On the Contextuality of Multi-Agent Quantum Paradoxes via Anomalous Weak Values

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In this thesis we investigate the role of quantum contextuality in multi-agent paradoxes by tracing the emergence of anomalous weak values (AWVs). Starting from the logical pre- and post-selection (LPPS) formulation of the Hardy paradox, we construct an explicit one-way LOCC protocol that reproduces its statistics and provides a proof of Kochen–Specker (KS) contextuality. We then embed this construction into a particular extended Wigner's friend argument known as Local Friendliness (LF). In a coarse-grained model where two friends perform joint two-qubit measurements, we show that the LF assumptions lead to a logical contradiction identical to that of the LPPS Hardy paradox, thereby providing a proof of the LF no-go theorem. We further develop two extensions of this model in which the intermediate measurements are implemented as weak interactions. In both cases, the anomalous weak value of −1 is preserved, confirming its robustness as a witness of contextuality in this setting.

A complementary fine-grained decomposition resolves each joint measurement into our LOCC-based construction. Although the same logical contradiction is recovered under the LF assumption, the weak measurement schemes in this decomposition no longer exhibit AWVs. This reveals an apparent tension between two seemingly equivalent descriptions, suggesting that the presence of AWVs may depend sensitively on how multi-agent scenarios are modeled.

Keywords: Quantum Contextuality, Hardy Paradox, Logical Pre- Post-Selection Paradoxes, Anomalous Weak Values, Extended Wigner's Friend Scenarios, Local Friendliness no-go Theorem, One-Way LOCC Channels.

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Contents

1 Introduction

Quantum mechanics defies the classical intuition that physical properties exist independently of the measurements that reveal them. The Kochen–Specker theorem [KS67] proves that, whenever three or more measurements can be measured jointly, no assignment of values independent of the set formed these compatible measurements is possible. Thereby it establishes quantum contextuality as a fundamental feature of nature.

A particularly sensitive probe of this non-classical phenomenon is provided by anomalous weak values. These arise from weak measurements, measurements that minimally disturb the quantum state, and take values outside the eigenvalue spectrum of the measured observable. Recent theorems have shown that every anomalous weak value is, in a precise sense, a witness of contextuality [Pus14].

Recently, attention has turned to multi-agent paradoxes such as the Frauchiger–Renner paradox [FR18] and the Local Friendliness argument [BUAG$^+$20]. These scenarios extend the original Wigner's Friend experiment, in which an observer (the friend) performs a measurement on a quantum system inside a sealed laboratory, while an external observer (Wigner) treats the entire lab, including the friend, as a quantum system evolving unitarily; leading to paradoxical conclusions. The recent extensions aim to incorporate the structure of well-known quantum paradoxes into multiple observers, each with potentially incompatible descriptions of a single quantum process, scenarios. Several recent works [WYWS24, NV25, WC25a, WBPW24, Mon23] have pointed out that some form of contextuality appears to play a central role in the emergence of these multi-agent paradoxes.

This thesis is in line with this research, proposes to investigate the relation between contextuality and multi-agent paradoxes via manifestation of anomalous weak values in the latest. We begin by reviewing the notion of contextuality and its formalization via the Kochen–Specker theorem. We then introduce the Hardy paradox and its logical pre- and post-selection (LPPS) variant, in which anomalous weak values arise as operational signatures of contextuality. Building on this, we extend the analysis of the LPPS Hardy paradox to Wigner's-friend-type scenarios, in particular, to the Local Friendliness argument.

2 Quantum Contextuality

In the seminal paper entitled "Can Quantum Mechanical Description of Physical Reality Be Considered Complete?" [EPR35], the authors define *elements of physical reality* as the predefined value that any physical observable possesses upon measurement. In this line of thought, randomness in quantum mechanics is merely a reflection of the incomplete description of reality that the theory addresses. A *Hidden Variable Theory* (HVT), therefore, is a mathematical model that seeks to explain the probabilistic nature of quantum mechanics by introducing additional, possibly inaccessible, *hidden variables*.

Experimental violations of Bell-type inequalities [CHSH69, AGR82] have shown that the predictions of quantum mechanics cannot be explained by any local HVT [Bel64]. These violations demonstrate that the principle of locality, the idea that operations performed on one particle cannot influence another spatially separated particle, is incompatible with quantum theory. In the present chapter, we will see that the probabilistic predictions of quantum mechanics cannot be explained by means of *nonlocal* HVTs, but also by *contextual*

HVTs.

2.1 Contextuality in a nutshell

In quantum mechanics, we mathematically model measurements as Hermitian operators, which we name *observables*. If no degeneracies are present, these operators define a unique basis where each element has an associated (eigen)value that we call the *measurement outcomes*. Therefore, in the case of commuting observables, multiple simultaneous measurements can be performed on the system. This notion allows us to define a *context* of a measurement. Suppose that we have observables A, B and C such that $[A, B] = 0$ and $[A, C] = 0$, but $[B, C] \neq 0$. Then, one could measure A alone, A and B or A and C simultaneously; this scenario defines three possible measurement contexts.

If we are given two commuting observables A and B, not only are we allowed to measure both of them simultaneously but any function of them, $f(A, B)$. Hence, if we prepare a system in state $|\psi\rangle$ such that $A|\psi\rangle = a|\psi\rangle$ and $B|\psi\rangle = b|\psi\rangle$, then $f(A, B)|\psi\rangle = f(a, b)|\psi\rangle$[1]. Then, motivated by this fact and guided by classical intuition, we may state the following postulate [per02]:

Postulate I: Unperformed measurements have pre-existing context-independent results

Even if $|\psi\rangle$ is not an eigenstate of the commuting operators A, B and $f(A, B)$, and even if these operators are not actually measured, one may still assume that the measurement outcomes (if these measurements were performed) would satisfy the same functional relationship as the operators.

This postulate asserts that quantum-mechanical observables have definite values at any given time, and that the values of these variables are intrinsic and independent of the device used to measure them. Moreover, in this line of thinking, we are free to reason about unperformed measurements, as they are assumed to carry a definite outcome even if they are not actually measured. This corresponds to the so-called *counterfactual thinking*[2], which in this context refers to make inferences from unperformed measurements.

However, the above assumptions lead to a contradiction in the framework of quantum mechanics, as shows the Peres-Mermin square [Mer90, Mer93, Per90, Per91, Per92]. Let us consider two spin-1/2 particles in any physical state and define a square array composed by nine measurements,

$$\begin{bmatrix} \mathbb{1} \otimes \sigma_z & \sigma_z \otimes \mathbb{1} & \sigma_z \otimes \sigma_z \\ \sigma_x \otimes \mathbb{1} & \mathbb{1} \otimes \sigma_x & \sigma_x \otimes \sigma_x \\ \sigma_x \otimes \sigma_z & \sigma_z \otimes \sigma_x & \sigma_y \otimes \sigma_y \end{bmatrix}. \tag{1}$$

Each measurement has two possible outcomes: ± 1, and the three measurements in each row and column define a context. Moreover, given any of these six contexts, we can

[1]This property also holds for noncommuting observables only sharing the eigenstate $|\psi\rangle$.

[2]An example: "If I had left home five minutes earlier, I would have caught the bus". This sentence asks about an event that did not actually happen—leaving home earlier—and speculates on its outcome (catching the bus).

build each individual measurement from the product of the two others, except for the third column context as

$$(\sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x) = -\sigma_y \otimes \sigma_y \tag{2a}$$
$$(\sigma_x \otimes \sigma_z)(\sigma_z \otimes \sigma_x) = \sigma_y \otimes \sigma_y \tag{2b}$$

If we insist on the classical intuition that Postulate I poses,

a) Each of the nine measurements has a definite outcome, regardless of which context the measurement is contained in.

b) Predefined outcomes follow the same functional relationship of their associated operators.

Then, if we define the value-assignment function $\nu(\cdot)$, mapping observables with their predefined measurement outcome,

$$\begin{aligned} \nu(\sigma_z \otimes \sigma_z)\nu(\sigma_x \otimes \sigma_x) &\overset{b)}{=} \nu((\sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x)) \overset{(2a),b)}{=} -\nu(\sigma_y \otimes \sigma_y) \\ \nu(\sigma_x \otimes \sigma_z)\nu(\sigma_z \otimes \sigma_x) &\overset{b)}{=} \nu((\sigma_x \otimes \sigma_z)(\sigma_z \otimes \sigma_x)) \overset{(2b)}{=} \nu(\sigma_y \otimes \sigma_y). \end{aligned} \tag{3}$$

Thus, we reach a contradiction[3,4] as we cannot consistently assign a value to all nine observables independently of the context. Therefore, we must reject one of the two main assumptions: Either we accept that there are no predefined measurement outcomes, thus no functional consistency of outcomes, or we acknowledge the fact that the value-assignment, the result of a measurement, must depend on which context the observable belongs to. The latter view is known as *quantum contextuality* (QC).

Remarkably, this phenomenon is state-independent, i.e., the contradiction arises purely from the operator algebra and does not rely on any statistical inference. While the previous proof involved a Hilbert space of dimension 4, we will now present the generalized result stating that QC arises in all spaces of dimension higher or equal to 3.

2.2 The Kochen-Specker Theorem

The *Kochen–Specker* (KS) theorem [KS67] states that no non-contextual HVT can reproduce the predictions of quantum theory when the dimension of the Hilbert space is at least 3. The mathematical form of the theorem, which is theory independent, follows by restricting our attention to a subclass of observables, i.e., to dichotomic observables, P_i. In quantum mechanics, these observables correspond to rank-1 projectors with eigenvalues $\{0, 1\}$, obtained from the spectral decomposition of a given non-degenerate observable. Hence, these rank-1 projectors sum to the identity[5] and form a context. These operators define a one dimensional subspace, the space onto which the state is projected under its action, corresponding to a ray spanned by the vector associated with the projector. Thus,

[3] Here the argument hinges on the logic of counterfactual reasoning, as the contradiction arises if we were actually performing the context-measurements of the third row and column at the same time; which are mutually exclusive.

[4] This contradiction illustrates the standard intuition behind the emergence of contextuality: Contextuality emerges when "a family of data is locally consistent but globally inconsistent" [?].

[5] As we will see, KS assumes functional consistency in the form of $\nu\left(\sum_i P_i\right) = \nu(\mathbb{1}) = 1$. Where $P_i = |i\rangle\langle i|$ and $A = \sum_i a_i |i\rangle\langle i|$.

the KS theorem treats dichotomic observables and their associated vector interchangeably, as there is a one-to-one correspondence between them.

Theorem I: Kochen-Specker (1967)

There exists a finite set $S \subset \mathbb{R}^3$ such that $\nexists \omega : S \to \{0, 1\}$ satisfying,

$$\nu(u) + \nu(v) + \nu(w) = 1$$

$\forall \{u, v, w\}$, a triad of mutually orthogonal vectors, in S.

For any dimension $d > 3$, the argument proceeds by generalizing the set S to contain vectors in a d-dimensional Hilbert space such that $\nexists \omega : S \to \{0, 1\}$[6] satisfying $\sum_{|\psi\rangle \in D} \nu(|\psi\rangle) = 1$, for any set $D \subset S$ consisting of d mutually orthogonal states. We remark that Dirac's notation is introduced here, as is customary in treatments involving general dimensions, to ease algebraic manipulations. However, the statement remains purely geometrical. In Appendix. A we provide a succinct introduction on the geometric view of the KS theorem.

3 The Hardy Paradox

The Hardy paradox [Har92, Har93] is a gedankenexperiment that provides what is widely regarded as the simplest proof of quantum nonlocality without inequalities. However, beyond its implications for nonlocality, its logical structure also serves as a proof of quantum contextuality [AB11, BBC+10, CBTCB13]. The scenario of the experiment involves two agents, Alice and Bob, who share the two-qubit state,

$$|\psi\rangle_{\text{Hardy}} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + |10\rangle_{AB} + |11\rangle_{AB}). \tag{4}$$

Each of them can measure in the computational, $\{|0\rangle, |1\rangle\}$, or diagonal, $\{|+\rangle, |-\rangle\}$, basis. Hence, this setting involves four possible measurement contexts corresponding to the different combination of measurement basis that each party can choose. The state of Eq.(4) corresponds to the case of Alice and Bob both measuring in the computational basis. In the three other combination of basis the state takes the following expressions

$$\begin{aligned} \text{Diagonal} - \text{Computational} : |\psi\rangle &= \sqrt{\frac{2}{3}}|{+}0\rangle + \frac{1}{\sqrt{6}}|{+}1\rangle - \frac{1}{\sqrt{6}}|{-}1\rangle, \\ \text{Computational} - \text{Diagonal} : |\psi\rangle &= \sqrt{\frac{2}{3}}|1{+}\rangle + \frac{1}{\sqrt{6}}|0{+}\rangle + \frac{1}{\sqrt{6}}|0{-}\rangle, \\ \text{Diagonal} - \text{Diagonal} : |\psi\rangle &= \frac{1}{\sqrt{12}}[|{+}{+}\rangle + |{+}{-}\rangle - |{-}{+}\rangle + |{-}{-}\rangle]. \end{aligned} \tag{5}$$

This scenario can be physically realized, see Fig. 1, with two superposed Mach-Zehnder interferometers with a positron and an electron as input particles on each arm. The state of the system is then described as the superposition of all possible paths that both particles

[6] We recall that earlier, ν was a value assignment function from observables to predefined measurement outcomes. In contrast, the KS theorem relies on the impossibility of defining a value assignment function ω that maps vectors in the set S to truth values.

can take. For instance, if the positron takes the left path, its state will be $|1\rangle_A$. Note that the state $|01\rangle_{AB}$ cannot occur, as the particles annihilate each other at the intersection point of both interferometers. Then, the expression of the state that Alice and Bob share depends on the basis that they choose to measure and Eq.(4) gives the form in the case of both measuring the computational basis.

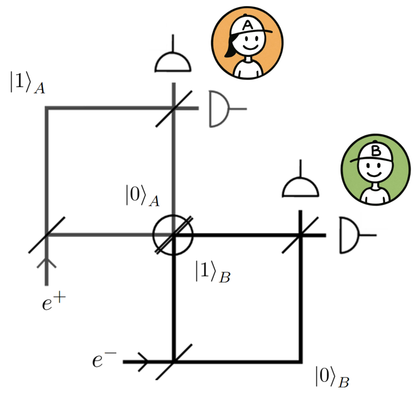

Figure 1: Physical setup of the Hardy paradox gedankenexperiment. An electron and a positron enter two superposed Mach-Zehnder interferometers. Alice and Bob, describe the state of the particle that enters their interferometer in terms of the physical path it took, and each measure in the computational or diagonal basis. An observer implement the latter by inserting a beam splitter at the crossing of the paths of their particle.

We denote Alice's (Bob's) measurement outcomes as the pair of binary[7] variables $A_{i\in\{0,1\}}$ ($B_{i\in\{0,1\}}$), where $i = 0$ denotes the computational basis and $i = 1$ the diagonal. For instance, $A_0 = 1$ corresponds to Alice having measured state $|1\rangle_A$ and $B_1 = 1$ to Bob observing state $|-\rangle_B$. Then, from the different expressions of the state given in Eq.(4) and Eq.(5), follows that

$$P(A_0 = 1|B_1 = 1) = P(A_0 = 0, B_0 = 1) = P(A_1 = 1|B_0 = 0) = 0, \tag{6}$$

recalling that $P(X|Y) = P(X,Y)/P(Y)$, and, e.g., $P(A_0 = 1, B_1 = 1) = |\langle\psi_{\text{Hardy}}|1-\rangle_{AB}|^2 = 0$.

Hence, from these probabilities it follows that if we assign a truth value of 1 (true) to the event $B_1 = 1$, then $A_0 = 0$ must also be assigned the value 1. This, in turn, implies that $B_0 = 0$ must be true, which leads to $A_1 = 0$ being assigned the value 1, i.e., $A_1 = 1$ must be 0 (false). Under the assumption of non-contextuality, we conclude that if $B_1 = 1$ is true, then $A_1 = 1$ must be false. However, this contradicts the quantum prediction that,

$$P(A_1 = 1, B_1 = 1) = 1/12 \neq 0. \tag{7}$$

The standard argument to resolve the Hardy paradox is to address the counterfactual nature implicit in its reasoning. In particular, one should note that in the previous sequence of implications we assumed that inferences from unperformed, because if performed the wave function would collapse, incompatible measurements could be made. Nonetheless, this paradox can be reformulated as a *Logical Pre- and Post-Selection* (LPPS) paradox, yielding a logical contradiction that is physically accessible through the so-called *Anomalous Weak Values* (AWV) [ABP+02].

[7]Throughout the text we will make de correspondence between classical outcomes: $+ \equiv 0$ and $- \equiv 1$, i.e., *outcomes* obtained measuring in the diagonal basis correspond to classical bits.

3.1 Mathematical framework of (Logical) Pre- and Post-Selection Paradoxes

In the setting of PPS paradoxes we assume that at an initial time t_i the system under study is in the (pre-selected) state $|\psi\rangle$. At a later intermediate time t, a projective measurement[8] $\{P_a\}_a$ is performed on the system. Finally, a second projective measurement $\{\Pi_{ok}, \Pi_{ok}^{\perp}\}$, where $\Pi_{ok} = |\phi\rangle\langle\phi|$, is performed at a time t_f. Only the runs in which Π_{ok} was observed are kept, successfully implementing the post-selection of state $|\phi\rangle$. Given this setup, the probability of observing the intermediate measurement outcome a conditioned on the pre-selected state $|\psi\rangle$ and post-selection $|\phi\rangle$ is given by [ABL64],

$$P(a|\,|\psi\rangle\,,|\phi\rangle) = \frac{|\,\langle\phi|\,P_a\,|\psi\rangle\,|^2}{\sum_{a'}|\,\langle\phi|\,P_{a'}\,|\psi\rangle\,|^2}, \tag{8}$$

widely known as the *Aharonov–Bergmann–Lebowitz* (ABL) rule.

PPS paradoxes are called *logical* when all ABL probabilities take binary values and $\langle\phi|\psi\rangle \neq 0$. In general, all LPPS paradoxes can give rise to a proof of contextuality by treating all the possible intermediate measurements as counterfactual alternatives [LS05b, LS05a]. However, if one interprets these intermediate measurements as slightly disturbing, we can perform them simultaneously with the pre-selection as the input state. Such measurements are known as *Weak Measurements* (WM).

3.2 Weak Measurements and Values

Following our brief review in Appendix. B on a particular quantum measurement framework, let us now consider the interaction Hamiltonian between the system and the measurement device:

$$H_{\text{int}} = gP_a \otimes \Gamma, \tag{9}$$

and take g, along with the contributions of $H_{\mathcal{S}}$ and $H_{\mathcal{M}}$ in Eq.(44), to be small. This setup models a weak measurement. This type of interaction is called a *von Neumann* interaction [vN96], and models the measurement device through the continuous position of its pointer, with conjugate momentum Γ coupled with strength parametrized by g to the observable P_a that one wishes to measure.

If after this interaction we post-select the system to the state $|\phi\rangle$, the position of the pointer shifts an amount $gtw(P_a|\,|\psi\rangle\,,|\phi\rangle)$, with t the time of the interaction and,

$$w(a|\,|\psi\rangle\,,|\phi\rangle) = \text{Re}\left(\frac{\langle\phi|\,P_a\,|\psi\rangle}{\langle\phi|\psi\rangle}\right), \tag{10}$$

the result of the weak measurement, i.e., the *Weak Value* (WV) of P_a. Note that, remarkably, WVs can sometimes lie outside the spectrum of the measured observable; we name these values as AWVs [AAV88]. Interestingly, it was shown that AWVs are a witness for contextuality [Pus14, KLP19].

It can be seen that in LPPS paradoxes follows that $P(a|\,|\psi\rangle\,,|\phi\rangle) = w(a|\,|\psi\rangle\,,|\phi\rangle)$ [AV91, PL15]. This implies that LPPS paradoxes always arise as AWVs if the intermediate measurements are considered to be weak. However, it must be noted that weak values should be understood as infinitesimal shifts in the meter's pointer, rather than as probabilities.

[8] Here we restric ourserlves to only consider PVMs, for the general case of POVMs one may consult [SGB+14].

3.3 The LPPS Hardy Paradox

We now proceed to reformulate the Hardy paradox in the framework of LPPS paradoxes. Following [Dou22], consider the pre-selection $|\psi_{\text{Hardy}}\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle + |10\rangle + |11\rangle)$, and the intermediate measurements $M_0 = \{P_0, P_0^{\perp}\}$ and $M_1 = \{P_1, P_1^{\perp}\}$, where $P_0 = |00\rangle\langle 00|$ and $P_1 = |11\rangle\langle 11|$. Note that these two measurements can be decomposed into a sum of projectors of the states $\{|00\rangle, |1+\rangle, |1-\rangle, |01\rangle\}$ for M_0 and $\{|11\rangle, |+0\rangle, |-0\rangle, |01\rangle\}$ for M_1. By the ABL rule, Eq.(8), if we post-select the state $|\phi\rangle = |--\rangle_{AB}$ after performing the measurement M_0 (M_1) on the state $|\psi_{\text{Hardy}}\rangle$, we obtain the outcome associated with P_0 (P_1) with certainty. A diagram of this procedure can be seen in Fig.(2).

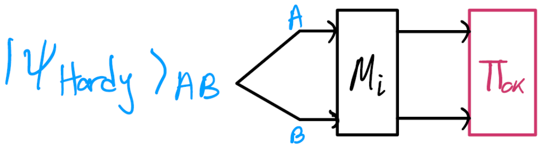

Figure 2: Diagram of the implementation of the joint 2-qubit measurements $M_{i\in\{0,1\}}$ on state $|\psi_{\text{Hardy}}\rangle$, followed by the post-selection $\Pi_{ok} = |--\rangle\langle--|^{AB}$.

Thus, if considered both measurements counterfactually, a logical contradiction arises when considering the 2-qubit computational basis; as both mutually orthogonal states $|00\rangle$ and $|11\rangle$ are predicted to happen with certainty. This contradiction constitutes a proof of the KS theorem, as no non-contextual value assignment can reproduce the previous outcomes possibilities.

If one considers these intermediate measurements to be weak, by Eq.(10), the contradiction manifest itself as an AWV with value -1, associated with the state $|10\rangle$, i.e., $w(|10\rangle \,|\, |\psi_{\text{Hardy}}\rangle, |--\rangle) = -1$. In Fig. 3 it is represented the proof of the KS theorem that can be constructed from this LPPS paradox [Dou22].

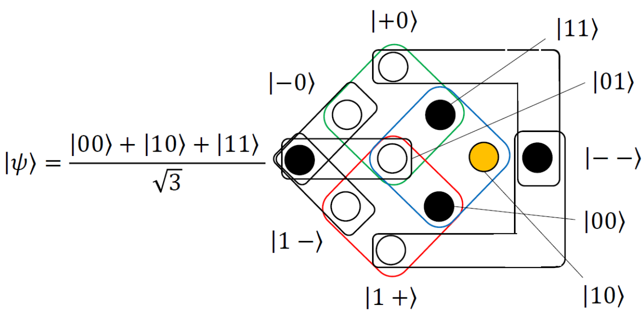

Figure 3: Hypergraph of the proof of the KS theorem based on the LPPS Hardy paradox. Black hyperedges correspond to the orthogonal relations due to the pre- and post-selection. Red (green) hyperedges represent a signaling from Alice (Bob) to Bob (Alice). In Blue the 2-qubit computational basis hyperedge. Black-colored nodes correspond to the pre- and post-selection and the states associated with the certain, by the ABL rule, projectors. The contradiction of two colored nodes in a same context is avoided by considering the AWV of -1 associated with the state in the orange node. The AWV makes the assigned values, colored nodes, in the 2-qubit computational basis context add up to 1, hence, removing the contradiction. Interpretations and figure extracted from [Dou22].

3.3.1 LPPS Hardy Paradox Measurement Statistics via Local Operations and Classical Communication

In [Dou22] it was identified that the intermediate measurements, M_0 and M_1, differ causally from one another. We make this explicit by framing each of these joint 2-qubit measurements occurring in a one-way *Local Operations and Classical Communication* (LOCC) scheme. In this scenario we also prepare the initial state $|\psi_{\text{Hardy}}\rangle_{AB}$, but we send each share to two new parties, Alice and Bob, connected by a classical channel. Each party will perform a computational (diagonal) basis measurement according to the binary configuration variable $x, y = 0$ $(x, y = 1)$, yielding the outcome a, b.

LOCC protocol for M_0

As can be seen in Fig.(4), the realization of M_0 relies on a direction of communication: $A \to B$ and, fixes Alice to first measure in her computational basis ($x = 0$), i.e., she will perform the measurement with associated projectors, $A_{a|x=0}$, $\{A_{0|0} = |0\rangle\langle 0|^A, A_{1|0} = |1\rangle\langle 1|^A\}$ on her qubit share. Conditioned on Alice's outcome, s.t. $y = a$, Bob will measure in the computational (diagonal) basis if $a = 0$ $(a = 1)$. Hence, performing measurement $\{B_{0|0} = |0\rangle\langle 0|^B, B_{1|0} = |1\rangle\langle 1|^B\}$ $(\{B_{0|1} = |+\rangle\langle +|^B, B_{1|1} = |-\rangle\langle -|^B\})$, with possible outcomes $b \in \{0, 1\}$ $(b \in \{+, -\} \equiv \{0, 1\})$.[9]

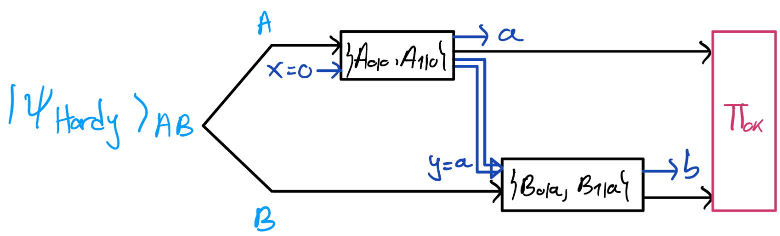

Figure 4: Diagram of the one-way LOCC protocol that reproduces the joint 2-qubit measurement M_0, followed by post-selection $\Pi_{ok} = |--\rangle\langle --|^{AB}$. Variables x and y denotes the basis choice of Alice and Bob respectively, with 0 (1) meaning computational (diagonal) basis. The thick arrow denotes the classical signaling of Alice's outcome, a, to Bob's basis choice, $y = a$.

By the fact that LOCC channels are separable channels [KW20], the associated Kraus operators of the aforementioned channel are the tensor product of the operators of each party given the pair of outcomes (a, b), conditioned on $y = a$ and $x = 0$,

$$\begin{aligned} K_{00} &= A_{0|0} \otimes B_{0|0} = P_0, & K_{01} &= A_{0|0} \otimes B_{1|0} = |01\rangle\langle 01| \\ K_{10} &= A_{1|0} \otimes B_{0|1} = |1+\rangle\langle 1+|, & K_{11} &= A_{1|0} \otimes B_{1|1} = |1-\rangle\langle 1-|. \end{aligned} \tag{11}$$

Which are also for the PVM $\{E_{\text{Yes}} = K_{00}^\dagger K_{00}, E_{\text{No}} = \sum_{\alpha \neq 00} K_\alpha^\dagger K_\alpha\} = \{|00\rangle\langle 00|, |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|\} = M_0$. Hence, proving that the defined LOCC protocol is equivalent to the joint measurement M_0.

Therefore, the un-normalised state after a run of this LOCC protocol with given outcomes (a, b), for any input state Ψ^{AB}, will be

$$\tilde{\rho}^{AB}_{(a,b)} = (A_{a|0} \otimes B_{b|a}) \Psi^{AB} (A_{a|0} \otimes B_{b|a})^\dagger, \tag{12}$$

[9] Note that we can relabel the outcomes in the diagonal basis as classical bits 0 or 1.

happening with probability,

$$P(a,b|x=0,y=a) = Tr[(A_{a|0}\otimes B_{b|a})\Psi^{AB}(A_{a|0}\otimes B_{b|a})^\dagger]. \tag{13}$$

If the initial state Ψ^{AB} is taken to be the pure state $|\psi_{\text{Hardy}}\rangle$, the LOCC protocol goes as follows,

- If $a=0$, the post-measurement state is $(A_{0|0}\otimes I)\,|\psi_{\text{Hardy}}\rangle = 1/\sqrt{3}\,|00\rangle$. Then, Alice classically signals to Bob her outcome such that he also measures in the computational basis ($y=0$), yielding $b=0$.
- If $a=1$, the post-measurement state is $(A_{1|0}\otimes I)\,|\psi_{\text{Hardy}}\rangle = \sqrt{\frac{2}{3}}\,|1+\rangle$. Alice classically signals to Bob her outcome such that he measures the diagonal basis ($y=1$), yielding $b=+\equiv 0$.

LOCC protocol for M_1

The realization of M_1 relies on a classical communication with the direction: $B\to A$ and, with Bob fixed to first measure in his computational basis ($y=0$), i.e., he will perform the measurement with associated projectors, $B_{b|y=0}$, $\{B_{0|0} = |0\rangle\!\langle 0|^B, B_{1|0} = |1\rangle\!\langle 1|^B\}$ on his qubit share. Conditioned on Bob's outcome, s.t. $x = 1\oplus b$, Alice will measure her computational (diagonal) basis if $b=1$ ($b=0$). Hence, performing the measurement $\{A_{0|0} = |0\rangle\!\langle 0|^A, A_{1|0} = |1\rangle\!\langle 1|^A\}$ ($\{A_{0|1} = |+\rangle\!\langle +|^A, A_{1|1} = |-\rangle\!\langle -|^A\}$).

Again, by the fact that LOCC channels are separable channels, the associated Kraus operators of the aforementioned channel are the tensor product of the operators of each party given (a,b) conditioned on $x=1\oplus b$ and $y=0$,

$$\begin{aligned} L_{11} &= A_{1|0}\otimes B_{1|0} = P_1, & L_{01} &= A_{0|0}\otimes B_{1|0} = |01\rangle\!\langle 01| \\ L_{00} &= A_{0|1}\otimes B_{0|0} = |{+0}\rangle\!\langle{+0}| & L_{10} &= A_{1|1}\otimes B_{0|0} = |{-0}\rangle\!\langle{-0}|, \end{aligned} \tag{14}$$

Which are also for the PVM $\{E_{\text{Yes}} = L_{11}^\dagger L_{11}, E_{\text{No}} = \sum_{\alpha\neq 11} L_\alpha^\dagger L_\alpha\} = M_1$. Hence, proving that the defined LOCC protocol is equivalent to the joint measurement M_1.

From this description and for any input state Ψ^{AB}, after a run of this LOCC protocol the un-normalised state, with given outcomes (a,b), will be

$$\tilde{\rho}^{AB}_{(a,b)} = (A_{a|1\oplus b}\otimes B_{b|0})\Psi^{AB}(A_{a|1\oplus b}\otimes B_{b|0})^\dagger, \tag{15}$$

with probability,

$$P(a,b|x=1\oplus b, y=0) = Tr[(A_{a|1\oplus b}\otimes B_{b|0})\Psi^{AB}(A_{a|1\oplus b}\otimes B_{b|0})^\dagger]. \tag{16}$$

If the initial state Ψ^{AB} is taken to be the pure state $|\psi_{\text{Hardy}}\rangle$, the LOCC protocol goes as follows,

- If $b=0$, the post-measurement state is $(I\otimes B_{0|0})\,|\psi_{\text{Hardy}}\rangle = \sqrt{\frac{2}{3}}\,|{+0}\rangle$. Then, Bob classically signals to Alice his outcome such that she measures in the diagonal basis ($x=1\oplus 0$), yielding $a=+\equiv 0$.
- If $b=1$, the post-measurement state is $(I\otimes B_{1|0})\,|\psi_{\text{Hardy}}\rangle = 1/\sqrt{3}\,|11\rangle$. Bob classically signals to Alice his outcome such that she measures in the computational basis ($x=1\oplus 1$), yielding $a=1$.

Therefore, we have seen that performing either two-qubit measurements M_i can be thought as a one-way LOCC channel between two parties. In order to recover the probabilities of the previous description of the paradox, we apply the measurement $\{\Pi_{ok} = |{-}{-}\rangle\langle{-}{-}|^{AB}, \Pi_{fail} = \Pi_{ok}^{\perp}\}$ with the aim of post-selecting the state $|{-}{-}\rangle_{AB}$. Hence, the general probability of obtaining the LOCC-pair of outcomes (a, b) given the pair of basis configurations (x, y) and a successful post-selection of state $|{-}{-}\rangle_{AB}$, denoted by the label "ok", can be expressed as[10]

$$P(a, b|ok, x, y) = \frac{P(a, b, ok|x, y)}{\sum_{a'b'} P(a', b', ok|x, y)} = \frac{Tr\left(\Pi_{ok}\tilde{\rho}_{(a,b)}\right)}{\sum_{a'b'} Tr(\Pi_{ok}\tilde{\rho}_{(a',b')})}, \tag{17}$$

where $x = 0$, $y = a$ and $\tilde{\rho}_{(a,b)}$ corresponds to Eq.(12) ($y = 0$, $x = 1 \oplus b$ and $\tilde{\rho}_{(a,b)}$ corresponds to Eq.(15)) if the intermediate measurement was M_0 (M_1).

From this expression we recover, deferring the explicit computations to Appendix. C, the 0/1 probabilities of the LPPS Hardy paradox,

$$\begin{aligned} P(a=0, b=0|ok, x=0, y=0, M_0) = 1, \quad P(a=1, b=1|ok, x=0, y=1, M_0) = 0, \\ P(a=1, b=1|ok, x=0, y=0, M_1) = 1, \quad P(a=0, b=0|ok, x=1, y=0, M_1) = 0, \end{aligned} \tag{18}$$

recalling the classical relabeling of outcomes $\{0 \equiv +, 1 \equiv -\}$ and noting that the second probability becomes 0 due to the orthogonality between K_{11} and $|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|$; remember that in the $A \to B$ LOCC run with $|\psi_{\text{Hardy}}\rangle$ chosen as the initial state, the branch $(1, 1)$ does not happen. While the last probability becomes 0 due to the orthogonality between Π_{ok} and $L_{00}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,L_{00}^{\dagger}$, here we see that in the $B \to A$ LOCC run with $|\psi_{\text{Hardy}}\rangle$ chosen as the initial state, the branch $(0, 0)$ *does* happen, but post-selection filters out this possibility.

4 The Wigner's Friend gedankenexperiment and its Extensions

The measurement problem is one of the most long-standing debates within the quantum foundations community [Leg05]. Although quantum mechanics refers to measurements in its fundamental postulates, it never addresses the question of which physical processes actually constitute a measurement. Two views of measurement can be considered:

The measurement problem

- *Projection postulate view*: Consider a quantum system to be in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathcal{H}_{\mathcal{S}}$. From the postulates of the theory, by measuring in the computational basis, $\{|0\rangle, |1\rangle\}$, the state will collapse to either $|0\rangle$ with probability $|\alpha|^2$ or to $|1\rangle$ with probability $|\beta|^2$.

- *Closed system view*: Consider a quantum system to be in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathcal{H}_{\mathcal{S}}$. Moreover, consider the measurement device as a further quantum system in space $\mathcal{H}_{\mathcal{M}}$, initially in the "ready" state $|M\rangle$. The interacting process between the system and the device is, then, described by the unitary evolution $|\psi\rangle_S \otimes |M\rangle_M \xrightarrow{U} \alpha|0\rangle_S \otimes |M_0\rangle_M + \beta|1\rangle_S \otimes |M_1\rangle_M \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{M}}$. Where, $|M_i\rangle$ are the eigenstates of the meter associated to each measurement outcome. This is the approach explained in Appendix. B.

[10] Recalling that $P(a|b) = P(a, b)/P(b)$ and $P(b) = \sum_a P(a)$.

While the first view results in a collapsed post-measurement state, the second one yields an entangled state. This highlights the tension between two a priori valid descriptions of a measurement process. This apparent contradiction only appears in the orthodox interpretation of quantum mechanics. Any valid interpretation of the theory solves the measurement problem.

Building on this, the Wigner's friend thought-experiment [Wig95, Deu85] promotes the measurement device to be an observer, the friend. The first view then corresponds to Wigner's friend measuring a quantum system in a lab. In contrast, the second view corresponds to the perspective of an external observer, Wigner, who describes both system and friend as quantum systems interacting via a unitary evolution. In this scenario, the previous space $\mathcal{H}_{\mathcal{M}}$ becomes $\mathcal{H}_F$, space spanned by all possible outcomes that Wigner's friend can observe. Hence, the friend's register serves as the memory register of the observed state. Restricting ourselves to qubit systems and the friend's "ready" state initialized to $|0\rangle_F$, the unitary interaction corresponds to a CNOT gate with the system qubit acting as the control,

$$U[(\alpha |0\rangle_S + \beta |1\rangle_S) \otimes |0\rangle_F] = \alpha |00\rangle_{SF} + \beta |11\rangle_{SF} . \tag{19}$$

Then, in the Wigner's friend thought experiment scenario the measurement problem becomes the discrepancy between an observer's observation on a quantum system and the observation of an observer observing another observer observing a quantum system, as depicted in Fig. 5.

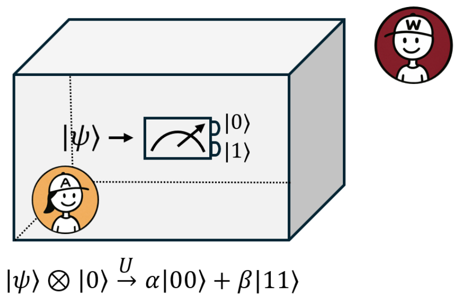

Figure 5: Representation of the Wigner friend thought experiment. Alice, Wigner's friend, measures the qubit system $|\psi\rangle_S = \alpha |0\rangle + \beta |1\rangle$ in the computational basis inside her lab. She observes either state $|0\rangle_S$ with probability $|\alpha|^2$ or the state $|1\rangle_S$ with probability $|\beta|^2$. Wigner, outside of the lab, regards Alice as another quantum system, which stores the state she has measured in her memory register living in $\mathcal{H}_{\mathcal{F}}$. From his point of view, the measurement process inside the lab is described by a unitary evolution acting on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{F}}$. Initializing the friend's memory qubit to $|0\rangle_F$, the unitary operator U corresponds to a CNOT gate with the system's qubit as the control. Therefore, while Alice observes a definite outcome, Wigner describes the composite system as an entangled state.

As in the case of the measurement problem, this tension is resolved in modern interpretations of quantum mechanics. For instance, by interpreting quantum states as states of knowledge about the system, rather than as states of reality, Wigner and his friend are understood to possess different levels of information about the measurement process. Hence, the apparent discrepancy arises only if they are incorrectly assumed to be on the same epistemic footing.

The original Wigner's friend thought experiment highlights an ambiguity in the description of the measurement process, but does not in itself yield a logical contradiction.

Extended Wigner's Friend (EWF) scenarios aim to formalize and build upon this idea, constructing scenarios where this ambiguity leads to a genuine paradox, i.e., a logical contradiction. In the spirit of Bell's and KS theorems, such contradictions give rise to new no-go theorems. In what follows, we focus on one particular extension: the *Local Friendliness* (LF) argument [BUAG+20].

4.1 The Local Friendliness no-go theorem

In this scenario, elements of a Bell setup is combined with the two distinct perspectives introduced in the Wigner's friend thought experiment. The paradigmatic setup [Bru18] is illustrated in Fig. 6.

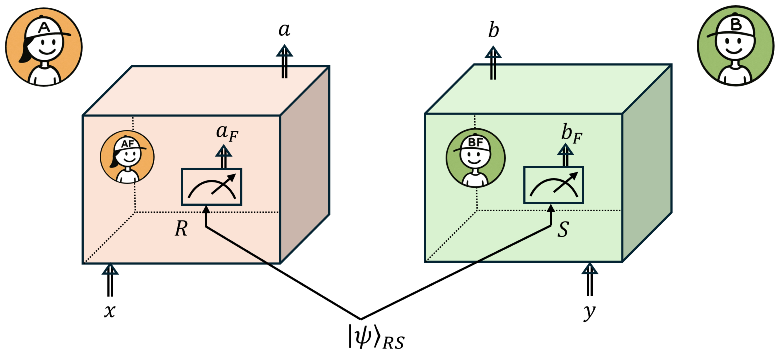

Figure 6: Representation of the paradigmatic LF setup. Alice and Bob are superobservers that have full control over their respective friend's labs. Each share of a bipartite state $|\psi\rangle_{RS}$ is sent to each lab. Inside them, each friend performs a measurement yielding an outcome a_F (b_F). Subsequently, Alice (Bob) performs a specific operation on their friend's lab according to the classical configuration variable x (y). This action produces the final classical outcome a (b).

The two main assumptions in the argument are defined as follows:

LF main assumptions

- *Absoluteness of Observed Events (AOE)*: An observed event is a real single event, and not relative to anything or anyone.
- *Local Agency (LA)*: If a measurement setting is freely chosen, then it is uncorrelated with any set of relevant events not in its future-light-cone.

The conjunction of these two metaphysical assumptions is the so-called Local Friendliness assumption. Implicitly, it is further assumed that *Universality of Unitarity*, i.e., the interactions between each system's share and the corresponding friend inside the lab can be modeled as a unitary evolution from the point of view of Alice and Bob, which we now name *superobservers*. This is justified by the further assumption that these external observers can apply and reverse any unitary evolution occurring on the labs under their control. One last implicit assumption is in order, *Operational Adequacy* of quantum mechanics, i.e., the validity of the Born rule.

Therefore, the LF no-go theorem can be stated as follows [BUAG+20]

Theorem II: Local Friendliness no-go theorem

If a superobserver can perform arbitrary quantum operations on an observer and its environment, then no physical theory compatible with quantum predictions can satisfy Local Friendliness.

That is the assumption of LF leads to a contradiction with quantum theory. Hence, the violation of the correlations imposed by LF in quantum mechanics forces one to abandon AOE in order to preserve a notion of relativistic locality. That is, "facts" are relative to observers. In interpretations of the theory such as Relational Quantum Mechanics or QBism, this idea of "observer-dependent facts" is naturally incorporated in their respective frameworks.

5 A LF scenario based on the LPPS Hardy Paradox

Building on the framework developed in [WYWS24], we translate the proof of the KS theorem constructed from the LPPS Hardy paradox into a LF argument and corresponding no-go theorem proof. We also investigate how the AWV appearing in the paradox translates into this scenario. Our analysis is presented in two equivalent formulations, which we refer to as: *Coarse-Grained* and *Fine-Grained*. Each corresponding to considering the intermediate measurements M_0 and M_1 as 2-qubit joint measurements or as one-way LOCC channels associated with each one.

5.1 Coarse-grained View

The scenario consists of a sequence of actions performed by three superobservers Alice, Bob0 and Bob1, where Alice has control over two observers: Friend0 and Friend1.

The protocol, as sketched in Fig. 7, starts by sending to Friend0, Bob0 and Bob1 the shares $\mathcal{H}_S = \mathbb{C}^2 \otimes \mathbb{C}^2$, $\mathcal{H}_{T_0} = \mathbb{C}^2$ and $\mathcal{H}_{T_1} = \mathbb{C}^2$ of the following prepared entangled state,

$$\begin{aligned} |\psi_0\rangle_{ST_0T_1} &= \mathrm{CNOT}_{ST_0}\mathrm{CNOT}_{ST_1}\,|\psi_{\mathrm{Hardy}}\rangle \otimes |00\rangle_{T_0T_1} \\ &= \frac{1}{\sqrt{3}}(|10\rangle_S \otimes |00\rangle_{T_0T_1} + |00\rangle \otimes |10\rangle_{T_0T_1} + |11\rangle \otimes |01\rangle_{T_0T_1}). \end{aligned} \tag{20}$$

Note that in defining this state we have employed the operator,

$$\mathrm{CNOT}_{ST_j} = (P_j^{\perp})_S \otimes I_{T_j} + (P_j)_S \otimes X_{T_j}, \tag{21}$$

which flips the flag qubit in system T_j if P_j fired for $j \in \{0,1\}$. Here X refers to the Pauli matrix σ_x and I to the identity operator.

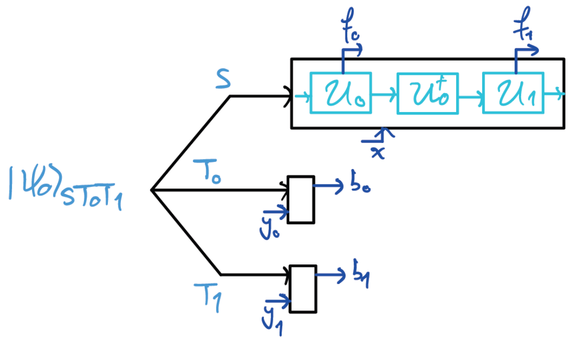

Figure 7: Diagram of the LF scenario based on the LPPS Hardy paradox in the coarse-grained view. A three-partite state $|\psi_0\rangle$ is shared among three superobservers, Alice, Bob0, and Bob1, with associated Hilbert spaces $\mathcal{H}_S$, $\mathcal{H}_{T_0}$, and $\mathcal{H}_{T_1}$, respectively. Alice controls two sequentially placed laboratories, where Friend0 and Friend1 perform measurements M_0 and M_1, respectively. From Alice's perspective, each measurement is modeled as a unitary evolution: U_0 for Friend0 and U_1 for Friend1, producing outcomes f_0 and f_1. Friend0 always performs measurement M_0. Then, depending on the setting $x \in \{0,1\}$, Alice proceeds as follows: for $x = 0$, she opens Friend0's lab, reveals the outcome f_0 to the outside world, and post-selects the state $|\phi\rangle = |- - -\rangle_{SF_0}$. In contrast, if she sets $x = 1$ after Friend0's measurement, she reverses the lab's evolution by applying $U_0^\dagger$; the restored share S then enters Friend1's lab, where a similar procedure takes place. Meanwhile, Bob0 and Bob1 perform computational (diagonal) measurements on their respective qubit shares, depending on the setting $y_i = 0$ ($y_i = 1$), yielding outcomes b_i.

We assume Universality of Unitary, which enables us to consider each friend as a quantum system. Therefore, we model each friend's measurement, as seen by Alice, on system S as the following CNOT gate

$$(U_i)_{SF_i} = (P_i^\perp)_S \otimes X_{F_i} + (P_i)_S \otimes I_{F_i}, \quad i \in \{0,1\}, \tag{22}$$

with $\mathcal{H}_{F_0}, \mathcal{H}_{F_1} = \mathbb{C}^2$, the Hilbert spaces associated with the two friends serving as memory registers for outcomes P_i ($P_i^\perp$) denoted by 0 (1)[11]. Note that the definiteness of the outcomes obtained by each friend is ensured by the AOE assumption.

The protocol on Alice's side proceeds as follows. Friend0 first performs the intermediate measurement M_0, yielding an outcome $f_0 \in \{0,1\}$. Since Alice is assumed to be a superobserver, she decides how the protocol continues by choosing a binary setting $x \in \{0,1\}$ after Friend0's measurement.

- If she chooses $x = 0$, Alice opens Friend0's lab[12], reveals the outcome f_0 to the outside world, post-selects onto the state $|\phi\rangle = |- - -\rangle_{SF_0}$, and terminates the protocol. In this case, Friend1's measurement does not take place.
- If she chooses $x = 1$, no post-selection is performed after Friend0's measurement. Instead, Alice reverses the measurement interaction by applying $U_0^\dagger$, and the system proceeds to Friend1's lab. There, Friend1 performs measurement M_1, yielding an outcome $f_1 \in \{0,1\}$, which is revealed before Alice applies the final post-selection onto the state $|\phi\rangle$.

[11] In both CNOTS defined in Eq.(21) and Eq.(22) we assume that the reference qubits living in spaces T_j and F_i are initialized in the $|0\rangle$ state. Furthermore, we stress that for the first CNOT the reference qubit is flagged with 1 if P_j fired, while for the second CNOT simply records the measurement outcome.

[12] This action collapses the entangled state in SF_0 to the definite observed outcome.

On Bob0 and Bob1's sides, space-like separated from Alice, one out of two basis measurements can be performed. If the binary measurement choice y_i is set to $y_i = 0$, the computational basis is measured on qubit T_i. We denote the absolute measurement outcome as $b_i = 0$ ($b_i = 1$) if $|0\rangle\langle 0|^{T_i}$ ($|1\rangle\langle 1|^{T_i}$) was observed. On the other hand, if $y_i = 1$, the diagonal basis is measured on qubit T_i. We denote the absolute measurement outcome as $b_i = +$ ($b_i = -$) if $|+\rangle\langle +|^{T_i}$ ($|-\rangle\langle -|^{T_i}$) was observed.

We now assume AOE, LA, Universality of Unitary and the Born rule in order to reach a contradiction similar to the one shown in Section. 3.3. Here we will explicitly differentiate between empirical probability distributions given by the Born rule, $\wp(\cdot)$, and the conditional probability distribution ensured by AOE,

$$P(f_0, f_1, b_0, b_1|x, y_0, y_1). \tag{23}$$

Notice that when $y_0 = y_1 = 1$ and $b_0 = b_1 = +$, the effective state in Alice's side is

$$\left|\tilde{\psi}_i\right\rangle = (I_S \otimes \langle ++|_{T_0T_1})\,|\psi_0\rangle_{ST_0T_1} = \frac{1}{2\sqrt{3}}(|10\rangle + |00\rangle + |11\rangle). \tag{24}$$

Hence, when the event $b_0 = b_1 = +$ occurs, which happens with probability $||\left|\tilde{\psi}_i\right\rangle||^2 = 1/4$, Alice's state becomes $|\psi_{\text{Hardy}}\rangle$.

From all of the above we compute, as we detail in Appendix. D.1, the following probabilities for the different possible observed events,

$$\wp(f_0 = 1|b_0 = b_1 = +, x = 0, y_0 = y_1 = 1) = 1. \tag{25}$$

It is important to note that we are implicitly assuming that this last probability, as well as all subsequent ones, are conditioned on Alice successfully post-selecting the state $|- - -\rangle_{SF_i}$ after opening the corresponding lab.

Analogously, for Friend1 measuring M_1,

$$\wp(f_1 = 1|b_0 = b_1 = +, x = 1, y_0 = y_1 = 1) = 1. \tag{26}$$

Furthermore, LA demands that

$$\wp(f_0 = 1|b_0 = b_1 = +, x = 0, y_0 = y_1 = 1) = \\ = P(f_0 = 1|b_0 = b_1 = +, x = 1, y_0 = y_1 = 1), \tag{27}$$

as f_0 is not in the future of the light cone of x, as can be seen in Fig. 8.

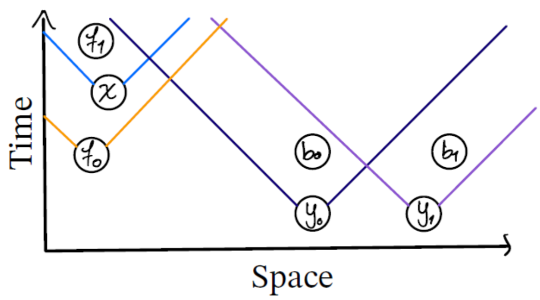

Figure 8: Space-time diagram illustrating the time ordering of the events within the coarse-grained view of the LF scenario based on the LPPS Hardy paradox.

It follows, then, that Eq.(26) and Eq.(27) constitute marginals of the probability distribution[13]

$$P(f_0 = f_1 = 1|b_0 = b_1 = +) = \frac{P(f_0 = f_1 = 1, b_0 = b_1 = +)}{\wp(b_0 = b_1 = +)}, \tag{28}$$

which is well-defined due to,

$$\wp(b_0 = b_1 = +) = |\langle\psi_0| I_S \otimes |{+}{+}\rangle\langle{+}{+}|^{T_1T_2} |\psi_0\rangle|^2 = 3/4 > 0. \tag{29}$$

By accounting these marginals, follows that,

$$P(f_0 = 1, f_1 = 1|b_0 = +, b_1 = +, x = 1, y_0 = 1, y_1 = 1) = 1. \tag{30}$$

However, this cannot be the case, as the projectors P_0 and P_1 are orthogonal to each other. In quantum theory, one would expect $P(f_0 = 1, f_1 = 1 \mid b_0 = +, b_1 = +, x = 1, y_0 = 1, y_1 = 1) = 0$, since $f_0 = 1$ implies that P_0 fired, and $f_1 = 1$ implies that P_1 fired. Therefore, assuming both outcomes simultaneously leads to a contradiction with the predictions of quantum mechanics under the LF assumptions.

5.1.1 Weak Measurements Version

We first consider that each lab is weakly coupled to a measurement device that records the measurement outcome obtained within the lab. Hence, from Eq.(10), we consider the following expression for the weak values when the measurements M_0 and M_1 are considered to be weak,

$$w(|ab\rangle \,|\, \left|\tilde{\psi}_i\right\rangle, |\phi\rangle) = \frac{{}_{SF_i}\langle - - -| \, U_i^{(a,b)} \left|\tilde{\psi}_i\right\rangle \otimes |0\rangle_{F_i}}{{}_{SF_I}\left\langle - - -\middle|\tilde{\psi}_i\right\rangle \otimes |0\rangle_{F_i}}, \tag{31}$$

where we implicitly assume that Bob0 and Bob1 obtained outcomes $b_0 = b_1 = +$, and thus the pre-selected state is proportional to the Hardy state. Moreover, $i \in \{0, 1\}$ labels the lab and we have defined in this equation the operator $U_i^{(a,b)} = |ab\rangle\langle ab|^S \, U_i$, as after the friend's measurement Alice opens the lab, thus collapsing the entangled state into the definite outcome seen by the friend, and post selects $|\phi\rangle = |- - -\rangle_{SF_i}$.

From Eq. (31), one can compute that the only non-zero weak value in the measurement context of M_0 (M_1) is equal to 1 and is associated with the state $|00\rangle$ ($|11\rangle$). If we consider the two-qubit computational basis decomposed into four operators of the form

$$V_a = (P_a^{\perp})_S \otimes X_{F_i} + (P_a)_S \otimes \mathbb{1}_{F_i},$$

with $P_a \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, we find that the weak value of 1 is associated with the states $|00\rangle$ and $|11\rangle$, while an AWV of -1 arises for the state $|10\rangle$. This exactly reproduces the weak values found in the LPPS Hardy paradox, as presented in Section 3.3.

We now propose a more thoughtful scenario, inspired by [MS20]. Let us consider that inside each lab, Alice's effective state is weakly coupled to a measurement device. In this setting, the lab contains two degrees of freedom, described by the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_{\mathcal{M}_i}$; where $i \in \{0, 1\}$ labels which friend we are referring to. However, from the external perspective of Alice, the total state includes a third degree of freedom, the memory of the friend,

[13] In order to alleviate notation, we omit writing the fixed settings $x = 1$ and $y_0 = y_1 = 1$ in the following expressions.

resulting in the extended space $\mathcal{H}_S \otimes \mathcal{H}_\mathcal{M} \otimes \mathcal{H}_{F_i}$. The only modification to our previous protocol is that the post-selection onto the state $|--\rangle_S$ is now performed by Alice's friend inside the corresponding sealed lab.

We denote the initial state to be weakly measured and post-selected by,

$$|\Psi_0\rangle = \left|\tilde{\psi}_i\right\rangle_S \otimes |m_0\rangle_{\mathcal{H}_\mathcal{M}} \otimes |0\rangle_{F_i}, \tag{32}$$

with $|m_0\rangle$ the "ready" state of the meter. For measurements M_0 and M_1 we define the following observables acting on $\mathcal{H}_S$,

$$\begin{aligned} A_0 &= |00\rangle\langle 00| + 2\,|1+\rangle\langle 1+| + 3\,|1-\rangle\langle 1-| + 4\,|01\rangle\langle 01|\,, \\ A_1 &= |11\rangle\langle 11| + 2\,|+0\rangle\langle +0| + 3\,|-0\rangle\langle -0| + 4\,|01\rangle\langle 01|\,. \end{aligned} \tag{33}$$

From these observables we define the unitary that describes Friend0's (Friend1's) weak intermediate measurement M_0 (M_1),

$$U_i^{\text{weak}} = e^{-igA_i\otimes\Gamma} \approx 1 - igA_i \otimes \Gamma_i, \tag{34}$$

...acting on $\mathcal{H}_S \otimes \mathcal{H}_{\mathcal{M}_i}$. Hence, the weak values of the measurement will be encoded in the meter's register. Furthermore, we describe a weak measurement performed by any friend on one of the projectors of the two-qubit computational basis as follows,

$$U_a^{\text{weak}} = e^{-igP_a\otimes\Gamma_i} \approx 1 - igP_a \otimes \Gamma_i. \tag{35}$$

where $P_a \in \{|00\rangle\langle 00|\,, |10\rangle\langle 10|\,, |01\rangle\langle 01|\,, |11\rangle\langle 11|\}$.

After the intermediate weak measurement performed by Friend0 or Friend1, the post-selection onto $|--\rangle_S$ is described by Alice as,

$$U_i^{ok} = |\phi\rangle\langle\phi|^S \otimes X_{F_i} + (I - |\phi\rangle\langle\phi|)^S \otimes I_{F_i}. \tag{36}$$

Then, in this scenario, the friend's memory register encodes whether the post-selection was successful or not: $|0\rangle_{F_i}$ denotes success, while $|1\rangle_{F_i}$ indicates failure.

As detailed in Appendix. D.2, the weak implementations of M_0, M_1, and the computational basis measurements in this protocol yield the same weak values and AWV as those found in the LPPS Hardy paradox, Section. 3.3. In this scenario, however, Alice observes an entangled state where weak values arising from post-selection onto the states asociated to $I - |\phi\rangle\langle\phi|$ appear. Upon successful post-selection by the friend onto the specific state $|\phi\rangle$, Alice opens the lab, causing the system to collapse to the definite outcome obtained by the friend. She then gains access to the meter's pointer reading, and thus, to the corresponding weak value.

5.2 Fine-grained View

We propose a reinterpretation of the joint measurements performed by Friend0 (Friend1) as a one-way LOCC channel between two, a pair for each former agent, new fine-grained agents Charlie0 (Charlie1) and Debbie0 (Debbie1) inside Alice's lab [14], as sketched in Fig.9. The former Friend0's coarse-grained measurement is equivalent to Charlie0 signaling his measurement outcome to Debbie0, and then she performing her measurement conditioned

[14]This enables us to consider communication between the two fine-grained agents.

on the message. Analogously, Friend1's measurement amounts to Debbie1 communicating to Charlie1.

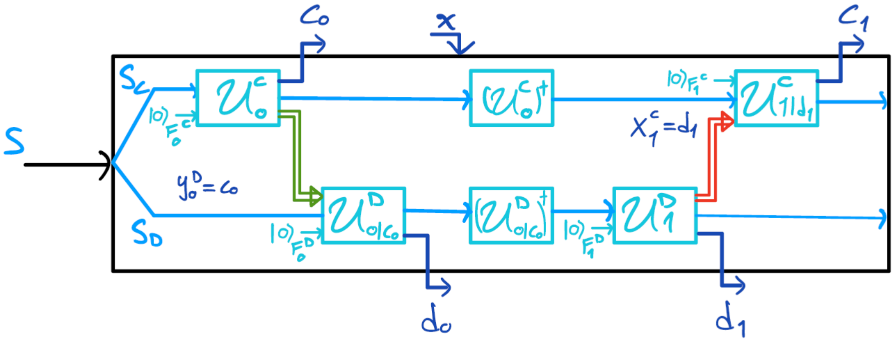

Figure 9: Diagram of the LF scenario based on the LPPS Hardy paradox in the fine-grained view. For simplicity, we only draw the protocol in Alice's side; as is the only part that has changed. Thick arrows colored in green (red) represent classical communication from Charlie to Debbie (Debbie to Charlie), in the sense of the LOCC protocol sketched in Fig. 4. The former classical outcome f_0 (f_1) is now spitted into two new ones: c_0 and d_0 (c_1 and d_1).

Now, we present how the protocol carries out for Friend0's step, analogous computations and reasoning will follow for Friend1 reversing the communication direction between fine-grained agents.

First, Charlie0 measures his share of the state $|\psi_0\rangle_{S_C S_D T_0 T_1}$, noting $S = S_S \otimes S_D$, in the computational basis ($x_0^C = 0$). Then,

- If $c_0 = 0$, the post-measurement state is $(|0\rangle\langle 0|^{S_C} \otimes I_{S_D} \otimes I_{T_0} \otimes I_{T_1}) |\psi_0\rangle = \frac{1}{\sqrt{3}} |00\rangle_{S_C S_D} |10\rangle_{T_0 T_1}$. Then, Debbie0 obtains outcome $d_0 = 0$.
- If $c_0 = 1$, the post-measurement state is $(|1\rangle\langle 1|^{S_C} \otimes I_{S_D} \otimes I_{T_0} \otimes I_{T_1}) |\psi_0\rangle = \sqrt{\frac{2}{3}} |1+\rangle_{S_C S_D} (|00\rangle_{T_0 T_1} + |01\rangle_{T_0 T_1})$. Then, Debbie0 obtains outcome $d_0 = + \equiv 0$.

Hence, the former outcome $f_0 = 1$, P_0 fired, is translated to Charlie0 and Debbie0 measuring $c_0 = d_0 = 0$. This can be expressed as, $f_0 = \overline{c_0 + d_0}$, i..e, a NOR logical operation between bits c_0 and d_0.

From Alice's perspective the first step of the LOCC protocol is represented by the action of the following unitary,

$$U_0^C = |1\rangle\langle 1|^{S_C} \otimes X_{F_0^C} + |0\rangle\langle 0|^{S_C} \otimes I_{F_0^C} \tag{37}$$

That is, Alice first sees Charlie0 measuring the computational basis on his qubit share in S_C and storing the measurement outcome in the memory register initialized to $|0\rangle_{F_0^C} \in F_0^C = \mathbb{C}^2$. Then, Chalie0 uses a classical channel to communicate his measurement outcome to Debbie0. Upon this, Debbie0 measures in the computational (diagonal) basis if Charlie0's outcome was $c_0 = 0$ ($c_0 = 1$). Debbie0's measurement is seen by Alice as the action of one of the following unitaries depending on c_0,

$$U_{0|c_0=0}^D = |1\rangle\langle 1|^{S_D} \otimes X_{F_0^D} + |0\rangle\langle 0|^{S_D} \otimes I_{F_0^D} \tag{38a}$$

$$U_{0|c_0=1}^D = |-\rangle\langle -|^{S_D} \otimes X_{F_0^D} + |+\rangle\langle +|^{S_D} \otimes I_{F_0^D}. \tag{38b}$$

Hence, from Alice's perspective, Debbie0's measurement outcome is stored in the agent's memory register[15] initialized to $|0\rangle_{F_0^D} \in F_0^D = \mathbb{C}^2$. In Appendix. D.3 we show that the unitaries in Eq.(37) and Eq.(38) are just the Stinespring dilation of the Kraus operators in Eq.(11). Besides, we show that the lab's evolution derived from this fine-grained scenario recovers the evolution of the coarse-grained view, i.e., $U_0 = U_{0|c_0}^D U_0^C$.

As in the coarse-grained interpretation, when Alice chooses the setting $x = 1$, she decides not to reveal either of the friends' outcomes and instead reverses the measurement interactions by applying $(U_0^C)^\dagger$ and $(U_{0|c_0}^D)^\dagger$. After this process, the protocol for Friend1 in the fine-grained view proceeds analogously to that of Friend0. The LOCC protocol for the state $|\psi_0\rangle_{S_C S_D T_0 T_1}$ then unfolds as follows:

First, Debbie1 measures his share of the state $|\psi_0\rangle_{S_C S_D T_0 T_1}$, noting $S = S_S \otimes S_D$, in the computational basis ($x_1^D = 0$). Then,

- If $d_1 = 0$, the post-measurement state is $(I_{S_C} \otimes |0\rangle\langle 0|^{S_D} \otimes I_{T_0} \otimes I_{T_1}) |\psi_0\rangle = \sqrt{\frac{2}{3}} |{+0}\rangle_{S_C S_D} (|10\rangle_{T_0 T_1} + |00\rangle_{T_0 T_1})$. Then, Charlie1 obtains outcome $c_1 = + \equiv 0$.
- If $d_1 = 1$, the post-measurement state is $(I_{S_C} \otimes |1\rangle\langle 1|^{S_D} \otimes I_{T_0} \otimes I_{T_1}) |\psi_0\rangle = \frac{1}{\sqrt{3}} |11\rangle_{S_C S_D} |01\rangle_{T_0 T_1}$. Then, Charlie1 obtains outcome $c_1 = 1$.

Hence, the former outcome $f_1 = 1$, P_1 fired, is translated to Debbie1 and Charlie1 measuring $d_1 = c_1 = 1$. This can be expressed as, $f_1 = c_1 \cdot d_1$, i.e., a AND logical operation between bits c_1 and d_1.

From Alice's perspective this LOCC protocol is represented by the sequential action of the following two unitaries,

$$\begin{aligned} U_1^D &= |1\rangle\langle 1|^{S_D} \otimes X_{F_1^D} + |0\rangle\langle 0|^{S_D} \otimes I_{F_1^D} \\ \text{If } d_1 = 0 \implies U_{1|d_1=0}^C &= |1\rangle\langle 1|^{S_C} \otimes X_{F_1^C} + |0\rangle\langle 0|^{S_C} \otimes I_{F_1^C} \\ \text{If } d_1 = 1 \implies U_{1|d_1=1}^D &= |-\rangle\langle -|^{S_C} \otimes X_{F_1^C} + |+\rangle\langle +|^{S_C} \otimes I_{F_1^C}. \end{aligned} \tag{39}$$

In this fine-grained view we reinterpret the probabilities in Eq.(27) and Eq.(26), from which Eq.(30) was derived. Recalling that the effective input state in Alice's side, conditioned on Bob0 and Bob1 measuring "+", is Eq.(24),

$$\wp(c_0 = 0, d_0 = 0 \mid x_0^C = 0, y_0^D = 0, x = 0, b_0 = b_1 = +, y_0 = y_1 = 1) = 1. \tag{40}$$

By applying LA, see Fig. 10, we obtain the fine-grained version of Eq.(27).

[15] Note that Debbie0's memory register is always 0 or 1, independently on which basis she measured. Here we take, again, the usual correspondence $0 \equiv +$ and $1 \equiv -$. The conjunction of c_0 with d_0 completely specifies which was the quantum state, in either basis, that Debbie0 observed.

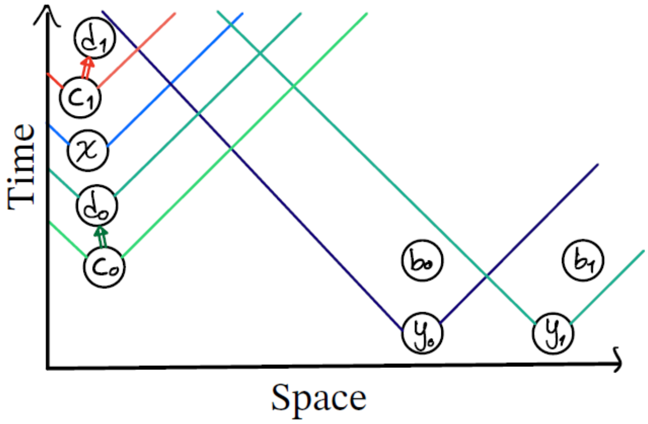

Figure 10: Space-time diagram illustrating the time ordering of the events within the fine-grained view of the LF protocol based on the LPPS Hardy paradox. The thick arrow in green (red) indicates the classical signaling from Charlie0 to Debbie0 (Debbie1 to Charlie1).

Similarly, the former Eq.(26) in this view bcomes,

$$P(c_1 = 1, d_1 = 1 \mid x_1^C = 0, y_1^D = 0, x = 1, b_0 = b_1 = +, y_0 = y_1 = 1) = 1. \tag{41}$$

As seen before the previous probabilities are marginals constraining the following probability, in the fine-grained view, to be a certainty,

$$P(c_0 = 0, d_0 = 0, c_1 = 1, d_1 = 1 | x_0^C = 0, y_0^D = 0, x_1^C = 0, y_1^D = 0, x = 1, b_0 = b_1 = +, y_0 = y_1 = 1) = 1 \tag{42}$$

It is important to note that we are implicitly assuming that all the probabilities above are conditioned on Alice successfully post-selecting the state $|- - - -\rangle_{S_C S_D F_0^C F_0^D}$ after opening the corresponding lab.

5.2.1 Weak Measurements Version

As we did in the coarse-grained interpretation, Section. 5.1.1, we first consider that each lab is weakly coupled to a measurement device that records the measurement outcome obtained within the lab, and condition to successfully post-selecting the state $|\phi\rangle = |- - - -\rangle_{S_C S_D F_C F_D}$. As derive in Appendix. , we recover the same weak values associated with measurements M_0 and M_1. However, surprisingly, in the 2-qubit computational basis the former AWV with value -1 in the LPPS Hardy paradox, Section. 3.3, now is not anomalous but just a weak value of 1. We comment on this results and a possible extension to the second weak measurement protocol presented in Section. 5.1.1 in the next section.

As in the coarse-grained interpretation discussed in Section. 5.1.1, we begin by considering that each lab is weakly coupled to a measurement device that records the measurement outcome obtained within the lab, conditional on successfully post-selecting the state $|\phi\rangle = |- - - -\rangle_{S_C S_D F_C F_D}$. As derived in Appendix. D.5, we recover the same weak values associated with the measurements M_0 and M_1. However, in the two-qubit computational basis, the AWV of -1 found in the LPPS Hardy paradox, Section 3.3, is no longer anomalous—it becomes a regular weak value of 1. We comment on these results and discuss a possible extension to the second weak measurement protocol presented in Section 5.1.1 in the next section.

6 Conclusions and Outlook

In this thesis, we first made explicit the causal asymmetry between the intermediate measurements M_0 and M_1 highlighted in [Dou22], by constructing a LOCC protocol involving a single round of classical communication. This protocol reproduces the probabilities found in the LPPS Hardy paradox and provides a constructive proof of the KS theorem.

Next, we developed a LF scenario grounded in the KS contextuality featured in the LPPS Hardy paradox, based on the general framework introduced in [WYWS24]. We analyzed this scenario from two complementary perspectives: the coarse-grained and the fine-grained views.

In the coarse-grained view, we considered three superobservers and two friends, each enclosed in a sealed laboratory. The friends perform the two-qubit measurements M_0 and M_1, respectively. By applying the LF assumptions, we derived a logical contradiction identical to that of the LPPS Hardy paradox, thereby establishing a proof of the LF no-go theorem. We then examined two possible extensions in which the intermediate measurements are weak. In the first, the entire lab is weakly coupled to a probe that records weak values. In the second, each friend weakly couples the system to a measurement device and then post-selects the desired outcome, keeping only successful runs. Upon opening the lab, the superobserver has access to the friend's meter reading and hence to the weak value. In both cases, we recovered the same weak values and the AWV of the original LPPS Hardy paradox. Therefore, in our coarse-grained LF scenario, AWVs remain valid witnesses of contextuality.

In the fine-grained view, we translated the LOCC implementation of the LPPS Hardy paradox into a setup similar to the coarse-grained one, now involving four friends (two for each original friend). Each of these observers yields an absolute observation, increasing the total number of observed outcomes. As before, we used the contextuality of the LPPS Hardy paradox to establish the LF no-go theorem in this setting. We then considered weak measurements in this view. When each lab is directly weakly coupled to an external meter, no AWV arises. We predict that adopting the second weak measurement scheme (where weak interactions and post-selection are performed inside the lab) would similarly yield no AWV, as it effectively amounts to computing weak values of single-qubit observables, which are known not to exhibit anomalous values [ABP+02]. This lasts results may suggest that the presence of AWVs could depend sensitively on how multi-agent scenarios are modeled.

These results motivate further investigation into whether AWVs remain reliable witnesses of contextuality in settings such as the fine-grained view. Additionally, this work could be extended to explore not only KS-type contextuality but also generalized contextuality, particularly in light of recent no-go theorems such as [WC25b]. Another promising direction is to compare the LF scenario developed here with both standard formulations and with the specific LF Hardy-type scenario proposed in [SYL23]. Advancing along these lines could ultimately enable a general construction that maps any LPPS paradox into an extended Wigner's Friend scenario.

Bibliography

[AAV88] Yakir Aharonov, David Z. Albert, and Lev Vaidman. How the result of a measurement of a component of the spin of a spin-1/2particle can turn out to be 100. *Physical Review Letters*, 60(14):1351–1354, April 1988.

[AB11] Samson Abramsky and Adam Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(11):113036, November 2011.

[ABL64] Yakir Aharonov, Peter G. Bergmann, and Joel L. Lebowitz. Time symmetry in the quantum process of measurement. *Physical Review*, 134(6B):B1410–B1416, June 1964.

[ABP+02] Yakir Aharonov, Alonso Botero, Sandu Popescu, Benni Reznik, and Jeff Tollaksen. Revisiting hardy's paradox: counterfactual statements, real measurements, entanglement and weak values. *Physics Letters A*, 301(3–4):130–138, August 2002.

[AGR82] Alain Aspect, Philippe Grangier, and Gérard Roger. Experimental realization of einstein-podolsky-rosen-bohmgedankenexperiment: A new violation of bell's inequalities. *Physical Review Letters*, 49(2):91–94, July 1982.

[AV91] Y Aharonov and L Vaidman. Complete description of a quantum system at a given time. *Journal of Physics A: Mathematical and General*, 24(10):2315–2328, May 1991.

[BBC+10] Piotr Badziag, Ingemar Bengtsson, Adán Cabello, Helena Granström, and Jan-Åke Larsson. Pentagrams and paradoxes. *Foundations of Physics*, 41(3):414–423, March 2010.

[Bel64] J. S. Bell. On the einstein podolsky rosen paradox. *Physics Physique* , 1(3):195–200, November 1964.

[Bru18] Časlav Brukner. A no-go theorem for observer-independent facts. *Entropy*, 20(5):350, May 2018.

[BUAG+20] Kok-Wei Bong, Aníbal Utreras-Alarcón, Farzad Ghafari, Yeong-Cherng Liang, Nora Tischler, Eric G. Cavalcanti, Geoff J. Pryde, and Howard M. Wiseman. A strong no-go theorem on the wigner's friend paradox. *Nature Physics*, 16(12):1199–1205, August 2020.

[CBTCB13] Adán Cabello, Piotr Badziag, Marcelo Terra Cunha, and Mohamed Bourennane. Simple hardy-like proof of quantum contextuality. *Physical Review Letters*, 111(18), October 2013.

[CHSH69] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. Proposed experiment to test local hidden-variable theories. *Physical Review Letters*, 23(15):880–884, October 1969.

[Cli93] Rob Clifton. Getting contextual and nonlocal elements-of-reality the easy way. *American Journal of Physics*, 61(5):443–447, May 1993.

[Deu85] David Deutsch. Quantum theory as a universal physical theory. *International Journal of Theoretical Physics*, 24(1):1–41, January 1985.

[Dou22] Hippolyte Dourdent. *Causal and logical loops from quantum theory.* Theses, Université Grenoble Alpes [2020-....], February 2022.

[EPR35] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, 47(10):777–780, May 1935.

[FR18] Daniela Frauchiger and Renato Renner. Quantum theory cannot consistently describe the use of itself. *Nature Communications*, 9(1), September 2018.

[Har92] Lucien Hardy. Quantum mechanics, local realistic theories, and lorentz-invariant realistic theories. *Physical Review Letters*, 68(20):2981–2984, May 1992.

[Har93] Lucien Hardy. Nonlocality for two particles without inequalities for almost all entangled states. *Physical Review Letters*, 71(11):1665–1668, September 1993.

[KLP19] Ravi Kunjwal, Matteo Lostaglio, and Matthew F. Pusey. Anomalous weak values and contextuality: Robustness, tightness, and imaginary parts. *Physical Review A*, 100(4), October 2019.

[KS67] Simon Kochen and E. Specker. The problem of hidden variables in quantum mechanics. *Indiana University Mathematics Journal*, 17(1):59–87, 1967.

[KW20] Sumeet Khatri and Mark M. Wilde. Principles of quantum communication theory: A modern approach, 2020.

[Leg05] A. J. Leggett. The quantum measurement problem. *Science*, 307(5711):871–872, February 2005.

[LS05a] M S Leifer and R W Spekkens. Logical pre- and post-selection paradoxes, measurement-disturbance and contextuality. *International Journal of Theoretical Physics*, 44(11):1977–1987, November 2005.

[LS05b] M. S. Leifer and Robert W. Spekkens. Pre- and post-selection paradoxes and contextuality in quantum mechanics. *Physical Review Letters*, 95(20), November 2005.

[Mer90] N D Mermin. Simple unified form for the major no-hidden-variables theorems. *Phys. Rev. Lett.*, 65(27):3373–3376, December 1990.

[Mer93] N. David Mermin. Hidden variables and the two theorems of john bell. *Reviews of Modern Physics*, 65(3):803–815, July 1993.

[Mon23] Sidiney B. Montanhano. Wigner and friends, a map is not the territory! contextuality in multi-agent paradoxes, 2023.

[MS20] A. Matzkin and D. Sokolovski. Wigner-friend scenarios with noninvasive weak measurements. *Physical Review A*, 102(6), December 2020.

[NV25] Nuriya Nurgalieva and V. Vilasini. Any theory that admits a wigner's friend type multi-agent paradox is logically contextual, 2025.

[Per90] Asher Peres. Incompatible results of quantum measurements. *Physics Letters A*, 151(3–4):107–108, December 1990.

[Per91] A Peres. Two simple proofs of the kochen-specker theorem. *Journal of Physics A: Mathematical and General*, 24(4):L175–L178, February 1991.

[Per92] Asher Peres. Recursive definition for elements of reality. *Foundations of Physics*, 22(3):357–361, March 1992.

[per02] *Quantum Theory: Concepts and Methods*. Springer Netherlands, 2002.

[PL15] Matthew F. Pusey and Matthew S. Leifer. Logical pre- and post-selection paradoxes are proofs of contextuality. *Electronic Proceedings in Theoretical Computer Science*, 195:295–306, November 2015.

[Pus14] Matthew F. Pusey. Anomalous weak values are proofs of contextuality. *Physical Review Letters*, 113(20), November 2014.

[SGB+14] Ralph Silva, Yelena Guryanova, Nicolas Brunner, Noah Linden, Anthony J. Short, and Sandu Popescu. Pre- and postselected quantum states: Density matrices, tomography, and kraus operators. *Physical Review A*, 89(1), January 2014.

[SYL23] David Schmid, Yìlè Yīng, and Matthew Leifer. A review and analysis of six extended wigner's friend arguments, 2023.

[vN96] John von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Springer Berlin Heidelberg, 1996.

[WBPW24] Laurens Walleghem, Rui Soares Barbosa, Matthew Pusey, and Stefan Weigert. A refined frauchiger–renner paradox based on strong contextuality, 2024.

[WC25a] Laurens Walleghem and Lorenzo Catani. An extended wigner's friend no-go theorem inspired by generalized contextuality, 2025.

[WC25b] Laurens Walleghem and Lorenzo Catani. An extended wigner's friend no-go theorem inspired by generalized contextuality, 2025.

[Wig95] E. P. Wigner. *Remarks on the Mind-Body Question*, page 247–260. Springer Berlin Heidelberg, 1995.

[WYWS24] Laurens Walleghem, Yìlè Yīng, Rafael Wagner, and David Schmid. Connecting extended wigner's friend arguments and noncontextuality, 2024.

A The Geometry of the Kochen-Specker Theorem.

The geometric view of the KS theorem can be constructed by noting that in a d-dimensional Hilbert space, any set, $\{P_i\}_{i=1}^d$, of rank-1 projectors obtained from a non-degenerate observable, i.e., a context, satisfies

$$\begin{aligned} &(\mathbf{O}\text{rthogonality}) \quad P_i P_j = 0 \;\; \forall i \neq j, \\ &(\mathbf{C}\text{ompleteness}) \quad \sum_{i=1}^{d} P_i = \mathbb{1}. \end{aligned}$$

This properties transfer to the boolean (represented in binary) variables, resulting from value assignment, $\{x_i = \nu(P_i)\}_{i=1}^d$ as

$$\begin{aligned} &(\mathbf{O'}) \quad x_i \wedge x_j = 0 \; \forall i \neq j, \text{ i.e., } x_i \text{ and } x_j \text{ cannot be both } 1, \\ &(\mathbf{C'}) \quad \exists! x_k = 1 \text{ from } \{x_i\}. \end{aligned}$$

If we choose to represent the vectors associated with each projector by connecting two orthogonal vectors, represented as nodes, with an edge we can interpret conditions (**O'**) and (**C'**) as coloring constraints on this orthogonality graph. As an example, let us consider the *probabilistic* proof of the KS theorem given by the representation in Fig. 11 [Cli93].

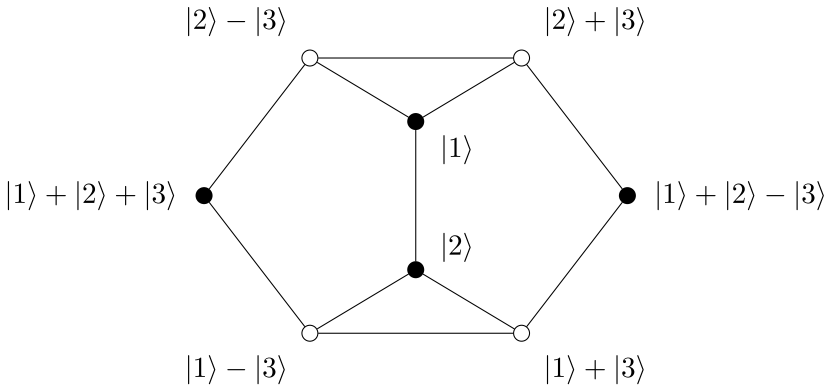

Figure 11: Diagrammatic-probabilistic proof of the KS theorem. Nodes represent vectors, each associated with a rank-1 projector, and edges their orthogonality relation. An assignment of value 1 corresponds to a colored node, while an assignment of value 0 to white. The proof involves eight 3-dimensional vectors written in the computational basis $\{|i\rangle\}_{i=1}^3$, e.g. $|1\rangle \doteq (1,0,0)^T$, and two contexts: $\{|2\rangle - |3\rangle, |2\rangle + |3\rangle, |1\rangle\}$ and $\{|1\rangle - |3\rangle, |1\rangle + |3\rangle, |2\rangle\}$.

In this figure black nodes represent a value assignment of 1, while white nodes have the assigned value of 0. The so-called “probabilistic" nature of the KS theorem proof of the diagram lies in the a priori assignment of the value 1 to the vectors $|1\rangle + |2\rangle + |3\rangle$ and $|1\rangle + |2\rangle - |3\rangle$. Once these nodes are fixed to have the unit value from the start, we apply the coloring rules: (**O'**) allows at most one node to be assigned the value 1 in any pair of orthogonal vectors, (**C'**) enforces us to only have a single colored node for any set of mutually orthogonal vectors, i.e., given any context, exactly one vector must be colored. Hence the coloring in Fig. 11 leads to a contradiction that proves, probabilistically, the KS theorem, as by rule (**O'**) nodes $|1\rangle$ and $|2\rangle$ cannot be both colored.

To fully prove the KS theorem, a more complex diagram has to be employed [KS67]. Nevertheless, the diagram in Fig. 11 serves as a foundational building block in this construction.

A diagrammatic proof of the KS theorem, in a slightly different convention, is presented in Fig.(12).

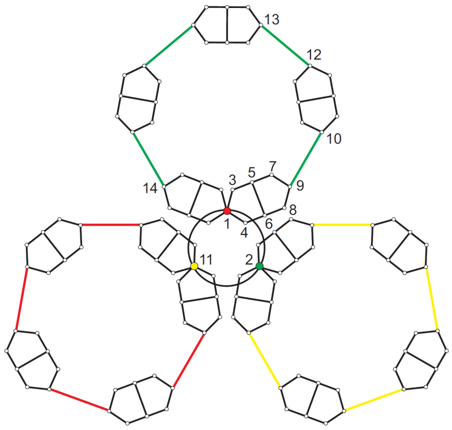

Figure 12: Diagrammatic proof of the KS theorem. It involves 117 vectors, nodes, and 118 contexts. Nodes in the same straight line or circumference form a context. If one chooses node 1 to have an assigned value of unity, by applying the coloring rules, node 14 must have also the value of the unit. Hence, the contradiction that proves the theorem.

B A mathematical framework for Quantum Measurements

A measurement process can be modeled by treating the measurement device as a quantum system that interacts with the system to be measured. Accordingly, we assign to the device a Hilbert space, spanned by the basis $\{|M_i\rangle\}_{i=0}^{d_M}$, which serves as the register, or the device's pointer, encoding the observed measurement outcome. Furthermore, we denote the intrinsic Hamiltonian of the device as $H_{\mathcal{M}}$. The total Hamiltonian comprising the system, the device, and the interactions modeling the actual measurement process is then given by,

$$H_{\mathcal{T}} = H_{\mathcal{S}} + H_{\mathcal{M}} + H_{\text{int}}. \tag{43}$$

Hence, the initial state to be measured corresponds to $|\psi\rangle_{\mathcal{S}} \otimes |M\rangle_{\mathcal{M}}$, with $|M\rangle_{\mathcal{M}}$ the "ready" state of the device. Assuming that the measurement process can be modeled as a unitary evolution,

$$U = \exp\left(-i/\hbar \int dt H_{\mathcal{T}}\right). \tag{44}$$

We restrict the unitary evolution to act on the initial state as,

$$|\psi\rangle_{\mathcal{S}} \otimes |M\rangle_{\mathcal{M}} \xrightarrow{U} \sum_{i=0}^{d_S} c_i \, |i\rangle_{\mathcal{S}} \otimes |M_i\rangle_{\mathcal{M}}\,, \tag{45}$$

expanding the system's state in its computational basis, we require $d_S \leq d_M$, i.e., the measurement device must have sufficient range to distinguish all possible outcomes of the system. From this, we see that the action of the unitary operator entangles the system with the measurement device, this marks the so-called *pre-measurement* stage. This is to be followed by the *read-out* stage, i.e., the post-selection of a particular state of the meter, which collapses the superposed state and thereby completes the quantum measurement process.

C Derivations in the LPPS Hardy Paradox

Here we make explicit the probabilities in Eq.(18),

$$
\begin{aligned}
P(a=0,b=0|ok,x=0,y=0,M_0) &= \frac{Tr(\Pi_{ok}K_{00}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,K_{00}^{\dagger})}{\sum_{a'b'}Tr(\Pi_{ok}K_{a'b'}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,K_{a'b'}^{\dagger})} \\
&= \frac{|1/\sqrt{3}\,\langle--|00\rangle\,|^2}{|1/\sqrt{3}\,\langle--|00\rangle\,|^2+\sqrt{2/3}\,\langle--|1+\rangle} \\
&= 1 \\
P(a=1,b=1|ok,x=0,y=1,M_0) &= \frac{Tr(\Pi_{ok}K_{11}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,K_{11}^{\dagger})}{\sum_{a'b'}Tr(\Pi_{ok}K_{a'b'}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,K_{a'b'}^{\dagger})} \\
&= 0 \\
P(a=1,b=1|ok,x=0,y=0,M_1) &= \frac{Tr(\Pi_{ok}L_{11}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,L_{11}^{\dagger})}{\sum_{a'b'}Tr(\Pi_{ok}L_{a'b'}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,L_{a'b'}^{\dagger})} \\
&= \frac{|1/\sqrt{3}\,\langle--|11\rangle\,|^2}{|1/\sqrt{3}\,\langle--|11\rangle\,|^2+\sqrt{2/3}\,\langle--|+0\rangle} \\
&= 1 \\
P(a=0,b=0|ok,x=1,y=0,M_1) &= \frac{Tr(\Pi_{ok}L_{00}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,L_{00}^{\dagger})}{\sum_{a'b'}Tr(\Pi_{ok}L_{a'b'}\,|\psi_{\text{Hardy}}\rangle\langle\psi_{\text{Hardy}}|\,L_{a'b'}^{\dagger})} \\
&= 0,
\end{aligned}
\tag{46}
$$

D Derivations in the LF scenario based on the LPPS Hardy Paradox

D.1 Probabilities in the coarse-grained view

Here we make explicit the expressions needed in order to compute the following probabilities,

$$\wp(f_0 = 1 \mid b_0 = b_1 = +, x = 0, y_0 = y_1 = 1) = \frac{Tr\left(|\phi\rangle\langle\phi|\, U_0^{(0,0)} \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| (U_0^{(0,0)})^\dagger\right)}{\sum_{ab} Tr\left(|\phi\rangle\langle\phi|\, U_0^{(a,b)} \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| (U_0^{(a,b)})^\dagger\right)} = 1, \tag{47}$$

where the sum takes values $(a,b) \in \{(0,0),(1,+),(1,-),(0,1)\}$ and $U_0^{(a,b)} \doteq |ab\rangle\langle ab|^{S_C S_D}\, U_0$, with U_0 as defined in Eq.(22).

Similarly,

$$\wp(f_1 = 1 \mid b_0 = b_1 = +, x = 1, y_0 = y_1 = 1) = \frac{Tr\left(|\phi\rangle\langle\phi|\, U_1^{(1,1)} \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| (U_1^{(1,1)})^\dagger\right)}{\sum_{ab} Tr\left(|\phi\rangle\langle\phi|\, U_1^{(a,b)} \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| (U_1^{(a,b)})^\dagger\right)} = 1, \tag{48}$$

where now the sum takes values $(a,b) \in \{(1,1),(+,0),(-,0),(0,1)\}$ and $U_1^{(a,b)} \doteq |ab\rangle\langle ab|^{S_C S_D}\, U_1$, with U_1 as defined in Eq.(22).

D.2 Weak Values derivations in the coarse-grained View

Here we detail the computations of the weak values mentioned in Section. 5.1.1. For the weak values associated to M_0 we first apply the weak intermediate measurement, Eq.(34) onto the initial state Eq.(32),

$$\begin{aligned} |\Psi_1\rangle &= U_0^{\text{weak}}(\left|\tilde{\psi}_i\right\rangle \otimes |m_0\rangle) \otimes |0\rangle \\ &= \left[\left|\tilde{\psi}_i\right\rangle \otimes |m_0\rangle - \frac{1}{2\sqrt{3}} ig \left(A_0\,|00\rangle + \frac{2}{\sqrt{2}} A_0\,|1+\rangle\right) \otimes \Gamma_0\,|m_0\rangle\right] \otimes |0\rangle\,. \end{aligned} \tag{49}$$

After this measurement, Friend0 post-selects $|\phi\rangle = |{-}{-}\rangle_S$, which is seen as Eq.(36) by Alice,

$$\begin{aligned} |\Psi_2\rangle &= U_0^{ok}\,|\Psi_1\rangle \\ &= \frac{1}{2\sqrt{3}} \langle{-}{-}|00\rangle \left(|{-}{-}\rangle - ig\,|{-}{-}\rangle \underbrace{\frac{\langle{-}{-}|\, A_0\,|00\rangle}{\langle{-}{-}|00\rangle}}_{w(A_0|\left|\tilde{\psi}_i\right\rangle,|\phi\rangle)=1} \Gamma_0\right) \otimes |m_0\rangle \otimes |1\rangle + \\ &\quad + \frac{1}{\sqrt{6}} (I - |\phi\rangle\langle\phi|)(1 - igA_0\Gamma_0)\,|1+\rangle \otimes |m_0\rangle \otimes |0\rangle\,. \end{aligned} \tag{50}$$

Similar computations for M_1 yield the weak value $w(A_1|\left|\tilde{\psi}_i\right\rangle, |\phi\rangle) = 1$.

For the 2-qubit computational basis performed by any of the two friends, $i \in \{0,1\}$, the weak values are derived as follows,

$$\begin{aligned}|\Psi_1\rangle &= U_a^{\text{weak}}(\left|\tilde{\psi}_i\right\rangle \otimes |m_0\rangle) \otimes |0\rangle \\ &= \left[\left|\tilde{\psi}_i\right\rangle \otimes |m_0\rangle - igP_a\left|\tilde{\psi}_i\right\rangle \otimes \Gamma|m_0\rangle\right] \otimes |0\rangle\,,\end{aligned} \tag{51}$$

to be followed by post-selection,

$$\begin{aligned}|\Psi_2\rangle &= U_i^{ok}|\Psi_1\rangle \\ &= \left\langle\phi\middle|\tilde{\psi}_i\right\rangle \Bigg(1 - ig\underbrace{\frac{\langle\phi|P_a\left|\tilde{\psi}_i\right\rangle}{\left\langle\phi\middle|\tilde{\psi}_i\right\rangle}}_{w(P_a|\left|\tilde{\psi}_i\right\rangle,|\phi\rangle)}\Gamma_i\Bigg) \otimes |\phi\rangle \otimes |m_0\rangle \otimes |1\rangle \\ &+ \left\langle ++\middle|\tilde{\psi}_i\right\rangle\left(1 - ig\frac{\langle ++|P_a\left|\tilde{\psi}_i\right\rangle}{\left\langle ++\middle|\tilde{\psi}_i\right\rangle}\Gamma_i\right) \otimes |++\rangle \otimes |m_0\rangle \otimes |0\rangle \\ &+ \left\langle +-\middle|\tilde{\psi}_i\right\rangle\left(1 - ig\frac{\langle +-|P_a\left|\tilde{\psi}_i\right\rangle}{\left\langle +-\middle|\tilde{\psi}_i\right\rangle}\Gamma_i\right) \otimes |+-\rangle \otimes |m_0\rangle \otimes |0\rangle \\ &+ \left\langle -+\middle|\tilde{\psi}_i\right\rangle\left(1 - ig\frac{\langle -+|P_a\left|\tilde{\psi}_i\right\rangle}{\left\langle -+\middle|\tilde{\psi}_i\right\rangle}\Gamma_i\right) \otimes |-+\rangle \otimes |m_0\rangle \otimes |0\rangle\,.\end{aligned} \tag{52}$$

From this, we note that the only valid weak value corresponds to the runs in which the friend's memory register qubit is flipped, indicating that the state $|\phi\rangle$ was successfully post-selected. The runs in which this post-selection fails are discarded by the friend.

D.3 coarse-grained and fine-grained Equivalency

We show that we recover the Kraus operators, Eq.(11), that implement the LOCC protocol between Cahrlie0 and Debbie0 from Alice's unitary sequence $U_0 \doteq U^D_{0|c_0}U^C_0$, this is just the equivalency between Kraus operators and Stinespring dilation of a channel, i.e.,

$$\rho \mapsto Tr_{F_0^C F_0^D}\left[U_0(\rho \otimes |00\rangle\langle 00|^{F_0^C F_0^D})U_0^\dagger\right] = \sum_\alpha K_\alpha \rho K_\alpha^\dagger, \quad \forall \rho, \tag{53}$$

¡ with $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ a basis spanning the environment, from Alice's point of view, $F_0^C \otimes F_0^D$, and K_α the kraus operators of this LOCC channel defined in Eq.(11) with $\alpha \in \{00, +1, 1+, 1-\}$. From the previous equation follows that,

$$K_\alpha = {}_{F_0^C F_0^D}\langle\alpha| U_0 |00\rangle_{F_0^C F_0^D}\,, \tag{54}$$

where $|\alpha\rangle_{F_0^C F_0^D} = |c_0, d_0\rangle_{F_0^C F_0^D}$, with $c_0.d_0 \in \{0,1\}$.

First, we note that

$$_{F_0^C}\langle c_0| U_0^C |0\rangle_{F_0^C} = |c_0\rangle\langle c_0|^{S_C}\,, \quad c_0 \in \{0,1\}, \tag{55}$$

as Charlie0 always measures his qubit on the computational basis. Once Debbie0 knows Charlie0's outcome she applies Eq.(38a) or Eq.(38b),

$$
{}_{F_0^D}\langle d_0|\, U^D_{0|c_0}\, |0\rangle_{F_0^D} = \\
= \begin{cases} c_0 = 0 \implies \begin{cases} |0\rangle\!\langle 0|^{S_D}\,, & d_0 = 0, \\ |1\rangle\!\langle 1|^{S_D}\,, & d_0 = 1 \end{cases} \\ c_0 = 1 \implies \begin{cases} |+\rangle\!\langle +|^{S_D}\,, & d_0 = 0 \equiv +, \\ |-\rangle\!\langle -|^{S_D}\,, & d_0 = 1 \equiv - \end{cases} \end{cases} \tag{56}
$$

Hence, the tensor product between these individual Kraus operators of each friend recovers the LOCC's Kraus operators of Eq.(11). The same computations follow for U_1 and the Kraus operators of Eq.(14).

Now, we show that $U_0 \doteq U^D_{0|c_0} U^C_0$ is not simply a definition but the consistency requirement between the coarse-grained, implemented by Eq.(22), and fine-grained interpretation. Let us consider the 2-qubit basis $\{|00\rangle, |01\rangle, |1+\rangle, |1-\rangle\}$, then the action of $U^D_{0|c_0} U^C_0$ on this basis is the following,

$$
\begin{aligned}
U_0\,|00\rangle_{S_C S_D} \otimes |00\rangle_{F_0^C F_0^D} &\to |00\rangle \otimes |00\rangle \\
U_0\,|01\rangle_{S_C S_D} \otimes |00\rangle_{F_0^C F_0^D} &\to |01\rangle \otimes |01\rangle \\
U_0\,|1+\rangle_{S_C S_D} \otimes |00\rangle_{F_0^C F_0^D} &\to |1+\rangle \otimes |10\rangle \\
U_0\,|1-\rangle_{S_C S_D} \otimes |00\rangle_{F_0^C F_0^D} &\to |1-\rangle \otimes |11\rangle\,.
\end{aligned} \tag{57}
$$

Hence, we recover the action of U_0 in Eq. (22) within the same basis by considering that the memory state of Friend0 corresponds to the logical OR between the registers of Charlie0 and Debbie0.

D.4 Probabilities in the fine-grained View

In this fine-grained view we interpret the former probabilities in Eq.(27) and Eq.(26), from which Eq.(30) was derived. The first of them is seen translates as follows by recalling that the effective input state in Alice's side, conditioned on Bob0 and Bob1 measuring "+", is Eq.(24),

$$
\begin{aligned}
&\wp(c_0 = 0, d_0 = 0 \mid x_0^C = 0, y_0^D = 0, x = 0, b_0 = b_1 = +, y_0 = y_1 = 1) = \\
&= \frac{Tr\left(|\phi\rangle\!\langle\phi|\, U_0^{(c_0,d_0)} \left|\tilde{\psi}_i\right\rangle\!\left\langle\tilde{\psi}_i\right| (U_0^{(c_0,d_0)})^\dagger\right)}{\sum_{c_0' d_0'} Tr\left(|\phi\rangle\!\langle\phi|\, U_0^{(c_0',d_0')} \left|\tilde{\psi}_i\right\rangle\!\left\langle\tilde{\psi}_i\right| (U_0^{(c_0',d_0')})^\dagger\right)} \\
&= \frac{\left|\langle\phi|U_0^{(0,0)}|\tilde{\psi}_i\rangle_{S_C S_D}\right|^2}{\left|\langle\phi|U_0^{(0,0)}|\tilde{\psi}_i\rangle_{S_C S_D}\right|^2 + \left|\langle\phi|U_0^{(1,+)}|\tilde{\psi}_i\rangle_{S_C S_D}\right|^2} = 1,
\end{aligned} \tag{58}
$$

where $|\phi\rangle_{S_C S_D F_0^C F_0^D} = |----\rangle$ is the state that Alice is post-selecting and $U_0^{c_0,d_0}$ is the unitary resulting from the composition $U^D_{0|c_0} U^C_0$ for the given outcomes (c_0, d_0). Note that last equation follows from $\left|\langle\phi|U_0^{(1,+)}|\tilde{\psi}_i\rangle_{S_C S_D}\right|^2 = 0$, as $\langle\phi|\,1+110\rangle_{S_C S_D F_0^C M F_0^D} = 0$ and that applying, again, LA we obtain the fine-grained counterpart of Eq.(27).

Similarly, we recover Eq.(26),

$$
\begin{aligned}
&P(c_1 = 1, d_1 = 1 \mid x_1^C = 0, y_1^D = 0, x = 1, b_0 = b_1 = +, y_0 = y_1 = 1) = \\
&= \frac{Tr\left(|\phi\rangle\langle\phi|\, U_1^{(c_1,d_1)} \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| (U_1^{(c_1,d_1)})^\dagger\right)}{\sum_{c_1' d_1'} Tr\left(|\phi\rangle\langle\phi|\, U_1^{(c_1',d_1')} \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| (U_1^{(c_1',d_1')})^\dagger\right)} \\
&= \frac{\left|\langle\phi|U_1^{(1,1)}|\tilde{\psi}_i\rangle_{S_C S_D}\right|^2}{\left|\langle\phi|U_1^{(1,1)}|\tilde{\psi}_i\rangle_{S_C S_D}\right|^2 + \left|\langle\phi|U_1^{(+,0)}|\tilde{\psi}_i\rangle_{S_C S_D}\right|^2} = 1,
\end{aligned}
\tag{59}
$$

with $U_1^{c_1,d_1}$ is the unitary resulting from the composition $U_{1|d_1}^C U_1^D$, as defined in Eq.(39), for the given outcomes (c_1, d_1). Note that last equation follows from $\left|\langle\phi|U_1^{(+,0)}|\tilde{\psi}_i\rangle_{S_C S_D}\right|^2 = 0$, as $\langle\phi| + 000\rangle_{S_C S_D F_1^C F_1^D} = 0$.

D.5 Weak Values derivations in the fine-grained View

We consider the intermediate measurements to be weak in the fine-grained view, and we analyze the weak values obtained for the projectors associated with M_0, specifically within the LOCC protocol from Charlie0 to Debbie0. We begin with the projector $|00\rangle\langle 00|^{S_C S_D}$, which corresponds to the classical outcomes $c_0 = d_0 = 0$,

$$
\begin{aligned}
w(c_0 = 0 = d_0|\phi, \psi) &= \frac{{}_{S_C S_D F_0^C F_0^D}\langle - - - -|\, U_0^{(0,0)} \left|\tilde{\psi}\right\rangle_{S_C S_D}}{{}_{S_C S_D F_0^C F_0^D}\left\langle - - - - \middle|\tilde{\psi}\right\rangle_{S_C S_D} \otimes |00\rangle_{F_0^C F_0^D}} \\
&= 1
\end{aligned}
\tag{60}
$$

For the projector $|1+\rangle\langle 1+|$ associated with $c_0 = 1$ and $d_0 = +$,

$$
\begin{aligned}
w(c_0 = 1, d_0 = +|\phi, \psi) &= \frac{{}_{S_C S_D F_0^C F_0^D}\langle - - - -|\, U_0^{(1,+)} \left|\tilde{\psi}\right\rangle_{S_C S_D}}{{}_{S_C S_D F_0^C F_0^D}\left\langle - - - - \middle|\tilde{\psi}\right\rangle_{S_C S_D} \otimes |00\rangle_{F_0^C F_0^D}} \\
&= 0.
\end{aligned}
\tag{61}
$$

For the projectors associated with M_1, i.e., in the LOCC protocol from Debbie1 to Charlie1. First for $|11\rangle\langle 11|$ associated with the classical outcomes $c_1 = d_1 = 1$,

$$
\begin{aligned}
w(c_1 = 1 = d_1|\phi, \psi) &= \frac{{}_{S_C S_D F_1^C F_1^D}\langle - - - -|\, U_1^{(1,1)} \left|\tilde{\psi}\right\rangle_{S_C S_D}}{{}_{S_C S_D F_1^C F_1^D}\left\langle - - - - \middle|\tilde{\psi}\right\rangle_{S_C S_D} \otimes |00\rangle_{F_1^C F_1^D}} \\
&= 1
\end{aligned}
\tag{62}
$$

For the projector $|+0\rangle\langle +0|$ associated with $c_1 = +$ and $d_1 = 0$,

$$
\begin{aligned}
w(c_1 = +, d_1 = 0|\phi, \psi) &= \frac{{}_{S_C S_D F_1^C F_1^D}\langle - - - -|\, U_1^{(+,0)} \left|\tilde{\psi}\right\rangle_{S_C S_D}}{{}_{S_C S_D F_1^C F_1^D}\left\langle - - - - \middle|\tilde{\psi}\right\rangle_{S_C S_D} \otimes |00\rangle_{F_1^C F_1^D}} \\
&= 0
\end{aligned}
\tag{63}
$$

If we now consider the projectors in the 2-qubit computational basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$. The stinnespring dilation of the measurement on this basis, is just two CNOT gates one acting on S_C as control qubit and F_C as target, and similarly for S_D and F_D, i.e., $V = CNOT_{S_C \to F_C} CNOT_{S_D \to F_D} = |1\rangle\langle 1|^{S_C} \otimes |1\rangle\langle 1|^{S_D} \otimes X_{F_C} \otimes X_{F_D} + |0\rangle\langle 0|^{S_C} \otimes |0\rangle\langle 0|^{S_D} \otimes I \otimes I$. Hence,

$$V \left|\tilde{\psi}\right\rangle = \frac{1}{2\sqrt{3}} (|0000\rangle + |1111\rangle + |1010\rangle)_{S_C S_D F_C F_D} \tag{64}$$

Then the weak values are,

$$\begin{aligned} w(|0101\rangle\langle 0101|\, |\phi, \psi) &= \frac{{}_{S_C S_D F^C F^D}\langle - - - -|\, V^{(0,1)} \left|\tilde{\psi}\right\rangle_{S_C S_D}}{{}_{S_C S_D F^C F^D}\left\langle - - - -\middle|\tilde{\psi}\right\rangle_{S_C S_D} \otimes |00\rangle_{F^C F^D}} \\ &= 0 \\ w(|1111\rangle\langle 1111|\, |\phi, \psi) &= \frac{{}_{S_C S_D F^C F^D}\langle - - - -|\, V^{(1,1)} \left|\tilde{\psi}\right\rangle_{S_C S_D}}{{}_{S_C S_D F^C F^D}\left\langle - - - -\middle|\tilde{\psi}\right\rangle_{S_C S_D} \otimes |00\rangle_{F^C F^D}} \\ &= 1 \\ w(|000\rangle\langle 0000|\, |\phi, \psi) &= \frac{{}_{S_C S_D F^C F^D}\langle - - - -|\, V^{(0,0)} \left|\tilde{\psi}\right\rangle_{S_C S_D}}{{}_{S_C S_D F^C F^D}\left\langle - - - -\middle|\tilde{\psi}\right\rangle_{S_C S_D} \otimes |00\rangle_{F^C F^D}} \\ &= 1 \\ w(|1010\rangle\langle 1010|\, |\phi, \psi) &= \frac{{}_{S_C S_D F^C F^D}\langle - - - -|\, V^{(1,0)} \left|\tilde{\psi}\right\rangle_{S_C S_D}}{{}_{S_C S_D F^C F^D}\left\langle - - - -\middle|\tilde{\psi}\right\rangle_{S_C S_D} \otimes |00\rangle_{F^C F^D}} \\ &= 1 \end{aligned} \tag{65}$$

Note that $V^{(a,b)}$ refers to having observed the effective Hardy state in the state $|ab\rangle$ where $a, b \in \{0, 1\}$, i.e., the four states in the computational basis. As the unitary evolution leaves the effective Hardy state in a superposition of all possible outcomes, but when we post-select and consider the weak value of a particular projector, the superposition collapses to a single one.