The role of pacemakers in the Kuramoto model

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Abstract: This work explores the dynamics of synchronization in a coupled system of oscillators, focusing on the role of a mobile pacemaker. Global synchronization, under certain conditions, may emerge when a pacemaker sequentially visits distinct oscillator groups, each with heterogeneous natural frequencies. This approach is applied to the so-called Bluegrass problem, inspired by the coordination of musicians with differing tempos. Numerical simulations reveal dependencies on the pacemaker's visiting order. Our findings establish that visiting first those communities with higher synchronization levels is optimal. Considering such ordering, convergence to the global synchronized state then follows a power law described by geometric progressions.

Keywords: Non-lineal dynamics, coupled oscillators, Kuramoto model, synchronization, *Bluegrass*
SDGs: Quality education (see page 6)

I. INTRODUCTION

Collective synchronization is a widespread phenomenon that emerges in many natural and engineered systems. In a few words, local interactions cause individual units to lock to a common frequency, despite their intrinsic distinct rhythms. The first to document this phenomenon was Christiaan Huygens. He observed that two of his pendulum clocks, mounted on a common support, would synchronize their swings over time, oscillating in opposite directions [1]. Biologic examples include neurons, exhibiting intrinsic circadian rhythms [2] which synchronize to an overall average period. Networks of pacemakers in the heart self-organize to produce a heart beat [3][4]. Natural swarms of Photinus carolinus in Southeast Asia, where thousands of fireflies flash in unison as a mating strategy [5]. In engineered systems, individual components or subsystems adjust their internal state to operate in unison through shared signals. This phenomenon is observed in Josephson junctions, where superconducting currents lock in phase [6], as well as in power grids, where phase coherence ensures stable electricity distribution [7].

One of the first to formally tackle this problem was Arthur T. Winfree [8]. Introducing a mathematical simplification of the problem, the essence of the problem is captured in one degree of freedom. Using phase responses, synchronization emergence was analyzed by mean-field interactions. His model is:

$$\dot{\theta}_i = \omega_i + \left(\sum_{j=1}^{N} X(\theta_j) \right) Z(\theta_i) \qquad (1)$$

where θ_i denotes the phase for the oscillator i and ω_i its natural frequency. Each oscillator j exerts a force modeled by $X(\theta_i)$ and responds with a sensitivity response function $Z(\theta_j)$. With this, he discovered that by decreasing the spread of the natural frequencies such populations could exhibit a phase transition from an unordered state to a partially or completely ordered one.

II. THE KURAMOTO MODEL

Yoshiki Kuramoto [9] worked on collective synchronization and contributed deeply to its analysis and refining. He began working on it in 1975, providing crucial results but raising even more questions. He first recognized that the mean-field treatment was of easier tractability with a purely sinusoidal, all-to-all interaction of N oscillators. Kuramoto's model is thus described by

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, .., N \qquad (2)$$

where $K \geq 0$ is the coupling strength and ω_i is the natural frequency of the i-th oscillator. The frequencies are distributed according a probability density $g(\omega)$. For simplicity, $g(\omega)$ is assumed to be unimodal and symmetric about its mean frequency.

The factor $1/N$ ensures the model is well behaved as $N \to \infty$ since the term $\sum_{j=1}^{N}$ scales as N. We will go back to this later.

From a dynamical perspective, the system's evolution is governed by two competing terms with opposing effects. The intrinsic frequency term ω_i promotes independent evolution of the oscillators, leading to an unsynchronized state and thus acts as a disordering force. In contrast, the interaction term stimulates coherence by driving the system toward a stable fixed point where all phases align. The resulting behavior emerges from the competition between these two forces, chaos and order.

A. Order parameter

The collective rhythm will be studied through the following complex order parameter:

$$re^{i\Psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \qquad (3)$$

It is a macroscopic quantity containing the crucial information of the internal collective rhythms of the system. The modulus $r(t)$ measures the phase coherence while $\Psi(t)$ measures the mean phase. It is highly useful to visualize each oscillator as a point in the complex circumference of radius unity. When most of them move clustered around the same phase, the modulus $r \approx 1$ and Ψ returns the mean phase. On the contrary, when the oscillators are scattered and unorganized around the circle, $r \approx 0$.

A system reaching a stable value of the order parameter contains a degree of synchronization if $r_\infty > 0$. In this context, synchronization refers to phase-locking synchronization, where all oscillators maintain a constant phase difference and evolve with a common frequency.

Expressing the sine function in the Kuramoto Model (2) as an imaginary exponential, one can introduce the complex order parameter in Kuramoto's equation.

$$\dot{\theta}_i = \omega_i - Kr\sin(\theta_i - \Psi), \quad i = 1,..,N \tag{4}$$

This is the mean-field representation of the Kuramoto Model. Note how no approximation has yet been done. The mean-field character is intrinsic to Kuramoto's model from the start in (2).

B. Conditions for synchronization

Solutions to equation (4) depend on the size of $|\omega_i|$ relative to Kr. The oscillators with $|\omega_i| < Kr$ will approach a stable fixed point

$$\omega_i = Kr\sin(\theta_i - \Psi). \tag{5}$$

where they will be phase-locked to the system.

Kuramoto [9] split the population of oscillators of the steady state in "locked" and "drifting". Locked oscillators fulfill condition (5) while drifting ones do not. When the frequency deviation from the mean is considerably small, the ordering effect of the interaction succeeds in phase-locking an oscillator. On the contrary, when the frequency is highly deviated from that of the frame, the coupling may not be strong enough and those will drift away.

Working on the limit for $N \to \infty$, he defined $\rho(\theta, t, \omega)d\theta$ to be the fraction of oscillators with frequency ω that lie in the phase θ and $\theta + d\theta$ in time t.

Using definition (3) for the order parameter, the self-consistency equation shows to be

$$r = Kr\int_{-\pi/2}^{\pi/2} cos^2\theta g(Kr\sin(\theta)d\theta. \tag{6}$$

Equation (6) has a trivial solution $r = 0$ for any K. A second branch of solutions continuously bifurcates from $r = 0$ at a value $K = K_c$ obtained by letting $r \to 0^+$:

$$K_c = \frac{2}{\pi g(0)} \tag{7}$$

$g(0)$ is the centre value of the frequency distribution $g(\omega)$. This is Kuramoto's critical coupling, separating the incoherent state from the coherent state.

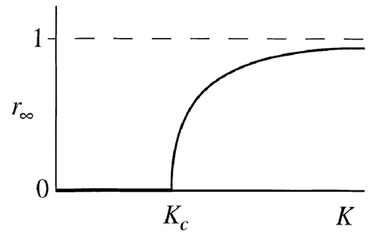

FIG. 1: Dependence of the steady state order parameter r_∞ on the coupling strength K. [10]

III. PACEMAKERS IN THE KURAMOTO MODEL

In many biologic and engineered systems, synchronization arises not only from the interaction of individual agents themselves, but from an external timing source known as a pacemaker. In the context of the Kuramoto model, a pacemaker is an external oscillator with a fixed frequency that interacts with the group and introduces external rhythms. An interesting analogy is the Bluegrass problem. Musicians in peripheral communities, lacking consistent local experts, overcome this challenge by creating festivals and jams where they can synchronize and entrain with visiting master musicians - the pacemaker. These circulating pacemakers transfer rhythms and facilitate intergroup entrainment, allowing geographically distant communities to learn from and eventually synchronize with each other not through direct interaction, but through the influence of recurring rhythmic participation guided by the circulating pacemaker.

It is of big importance to highlight that the *natural frequency* of a group is not merely a tempo, it is a way to encode internal characteristics such as skills and style. The most skilled agent is the pacemaker! The aim of this paper is to create the optimal strategy so that all groups learn as fast as possible from the master. How can the master optimize its time and teach efficiently? What should the master expect at each tour? How fast will the groups learn?

In this approach, we want to scale the pacemaker's effect to be that of a whole group. Each group g contains N_g conventional oscillators. Having l groups, the total amount of conventional agents is $N_T = \sum_{g=1}^{l} N_g$.

All oscillators in group g have an internal rhythm of ω_g. Kuramoto's equation (2) for the i-th oscillator in group g connected to the pacemaker is modified in the following fashion:

$$\dot{\theta}_i^g = \omega_g + \frac{1}{N_g}\sum_{j=1}^{N_g+1} K_j \sin(\theta_j - \theta_i), \quad i = 1,...,N_g \tag{8}$$

where the interaction term sums over the N_g oscillators in group g, denoted $\mathcal{G}_g$, plus one pacemaker. Coupling constant $K_j = K$ for $j \in \mathcal{G}_g$ and $K_j = K_{pm}$ for the interaction with the pacemaker. The pronounced effect of the pacemaker is encoded in the coupling constant K_{pm} scaling as N_g.

By defining $K_{pm} = QN_g$, the pacemaker induces an effect comparable that to of the whole group but with coupling constant Q. On the contrary, the equation of motion of the pacemaker when connected to $\mathcal{G}_g$ is simply:

$$\dot{\theta}_{pm} = \omega_{pm} + \frac{K}{N_g} \sum_{j \in \mathcal{G}_g} \sin(\theta_{pm} - \theta_i) \tag{9}$$

Where ω_{pm} is the natural rhythm of the pacemaker. Note how the interaction is symmetrical only when $K = Q$.

Once the static parameters are set, the focus of attention becomes the dynamics of the process. The Bluegrass problem concerns a master who helps l groups to synchronize with each other. Information from group 1 must travel to group 2 and vice versa. The driving force behind global synchronization is the pacemaker, therefore it must retain and "remember" its interaction with $\mathcal{G}_1$ when it moves to $\mathcal{G}_2$. In physics, this is what is known as inertia, an inner drive to continue the former motion one had. For terms of simplicity, as inertia entails higher order derivatives, the system is modeled in the following fashion.

Each interaction results in a changed $\dot{\theta}$ for all oscillators involved. The natural rhythms ω_i change as a cause of an interaction. Otherwise, once decoupled, the oscillators would return to their original dynamics, and no synchronization would occur. A straightforward choice is for ω_i to be $\dot{\theta}_i$ right before decoupling. Hence, the pacemaker visits all l groups, introducing changes in their rhythms and also conveying information from one to another. The pacemaker encodes the influence of $\mathcal{G}_1$ into its frequency ω_{pm}, which it then communicates to $\mathcal{G}_2$.

After visiting all groups, one *tour* has been completed. The master then returns to its master group, where it interacts as a conventional oscillator and recovers its original frequency. Meanwhile, the frequency shifts induced in each group persist. At the beginning of each new tour, this reset can be understood as ω_{pm} returning to its original value.

A. Analytical Approach

All analytical derivations presented in this section are original contributions of this work and have not been previously reported in the literature.

Motion equations (8) and (9) will provide a theoretical basis for the local and global interactions. It is assumed that each group remains internally synchronized while interacting with the pacemaker, and thus behaves as a unified entity.

Then, equations (8) and (9) when $\theta_i = \theta_j \quad \forall i, j \in \mathcal{G}$ simplify to:

$$\begin{cases} \dot{\theta}_i = \omega_g + Q \sin(\theta_{pm} - \theta_i) \quad i = 1, ..., N_g \\ \dot{\theta}_{pm} = w_{pm} + K \sin(\theta_i - \theta_{pm}) \end{cases} \tag{10}$$

1. *Local coupling dynamics*

This section is concerned with the group-pacemaker coupling. Group $\mathcal{G}_g$ is initialized with random initial phases ranging from 0 to 2π. With the suitable parameters for ω_g and K, the group will perfectly synchronize after some transitory period. When the internal synchronization of the group is achieved, the pacemaker is coupled. We will explore how the phase and effective frequency differences from the group to the pacemaker evolve in time.

The difference variable is introduced as $\delta = \theta_{pm} - \theta_i$. Using the previous assumption, δ does not depend on i. The effective frequency difference from one to another is described then by $\dot{\delta} = \dot{\theta}_{pm} - \dot{\theta}_i$. Subtracting equations (10), we get the equation of motion for group g for the difference variable.

$$\dot{\delta}_g = \Delta\omega_g - (Q + K) \sin(\delta) \tag{11}$$

where $\Delta\omega_g = \omega_{pm} - \omega_g$. Approximating for small differences $sin\delta \approx \delta$ and solving the differential equation, the solution is

$$\delta_g(t) = \frac{\Delta\omega_g}{Q + K} \left(1 - e^{-(Q+K)t}\right) + \delta_0 e^{-(Q+K)t} \tag{12}$$

where δ_0 is the initial difference. For $t \to \infty$, $\delta(t) \to \frac{\Delta\omega}{Q+K}$. This constant phase difference in the steady state will be defined as δ^*. Differentiating equation (12), one obtains

$$\dot{\delta}_g(t) = (\Delta\omega_g - \delta_0(Q + K))e^{-(Q+K)t}. \tag{13}$$

In this case, as t grows large, $\dot{\delta}(t) \to 0$. Equations directly show how the system tends to reach equal frequencies even if there is a constant phase difference. This result is in great accordance with the literature [11].

Once it is clear the conventional group + pacemaker system reaches a global effective frequency, we shall now be concerned with what this frequency is.

2. *Local effective frequency*

For the sake of simplicity and applicability of the equations, this section will concretely refer to the case in which $Q = K \equiv K$. Adding equations of motion (10) and further adding it to equation (13), one can isolate the evolution of the frequency in time

$$\dot{\theta}_i = \frac{w_g + w_{pm}}{2} + \frac{\Delta\omega_g - 2K\delta_0}{2} e^{-2Kt}, \tag{14}$$

concluding that the frequency exponentially relaxes to the half-sum $\Omega = \frac{w_g + w_{pm}}{2}$.

The fact that for large t $\dot{\delta}(t) \to 0$ implies that in the steady state $\dot{\theta}_i = \dot{\theta}_{pm} \equiv \Omega$.

3. Conditions for synchronization

As mentioned in section II B, not all parameters lead to a fully synchronized steady regime. The aim of this section is to provide a general expression for this condition. In the same fashion as the previous section, one can subtract both equations of motion in the steady state. Rearranging terms, we can see

$$\Delta\omega = (Q + K)\sin(\delta^*) \quad \Rightarrow \quad \Delta\omega \leq (Q + K). \tag{15}$$

The difference in the suitable natural frequencies has an upper bound dictated by the coupling constants.

It is of great significance that none of the results regarding the pacemaker coupling depend on N_g. This further confirms our hypothesis in which, when the effect of the pacemaker escalates as N, the group and the pacemaker interact one-to-one as the group moves as a unified oscillator.

B. Strategies

The order in which the pacemaker interacts with the different groups influences the overall synchronization dynamics. Since each interaction modifies the pacemaker's state, the sequence of visits impacts the efficiency of the convergence process. This section is dedicated to exploring how different ordering strategies affect the evolution toward synchronization.

One possible strategy, referred to as the *worst-first* method, consists of visiting first the group with the largest $\Delta\omega_g$ and proceeding towards those with smaller discrepancies. On the contrary, the *best-first* method starts with the groups closest to synchronization and proceeds towards the least synchronized ones.

Physical intuition suggests that the best-first strategy is preferable. By first visiting the most synchronized groups, the pacemaker's frequency is preserved closer to its initial value, thereby constraining the entire system to remain near the target rhythm. This hypothesis matches numerical simulations.

The total time required for the pacemaker to visit all groups is denoted by T_{tour}, with equal residence time in each group. To validate the strategy, the simulations are performed over a range of T_{tour} values.

The synchronization time is determined based on a threshold criterion. Synchronization is considered achieved when the frequencies of all groups differ from that of the pacemaker by less than a predefined threshold. For the simulations presented, this threshold was set to 0.0001.

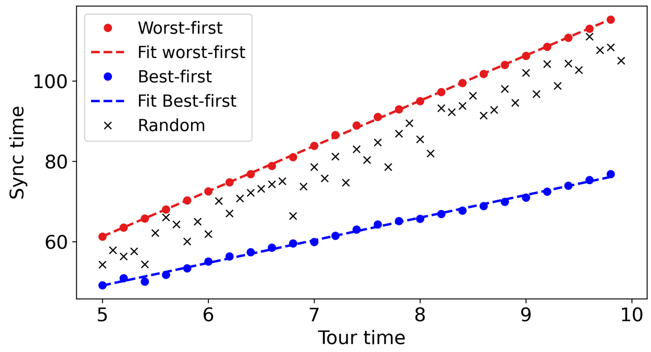

FIG. 2: Synchronization time as a function of the tour time for two distinct ordering methods. Random ordering at each iteration has been added.

As evidenced by Figure 2, the best-first scheme gives faster rhythm locking, with synchronization times consistently shorter than those observed in the worst-first strategy. The rest of all possible orders give results bounded by the latter two methods.

C. Frequency convergence behavior

Once the proper ordering method has been chosen, we proceed to study how the system achieves such a synchronized state. For simplicity, further derivations correspond to the case in which both coupling constants are equal, $Q = K$. Equation (14) therefore states that the achieved stationary frequency lies exactly in the midpoint of the two starting ones. The succession of natural frequencies a group has is represented by w_n^g, where n is the number of tours performed and g represents the group.

The first visited group in the tour updates its frequency w_n^0 as predicted by equation (14). When considering the frequency increment at each step $\Delta\omega_n = \omega_n - \omega_{n-1}$, one can find the recursive relation

$$\Delta\omega^0 = \frac{\Delta\omega_{n-1}^0}{2} \tag{16}$$

which is exactly a geometric progression of ratio $r = 1/2$. The frequency increment as a function of n therefore follows

$$\Delta\omega_n^0 = \Delta\omega_0^0 \cdot 2^{-n} \tag{17}$$

We proceed to apply an analogous derivation for the second group visited by the pacemaker, expressing its frequency increment in terms of n. The resulting expression will differ due to the fact that the pacemaker's frequency is no longer constant but evolves with each tour as predicted by the previous result (17).

The next group to be visited updates its natural frequency as ω_n^1 as

$$\omega_n^1 = \frac{\omega_{n-1}^1 + \omega_n^0}{2} \tag{18}$$

since the pacemaker's frequency becomes updated to that achieved by the former group. The evolution of $\Delta\omega_n^0$ allows for this frequency increment to be solved. After a series of rearrangements, one can show that

$$\Delta\omega_n^0 = \left(\frac{\Delta\omega_0^0}{2} n + \Delta\omega_0^1\right) 2^{-n}. \tag{19}$$

This expression maintains a geometric form but incorporates an $n-$dependent correction to the initial value, resulting in a distinct behaviour. Since the procedure is analogous to subsequent groups, a detailed derivation will not be provided.

In order to validate such calculations, a simulation has been performed.

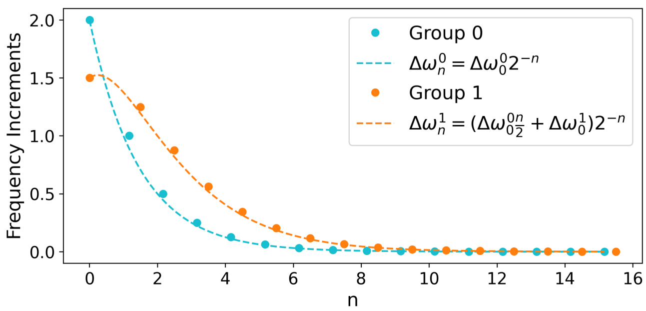

FIG. 3: Frequency increments for group 0, $\omega_0^0 = 2$ and group 1, $\omega_0^1 = 1$ in contact with a pacemaker of frequency $w_{pm} = 6$, along with the theoretical predictions.

One might ask, what would the convergence behave like if the time spent in each group is not sufficient for the steady state to be achieved. The frequency at the time of decoupling is found by evaluating equation (14) at $t = T_{round}$. Therefore, the geometric progression will not be of ratio $1/2$ but of one depending on the system parameters $\Delta\omega, K, \delta_0$ and T_{round}.

IV. CONCLUSIONS

Both analytical and numerical results show conclusions that can be inferred to the real intergroup entrainment of the bluegrass musicians. Analytical derivations describe the exponential approach the group and the pacemaker have in their respective abilities, with the final state always lying between the original ones. We can therefore conclude that the group will not fully learn at a master level in the first interaction; it will rather need several visits to adjust to the desired state. Due to the inertia carried by the master, the order proved to be of importance when optimizing global synchronization. The best and worst orders were found and described, thus bounding the overall efficiency. The master must start with the best groups, and sequentially move into the worse ones, thus maximizing its mastery throughout the tour.

Each group approaches the final rhythm following a power-law, precisely a geometric-like progression with a ratio dependent on the system parameters and the group considered. Finally, numerical results prove the internal coherence of each group when playing in front of the master, highlighting how each peripheral community maintains its inner alignment.

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[1] A. R. Willms, P. M. Kitanov, and W. F. Langford, "Huygens' clocks revisited," Royal Society Open Science, **vol. 4, no. 9** (2017).

[2] C. Liu, D.R. Weaver, S.H. Strogatz, S.M. Reppert, Cell **91** (1997) 855.

[3] C.S. Peskin, Mathematical Aspects of Heart Physiology, Courant Institute of Mathematical Science Publication, New York (1975) pp. 268–278.

[4] D.C. Michaels, E.P. Matyas, J. Jalife, Circulation Res. **61** (1987) 704.

[5] J. Buck, E. Buck, Scientific Am. **234** (1976) 74.

[6] K. Wiesenfeld, P. Colet, Steven H. Strogatz, Freq. locking in J. arrays, Phys. Rev. E, **57** (1998) 1563–1569.

[7] Filatrella. Giovanni and Nielsen, A. and Pedersen. Niels, A. power grid using K. M., The European Physical Journal B - Condensed Matter and Complex Systems, **61** (2008) 485-491.

[8] A.T. Winfree, J. Theoret. Biol. **16** (1967)

[9] Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, Springer, Berlin, 1984.

[10] S. Strogatz, From Kuramoto to Crawford, Physica D: Nonlinear Phenomena, **143** (200)

[11] Y. Wang and F. J. Doyle,"Exp. Sync. in the Presence of a Pacemaker", in IEEE Transactions on Automatic Control, **vol. 58, no. 4, pp. 989-994** 2013

El paper dels "pacemakers" en el model de Kuramoto

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Resum: Aquest treball explora la dinàmica de sincronització en un sistema d'oscil·ladors acoblats, centrant-se en el paper d'un *pacemaker* mòbil. La sincronització global, sota certes condicions, pot succeir quan un *pacemaker* visita seqüencialment grups d'oscil·ladors diferents, cadascun amb freqüències naturals heterogènies. Aquest enfocament s'aplica al conegut problema del *Bluegrass*, inspirat en la coordinació de músics amb diferents tempos. Les simulacions numèriques apunten a una dependència respecte de l'ordre de visita del *pacemaker*. Els resultats estableixen que és òptim visitar primer aquelles comunitats amb un nivell de sincronització més alt. Considerant aquest ordre, la convergència cap a l'estat sincronitzat global segueix una llei de ptències descrita per progressions geomètriques.
Paraules clau: Dinàmiques no lineals, oscil·ladors acoblats, Model de Kuramoto, sincronització, *Bluegrass*
ODSs: Educació de qualitat

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de les desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			