A kinetic view of nonlocal self-avoiding processes

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Abstract: Self-avoiding walks (SAWs) are central to modeling excluded-volume effects in polymers and related systems. SAWs are typically defined with local, nearest-neighbor steps, and their statistical behavior is well studied. In contrast, less is known about SAWs involving nonlocal motion. We study how introducing nonlocality — in the form of fixed-length jumps inspired by the knight's move in chess — affects kinetic self-avoiding walks (GSAWs), where paths grow irreversibly. The resulting model, called the Self-Avoiding Random Knight (SARK), replaces local propagation with constrained long-range steps. Using large-scale simulations, we examine how this dynamic influences key properties of the walk. We find that nonlocality increases the walker's lifetime and spatial extent. The end-to-end distance shows a crossover in scaling behavior, approaching that of the GSAW at long times. Clustering analysis reveals a dominant connected component in small lattices, which vanishes in larger ones. These results offer insight into how nonlocal constraints shape the geometry and growth of SAWs, with possible applications in ecological foraging and transport in constrained environments.
Keywords: Random walks, scaling laws, universality, lattice models.
SDGs: Life below water and life on earth.

I. INTRODUCTION

The random walk framework was first introduced by Karl Pearson in 1905 to describe the random migration of insects [1]. Since then, it has become central to the study of stochastic processes such as diffusion, search behavior, and even financial markets. For instance, in mathematical ecology it is used to model animal foraging and migration. Despite its conceptual simplicity, the random walk gives rise to remarkably rich behavior.

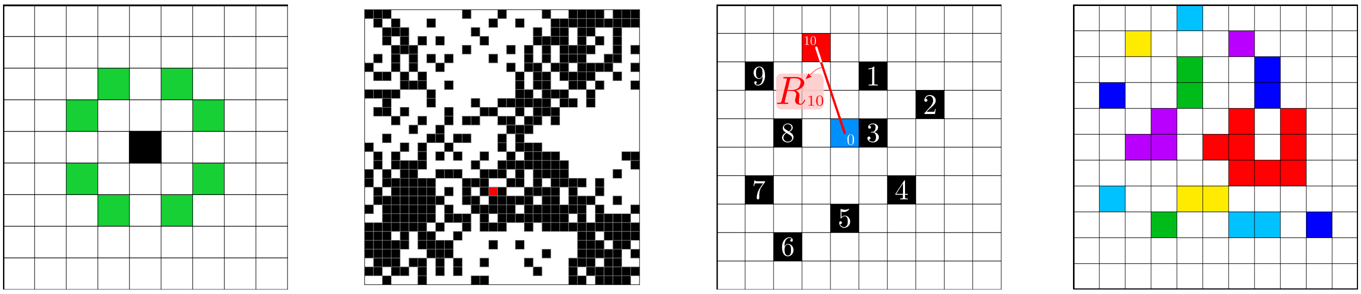
Among its many variants, the self-avoiding random walk (SAW) plays a significant role in understanding systems with excluded volume effects, such as polymer configurations [2]. Unlike the classical random walk, a SAW forbids revisiting previously visited sites, introducing strong memory effects and long-range correlations in the trajectory. This self-avoidance alters the geometry of the path, resulting in spatially complex structures with a characteristic fractal dimension [3].

It is important to distinguish between the equilibrium formulation of the SAW — as typically treated in mathematical combinatorics — and the growing self-avoiding walk (GSAW), also known as the kinetic SAW [4]. The canonical SAW is defined by the ensemble of all self-avoiding paths of a given length on a lattice, with each path typically assigned equal statistical weight. In contrast, the GSAW is a stochastic, irreversible process in which the path is constructed step-by-step: at each step, the walker selects randomly among the unvisited sites accessible from its current position, and the walk terminates when no further moves are possible (trapping). While the set of valid paths of a given length is the same in both models, the statistical weights assigned to those paths differ. As a result, certain properties — such as scaling exponents — can differ from those of the classical SAW. This kinetic version is the framework adopted in this work.

In this study, we explore a non-traditional version of the kinetic SAW, in which the walker takes non-local, fixed-length steps modeled after the knight's movement in chess — L-shaped jumps that skip over immediate neighbors. The idea originated from the classical Knight's Tour problem [5], a long-standing combinatorial puzzle where a knight must visit every square of a chessboard exactly once. By interpreting knight tours as self-avoiding paths on a lattice, we recast this problem in a statistical physics context and investigate its properties using tools from random walk theory.

Our model, which we refer to as the Self-Avoiding Random Knight (SARK), combines self-avoidance with structured non-locality: the walker selects randomly among allowed knight moves that lead to unvisited sites. Unlike nearest-neighbor SAWs, the SARK can reach distant sites in a single move, leading to qualitatively different exploration patterns.

Because self-avoiding walks are already difficult to study analytically, the additional complexity introduced by non-local, knight-like steps makes exact treatment even more challenging. For this reason, we adopt a computational approach. By simulating a large number of SARKs on finite lattices, we study several observables: the distribution of trapping times, which reveals how long the walker typically survives before getting stuck; the end-to-end distance scaling, from which we extract an effective Flory exponent and estimate the fractal dimension of the paths; and the cluster structure of visited sites, to investigate whether percolation-like behavior emerges in the spatial footprint of the walk.



(a) Knight steps (b) Trapping (c) End-to-end distance (d) Cluster structure

FIG. 1: (a) Knight steps pattern and main observables of the SARK: (b) trapping times, (c) end-to-end distance, and (d) spatial clustering.

II. THE SELF-AVOIDING RANDOM KNIGHT

The random walks studied in this work are models defined on discrete lattices, where a walker moves from site to site according to a set of rules. In our case, the underlying space is a two-dimensional square lattice of lateral size L, equipped with periodic boundary conditions. This means the lattice has a toroidal topology: the site at position (x, y) is identified with those at $(x \pm L, y)$ and $(x, y \pm L)$. As a result, all sites are equivalent, and boundary effects are eliminated.

In the classical GSAW, the walker starts from an initial site and makes local, nearest-neighbor steps — that is, displacements of the form $(\pm 1, 0)$ or $(0, \pm 1)$ — while avoiding previously visited sites. This constraint introduces memory into the system and alters its scaling behavior compared to the unconstrained random walk.

In our model, we modify this dynamics by allowing the walker to perform non-local, fixed-length moves corresponding to the knight's movement in chess. At each time step, the walker attempts to move to one of the eight possible target sites at displacements $(\pm 1, \pm 2)$ or $(\pm 2, \pm 1)$ (see FIG. 1a). Among these, only those not yet visited are considered valid. One of the valid options is selected uniformly at random, and the walker proceeds. The process terminates when no legal move remains — a condition we refer to as trapping (FIG. 1b).

We simulate many independent realizations of the Self-Avoiding Random Knight (SARK) to estimate key statistical properties. A central observable is the survival time (τ), defined as the number of steps the walker takes before becoming trapped. Unlike the classical SAW framework, where one studies the ensemble of all self-avoiding configurations of a fixed length, kinetic walks like the SARK and the GSAW are dynamic and once the walker is trapped, the process ends. As a result, survival time is a fundamental quantity that determines how long the walk can grow and how far it can explore the lattice on average. For the standard GSAW in two dimensions, the average survival time is approximately 71 steps [6], which effectively sets a limit on the maximum length of walks that can be realistically generated. This constraint inherently localizes the process in space. By studying the survival time statistics of the SARK, we aim to understand two things: first, how the non-local and constrained nature of the knight's movement influences the mechanisms of trapping; and second, how large (in both time and space) typical walks can grow, which determines the degree of localization of the process.

Another central observable is the end-to-end distance (R_τ), defined as the Euclidean distance between the walker's current position and its starting point (FIG. 1c). The average end-to-end distance typically scales as

$$\langle R_\tau \rangle \sim \tau^{\nu}, \tag{1}$$

where ν is sometimes called the Flory exponent [2]. This exponent characterizes how the spatial extent of the walk grows with time. It is inversely related to the fractal dimension D_f of the walk via

$$D_f = \frac{1}{\nu}. \tag{2}$$

For the SAW in two dimensions, $\nu = 3/4$, yielding $D_f = 4/3$. For the GSAW, the exponent is lower: ν is approximately 0.68 [4], corresponding to a more compact walk with D_f around 1.47. One of the goals of this work is to determine how the non-local nature of the SARK affects these exponents.

Finally, we analyze the geometric structure of the region visited by the walker by studying its cluster properties. At a given time step, we identify all lattice sites visited so far and apply a clustering algorithm to group together connected components. Here, connectivity is defined in terms of nearest-neighbor adjacency on the lattice (not based on knight-move reachability). This distinction is important: while the walker moves via the knight's functional network, a non-local connectivity graph, the spatial clusters are measured using the local lattice adjacency (FIG. 1d). In this way, we aim to understand how dynamics on the functional network translates into clustering properties on the underlying lattice.

Unlike the classical GSAW, which grows a single connected cluster due to local steps, the SARK can create new disconnected clusters at each step by jumping over unvisited regions. This gives rise to richer spatial structures. We track the sizes of the largest clusters at specified time steps. In particular, the size of the largest connected component can be informative: in percolation theory, the emergence of a dominant or infinite cluster signals a connectivity phase transition. By observing whether such a component emerges in the SARK, we gain insight into the connectedness of the walk's spatial footprint.

III. RESULTS

A. Survival time

In finite systems, survival time is affected not only by the walker's intrinsic dynamics but also by the size of the lattice. For small system sizes L, the knight walker is more likely to become trapped early, not because of the self-avoidance constraint alone, but due to the limited number of available sites and the increased likelihood of re-entering already explored regions.

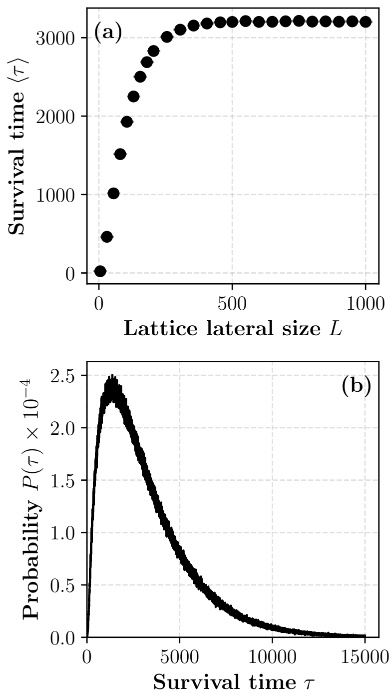


FIG. 2: Survival time analysis of the SARK. (a) Mean survival time $\langle\tau\rangle$ as a function of lattice lateral size L. The curve shows saturation for large L, indicating that trapping is intrinsic and not driven by finite-size effects. (b) Distribution of survival times τ for $N = 10^7$ SARKs on a lattice of size $L = 10^6$. The distribution has a mean $\langle\tau\rangle = 3209.0 \pm 0.8$ and an exponentially decaying tail. The plot is truncated at $\tau = 15{,}000$ for clarity, as very few walks exceed this value; the longest recorded walk reached $\tau = 36{,}236$.

To study the intrinsic trapping behavior of the SARK, independent of finite-size effects, we must ensure that the walker has access to a space that is effectively infinite. To identify the minimum lattice size required to suppress these finite-size effects, we simulate $N = 500,000$ walks on lattices of increasing size L and compute the mean survival time $\langle\tau\rangle$ for each case. Results are shown in FIG. 2a.

Since $\langle\tau\rangle$ saturates as L increases, this indicates that the knight walker typically becomes trapped due to its intrinsic dynamics. In this regime, $\langle\tau\rangle$ becomes an intensive quantity: giving the walker more space no longer significantly changes its expected lifetime.

Once a suitable lattice size is identified — that is, once $\langle\tau\rangle$ stops growing significantly with L — we perform high-statistics simulations to characterize the survival time distribution in the asymptotic regime. We simulate $N = 10^7$ walks on a lattice with size $L = 10^6$, which is well beyond the saturation threshold and thus effectively corresponds to an infinite system. We compute the survival time τ for each walk and obtain the full distribution. Results are shown in FIG. 2b. The mean survival time is found to be $\langle\tau\rangle = 3209.0 \pm 0.8$.

The survival time distribution shows an exponential tail, indicating that although long-lived walks are possible, they become increasingly rare. Compared to the kinetic SAW, where the average survival time is approximately 71 steps [6], the SARK exhibits a dramatic extension in lifetime. This highlights the effect of non-locality: the ability to make knight-like jumps allows the walker to bypass crowded regions and escape local traps more efficiently, thereby postponing the onset of confinement. In contrast to the GSAW, where the walker typically becomes trapped in a compact region, the SARK explores a much larger area before terminating. This illustrates how the introduction of non-local movement patterns alters the fundamental trapping mechanism of kinetic self-avoiding walks.

B. End-to-end distance and fractal dimension

In the same simulation used to obtain the survival time distribution (with $N = 10^7$ walks on a lattice of size $L = 10^6$), we compute the end-to-end distance R_τ at each step of the walk. Specifically, we measure the Euclidean distance between the walker's current position and its starting point after τ steps. After collecting this data across all walks, we compute the average value $\langle R_\tau\rangle$ for each τ.

To extract the scaling behavior of the walk, we plot $\langle R_\tau\rangle$ as a function of τ in a log-log plot. The scaling relation is expected to follow (1). The slope of the curve in the log-log plot provides an estimate of ν, which is related to the fractal dimension of the walk through (2). The results are shown in FIG. 3.

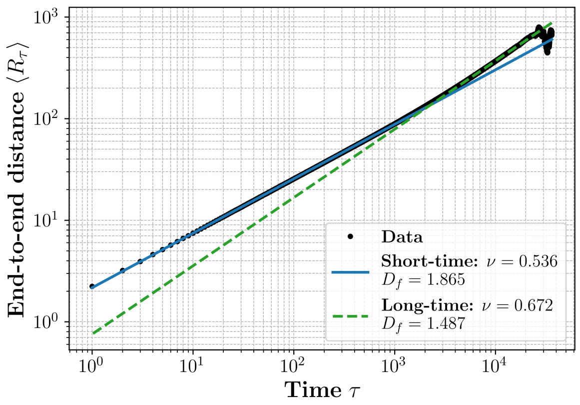


FIG. 3: Log-log plot of the mean end-to-end distance $\langle R_\tau \rangle$ as a function of survival time τ, measured over $N = 10^7$ walks on a lattice of size $L = 10^6$. A crossover between two scaling regimes is observed. The short-time regime ($\tau \lesssim 600$) yields an exponent $\nu = 0.536$ with $R^2 = 0.99999$, while the long-time regime ($5000 < \tau \lesssim 15000$) yields $\nu = 0.672$ with $R^2 = 0.9998$.

The resulting log-log plot reveals the presence of a crossover between two distinct scaling regimes. For short walks (typically up to a few hundred steps), the extracted ν exponent is lower than that of the GSAW ($\nu_{GSAW} \approx 0.68$), indicating a higher effective fractal dimension. In this regime, the non-local character of the knight moves has a strong influence on the geometry of the path, leading to denser, more space-filling configurations.

In contrast, for longer walks — particularly those exceeding the mean survival time — the scaling behavior converges toward that of the GSAW, suggesting the emergence of a universal behavior at long times, largely independent of the microscopic details of the step dynamics. It is known that the GSAW and SAW share the same universality class in the asymptotic limit [7, 8], although this convergence requires extremely long walks that are difficult to generate due to the short survival time of the GSAW. Whether the SARK also belongs to the same universality class as the SAW for very long trajectories remains an open question, given the computational limits on producing such long walks. Nonetheless, the numerical evidence provided suggests that this could indeed be the case.

C. Clustering properties

To better understand the spatial organization of visited sites, we analyze the formation and growth of clusters during the walk. At selected time steps, we take a snapshot of the set of all visited lattice sites and group them into connected components based on nearest-neighbor adjacency. Specifically, two sites are considered to belong to the same cluster if they are connected via a sequence of adjacent lattice sites with displacements of the form $(\pm 1, 0)$ or $(0, \pm 1)$. This definition is independent of the knight's movement pattern and instead reflects standard spatial connectivity on the lattice.

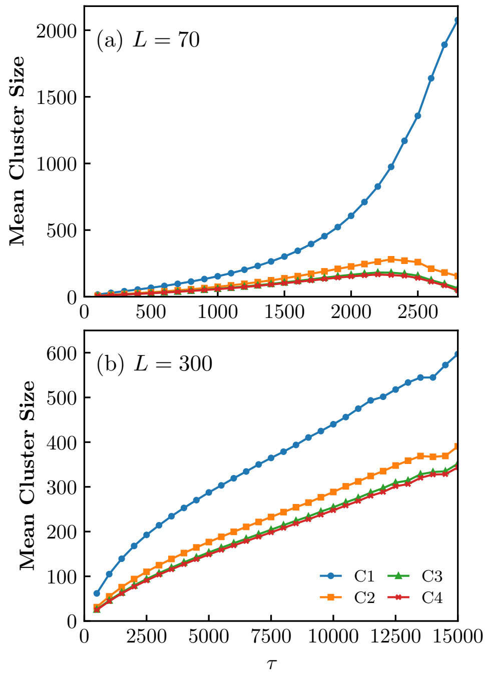


FIG. 4: Size of the largest four clusters as a function of time for the SARK. (a) Results for a small lattice (finite-size regime) show the emergence of a dominant cluster, with the remaining ones vanishing, indicating a connectivity transition. (b) In contrast, results for a large lattice show all clusters growing independently, with no clear dominance, suggesting that the transition observed in (a) is a finite-size effect.

For each sampled configuration, we keep track of the the sizes of the four largest clusters. The goal is to identify whether a connectivity transition takes place during the evolution of the walk, in which many initially disconnected clusters merge into a dominant, system-spanning component. A hallmark of such a transition is the sudden emergence of a giant component, characterized by a rapid increase in the size of the largest cluster and a concurrent decrease in the sizes of the remaining ones.

To investigate this, we compare simulations performed on both small and large lattices. In both cases 500,000 walks were simulated. The small lattice has linear size $L = 70$, where finite-size effects are expected to play a significant role, while the large lattice has size $L = 300$, in which finite-size effects are largely reduced, as evidenced by the saturation behavior observed in the survival time analysis. Results are shown in FIG. 4.

In small systems, we observe a clear transition: after a certain number of steps, the largest cluster grows rapidly and overtakes all others, which vanish in comparison. This indicates that the disconnected components generated by the non-local knight dynamics eventually become connected into a single dominant structure — a behavior consistent with a percolation-like transition.

However, in simulations on large lattices (where finite-size effects are suppressed, as discussed in previous sections), this behavior is no longer observed. Instead, all clusters appear to continue growing independently until the walker becomes trapped. The size of the largest cluster increases steadily, but no abrupt emergence of a dominant component is observed. This suggests that in the asymptotic regime, the system does not spontaneously self-connect within the typical survival time of the walk. In other words, the walker generates a fragmented spatial footprint, and the merging of clusters — if it occurs at all — happens on timescales longer than the accessible lifetime of the process.

These findings suggest that the transition observed in small systems is likely a finite-size effect rather than an intrinsic feature of the walk dynamics. Whether a genuine connectivity transition exists in the infinite-lattice limit for extremely long walks remains an open question, currently inaccessible due to the computational difficulty of generating such long-lived trajectories.

IV. CONCLUSIONS

- We have studied a variation of the GSAW in which the walker moves using knight-like, non-local steps on a two-dimensional square lattice. Through extensive simulations, we analyzed three main aspects of the process: the survival time, the end-to-end distance scaling, and the spatial clustering of visited sites.

- Our analysis of survival time reveals that the non-locality of the step rule significantly increases the typical lifetime of the walker compared to the classical GSAW. This allows the SARK to explore much larger regions before becoming trapped. Although the survival time distribution retains an exponential tail — indicating that very long walks remain rare — the overall increase in accessible space and duration highlights how non-locality alters the trapping dynamics by enhancing the walker's exploratory capacity.

- In the study of end-to-end scaling, we observed a crossover between two regimes. At short times, the walk is denser than the GSAW, with a higher effective fractal dimension. At longer times, however, the scaling behavior converges to that of the GSAW, suggesting that a form of universal behavior emerges at large scales, regardless of the microscopic step rule.

- Finally, our cluster analysis shows a striking contrast between small and large lattice sizes. While small systems display the emergence of a dominant cluster, this behavior disappears in larger lattices. In the large-size limit, no clear connectivity transition is observed within the typical lifetime of the process.

- These results highlight how non-locality modifies the local and global features of self-avoiding walks, while also suggesting that universal scaling laws may still govern the asymptotic behavior. An open question remains as to whether a true connectivity transition or convergence to the SAW universality class emerges in even longer walks, which are currently beyond reach due to computational limitations. It would be interesting to explore methods such as those proposed in [9], which allow for indefinitely growing walks; adapting such techniques to the SARK could offer access to the asymptotic regime. Additionally, an analytical treatment of the model could yield further insight into its scaling behavior and universality.

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Una visió cinètica dels processos autoevitants no locals

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Resum: Els caminants autoevitants (SAWs) són emprats per modelitzar els efectes d'exclusió de volum en polímers entre altres sistemes. Normalment es defineixen amb passos locals entre primers veïns, i el seu comportament estadístic ha sigut àmpliament estudiat. En canvi, se sap molt menys sobre els SAWs que incorporen moviments no locals. En aquest treball estudiem com la introducció de no-localitat —en forma de salts de longitud fixa inspirats en el moviment del cavall dels escacs— afecta els caminants autoevitants cinètics (GSAWs), en els quals el camí creix de manera irreversible. El model resultant, el qual anomenem cavaller autoevitant aleatori (SARK), substitueix la propagació local per passos llargs i restringits. Mitjançant simulacions a gran escala, examinem com aquesta dinàmica influeix en diverses propietats del camí. Trobem que la no-localitat incrementa significativament la vida mitjana i l'abast espacial del procés. L'escalament de la distància *end-to-end* mostra una transició que tendeix cap al comportament dels GSAW en temps llargs. L'anàlisi de clústers revela una component dominant en xarxes petites que desapareix en xarxes més grans. Aquests resultats aporten nova comprensió sobre com les restriccions no locals afecten la geometria i el creixement dels SAWs, amb possibles aplicacions en ecològia i en fenòmens de transport en entorns limitats.
Paraules clau: Caminants aleatòris, lleis d'escala, universalitat, modèls de xarxa.
ODSs: Vida submarina i vida terrestre

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de les desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat		13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	X
6. Aigua neta i sanejament		15. Vida terrestre	X
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			