Enhancing Quantum Teleportation with Bayesian Inference

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Abstract: We present an algorithm to enhance the performance of quantum teleportation when utilizing non-maximally entangled states. We adopt a proposed entanglement measure and calculate its values for several well-known states. After outlining the quantum teleportation process, we introduce fidelity functions that can be employed when the considered state fails to meet certain conditions. These functions are optimized using a Bayesian inference method, known as Multiple Correlated-Try Metropolis algorithm, by generating Markov chains of length $3 \cdot 10^5$. The results are applied to analyze a family of two-qubit states parameterized by a single variable, observing a notable improvement in teleportation performance. Ultimately, we identify a strong correlation between the proposed entanglement measure and the success rate of quantum teleportation.
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SDGs: Educational development (SDG 4), Industry, innovation and infrastructure (ODS 9).

I. INTRODUCTION

The utility of quantum mechanics in technology has been evident since the 20th century. In 1993, it was experimentally verified that, using Bell states —maximally entangled (m.e.) states of two qubits— it is possible to "teleport" a qubit from one point to another, without being concerned about distance. This is the well-known quantum teleportation (QT) phenomenon. Recently, QT was shown to coexist with internet signals [1], one step toward establishing a global-scale quantum network.

A robust measure of entanglement cannot increase under local operations [2] and must provide a clear definition of maximal entanglement —a crucial requirement for achieving successful QT. Over time, various methods for quantifying entanglement have emerged [3]. The paper [4] proposes a measure defined by an observable, $\mathcal{J}_n$. This is based on the Segre embeddings, offering a geometric perspective on quantum entanglement.

The Markov chain Monte Carlo (MCMC) methods experienced significant growth in the 1980s. Based on Bayesian inference, these methods can model density functions without requiring normalization [5]. This approach may be relevant for studying non-m.e. states in QT. The insights gained from this method could improve the performance of such protocol, while also offering a deeper understanding of the relationship between QT and entanglement.

Regarding the structure of this work, in Sec. II we mathematically describe qubits, entanglement and the method used to measure entanglement in any n-qubit state. In Sec. III, we explain the QT algorithm for a single qubit. Sec. IV starts by considering states different from Bell's when doing QT. Then, we explain and test a version of an MCMC algorithm that can be efficient in those cases. In Sec. V, we apply these concepts to study specific states parameterized by an angle $\theta \in [0, \pi)$. Finally, in Sec. VI we comment the results obtained and present options about how this paper may be continued.

II. CHARACTERIZATION OF QUANTUM ENTANGLEMENT

In this work, we will only deal with mechanical systems formed by $n \geq 1$ fixed 2-level particle states (*qubits*). Their quantum states are described by unitary complex vectors lying in a 2^n-dimensional Hilbert space, $\mathcal{H}^{2^n}$.

The general state of a system consisting of $n \geq 1$ qubits is of the form

$$|\psi\rangle = a_1 |0\overset{(n)}{\cdots}0\rangle + a_2 |0\overset{(n-1)}{\cdots}01\rangle + \cdots + a_{2^n} |1\overset{(n)}{\cdots}1\rangle , \quad (1)$$

where $a_i \in \mathbb{C}$, $\forall i = 1, \dots, 2^n$, and $|a_1|^2 + \cdots + |a_{2^n}|^2 = 1$. The states $|i_1 \cdots i_n\rangle \equiv |i_1\rangle_{\mathcal{O}_1} \otimes \cdots \otimes |i_n\rangle_{\mathcal{O}_n}$ with $i_j \in \{0, 1\}$ form an orthonormal basis of $\mathcal{H}^{2^n}$. The symbol $\mathcal{O}_j$ represents the observer associated to the j-th qubit. Some examples of 2-qubit states are the well-known *Bell states*:

$$|\phi_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}, \quad |\phi_1\rangle = (|01\rangle + |10\rangle)/\sqrt{2},$$
$$|\phi_2\rangle = (|01\rangle - |10\rangle)/\sqrt{2}, \quad |\phi_3\rangle = (|00\rangle - |11\rangle)/\sqrt{2}.$$

A state $|\psi\rangle$ of $n \geq 2$ qubits is said to be *product* if there exist two states $|\psi_1\rangle \in \mathcal{H}^{N_1}$ and $|\psi_2\rangle \in \mathcal{H}^{N_2}$ such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. On the contrary, the state is *entangled*. It must be satisfied that $N_i = 2^{n_i}$ and $n_1 + n_2 = n$ [6].

Given a product state, it is *q-partite* if it can be written as a product of $q > 1$ states. When q is maximal, i.e. $q = n - 1$, then the state is called *separable*. In this context, entangled states are also known as *1-partite*.

Let us consider a state $|\psi\rangle$ of a system composed by $n \geq 2$ particles. We desire to know if this one is entangled, so we must consider all possible states that could satisfy $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. If we denote $n_1 = l$ where $1 \leq l < n$, then $n_2 = n - l$. According to the notation used in Eq. (1), we can think our system where the particles are arranged in an orderly row.

$$\underset{\bullet}{\mathcal{O}_1} \quad \underset{\bullet}{\mathcal{O}_2} \quad \cdots \quad \underset{\bullet}{\mathcal{O}_{n-1}} \quad \underset{\bullet}{\mathcal{O}_n}$$

We are interested in looking for a decomposition of our state in the form $|\psi\rangle_{\mathcal{O}_1\cdots\mathcal{O}_n} = |\psi_1\rangle_{\mathcal{O}_1\cdots\mathcal{O}_l} \otimes |\psi_1\rangle_{\mathcal{O}_{l+1}\cdots\mathcal{O}_n}$. Another different decomposition one may contemplate would be obtained by permuting at least two particles, thereby altering the position of the observers. This corresponds to permuting the Hilbert basis. Nevertheless, this would probably represent a different state from $|\psi\rangle$. An example of this would be for the state $|B_1\rangle$ from Tab. I: by permuting the basis, we obtain two different states $|B_2\rangle$ and $|B_3\rangle$.

All in all, to characterize the entanglement of our original state, we must contemplate all the possible bipartitions of the system. They are parameterized with l and there are $n-1$ different ones.

There exists a family of observables $\{\mathcal{J}_{n,l}\}$ —given by an $(n-1)$-*hypercube of Segre embeddings* from [4]— the expected values of which on the state $|\psi\rangle$, $\langle\mathcal{J}\rangle_\psi$, are defined as

$$\mathcal{J}_{n,l}(\psi) := 2 - \left(\tfrac{1}{2^{l-1}} \sum_{i_1,\dots,i_l}^{3} |\langle\psi|\, \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_l} \otimes \mathbb{I}_{2^{n-l}}\, |\psi\rangle|^2\right) \tag{2}$$

where $\sigma_0, \dots, \sigma_3$ are the Pauli operators (see **S.M.1**). Their values in general stay in $[0,1]$. In [4], one will find that, given a partition l, from these observables it follows the next result:

$$\mathcal{J}_{n,l}(\psi) = 0 \Leftrightarrow |\psi\rangle_{\mathcal{O}_1\cdots\mathcal{O}_n} = |\psi_1\rangle_{\mathcal{O}_1\cdots\mathcal{O}_l} |\psi_2\rangle_{\mathcal{O}_{l+1}\cdots\mathcal{O}_n}.$$

From this, it can be shown that if η is the number of partitions such as $\mathcal{J}_{n,l}(\psi) = 0$, then the state is $(\eta+1)$-partite. Furthermore, there is proposed a measurement of entanglement defined by the observables in Eq. (2):

$$\mathcal{J}_n(\psi) := \frac{1}{n-1} \sum_{l=1}^{n-1} \mathcal{J}_{n,l}(\psi). \tag{3}$$

The larger its value, the greater the entanglement. From this, the following physical interpretation arises:

$\mathcal{J}_n(\psi) = 0 \Leftrightarrow |\psi\rangle$ separable state,
$\mathcal{J}_n(\psi) = 1 \Leftrightarrow |\psi\rangle$ *maximally entangled (m.e.)* state.

TABLE I: Table with values of $\{\mathcal{J}_{n,l}\}$ and $\mathcal{J}_n$ (computed in [7]) for some well-known states of $n = 2,3,4$ qubits. States $|\phi_i\rangle$ and $|\text{GHZ}\rangle$ are recognized in the literature as m.e..

State	Observable	Value	$\mathcal{J}_n$
Bell states $\|\phi_i\rangle$	$\mathcal{J}_2(\phi_i)$	1	
$\|\text{Sep}\rangle$	$\mathcal{J}_{3,1}(\text{Sep})$	0	$\mathcal{J}_3(\text{Sep})$
$\|000\rangle$	$\mathcal{J}_{3,2}(\text{Sep})$	0	0
$\|B_1\rangle$	$\mathcal{J}_{3,1}(B_1)$	0	$\mathcal{J}_3(B_1)$
$\frac{1}{\sqrt{2}}(\|000\rangle + \|011\rangle)$	$\mathcal{J}_{3,2}(B_1)$	1	1/2
$\|W\rangle$	$\mathcal{J}_{3,1}(W)$	8/9	$\mathcal{J}_3(W)$
$\frac{1}{\sqrt{3}}(\|001\rangle + \|010\rangle + \|100\rangle)$	$\mathcal{J}_{3,2}(W)$	8/9	8/9
$\|\text{GHZ}\rangle$	$\mathcal{J}_{3,1}(\text{GHZ})$	1	$\mathcal{J}_3(\text{GHZ})$
$\frac{1}{\sqrt{2}}(\|000\rangle + \|111\rangle)$	$\mathcal{J}_{3,2}(\text{GHZ})$	1	1
$\|D_{4,1}\rangle$	$\mathcal{J}_{4,1}(D_{4,1})$	3/4	$\mathcal{J}_3(D_{4,1})$
$\frac{1}{2}(\|0001\rangle + \|0010\rangle)+$	$\mathcal{J}_{4,2}(D_{4,1})$	1	5/6
$+\frac{1}{2}(\|0100\rangle + \|1000\rangle)$	$\mathcal{J}_{4,3}(D_{4,1})$	3/4	

III. QUANTUM TELEPORTATION (QT)

Let us consider three people: Alice, Bob and Charlie. The first two are connected via a classical channel, through which only bits of information can be sent. In contrast, Alice and Charlie can only transfer qubits of information through a quantum channel. Alice and Bob prepare and share the Bell state $|\phi_0\rangle_{AB}$, which is m.e.. The labels A and B represent two different observers [8]. After that, Alice takes one qubit whereas Bob the other one, and then both move away (see page 6).

Charlie prepares a single qubit $|\psi\rangle_C$ with the aim of sending it to Bob, but they are not connected in any way. Therefore, Charlie decides to transmit his qubit to Alice, who can make contact with Bob. Unfortunately, it is impossible for her to get to know this state. Suppose then that Charlie's qubit is $|\psi\rangle_C = a_1|0\rangle + a_2|1\rangle$, where $a_i \in \mathbb{C}$ such that $a_1^2 + a_2^2 = 1$.

Let us contemplate the state of the whole system: $|\Psi\rangle_{CAB} = |\psi\rangle_C \otimes |\phi_0\rangle_{AB}$. Because of the fact that at this moment Alice is in possession of two qubits, then she can perform a measurement in the Bell basis —the four Bell states form an orthonormal basis of $\mathcal{H}^4$—. After that, the state of the system formed by the qubits A and C will be projected onto one Bell state. Thus, it may be convenient to write $|\Psi\rangle_{CAB}$ in terms of the four Bell states:

$$\begin{aligned}|\Psi\rangle_{CAB} = \frac{1}{2}\big[&|\phi_0\rangle_{CA}|\psi\rangle_B + |\phi_1\rangle_{CA}\,\sigma_1^B|\psi\rangle_B \\ &+ |\phi_2\rangle_{CA}(-i\sigma_2^B)|\psi\rangle_B + |\phi_3\rangle_{CA}\,\sigma_3^B|\psi\rangle_B\big].\end{aligned} \tag{4}$$

Once the Bell measurement is carried out, the state of Alice's system is $|\phi_i\rangle_{CA}$ for some $i \in \{0,1,2,3\}$. Then, Bob's qubit transforms into $|\varphi\rangle_B = \frac{1}{\sqrt{p_i}}{}_{CA}\langle\phi_i|\Psi\rangle_{CAB}$. It is easy to see that $p_i = 1/4$, which matches with the probability of Alice obtaining $|\phi_i\rangle$. Therefore, Bob's state is one of the following: $|\varphi_0\rangle = |\psi\rangle, |\varphi_1\rangle = \sigma_1|\psi\rangle, |\varphi_2\rangle = (-i\sigma_2)|\psi\rangle$ or $|\varphi_3\rangle = \sigma_3|\psi\rangle$.

At this point, Alice's state is $|\phi_i\rangle_{CA}$ for some i, which corresponds to the i-th measurement outcome. Alice then communicates her outcome to Bob using 2 bits of classical information. With this message, Bob can identify Alice's state and, consequently, his own state, which is $|\varphi_i\rangle_B$. Considering that the Pauli operators satisfy $\sigma_k\sigma_k = \mathbb{I}$, $\forall k \in \{0,1,2,3\}$, Bob must apply the transformation σ_i to his qubit. This results in a new state $|\widetilde{\varphi}_i\rangle_B = \sigma_i^B|\varphi_i\rangle_B$ s. t. $|\langle\widetilde{\varphi}_i \mid \psi\rangle|^2 = 1$. In other words, the probability of Bob's state matching Charlie's is one, so the experiment has been successfully completed.

One important observation is that, until the teleportation is completed, neither Alice nor Bob obtains no information about the state of Charlie's particle. This leads us to conclude that the state $|\phi_0\rangle_{AB}$ can teleport any qubit state prepared by Charlie. Indeed, it is known that for teleportation to be successfully completed, the only requirement for the shared state is that it must be m.e.. However, if this state is not a Bell state, the procedure might differ slightly.

Instead of Alice and Bob preparing one Bell state, let us suppose that the shared state, $|\Phi\rangle_{AB}$, is an arbitrary state of two qubits: it could be product, entangled or m.e.. According to Eq. (1), let be $|\Phi\rangle = b_1|00\rangle + \cdots + b_4|11\rangle$ for some $b_i \in \mathbb{C}$. We will continue to assume that Alice always performs a Bell measurement on her particle and Charlie's. Hence, it is still convenient expressing the state of the entire system in terms of the four Bell states:

$$|\Upsilon\rangle_{CAB} = \sum_{k=0}^{3} c_i\,|\phi_i\rangle_{CA}\,|\varphi_i\rangle_{B}\,. \tag{5}$$

After the measurement, Alice will obtain the outcome $|\phi_j\rangle$ with probability $p_j = |c_j|^2$ for some $j \in \{0,1,2,3\}$. Bob's state will change then to $|\varphi_j\rangle$. The unclear point now is the transformation Bob may apply to his qubit once he learns of Alice's outcome. If it is a Bell state or one satisfying some specific conditions proved in [9],

$$\begin{cases} |b_1|^2 + |b_2|^2 = |b_3|^2 + |b_4|^2 = \frac{1}{2} \\ b_1\bar{b}_3 = -b_2\bar{b}_4 \end{cases}, \tag{6}$$

then this is not an issue. The thing we already can ensure is that QT may not be one hundred per cent successful, as the requirement of the shared state to be m.e. might no longer hold. That is, the state Bob receives at the end of the process $|\widetilde{\varphi}_j\rangle_B$ will have a certain probability (less than or equal to one) of being the state of Charlie's $|\psi\rangle_C$. This probability is referred to as *fidelity* and, for the j-th outcome, is computed as

$$F_j^{\psi} := |_B\langle\widetilde{\varphi}_j|\psi\rangle_C|^2. \tag{7}$$

From now on, we will refer to the set of transformations $\Lambda = \{\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\}$ as the *instructions* Bob must follow to complete QT. If Alice has obtained the Bell state $|\phi_j\rangle$, then Bob will use the transformation Λ_j, so his state at the end of the experiment will be $|\widetilde{\varphi}_j\rangle_B = \Lambda_j\,|\varphi_j\rangle_B$. Note that these instructions must only depend on the shared state $|\Phi\rangle_{AB}$ and must be the same for every $|\psi\rangle_C$, so we will often write Λ^{Φ}.

Assuming these instructions are known, we provide a way for characterizing how effective $|\Phi\rangle_{AB}$ is for teleporting a single qubit $|\psi\rangle_C$. This is done by considering the following weighted arithmetic mean:

$$\mathcal{F}_{\Phi}(\psi) := \sum_{i=0}^{3} p_i F_i^{\psi} = \sum_{i=0}^{3} p_i |_B\langle\varphi_i|(\Lambda_i^{\Phi})^{\dagger}|\psi\rangle_C|^2. \tag{8}$$

From this, we will say that the state $|\Phi\rangle$ is *perfect for quantum teleportation (PQT)* if $\mathcal{F}_{\Phi}(\psi) = 1$, $\forall\,|\psi\rangle \in \mathcal{H}^2$. For example, we already know that the four Bell states are PQT and the corresponding instructions are

$$\begin{aligned}
\Lambda^{\phi_0} &= \{\Lambda_0 = \sigma_0,\, \Lambda_1 = \sigma_1,\, \Lambda_2 = \sigma_2,\, \Lambda_3 = \sigma_3\},\\
\Lambda^{\phi_1} &= \{\Lambda_0 = \sigma_1,\, \Lambda_1 = \sigma_0,\, \Lambda_2 = \sigma_3,\, \Lambda_3 = \sigma_2\},\\
\Lambda^{\phi_2} &= \{\Lambda_0 = \sigma_2,\, \Lambda_1 = \sigma_3,\, \Lambda_2 = \sigma_0,\, \Lambda_3 = \sigma_1\},\\
\Lambda^{\phi_3} &= \{\Lambda_0 = \sigma_3,\, \Lambda_1 = \sigma_2,\, \Lambda_2 = \sigma_1,\, \Lambda_3 = \sigma_0\}.
\end{aligned}$$

IV. OPTIMIZING INSTRUCTIONS FOR QT

The main challenge when attempting to conduct QT with a given shared state $|\Phi\rangle_{AB}$ is determining the exact instructions Λ^{Φ}. The transformations Λ_j^{Φ} are 2×2 unitary matrices (see **S.M.2**), so their general form is

$$\Lambda_j^{\Phi}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{i\alpha}\cos\gamma & e^{i\beta}\sin\gamma \\ -e^{-i\beta}\sin\gamma & e^{-i\alpha}\cos\gamma \end{pmatrix} \tag{9}$$

$\forall j \in \{0,1,2,3\}$. Thus, each Λ_j^{Φ} is fully determined, up to a global phase, by a set of points $\chi_j^{\Phi} = \{(\alpha,\beta,\gamma)_k\}$ where $\alpha,\beta,\gamma \in [0,2\pi)$. The QT fidelity from Eq. (7) may depend on these three parameters like

$$F_j^{\psi}(\alpha,\beta,\gamma) = |_B\langle\varphi_j \mid (\Lambda_j^{\Phi}(\alpha,\beta,\gamma))^{\dagger} \mid \psi\rangle_C|^2. \tag{10}$$

The instructions Λ^{Φ} must be those that yield the best teleportation results. If $|\Phi\rangle$ satisfy Eq. (6), then the state is PQT [9], its instructions are perfectly known and $\mathcal{J}_2(\Phi) = 1$ (see **S.M.3**). More generally, if $|\Phi\rangle$ is m.e., finding them is likely straightforward. But if $\mathcal{J}_2(\Phi) < 1$ or the number of qubits increases, this task can become significantly more complex, potentially requiring a brute force method. In such cases, let us consider a set of m arbitrary qubits $\Omega = \{|\psi_1\rangle_C, \ldots, |\psi_m\rangle_C\}$. The next function, defined using fidelities of the form Eq. (10),

$$F_j(\alpha,\beta,\gamma) := \prod_{k=1}^{m} F_j^{\psi_k}(\alpha,\beta,\gamma) \tag{11}$$

may give an estimation of the value of (α,β,γ) characterizing the transformation Λ_j^{Φ} that Bob must follow when Alice gets the j-th outcome.

We are interested in finding the global maximums of the latter function. Nevertheless, since Eq. (11) provides a statistical value of the parameters because of Ω, then F_j is a kind of an *error* function. This is the reason why is more convenient finding the distributing values of (α,β,γ) in respect of F_j.

A. Multiple Correlated-Try Metropolis (MCTM)

Let us consider a fixed outcome j and let be $x = (\alpha,\beta,\gamma)$, where $\alpha,\beta,\gamma \in [0,2\pi)$. Instead of using Eq. (11), we will take $f_j(x) := -\ln F_j(x) \in [0,\infty)$. By following [10], there exists a density function $\rho_T(x) \propto \exp(-f_j(x)/T) =: g_T(x)$, where $T > 0$ is a scale parameter. Note that the modes of ρ_T (the points with higher density) match the minimum values of f_j. The parameter T does not change these modes, but the lower its value, the narrower the density at these points.

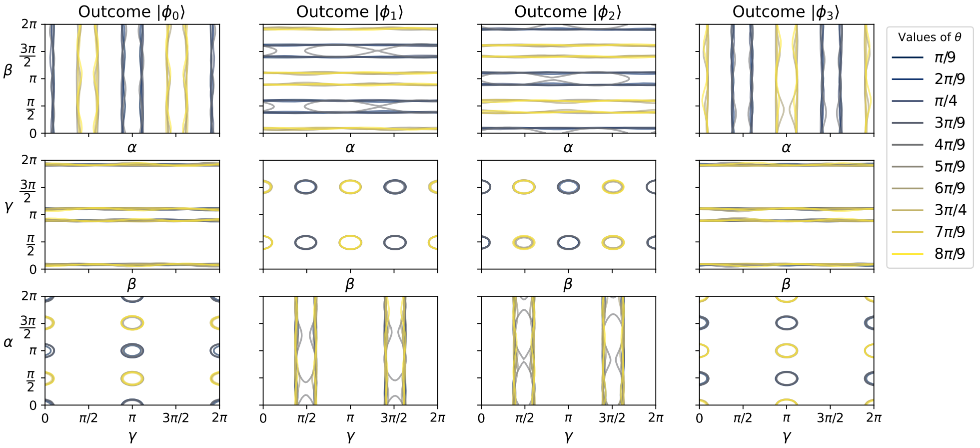

FIG. 3: Contours of the modes from the densities $\rho_{\alpha\beta}, \rho_{\beta\gamma}$ and $\rho_{\gamma\alpha}$. The data has been obtained implementing the MCTM algorithm to the states $|\Phi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$ for ten different values of $\theta \in [0, \pi)$. For $\theta \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$ the states are m.e., whereas the remaining states are entangled but not maximally (see Fig. 2).

The main objective here is to look for the form of the target density ρ_T. We could try to compute $\int_{\mathbb{R}^3} g_T(x)dx$, but this integral is too complex. Fortunately, [11] provides a method for estimating ρ_T without computing any integral at all. This algorithm is called *Multiple Correlated-Try Metropolis* (MCTM). Being one MCMC method, it generates random numbers from the known function g_T, which produces a Markov chain $\{x_t\}_{t=1}^N$ such that its invariant density function matches with our target density $\rho \propto g_T$. The generation of such Markov chain using a particular version of the MCTM algorithm is described in **S.M.4**.

We will use large values of N, like $3\cdot10^5$, and a number of $m = 50$ random qubits for a better convergence to ρ_T. Fixed these values, we will need to choose an appropriate value of T following the statement mentioned in [5]: the optimal acceptance rate of points during the algorithm should be around $1/4$. We have seen that values of T around $2.5 - 3$ are good ones. On the other hand, the points of the resulting Markov chain after the algorithm theoretically have been generated from the density function ρ_T. But for this, it is required a certain period of convergence. This is the reason why it is necessary to apply what is called some *burn-in period*: removing from the Markov chain the points at the begging that have not already converged. We have observed that a burn-in period of $\mathfrak{b} = 300$ points is more than enough.

Once we get such set of points $\{(\alpha,\beta,\gamma)_t\}_{\mathfrak{b}}^N$, thanks to the function `KernelDensity` from the Python's library **`sklearn`** —see [7]— we can plot the 2-dimensional marginal densities $\rho_{\alpha\beta}$, $\rho_{\beta\gamma}$ and $\rho_{\gamma\alpha}$ which better fit the

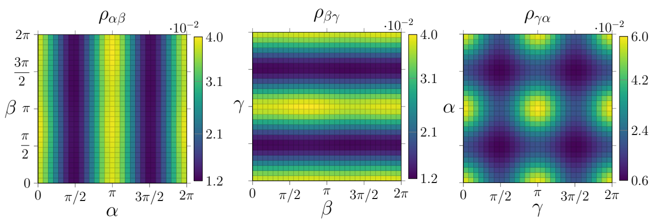

FIG. 1: Plot of the 2D marginal densities $\rho_{\alpha\beta}$, $\rho_{\beta\gamma}$ and $\rho_{\gamma\alpha}$ associated to the $j = 0$ outcome when the shared state is $|\phi_0\rangle$. From their modes it is obtained that $\chi_0^{\phi_0} = \{(0,\beta_1,0),(0,\beta_2,\pi),(\pi,\beta_3,0),(\pi,\beta_4,\pi)\}$, where β_i can be any in $[0, 2\pi)$.

simulated data. By combining the modes of each 2D density, we obtain the modes of the 3D density ρ_T. For instance, we have performed the MCTM method on the shared Bell state $|\phi_0\rangle$. For each outcome j, the modes of the associated 3D density, $\chi_j^{\phi_0}$, are effectively the unique points defining the transformation $\Lambda_j^{\phi_0}$ from Λ^{ϕ_0} (see Fig. 1 or page 6).

V. CONNECTING QUANTUM TELEPORTATION AND ENTANGLEMENT

In this section, we focus on states of the form $|\Phi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$, where $\theta \in [0, \pi)$. See in Fig. 2 that, depending on the angle, the state may be non-ideal for QT because the observable from Eq. (3) varies from 0 to 1, so it can be non-m.e.. Therefore, we will implement the MCTM algorithm to determine their optimal instructions and evaluate their efficacy in QT. Furthermore, we will analyze the relationship between the resulting success rate of QT and the observable $\mathcal{J}_2$.

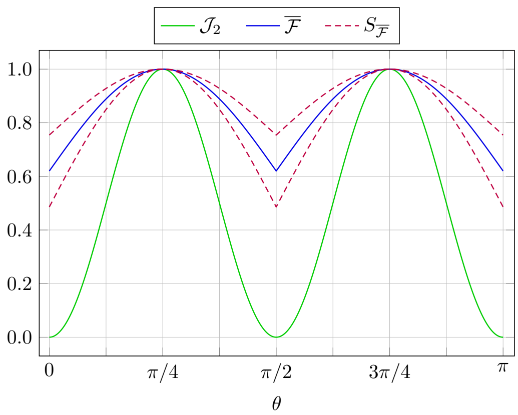

FIG. 2: Representation of the $\mathcal{J}_2$ values, Eq. (3), and the arithmetic mean values $\overline{\mathcal{F}}_\theta$ of the set $\{\mathcal{F}_{\Phi(\theta)}(\psi)\}_\psi$, Eq. (8), with its corresponding standard deviation $S_{\overline{\mathcal{F}}}$, for each state of the form $|\Phi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$. These mean values have been computed by taking 10^5 arbitrary qubits $|\psi\rangle$.

For $\theta \notin \{\frac{\pi}{4}, \frac{3\pi}{4}\}$, the corresponding states are not m.e.. After applying the MCTM algorithm to selected angles (see Fig. 3), we find that $\Lambda^{\Phi(\theta_1)} = \Lambda^{\phi_0}$ for all $\theta_1 \in \left(0, \frac{\pi}{2}\right)$, whereas $\Lambda^{\Phi(\theta_2)} = \Lambda^{\phi_3}$ for all $\theta_2 \in \left(\frac{\pi}{2}, \pi\right)$. For angles $\theta_3 \in \left\{0, \frac{\pi}{2}\right\}$, we have obtained $\Lambda^{\Phi(\theta_3)}$ can be either Λ^{ϕ_0} or Λ^{ϕ_3}.

When $\theta \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$, the states correspond to $|\phi_0\rangle$ and $|\phi_3\rangle$, respectively, which are m.e.. The results (see Fig. 2), $\mathcal{J}_2(\Phi(\theta)) = 1$ and $\overline{\mathcal{F}}_\theta = 1$, with a negligible error, align with our expectations. In contrast, for $\theta \in \left\{0, \frac{\pi}{2}\right\}$, we find $\mathcal{J}_2(\Phi(\theta)) = 0$, together with the minimum fidelity $\overline{\mathcal{F}}_\theta = 0.62$ and the maximum standard deviation $S_{\overline{\mathcal{F}}_\theta} = 0.13$ across all angles in $[0, \pi)$. This result is consistent, as these states correspond to $|00\rangle$ and $|11\rangle$, which are product states. For the remaining angles, even with the MCTM method, QT is not fully successful: the fidelity values $\overline{\mathcal{F}}_\theta + S_{\overline{\mathcal{F}}_\theta}$ do not reach the maximum value for any angle. This suggests that these states are not m.e., which is also supported by $\mathcal{J}_2$.

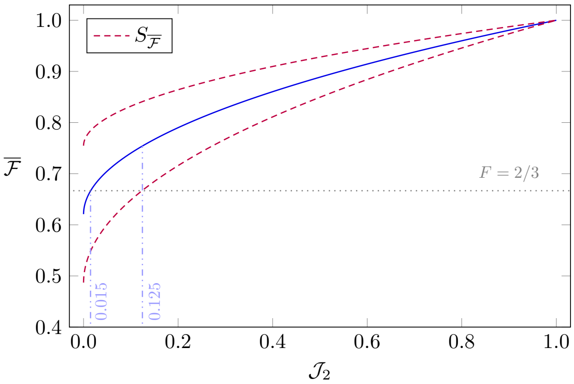

FIG. 4: Plot of the mean values $\overline{\mathcal{F}}$ and the std. deviation $S_{\overline{\mathcal{F}}}$, depending on $\mathcal{J}_2$. The corresponding data is from Fig. 2.

The dependence of QT success on entanglement is evidenced by observing a continuous and symmetric evolution of $\overline{\mathcal{F}}_\theta$ and $\mathcal{J}_2$ with respect to $\theta = \frac{\pi}{2}$. As seen in Fig. 4, a clear correlation exists between $\mathcal{J}_2$ and $\overline{\mathcal{F}}$. When the shared state is less entangled, QT success decreases, resulting in higher fidelity errors. At the same time, the standard deviation increases, indicating that QT success becomes more sensitive to the qubit being teleported.

Finally, implementing the MCTM algorithm for states with $\mathcal{J}_2 > 0.125$ improves QT performance compared to other methods. For instance, [8] introduces an alternative such that the best possible fidelity result is $\frac{2}{3}$.

VI. SUMMARY AND CONCLUSIONS

In this work, we explored how well the proposal entanglement measure $\mathcal{J}_2$ relates with the QT efficiency. We first introduced physical interpretations for the values of $\mathcal{J}_2$. To analyze QT using non-full entangled states ($\mathcal{J}_2 < 1$) of the form $|\Phi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$, we implemented the MCTM algorithm. Using the parameters $N = 3 \cdot 10^5, m = 50$, $T = 2.5$ and $\mathfrak{b} = 300$, we obtained optimal density distributions which describe the set of transformations $\Lambda^{\Phi(\theta)}$ for QT. The results support the conclusion that if $|\Phi\rangle$ is PQT, then $\mathcal{J}_2(\Phi) = 1$. Additionally, the MCTM algorithm provided better fidelity outcomes compared to alternative approaches. Future work could focus on investigating alternative measures beyond Bell's, which might help establish a stronger connection between $\mathcal{J}_n$ and QT. Finally, the MCTM algorithm used in this study can be perfectly adapted and potentially applied to states with more qubits, where problem complexity increases significantly. This could enable teleporting multiple qubits, thereby improving the rate of information transmission via QT.

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Optimitzant la Teleportació Quàntica amb Inferència Bayesiana

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Resum: Presentem un algoritme per millorar el rendiment de la teleportació quàntica quan s'utilitzen estats no màximament entrellaçats. Adoptem una proposta de mesura d'entrellaçament i calculem els seus valors per a diversos estats coneguts. Després de descriure el procés de teleportació quàntica, introduïm funcions de fidelitat que es poden utilitzar quan l'estat considerat no compleix certes condicions. Aquestes funcions s'optimitzen mitjançant un mètode d'inferència Bayesiana, conegut com *Multiple Correlated-Try Metropolis algorithm*, generant cadenes de Markov de longitud $3 \cdot 10^5$. Els resultats s'apliquen per analitzar una família d'estats de dos qubits parametritzats per una sola variable, observant una millora notable en el rendiment de la teleportació. Finalment, identifiquem una forta correlació entre la mesura d'entrellaçament proposada i la taxa d'èxit de la teleportació quàntica.
Paraules clau: Entrellaçament, cadena de Markov, Densitat, Qubit, Transformació unitària
ODSs: Aquest TFG està relacionat amb els Objectius de Desenvolupament Sostenible (SDGs)

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de les desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures	X		

Aquest TFG d'un grau universitari de Física es relaciona amb l'ODS 9 d'indústria, innovació i infraestructures: contribueix al coneixement fonamental per avançar en les tecnologies de comunicació quàntica. Més concretament, el desenvolupament d'algoritmes que milloren els protocols de teleportació quàntica incentiva la innovació en xarxes de comunicació més segures i efectives. A més, promou el desenvolupament educatiu de qualitat (ODS 4), ja que implica metodologies avançades de resolució de problemes rellevants per a la física i les ciències computacionals.

GRAPHICAL ABSTRACT

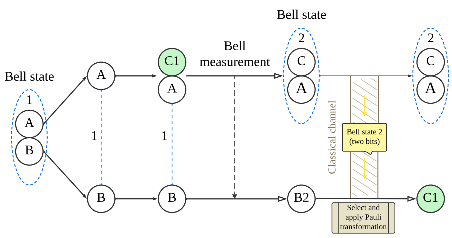

Figure 5: A schematic representation of the global quantum teleportation process.

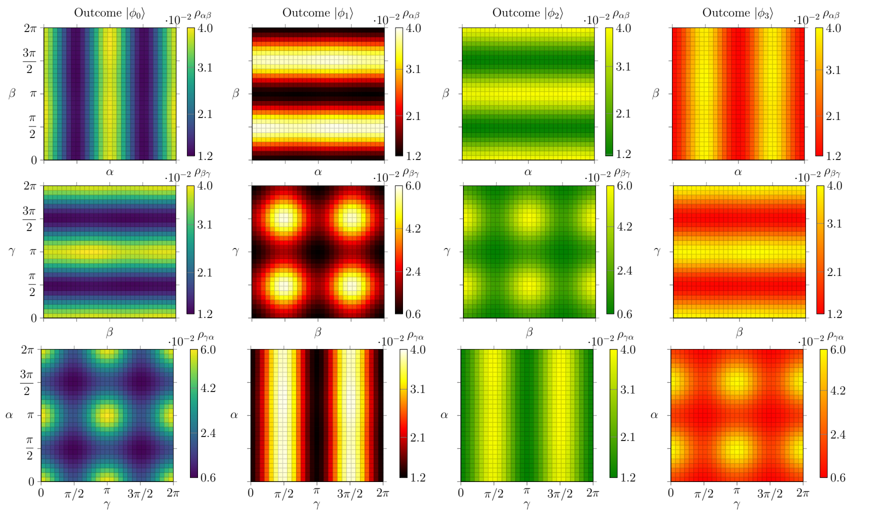

Figure 6: Resulting 2D marginal densities after implementing the MCTM method to the Bell state $|\phi_0\rangle = (|00\rangle+|11\rangle)/\sqrt{2}$

SUPPLEMENTARY MATERIAL

S.M.1 The Pauli matrices used in this work are

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ \text{and} \ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

S.M.2 A unitary transformation on $\mathcal{H}^2$ is a 2×2 complex matrix $\mathcal{U}$ such as $(\mathcal{U})^\dagger = (\mathcal{U})^{-1}$. These type of matrices preserve the norm of two-dimensional vectors, so if $|\psi\rangle$ is a physical state in $\mathcal{H}^2$, then so is $\mathcal{U}|\psi\rangle$. The general form of $\mathcal{U}$ can be written as

$$\mathcal{U} = e^{i\theta} \begin{pmatrix} e^{i\alpha}\cos\gamma & e^{i\beta}\sin\gamma \\ -e^{-i\beta}\sin\gamma & e^{-i\alpha}\cos\gamma \end{pmatrix},$$

where α, β, γ and θ are four real parameters.

S.M.3 Let $|\Phi\rangle = a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle$ be a 2-qubit state.

$$\text{If} \quad \left.\begin{array}{l} |a_1|^2 + |a_2|^2 = |a_3|^2 + |a_4|^2 = \frac{1}{2} \\ a_1\bar{a}_3 = -a_2\bar{a}_4 \end{array}\right\} \implies |\Phi\rangle \ \text{is PQT and} \ \mathcal{J}_2(\Phi) = 1.$$

In this case, the instructions Λ^Φ are of the form (up to a global phase)

$$\Lambda_0^\Phi = \sqrt{2}\begin{pmatrix} \bar{a}_1 & \bar{a}_2 \\ \bar{a}_3 & \bar{a}_4 \end{pmatrix} \qquad \Lambda_1^\Phi = \sqrt{2}\begin{pmatrix} \bar{a}_3 & \bar{a}_4 \\ \bar{a}_1 & \bar{a}_2 \end{pmatrix}$$

$$\Lambda_2^\Phi = \sqrt{2}\begin{pmatrix} \bar{a}_3 & \bar{a}_4 \\ -\bar{a}_1 & -\bar{a}_2 \end{pmatrix} \qquad \Lambda_3^\Phi = \sqrt{2}\begin{pmatrix} \bar{a}_1 & \bar{a}_2 \\ -\bar{a}_3 & -\bar{a}_4 \end{pmatrix}$$

and the outcomes of the Bell measurement are all equiprobable, i.e. $p_j = 1/4$ for all $j = 0, 1, 2, 3$ and for any teleported state $|\psi\rangle \in \mathcal{H}^2$.

S.M.4 The version of the MCTM method we have used in this work generates a Markov chain $\{x_t\}_1^N$ following these steps:

1. We choose a random point in V as the first member of the chain, x_1, such as $f_j(x_1) \neq \infty$.
2. For $t \geq 1$, let $x := x_t$ be the last point of the chain. From a Gaussian distribution $\mathcal{N}((x^T, \overset{(k)}{\ldots}, x^T)^T, \Sigma_{3k})$, we generate k trial proposals $y_1, \ldots, y_k$. The covariance matrix Σ_{3k} is of the form

$$\Sigma_{3k} = \begin{pmatrix} \Sigma & \Gamma & \cdots & \Gamma \\ \Gamma & \Sigma & \Gamma & \Gamma \\ \cdots & \cdots & \cdots & \cdots \\ \Gamma & \Gamma & \cdots & \Sigma \end{pmatrix}$$

   where $\Sigma = \sigma^2\mathbb{I}_3$ and $\Gamma = \frac{\sigma^2}{1-k}\mathbb{I}_3$ for some σ^2. We will use values of σ^2 around $10 - 20$. Moreover, the work [11] indicates that, for this version of the MCTM, a number of $k = 7$ trial proposals is a good one.
3. We compute $g_T(y_l)$ for each $l = 1, \ldots, k$, and then we select one of these points y with probability proportional to $g_T(y)$.
4. We generate $\widetilde{x}_1, \ldots, \widetilde{x}_k$ points from the same Gaussian distribution from before, but with y instead of x and conditioning that $\widetilde{x}_k = x$.
5. We compute the *acceptance probability*:

$$\mathcal{A}(y, x_t) = \min\left\{1, \frac{g_T(y_1) + \cdots + g_T(y_k)}{g_T(\widetilde{x}_1) + \cdots + g_T(\widetilde{x}_k)}\right\}.$$

6. The following point of the chain x_{t+1} is y with probability $\mathcal{A}(y, x)$ or x with probability $1 - \mathcal{A}(y, x)$.
7. The steps $2 - 6$ are repeated with the last point of the chain x_{t+1}, until x_N is reached.