

Watts-Strogatz graphs: Watts-Strogatz (WS) random graphs are small-world graphs generated by first constructing a regular ring lattice of degree k_0 and then randomly rewiring the edges with a certain probability p. It is a random graph, which is a family of graphs that in particular satisfy the inequality

$$\langle k \rangle \le \lambda_{PF} \le k_{max} \tag{C1}$$

where $\langle k \rangle$ is the expected value of the degree and k_{max} the largest degree (a proof of this inequality can be found in [5]).

The Watts-Strogatz graphs are constructed maintaining $\langle k \rangle = k_0$, so by (C1) this gives us a lower bound of λ_{PF} independently of p. However, as p increases, the degree variability increases, and the maximum degree k_{max} tends to increase on average. Thus, one can expect λ_{PF} to increase with p on average. The degree variability also increases with N, which implies that the average λ_{PF} also increases with N.

Scale-Free graph: An important random graph for network science, widely used to describe social networks, is the scale-free graph, also known as Barabási-Albert (BA) graph. This graph is characterized by a small number of high-degree nodes (hubs), while most nodes have a low degree. These networks are constructed by first considering a small number of nodes, and then adding one by one new nodes with a certain number of connections m, which preferentially connect with already existing high degree nodes. The degree distribution of the scale-free graph follows a power law: $P(k) \sim k^{-\sigma}$, where σ is a power-law exponent, which in real-world networks lies in the range $2 \le \sigma \le 3$.

Scale-free networks are generated by two mechanisms, namely, growth and preferential attachment. Specifically, these networks are constructed by first considering a small number of nodes, and then adding one by one new nodes with a certain number of connections m. The newly added nodes preferentially connect with already existing high degree nodes i, with a probability $p_i = \frac{k_i}{\sum_j k_j}$.

For a scale-free graph with sufficiently large N, the Perron–Frobenius eigenvalue asymptotically behaves as:

$$\lambda_{PF} \sim \begin{cases} \frac{\langle k^2 \rangle}{\langle k \rangle}, & \text{if } 2 \le \sigma < 2.5. \\ \sqrt{k_{max}}, & \text{if } 2.5 < \sigma \le 3, \end{cases} \tag{C2}$$

Additionally, in the considered range of values for σ, the maximum degree behaves as $k_{max} \sim N^{1/2}$ (see [5] for a detailed study of these behaviours).

Appendix D: Wheel Graph

When we consider a wheel graph of N agents, we assume that the $n = N - 1$ peripheral agents described in the star graph are also connected to their first neighbours, as in a ring lattice. In this case, we have (see for example §5.6 in [4])):

$$\lambda_{PF} = 1 + \sqrt{n+1} \tag{D1}$$

$$u^* = \frac{d}{\alpha + \gamma(1 + \sqrt{n+1})}, \tag{D2}$$

$$\mathbf{v}^* = K(n)(\frac{n}{\lambda_{PF}(n)}, 1, \ldots, 1), \tag{D3}$$

$$K(n) = \frac{\lambda_{PF}(n)}{n}\sqrt{\frac{n}{n + \lambda_{PF}^2(n)}}. \tag{D4}$$

The qualitative analysis of the system's behaviour is equivalent to that performed for the star graph. However, in this case, the ratio between stationary opinions is smaller: $x_0^*/x_i^* = \frac{n}{\lambda_{PF}(n)} < \frac{n}{\sqrt{n}} = \sqrt{n}$, reflecting the fact that the peripheral agents are now also socially influenced by their neighbours.

Concerning the discussion in the subsection Disrupting Opinion Hubs regarding the introduction of bias into the model, for the case of a wheel graph on $n+1$ nodes, the behaviour is very similar to that of the star graph discussed in the section, but with $a_0 \equiv \left(\frac{b_0}{b} - 1\right)^2 - 1$. This expression is obtained directly from (D3), using the same reasoning as for the star graph. Observe that this value of a_0 is smaller than in the star graph, reflecting the fact that in the wheel graph, the peripheral agents receive opinion feedback from their neighbours.

Appendix E: Curve Fitting of Computational Data

Here, we present the fits to the computationally obtained graphs.

FIG. 3: The fitted curve $\langle u^* \rangle(p)$ in the FIG. 3 corresponds to a sigmoid model for the PF eigenvalue of the WS graph:

$$u^* = \frac{2}{1 + \lambda_{PF}(p)/10} \tag{E1}$$

$$\lambda_{PF}(p) = \frac{0.992}{1 + \exp\big(-3.932p\big)} + 9.519, \tag{E2}$$

with $R^2 = 0.9986$ and RMSE $= 0.0003$.

FIG. 4: The fitted curve $\langle u^* \rangle(N)$ in the FIG. 4 corresponds to a power-law model for the PF eigenvalue of the scale-free graph:

$$u^* = \frac{2}{1 + 2.274N^{0.24}/10}, \tag{E3}$$

with $R^2 = 0.9837$ and RMSE $= 0.0117$.