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Master in Pure and Applied Logic

Master's Thesis

Structural reflection for large cardinal partition properties

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Abstract

In the theory of large cardinals, the Structural Reflection research program has the ultimate goal of providing a uniform way of characterizing any large cardinal notion in terms of structural reflection principles. In the present work, we study and provide such a characterization for Erdős, Ramsey, Rowbottom and Jónsson cardinals, which are large cardinal notions commonly defined in terms of partition properties and contained in the region below the first measurable cardinal. We introduce three new families of structural reflection principles: the invariant structural reflection principles, which characterize Erdős and Ramsey cardinals; the two-cardinal structural reflection principles, which characterize Rowbottom cardinals; and the proper structural reflection principles, which characterize Jónsson cardinals. Finally, we show how a particular generalization of a proper structural reflection principle yields a characterization of exacting cardinals.

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Chapter 1

Introduction

The main and most prominent first-order axiomatization of set theory, the ZFC theory, allows us to formalize all of mathematics. However, it is a well-known fact that, as a consequence of Gödel's Incompleteness Theorems, if ZFC is a consistent theory, then it is also an incomplete theory. This means that there are statements expressible in the language of ZFC that are independent of ZFC; *i.e.* there are statements that, under the assumption that ZFC is consistent, neither they nor their negations are theorems of ZFC. Consequently, there are plenty of questions, either purely set-theoretical or belonging to any thinkable area of mathematics (*e.g.* arithmetics, geometry, topology, real analysis, etc.) that ZFC cannot settle (Bagaria, 2023b).

It was also Gödel who proposed that ZFC can be enriched with extra axioms that allow to decide those questions that ZFC does not provide an answer for (Gödel, 1947). Among the different families of new possible axioms for set theory, the so-called *large cardinal axioms* postulate the existence (which cannot be proven in ZFC) of infinite cardinal sets (typically known as *large cardinals*) that satisfy set-theoretic properties that "prescribe their own transcendence over smaller cardinals and provide a superstructure for the analysis of strong propositions" (Kanamori, 2003).

Thus, the *theory of large cardinals* is one of the most central and relevant areas of research in modern set theory. It studies how large cardinal axioms form a linear hierarchy of increasingly strong set-theoretic systems. Furthermore, it plays a crucial role in the task of both successfully settling many relevant independent questions and providing a way to compare different mathematical systems in terms of consistency strength (Koellner, 2011).

The success of the theory of large cardinals in providing a framework that gives answer to a high number of important questions belonging to many different mathematical fields allows us to argue that large cardinal axioms are indeed acceptable and well-justified axioms of set theory; *i.e.* that they can be legitimately added to the axioms of ZFC to form a richer theory of sets. However, it has also been argued that this kind of *extrinsic justification* (since it appeals to consequences yielded by the new axioms in areas of mathematics that are, strictly speaking, external to set theory) is by itself an insufficient argument for the status of large cardinal axioms as real and valid axioms of set theory (Tait, 2001). Thus, forms of *intrinsic justification* (*i.e.* arguments that present and appeal to purely set-theoretical evidence and reasons) of the validity of large cardinal axioms have also been proposed and studied (Koellner, 2009).

The distinction between extrinsic and intrinsic justifications was pointed out by Gödel (1947), who argued in favor of the higher importance of the intrinsic evidence and, in particular, of the *reflection principles* studied by Lévy (1960a,b, 1962) as being the main source of (intrinsic) justification for set-theoretic axioms (Wang, 1997). Nevertheless, the family of reflection principles analogous to Lévy's presents relevant weaknesses and limitations (Koellner, 2009, p. 217–218; Bagaria, 2023a, p. 22–28), which has led to the search of alternative forms of reflection that provide a better and stronger justification of the naturalness of the large cardinal hierarchy.

It is in this context that the *Structural Reflection* research program has been proposed and developed in recent years (Bagaria, 2023a). The ultimate goal of the program is to *provide a uniform way of characterizing any large cardinal notion*, which not only aims at any currently known kind of large cardinal already present in the large cardinal hierarchy (*e.g.* inaccessible cardinals, Mahlo cardinals, weakly compact cardinals, measurable cardinals, supercompact cardinals, huge cardinals, etc.), but also allows devising new natural large cardinal notions as yet unknown. Thus, the Structural Reflection program aims to, among other relevant questions, show internal relational patterns between different regions of the large cardinal hierarchy, explain "the fact that all known large cardinals line up into a well-ordered hierarchy of consistency strength" (Bagaria and Lücke, 2024, p. 3398) and eventually "fill up an outstanding, and somewhat embarrassing, definitional void in the theory of large cardinals; *i.e.* the definition of 'large cardinal' itself" (Bagaria and Ternullo, 2025, p. 34).

In Lévy's kind of conception, the phenomenon of reflection consists, generally speaking, in the non-existence of a set-theoretical statement (*i.e.* a formula written in the language of set theory) that is satisfied by the universe of all sets and only by the universe of all sets. Thus, the original Montague-Lévy reflection principle, which is a theorem of the ZF theory (*i.e.* ZFC minus the Axiom of Choice), says that, if a formula in the language of ZF is true in the universe of all sets (which is not a set, but a proper class), then there is a set-sized initial segment of the universe where that formula is also satisfied. Consequently, there is no formula in the language of set theory that defines the universe of all sets.

This principle of the indefinability of the set-theoretic universe can be traced back to Cantor (1883), that is, to the very origin of set theory, and it was also accepted and elaborated by Gödel (Wang, 1997):

> The universe of sets cannot be uniquely characterized (*i.e.* distinguished from all its initial segments) by any internal structural property of the membership relation in it which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number.

In order to transcend the limitations of Lévy's kind of reflection, the structural conception of reflection relies on Gödel's understanding of the phenomenon of indefinability and postulates that what is to be reflected are not the formulas, but the *internal structural properties of the set-membership relation.* In other words, the main original idea of the Structural Reflection program is to reflect the structural content of the universe of all sets, instead of reflecting its theory (Bagaria, 2023a, p. 29–30).

In order to state this structural conception of reflection in a precise, mathematical form, the notion of "a structural property of the set-membership relation" is formalized in terms of a formula φ (possibly with parameters) in the first-order language of set theory that defines a (possibly proper) class $\mathcal{C}$ of structures of the same type. Thus, the class $\mathcal{C}$ materializes the structural property expressed by the formula φ. Then, the reflection of such a structural property consists in the following phenomenon: there exists a set-sized initial segment of the set-theoretic universe which, for every structure $\mathcal{A}$ belonging to $\mathcal{C}$, contains a structure $\mathcal{B}$ that also belongs to $\mathcal{C}$ and that *is like* $\mathcal{A}$. This "being like" relation denotes a resemblance between both structures that could ideally be formalized in terms of the existence of an isomorphic map between them. However, since the cardinality of $\mathcal{A}$ may very well be much larger than the cardinality of $\mathcal{B}$, what is required by the structural conception of reflection is that $\mathcal{B}$ can be elementarily embedded into $\mathcal{A}$. Therefore, *the general principle of structural reflection* is as follows (Bagaria, 2023a, p. 30):

> For every definable, in the first-order language of set theory, possibly with parameters, class $\mathcal{C}$ of relational structures of the same type, there exists an ordinal α that *reflects* $\mathcal{C}$; *i.e.* for every $\mathcal{A}$ in $\mathcal{C}$, there exists $\mathcal{B}$ in $\mathcal{C}$ of rank less than α and an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$.

Starting with Bagaria (2012) and Bagaria et al. (2015), the Structural Reflection program has so far shown that many different large cardinal notions are equivalent to some concretion or variant of the general principle of structural reflection quoted in the previous paragraph (see section 2.3 of the present work for a brief sample), both in lower (Bagaria and Väänänen, 2016) and higher (Bagaria and Lücke, 2023) regions of the large cardinal hierarchy. Most recently, its ability to produce and justify new large cardinal notions at the highest levels of the hierarchy has proved instrumental in the current research and discussion on the configuration of the set-theoretic universe and Woodin's HOD and Ultimate-L Conjectures (Aguilera et al., 2024). The program is fully alive and there is yet much work to do in it. Currently, there are regions in the large cardinal hierarchy whose characterization in terms of structural reflection principles is yet to be devised.

The goal of this work is to study and provide such a characterization for the large cardinal notions that are commonly defined in terms of partition properties and contained in the region below the first measurable cardinal; to wit, the well-known notions of Erdős, Ramsey, Rowbottom and Jónsson cardinals.

To that end, the present work is structured as follows. In chapter 2, we fix our framework and notation, and we briefly present a few tools that will be of use in the subsequent chapters. The reader not familiar with the Structural Reflection program should not skip section 2.3. The contributions of the present work are found in the next three chapters. In chapter 3, we introduce the family of Erdős and Ramsey cardinals and, in section 3.3, we define new principles of structural reflection that characterize Erdős and Ramsey cardinals. We proceed analogously in chapter 4 with the family of Rowbottom cardinals and, in section 4.3, we provide its characterization in terms of new principles of structural reflection. In chapter 5, we present Jónsson cardinals, we characterize them by means of new structural reflection principles in section 5.2 and we briefly discuss a particular strengthening of such a characterization in connection with the notion of an *exacting cardinal* in section 5.3. Finally, we draw our conclusions and propose a few open questions for further work in chapter 6.

Chapter 2

Preliminaries

This chapter includes three brief sections where we introduce some set-theoretic and model-theoretic tools that will be extensively employed in chapters 3 to 5. In section 2.1, we present Skolem functions and Skolem hulls. In section 2.2, we introduce basic definitions and facts about relative constructibility, as well as the well-known Mostowski's Collapsing Theorem and Gödel's Condensation Lemma (both the original and the generalized version). Finally, a quick review of some structural reflection principles is provided in section 2.3. Since all these tools are available in the literature, no proofs are included in this chapter, but the corresponding references for the interested reader.

We work in ZFC. Except for specific points where we explicitly indicate the reference we are working with (and referring the reader to), we follow Jech (2003) for definitions and proofs of basic set-theoretic notions and results, Kanamori (2003) for definitions and proofs of known facts and results in large cardinal theory, and Tent and Ziegler (2012) for definitions and proofs of basic model-theoretic notions and results. We next fix some basic notation for the rest of the document.

Notation.

- 'iff' stands for 'if and only if'.
- $\emptyset$ denotes the empty set.
- $f``A$ denotes the image of the set A under the function f.
- $A \sim B$ denotes that the sets A and B are bijectable.

- $f\colon A \sim B$ denotes that the function $f\colon A \to B$ is a bijection.
- id_A denotes the identity function with domain A.
- Ord denotes the class of ordinals.
- Card denotes the class of cardinals.
- For α, β ordinals, $\alpha < \beta$ denotes $\alpha \in \beta$, $\alpha \leq \beta$ denotes $\alpha \subseteq \beta$, $\alpha > \beta$ denotes $\beta \in \alpha$, and $\alpha \geq \beta$ denotes $\beta \subseteq \alpha$.
- $\mathrm{cf}(\alpha)$ denotes the cofinality of the ordinal α.
- $\mathrm{ot}(A, \lhd)$ denotes the order-type of the well-order $(A, \lhd)$.
- V_α denotes the initial segment of rank α of the cumulative universe of all sets.
- $\mathrm{rk}(A)$ denotes the rank of the set A.
- $\mathrm{tc}(A)$ denotes the transitive closure of the set A.
- H_κ denotes the set of sets hereditarily of cardinality less than κ.
- A *language* is a countable first-order language with equality.
- $\mathcal{L}_\in$ denotes the language of ZFC; *i.e.* the language with one single primitive non-logical symbol, $\in$, whose interpretation is the set-membership relation.
- $\mathcal{L}(\dot{c})$, $\mathcal{L}(\dot{f})$, $\mathcal{L}(\dot{P})$ denote the languages resulting from augmenting the language $\mathcal{L}$ by the constant symbol $\dot{c}$, the function symbol $\dot{f}$, the predicate symbol $\dot{P}$, respectively.
- An *$\mathcal{L}$-formula* is a formula written in the language $\mathcal{L}$.
- $\varphi(x_1, \ldots, x_n) \in \mathcal{L}$ denotes that φ is an $\mathcal{L}$-formula whose free variables are x_1, ..., x_n.
- $|\mathcal{L}|$ denotes the cardinality of the set of $\mathcal{L}$-formulas.
- A *structure* is a structure for some language.
- An *$\mathcal{L}$-structure* is a structure for the language $\mathcal{L}$.
- Calligraphic letters ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$, $\mathcal{N}$, etc.) denote structures and, unless stated otherwise, the same non-calligraphic letters (A, B, M, N, etc.) denote their respective universes.
- The *cardinality of a structure* is the cardinality of its universe.

- The *rank of a structure* is the rank of its universe.
- $\mathcal{L}_\mathcal{A}$ denotes the language of the structure $\mathcal{A}$; *i.e.* $\mathcal{A}$ is an $\mathcal{L}_\mathcal{A}$-structure.
- $\mathcal{A} \upharpoonright \mathcal{L}$ denotes the reduct of the structure $\mathcal{A}$ to the language $\mathcal{L}$, where typically $\mathcal{L} \subseteq \mathcal{L}_\mathcal{A}$.
- $\mathcal{A} \prec \mathcal{B}$ denotes that the structure $\mathcal{A}$ is an elementary substructure of the structure $\mathcal{B}$.
- $\mathcal{A} \prec_n \mathcal{B}$ denotes that the structure $\mathcal{A}$ is a Σ_n-elementary substructure of the structure $\mathcal{B}$.
- $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$ denotes that the function $f\colon A \to B$ is an embedding from the structure $\mathcal{A}$ into the structure $\mathcal{B}$.
- $f\colon \mathcal{A} \cong \mathcal{B}$ denotes that the function $f\colon A \to B$ is an isomorphism between the structures $\mathcal{A}$ and $\mathcal{B}$.

2.1 Skolem functions and Skolem hulls

Provided a structure $\mathcal{M}$ and a set $X \subseteq M$ with $|X| < |M|$, a very natural question arises:

> Is there a proper elementary substructure of $\mathcal{M}$ whose universe contains X as a subset?

It is a well-known model-theoretic fact that the answer to that question is affirmative (Tent and Ziegler, 2012, p. 18; Kanamori, 2003, p. 8; Jech, 2003, p. 156):

> Yes, the Skolem hull of X in $\mathcal{M}$ with respect to some complete set of Skolem functions for $\mathcal{M}$.

The tools typically employed to prove this fact, which we next briefly present, include the well-known Tarski-Vaught criterion.

Definition 2.1.1. Let $\mathcal{M}$ be a structure. Let $\varphi(x, x_1, \ldots, x_n) \in \mathcal{L}_\mathcal{M}$. Then, $f_\varphi\colon M^n \to M$ is a *Skolem function for* φ iff, for every $a_1, \ldots, a_n \in M$,

$$\mathcal{M} \models \exists x \varphi[a_1, \ldots, a_n] \Longrightarrow \mathcal{M} \models \varphi[f_\varphi(a_1, \ldots, a_n), a_1, \ldots, a_n].$$

If a well-ordering of the universe of the structure is available, then Skolem functions can be defined by taking least elements as witnesses. Since we work in ZFC, we can always provide such functions.

Definition 2.1.2. A *complete set of Skolem functions for a structure* $\mathcal{M}$ is the closure of some set $\{f_\varphi \colon \varphi \in \mathcal{L}_\mathcal{M}\}$ of Skolem functions under functional composition.

Any complete set of Skolem functions for a structure is equinumerous with the set of formulas in the language of the structure. Hence, in our setting, the cardinality of such a set is $\aleph_0$.

Definition 2.1.3. Let $\mathcal{M}$ be a structure. Let $X \subseteq M$. Let $\{f_\alpha \colon \alpha < |\mathcal{L}_\mathcal{M}|\}$ be a complete set of Skolem functions for $\mathcal{M}$. The *Skolem hull of* X *in* $\mathcal{M}$ is the substructure of $\mathcal{M}$ whose universe is

$$\{f_\alpha(a_1, \dots, a_n) \colon \alpha < |\mathcal{L}_\mathcal{M}|,\ a_1, \dots, a_n \in X\}.$$

Notation. We denote the Skolem hull of X in $\mathcal{M}$ by $\mathcal{H}_\mathcal{M}(X)$, or just $\mathcal{H}(X)$ if the context is clear. Moreover, we denote the universe of $\mathcal{H}(X)$ by $H(X)$.

Clearly, $X \subseteq H(X)$. Furthermore, the cardinality of the Skolem hull of X in $\mathcal{M}$ is $|X| + |\mathcal{L}| + \aleph_0$. Hence, in our setting, $|H(X)| = \max\{|X|, \aleph_0\}$.

Finally, by the *Tarski-Vaught criterion*, which we next present, the Skolem hull of X in $\mathcal{M}$ is an elementary substructure of $\mathcal{M}$; *i.e.* $\mathcal{H}_\mathcal{M}(X) \prec \mathcal{M}$.

Theorem 2.1.4 (Tarski and Vaught, 1957). *Let* $\mathcal{M}$ *be a structure. Let* $X \subseteq M$*. The following are equivalent:*

(1) X *is the universe of a elementary substructure of* $\mathcal{M}$*.*

(2) For every $\varphi(x, x_1, \dots, x_n) \in \mathcal{L}_\mathcal{M}$ *and for every* $a_1, \dots, a_n \in X$,

$$\mathcal{M} \models \exists x \varphi[a_1, \dots, a_n] \implies \exists a \in X \text{ such that } \mathcal{M} \models \varphi[a, a_1, \dots, a_n].$$

Proof. See Theorem 2.1.2 in Tent and Ziegler (2012, p. 18). ■

2.2 Relative constructibility

The hierarchy of constructible sets was introduced by Gödel (1940) in his seminal proof of the consistency (relative to ZF) of the Axiom of Choice and the Generalized Continuum Hypothesis. In this section, we mainly follow Devlin (2016, p. 102–106) and Jech (2003, p. 67–69) to briefly present the related concepts of relative constructibility and the Generalized Condensation Lemma, which will prove useful in chapters 3 to 5.

First, we provide definitions for the hierarchy of sets constructible relative to some set.

Definition 2.2.1 (Definability over a structure).

- Let $\mathcal{M}$ be a structure. A set $A \subseteq M$ is *definable over* $\mathcal{M}$ iff there is an $\mathcal{L}_{\mathcal{M}}$-formula $\varphi(x)$ such that $A = \{a \in A \colon \mathcal{M} \models \varphi[a]\}$.
- Let X, M be sets. $\mathrm{Def}_X(M)$ denotes the set of subsets of M that are definable over the $\mathcal{L}_{\in}(\dot{X})$-structure $\langle M, \in, X \cap M\rangle$, where $\dot{X}$ is a unary predicate symbol; *i.e.*

$$\mathrm{Def}_X(M) := \{A \subseteq M \colon A \text{ is definable over } \langle M, \in, X \cap M\rangle\}.$$

Definition 2.2.2 (Universe of sets constructible relative to *X*). Let X be a set.

- The *hierarchy of sets constructible relative to* X is defined by the following recursion on the ordinals:

$$\begin{aligned} &L_0[X] := \emptyset; \\ &L_{\alpha+1}[X] := \mathrm{Def}_X(L_\alpha[X]); \\ &L_\alpha[X] := \bigcup_{\beta<\alpha} L_\beta[X], \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

- The *universe of sets constructible relative to* X is the proper class defined as

$$L[X] := \bigcup_{\alpha \in \mathsf{Ord}} L_\alpha[X].$$

Observation 2.2.3. The proper class $L[X]$ is Δ_1-definable in ZFC with X as a parameter. See Lemmas 13.12 and 13.14 in Jech (2003, p. 186–187) for more details.

Notation. We write L to denote $L[\emptyset]$ (*i.e.* Gödel's original formulation of the universe of constructible sets) and L_α to denote $L_\alpha[\emptyset]$.

Next, we provide a few useful properties of the hierarchy of sets constructible relative to some set.

Proposition 2.2.4. *Let X be a set.*

1. *$L_\alpha[X]$ is a transitive set, for every ordinal α.*
2. *$\alpha = \mathsf{Ord} \cap L_\alpha[X]$, for every ordinal α.*
3. *$|L_\alpha[X]| = |\alpha|$, for every infinite ordinal α.*
4. *$L_\alpha[X] \subseteq L_\beta[X]$, for any ordinals $\alpha \leq \beta$.*
5. *$L_\alpha[X] \subseteq V_\alpha$, for every ordinal α.*
6. *$L_\alpha[X] \in H_{|\alpha|^+}$, for every infinite ordinal α.*

Proof. Easily, by induction on α. See the proof of Lemma 1.1 in Devlin (2016, p. 58–60) for more details. ■

We now introduce the well-known notions of transitive collapse, well-founded extensional relations, extensional classes, and Mostowski's Collapsing Theorem, which is an essential ingredient in the Condensation Lemmas. These notions and results show how every extensional class is $\in$-isomorphic to some transitive class.

Definition 2.2.5. Let A be a class. Let E be a well-founded set-like (*i.e.* for every $x \in A$, $\{y \in A \colon yEx\}$ is a set) binary relation on A.

- The *transitive collapse* of (A, E) is the following function π defined by well-founded recursion on E: for every $x \in A$,

$$\pi(x) := \{\pi(y) \colon yEx\}.$$

- The relation E is *extensional* iff, for every $x, y \in A$, if $x \neq y$, then

$$\{z \in A \colon zEx\} \neq \{z \in A \colon zEy\}.$$

- The class A is *extensional* iff the relation $\in$ on A is extensional; *i.e.* iff $x \cap A \neq y \cap A$ for every $x, y \in A$ such that $x \neq y$.

Observation 2.2.6.

1. The range of the transitive collapse π of (A, E) is a transitive class and, moreover, for every $x, y \in A$, if xEy, then $\pi(x) \in \pi(y)$.
2. By the Axiom of Foundation, the set-like binary relation $\in$ is well-founded on any class.

Theorem 2.2.7 (Mostowski's Collapsing Theorem). *Let A be a class. Let E be a well-founded extensional relation on A.*

1. *There is a unique transitive class M and a unique transitive collapse $\pi\colon (A, E) \cong (M, \in)$.*
2. *Consequently, if A is an extensional class, then there is a unique transitive class M such that $\pi\colon (A, \in) \cong (M, \in)$. Moreover, for every transitive $B \subseteq A$, we have that $\pi \restriction B = \mathrm{id}_B$.*

Proof. See the proof of Theorem 6.15 in Jech (2003, p. 69). ■

Finally, we close this section by introducing both Gödel's original formulation of his well-known Condensation Lemma (*i.e.* restricted to limit ordinals and initial segments of the constructible universe L) and its generalized version to infinite ordinals and initial segments of relative constructible universes. These lemmas are extremely useful. As already pointed in Proposition 2.2.4, the initial segments of universes of constructible sets are transitive, highly well-behaved sets. Condensation shows, via Mostowski's transitive collapse, how elementary substructures of initial segments of universes of constructible sets are isomorphic to initial segments of universes of constructible sets.

Lemma 2.2.8 (Gödel's Condensation Lemma). *For every limit ordinal α, if $\langle M, \in\rangle \prec_1 \langle L_\alpha, \in\rangle$, then there is $\pi\colon \langle M, \in\rangle \cong \langle L_\beta, \in\rangle$ for some limit ordinal $\beta \leq \alpha$.*

Proof. See the proof of Theorem 2.5 in Devlin (2016, p. 80–82). ■

Lemma 2.2.9 (Generalized Gödel's Condensation Lemma). *For every infinite ordinal α, if $\langle M, \in, X \cap M\rangle \prec_1 \langle L_\alpha[X], \in, X \cap L_\alpha[X]\rangle$, then there is $\pi\colon \langle M, \in, X \cap M\rangle \cong \langle L_\beta[X'], \in, X'\rangle$ for some infinite ordinal $\beta \leq \alpha$, with $X' = \pi``(X \cap M)$.*

Proof. See the proof of Lemma 1.3 in Magidor (1990, p. 96–97). ■

2.3 Structural reflection principles

We start by introducing the three "basic" formal versions of the general principle of structural reflection (see chapter 1). We follow, although not literally, Bagaria and Lücke (2024) in the next definitions.

Notation. For the rest of this work, the calligraphic letter $\mathcal{C}$ denotes some arbitrary (possibly proper) class of structures definable by a $\mathcal{L}_\in$-formula (possibly with parameters).

Definition 2.3.1 (SR). The SR($\mathcal{C}$) principle states: There is an ordinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| < \alpha$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Definition 2.3.2 (HSR). The HSR($\mathcal{C}$) principle states: There is an infinite cardinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C} \cap H_\alpha$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Definition 2.3.3 (VSR). The VSR($\mathcal{C}$) principle states: There is an ordinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C} \cap V_\alpha$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Notation. For any principle P of structural reflection, we write $\alpha \models \mathrm{P}(\mathcal{C})$ to denote that the ordinal α witnesses the principle P($\mathcal{C}$); *e.g.* $\alpha \models \mathrm{SR}(\mathcal{C})$ denotes that α witnesses SR($\mathcal{C}$).

Observation 2.3.4. If κ is an infinite cardinal, then $\kappa \models \mathrm{HSR}(\mathcal{C})$ implies both $\kappa \models \mathrm{SR}(\mathcal{C})$ and $\kappa \models \mathrm{VSR}(\mathcal{C})$. Furthermore, if κ is a fixed point of the Beth function, then $V_\kappa = H_\kappa$, whence $\kappa \models \mathrm{VSR}(\mathcal{C})$ iff $\kappa \models \mathrm{HSR}(\mathcal{C})$; and if moreover the class $\mathcal{C}$ of structures is closed under isomorphic images, then $\kappa \models \mathrm{SR}(\mathcal{C})$ implies $\kappa \models \mathrm{HSR}(\mathcal{C})$. Thus, if $\mathcal{C}$ is closed under isomorphic images and if $\kappa \models \beth_\kappa$, then $\kappa \models \mathrm{SR}(\mathcal{C})$ iff $\kappa \models \mathrm{HSR}(\mathcal{C})$ iff $\kappa \models \mathrm{VSR}(\mathcal{C})$.

Now, we introduce the generalized versions of the previous principles for any Σ_n-definable or Π_n-definable, with and without parameters, class of structures, for some natural number n. We follow, although not literally, Bagaria (2023a, p. 31) in the next definition.

Definition 2.3.5. Let $n \in \omega$.

- The Σ_n-SR principle states: There is an ordinal α such that, for every Σ_n-definable, without parameters, class $\mathcal{D}$ of structures, $\alpha \models \mathrm{SR}(\mathcal{D})$.
- For A a set, the $\Sigma_n(A)$-SR principle states: There is an ordinal α such that, for every Σ_n-definable, with parameters in A, class $\mathcal{D}$ of structures, $\alpha \models \mathrm{SR}(\mathcal{D})$.

- The $\mathbf{\Sigma_n}$-SR principle states: There is a proper class of ordinals α such that $\alpha \models \Sigma_n(V_\alpha)$-SR.

Observation 2.3.6. The following principles are defined in the same way as in Definition 2.3.5:

- Π_n-SR, $\Pi_n(A)$-SR, $\mathbf{\Pi_n}$-SR;
- Σ_n-HSR, $\Sigma_n(A)$-HSR, $\mathbf{\Sigma_n}$-HSR, Π_n-HSR, $\Pi_n(A)$-HSR, $\mathbf{\Pi_n}$-HSR;
- Σ_n-VSR, $\Sigma_n(A)$-VSR, $\mathbf{\Sigma_n}$-VSR, Π_n-VSR, $\Pi_n(A)$-VSR, $\mathbf{\Pi_n}$-VSR.

Conceptually speaking, the theorems we present next can be considered the first results of the Structural Reflection program. On the one hand, the principles $\mathbf{\Sigma_0}$-VSR and $\mathbf{\Sigma_1}$-VSR happen to be equivalent theorems of ZFC. Hence, they do not yield any large cardinal notion.

Definition 2.3.7. Let $n \in \omega$. $C^{(n)} := \{\alpha \in \mathsf{Ord} \colon (V_\alpha, \in) \prec_n (V, \in)\}$.

Observation 2.3.8. $C^{(0)} = \mathsf{Ord}$. For every $n \in \omega \setminus \{0\}$, $C^{(n)}$ is a Π_n-definable club proper class of cardinals. Every element in $C^{(1)}$ is an uncountable cardinal and a fixed point of the $\beth$ function. See Bagaria (2012) for more details.

Theorem 2.3.9. *Let α be an ordinal. The following are equivalent:*

(1) $\alpha \models \Sigma_0(V_\alpha)$*-VSR.*

(2) $\alpha \models \Sigma_1(V_\alpha)$*-VSR.*

(3) $\alpha \in C^{(1)}$.

Proof. See the proof of Proposition 3.1 in Bagaria (2023a, p. 31–32). ■

On the other hand, the principles Π_1-VSR and Σ_2-VSR (and also their boldfaced versions) are also equivalent and, moreover, yield an equivalent characterization of the existence of a supercompact cardinal, a very strong large cardinal notion. We refer the reader to Bagaria (2012) and Bagaria et al. (2015) for the details of the proofs of the next theorems.

Theorem 2.3.10. *Let κ be a cardinal. The following are equivalent:*

(1) κ is either a supercompact cardinal or a limit of supercompact cardinals.

(2) $\kappa \models \Sigma_2(V_\kappa)$*-SR.*

(3) $\kappa \models \Sigma_2(V_\kappa)$*-HSR.*

(4) $\kappa \models \Sigma_2(V_\kappa)$*-VSR.*

Theorem 2.3.11. *The following are equivalent:*

(1) Π_1*-VSR.*

(2) Σ_2*-VSR.*

(3) There exists a supercompact cardinal.

Theorem 2.3.12. *The following are equivalent:*

(1) $\boldsymbol{\Pi_1}$*-VSR.*

(2) $\boldsymbol{\Sigma_2}$*-VSR.*

(3) There exists a proper class of supercompact cardinals.

Thus, the main goal of the Structural Reflection program is to provide equivalent characterizations of every large cardinal notion, similarly to the ones provided for supercompact cardinals in the previous theorems, in the form of combinations of (i) variations (either strengthenings or weakenings) of the basic structural reflection principles of Definitions 2.3.1 to 2.3.3 and (ii) different definable classes, or definable families of classes, of structures.

Section 9 of Bagaria (2023a, 65–68) provides a wide summary of equivalent characterizations of a broad variety of large cardinals in terms of different principles of structural reflection:

- The region between supercompactness and Vopěnka's Principle is characterized by means of Σ_n and Π_n VSR principles.
- The region beyond Vopěnka's Principle is characterized by means of the family of Exact Structural Reflection (ESR) principles.
- The region between strong cardinals and "Ord is Woodin" is characterized by means of the Product Structural Reflection (PSR) principles, variations of which also allow us to characterize measurable and globally superstrong cardinals.
- The families of "sharps" and "daggers" principles are characterized by the VSR principle applied to specific families of classes of structures whose universes are initial segments of relative constructible universes.

- The region below “$0^{\#}$ exists” is characterized by means of restricted versions of VSR and also by the Generic Structural Reflection (GSR) principles.
- And so forth.

Chapter 3

Structural reflection for Erdős and Ramsey cardinals

In this chapter, we provide equivalent characterizations of Erdős and Ramsey cardinals (and also of Almost Ramsey cardinals) in terms of new structural reflection principles. To begin with, we introduce the partition properties commonly associated with the field of Ramsey combinatorics in section 3.1, after which we briefly present the large cardinal notions targeted in this chapter in section 3.2. Lastly, we define a new family of structural reflection principles and show how it characterizes Erdős, Ramsey and Almost Ramsey cardinals in section 3.3.

3.1 Ordinary partition properties

We first define the family of partition properties that are commonly employed to define Erdős and Ramsey cardinals, as we will see in section 3.2. We follow Kanamori (2003, p. 71 and 80), although not literally, since we provide our definitions in a slightly more general setting.

Definition 3.1.1. Let $(A, \lhd)$ be a well-order and α be an ordinal.

- $[A]^\alpha \coloneqq \{X \subseteq A \colon \mathrm{ot}(X, \lhd) = \alpha\}$. If A is a set of ordinals, then $\lhd$ is $\in$ by default.
- $[A]^{<\alpha} \coloneqq \bigcup_{\beta<\alpha} [A]^\beta$.

Definition 3.1.2 (Ordinary partition properties). Let $(A, \lhd)$ be a well-order, C be a set and β, δ be ordinals.

- $A \longrightarrow (\beta)^{\delta}_{C}$ denotes the following: For every $f\colon [A]^{\delta} \to C$, there is $H \in [A]^{\beta}$ such that $|f``[H]^{\delta}| = 1$.
- $A \longrightarrow (\beta)^{<\delta}_{C}$ denotes the following: For every $f\colon [A]^{<\delta} \to C$, there is $H \in [A]^{\beta}$ such that, for every $\eta < \delta$, $|f``[H]^{\eta}| = 1$.

Notation. In the context of an ordinary partition property, we say that the set H is *f-homogeneous*. Also, we write $\not\longrightarrow$ when the property fails.

Next, we show that ordinary partition properties are preserved if the order-type of $(A, \lhd)$ and the cardinality of C are preserved.

Proposition 3.1.3. *Let $(A, \lhd)$ be a well-order, C be a set and β, δ be ordinals.*

1. *If $A \longrightarrow (\beta)^{\delta}_{C}$, then $A' \longrightarrow (\beta)^{\delta}_{C'}$ for any well-order $(A', \lhd') \cong (A, \lhd)$ and for any set $C' \sim C$.*
2. *If $A \longrightarrow (\beta)^{<\delta}_{C}$, then $A' \longrightarrow (\beta)^{<\delta}_{C'}$ for any well-order $(A', \lhd') \cong (A, \lhd)$ and for any set $C' \sim C$.*

Proof. 1. Let us assume $A \longrightarrow (\beta)^{\delta}_{C}$. Let $i\colon (A, \lhd) \cong (A', \lhd')$, $j\colon C \sim C'$ and $f\colon [A']^{\delta} \to C'$. Let $g\colon [A]^{\delta} \to C$ be defined as follows:

$$g \coloneqq \{\langle x, y\rangle \in [A]^{\delta} \times C\colon \langle i``x, j(y)\rangle \in f\}.$$

By assumption, there is $H \in [A]^{\beta}$ g-homogeneous. Then, $i``H \in [A']^{\beta}$ is f-homogeneous.

2. By the same argument exhibited in point 1. ■

Observation 3.1.4. Most often, ordinary partition properties are particularized for α and γ ordinals, instead of $(A, \lhd)$ and C, respectively. In such a context, we always assume $\alpha \geq \beta \geq \delta > 0$ and $\gamma \geq 2$ to avoid trivialities and nonsensical scenarios.

We close this section by showing, first, the preservation properties of the ordinary partition properties in terms of ordinals and, second, the conditions under which the least ordinal α that satisfies an ordinary partition property, if any, is a cardinal.

Proposition 3.1.5. *Let α, β, γ, δ be ordinals.*

1. *If $\alpha \longrightarrow (\beta)^{\delta}_{\gamma}$, then $\alpha' \longrightarrow (\beta')^{\delta'}_{\gamma'}$ for any ordinals $\alpha' \geq \alpha$, $\beta' \leq \beta$, $\gamma' \leq \gamma$ and $\delta' \leq \delta$ with $\delta' < \mathrm{cf}(\beta)$.*
2. *If $\alpha \longrightarrow (\beta)^{<\delta}_{\gamma}$, then $\alpha' \longrightarrow (\beta')^{<\delta'}_{\gamma'}$ for any ordinals $\alpha' \geq \alpha$, $\beta' \leq \beta$, $\gamma' \leq \gamma$ and $\delta' \leq \delta$.*

Proof. 1. Let us assume $\alpha \longrightarrow (\beta)^{\delta}_{\gamma}$. Let $\alpha' \geq \alpha$ and $f\colon [\alpha']^{\delta} \to \gamma$. Since $\alpha \subseteq \alpha'$, any $(f \restriction [\alpha]^{\delta})$-homogeneous set $H \in [\alpha]^{\beta} \subseteq [\alpha']^{\beta}$, which exists by assumption, is f-homogeneous. Let $\beta' \leq \beta$ and $f\colon [\alpha]^{\delta} \to \gamma$. By assumption, there is $H \in [\alpha]^{\beta}$ f-homogeneous, whence any $H' \in [H]^{\beta'}$ is f-homogeneous. Let $\gamma' \leq \gamma$ and $f\colon [\alpha]^{\delta} \to \gamma'$. Since $\gamma' \subseteq \gamma$, we have that f is a function from $[\alpha]^{\delta}$ into γ, whence, by assumption, there is $H \in [\alpha]^{\beta}$ f-homogeneous. Let $\delta' \leq \delta$ with $\delta' < \mathrm{cf}(\beta)$ and $f\colon [\alpha]^{\delta'} \to \gamma$. We define $g\colon [\alpha]^{\delta} \to \gamma$ as follows: $g(\{\xi_i\colon i < \delta\}) \coloneqq f(\{\xi_i\colon i < \delta'\})$ for every $\{\xi_i\colon i < \delta'\} \in [\alpha]^{\delta}$, with $\xi_i < \xi_j$ for every $i < j < \delta$. By assumption, there is $H \in [\alpha]^{\beta}$ g-homogeneous. Let $\{\xi_i\colon i < \delta'\} \in [H]^{\delta'}$, with $\xi_i < \xi_j$ for every $i < j < \delta'$. Since $\delta' < \mathrm{cf}(\beta)$, there is $\{\xi_i\colon i < \delta\} \in [H]^{\delta}$ such that $\xi_i < \xi_j$ for every $i < \delta' \leq j < \delta$. Hence, by construction of g, we have that $f(\{\xi_i\colon i < \delta'\}) = g(\{\xi_i\colon i < \delta\})$. Therefore, H is f-homogeneous.

2. Let us assume $\alpha \longrightarrow (\beta)^{<\delta}_{\gamma}$. The cases $\alpha' \geq \alpha$, $\beta' \leq \beta$ and $\gamma' \leq \gamma$ are shown by the same arguments exhibited in point 1. Let $\delta' \leq \delta$ and $f\colon [\alpha]^{<\delta'} \to \gamma$. Since $\delta' \subseteq \delta$, we define $g\colon [\alpha]^{<\delta} \to \gamma$ extending f; *i.e.* $g \restriction [\alpha]^{<\delta'} = f$. Hence, any g-homogeneous set $H \in [\alpha]^{\beta}$, which exists by assumption, is f-homogeneous. ∎

Proposition 3.1.6. *Let α, γ be ordinals and μ be a cardinal.*

1. *If $\alpha \longrightarrow (\mu)^{n}_{\gamma}$ with $n \in \omega$, then the least κ such that $\kappa \longrightarrow (\mu)^{n}_{\gamma}$ is a cardinal.*
2. *If $\alpha \longrightarrow (\mu)^{<\omega}_{\gamma}$, then the least κ such that $\kappa \longrightarrow (\mu)^{<\omega}_{\gamma}$ is a cardinal.*

Proof. 1. Let us assume $\alpha \longrightarrow (\mu)^{n}_{\gamma}$, with $n \in \omega$. Let $\kappa = |\alpha|$ and $f\colon [\kappa]^{n} \to \gamma$. Let $h\colon \alpha \sim \kappa$. We define $g\colon [\alpha]^{n} \to \gamma$ as follows: $g(a) = f(h``a)$, for every $a \in [\alpha]^{n}$. By assumption, there is $H \in [\alpha]^{\mu}$ $g$-homogeneous. Since $\mu$ is a cardinal, we have that $\mathrm{ot}(h``H) \geq \mu$. Let $H' \in [h``H]^{\mu}$. By construction of g, we have that H' is f-homogeneous.

2. By the same argument exhibited in point 1. ∎

3.2 Erdős and Ramsey cardinals

We next provide a brief introduction to the notions of Erdős, Ramsey and Almost Ramsey cardinals. More detailed accounts of these notions can be found in Kanamori (2003, p. 70-84), for Erdős and Ramsey cardinals, and in Vickers and Welch (2001), for Almost Ramsey cardinals.

The family of Erdős cardinals was introduced by Erdős and Hajnal (1958) as a result of their study on ordinary partition properties.

Definition 3.2.1.

- Let β be an infinite ordinal. A cardinal κ is a *β-Erdős cardinal* iff $\kappa \longrightarrow (\beta)^{<\omega}_2$. Moreover, $\eta(\beta)$ denotes the least β-Erdős cardinal.
- κ is an *Erdős cardinal* iff $\kappa = \eta(\beta)$ for some β (*i.e.* κ is the least β-Erdős cardinal for some β).

The Ramsey cardinals are the fixed points in the sequence of Erdős cardinals.

Definition 3.2.2. κ is a *Ramsey cardinal* iff $\kappa \longrightarrow (\kappa)^{<\omega}_2$. In other words, κ is a Ramsey cardinal iff $\kappa = \eta(\kappa)$.

Furthermore, we also consider the Almost Ramsey cardinals, which are just a weakening of Ramsey cardinals in terms of the order-type of the homogeneous set in the partition property.

Definition 3.2.3. κ is an *Almost Ramsey cardinal* iff $\kappa \longrightarrow (\beta)^{<\omega}_2$ for every ordinal $\beta < \kappa$.

It is well-known that Erdős cardinals are strongly inaccessible and that the existence of $\eta(\omega_1)$ implies that $V \neq L$ (Silver, 1966). Also, Ramsey cardinals are weakly compact (Kanamori, 2003, p. 81) and measurable cardinals are Ramsey (Erdős and Hajnal, 1958). Moreover, the least Almost Ramsey cardinal, if it exists, has countable cofinality and is larger than $\eta(\omega_1)$ (Vickers and Welch, 2001).

Next, we show Silver's Theorem, which is a well-known model-theoretic equivalent characterization of the purely combinatorial ordinary partition property employed in the definition of Erdős, Ramsey and Almost Ramsey cardinals. To that end, we first need to introduce the notion of indiscernibility for a structure.

Definition 3.2.4. Let $(X, \lhd)$ be a linear order. X is a *set of $\lhd$-indiscernibles for a structure $\mathcal{M}$* with $X \subseteq M$ iff, for every $a_1, \dots, a_n, b_1, \dots, b_n \in X$ such that $a_1 \lhd \dots \lhd a_n$ and $b_1 \lhd \dots \lhd b_n$, and for every $\varphi(x_1, \dots, x_n) \in \mathcal{L}_{\mathcal{M}}$,

$$\mathcal{M} \models \varphi[a_1, \dots, a_n] \Longleftrightarrow \mathcal{M} \models \varphi[b_1, \dots, b_n].$$

Proposition 3.2.5. *If X is a set of $\lhd$-indiscernibles for a structure $\mathcal{M}$, then X is a set of $\lhd$-indiscernibles for any structure $\mathcal{M} \restriction \mathcal{L}$, where $\mathcal{L} \subseteq \mathcal{L}_{\mathcal{M}}$.*

Proof. Because M is the universe of both $\mathcal{M}$ and $\mathcal{M} \restriction \mathcal{L}$, and because, since $\mathcal{L} \subseteq \mathcal{L}_{\mathcal{M}}$, every $\mathcal{L}$-formula is an $\mathcal{L}_{\mathcal{M}}$-formula. ■

Theorem 3.2.6 (Silver, 1966, 1971). *Let κ be an infinite cardinal and β be a limit ordinal. The following are equivalent:*

(1) $\kappa \longrightarrow (\beta)_2^{<\omega}$.

(2) For every structure $\mathcal{M}$ with $\kappa \subseteq M$, there is a set of $\in$-indiscernibles $X \in [\kappa]^\beta$ for $\mathcal{M}$.

Proof. (1)⇒(2): Let us assume (1). Let $\mathcal{M}$ be a structure with $\kappa \subseteq M$. Let $\{\varphi_n : n \in \omega\}$ be an enumeration of the $\mathcal{L}_{\mathcal{M}}$-formulas such that $n \geq k(n)$ for every $n \in \omega$, where $k(n)$ is the number of free variables of the formula φ_n. Let $f : [\kappa]^{<\omega} \to \{0, 1\}$ be defined as follows: for every $n \in \omega$,

$$f(\{\xi_1, \dots, \xi_n\}) = 0, \text{ if } \mathcal{M} \models \varphi_n[\xi_1, \dots, \xi_{k(n)}];$$
$$f(\{\xi_1, \dots, \xi_n\}) = 1, \text{ otherwise;}$$

where $\xi_1 \in \dots \in \xi_n$. By (1), there is $X \in [\kappa]^\beta$ f-homogeneous. Let $n \in \omega$. Let $\varphi_n \in \mathcal{L}_{\mathcal{M}}$. Let $\xi_1, \dots, \xi_{k(n)} \in X$ be such that $\xi_1 \in \dots \in \xi_{k(n)}$. Since β is a limit ordinal, let $\xi_{k(n)+1}, \dots, \xi_n \in X$ be such that $\xi_{k(n)} \in \xi_{k(n)+1} \in \dots \in \xi_n$. Hence, we have that $\mathcal{M} \models \varphi[\xi_1, \dots, \xi_{k(n)}]$ iff $f(\{\xi_1, \dots, \xi_n\}) = 0$. Therefore, since X is f-homogeneous, X is a set of $\in$-indiscernibles for $\mathcal{M}$.

(2)⇒(1): Let us assume (2). Let $f : [\kappa]^{<\omega} \to \{0, 1\}$. By (2), let $X \in [\kappa]^\beta$ be a set of $\in$-indiscernibles for the structure $\langle \kappa, \in, f \restriction [\kappa]^n \rangle_{n \in \omega}$. Therefore, X is f-homogeneous. ■

We close this section by showing a slight generalization of Silver's Theorem 3.2.6 that will be useful in the next section.

Theorem 3.2.7. *Let α be an infinite ordinal and β be a limit ordinal. The following are equivalent:*

(1) $\alpha \longrightarrow (\beta)^{<\omega}_2$.

(2) For every structure $\mathcal{M}$ with $\text{ot}(\mathsf{Ord} \cap M) \geq \alpha$, *there is a set of $\in$-indiscernibles $X \in [\mathsf{Ord} \cap M]^\beta$ for $\mathcal{M}$.*

Proof. (1)⇒(2): Let us assume (1). Let $\mathcal{M}$ be a structure with $\text{ot}(\mathsf{Ord} \cap M) = \alpha' \geq \alpha$. Hence, $(\mathsf{Ord} \cap M, \in) \cong (\alpha', \in)$. By Propositions 3.1.3 and 3.1.5, we have that $(\mathsf{Ord} \cap M) \longrightarrow (\beta)^{<\omega}_2$. By the same argument exhibited in the proof of (1)⇒(2) of Theorem 3.2.6, there is $X \in [\mathsf{Ord} \cap M]^\beta$ a set of indiscernibles for $\mathcal{M}$.

(2)⇒(1): By the proof of (2)⇒(1) of Theorem 3.2.6. ■

3.3 Structural reflection principles for Erdős and Ramsey cardinals

In this section, we provide the first main contributions of the present work in the form of different equivalent characterizations of Erdős, Ramsey and Almost Ramsey cardinals in terms of principles of structural reflection. In subsection 3.3.1, we introduce a new family of principles: the invariant structural reflection principles. We show different definable classes of structures for which this new family of principles characterizes our targeted large cardinal notions. In subsection 3.3.2, we connect the results obtained in the previous section with the level-by-level principle of structural reflection introduced by Hou (2024). We provide level-by-level versions of the invariant structural reflection principles and show that they also allow us to characterize Erdős, Ramsey and Almost Ramsey cardinals. Finally, in subsection 3.3.3, we introduce the notion of for-all-levels structural reflection, define for-all-levels versions of the invariant structural reflection principles and show how our targeted large cardinal notions can also be characterized by them.

3.3.1 Invariant structural reflection

We first provide a result that leads to the definition of a new family of structural reflection principles that allow us to characterize Erdős and Ramsey cardinals. To begin with, we define the property of being an "invariant" elementary embedding with respect to some linear order.

Definition 3.3.1. Let $(A, \lhd_A)$, $(B, \lhd_B)$ be partial orders. A map $f\colon A \to B$ is $\lhd$-*preserving* iff, for every $x, y \in A$, if $x \lhd_A y$ then $f(x) \lhd_B f(y)$.

Definition 3.3.2. Let $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{L}$-structures, $j\colon \mathcal{M} \hookrightarrow \mathcal{N}$ be elementary, and $(X, \lhd)$ be a linear order where $X \subseteq N$. We say that j is $(X, \lhd)$-*invariant* iff $|X \cap \operatorname{ran}(j)| \geq \omega$ and, for every $n \in \omega$, for every $\varphi(x_1, \ldots, x_n) \in \mathcal{L}$, for every $a_1, \ldots, a_n \in M$ with $j``\{a_1, \ldots, a_n\} \subseteq X$, and for every $\lhd$-preserving map $h\colon j``\{a_1, \ldots, a_n\} \to X$,

$$\mathcal{M} \models \varphi[a_1, \ldots, a_n] \Longleftrightarrow \mathcal{N} \models \varphi[h(j(a_1), \ldots, h(j(a_n))].$$

The following lemma shows that the relation between ordinary partition properties and indiscernibility for structures pointed out in the previous section implies that the existence of Erdős and Ramsey cardinals is equivalent to the existence of invariant elementary embeddings between structures that satisfy the conditions of structure $\mathcal{M}$ in Theorem 3.2.7.

Lemma 3.3.3. *Let α be an infinite ordinal and β be a limit ordinal. The following are equivalent:*

(1) $\alpha \longrightarrow (\beta)^{<\omega}_2$.

(2) For every structure $\mathcal{A}$ with $\operatorname{ot}(\mathsf{Ord} \cap A) \geq \alpha$ and for every infinite cardinal $\mu \leq \alpha$, there is a structure $\mathcal{B}$ with $|\mathsf{Ord} \cap B| = |B| = \mu$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Proof. (1)⇒(2): Let us assume (1). Let $\mathcal{A}$ be a structure with $\operatorname{ot}(\mathsf{Ord} \cap A) \geq \alpha$. By Theorem 3.2.7, there is $X \in [\mathsf{Ord} \cap A]^\beta$ a set of $\in$-indiscernibles for $\mathcal{A}$. Let $\mu \leq \alpha$ be an infinite cardinal. Let $Y \in [A]^\mu$ be such that $|\mathsf{Ord} \cap Y| = \mu$ and $|X \cap Y| \geq \omega$. Let $\mathcal{B} := \mathcal{H}(Y)$, the Skolem hull of Y in $\mathcal{A}$. Therefore, we have that $\mathcal{B} \prec \mathcal{A}$, with $Y \subseteq B$ and $|B| = |Y| = \mu$. Moreover, since $|\mathsf{Ord} \cap Y| = \mu$, we have that $|\mathsf{Ord} \cap B| = \mu$. Hence, the identity map $j\colon B \to A$ is an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$ with $Y \subseteq \operatorname{ran}(j)$. Furthermore, since $|X \cap Y| \geq \omega$, we have that $|X \cap \operatorname{ran}(j)| \geq \omega$. Let $n \in \omega$. Let $\xi_1, \ldots, \xi_n \in B$ be such that $j``\{\xi_1, \ldots, \xi_n\} \subseteq X$ and $\xi_1 \in \ldots \in \xi_n$. Let $\varphi(x_1, \ldots, x_n) \in \mathcal{L}_{\mathcal{A}}$. Let $h\colon \{\xi_1, \ldots, \xi_n\} \to X$ be $\in$-preserving. By the elementarity of j and the indiscernibility of X for $\mathcal{A}$, we have that

$$\begin{aligned} \mathcal{B} \models \varphi[\xi_1, \ldots, \xi_n] &\Longleftrightarrow \mathcal{A} \models \varphi[\xi_1, \ldots, \xi_n] \\ &\Longleftrightarrow \mathcal{A} \models \varphi[h(\xi_1), \ldots, h(\xi_n)]. \end{aligned}$$

Therefore, j is $(X, \in)$-invariant.

(2)$\Rightarrow$(1): Let us assume (2). Let $f\colon [\alpha]^{<\omega} \to \{0,1\}$. Let $\mathcal{A} \coloneqq \langle \alpha, \in, f_n\rangle_{n\in\omega}$, where $f_n(\xi_1, \dots, \xi_n) \coloneqq f(\{\xi_1, \dots, \xi_n\})$ for all $n \in \omega$. Let $\mu \leq \alpha$ be an infinite cardinal. By assumption, there is a structure $\mathcal{B} = \langle B, \in, g_n\rangle_{n\in\omega}$ such that $|\mathsf{Ord} \cap B| = |B| = \mu$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\alpha]^\beta$ with $|X \cap \operatorname{ran}(j)| \geq \omega$. Let $n \in \omega$. Let $\xi_1, \dots, \xi_n \in X \cap \operatorname{ran}(j)$ be such that $\xi_1 \in \dots \in \xi_n$. Let $c \coloneqq f(\{\xi_1, \dots, \xi_n\})$. Therefore, we have that $\mathcal{A} \models$ "$f_n(\xi_1, \dots, \xi_n) = c$". Let $\varsigma_1, \dots, \varsigma_n \in X$. Let $h\colon \{\xi_1, \dots, \xi_n\} \to \{\varsigma_1, \dots, \varsigma_n\}$ be $\in$-invariant. Since j is elementary, we have $j^{-1}(c) = c$, $j^{-1}(\xi_1) \in \dots \in j^{-1}(\xi_n)$ and $\mathcal{B} \models$ "$g_n(j^{-1}(\xi_1), \dots, j^{-1}(\xi_n)) = c$". Therefore, since j is $(X, \in)$-invariant, we have that

$$\mathcal{B} \models \text{“}g_n(j^{-1}(\xi_1), \dots, j^{-1}(\xi_n)) = c\text{”} \iff \mathcal{A} \models \text{“}f_n(h(\xi_1), \dots, h(\xi_n)) = c\text{”}.$$

Hence, we have that $f(\{\varsigma_1, \dots, \varsigma_n\}) = f_n(h(\xi_1), \dots, h(\xi_n)) = c$. Therefore, X is f-homogeneous. ■

Relying on this result, we next define the following class of structures.

Definition 3.3.4. $\mathcal{C}_{\mathsf{Ord}}$ denotes the class of structures whose universe M is such that $|\mathsf{Ord} \cap M| = |M|$.

Remark 3.3.5. The class $\mathcal{C}_{\mathsf{Ord}}$ is Σ_1-definable.

Proof. Both the condition $|\mathsf{Ord} \cap M| = |M|$ and the fact that the type of the structure is countable are Σ_1-expressible; the rest (*i.e.* the fact that every other element in the structure is either an element of M or a function or a relation on M) is Δ_0-expressible. ■

Now, we are in a position to define the following *invariant* structural reflection principle, which yields equivalent characterizations of our targeted large cardinal notions when applied to the class $\mathcal{C}_{\mathsf{Ord}}$ of structures.

Definition 3.3.6 (ISR). For β an ordinal, the $\mathrm{ISR}(\mathcal{C}, \beta)$ principle states: There is an ordinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$ with $|A| \geq \alpha$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| < \alpha$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Observation 3.3.7. If we compare the SR (recall Definition 2.3.1) and ISR principles, we observe two differences. The first and most obvious one is that, while SR requires the existence of just an elementary embedding, ISR adds to that requirement that the elementary embedding must satisfy an extra property (namely, the property of being invariant with respect to some well-order). Moreover, due to this extra property, the ISR principle has a parametric definition: the parameter β is the order-type of the well-order. The second difference is that, while SR does not require the structure $\mathcal{A}$ to satisfy any property other than belonging to the class $\mathcal{C}$ of structures, ISR adds the extra requirement that $|A| \geq \alpha$. We want to stress the fact that the second difference is a direct consequence of the first one. If we just require the existence of an elementary embedding, then the case $|A| < \alpha$ becomes trivial by picking j the identity map, whence $\mathcal{B} = \mathcal{A}$. Then, of course, the really interesting case in the structural reflection principle is $|A| \geq \alpha$. However, if we require the elementary embedding to satisfy some extra non-trivial property, then we must block the $|A| < \alpha$ case by explicitly restricting the principle to the case $|A| \geq \alpha$, since the previous trivial solution is no longer available.

Theorem 3.3.8. *Let κ be a cardinal and β be a limit ordinal. The following are equivalent:*

(1) κ is a β-Erdős cardinal.

(2) $\kappa \models \mathrm{ISR}(\mathcal{C}_{\mathsf{Ord}}, \beta)$.

Proof. By Definition 3.2.1 and Lemma 3.3.3 with $\mu < \kappa$. ■

Our next step is to observe that the ISR principle has natural counterparts in the form of invariant HSR and VSR principles. We point out that Observation 3.3.7 evidently applies to these counterparts as well.

Definition 3.3.9 (IHSR). For β an ordinal, the IHSR$(\mathcal{C}, \beta)$ principle states: There is an infinite cardinal α such that, for every structure $\mathcal{A} \in \mathcal{C} \setminus H_\alpha$, there is a structure $\mathcal{B} \in \mathcal{C} \cap H_\alpha$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Definition 3.3.10 (IVSR). For β an ordinal, the IVSR$(\mathcal{C}, \beta)$ principle states: There is an ordinal α such that, for every structure $\mathcal{A} \in \mathcal{C} \setminus V_\alpha$, there is a structure $\mathcal{B} \in \mathcal{C} \cap V_\alpha$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

And then we also observe that, if we restrict the class $\mathcal{C}_{\mathsf{Ord}}$ to structures with transitive universes, then we obtain more equivalent characterizations of our targeted large cardinal notions in terms of IHSR and IVSR. To that end, we first prove the following lemma.

Lemma 3.3.11. *Let α be an infinite ordinal and β be a limit ordinal. The following are equivalent:*

(1) $\alpha \longrightarrow (\beta)^{<\omega}_2$.

(2) For every structure $\mathcal{A}$, with A transitive and $\operatorname{ot}(\mathsf{Ord} \cap A) \geq \alpha$*, and for every infinite cardinal $\mu \leq \alpha$, there is a structure $\mathcal{B}$, with B transitive and $|\mathsf{Ord} \cap B| = |B| = \mu$, and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.*

Proof. (1)⇒(2): Let us assume (1). Let $\mathcal{A}$ be a structure with A transitive and $\operatorname{ot}(\mathsf{Ord} \cap A) \geq \alpha$. Let $\mathcal{A}' := \langle \mathcal{A}, \in \rangle$. Let us note that $\mathcal{L}_\in, \mathcal{L}_\mathcal{A} \subseteq \mathcal{L}_{\mathcal{A}'}$. By Theorem 3.2.7, there is $X \in [\mathsf{Ord} \cap A]^\beta$ a set of $\in$-indiscernibles for $\mathcal{A}'$ and, by Proposition 3.2.5, also for $\mathcal{A}$. Let $\mu \leq \alpha$ be an infinite cardinal. Let $Y \in [A]^\mu$ be such that $|\mathsf{Ord} \cap Y| = \mu$ and $|X \cap Y| \geq \omega$. Let $\mathcal{H}(Y)$ be the Skolem hull of Y in $\mathcal{A}'$. Hence, we have that $\mathcal{H}(Y) \prec \mathcal{A}'$, with $Y \subseteq H(Y)$ and $|H(Y)| = |Y| = \mu$. Moreover, since $|\mathsf{Ord} \cap Y| = \mu$, we have that $|\mathsf{Ord} \cap H(Y)| = \mu$. Therefore, by Theorem 2.2.7 (Mostovski's Collapsing Theorem), we have an isomorphism $j \colon \langle B, \in \rangle \cong \mathcal{H}(Y) \restriction \mathcal{L}_\in$, with B transitive, $j^{-1}``Y \subseteq \mathsf{Ord}$ and $|j^{-1}``Y| = |B| = \mu$, whence $|\mathsf{Ord} \cap B| = \mu$. We define the $\mathcal{L}_\mathcal{A}$-structure $\mathcal{B}$ with universe B induced by j. Thus, we have $j \colon \mathcal{B} \cong (\mathcal{H}(Y) \restriction \mathcal{L}_\mathcal{A}) \prec \mathcal{A}$, whence $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ is elementary with $Y \subseteq \operatorname{ran}(j)$. Furthermore, since $|X \cap Y| \geq \omega$, we have $|X \cap \operatorname{ran}(j)| \geq \omega$. Let $n \in \omega$. Let $\xi_1, \ldots, \xi_n \in B$ be such that $j``\{\xi_1, \ldots, \xi_n\} \subseteq X$ and $\xi_1 \in \ldots \in \xi_n$. Let $\varphi(x_1, \ldots, x_n) \in \mathcal{L}_\mathcal{A}$. Let $h \colon j``\{\xi_1, \ldots, \xi_n\} \to X$ be $\in$-preserving. By the elementarity of j and the indiscernibility of X for $\mathcal{A}$, we have that

$$\begin{aligned}\mathcal{B} \models \varphi[\xi_1, \ldots, \xi_n] &\iff \mathcal{A} \models \varphi[j(\xi_1), \ldots, j(\xi_n)] \\ &\iff \mathcal{A} \models \varphi[h(j(\xi_1)), \ldots, h(j(\xi_n))].\end{aligned}$$

Therefore, j is $(X, \in)$-invariant.

(2)⇒(1): By the proof of (2)⇒(1) of Lemma 3.3.3. ■

The previous lemma leads to the definition of the following class of structures.

Definition 3.3.12. $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$ denotes the subclass of structures of $\mathcal{C}_{\mathsf{Ord}}$ (see Definition 3.3.4) whose universe is transitive.

Remark 3.3.13. The class $\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}$ is Σ_1-definable.

Proof. By Remark 3.3.5, plus the fact that the predicate "M is transitive" is Δ_0-expressible. ■

And now we obtain characterizations of our targeted large cardinal notions also in terms of the IHSR and IVSR principles.

Theorem 3.3.14. *Let κ be a cardinal and β be a limit ordinal. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}$. The following are equivalent:*

(1) κ *is a β-Erdős cardinal.*

(2) $\kappa \models \text{ISR}(\mathcal{D}, \beta)$.

(3) $\kappa \models \text{IHSR}(\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}, \beta)$.

(4) $\kappa \models \text{IVSR}(\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}, \beta)$.

Proof. For the ISR cases, by Theorem 3.3.8 and Lemma 3.3.11 with $\mu < \kappa$.

For the IHSR case: Since $\mathcal{A} \notin H_\kappa$ is transitive, we have $|A| \geq \kappa$, whence $|\mathsf{Ord} \cap A| \geq \kappa$, whence, by Lemma 3.3.11, we have $\mathcal{B}$ with B transitive and $|\mathsf{Ord} \cap B| = |B| = \mu < \kappa$, whence $\mathcal{B} \in H_\kappa$. And conversely, $\langle \kappa, \in, f_n \rangle_{n \in \omega} \notin H_\kappa$.

For the IVSR case: Since $\mathcal{A} \notin V_\kappa$, we have that $\mathcal{A} \notin H_\kappa$. By the IHSR case, we have $\mathcal{B} \in H_\kappa \subseteq V_\kappa$. And conversely, $\langle \kappa, \in, f_n \rangle_{n \in \omega} \notin V_\kappa$. ■

Finally, in addition to these results, we can as well characterize our targeted large cardinal notions by applying the invariant structural reflection principles to a much narrower class of structures. In order to do so, we first observe the following lemma.

Lemma 3.3.15. *Let α be an infinite ordinal and β be a limit ordinal. The following are equivalent:*

(1) $\alpha \longrightarrow (\beta)^{<\omega}_2$.

(2) For every structure $\mathcal{A}$ with universe $L_\gamma[X]$ for $\gamma \geq \alpha$ and X some set, and for every infinite cardinal $\mu \leq \alpha$, there is a structure $\mathcal{B}$ with universe $L_\delta[X']$ for $\delta \leq \gamma$ such that $|\delta| = \mu$ and X' some set, and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(Y, \in)$-invariant for some $Y \in [\gamma]^\beta$.

Proof. (1)⇒(2): Let us assume (1). Let $\mathcal{A}$ be a structure with universe $L_\gamma[X]$ for $\gamma \geq \alpha$ and X some set. Let $\mathcal{A}' \coloneqq \langle \mathcal{A}, \in, X \cap L_\gamma[X] \rangle$. Let us note that $\mathcal{L}_\in \subsetneq \mathcal{L}_\in(\dot{X}) \subseteq \mathcal{L}_{\mathcal{A}'}$ and $\mathcal{L}_\mathcal{A} \subseteq \mathcal{L}_{\mathcal{A}'} = \mathcal{L}_\mathcal{A}(\dot{\in}, \dot{X})$, where $\dot{\in}$ is the binary relation symbol for the set-membership relation $\in$ and $\dot{X}$ is a unary predicate symbol. By Proposition 2.2.4, $\gamma = \mathsf{Ord} \cap L_\gamma[X]$. Hence, by Theorem 3.2.7, there is $Y \in [\gamma]^\beta$ a set of $\in$-indiscernibles for $\mathcal{A}'$ and, by Proposition 3.2.5, also for $\mathcal{A}$. Let $\mu \leq \alpha$ be an infinite cardinal. Let $Y' \in [L_\gamma[X]]^\mu$ be such that $|Y \cap Y'| \geq \omega$. Let $\mathcal{H}(Y')$ be the Skolem hull of Y' in $\mathcal{A}'$. Hence, we have that $\mathcal{H}(Y') \prec \mathcal{A}'$, with $Y' \subseteq H(Y')$ and $|H(Y')| = |Y'| = \mu$. Therefore, by Lemma 2.2.9 (Generalized Gödel's Condensation Lemma), we have an isomorphism $j\colon \langle L_\delta[X'], \in, X' \rangle \cong \mathcal{H}(Y') \upharpoonright \mathcal{L}_\in(\dot{X})$, with $\delta \leq \gamma$ an infinite ordinal and $X' = j^{-1}``(X \cap H(Y'))$. By Proposition 2.2.4, $\delta = \mathsf{Ord} \cap L_\delta[X']$ and $|L_\delta[X']| = |\delta|$. Therefore, since $H(Y') \sim L_\delta[X']$ and $|H(Y')| = \mu$, we have that $|\delta| = \mu$. We define the $\mathcal{L}_\mathcal{A}$-structure $\mathcal{B}$ with universe $L_\delta[X']$ induced by j. Thus, we have that $j\colon \mathcal{B} \cong (\mathcal{H}(Y') \upharpoonright \mathcal{L}_\mathcal{A}) \prec \mathcal{A}$, whence $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ is elementary with $Y' \subseteq \operatorname{ran}(j)$. Furthermore, since $|Y \cap Y'| \geq \omega$, we have that $|Y \cap \operatorname{ran}(j)| \geq \omega$. By the same argument exhibited in the last part of the proof of (1)⇒(2) of Lemma 3.3.11, we have that j is $(Y, \in)$-invariant.

(2)⇒(1): Let us assume (2). Let $f\colon [\alpha]^{<\omega} \to \{0,1\}$. Let $\mathcal{A} \coloneqq \langle L_\alpha, \in, f_n \rangle_{n\in\omega}$, where

$$f_n(a_1, \dots, a_n) \coloneqq f(\{a_1, \dots, a_n\}), \text{ if } a_1, \dots, a_n \in \alpha;$$
$$f_n(a_1, \dots, a_n) \coloneqq 0, \text{ otherwise.}$$

Let $\mu \leq \alpha$ be an infinite cardinal. By assumption, there is a structure $\mathcal{B} = \langle L_\delta[X], \in, g_n \rangle_{n\in\omega}$, with $\delta \leq \alpha$ such that $|\delta| = \mu$ and X some set, and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(Y, \in)$-invariant for some $Y \in [\alpha]^\beta$ with $|Y \cap \operatorname{ran}(j)| \geq \omega$. By the same argument exhibited in the last part of the proof of (2)⇒(1) of Lemma 3.3.3, we have that Y is f-homogeneous. ■

Observation 3.3.16. The proof of Lemma 3.3.15 shows that, if $X = \emptyset$, then $X' = \emptyset$.

By the previous lemma, we observe that the following classes of structures allow us to characterize our targeted large cardinal notions by means of any of the three invariant structural reflection principles previously defined.

Definition 3.3.17. $\mathcal{C}_{L[\cdot]}$ denotes the class of structures whose universe is $L_\alpha[X]$ for some ordinal α and some set X (see Definition 2.2.2). Moreover,

$\mathcal{C}_L$ denotes the subclass of structures of $\mathcal{C}_{L[\cdot]}$ whose universe is L_α for some ordinal α.

Observation 3.3.18. Clearly, $\mathcal{C}_L \subsetneq \mathcal{C}_{L[\cdot]} \subsetneq \mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}} \subsetneq \mathcal{C}_{\mathsf{Ord}}$.

Remark 3.3.19. The classes $\mathcal{C}_{L[\cdot]}$ and $\mathcal{C}_L$ are Σ_1-definable.

Proof. By Remark 3.3.5, plus the fact that the class function $\alpha \mapsto L_\alpha[X]$ is Δ_1-definable with X as a parameter (see Observation 2.2.3). ■

Finally, we provide a full characterization of Erdős, Ramsey and Almost Ramsey cardinals in terms of the ISR, IHSR and IVSR principles, combined with the increasingly narrower classes $\mathcal{C}_{\mathsf{Ord}}$, $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ and $\mathcal{C}_L$ of structures.

Theorem 3.3.20. *Let κ be a cardinal and β be a limit ordinal. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is a β-Erdős cardinal.

(2) $\kappa \models \mathrm{ISR}(\mathcal{D}, \beta)$.

(3) $\kappa \models \mathrm{IHSR}(\mathcal{C}^{\mathrm{tr}}, \beta)$.

(4) $\kappa \models \mathrm{IVSR}(\mathcal{C}^{\mathrm{tr}}, \beta)$.

Proof. For the ISR cases: By Theorems 3.3.8 and 3.3.14. And by Definition 3.2.1 and Lemma 3.3.15 with $\mu < \kappa$, since, by Proposition 2.2.4, we have that $|L_\alpha[X]| = |\alpha|$ for every infinite ordinal α.

For the IHSR and IVSR cases: By Theorem 3.3.14. And by Definition 3.2.1 and Lemma 3.3.15: we pick $\mu < \kappa$ and we note that, by Proposition 2.2.4, for any ordinal $\alpha \geq \kappa$, we have that $L_\alpha[X]$ and $L_\delta[X']$ are transitive with $L_\alpha[X] \in V_{\alpha+1} \setminus V_\alpha$ and $|L_\delta[X']| = |\delta| = \mu < \kappa$, whence $L_\alpha[X] \notin V_\kappa \supseteq H_\kappa$ and $L_\delta[X'] \in H_\kappa \subseteq V_\kappa$. ■

Corollary 3.3.21. *Let κ be a cardinal. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is a Ramsey cardinal.

(2) $\kappa \models \mathrm{ISR}(\mathcal{D}, \kappa)$.

(3) $\kappa \models \mathrm{IHSR}(\mathcal{C}^{\mathrm{tr}}, \kappa)$.

(4) $\kappa \models \mathrm{IVSR}(\mathcal{C}^{\mathrm{tr}}, \kappa)$.

Proof. By Definition 3.2.2 and Theorems 3.3.8, 3.3.14 and 3.3.20. ∎

Corollary 3.3.22. *Let κ be a cardinal. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) *κ is an Almost Ramsey cardinal.*

(2) $\kappa \models \mathrm{ISR}(\mathcal{D}, \beta)$, *for every limit ordinal $\beta < \kappa$.*

(3) $\kappa \models \mathrm{IHSR}(\mathcal{C}^{\mathrm{tr}}, \beta)$, *for every limit ordinal $\beta < \kappa$.*

(4) $\kappa \models \mathrm{IVSR}(\mathcal{C}^{\mathrm{tr}}, \beta)$, *for every limit ordinal $\beta < \kappa$.*

Proof. By Definition 3.2.3, Proposition 3.1.5(2) and Theorems 3.3.8, 3.3.14 and 3.3.20. ∎

We close this subsection by pointing out the following observation.

Observation 3.3.23. In Lemmas 3.3.3, 3.3.11 and 3.3.15, the condition for the infinite cardinal μ is to be less than *or equal* to α. This fact allows the particular case $\mu = \alpha$ that has not been included in the invariant structural principles defined in this subsection, since it is not contemplated in the definitions of the basic SR, HSR and VSR principles originally provided by Bagaria and Lücke (2024) (see Definitions 2.3.1 to 2.3.3). However, we can define the variants SR* and ISR* (with $|B| \leq \alpha$), HSR* and IHSR* (with $\mathcal{B} \in H_{\alpha^+}$) and VSR* and IVSR* (with $B \in V_{\alpha+1}$). We want to highlight the fact that, if we allowed for these variants, then Theorems 3.3.8, 3.3.14 and 3.3.20, as well as Corollaries 3.3.21 and 3.3.22, would also include the ISR*, IHSR* and IVSR* principles.

3.3.2 Level-by-level structural reflection

We begin this subsection by highlighting the following observation.

Observation 3.3.24. The ISR, IVSR and IHSR principles do not exploit all the power of Lemmas 3.3.3, 3.3.11 and 3.3.15, which provide the existence of invariant elementary embeddings *for every* infinite cardinal $\mu \leq \alpha$.

We notice that the fact pointed out in the previous observation allows us to connect the results obtained in the previous subsection with the *level-by-level* kind of structural reflection proposed by Hou (2024).

Hou's definition of the level-by-level structural reflection is made in terms of the ranks of the structures of the class involved in the reflection, that is to say, in terms of the VSR principle. Therefore, although Lemmas 3.3.3, 3.3.11 and 3.3.15 do not yield a level-by-level variant of invariant elementary embeddings in terms of ranks, they do yield such a variant in terms of the cardinalities of the universes of structures.

Thus, as a first step, we next present the level-by-level structural reflection principle proposed by Hou and rename it as "Level-by-level VSR".

Definition 3.3.25 (LVSR). The LVSR($\mathcal{C}$) principle states: There is an ordinal α such that, for every ordinal β, there is an ordinal $\gamma < \alpha$ such that, for every structure $\mathcal{A} \in \mathcal{C} \cap (V_{\beta+1} \setminus V_\beta)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (V_{\gamma+1} \setminus V_\gamma)$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Secondly, we next propose analogue counterparts of the LVSR principle in terms of SR and HSR; *i.e.* we define level-by-level versions of the SR and HSR principles.

Definition 3.3.26 (LSR). The LSR($\mathcal{C}$) principle states: There is an ordinal α such that, for every cardinal κ, there is a cardinal $\mu < \alpha$ such that, for every structure $\mathcal{A} \in \mathcal{C}$ with $|A| = \kappa$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| = \mu$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Definition 3.3.27 (LHSR). The LHSR($\mathcal{C}$) principle states: There is an infinite cardinal α such that, for every infinite cardinal κ, there is an infinite cardinal $\mu < \alpha$ such that, for every structure $\mathcal{A} \in \mathcal{C} \cap (H_{\kappa^+} \setminus H_\kappa)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (H_{\mu^+} \setminus H_\mu)$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

We are now in a position to extend the invariant structural reflection principles to their respective level-by-level versions, which is precisely what we do next.

Definition 3.3.28 (LISR). For β an ordinal, the LISR($\mathcal{C}, \beta$) principle states: There is an ordinal α such that, for every cardinal $\kappa \geq \alpha$, there is a cardinal $\mu < \alpha$ such that, for every structure $\mathcal{A} \in \mathcal{C}$ with $|A| = \kappa$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| = \mu$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Definition 3.3.29 (LIHSR). For β an ordinal, the LIHSR($\mathcal{C}, \beta$) principle states: There is an infinite cardinal α such that, for every cardinal $\kappa \geq \alpha$, there is an infinite cardinal $\mu < \alpha$ such that, for every structure $\mathcal{A} \in \mathcal{C} \cap (H_{\kappa^+} \setminus H_\kappa)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (H_{\mu^+} \setminus H_\mu)$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Definition 3.3.30 (LIVSR). For β an ordinal, the LIVSR$(\mathcal{C}, \beta)$ principle states: There is an ordinal α such that, for every ordinal $\gamma \geq \alpha$, there is an ordinal $\delta < \alpha$ such that, for every structure $\mathcal{A} \in \mathcal{C} \cap (V_{\gamma+1} \setminus V_\gamma)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (V_{\delta+1} \setminus V_\delta)$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Thus, we finally present the equivalent characterizations of Erdős, Ramsey and Almost Ramsey cardinals yielded by the LISR and LIHSR principles.

Theorem 3.3.31. *Let κ be a cardinal and β be a limit ordinal. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is a β-Erdős cardinal.

(2) $\kappa \models$ LISR$(\mathcal{D}, \beta)$.

(3) $\kappa \models$ LIHSR$(\mathcal{C}^{\mathrm{tr}}, \beta)$.

Proof. By Theorem 3.3.20 and Observation 3.3.24. ■

Corollary 3.3.32. *Let κ be a cardinal. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is a Ramsey cardinal.

(2) $\kappa \models$ LISR$(\mathcal{D}, \kappa)$.

(3) $\kappa \models$ LIHSR$(\mathcal{C}^{\mathrm{tr}}, \kappa)$.

Proof. By Definition 3.2.2 and Theorem 3.3.31. ■

Corollary 3.3.33. *Let κ be a cardinal. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is an Almost Ramsey cardinal.

(2) $\kappa \models$ LISR$(\mathcal{D}, \beta)$, for every limit ordinal $\beta < \kappa$.

(3) $\kappa \models$ LIHSR$(\mathcal{C}^{\mathrm{tr}}, \beta)$, for every limit ordinal $\beta < \kappa$.

Proof. By Definition 3.2.3, Proposition 3.1.5(2) and Theorem 3.3.31. ■

We close this subsection by getting back to Observation 3.3.23, at the end of the previous subsection.

Observation 3.3.34. Analogously to the basic structural reflection principles, all the level-by-level principles revisited or newly defined in this subsection allow for * variants that exploit the particular case $\mu = \alpha$ in Lemmas 3.3.3, 3.3.11 and 3.3.15 pointed out in Observation 3.3.23. We want to highlight the fact that, in case we allowed for these variants, then both Theorem 3.3.31 and Corollary 3.3.32 would also include the LISR* and LIHSR* principles.

3.3.3 For-all-levels structural reflection

We close this chapter by pointing out that Observation 3.3.24 allows to go even further than the level-by-level principles defined in the previous subsection. In all the previous level-by-level principles (invariant or not), for every level of the structure $\mathcal{A}$, *there is some* level below the reflection point for the structure $\mathcal{B}$. However, Observation 3.3.24 suggests that we can actually work with a kind of "for all" level-by-level principles; *i.e.* level-by-level principles with a structure $\mathcal{B}$ and an elementary embedding *for every* level below the reflection point that can in principle admit such a scenario.

Observation 3.3.35. There are some restrictions that we must accommodate in the definitions of any conceivable "for-all-levels" structural reflection principles:

1. Finitary levels both for structures $\mathcal{A}$ and $\mathcal{B}$ must be counted out, otherwise any "for-all-levels" principle would be trivially false, since evidently a structure with a universe whose cardinality is finite neither can be elementarily embedded into nor admits an elementary embedding from a structure whose universe is of any other different cardinality.
2. Moreover, "for-all-levels" principles must also restrict the levels for the structure $\mathcal{A}$ to those larger than or equal to the reflection point α. Otherwise, we would be allowing for combinations of levels where the cardinality of the structure $\mathcal{B}$ would be larger than the cardinality of the structure $\mathcal{A}$, which would make the existence of the elementary embedding impossible and whence yield the principle trivially false.

3. Finally, the reflection point α itself must be restricted to be larger than ω, otherwise the principle would be vacuously witnessed by any $\alpha \leq \omega$.

Thus, keeping this observation in mind, we next define the *for-all-levels* versions of the SR, HSR and VSR principles.

Definition 3.3.36 (∀SR). The SR($\mathcal{C}$) principle states: There is an ordinal $\alpha > \omega$ such that, for every cardinal $\kappa \geq \alpha$ and for every infinite cardinal $\mu < \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C}$ with $|A| = \kappa$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| = \mu$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Definition 3.3.37 (∀HSR). The HSR($\mathcal{C}$) principle states: There is an uncountable cardinal α such that, for every cardinal $\kappa \geq \alpha$ and for every infinite cardinal $\mu < \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C} \cap (H_{\kappa^+} \setminus H_\kappa)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (H_{\mu^+} \setminus H_\mu)$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Definition 3.3.38 (∀VSR). The VSR($\mathcal{C}$) principle states: There is an ordinal $\alpha > \omega$ such that, for every ordinal $\beta \geq \alpha$ and for every infinite ordinal $\gamma < \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C} \cap (V_{\beta+1} \setminus V_\beta)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (V_{\gamma+1} \setminus V_\gamma)$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Now, we proceed to extend the invariant structural reflection principles to their respective for-all-levels versions.

Definition 3.3.39 (∀ISR). For β an ordinal, the ∀ISR($\mathcal{C}, \beta$) principle states: There is an ordinal $\alpha > \omega$ such that, for every cardinal $\kappa \geq \alpha$ and for every infinite cardinal $\mu < \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C}$ with $|A| = \kappa$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| = \mu$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Definition 3.3.40 (∀IHSR). For β an ordinal, the ∀IHSR($\mathcal{C}, \beta$) principle states: There is an uncountable cardinal α such that, for every cardinal $\kappa \geq \alpha$ and for every infinite cardinal $\mu < \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C} \cap (H_{\kappa^+} \setminus H_\kappa)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (H_{\mu^+} \setminus H_\mu)$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Definition 3.3.41 (∀IVSR). For β an ordinal, the ∀IVSR($\mathcal{C}, \beta$) principle states: There is an ordinal $\alpha > \omega$ such that, for every ordinal $\gamma \geq \alpha$ and for every infinite ordinal $\delta < \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C} \cap (V_{\gamma+1} \setminus V_\gamma)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (V_{\delta+1} \setminus V_\delta)$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary and $(X, \in)$-invariant for some $X \in [\mathsf{Ord} \cap A]^\beta$.

Thus, we finally present the equivalent characterizations of Erdős, Ramsey and Almost cardinals yielded by the $\forall$ISR and $\forall$IHSR principles.

Theorem 3.3.42. *Let κ be a cardinal and β be a limit ordinal. Let $\mathcal{C}^{\text{tr}}$ denote either $\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\text{tr}}$. The following are equivalent:*

(1) κ *is a β-Erdős cardinal.*

(2) $\kappa \models \forall\text{ISR}(\mathcal{D}, \beta)$.

(3) $\kappa \models \forall\text{IHSR}(\mathcal{C}^{\text{tr}}, \beta)$.

Proof. By Theorem 3.3.31 and Observation 3.3.24. ■

Corollary 3.3.43. *Let κ be a cardinal. Let $\mathcal{C}^{\text{tr}}$ denote either $\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\text{tr}}$. The following are equivalent:*

(1) κ *is a Ramsey cardinal.*

(2) $\kappa \models \forall\text{ISR}(\mathcal{D}, \kappa)$.

(3) $\kappa \models \forall\text{IHSR}(\mathcal{C}^{\text{tr}}, \kappa)$.

Proof. By Definition 3.2.2 and Theorem 3.3.42. ■

Corollary 3.3.44. *Let κ be a cardinal. Let $\mathcal{C}^{\text{tr}}$ denote either $\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}$, $\mathcal{C}_{L[\cdot]}$ or $\mathcal{C}_L$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\mathsf{Ord}}$ or $\mathcal{C}^{\text{tr}}$. The following are equivalent:*

(1) κ *is an Almost Ramsey cardinal.*

(2) $\kappa \models \forall\text{ISR}(\mathcal{D}, \beta)$*, for every limit ordinal* $\beta < \kappa$.

(3) $\kappa \models \forall\text{IHSR}(\mathcal{C}^{\text{tr}}, \beta)$*, for every limit ordinal* $\beta < \kappa$.

Proof. By Definition 3.2.3, Proposition 3.1.5(2) and Theorem 3.3.42. ■

We close this subsection (and this chapter) by pointing out an observation analogous to Observation 3.3.34, at the end of the previous subsection.

Observation 3.3.45. Analogously to the level-by-level structural reflection principles, all the for-all-levels principles defined in this subsection allow for * variants that exploit the particular case $\mu = \alpha$ in Lemmas 3.3.3, 3.3.11 and 3.3.15 pointed out in Observation 3.3.23. Again, if we allowed for these variants, then both Theorem 3.3.42 and Corollary 3.3.43 would also include the $\forall$ISR* and $\forall$IHSR* principles.

Chapter 4

Structural reflection for Rowbottom cardinals

In this chapter, we provide equivalent characterizations for the family of Rowbottom cardinals in terms of new structural reflection principles. We proceed analogously to chapter 3 and we start by introducing the family of square-bracket partition properties in section 4.1. Subsequently, we briefly present the large cardinal notions targeted in this chapter in section 4.2. Lastly, we define a new family of structural reflection principles and show how it characterizes Rowbottom cardinals in section 4.3.

4.1 Square-bracket partition properties

The kind of partition property that can be employed to define Rowbottom cardinals is a weakening of the ordinary partition properties presented in section 3.1. Following Kanamori (2003, p. 85–86), we call this family of partition relations "square-bracket partition properties" and we next provide our definitions adopting the same slightly more general setting that we employed in section 3.1.

Definition 4.1.1 (Square-bracket partition properties). Let $(A, \lhd)$ be a well-order, C be a set and β, δ, η be ordinals.

- $A \longrightarrow [\beta]^{\delta}_{C}$ denotes the following: For every $f\colon [A]^{\delta} \to C$, there is $H \in [A]^{\beta}$ such that $f``[H]^{\delta} \neq C$.
- $A \longrightarrow [\beta]^{\delta}_{C,<\eta}$ denotes the following: For every $f\colon [A]^{\delta} \to C$, there is $H \in [A]^{\beta}$ such that $|f``[H]^{\delta}| < \eta$.

- $A \longrightarrow [\beta]^{<\delta}_C$ denotes the following: For every $f\colon [A]^{<\delta} \to C$, there is $H \in [A]^\beta$ such that $f``[H]^{<\delta} \neq C$.
- $A \longrightarrow [\beta]^{<\delta}_{C,<\eta}$ denotes the following: For every $f\colon [A]^{<\delta} \to C$, there is $H \in [A]^\beta$ such that $|f``[H]^{<\delta}| < \eta$.

Notation. In the context of a square-bracket partition property, we say that the set H is *f-homogeneous*. Also, we write $\not\longrightarrow$ when the property fails.

Next, we show that square-bracket partition properties are also preserved as long as the order-type of $(A, \lhd)$ and the cardinality of C are preserved.

Proposition 4.1.2. *Let $(A, \lhd)$ be a well-order, C be a set and β, δ, η be ordinals.*

1. *If $A \longrightarrow [\beta]^\delta_C$, then $A' \longrightarrow [\beta]^\delta_{C'}$ for any well-order $(A', \lhd') \cong (A, \lhd)$ and for any set $C' \sim C$.*
2. *If $A \longrightarrow [\beta]^\delta_{C,<\eta}$, then $A' \longrightarrow [\beta]^\delta_{C',<\eta}$ for any well-order $(A', \lhd') \cong (A, \lhd)$ and for any set $C' \sim C$.*
3. *If $A \longrightarrow [\beta]^{<\delta}_C$, then $A' \longrightarrow [\beta]^{<\delta}_{C'}$ for any well-order $(A', \lhd') \cong (A, \lhd)$ and for any set $C' \sim C$.*
4. *If $A \longrightarrow [\beta]^{<\delta}_{C,<\eta}$, then $A' \longrightarrow [\beta]^{<\delta}_{C',<\eta}$ for any well-order $(A', \lhd') \cong (A, \lhd)$ and for any set $C' \sim C$.*

Proof. By the proof of Proposition 3.1.3. ■

Observation 4.1.3. Most often, square-bracket partition properties are particularized for α and γ ordinals, instead of $(A, \lhd)$ and C, respectively. In such a context, we always assume $\alpha \geq \beta \geq \delta > 0$ and $\gamma \geq \eta \geq 2$ to avoid trivialities and nonsensical scenarios.

To close this section, we next show the preservation properties of the square-bracket partition properties in terms of ordinals, as well as the conditions under which the least ordinal α that satisfies a square-bracket partition property, if any, is a cardinal.

Proposition 4.1.4. *Let α, β, γ, δ, η be ordinals.*

1. *If $\alpha \longrightarrow [\beta]^\delta_\gamma$, then $\alpha' \longrightarrow [\beta']^{\delta'}_{\gamma'}$ for any ordinals $\alpha' \geq \alpha$, $\beta' \leq \beta$, $\gamma' \geq \gamma$ and $\delta' \leq \delta$ with $\delta' < \operatorname{cf}(\beta)$.*

2. *If* $\alpha \longrightarrow [\beta]^{\delta}_{\gamma,<\eta}$*, then* $\alpha' \longrightarrow [\beta']^{\delta'}_{\gamma',<\eta'}$ *for any ordinals* $\alpha' \geq \alpha$*,* $\beta' \leq \beta$*,* $\gamma' \leq \gamma$*,* $\delta' \leq \delta$ *with* $\delta' < \mathrm{cf}(\beta)$*, and* $\eta' \geq \eta$*.*

3. *If* $\alpha \longrightarrow [\beta]^{<\delta}_{\gamma}$*, then* $\alpha' \longrightarrow [\beta']^{<\delta'}_{\gamma'}$ *for any ordinals* $\alpha' \geq \alpha$*,* $\beta' \leq \beta$*,* $\gamma' \geq \gamma$ *and* $\delta' \leq \delta$*.*

4. *If* $\alpha \longrightarrow [\beta]^{<\delta}_{\gamma,<\eta}$*, then* $\alpha' \longrightarrow [\beta']^{<\delta'}_{\gamma',<\eta'}$ *for any ordinals* $\alpha' \geq \alpha$*,* $\beta' \leq \beta$*,* $\gamma' \leq \gamma$*,* $\delta' \leq \delta$ *and* $\eta' \geq \eta$*.*

Proof. 1. Let us assume $\alpha \longrightarrow [\beta]^{\delta}_{\gamma}$. The cases $\alpha' \geq \alpha$, $\beta' \leq \beta$ and $\delta' \leq \delta$ with $\delta' < \mathrm{cf}(\beta)$ are shown by the same arguments exhibited in the proof of point 1 of Proposition 3.1.5. Let $\gamma' \geq \gamma$ and $f\colon [\alpha]^{\delta} \to \gamma'$. We define $g\colon [\alpha]^{\delta} \to \gamma$ as follows: $g(x) = f(x)$ if $f(x) \in \gamma$; $g(x) = 0$ if $f(x) \notin \gamma$. By assumption, there is $H \in [\alpha]^{\beta}$ g-homogeneous. Hence, $\gamma \setminus g``[H]^{\delta} \neq \emptyset$. Let $x \in [H]^{\delta}$. If $f(x) \in \gamma$, then $f(x) = g(x) \in g``[H]^{\delta}$, whence $f(x) \notin \gamma \setminus g``[H]^{\delta}$; if $f(x) \notin \gamma$, then $f(x) \notin \gamma \setminus g``[H]^{\delta}$. Therefore, we have that $f``[H]^{\delta} \cap (\gamma \setminus g``[H]^{\delta}) = \emptyset$, whence $\gamma \setminus g``[H]^{\delta} \subseteq \gamma' \setminus f``[H]^{\delta}$, whence $\gamma' \setminus f``[H]^{\delta} \neq \emptyset$. Thus, H is f-homogeneous.

2. Let us assume $\alpha \longrightarrow [\beta]^{\delta}_{\gamma,<\eta}$. The cases $\alpha' \geq \alpha$, $\beta' \leq \beta$, $\gamma' \leq \gamma$ and $\delta' \leq \delta$ with $\delta' < \mathrm{cf}(\beta)$ are shown by the same arguments exhibited in the proof of point 1 of Proposition 3.1.5. Let $\eta' \geq \eta$ and $f\colon [\alpha]^{\delta} \to \gamma$. By assumption, there is $H \in [\alpha]^{\beta}$ such that $|f``[H]^{\delta}| < \eta \leq \eta'$.

3. The cases $\alpha' \geq \alpha$ and $\beta' \leq \beta$ are shown by the same arguments exhibited in the proof of point 1 of Proposition 3.1.5; the case $\gamma' \geq \gamma$ is shown by the same argument exhibited in point 1; the case $\delta' \leq \delta$ is shown by the same argument exhibited in the proof of point 2 of Proposition 3.1.5.

4. The cases $\alpha' \geq \alpha$, $\beta' \leq \beta$ and $\gamma' \leq \gamma$ are shown by the same arguments exhibited in the proof of point 1 of Proposition 3.1.5; the case $\delta' \leq \delta$ is shown by the same argument exhibited in the proof of point 2 of Proposition 3.1.5; the case $\eta' \geq \eta$ is shown by the same argument exhibited in point 2. ■

Proposition 4.1.5. *Let* α*,* γ*,* η *be ordinals and* μ *be a cardinal.*

1. *If* $\alpha \longrightarrow [\mu]^{n}_{\gamma}$ *with* $n \in \omega$*, then the least* κ *such that* $\kappa \longrightarrow [\mu]^{n}_{\gamma}$ *is a cardinal.*

2. *If* $\alpha \longrightarrow [\mu]^{n}_{\gamma,<\eta}$ *with* $n \in \omega$*, then the least* κ *such that* $\kappa \longrightarrow [\mu]^{n}_{\gamma,<\eta}$ *is a cardinal.*

3. *If* $\alpha \longrightarrow [\mu]^{<\omega}_{\gamma}$*, then the least* κ *such that* $\kappa \longrightarrow [\mu]^{<\omega}_{\gamma}$ *is a cardinal.*

4. *If* $\alpha \longrightarrow [\mu]^{<\omega}_{\gamma,<\eta}$*, then the least* κ *such that* $\kappa \longrightarrow [\mu]^{<\omega}_{\gamma,<\eta}$ *is a cardinal.*

Proof. By the proof of Proposition 3.1.6. ■

4.2 Rowbottom cardinals

In this section, we provide a brief introduction to the large cardinal family of ν-Rowbottom cardinals, which were defined as a result of the investigation on the partition properties of measurable cardinals carried out by Rowbottom (1964). More detailed accounts of this large cardinal notion can be found in Kanamori (2003, p. 85–92).

Definition 4.2.1. Let $\kappa > \nu$ be uncountable cardinals.

- κ is a *ν-Rowbottom cardinal* iff $\kappa \longrightarrow [\kappa]^{<\omega}_{\lambda,<\nu}$ for every cardinal $\lambda < \kappa$.
- κ is a *Rowbottom cardinal* iff κ is ω_1-Rowbottom.

Proposition 4.2.2.

1. *If κ is a ν-Rowbottom cardinal, then κ is a ν'-Rowbottom cardinal for every $\nu' \geq \nu$.*
2. *κ is a ν-Rowbottom cardinal iff $\kappa \longrightarrow [\kappa]^{<\omega}_{\lambda,<\nu}$ for every cardinal λ such that $\nu \leq \lambda < \kappa$.*

Proof. 1. By Proposition 4.1.4(4). 2. Since the ν-Rowbottom partition property is trivially true for any $\lambda < \nu$. ■

Well-known facts are that ν-Rowbottom cardinals are either weakly inaccessible or singular with cofinality less than ν (Kanamori, 2003, p. 90), that Ramsey cardinals are Rowbottom (Kanamori, 2003, p. 81–82) and that the existence of a ν-Rowbottom cardinal implies that $V \neq L$ (Kanamori, 2003, p. 88). Moreover, the consistency of "$\aleph_\omega$ is Rowbottom" is a famous still unsolved question.

It was Rowbottom's discovery (see Theorem 4.2.5) that square-bracket partition properties are the purely combinatorial, set-theoretic counterpart of the two-cardinal versions of the Downward Löwenheim-Skolem Theorem, which is a well-known model-theoretic notion, that we next introduce in Definition 4.2.4.

Definition 4.2.3. Let $\mathcal{M}$ be an $\mathcal{L}(\dot{P})$-structure, where $\mathcal{L}$ is some language and $\dot{P}$ is a unary predicate symbol.

- The *type* of $\mathcal{M}$ is $\langle \kappa, \lambda \rangle$ iff $|M| = \kappa$ and $|\dot{P}^{\mathcal{M}}| = \lambda$.
- Moreover, we allow for the notation of the type to support variants with the symbols $<$, $\leq$, $>$ and $\geq$; *e.g.* the type of $\mathcal{M}$ is $\langle \geq \kappa, < \lambda \rangle$ iff $|M| \geq \kappa$ and $|\dot{P}^{\mathcal{M}}| < \lambda$.

Definition 4.2.4 (Double arrow notation).

- $\langle \kappa, \lambda \rangle \twoheadrightarrow \langle \mu, \nu \rangle$ denotes the following property:

  Every structure of type $\langle \kappa, \lambda \rangle$ has an elementary substructure of type $\langle \mu, \nu \rangle$.
- Moreover, we allow the double arrow notation to support variants with the symbols $<$, $\leq$, $>$ and $\geq$; *e.g.* $\langle \geq \kappa, < \lambda \rangle \twoheadrightarrow \langle > \mu, < \nu \rangle$ denotes the following property:

  Every structure of type $\langle \geq \kappa, < \lambda \rangle$ has an elementary substructure of type $\langle > \mu, < \nu \rangle$.

Theorem 4.2.5 (Rowbottom, 1964, 1971). *Let κ, μ, λ, ν be cardinals such that $\kappa > \lambda$, $\kappa \geq \mu \geq \omega$ and $\nu > \omega$. The following are equivalent:*

(1) $\kappa \longrightarrow [\mu]^{<\omega}_{\lambda, <\nu}$.

(2) $\langle \kappa, \lambda \rangle \twoheadrightarrow \langle \mu, < \nu \rangle$.

Proof. See the proof of Theorem 8.1 in Kanamori (2003, p. 86). ■

We close this section by showing the model-theoretic characterization of ν-Rowbottom cardinals immediately derivable from Theorem 4.2.5, as well as a slight generalization of it that will be useful in the next section and that immediately follows from the preservation properties of the square-bracket partition relations.

Theorem 4.2.6. *Let $\kappa > \nu$ be uncountable cardinals. The following are equivalent:*

(1) κ is a ν-Rowbottom cardinal.

(2) $\langle \kappa, < \kappa \rangle \twoheadrightarrow \langle \kappa, < \nu \rangle$.

(3) $\langle \geq \kappa, < \kappa \rangle \twoheadrightarrow \langle \mu, < \nu' \rangle$ *for every infinite cardinal $\mu \leq \kappa$ and every cardinal $\nu' \geq \nu$.*

Proof. (1)⇔(2): By Definition 4.2.1 and Theorem 4.2.5.

(1)⇔(3): By Definition 4.2.1, Proposition 4.1.4(4) and Theorem 4.2.5. ■

Corollary 4.2.7. *Let κ be an uncountable cardinal. The following are equivalent:*

(1) *κ is a Rowbottom cardinal.*

(2) $\langle \kappa, < \kappa \rangle \twoheadrightarrow \langle \kappa, < \omega_1 \rangle$.

(3) $\langle \kappa, \lambda \rangle \twoheadrightarrow \langle \kappa, \omega \rangle$ *for every infinite cardinal* $\lambda < \kappa$.

(4) $\langle \geq \kappa, < \kappa \rangle \twoheadrightarrow \langle \mu, < \nu \rangle$ *for every infinite cardinal* $\mu \leq \kappa$ *and every uncountable cardinal* ν.

Proof. (1)⇔(2) and (1)⇔(4): By Theorem 4.2.6.

(1)⇔(3): By Definition 4.2.1 and Theorem 4.2.5. ■

4.3 Structural reflection principles for Rowbottom cardinals

In this section, we provide the main contributions of the present work concerning the equivalent characterization of the family of Rowbottom cardinals in terms of principles of structural reflection. In subsection 4.3.1, we introduce a new family of principles: the two-cardinal structural reflection principles. We show different definable classes of structures for which this new family of principles characterizes ν-Rowbottom cardinals. And, analogously to subsection 3.3.3, we introduce in subsection 4.3.2 the for-all-levels versions of the two-cardinal structural reflection principles and show how ν-Rowbottom cardinals can also be characterized by them.

4.3.1 Two-cardinal structural reflection

In light of Theorem 4.2.6 and Corollary 4.2.7, we can already define two different classes of structures and a new structural reflection principle that, applied to any of those two classes of structures, yields equivalent characterizations of ν-Rowbottom cardinals.

Definition 4.3.1. Let κ be a cardinal. $\mathcal{C}_{\geq\kappa,<\kappa}$ denotes the class of structures of the form $\langle M, S, \ldots \rangle$, where M is the universe of the structure, $|M| \geq \kappa$, $S \subseteq M$ and $|S| < \kappa$. Moreover, $\mathcal{C}_{\kappa,<\kappa}$ denotes the subclass of structures of $\mathcal{C}_{\geq\kappa,<\kappa}$ whose universe has cardinality κ.

Remark 4.3.2. The classes $\mathcal{C}_{\kappa,<\kappa}$ and $\mathcal{C}_{\geq\kappa,<\kappa}$ are Σ_1-definable with a cardinal κ as a parameter.

Proof. The conditions $|M| \geq \kappa$, $|S| < \kappa$ and $|M| = \kappa$ are Σ_1-expressible with a cardinal κ as a parameter. The fact that the type of the structure is countable is Σ_1-expressible and the rest is Δ_0-expressible. ■

Furthermore, the SR principle introduced in Definition 2.3.1 can be extended in a very natural way into the following *two-cardinal* version.

Definition 4.3.3 (2CSR). For β an ordinal, the $\mathrm{2CSR}(\mathcal{C}, \beta)$ principle states: There is an ordinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C}$ with type $\langle <\alpha, <\beta\rangle$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

And next, we give the characterization of ν-Rowbottom cardinals by means of the 2CSR principle.

Theorem 4.3.4. *Let $\kappa > \nu$ be uncountable cardinals. Let $\mathcal{D}$ denote either $\mathcal{C}_{\kappa,<\kappa}$ or $\mathcal{C}_{\geq\kappa,<\kappa}$. The following are equivalent:*

(1) κ is a ν-Rowbottom cardinal.

(2) $\kappa + 1 \models \mathrm{2CSR}(\mathcal{D}, \nu)$.

Proof. This is equivalent to Theorem 4.2.6. ■

In contrast to the invariant structural reflection principles studied in subsection 3.3.1, the 2CSR principle has no straight counterparts in the form of the HSR and VSR principles respectively introduced in Definitions 2.3.2 and 2.3.3, since 2CSR is all about the type of structure $\mathcal{B}$ and then types are about two different cardinalities: that of the universe of the structure and that of the interpretation in the structure of some designated unary predicate symbol $\dot{P}$. The only "natural" moves that we are able to conceive here are the following:

1. As for an HSR-version of the 2CSR principle, we require for B and $\dot{P}^{\mathcal{B}}$ to be hereditarily of cardinality less than α and β, respectively.
2. As for a VSR-version of the 2CSR principle, we require for B and $\dot{P}^{\mathcal{B}}$ to be of rank less than α and β, respectively.

Thus, we next define the notions of "H-type" and "V-type" of a structure, followed by our definitions of the 2CHSR and 2CVSR principles.

Definition 4.3.5. Let $\mathcal{M}$ be an $\mathcal{L}(\dot{P})$-structure, where $\mathcal{L}$ is some language and $\dot{P}$ is a unary predicate symbol.

- The *H-type* of $\mathcal{M}$ is $\langle \kappa, \lambda \rangle$ iff $M \in H_{\kappa+} \setminus H_\kappa$ and $\dot{P}^{\mathcal{M}} \in H_{\lambda+} \setminus H_\lambda$ (or equivalently: M and $\dot{P}^{\mathcal{M}}$ are hereditarily of cardinality κ and λ, respectively).
- Moreover, we allow the notation of the type to support variants with the symbols $<$, $\leq$, $>$ and $\geq$; *e.g.* the H-type of $\mathcal{M}$ is $\langle < \kappa, < \lambda \rangle$ iff $M \in H_\kappa$ and $\dot{P}^{\mathcal{M}} \in H_\lambda$ (or equivalently: M and $\dot{P}^{\mathcal{M}}$ are hereditarily of cardinality less than κ and less than λ, respectively).

Definition 4.3.6. Let $\mathcal{M}$ be an $\mathcal{L}(\dot{P})$-structure, where $\mathcal{L}$ is some language and $\dot{P}$ is a unary predicate symbol.

- The *V-type* of $\mathcal{M}$ is $\langle \alpha, \beta \rangle$ iff $M \in V_{\alpha+1} \setminus V_\alpha$ and $\dot{P}^{\mathcal{M}} \in V_{\beta+1} \setminus V_\beta$ (or equivalently: $\mathrm{rk}(M) = \alpha$ and $\mathrm{rk}(\dot{P}^{\mathcal{M}}) = \beta$).
- Moreover, we allow the notation of the type to support variants with the symbols $<$, $\leq$, $>$ and $\geq$; *e.g.* the V-type of $\mathcal{M}$ is $\langle < \alpha, < \beta \rangle$ iff $M \in V_\alpha$ and $\dot{P}^{\mathcal{M}} \in V_\beta$ (or equivalently: $\mathrm{rk}(M) < \alpha$ and $\mathrm{rk}(\dot{P}^{\mathcal{M}}) < \beta$).

Definition 4.3.7 (2CHSR). For β an infinite cardinal, the $\mathrm{2CHSR}(\mathcal{C}, \beta)$ principle states: There is an infinite cardinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C}$ with H-type $\langle < \alpha, < \beta \rangle$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Definition 4.3.8 (2CVSR). For β an ordinal, the $\mathrm{2CVSR}(\mathcal{C}, \beta)$ principle states: There is an ordinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C}$ with V-type $\langle < \alpha, < \beta \rangle$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Now, we observe that the transitive versions of the classes $\mathcal{C}_{\kappa,<\kappa}$ and $\mathcal{C}_{\geq\kappa,<\kappa}$ of structures yield equivalent characterizations of Rowbottom cardinals in terms of 2CSR and 2CHSR. To that end, we first prove the following lemma.

Lemma 4.3.9. *Let κ, μ, λ, ν be cardinals with $\kappa > \lambda$, $\kappa \geq \mu \geq \omega$ and $\nu > \omega$. The following are equivalent:*

(1) $\kappa \longrightarrow [\mu]^{<\omega}_{\lambda,<\nu}$.

(2) For every structure $\mathcal{A} = \langle A, S, \ldots \rangle$, with $S \subseteq A$ transitive sets, $|A| = \kappa$ and $|S| = \lambda$, there is a structure $\mathcal{B} = \langle B, Z, \ldots \rangle$, with $Z \subseteq B$ transitive sets, $|B| = \mu$ and $|Z| < \nu$, and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Proof. (1)⇒(2): Let us assume (1). Let $\mathcal{A} := \langle A, S, \dots\rangle$ be a structure with $S \subseteq A$ transitive sets, $|A| = \kappa$ and $|S| = \lambda$. Let $\mathcal{A}' := \langle \mathcal{A}, \in\rangle$. Let us note that $\mathcal{L}_\in, \mathcal{L}_\mathcal{A} \subseteq \mathcal{L}_{\mathcal{A}'}$. By Theorem 4.2.5, there is $\mathcal{M} \prec \mathcal{A}'$, with $|M| = \mu$ and $|S \cap M| < \nu$. Therefore, by Theorem 2.2.7 (Mostovski's Collapsing Theorem), we have an isomorphism $j\colon \langle B, \in\rangle \cong \mathcal{M} \restriction \mathcal{L}_\in$, with B transitive. We define the $\mathcal{L}_{\mathcal{A}'}$-structure $\mathcal{B}'$ with universe B induced by j. Thus, we have that $j\colon \mathcal{B}' \cong \mathcal{M} \prec \mathcal{A}'$, whence $j\colon \mathcal{B}' \hookrightarrow \mathcal{A}'$ is elementary. Furthermore, since $|S \cap M| < \nu$, we have that $|Z| < \nu$. By the elementarity of j and since A, B, S are transitive, we have that Z is transitive. Let $\mathcal{B} := \mathcal{B}' \restriction \mathcal{L}_\mathcal{A}$. Therefore, since $\mathcal{A} = \mathcal{A}' \restriction \mathcal{L}_\mathcal{A}$, we have that $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ is elementary.

(2)⇒(1): Let us assume (2). Let $f\colon [\kappa]^{<\omega} \to \lambda$. Let $\mathcal{A} := \langle \kappa, \lambda, \in, f \restriction [\kappa]^n\rangle_{n\in\omega}$. By (2), there is a structure $\mathcal{B} = \langle B, Z, \in, g_n\rangle_{n\in\omega}$ with $Z \subseteq B$ transitive sets, $|B| = \mu$ and $|Z| < \nu$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary. By the elementarity of j, we have that $j``B$ is f-homogeneous. ∎

The previous lemma leads to the definition of the following classes of structures.

Definition 4.3.10. Let κ be a cardinal. $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,<\kappa}$ and $\mathcal{C}^{\mathrm{tr}}_{\kappa,<\kappa}$ denote the respective subclasses of structures of $\mathcal{C}_{\geq\kappa,<\kappa}$ and $\mathcal{C}_{\kappa,<\kappa}$ (see Definition 4.3.1) where both M and S are transitive sets.

Observation 4.3.11. Clearly, $\mathcal{C}^{\mathrm{tr}}_{\kappa,<\kappa} \subsetneq \mathcal{C}_{\kappa,<\kappa}, \mathcal{C}^{\mathrm{tr}}_{\geq\kappa,<\kappa} \subsetneq \mathcal{C}_{\geq\kappa,<\kappa}$.

Remark 4.3.12. The classes $\mathcal{C}^{\mathrm{tr}}_{\kappa,<\kappa}$ and $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,<\kappa}$ are Σ_1-definable with a cardinal κ as a parameter.

Proof. By Remark 4.3.2, plus the fact that the predicate "x is transitive" is Δ_0-expressible. ∎

And now we obtain a characterization of ν-Rowbottom cardinals also in terms of the 2CHSR principle.

Theorem 4.3.13. *Let $\kappa > \nu$ be uncountable cardinals. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\kappa,<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,<\kappa}$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\kappa,<\kappa}$, $\mathcal{C}_{\geq\kappa,<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is a ν-Rowbottom cardinal.

(2) $\kappa + 1 \models \mathrm{2CSR}(\mathcal{D}, \nu)$.

(3) $\kappa^+ \models \mathrm{2CHSR}(\mathcal{C}^{\mathrm{tr}}, \nu)$.

Proof. For the 2CSR cases: By Theorem 4.3.4, and by Definition 4.2.1, Proposition 4.1.4(4) and Lemma 4.3.9.

For the 2CHSR cases: By Definition 4.2.1, Proposition 4.1.4(4) and Lemma 4.3.9 (the structure $\mathcal{B} \in \mathcal{C}^{\text{tr}}$ with $Z \subseteq B$ transitive sets has H-type $\langle < \kappa^+, < \nu \rangle$ iff $|B| = \kappa$ and $|Z| < \nu$). And conversely, $\langle \kappa, \lambda, \in, f_n \rangle_{n \in \omega} \in \mathcal{C}^{\text{tr}}$. ∎

It is worth pointing out that κ being ν-Rowbottom renders the structural reflection of the classes $\mathcal{C}^{\text{tr}}_{\kappa,<\kappa}$ and $\mathcal{C}^{\text{tr}}_{\geq\kappa,<\kappa}$ in terms of 2CVSR, but not the converse, since the fact that $\text{rk}(S) < \nu$ for S a transitive set does not imply that $|S| < \nu$, which is imperative to obtain the ν-Rowbottom partition property.

Lemma 4.3.14. *Let $\mathcal{C}^{\text{tr}}$ denote either $\mathcal{C}^{\text{tr}}_{\kappa,<\kappa}$ or $\mathcal{C}^{\text{tr}}_{\geq\kappa,<\kappa}$. If κ is a ν-Rowbottom cardinal, then $\kappa^+ \models$ 2CVSR$(\mathcal{C}^{\text{tr}}, \nu)$.*

Proof. Let us assume that κ is ν-Rowbottom. By Theorem 4.3.13, we have that $\kappa^+ \models$ 2CHSR$(\mathcal{C}^{\text{tr}}, \nu)$, whence $\kappa^+ \models$ 2CVSR$(\mathcal{C}^{\text{tr}}, \nu)$, since $\mathcal{B} \in \mathcal{C}^{\text{tr}}$ with H-type $\langle < \alpha, < \beta \rangle$ implies $\mathcal{B}$ with V-type $\langle < \alpha, < \beta \rangle$. ∎

We can, however, obtain a full equivalent characterization in terms of 2CVSR, if we require the interpretation of the designated unary predicate symbol to be an ordinal.

Lemma 4.3.15. *Let κ, μ, λ, ν be cardinals with $\kappa > \lambda$, $\kappa \geq \mu \geq \omega$ and $\nu > \omega$. The following are equivalent:*

(1) $\kappa \longrightarrow [\mu]^{<\omega}_{\lambda,<\nu}$.

(2) For every structure $\mathcal{A} = \langle A, \alpha, \ldots \rangle$, with A transitive, $|A| = \kappa$ and $\alpha \subseteq A$ an ordinal such that $|\alpha| = \lambda$, there is a structure $\mathcal{B} = \langle B, \beta, \ldots \rangle$, with B transitive, $|B| = \mu$ and $\beta \subseteq B$ an ordinal such that $\beta < \nu$, and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Proof. By Lemma 4.3.9, plus the fact that, in (1)⇒(2), β is an ordinal by the elementarity of j. ∎

So, we add the extra requirement that the interpretation of the designated unary predicate symbol is an ordinal in the definition of the classes of transitive structures.

Definition 4.3.16. Let κ be a cardinal. $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,\alpha<\kappa}$ and $\mathcal{C}^{\mathrm{tr}}_{\kappa,\alpha<\kappa}$ denote the respective subclasses of structures of $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,<\kappa}$ and $\mathcal{C}^{\mathrm{tr}}_{\kappa,<\kappa}$ (see Definition 4.3.10) where S is some ordinal $\alpha < \kappa$.

Observation 4.3.17. Clearly, $\mathcal{C}^{\mathrm{tr}}_{\kappa,\alpha<\kappa} \subsetneq \mathcal{C}^{\mathrm{tr}}_{\kappa,<\kappa}, \mathcal{C}^{\mathrm{tr}}_{\geq\kappa,\alpha<\kappa} \subsetneq \mathcal{C}^{\mathrm{tr}}_{\geq\kappa,<\kappa}$.

Remark 4.3.18. The classes $\mathcal{C}^{\mathrm{tr}}_{\kappa,\alpha<\kappa}$ and $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,\alpha<\kappa}$ are Σ_1-definable with a cardinal κ as a parameter.

Proof. By Remark 4.3.12. ■

And we obtain a new characterization of ν-Rowbottom cardinals that includes the 2CVSR principle.

Theorem 4.3.19. *Let $\kappa > \nu$ be uncountable cardinals. Let $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\kappa,\alpha<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,\alpha<\kappa}$. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\kappa,<\kappa}$, $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\kappa,<\kappa}$, $\mathcal{C}_{\geq\kappa,<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is a ν-Rowbottom cardinal.

(2) $\kappa + 1 \models \mathrm{2CSR}(\mathcal{D}, \nu)$.

(3) $\kappa^+ \models \mathrm{2CHSR}(\mathcal{C}^{\mathrm{tr}}, \nu)$.

(4) $\kappa^+ \models \mathrm{2CVSR}(\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}, \nu)$.

Proof. For the 2CSR and 2CHSR cases: By Theorem 4.3.13, and by Definition 4.2.1, Proposition 4.1.4(4) and Lemma 4.3.15.

For the 2CVSR cases: By the 2CHSR case, we get $\mathcal{B} \in \mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$ with H-type $\langle <\kappa^+, <\nu\rangle$, which implies $\mathcal{B}$ with V-type $\langle <\kappa^+, <\nu\rangle$. And conversely, we have $\langle \kappa, \lambda, \in, f_n\rangle_{n\in\omega} \in \mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$ and we get $\mathcal{B} = \langle B, \beta, \ldots\rangle \in \mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$ with V-type $\langle <\kappa^+, <\nu\rangle$ and $j\colon \mathcal{B} \to \mathcal{A}$ elementary, whence $|B| = \kappa$ and $\beta < \nu$. ■

In addition to the classes of structures so far defined in this subsection, we can narrow the classes $\mathcal{C}^{\mathrm{tr}}_{\kappa,\alpha<\kappa}$ and $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,\alpha<\kappa}$ to the respective subclasses where the universe of the structure is an initial segment of some universe of sets constructible relative to a set. As a result, we obtain new equivalent characterizations of ν-Rowbottom cardinals in terms of the three 2C structural reflection principles with respect to those new classes of structures.

To that end, we first prove the following lemma, where, analogously to what we did in Lemma 3.3.15, we go by a condensation argument to obtain a characterization of the square-bracket partition property similar to the one obtained in Lemma 4.3.15, but here restricted to initial segments of some universe of sets constructible relative to a set.

Lemma 4.3.20. *Let κ, μ, λ, ν be cardinals with $\kappa > \lambda$, $\kappa \geq \mu \geq \omega$ and $\nu > \omega$. The following are equivalent:*

(1) $\kappa \longrightarrow [\mu]^{<\omega}_{\lambda,<\nu}$.

(2) For every structure $\mathcal{A} = \langle L_\alpha[X], \beta, \ldots \rangle$, with α, β ordinals, $|\alpha| = \kappa$, $|\beta| = \lambda$ and X some set, there is a structure $\mathcal{B} = \langle L_\gamma[X'], \delta, \ldots \rangle$, with γ, δ ordinals, $|\gamma| = \mu$, $\delta < \nu$ and X' some set, and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Proof. (1)⇒(2): Let us assume (1). Let $\mathcal{A} \coloneqq \langle L_\alpha[X], \beta, \ldots \rangle$, with α, β ordinals, $|\alpha| = \kappa$, $|\beta| = \lambda$ and X some set. Let $\mathcal{A}' \coloneqq \langle \mathcal{A}, \in, X \cap L_\alpha[X] \rangle$. Let us note that $\mathcal{L}_\in \subsetneq \mathcal{L}_\in(\dot{X}) \subsetneq \mathcal{L}_{\mathcal{A}'}$ and $\mathcal{L}_\mathcal{A} \subseteq \mathcal{L}_{\mathcal{A}'} = \mathcal{L}_\mathcal{A}(\dot{\in}, \dot{X})$, where $\dot{\in}$ is the binary relation symbol for the set-membership relation $\in$ and $\dot{X}$ is a unary predicate symbol. By Theorem 4.2.5, there is a structure

$$\mathcal{M} = \langle M, \beta \cap M, \in, X \cap L_\alpha \cap M, \ldots \rangle \prec \mathcal{A}',$$

with $|M| = \mu$. Therefore, by Lemma 2.2.9 (Generalized Gödel's Condensation Lemma), we have an isomorphism $j \colon \langle L_\gamma[X'], \in, X' \rangle \cong \mathcal{M} \restriction \mathcal{L}_\in(\dot{X})$, with $\gamma \leq \alpha$ an infinite ordinal and $X' = j^{-1}``(X \cap L_\alpha \cap M)$. By Proposition 2.2.4, $|L_\gamma[X']| = |\gamma|$. Therefore, since $M \sim L_\gamma[X]$, we have that $|\gamma| = \mu$. We define the $\mathcal{L}_{\mathcal{A}'}$-structure $\mathcal{B}' \coloneqq \langle L_\gamma[X'], j^{-1}``(\beta \cap M), \in, X', \ldots \rangle$ induced by j. Thus, we have that $j \colon \mathcal{B}' \cong \mathcal{M} \prec \mathcal{A}'$, whence $j \colon \mathcal{B}' \hookrightarrow \mathcal{A}'$ is elementary. Therefore, since $\beta \in \mathcal{A}'$ is an ordinal, we have that $j^{-1}``(\beta \cap M)$ is an ordinal. Let therefore $\delta \coloneqq j^{-1}``(\beta \cap M)$. Moreover, since $|\beta \cap M| < \nu$, we have that $\delta < \nu$. Let $\mathcal{B} \coloneqq \mathcal{B}' \restriction \mathcal{L}_\mathcal{A}$. Therefore, since $\mathcal{A} = \mathcal{A}' \restriction \mathcal{L}_\mathcal{A}$, we have that $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ is elementary.

(2)⇒(1): Let us assume (2). Let $f \colon [\kappa]^{<\omega} \to \lambda$. Let $\mathcal{A} \coloneqq \langle L_\kappa, \lambda, \in, f_n \rangle_{n\in\omega}$, where

$$\begin{aligned} f_n(a_1, \ldots, a_n) &\coloneqq f(\{a_1, \ldots, a_n\}), \text{ if } a_1, \ldots, a_n \in \kappa; \\ f_n(a_1, \ldots, a_n) &\coloneqq 0, \text{ otherwise.} \end{aligned}$$

By assumption, there is a structure $\mathcal{B} = \langle L_\gamma[X], \delta, \in, g_n \rangle_{n\in\omega}$, with γ, δ ordinals, $|\gamma| = \mu$, $\delta < \nu$ and X some set, and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary. Since $\mu \subseteq \gamma \subseteq L_\gamma[X]$, we have that $j``\mu \in [\kappa]^\mu$. Let $n \in \omega$. Let $\xi_1, \ldots, \xi_n \in j``\mu$. By the elementarity of j, we have that $f_n(\xi_1, \ldots, \xi_n) =$

$j(g_n(j^{-1}(\xi_1), \dots, j^{-1}(\xi_n)))$. Since $\text{ran}(g_n) \subseteq \delta$, we have that $f_n(\xi_1, \dots, \xi_n) \in j``\delta$. Therefore, $f``[j``\mu]^{<\omega} \subseteq j``\delta$. Since $|j``\delta| = |\delta| < \nu$, we have that $|f``[j``\mu]^{<\omega}| < \nu$. Hence, $j``\mu$ is f-homogeneous. ■

Observation 4.3.21. The proof of Lemma 4.3.20 shows that, if $X = \emptyset$, then $X' = \emptyset$.

Now, we define the "L-versions" of the classes $\mathcal{C}^{\text{tr}}_{\kappa,\alpha<\kappa}$ and $\mathcal{C}^{\text{tr}}_{\geq\kappa,\alpha<\kappa}$ of structures.

Definition 4.3.22. Let κ be a cardinal. $\mathcal{C}_{L_{\geq\kappa}[\cdot],<\kappa}$ and $\mathcal{C}_{L_{\geq\kappa},<\kappa}$ denote the subclasses of structures of $\mathcal{C}^{\text{tr}}_{\geq\kappa,<\kappa}$ (see Definition 4.3.10) whose respective universes are $L_\alpha[X]$ and L_α for some ordinal $\alpha \geq \kappa$ and some set X, and where S is some ordinal $\beta < \kappa$. Moreover, $\mathcal{C}_{L_\kappa[\cdot],<\kappa}$ and $\mathcal{C}_{L_\kappa,<\kappa}$ denote the respective subclasses of structures of $\mathcal{C}_{L_{\geq\kappa}[\cdot],<\kappa}$ and $\mathcal{C}_{L_{\geq\kappa},<\kappa}$ where $|\alpha| = \kappa$.

Observation 4.3.23. Clearly, $\mathcal{C}_{L_\kappa,<\kappa} \subsetneq \mathcal{C}_{L_{\geq\kappa},<\kappa}, \mathcal{C}_{L_\kappa[\cdot],<\kappa} \subsetneq \mathcal{C}_{L_{\geq\kappa}[\cdot],<\kappa}$.

Remark 4.3.24. The classes $\mathcal{C}_{L_\kappa,<\kappa}$, $\mathcal{C}_{L_{\geq\kappa},<\kappa}$, $\mathcal{C}_{L_\kappa[\cdot],<\kappa}$ and $\mathcal{C}_{L_{\geq\kappa}[\cdot],<\kappa}$ are Σ_1-definable with a cardinal κ as a parameter.

Proof. By Remarks 3.3.19 and 4.3.18. ■

And finally, we complete our characterization of the family of Rowbottom cardinals in terms of the 2C structural reflection principles by adding the L-classes of structures.

Theorem 4.3.25. *Let $\kappa > \nu$ be uncountable cardinals. Let $\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}$ denote either $\mathcal{C}^{\text{tr}}_{\kappa,\alpha<\kappa}$, $\mathcal{C}^{\text{tr}}_{\geq\kappa,\alpha<\kappa}$, $\mathcal{C}_{L_\kappa,<\kappa}$, $\mathcal{C}_{L_{\geq\kappa},<\kappa}$, $\mathcal{C}_{L_\kappa[\cdot],<\kappa}$ or $\mathcal{C}_{L_{\geq\kappa}[\cdot],<\kappa}$. Let $\mathcal{C}^{\text{tr}}$ denote either $\mathcal{C}^{\text{tr}}_{\kappa,<\kappa}$, $\mathcal{C}^{\text{tr}}_{\geq\kappa,<\kappa}$ or $\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\kappa,<\kappa}$, $\mathcal{C}_{\geq\kappa,<\kappa}$ or $\mathcal{C}^{\text{tr}}$. The following are equivalent:*

(1) κ is a ν-Rowbottom cardinal.

(2) $\kappa + 1 \models \text{2CSR}(\mathcal{D}, \nu)$.

(3) $\kappa^+ \models \text{2CHSR}(\mathcal{C}^{\text{tr}}, \nu)$.

(4) $\kappa^+ \models \text{2CVSR}(\mathcal{C}^{\text{tr}}_{\mathsf{Ord}}, \nu)$.

Proof. By Theorem 4.3.19 and Lemma 4.3.20. ■

Corollary 4.3.26. *Let κ be an uncountable cardinal. Let $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\kappa,\alpha<\kappa}$, $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,\alpha<\kappa}$, $\mathcal{C}_{L_\kappa,<\kappa}$, $\mathcal{C}_{L_{\geq\kappa},<\kappa}$, $\mathcal{C}_{L_\kappa[\cdot],<\kappa}$ or $\mathcal{C}_{L_{\geq\kappa}[\cdot],<\kappa}$. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\kappa,<\kappa}$, $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa,<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}}$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\kappa,<\kappa}$, $\mathcal{C}_{\geq\kappa,<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) *κ is a Rowbottom cardinal.*

(2) $\kappa+1 \models 2\mathrm{CSR}(\mathcal{D},\omega_1)$.

(3) $\kappa^+ \models 2\mathrm{CHSR}(\mathcal{C}^{\mathrm{tr}},\omega_1)$.

(4) $\kappa^+ \models 2\mathrm{CVSR}(\mathcal{C}^{\mathrm{tr}}_{\mathsf{Ord}},\omega_1)$.

Proof. By Definition 4.2.1 and Theorem 4.3.25. ■

We close this subsection by observing the following point.

Observation 4.3.27. In Theorem 4.3.25 and Corollary 4.3.26, the ordinal that witnesses the structural reflection principles is never κ, which is what one intuitively would expect to obtain. The reason for this lies in Theorem 4.2.6(2): in order to render the Rowbottom partition property, the cardinality of the universe of the structure $\mathcal{B}$ *must* be κ. Since the 2C structural reflection principles defined in this subsection have been conceived as two-cardinal variants of the SR, HSR and VSR principles originally provided by Bagaria and Lücke (2024) (see Definitions 2.3.1 to 2.3.3), the restriction on the type/H-type/V-type of $\mathcal{B}$ is $\langle <\alpha, <\beta\rangle$, which puts a *less than* α condition on B. Thus, if κ is ν-Rowbottom, then κ can never witness the 2C principle with parameter ν, since that would imply that the type/H-type/V-type of $\mathcal{B}$ is $\langle <\kappa, <\nu\rangle$, whence $|B| < \kappa$. Nonetheless, we can define the variants 2CSR* and 2CHSR*, where the type/H-type of $\mathcal{B}$ is $\langle \leq\kappa, <\nu\rangle$, and thus obtain variants of Theorem 4.3.25 and Corollary 4.3.26 with the 2CSR* and 2CHSR* principles with parameter ν witnessed by κ for all the classes of structures defined in this subsection.

After this observation, it is worth noticing that defining a variant 2CVSR* in the same fashion as 2CSR* and 2CHSR* would be pointless, since, working with V-types (*i.e.* in terms of ranks), the ordinal that would witness the hypothetical 2CVSR* principle with parameter ν would still be κ^+.

4.3.2 For-all-levels two-cardinal structural reflection

To close this chapter, we begin this subsection by highlighting the following observation.

Observation 4.3.28. The 2C structural reflection principles defined in the previous subsection do not exploit all the power of Theorem 4.2.6 and Lemmas 4.3.9, 4.3.15 and 4.3.20. Thanks to the preservation properties of the square-bracket partition relation (see Proposition 4.1.4(4)), the fact that κ is ν-Rowbottom yields that any structure of type $\langle \kappa', < \kappa \rangle$ *for any* cardinal $\kappa' \geq \kappa$ has embedded a substructure of type $\langle \mu, < \nu \rangle$ *for any* infinite cardinal $\mu \leq \kappa$, and vice versa (see Theorem 4.2.6(3)).

The previous observation indicates that Rowbottom cardinals can be characterized in terms of for-all-levels 2C structural reflection principles, conceived as two-cardinal variants of the ∀SR, ∀HSR and ∀VSR principles introduced in subsection 3.3.3 (see Definitions 3.3.36 to 3.3.38). Let us hence recall that, by Observation 3.3.35, these principles restrict the levels for the structure $\mathcal{A}$ to those larger than or equal to the reflection point α.

Therefore, by Observation 4.3.27, we next define the for-all-levels versions of the 2C structural reflection principles by taking as reference not the ∀SR, ∀HSR and ∀VSR principles just as introduced in Definitions 3.3.36 to 3.3.38, but their * variants mentioned in Observation 3.3.45.

Definition 4.3.29 (∀2CSR*). For β an ordinal, the ∀2CSR*$(\mathcal{C}, \beta)$ principle states: There is an infinite ordinal α such that, for every cardinal $\kappa \geq \alpha$ and for every infinite cardinal $\mu \leq \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C}$ with $|A| = \kappa$, there is a structure $\mathcal{B} \in \mathcal{C}$ with type $\langle \mu, < \beta \rangle$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Definition 4.3.30 (∀2CHSR*). For β an infinite cardinal, the ∀2CHSR* $(\mathcal{C}, \beta)$ principle states: There is an infinite cardinal α such that, for every cardinal $\kappa \geq \alpha$ and for every infinite cardinal $\mu \leq \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C} \cap (H_{\kappa^+} \setminus H_\kappa)$, there is a structure $\mathcal{B} \in \mathcal{C}$ with H-type $\langle \mu, < \beta \rangle$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Definition 4.3.31 (∀2CVSR*). For β an ordinal, the ∀2CVSR*$(\mathcal{C}, \beta)$ principle states: There is an infinite ordinal α such that, for every ordinal $\gamma \geq \alpha$ and for every infinite ordinal $\delta \leq \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C} \cap (V_{\gamma+1} \setminus V_\gamma)$, there is a structure $\mathcal{B} \in \mathcal{C}$ with V-type $\langle \mu, < \beta \rangle$ and there is $j \colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary.

Before providing an equivalent characterization of ν-Rowbottom cardinals in terms of ∀2C structural reflection principles, we observe three relevant points.

Observation 4.3.32.

1. Theorem 4.2.6 and Lemmas 4.3.9, 4.3.15 and 4.3.20 yield a for-all-levels variety of elementary embeddings in terms of the cardinalities of the universes of structures, but they do not yield such a variety in terms of ranks. Hence, we will be able to characterize ν-Rowbottom cardinals by means of ∀2CSR* and ∀2CHSR*, but not ∀2CVSR*.

2. The for-all-levels 2C principles, as previously defined, allow for the classes of structures involved in the structural reflection to have universes of arbitrary cardinality (*i.e.* not restricted by some parameter κ). As a consequence, we will define new versions of the classes of structures defined in the previous subsection.

3. Even so, the new classes of structures will be defined still using κ as a parameter. The reason for this is that, by Definition 4.2.1 and Proposition 4.1.4, all the variety of square-bracket partition properties that hold for κ a ν-Rowbottom cardinal requires for the type of structure $\mathcal{A}$ to be $\langle \kappa', < \kappa \rangle$, even when $\kappa' > \kappa$. Otherwise, the ν-Rowbottom partition property cannot yield an elementary substructure of the desired type.

Thus, we next define the new versions of the classes of structures introduced in the previous subsection.

Definition 4.3.33. Let κ be a cardinal. $\mathcal{C}_{\cdot,<\kappa}$, $\mathcal{C}^{\text{tr}}_{\cdot,<\kappa}$ and $\mathcal{C}^{\text{tr}}_{\cdot,\alpha<\kappa}$ denote the respective superclasses of structures of $\mathcal{C}_{\geq\kappa,<\kappa}$, $\mathcal{C}^{\text{tr}}_{\geq\kappa,<\kappa}$ and $\mathcal{C}^{\text{tr}}_{\geq\kappa,\alpha<\kappa}$ (see Definitions 4.3.1, 4.3.10 and 4.3.16) whose universes have arbitrary cardinality. Moreover, $\mathcal{C}_{L[\cdot],<\kappa}$ and $\mathcal{C}_{L,<\kappa}$ denote the respective superclasses of structures of $\mathcal{C}_{L_{\geq\kappa}[\cdot],<\kappa}$ and $\mathcal{C}_{L_{\geq\kappa},<\kappa}$ (see Definition 4.3.22) where α is some arbitrary ordinal.

Observation 4.3.34. Clearly, $\mathcal{C}_{L,<\kappa} \subsetneq \mathcal{C}_{L[\cdot],<\kappa} \subsetneq \mathcal{C}^{\text{tr}}_{\cdot,\alpha<\kappa} \subsetneq \mathcal{C}^{\text{tr}}_{\cdot,<\kappa} \subsetneq \mathcal{C}_{\cdot,<\kappa}$.

Remark 4.3.35. The classes $\mathcal{C}_{\cdot,<\kappa}$, $\mathcal{C}^{\text{tr}}_{\cdot,<\kappa}$, $\mathcal{C}^{\text{tr}}_{\cdot,\alpha<\kappa}$, $\mathcal{C}_{L[\cdot],<\kappa}$ and $\mathcal{C}_{L,<\kappa}$ are Σ_1-definable with a cardinal κ as a parameter.

Proof. By Remarks 4.3.2, 4.3.12, 4.3.18 and 4.3.24. ■

And finally, we close this subsection (and this chapter) by giving the equivalent characterizations of the family of Rowbottom cardinals in terms of the ∀2CSR* and ∀2CHSR* principles.

Theorem 4.3.36. *Let $\kappa > \nu$ be uncountable cardinals. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\cdot,<\kappa}$, $\mathcal{C}^{\mathrm{tr}}_{\cdot,\alpha<\kappa}$, $\mathcal{C}_{L[\cdot],<\kappa}$ or $\mathcal{C}_{L,<\kappa}$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\cdot,<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is a ν-Rowbottom cardinal.

(2) $\kappa \models \forall 2CSR^*(\mathcal{D}, \nu)$.

(3) $\kappa \models \forall 2CHSR^*(\mathcal{C}^{\mathrm{tr}}, \nu)$.

Proof. By Theorem 4.2.6(3) plus Theorems 4.3.4, 4.3.13, 4.3.19 and 4.3.25. ■

Corollary 4.3.37. *Let κ be an uncountable cardinal. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\cdot,<\kappa}$, $\mathcal{C}^{\mathrm{tr}}_{\cdot,\alpha<\kappa}$, $\mathcal{C}_{L[\cdot],<\kappa}$ or $\mathcal{C}_{L,<\kappa}$. Let $\mathcal{D}$ denote either $\mathcal{C}_{\cdot,<\kappa}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) κ is a Rowbottom cardinal.

(2) $\kappa \models \forall 2CSR^*(\mathcal{D}, \omega_1)$.

(3) $\kappa \models \forall 2CHSR^*(\mathcal{C}^{\mathrm{tr}}, \omega_1)$.

Proof. By Definition 4.2.1 and Theorem 4.3.36. ■

Chapter 5

Structural reflection for Jónsson cardinals

In this chapter, we provide equivalent characterizations of Jónsson cardinals in terms of new structural reflection principles. Jónsson cardinals can be characterized by means of the square-bracket partition properties we already introduced in section 4.1. Thus, we start by briefly introducing the notion of a Jónsson cardinal and its most prominent features in section 5.1. Then, we define a new family of structural reflection principles in section 5.2 and show how it characterizes Jónsson cardinals. Finally, in section 5.3, we show the connection between one of the new structural reflection principles introduced in the previous section and the notion of exacting cardinals.

5.1 Jónsson algebras and Jónsson cardinals

We next provide a brief introduction to Jónsson cardinals. More detailed accounts of this large cardinal notion can be found in Kanamori (2003, p. 92–98). Since Jónsson cardinals were originally introduced in an algebraic setting, we start by giving the definition of the notions of algebra and subalgebra.

Definition 5.1.1.

- An *algebra* is any structure of the form $\langle M, f_i \rangle_{i \in I}$, where $|I| \leq \omega$ and, for every $i \in I$, $f_i \colon M^n \to M$ for some $n \in \omega$.

- Given an algebra $\mathcal{M} = \langle M, f_i \rangle_{i \in I}$, a *subalgebra* of $\mathcal{M}$ is any structure of the form $\langle M', f_i \upharpoonright M' \rangle_{i \in I}$, where $M' \subseteq M$ and, for every $i \in I$, $f_i \upharpoonright M' \subseteq M'$.

By all accounts, Bjarni Jónsson asked in 1962 the following question concerning finite algebras (Nation, 2018):

> For which cardinals κ can one find an algebra $\mathcal{A}$ of finite type with $|A| = \kappa$, but $|B| < \kappa$ for every proper subalgebra $\mathcal{B} < \mathcal{A}$?

Although he wrote no papers on the subject, the question was posed in the context of his works on transfinite universal algebra (Jónsson, 1972) and the notion of a "Jónson algebra" was quickly generalized to a standard model-theoretic setting with countable languages.

Definition 5.1.2. A *Jónsson algebra* is an algebra without a proper subalgebra of the same cardinality.

And then we get to define the notion of a Jónsson cardinal.

Definition 5.1.3. κ is a *Jónsson cardinal* iff there are no Jónsson algebras of cardinality κ (*i.e.* iff every algebra of cardinality κ has a proper subalgebra of the same cardinality).

We next show how the algebraic conception of Jónsson cardinals can be equivalently characterized both in the strictly combinatorial terms of a square-bracket partition property (Erdős and Hajnal, 1966) and in model-theoretical terms similar to those employed to characterize Rowbottom cardinals (Keisler and Rowbottom, 1965). This result is the one that will be useful in the next section in order to obtain equivalent characterizations of Jónsson cardinals in terms of structural reflection principles.

Theorem 5.1.4. *Let κ be a cardinal. The following are equivalent:*

(1) κ is a Jónsson cardinal.

(2) $\kappa \longrightarrow [\kappa]^{<\omega}_{\kappa}$.

(3) Every structure of cardinality κ has a proper elementary substructure of the same cardinality

Proof. See Kanamori (2003, p. 93). ∎

We close this section with a brief summary of some well-known facts about Jónsson cardinals and their relation with both Rowbottom and Ramsey cardinals. On the one hand, as shown by Kanamori (2003) in Proposition 8.14, Jónsson cardinals are uncountable (*i.e.* ω is not Jónsson), the least Jónsson cardinal (if it exists) is either weakly inaccessible or singular with cofinality ω and, if κ is not Jónsson, then neither is κ^+ (*i.e.* the first possible candidate to be consistently Jónsson is $\aleph_\omega$). Moreover, ν-Rowbottom cardinal are Jónsson (Keisler and Rowbottom, 1965) and the least Jónsson cardinal (if it exists) is a ν-Rowbottom cardinal for some $\nu < \kappa$ (Kleinberg, 1973). Furthermore, if κ is a Jónsson cardinal, then $x^{\#}$ exists for every $x \in V_\kappa$ (Kanamori, 2003, p. 277 and 282), whence the existence of a Jónsson cardinal implies that $V \neq L$ (Keisler and Rowbottom, 1965). And on the other hand, the existence of a Jónsson cardinal is equiconsistent both to the existence of a Rowbottom cardinal (Kleinberg, 1979) and to the existence of a Ramsey cardinal (Mitchell, 1999).

5.2 Structural reflection principles for Jónsson cardinals

In terms of structural reflection principles that may provide an equivalent characterization of Jónsson cardinals, our first observation is that the situation is highly analogous to the one with Rowbottom cardinals. By Theorem 5.1.4, we observe that the Jónsson partition property requires that the structures involved in the elementary embedding are of the same cardinality. Therefore, we conceive two possible strategies here:

1. In the definitions of the classes of structures to be reflected, we limit the cardinality of the structures to be at least κ. Then, a slight modification of the basic structural reflection principles (namely, we require that the range of j is a proper subset of A) will do the job. We follow this strategy in subsection 5.2.1.

2. We work with "for-all-levels" versions of the structural reflection principles employed in the first strategy, whence the cardinality of the structures need no longer be limited by κ in the definitions of the classes of structures to be reflected. We follow this strategy in subsection 5.2.2.

5.2.1 Proper structural reflection

Given that the situation is analogous to the one that we have already studied in subsection 4.3.1 with respect to Rowbottom cardinals, we directly provide the definitions of the classes of structures involved in this subsection. We point out that they are the same classes defined in subsection 4.3.1 but without the requirement of the presence of a designated unary predicate symbol in the language of the structures.

Definition 5.2.1. Let κ be a cardinal. $\mathcal{C}_{\geq\kappa}$, $\mathcal{C}^{\text{tr}}_{\geq\kappa}$, $\mathcal{C}^{\text{tr}}_{\kappa}$ and $\mathcal{C}_{\kappa}$ denote the respective superclasses of structures of $\mathcal{C}_{\geq\kappa,<\kappa}$, $\mathcal{C}^{\text{tr}}_{\geq\kappa,<\kappa}$, $\mathcal{C}^{\text{tr}}_{\kappa,<\kappa}$ and $\mathcal{C}_{\kappa,<\kappa}$ (see Definitions 4.3.1 and 4.3.10) defined without the bits $S \subseteq M$ and $|S| < \kappa$. Moreover, $\mathcal{C}_{L_{\geq\kappa}[\cdot]}$, $\mathcal{C}_{L_{\kappa}[\cdot]}$, $\mathcal{C}_{L_{\geq\kappa}}$ and $\mathcal{C}_{L_{\kappa}}$ denote the respective superclasses of structures of $\mathcal{C}_{L_{\geq\kappa}[\cdot],<\kappa}$, $\mathcal{C}_{L_{\kappa}[\cdot],<\kappa}$, $\mathcal{C}_{L_{\geq\kappa},<\kappa}$ and $\mathcal{C}_{L_{\kappa},<\kappa}$ (see Definition 4.3.22) defined without the bit $\beta < \kappa$.

Observation 5.2.2. Clearly, $\mathcal{C}_{\kappa} \subsetneq \mathcal{C}_{\geq\kappa}, \mathcal{C}^{\text{tr}}_{\kappa} \subsetneq \mathcal{C}^{\text{tr}}_{\geq\kappa}$ and $\mathcal{C}_{L_{\kappa}} \subsetneq \mathcal{C}_{L_{\geq\kappa}}, \mathcal{C}_{L_{\kappa}[\cdot]} \subsetneq \mathcal{C}_{L_{\geq\kappa}[\cdot]} \subsetneq \mathcal{C}^{\text{tr}}_{\geq\kappa}$.

Remark 5.2.3. The classes $\mathcal{C}_{\kappa}$, $\mathcal{C}_{\geq\kappa}$, $\mathcal{C}^{\text{tr}}_{\kappa}$, $\mathcal{C}^{\text{tr}}_{\geq\kappa}$, $\mathcal{C}_{L_{\kappa}}$, $\mathcal{C}_{L_{\geq\kappa}}$, $\mathcal{C}_{L_{\kappa}[\cdot]}$ and $\mathcal{C}_{L_{\geq\kappa}[\cdot]}$ are Σ_1-definable with a cardinal κ as a parameter.

Proof. By Remarks 4.3.2, 4.3.12 and 4.3.24. ■

Next, we directly define the new family of structural reflection principles involved in this subsection. They are *proper* versions of the basic principles originally defined by Bagaria and Lücke (2024) (see Definitions 2.3.1 to 2.3.3); *i.e.* just like those, but with the addition of the requirement that the range of j is a proper subset of A, which is the "properness" that Jónsson cardinals require when the structures involved in the elementary embedding share the same cardinality.

Definition 5.2.4 (PrSR). The PrSR($\mathcal{C}$) principle states: There is an ordinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| < \alpha$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary with $\operatorname{ran}(j) \subsetneq A$.

Definition 5.2.5 (PrHSR). The PrHSR($\mathcal{C}$) principle states: There is an infinite cardinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C} \cap H_{\alpha}$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary with $\operatorname{ran}(j) \subsetneq A$.

Definition 5.2.6 (PrVSR). The PrVSR($\mathcal{C}$) principle states: There is an ordinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$, there is a structure $\mathcal{B} \in \mathcal{C} \cap V_{\alpha}$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary with $\operatorname{ran}(j) \subsetneq A$.

And finally, we provide the characterization of Jónsson cardinals in terms of proper structural reflection principles applied to the classes of structures previously defined.

Theorem 5.2.7. *Let κ be a cardinal. Let $\mathcal{C}^{\mathrm{tr}}$ denote either $\mathcal{C}^{\mathrm{tr}}_{\kappa}$, $\mathcal{C}^{\mathrm{tr}}_{\geq\kappa}$, $\mathcal{C}_{L_\kappa}$, $\mathcal{C}_{L_{\geq\kappa}}$, $\mathcal{C}_{L_\kappa[\cdot]}$ or $\mathcal{C}_{L_{\geq\kappa}[\cdot]}$. Let $\mathcal{D}$ denote either $\mathcal{C}_\kappa$, $\mathcal{C}_{\geq\kappa}$ or $\mathcal{C}^{\mathrm{tr}}$. The following are equivalent:*

(1) *κ is a Jónsson cardinal.*

(2) $\kappa + 1 \models \mathrm{PrSR}(\mathcal{D})$.

(3) $\kappa^+ \models \mathrm{PrHSR}(\mathcal{C}^{\mathrm{tr}})$.

(4) $\kappa^+ \models \mathrm{PrVSR}(\mathcal{C}^{\mathrm{tr}})$.

Proof. By Theorem 5.1.4 and Proposition 4.1.4(3), we obtain modified versions of Lemmas 4.3.9 and 4.3.20 where the "$< \nu$ condition" (with respect to the interpretations of the designated unary predicate symbol in the language of the structures) is substituted with the "proper condition" (*i.e.* the fact that $\mathrm{ran}(j) \subsetneq A$) and the rest is just the same. And then we obtain the present equivalencies in terms of Jónsson cardinals and proper structural reflection principles by Theorems 4.3.4, 4.3.13, 4.3.19 and 4.3.25. ■

Similarly to what we did in subsection 4.3.1, we close this subsection by stressing a point analogous to Observation 4.3.27.

Observation 5.2.8. We can define the variants PrSR* and PrHSR*, with $|B| \leq \alpha$ and $\mathcal{B} \in H_{\alpha^+}$, respectively, and thus obtain a variant of Theorem 5.2.7 with the PrSR* and PrHSR* principles witnessed by κ for all the classes of structures defined in this subsection. And, again, this would imply no changes in the case of PrVSR.

5.2.2 For-all-levels proper structural reflection

We proceed in this subsection just like in the previous one, since the situation is similarly analogous to that of subsection 4.3.2 with respect to Rowbottom cardinals and the for-all-level versions of the structural reflection principles previously defined.

Thus, we first directly define the classes of structures involved in this subsection. Since we are also going to use them here, let us recall that the classes $\mathcal{C}_{L[\cdot]}$ and $\mathcal{C}_L$ were already introduced in Definition 3.3.17.

Definition 5.2.9. The class $\mathcal{C}_\cdot$ denotes the class of all structures (*i.e.* the class of first-order structures of countable type). Moreover, the class $\mathcal{C}_\cdot^{\mathrm{tr}}$ denotes the class of structures whose universe is transitive.

Observation 5.2.10. Clearly, $\mathcal{C}_L \subsetneq \mathcal{C}_{L[\cdot]} \subsetneq \mathcal{C}_\cdot^{\mathrm{tr}} \subsetneq \mathcal{C}_\cdot$.

Remark 5.2.11. The classes $\mathcal{C}_\cdot$ and $\mathcal{C}_\cdot^{\mathrm{tr}}$ are Σ_1-definable.

Proof. The fact that the type of the structure is countable is Σ_1-expressible and the rest is Δ_0-expressible. ■

Next, we directly define the for-all-levels versions of the proper structural reflection principles presented in the previous subsection. Again, we include the point made in Observation 5.2.8 in our definitions and we hence take as a reference the ∀2CSR*, ∀2VCSR* and ∀2HCSR* principles defined in subsection 4.3.2.

Definition 5.2.12 (∀PrSR*). The ∀PrSR*($\mathcal{C}$) principle states: There is an infinite ordinal α such that, for every cardinal $\kappa \geq \alpha$ and for every infinite cardinal $\mu \leq \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C}$ with $|A| = \kappa$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| = \mu$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary with $\mathrm{ran}(j) \subsetneq A$.

Definition 5.2.13 (∀PrHSR*). The ∀PrHSR*($\mathcal{C}$) principle states: There is an infinite cardinal α such that, for every cardinal $\kappa \geq \alpha$ and for every infinite cardinal $\mu \leq \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C} \cap (H_{\kappa^+} \setminus H_\kappa)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (H_{\mu^+} \setminus H_\mu)$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary with $\mathrm{ran}(j) \subsetneq A$.

Definition 5.2.14 (∀PrVSR*). The ∀PrVSR*($\mathcal{C}$) principle states: There is an infinite ordinal α such that, for every ordinal $\gamma \geq \alpha$ and for every infinite ordinal $\delta \leq \alpha$, we have that, for every structure $\mathcal{A} \in \mathcal{C} \cap (V_{\gamma+1} \setminus V_\gamma)$, there is a structure $\mathcal{B} \in \mathcal{C} \cap (V_{\delta+1} \setminus V_\delta)$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary with $\mathrm{ran}(j) \subsetneq A$.

Finally, we close this subsection by providing the characterization of Jónsson cardinals in terms of proper structural reflection principles applied to the classes of structures previously defined. For the same reasons already explained in Observation 4.3.32, we next characterize Jónsson cardinals by means of ∀PrSR* and ∀PrHSR*, but not ∀PrVSR*

Theorem 5.2.15. *Let κ be a cardinal. Let $\mathcal{C}^{\text{tr}}$ denote either $\mathcal{C}^{\text{tr}}_{.}$, $\mathcal{C}_L$ or $\mathcal{C}_{L[\cdot]}$. Let $\mathcal{D}$ denote either $\mathcal{C}_{.}$ or $\mathcal{C}^{\text{tr}}$. The following are equivalent:*

(1) *κ is a Jónsson cardinal.*

(2) $\kappa \models \forall PrSR^*(\mathcal{D})$.

(3) $\kappa \models \forall PrHSR^*(\mathcal{C}^{\text{tr}})$.

Proof. By Proposition 4.1.4 and Theorem 5.1.4. ■

5.3 Strong *Jónssonness*

To close this chapter, we connect the PrSR* principle defined in Observation 5.2.8 with the notion of an exacting cardinal, which is a very strong large cardinal notion introduced by Aguilera et al. (2024, p. 6–10).

Definition 5.3.1. A cardinal κ is *exacting* iff, for every $\zeta > \kappa$, there is $X \prec V_\zeta$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and there is $j\colon X \hookrightarrow V_\zeta$ elementary with $j(\kappa) = \kappa$ and $j \restriction \kappa \neq \mathrm{id}_\kappa$.

The existence of exacting cardinals is consistent under the existence of an I0 embedding (Aguilera et al., 2024, p. 10–12), which is one of the strongest large cardinal notions not known to be inconsistent with ZFC. Interestingly, the existence of an exacting cardinal implies that $V \neq HOD$ (Aguilera et al., 2024, p. 12). And, moreover, the consistency of ZFC with an exacting cardinal above an extendible cardinal refutes Woodin's HOD Conjecture and Ultimate-L Conjecture (Aguilera et al., 2024, p. 34–40).

The reason for our interest in exacting cardinals is that they happen to be equivalent to a strong form of *Jónssonness*, as the next theorem shows.

Theorem 5.3.2. *Let $\kappa \in C^{(1)}$. The following are equivalent:*

(1) *κ is an exacting cardinal.*

(2) *For every class $\mathcal{D}$ of structures that is definable by a formula with parameters contained in $V_\kappa \cup \{\kappa\}$, every structure of cardinality κ in $\mathcal{D}$ contains a proper elementary substructure of cardinality κ that is isomorphic to a structure in $\mathcal{D}$.*

Proof. See Corollary 2.8 and the proofs of Propositions 2.6 and 2.7 in Aguilera et al. (2024, p. 8–10). ■

We notice that the statement (2) in the previous theorem is basically the model-theoretic characterization of a Jónsson cardinal κ generalized to any class of structures definable with parameters in $V_\kappa \cup \{\kappa\}$. Thus, we next define a version of the PrSR* principle *restricted to structures of the same cardinality* and then we generalize it by applying the same kind of generalization of the SR principle introduced in Definition 2.3.5.

Definition 5.3.3 (PrSR$^-$). The PrSR$^-$($\mathcal{C}$) principle states: There is a cardinal α such that, for every structure $\mathcal{A} \in \mathcal{C}$ with $|A| = \alpha$, there is a structure $\mathcal{B} \in \mathcal{C}$ with $|B| = \alpha$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary with $\mathrm{ran}(j) \subsetneq A$.

Definition 5.3.4 ($\Sigma_n(A)$-PrSR$^-$). For $n \in \omega$ and A a set, the principle $\Sigma_n(A)$-PrSR$^-$ states: There is an ordinal α such that, for every Σ_n-definable, with parameters in A, class $\mathcal{D}$ of structures, $\alpha \models$ PrSR$^-$($\mathcal{D}$).

And now, we show that the $\Sigma_n(A)$-PrSR$^-$ principle yields an equivalent characterization of exacting cardinals.

Theorem 5.3.5. *Let $\kappa \in C^{(1)}$. The following are equivalent:*

(1) κ is an exacting cardinal.

(2) $\kappa \models \Sigma_n(V_\kappa \cup \{\kappa\})$-PrSR$^-$, for every $n \in \omega$.

Proof. (1)⇒(2): Let us assume (1). Let $n \in \omega$. Let $\mathcal{C}$ be a Σ_n-definable, with parameters in $V_\kappa \cup \{\kappa\}$, class of structures. Let $\mathcal{A} \in \mathcal{C}$ be with $|A| = \kappa$. By Theorem 5.3.2, there is $\mathcal{A}' \prec \mathcal{A}$, with $|A'| = \kappa$ and $A' \subsetneq A$, and there is $\mathcal{B} \in \mathcal{C}$ such that $\mathcal{B} \cong \mathcal{A}'$. The isomorphic map between $\mathcal{B}$ and $\mathcal{A}'$ is an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$ whose range is A'.

(2)⇒(1): Let us assume (2). Let $\mathcal{C}$ be a class of structures defined by a formula with parameters in $V_\kappa \cup \{\kappa\}$. Let $\mathcal{A} \in \mathcal{C}$ be with $|A| = \kappa$. Let Σ_n be the complexity of the formula that defines $\mathcal{C}$, for some $n \in \omega$. By (2), there is $\mathcal{B} \in \mathcal{C}$ with $|B| = \kappa$ and there is $j\colon \mathcal{B} \hookrightarrow \mathcal{A}$ elementary with $\mathrm{ran}(j) \subsetneq A$. Hence, $\mathrm{ran}(j)$ is the universe of a proper elementary substructure $\mathcal{A}'$ of $\mathcal{A}$ of cardinality κ and $j\colon \mathcal{B} \cong \mathcal{A}'$. Therefore, by Theorem 5.3.2, κ is an exacting cardinal. ■

This result clearly suggests the interest of the study of possible generalizations of the family of proper structural reflection principles introduced in this chapter, as they might yield new large cardinal notions of high consistency strength and interesting properties. Moreover, the study of such generalizations could also be extended to the families of invariant and two-cardinal structural reflection principles introduced in chapters 3 and 4, respectively, as they might also lead to interesting new strong forms of both *Ramseyness* and *Rowbottomness*.

Chapter 6

Conclusions and open questions

In the present work, we have shown how families of large cardinals defined in terms of partition properties (namely, Erdős, Ramsey, Rowbottom and Jónsson cardinals) are equivalent to different kinds of structural reflection principles applied to a variety of classes of structures. In particular, we have introduced three new families of structural reflection principles:

1. Invariant structural reflection principles yield equivalent characterizations of Erdős and Ramsey cardinals.
2. Two-cardinal structural reflection principles yield equivalent characterizations of Rowbottom cardinals.
3. Proper structural reflection principles yield equivalent characterizations of Jónsson cardinals.

Furthermore, at the end of the work, we have shown that a particular generalization of a proper structural reflection principle yields an equivalent characterization of exacting cardinals, which are equivalent to a sort of "strong Jónsson" cardinals.

Interestingly, we have observed that every large cardinal notion studied in this work fails to yield the level-by-level and for-all-levels versions of their respective structural reflection principles in terms of ranks (*i.e.* the V-versions of the principles). We consider that this fact suggests a few questions regarding such principles. Furthermore, we also consider that the characterization of exacting cardinals observed at the end of the work suggests a few more questions that may lead further research in the Structural Reflection program. We next pose all of such open questions.

Question 6.1. What Erdős/Ramsey-like large cardinal notion (if any) is yielded by the LIVSR principle?

Question 6.2. What Erdős/Ramsey-like large cardinal notion (if any) is yielded by the ∀IVSR principle?

Question 6.3. What Rowbottom-like large cardinal notion (if any) is yielded by the ∀2CVSR* principle?

Question 6.4. What Jónsson-like large cardinal notion (if any) is yielded by the ∀PrVSR* principle?

Question 6.5. Let us consider a $\Sigma_n(A)$-2CSR$^-$ principle analogous to the $\Sigma_n(A)$-PrSR$^-$ principle. Does the former yield a strong form of *Rowbottomness* analogous to the strong form of *Jónssonness* yielded by the latter?

Question 6.6. Let us consider a $\Sigma_n(A)$-ISR$^-$ principle analogous to the $\Sigma_n(A)$-PrSR$^-$ principle. Does the former yield a strong form of *Ramseyness* analogous to the strong form of *Jónssonness* yielded by the latter?

Question 6.7. What large cardinal notions (if any) are yielded by the Σ_n and Π_n (without and with parameters) generalizations of the LSR and LHSR principles?

Question 6.8. What large cardinal notions (if any) are yielded by the Σ_n and Π_n (without and with parameters) generalizations of the ∀SR, ∀HSR and ∀VSR principles?

Question 6.9. What large cardinal notions (if any) are yielded by the Σ_n and Π_n (without and with parameters) generalizations of the LISR, LIHSR and LIVSR principles?

Question 6.10. What large cardinal notions (if any) are yielded by the Σ_n and Π_n (without and with parameters) generalizations of the ∀ISR, ∀IHSR and ∀IVSR principles?

Question 6.11. What large cardinal notions (if any) are yielded by the Σ_n and Π_n (without and with parameters) generalizations of the ∀2CSR*, ∀2CHSR* and ∀2CVSR* principles?

Question 6.12. What large cardinal notions (if any) are yielded by the Σ_n and Π_n (without and with parameters) generalizations of the $\forall$PrSR*, $\forall$PrHSR* and $\forall$PrVSR* principles?

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