Characterization of the local Stokes parameters of a highly focused beam

Author: Rosa Minguella Rafecas, rmingura18@alumnes.ub.edu
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Artur Carnicer, artur.carnicer@ub.edu

Abstract: The polarisation state of a transverse electromagnetic wave is described by the Stokes parameters. The goal of this project is to describe the polarization state of light taking into account the three-dimensional nature of the electric field, present in highly focused beams. To achieve this, a modified version of the Stokes parameters, the local Stokes parameters, has been calculated. The intensity of the field in the focal area and the local Stokes parameters are shown throughout the project for different types of incident beams.
Keywords: Stokes parameters, microscopy.
SDGs: Health and wellbeing.

I. INTRODUCTION

The Stokes parameters are used to describe polarization of an electromagnetic wave, by analysing the transverse components of the electric field. While they are highly useful for characterizing transverse beams, they cannot accurately represent waves that have a longitudinal component.

This problem arises in highly focused beams, described by the Richards-Wolf integral [1]. In these cases the field distribution in the focal area contains both transverse and longitudinal components. When attempting to describe the polarization, the Stokes parameters are unable to fully describe the polarization state of the focused field [2]. To address this, a modified version of the Stokes parameters was introduced in Ref. [3] which accounts for the 3D characteristics of the light beam. Other methods have also been described in Ref. [4, 5], but the method described in Ref. [3] does not abandon the conceptual appeal involved in the usual formulation of 2D Stokes parameters. In this project, we aim to calculate these Stokes parameters in order to have an accurate description of the field distribution in the focal region.

We will study different cases, with different incident beams. First, we will study two polarization states that remain constant for each point in space: a linearly polarized and a circularly polarized incident beam. We will continue studying a circularly polarized with topological charge beam. Finally, for their interest in microscopy, we will examine two beams with cylindrical symmetry in polarization [6]: a radially polarized beam and an azimuthally polarized beam.

To obtain these results, we have used the *Scipy* library within the *Python* programming language [7].

This paper is structured as follows: Section II we introduce theoretical background of highly focused beams and the local Stokes parameters. In Section III, we show the results obtained for different polarization states of the incident beam. Finally, in Section IV, we present the conclusions.

II. THEORETICAL BACKGROUND

A. Highly focused beams

Let us consider a laser beam that propagates in the z direction in an aplanatic lens system. For an aplanatic lens system, two rules of geometrical optics have to be followed.

- The *sine condition* states that each optical ray which emerges from or converges to the focus of an aplanatic optical system intersects its conjugate ray on a sphere of radius f, where f is the focal length of the lens.
- The *intensity law*,a consequence of the Fresnel laws [8], states that the energy incident on the aplanatic lens equals the energy that leaves the lens. Because of that, the fields of non-magnetic materials before and after the refraction must fulfill

$$|\mathbf{E}_1| = |\mathbf{E}_2|\sqrt{\frac{n_1}{n_2}}(\cos\theta)^{1/2}. \quad (1)$$

The angular spectrum representation of the focal field follows the Richards-Wolf integral:

$$\mathbf{E}(\rho,\varphi,z) = A\int_0^{\theta_0}\int_0^{2\pi}\mathbf{E}_\infty(\theta,\phi)e^{[ikz\cos\theta]}e^{[ik\rho\sin\theta\cos(\phi-\varphi)]}\sin\theta\, d\phi\, d\theta, \quad (2)$$

where A is a constant, θ_0 is defined by the numerical aperture of the microscope objective (NA $= n\sin\theta_0$), ρ, φ and z are the cylindrical coordinates, and θ and ϕ are the spherical coordinates. The results will be evaluated at $z=0$, the focal plane. $\mathbf{E}_\infty$ is the field in the surface of the reference sphere of the focus, and it can be expressed as

$$\mathbf{E}_\infty = [t^s(\mathbf{E}_{inc}\cdot\mathbf{e}_1)\mathbf{e}_1 + t^p(\mathbf{E}_{inc}\cdot\mathbf{e}_2')\mathbf{e}_2]\sqrt{\frac{n_1}{n_2}\cos\theta}, \quad (3)$$

where t^s and t^p are the transmission coefficients.

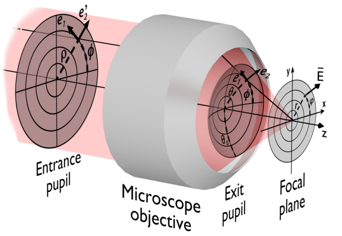


FIG. 1: Diagram of the microscope objective and definitions of the coordinates and variables involved in the process [3].

If we express the unit vectors $\mathbf{n}_\phi$, $\mathbf{n}_\rho$ and $\mathbf{n}_\theta$ in terms of the Cartesian unit vectors, assuming $t^s = t^p = 1$ and describing $\mathbf{E}_{inc}$ as $E_{inc,x}\mathbf{n}_x + E_{inc,y}\mathbf{n}_y$, we can simplify the farfield to

$$\mathbf{E}_\infty = \left\{ \frac{E_{inc,x}}{2} \begin{pmatrix} (1+\cos\theta) - (1-\cos\theta)\cos 2\phi \\ -(1-\cos\theta)\sin 2\phi \\ -2\cos\phi\sin\theta \end{pmatrix} + \frac{E_{inc,y}}{2} \begin{pmatrix} -(1-\cos\theta)\sin 2\phi \\ (1+\cos\theta) + (1-\cos\theta)\cos 2\phi \\ -2\sin\phi\sin\theta \end{pmatrix} \right\} \sqrt{\frac{n_1}{n_2}}\cos\theta. \tag{4}$$

We will describe each incident beam by its Jones vector, being

$$\mathbf{J} = \frac{1}{\sqrt{|E_x|^2 + |E_y|^2}} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} J_x \\ J_y \end{pmatrix}. \tag{5}$$

We can see that the Jones vector is unitary. Because of that, the intensity of each incident beam point will have a maximum value of 1.

Assuming that the profile of the incident beam is Gaussian, with the function,

$$f(\theta) = e^{-\frac{1}{f_0^2}\frac{\sin^2\theta}{\sin^2\theta_0}}, \tag{6}$$

where f_0 is the *filling factor* ($f_0 = \frac{\omega_0}{f\sin\theta_0}$), we will compute the results taking the values NA $= 1.4$, $n_2 = 1.518$, as it is a common value for microscopes, and $f_0 = 100$, to make the beam almost a plane wave.

B. Stokes parameters

The usual transverse Stokes parameters are described by the following equations [8]:

$$S_0 = |E_x|^2 + |E_y|^2, \tag{7}$$
$$S_1 = |E_x|^2 - |E_y|^2, \tag{8}$$
$$S_2 = 2\,Re[E_x^* E_y], \tag{9}$$
$$S_3 = 2\,Im[E_x^* E_y]. \tag{10}$$

The electrical field vector can be described as a combination of two vectors, $\mathbf{E}_r$ and $\mathbf{E}_i$ [3], being

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_r(\mathbf{r}) + i\mathbf{E}_i(\mathbf{r}), \tag{11}$$

where both $\mathbf{E}_r(\mathbf{r})$ and $\mathbf{E}_i(\mathbf{r})$ are real vectors at any given point in space. The two vectors are contained within a plane, where we will find a local, orthogonal basis set:

$$\mathbf{P}(\mathbf{r}) = \cos\alpha(\mathbf{r})\,\mathbf{E}_r(\mathbf{r}) + \sin\alpha(\mathbf{r})\,\mathbf{E}_i(\mathbf{r}), \tag{12}$$
$$\mathbf{Q}(\mathbf{r}) = -\sin\alpha(\mathbf{r})\,\mathbf{E}_r(\mathbf{r}) + \cos\alpha(\mathbf{r})\,\mathbf{E}_i(\mathbf{r}), \tag{13}$$

where the spatially dependent rotation angle ($\alpha(\mathbf{r})$) is defined as

$$\tan 2\alpha(\mathbf{r}) = \frac{2\mathbf{E}_r(\mathbf{r})\cdot\mathbf{E}_i(\mathbf{r})}{||\mathbf{E}_r(\mathbf{r})||^2 - ||\mathbf{E}_i(\mathbf{r})||^2}. \tag{14}$$

We can describe the plane by its normal vector:

$$\mathbf{N}(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \times \mathbf{Q}(\mathbf{r}) = \mathbf{E}_r(\mathbf{r}) \times \mathbf{E}_i(\mathbf{r}). \tag{15}$$

With this vectors, the local Stokes parameters are defined as follows.

$$\tilde{S_0}(\mathbf{r}) = ||\mathbf{P}(\mathbf{r})||^2 + ||\mathbf{Q}(\mathbf{r})||^2, \tag{16}$$
$$\tilde{S_1}(\mathbf{r}) = ||\mathbf{P}(\mathbf{r})||^2 - ||\mathbf{Q}(\mathbf{r})||^2, \tag{17}$$
$$\tilde{S_2}(\mathbf{r}) = 0, \tag{18}$$
$$\tilde{S_3}(\mathbf{r}) = 2||\mathbf{N}(\mathbf{r})||. \tag{19}$$

We can see that only three parameters are needed to describe the polarization, instead of four, as in this formalism $\tilde{S_2}$ is always defined as 0.

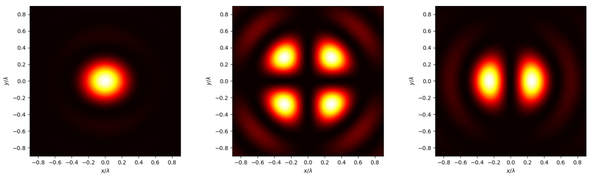
III. RESULTS

A. Linearly polarized incident beam

The first case we are going to study is a linearly polarized in the x direction incident beam, with the Jones vector:

$$\mathbf{J}_{lineal} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

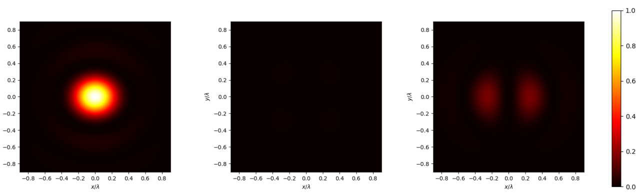
In this case, the values of $|E_x|^2$ follow a 2D Gaussian distribution, as we can see in figure 2. This Gaussian is not completely symetrical, as it is slightly narrower in the y direction than in the x direction. This is hard to appreciate, but it can be seen more in detail in Fig. 4 (a).



(a) $|E_x|^2$ (b) $|E_y|^2$ (c) $|E_z|^2$

FIG. 2: Magnitude of $|E_x|^2$, $|E_y|^2$ and $|E_z|^2$ in the focal plane ($z = 0$) for a linearly polarized incident beam. This image is autoscaled, meaning the lightest pixels correspond to the maximum values and the darkest pixels (black) correspond to the minimum values.

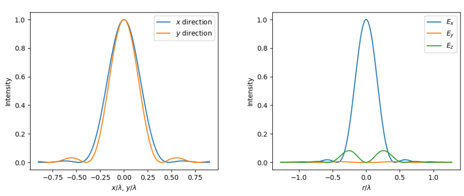
The results shown in Fig. 2 can be misleading, as they are not normalized, and they are scaled to the minimum and maximum values of the data from each image. In this way, we can see the distributions of each component of the electric field in the focus, but cannot properly see their real contributions to the overall field. This is addressed in Fig. 3, as the results are normalized to the maximum intensity of the electric field ($|E_x|^2+|E_y|^2+|E_z|^2$). From this point on, all the results for the field of the focused beam will be shown normalized.



(a) $|E_x|^2$ (b) $|E_y|^2$ (c) $|E_z|^2$

FIG. 3: Magnitude of $|E_x|^2$, $|E_y|^2$ and $|E_z|^2$ in the focal plane ($z = 0$) for a linearly polarized incident beam. This image is scaled to the maximum value of $|E_x|^2 + |E_y|^2 + |E_z|^2$.

As we can see in Fig. 3 and 4 (b), the y component has very small values and is not noticeable to the overall field. On the other hand, the z component does have its importance at certain points, even though its contribution is not as big as the x component.

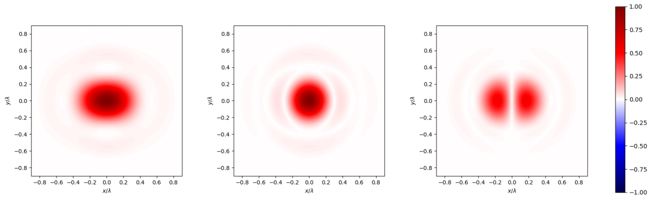


(a) (b)

FIG. 4: Horizontal and vertical profiles of $|E_x|^2$ (a) and diagonal of $|E_x|^2$, $|E_y|^2$ and $|E_z|^2$ (b).

With the results of the field in the focal plane, we can obtain the local Stokes parameters. In order to properly show the results, all the Stokes parameters are normalized to the maximum of $\tilde{S}_0$.

In Fig. 5 we can see the results for the linearly polarized incident beam. It is clear that $\tilde{S}_0$ is the total intensity of the electrical field, and in the points where it is 0, the rest of the Stokes parameters are also 0. We can also see that in this case, $\tilde{S}_3$ is greater than 0 only in the points where $|\mathbf{E}_x|^2$ and $|\mathbf{E}_z|^2$ are greater than zero, since for the rest of the points the polarization is linear.



(a) $\tilde{S}_0$ (b) $\tilde{S}_1$ (c) $\tilde{S}_3$

FIG. 5: 3D local Stokes parameters for the results of a lineal incident beam.

B. Circularly polarized incident beam

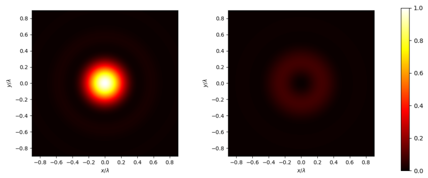
The next case is a right handed circularly polarized incident beam, with the Jones vector:

$$\mathbf{J}_{circular} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

From this point on, the results of the field will be shown with two images: their transverse components and their longitudinal components, as showing $|Ex|^2$ and $|E_y|^2$ separately, does not give any relevant information.

In this case, the values of the transverse field follow a 2D gaussian distribution. This time, it is a symetrical distribution. The longitudinal field has a distribution with the shape of a ring, and its intensity is small compared to the transverse intensity, but it is noticeable.

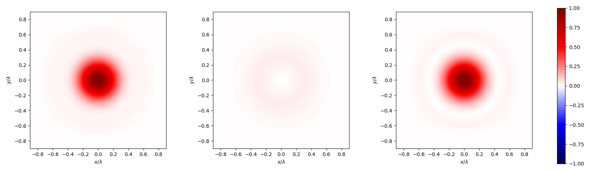
The results of this incident beam can be obtained from the results of the linear beam. The transverse intensity can be obtained from the x component of the linear beam results, by adding the image to itself, but rotated 90°. The same procedure can be made from the z component of the linear beam in order to obtain the longitudinal results of the circular beam.



(a) $|E_x|^2 + |E_y|^2$ (b) $|E_z|^2$

FIG. 6: Magnitude of $|E_x|^2 + |E_y|^2$ and $|E_z|^2$ in the focal plane ($z = 0$) for a circularly polarized incident beam.

In Fig. 7 we can see the local Stokes parameters for the circularly polarized incident beam. Again, we can see that $\tilde{S}_0$ is the total intensity of the electrical field. In this case, $\tilde{S}_1 \approx 0$, which means the polarization is circular.



(a) $\tilde{S}_0$ (b) $\tilde{S}_1$ (c) $\tilde{S}_3$

FIG. 7: 3D local Stokes parameters for the results of a circular incident beam.

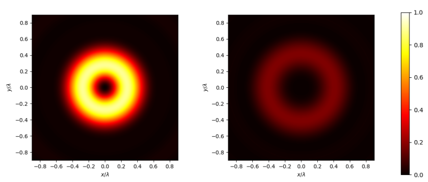
C. Circularly polarized with topological charge incident beam

The next case is a right handed circularly polarized with topological charge incident beam, with the Jones vector:

$$\mathbf{J}_{lineal} = \frac{e^{i\phi}}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

In this case, both transverse and longitudinal field distributions have the shape of a ring, with higher intensity in the transverse components.

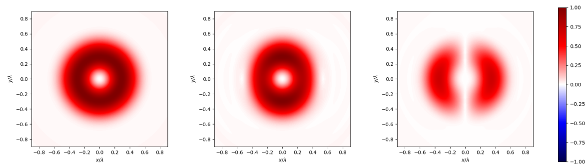
This beam is useful for applications that require a null electrical field in the focal point, such as STED microscopy.



(a) $|E_x|^2 + |E_y|^2$ (b) $|E_z|^2$

FIG. 8: Magnitude of $|E_x|^2 + |E_y|^2$ and $|E_z|^2$ in the focal plane ($z = 0$) for a circularly polarized with topological charge incident beam.

In this case, the polarization is harder to interpret, as both $\tilde{S}_1$ and $\tilde{S}_3$ have strange shapes. To analyse what each parameter describes about the field we wold have to look in detail the real and imaginary values of each component of the electrical field.



(a) $\tilde{S}_0$ (b) $\tilde{S}_1$ (c) $\tilde{S}_3$

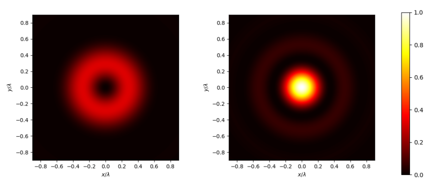
FIG. 9: 3D local Stokes parameters for the results of a circular incident beam.

D. Radially polarized incident beam

The next case is a radially polarized incident beam, with the Jones vector:

$$\mathbf{J}_{radial} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}.$$

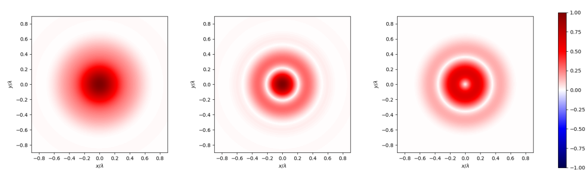
The radially polarized incident beam has the longitudinal contribution of the field larger than the transverse contribution. As it can be seen in Fig. 10, the longitudinal field has a peak at the point (0,0), and the transverse field has a ring shape.



(a) $|E_x|^2 + |E_y|^2$ (b) $|E_z|^2$

FIG. 10: Magnitude of $|E_x|^2 + |E_y|^2$ and $|E_z|^2$ in the focal plane ($z = 0$) for a circularly polarized incident beam.

We can see that the local Stokes parameters have cylindrical symmetry, similar to the incident beam.



(a) $\tilde{S}_0$ (b) $\tilde{S}_1$ (c) $\tilde{S}_3$

FIG. 11: 3D local Stokes parameters for the results of a radial incident beam.

E. Azimuthally polarized incident beam

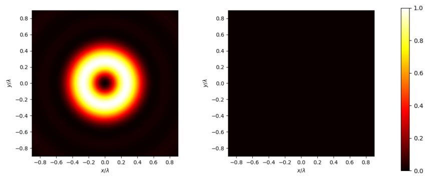
The next case is an azimuthally polarized incident beam, with the Jones vector:

$$\mathbf{J}_{azimuthal} = \begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix}.$$

Being the complete opposite to the radially polarized beam, the azimuthally polarized beam has a null longitudinal component. $\mathbf{E}_\infty$ (4) results to a z component equal to 0. It is the only case in this project where the focalized beam is a transverse wave, and could potentially be described with the usual transverse Stokes parameters.

The transverse components of the field have a ring shape.

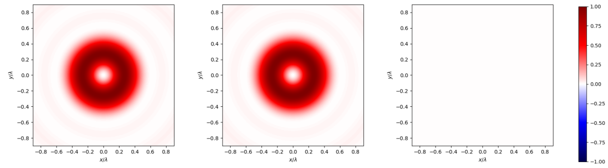
Like the circularly polarized with topological charge incident beam, the azimuthally polarized incident beam can be used for applications that require a null electrical field in the focal point, such as STED microscopy.



(a) $|E_x|^2 + |E_y|^2$ (b) $|E_z|^2$

FIG. 12: Magnitude of $|E_x|^2+|E_y|^2$ and $|E_z|^2$ in the focal plane ($z = 0$) for a circularly polarized incident beam.

Here we can see that $\tilde{S}_3 = 0$ for all points. Because of that, we can say that the polarization is linear.



(a) $\tilde{S}_0$ (b) $\tilde{S}_1$ (c) $\tilde{S}_3$

FIG. 13: 3D local Stokes parameters for the results of an azimuthal incident beam.

IV. CONCLUSIONS

We have managed to describe de polarization of highly focused beams in an efficient way, by computing the field in the focus with different input fields and calculating the local Stokes parameters.

The local Stokes parameters are used to describe the polarization of the focal field taking into account all its components, but its interpretation is not too intuitive.

As the results for this project have been found computationally, we can see that having strong calculus tools is a very useful aspect for the development of microscopy.

We could have gone further by computing the Richards-Wolf integral with Fourier transforms. This would have allowed us to obtain results from incident beams that do not allow the two-dimensional integral to be described as one-dimensional integrals. At a certain point, the decision was made to advance in the calculation of the local Stokes parameters, instead of finding other ways to compute the focused field.

[1] Wolf, E. “Electromagnetic Diffraction in Optical Systems. I. An Integral Representation of the Image Field.” Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 253, no. 1274 (1959): 349–57. http://www.jstor.org/stable/100739.

[2] David Maluenda, Rosario Martínez-Herrero, Ignasi Juvells, and Artur Carnicer, ”Synthesis of highly focused fields with circular polarization at any transverse plane,” Opt. Express 22, 6859-6867 (2014)

[3] Rosario Martínez-Herrero, David Maluenda, Marcos Aviñoá, Artur Carnicer, Ignasi Juvells, and Ángel S. Sanz, ”Local characterization of the polarization state of 3D electromagnetic fields: an alternative approach,” Photon. Res. 11, 1326-1338 (2023)

[4] T. Setälä, A. Shevchenko, M. Kaivola, and A. T. Friberg, “Degree of polarization for optical near fields,” Phys. Rev. E 66, 016615 (2002).

[5] C. J. R. Sheppard, “Jones and Stokes parameters for polarization in three dimensions,” Phys. Rev. A 90, 023809 (2014).

[6] Qiwen Zhan, ”Cylindrical vector beams: from mathematical concepts to applications,” Adv. Opt. Photon. 1, 1-57 (2009)

[7] Pauli Virtanen, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, Pearu Peterson, Warren Weckesser, Jonathan Bright, Stéfan J. van der Walt, Matthew Brett, Joshua Wilson, K. Jarrod Millman, Nikolay Mayorov, Andrew R. J. Nelson, Eric Jones, Robert Kern, Eric Larson, CJ Carey, İlhan Polat, Yu Feng, Eric W. Moore, Jake VanderPlas, Denis Laxalde, Josef Perktold, Robert Cimrman, Ian Henriksen, E.A. Quintero, Charles R Harris, Anne M. Archibald, Antônio H. Ribeiro, Fabian Pedregosa, Paul van Mulbregt, and SciPy 1.0 Contributors. (2020) SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. Nature Methods, 17(3), 261-272.

[8] Novotny L, Hecht B. Principles of Nano-Optics. 2nd ed. Cambridge University Press; 2012.

Càlcul dels paràmetres d'Stokes locals d'un feix altament enfocat

Author: Rosa Minguella Rafecas, rmingura18@alumnes.ub.edu
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Artur Carnicer, artur.carnicer@ub.edu

Resum: L'estat de polarització d'una ona electromagnètica transversal es descriu amb els paràmetres de Stokes. L'objectiu d'aquest treball és descriure l'estat de polarització de la llum tenint en compte el comportament tridimensional del camp elèctric, present en feixos altament enfocats. Per assolir-ho, hem calculat una versió modificada dels paràmetres de Stokes, els paràmetres de Stokes locals. En el treball es mostren els resultats de la intensitat del feix a la regió focal i els resultats dels paràmetres de Stokes locals per a diferents tipus de feixos incidents.
Paraules clau: Paràmetres de Stokes, microscopia.
ODSs: Aquest TFG està relacionat amb els Objectius de Desenvolupament Sostenible (SDGs) número 3, corresponent a salut i benestar.

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar	X	12. Consum i producció responsables	
4. Educació de qualitat		13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

El contingut d'aquest TFG es relaciona amb l'ODS 3, salut i benestar, per les seves aplicacions en microscopia. La microscopia ajuda molt a l'anàlisi i a la recerca relacionada amb malalties diverses.