Brain Connectomes Navigability in Hyperbolic Space

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Abstract: This study evaluates navigability in brain connectomes embedded in 2D hyperbolic geometry using greedy routing—a decentralized navigation protocol. Applying this framework to four empirical connectomes—spanning species, scales, and node types—we assess navigability via success rate, topological stretch, and two geometrical stretch variants. We introduce a novel geometrical stretch definition based on accumulated hyperbolic distance along minimal-cost paths, demonstrating improved consistency with topological stretch and potentially better reflection of the underlying geometry. Our results confirm hyperbolic geometry's effectiveness as a latent space for efficient brain information transmission, providing further perspective on network science, complex systems, and brain connectivity.

Keywords: brain connectomes, hyperbolic geometry, greedy routing, network science, complex systems, navigability
SDGs: 4.4: Skills for work, 3.4: Health and well-being, 9.5: Research and technological capabilities

I. INTRODUCTION

The human brain is one of the most complex systems in nature, and understanding how its structure gives rise to cognition and behavior remains a fundamental challenge in science. From a network science perspective, the brain can be modeled as a *connectome*—a graph where nodes represent brain regions (or neurons) and edges represent structural connections, such as axonal projections or white matter fiber tracts that physically link distinct areas of the nervous system [1]. Like many other complex networks, brain networks exhibit characteristic topological features that support efficient communication, robustness, and functional specialization. These include regions that are more densely interconnected internally than with the rest of the network (*modularity*); a low average shortest-path length between nodes, meaning that any two regions are separated by only a few steps (*small-world property*); a highly uneven distribution of connections, where a few regions act as hubs while most have few links (*heterogeneous degree distribution*); and a tendency for connected nodes to share common neighbors, forming tightly interconnected groups (*high clustering*). [2].

These same features are naturally captured by the concept of *network geometry*, and in particular, a model with hyperbolic geometry—a non-Euclidean geometry in which distances encode both similarity and hierarchical relationships between nodes. In this model, the probability of connection between brain regions depends on distances in the underlying space [3]. Unlike Euclidean space, where volume grows polynomially with radius, hyperbolic space expands exponentially: in two dimensions, the circumference and area of a disk grow as $\sim e^r$ and $\sim e^{2r}$, respectively. This exponential expansion makes hyperbolic geometry especially suitable for embedding hierarchical, tree-like structures such as brain networks [4]. Connectomes, however, are not purely trees—they also contain loops and long-range projections that support global integration, resilience, and feedback. Hyperbolic space accommodates both aspects: it preserves branching hierarchies while allowing efficient shortcuts, making it a compact and biologically plausible framework for modeling brain connectivity.

In the context of networks, and in particular of brain networks, hyperbolic embeddings can be interpreted as an *effective geometry*—an abstract latent space that better captures the underlying principles of network organization than anatomical (Euclidean) coordinates. While the brain is physically embedded in Euclidean space, such distances often fail to fully explain its observed wiring patterns [5]. Hyperbolic geometry, by contrast, reconciles local clustering with global efficiency and thus provides a more faithful representation of the brain's structural and communication architecture.

This study investigates the efficiency of decentralized navigation in hyperbolic maps of structural brain networks using a *greedy routing* (GR) navigation protocol. While GR typically underperforms in Euclidean space [5], previous work has shown that hyperbolic embeddings enable near-optimal routing, revealing a strong alignment between hyperbolic distance and brain connectivity. In the context of network navigation, efficiency is achieved when signals can reach their targets by following paths that are both successful and close in length to the shortest possible routes—minimizing detours and communication costs.

Building on these insights, we implemented GR in various empirical connectomes embedded in hyperbolic space. In addition to standard metrics, such as success rate and topological stretch, we propose a new formulation of geometrical stretch that uses, as a reference, the path with minimal cumulative hyperbolic distance among all topological paths connecting a given node pair. We compared this alternative with the standard definition based on direct geodesic distance between source and target. This comparison allows us to explore which formulation better captures the efficiency of greedy routing and offers a more meaningful geometric interpretation of communication in brain networks.

II. DATA AND METHODS

A. Connectome Datasets

This study analyzes a subset of the structural connectome datasets used in Allard et al. [5], specifically those corresponding to Mouse, Macaque, and Human brains. These datasets represent anatomical connectivity between brain regions or individual neurons and were preprocessed in the original study to enable cross-species and cross-scale comparisons.

Four connectomes were selected to capture variability in species, scale, and network unit type:

- **Human 2 (H2):** Single-hemisphere cortical network; 496 nodes (brain regions).
- **Human 5 (H5):** Full cortical network; 1024 nodes (both hemispheres).
- **Macaque 1 (Ma1):** Low-resolution inter-regional network; 94 nodes.
- **Mouse 2 (Mo2):** High-resolution retinal network; 916 nodes (individual neurons).

These datasets span macroscopic to microscopic scales, enabling navigability analysis across levels of biological organization. Comparing human and non-human connectomes also aids in identifying generalizable structural features.

The data were obtained from public repositories such as `https://openconnecto.me` and `https://icon.colorado.edu`, or directly from the original authors [6–9]. Each dataset includes a binary connectivity matrix and spatial coordinates in Euclidean space.

For consistency, all networks are treated as undirected and unweighted, following previous work [5]. Further methodological details are available in the S1 Appendix of Allard et al. [5].

A. Hyperbolic Embeddings

To explore the latent geometry underlying brain connectivity, each connectome was embedded into a two-dimensional hyperbolic space using the S^1/H^2 model framework [10]. In this model, each node is assigned two coordinates: a radial coordinate, which reflects its popularity or connectivity (linked to expected degree), and an angular coordinate, which encodes similarity or affinity to other nodes. These embeddings do not preserve anatomical distances but instead aim to recover an *effective geometry*, in which spatial proximity reflects the likelihood of forming a connection.

The S^1 and H^2 models are formally isomorphic: they define equivalent probabilistic models for network generation, differing only in the metric used to interpret distances. In the S^1 formulation, nodes are placed on a circle and assigned a hidden degree parameter κ_i that governs their connection probability. In the H^2 model, these hidden degrees are transformed into radial coordinates r_i, such that higher-degree nodes lie closer to the origin in the hyperbolic plane. Angular coordinates are preserved in both cases and measure similarity. While the S^1 model is often used for analytic derivations and inference due to its tractable connection probability structure, the H^2 version is preferred for studying navigation and geometry, as it allows meaningful measurement of distances between nodes in hyperbolic space [11].

The embedding is performed via maximum likelihood estimation using the Metropolis–Hastings algorithm described in García-Pérez et al. [12]. This method infers the coordinates $\{r_i\}, \{\theta_i\}$ that maximize the likelihood that the observed network structure was generated by the model. The connection probability between two nodes i and j in the S^1 model formulation is given by:

$$p_{ij} = \frac{1}{1 + \left(\frac{x_{ij}}{R\mu\kappa_i\kappa_j}\right)^{\beta}}, \quad (1)$$

where x_{ij} is the hyperbolic distance between nodes i and j, R is the radius of the circle representing the similarity space, κ_i and κ_j are their hidden degrees (or expected degrees), and β controls the coupling between network structure and the underlying geometry, that is, clustering (triangles in the network structure) as a manifestation of the triangle inequality in the underlying space.

In the isomorphic H^2 formulation, the hidden degree κ_i is converted into a radial coordinate via $r_i = R - 2\ln\left(\frac{\kappa_i}{\kappa_{\min}}\right)$. The connection probability then depends solely on the hyperbolic distance between nodes and takes the form:

$$p_{ij} = \frac{1}{1 + e^{\frac{\beta}{2}(x_{ij} - R)}}. \quad (2)$$

The projection into a 2D hyperbolic disk is not meant to reflect physical space, but to reveal a latent geometry that better aligns with the topological structure of the brain. Hyperbolic embeddings have been shown to reproduce key features of real-world networks, such as heavy-tailed degree distributions, strong clustering, modularity, hierarchical organization and also hidden symmetries—properties that are also present in empirical connectomes [4].

The resulting hyperbolic maps provide a compact and information-rich geometric representation of network structure. In the results section, we evaluate their utility by testing how well they support efficient decentralized communication through greedy routing.

B. Greedy Routing

To evaluate how well the hyperbolic embedding captures the structure of a connectome, we apply a decen-

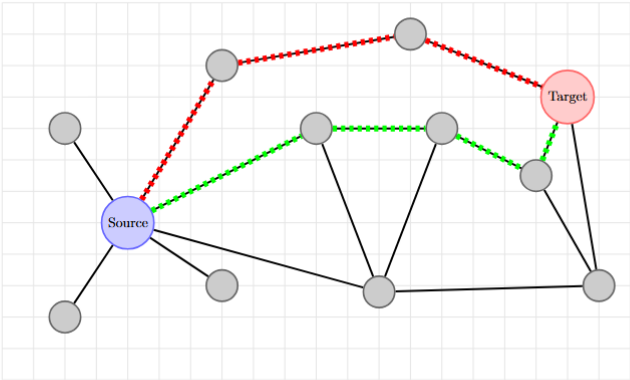

FIG. 1: Illustration of a successful greedy path (green) and the topological shortest path (red) on a network embedded in the 2D-plane.

tralized navigation strategy known as greedy routing (GR). This method not only tests the geometric quality of the embedding but also serves as a model for exploring network communicability under local decision-making constraints. In this process, a signal is sent from a source node to a target node by successively selecting the neighbor that lies closest to the target in hyperbolic space. The signal progresses step by step based on local geometric decisions, relying only on the spatial positions of neighboring nodes.

The routing attempt terminates when the target is reached (success) or when no neighbor is closer to the target than the current node, and the process enters into an endless loop (failure).Figure 1 illustrates an example of a successful greedy route between a source and a target node and its corresponding shortest path.

Greedy routing serves as a functional probe of the embedding. If most source–target pairs can be connected through greedy paths that closely resemble the shortest paths, it indicates that the spatial configuration of node's coordinates effectively encodes the network's connectivity. To quantify this, we compute three main performance metrics:

- **Success rate** (S): the fraction of all node pairs (i, j) for which greedy routing reaches the target:

$$S = \frac{1}{N(N-1)} \sum_{i \neq j} \delta_{ij}, \tag{3}$$

where $\delta_{ij} = 1$ if the path from i to j is successful, and $\delta_{ij} = 0$ otherwise.

- **Topological stretch** ($\sigma_{ij}^{\mathrm{topo}}$): the ratio between the number of hops in the greedy path and the number of hops in the topological shortest path:

$$\sigma_{ij}^{\mathrm{topo}} = \frac{L_{ij}^{\mathrm{GR}}}{L_{ij}^{\mathrm{SP}}}, \tag{4}$$

where L_{ij}^{GR} is the number of links (hops) in the greedy path, and L_{ij}^{SP} is the same count but for the topological shortest path.

- **Geometrical stretch** ($\sigma_{ij}^{\mathrm{geo}}$): the ratio between the total hyperbolic length of the greedy path and that of a reference path:

$$\sigma_{ij}^{\mathrm{geo}} = \frac{D_{ij}^{\mathrm{GR}}}{D_{ij}^{\mathrm{ref}}}, \tag{5}$$

where D_{ij}^{GR} is the sum of geodesic (hyperbolic) distances between consecutive nodes along the greedy path, and D_{ij}^{ref} is the total length of a reference path.

Note on the Definition of Geometrical Stretch

There are multiple ways to define the denominator D_{ij}^{ref} in geometrical stretch. Below, we describe the three main variants considered:

(1) The version used in Allard et al. [5], where D_{ij}^{ref} is the total hyperbolic distance along the *topological shortest path* between nodes i and j.

(2) The standard version often used in the literature, in which D_{ij}^{ref} is defined as the direct hyperbolic distance between the source and target:

$$D_{ij}^{\mathrm{ref}} = d_{\mathbb{H}}(i, j). \tag{6}$$

(3) A new variant proposed in this study, where D_{ij}^{ref} corresponds to the hyperbolic length of the path with the *least total geodesic distance* among all topological paths connecting i and j in the network. This "minimal-geodesic" stretch is computationally tractable and potentially offers a more accurate estimate of geometrical efficiency.

Together, the success rate and the three stretch metrics offer complementary perspectives: the former measures whether greedy navigation is feasible, while the latter reveals how direct the resulting routes are. These metrics jointly assess how well the hyperbolic embedding supports decentralized communication in brain networks.

III. RESULTS AND DISCUSSION

This section presents the results of applying greedy routing to the hyperbolically embedded brain connectomes of four networks: Human2, Human5, Macaque1, and Mouse2. We evaluated the navigability of each network using three main metrics: success rate, topological stretch, and two variants of geometrical stretch (see Section II B). Among the latter, version (2) serves as the standard reference in the literature, while version (3) is the alternative proposed in this work. In addition to these global metrics, we also compared the distributions of path lengths obtained through greedy routing versus shortest paths, both in terms of topological distance (number of hops) and hyperbolic (geodesic) distance.

A. Global Navigability Metrics

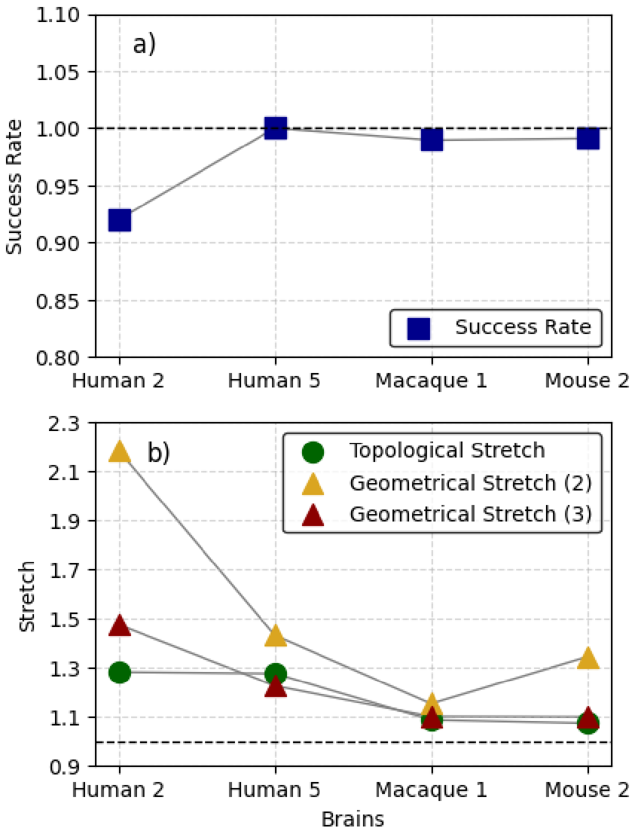

FIG. 2: Comparison of greedy routing performance across the studied connectomes. (a) Success rates. (b) Topological and geometrical stretch metrics (versions 2 and 3). Exact numerical values are provided in Table 1.

Figure 2 and Table 1 summarize the global navigation metrics across the four connectomes. The success rates (SR) are exceptionally high across all networks, ranging from 0.92 in Human2 to perfect success (1.0) in Human5. Macaque1 and Mouse2 also exhibit near-optimal values (0.9895 and 0.9910, respectively). These results indicate that almost all source–target pairs are reachable via greedy routing based solely on hyperbolic proximity, suggesting strong congruence between the network topology and the inferred hyperbolic geometry. The average topological stretch remains low in all cases (1.07–1.28), showing that greedy paths closely follow topological shortest paths. This confirms that the routing not only succeeds frequently, but also finds routes that are nearly optimal in terms of hop count.

Regarding geometrical stretch, both the standard and proposed definitions reveal important insights. Using the *standard version* (2), the obteined values are slightly higher—especially in Human2 (2.19)—suggesting some deviation from the most direct metric path. However, this reference path does not necessarily correspond to any actual route allowed by the network topology, which limits its interpretability in terms of real communication paths.

TABLE I: Metric values for the studied connectomes, with associated errors calculated as $\frac{\sigma}{\sqrt{n}}$, where n is the number of successful paths used in the computation. Note that the success rate (SR) is not accompanied by an error value, as it is not a mean-based measure and was computed over all possible source–target pairs in each network.

	SR	TS	GS 2	GS 3
H2	0.92	1.2828 ± 0.0008	1.4759 ± 0.0016	2.1859 ± 0.0012
H5	1.00	1.2766 ± 0.0002	1.2267 ± 0.0003	1.4326 ± 0.0005
Ma1	0.99	1.0878 ± 0.0012	1.1015 ± 0.0023	1.1535 ± 0.0031
Mo2	0.99	1.0744 ± 0.0022	1.0998 ± 0.0011	1.3452 ± 0.0019

In contrast, the *proposed minimal-geodesic version* (3), provides consistently lower stretch values (1.10–1.47), with particularly close agreement across Macaque1 and Mouse2. Notably, this version yields geometrical stretch values that are much closer to the corresponding topological stretch values compared to version (2). This suggests that it better captures the interplay between geometry and topology in the network, and may offer a more realistic and interpretable metric for assessing routing efficiency.

These global results are consistent with prior findings [3, 5], reinforcing that hyperbolic geometry provides an effective latent space for efficient decentralized communication in brain networks across scales, whether the connectome represents brain regions or neurons. This motivates a detailed comparison of greedy routing paths to shortest paths—particularly their length distributions—to quantify how well hyperbolic embeddings optimize information flow.

B. Path Length Distributions

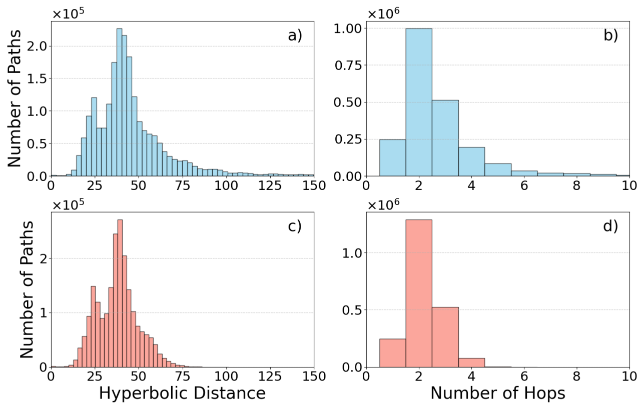

FIG. 3: Path length distributions for greedy navigation (blue) versus shortest paths (red) across four connectomes. Histograms compare: (a) Hyperbolic length of greedy paths, (c) Hyperbolic length of shortest paths, (b) Topological length (hops) of greedy paths, (d) Topological length of shortest paths. All panels aggregate all source–target pairs across connectomes.

To further examine how closely greedy paths approximate shortest paths, we analyzed the distribution of path lengths across all source–target pairs, aggregated over the four connectomes. Figure 3 presents histograms comparing greedy routing and shortest paths, both in terms of topological distance (number of hops) and hyperbolic (geodesic) distance. In both representations, the distributions of greedy path lengths closely resemble those of the corresponding shortest paths. This similarity suggests that the hyperbolic embeddings preserve not only global navigability but also the local structure of optimal routing paths.

Greedy path distributions, however, consistently exhibit slightly heavier tails, particularly in the number of hops. This behavior is expected, as greedy routing relies solely on local information and may occasionally lead to suboptimal detours. Nevertheless, the deviations remain small, indicating that greedy routing is remarkably effective at identifying low-cost paths despite its decentralized nature. This reinforces the conclusion that hyperbolic space effectively captures both the hierarchical and geometric organization of brain connectivity.

IV. CONCLUSIONS

This study demonstrates that hyperbolic embeddings provide an effective and biologically plausible framework for modeling communication in structural brain networks. By evaluating greedy routing across a diverse set of connectomes—including different species (mouse, macaque, human), spatial scales, and node types (brain regions vs. individual neurons)—we show that decentralized navigation is consistently successful and efficient. The high success rates and low stretch values observed across all cases indicate that hyperbolic geometry captures core structural principles of brain organization in a way that generalizes beyond specific datasets.

A central contribution of this work is the introduction and evaluation of a new definition of geometrical stretch, which uses the path with minimal cumulative hyperbolic distance as a reference. This variant yields lower and more stable stretch values than the standard definition and shows a closer alignment with topological stretch, suggesting a better agreement between network topology and spatial embedding. Its robustness across networks and its improved interpretability position it as a promising metric for future studies.

Further investigation is needed to fully assess the potential of this new stretch formulation. A more detailed analysis of its mathematical properties, biological plausibility, and relation to topological and geometric features of the connectome could provide valuable insights into how brains optimize communication under structural constraints.

Overall, our findings reinforce the idea that the brain's architecture is well-suited for efficient local navigation, and that hyperbolic embeddings offer a compact and powerful way to reveal the latent geometric organization underlying structural connectivity.

Acknowledgments

I would like to thank my advisor, Mª Àngels, for introducing me to this fascinating field and for the insightful meetings that sparked a genuine passion in me. I am also grateful to my colleagues and family for their constant support and encouragement throughout this journey.

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Navegabilitat dels connectomes cerebrals en l'espai hiperbòlic

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Resum: Aquest estudi avalua la navegabilitat en connectomes cerebrals incrustats en geometria hiperbòlica 2D mitjançant el greedy routing, un protocol de navegació descentralitzat. Aplicant aquest marc a quatre connectomes empírics —que abasten espècies, escales i tipus de nodes—, mesurem la navegabilitat mitjançant la taxa d'èxit, l'stretch topològic i dues variants de l'stretch geomètric. Introduïm una nova definició d'stretch geomètric basada en la distància hiperbòlica acumulada al llarg de camins de cost mínim, que mostra una millor consistència amb l'stretch topològic i reflecteix potencialment millor la geometria subjacent. Els nostres resultats confirmen l'eficàcia de la geometria hiperbòlica com a espai latent per a un transport de la informació eficient, aportant noves perspectives a la ciència de xarxes, els sistemes complexos i la connectivitat cerebral.
Paraules clau: connectomes cerebrals, geometria hiperbòlica, greedy routing, ciència de xarxes, sistemes complexos, navegabilitat
ODSs: 4.4: Habilitats per a l'ocupació, 3.4: Salut i benestar, 9.5: Recerca i capacitats tecnològiques

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar	X	12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures	X		

El contingut d'aquest Treball de Fi de Grau, realitzat en el marc del grau de Física, es relaciona principalment amb l'**ODS 4**, i en particular amb la **fita 4.4**, ja que contribueix a l'educació científica i tecnològica a nivell universitari. També es vincula amb l'**ODS 3**, **fita 3.4**, atès que el coneixement fonamental sobre la connectivitat cerebral pot contribuir, a llarg termini, a millorar la comprensió, la prevenció i el tractament de trastorns neurològics. A més, el treball promou la recerca en ciència bàsica i el desenvolupament de models innovadors dins del camp dels sistemes complexos, en línia amb l'**ODS 9**, **fita 9.5**, que fomenta la investigació científica i la millora de la capacitat tecnològica.