Mean-field Burridge-Knopoff model for understanding earthquakes

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Abstract: In this paper, I study avalanches in a mean-field spring-block model to simulate earthquake dynamics. The model is implemented in *Fortran90*, and the equations of motion are solved using a fourth-order Runge-Kutta method. The study begins with the analysis of a single block, where the system behaves deterministically, and the elastic rebound theory is recovered. Later, I consider a system of two interacting blocks. The introduction of interactions leads to the emergence of complex dynamics, and a period-doubling bifurcation appears as heterogeneity increases. Because of complex behaviour, a statistical analysis of earthquakes is performed using up to 400 blocks. Then, I observe that the magnitude distribution, related to the logarithm of the released energy, exhibits scale invariance, consistent with a power-law behaviour. In contrast, the recurrence time between earthquakes follows an exponential distribution, which is characteristic of a Poisson process, suggesting that earthquakes are statistically independent.
Keywords: Criticallity, mean-field, power-law behaviour, transition to chaos, RK4 method solving
SDGs: 3. Health and well-being, 4. Quality education.

I. INTRODUCTION

Earthquakes are a direct consequence of Earth's crust deformation. In some areas, fractures arise with deformation leading to faults. Earthquakes are mainly associated to the stick-slip behaviour of those stressed faults. The Earth's crust is a complex highly heterogeneous and interacting system with non-linear dynamics, and so complexity is an inherent part of it. Because internal details are hard to assess we cannot pretend to carry out deterministic predictions of earthquakes. Consequently, statistical models based on empirical observations are often employed. It is well known that earthquake statistics follow simple laws. One of the most famous is the Gutenberg-Ritcher law, introduced by seismologists Beno Gutenberg and Charles Francis Ritcher in 1956. This law states that the frequency of occurrence of earthquakes in terms of their magnitude in a given region or fault follows the power-law

$$ln\dot{N} = -bm + ln\dot{a}, \quad (1)$$

where $\dot{N}$ is the number of earthquakes per unit time with a magnitude greater than m and $\dot{a}$ and b are constants. Given eq. (1), the same behaviour can be obtained for the rate of events that fall in the magnitude range $[m, m + dm]$. The magnitude m of an event is $m = lnM$, where M is its moment and it is calculated as the sum of total displacement of slipping blocks during and event, $M = \sum_{i=1}^{N} \Delta y_i$. Thus, the magnitude is related to the logarithm of energy and that is why exhibits scale-invariance, because energy follows indeed a power-law. That is a sign of Self-Organized Criticallity (SOC), by which the system tends to be in a critical state. Therefore, there is no typical event size, and a wide range of magnitudes can occur.

A deterministic approach is provided by the elastic rebound theory, established by Henry Fielding Reid, and primarily based on his observations of the displacement of the ground surface after the 1906 San Francisco earthquake. According to it, opposite plates in a fault are pushed together and forced to move in opposite directions along their line of contact [3], generating compressive and shear stresses. These stresses accumulate until the local stress exceeds the static friction and it is suddenly released, resulting in an earthquake event. However, because deterministic models fail to reproduce the statistical features of seismic activity, alternative approaches are needed. The Burridge-Knopoff (BK) model used in this paper incorporate interactions, which give rise to complex dynamics. As will be shown below, complexity emerges in the two-block BK model as heterogeneity is increased: the recurrence time of events goes to chaos through period-doubling bifurcations. Moreover, for a large number of blocks, the system exhibits power-law behaviour. Another approach is the Olami-Feder-Christensen model (OFC), based on the idea of cellular automaton and dated from 1992. It is derives from the two-dimensional BK and elements are arranged in a square grid, where each cell represents a block. The difference with the BK model is that in the OFC model slipping is discrete, which means that slipping blocks first jump to their final stable position and then, the stress on their neighbours is recomputed by distributing the original strain stored in the slipped blocks. For a large system, this model shows SOC, leading to a power-law behaviour.

The following sections are structured as follows. Section 2 introduces the Burridge–Knopoff (BK) model, including its dynamic equations, friction law, and dimensionless units. It also describes how numerical simulations are implemented. Section 3 covers the simulation results, including deterministic dynamics (single block), the transition to chaos (two blocks), more complex interactions (few blocks), and statistical behaviour (many blocks). Finally, Section 4 presents the conclusions drawn from the study.

II. THE MODEL

The BK model is a simple deterministic model proposed by R. Burridge and L. Knopoff in 1967 [1] representing an earthquake fault. The one-dimensional BK model consists of a chain of blocks of mass m lying on a rough surface and connected to each other by mean of springs with stiffness k_c and to a driving plate by springs of stiffness k_p as illustrated in Fig. 1. The driving plate moves at a constant slow velocity v' loading the system, simulating the external forces acting on a fault. The compressive and shear stresses applied are then stored as elastic energy in the springs until it is dissipated by the friction force when a block slips. In the mean-field model studied in this paper, all blocks are connected to each other beyond the neighbouring blocks and the stiffness constant between blocks k_c is normalized by $N-1$.

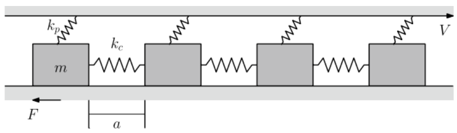

FIG. 1: The one-dimensional Burridge-Knopoff model. This picture is from [5].

The equation of motion for a bloc i is

$$m\ddot{Y_i} = k_p(v't' - Y_i) + \frac{k_c}{N-1}\left(\sum_{i=1}^{N} Y_i - NY_i\right) - F_i(\dot{Y_i}), \tag{2}$$

where Y_i is the displacement of a block from the initial position and F_i the friction force.

$$F_i(\dot{Y_i}) = \begin{cases} (-\infty, F_{0,i}], & \dot{Y_i} \leq 0, \\ \frac{F_{0,i}(1-\sigma)}{1+2\mu\dot{Y_i}/(1-\sigma)}, & \dot{Y_i} > 0. \end{cases} \tag{3}$$

The only nonlinear term of eq. (2) comes from the stick-slip friction law, which in this case is the so-called *velocity-weakening* friction in its asymmetric form, *i.e.*, with back-slip $\dot{Y} < 0$ prevented. In the mean-field BK model with all blocks coupled, to obtain chaos it is necessary to introduce disorder by setting different static friction values $F_{0,i}$ per each block. During stick, the net sum of the elastic forces is totally balanced by the static friction but when it exceeds $F_{0,i}$ a block starts slipping. Then it turns into dynamic friction. Some parameters are taken into account: σ is the acceleration of a block when it starts slipping [3] and it ensures that a block starts to slip with a certain velocity. Conversely, the block would initiate slipping with an indefinitely slow velocity and that is a problem when doing numerical calculations. In this paper, the parameter σ is set to be 10^{-2}. On the other hand, parameter α determines how quickly dynamic friction decays as velocity is increased.

A. Dimensionless units

Eq. (2) can be expressed in dimensionless units: $\omega = \sqrt{k_p/m}$ is the frequency of a block attached to a spring of the driving plate. $L = F_0/k_p$ is the maximum elongation of the spring attached to the driving plate before a block with a reference static friction of F_0 slips, when neglecting the elastic forces due to the rest of the blocks. It is also half of the distance travelled by that block until it sticks if there is no friction and it is not joined to other blocks. $t_0 = T/2\pi = \omega^{-1}$ is a characteristic time, where T is the period of a block attached only to the driving plate. $v_0 = L/t_0 = L\omega$ is a characteristic speed. F_0 is a characteristic static friction. Finally, $\alpha = k_c/k_p$ is the ratio between the two elastic coefficients. By expressing the former variables in terms of these units, dimensionless quantities are obtained: $y_i = Y_i/L$ for displacement, $t = t'/t_0 = t'\omega$ for time, $v = v'/v_0 = v'/L\omega$ for velocity and $\beta_i\Phi(\dot{y_i}) = F_i(\dot{Y_i})/F_0$ for friction, where $\beta_i = F_{0,i}/F_0$ is uniformly spaced in range $[1, 1+R]$ and defined as $\beta_i = 1 + \frac{i-1}{N-1}R$. Then, eq. (2) results in

$$\ddot{y_i} = vt - y_i + \frac{\alpha}{N-1}\left(\sum_{i=1}^{N} y_i - Ny_i\right) - \beta_i\Phi_i(\dot{y_i}). \tag{4}$$

B. Numerical simulation and time evolution

The BK model is a deterministic dissipative system which only contains quenched disorder, but it cannot be solved analytically because of the nonlinear friction term. Thus, I solved it by mean of a fourth-order Runge-Kutta method (RK4) using *Fortran90*. Unless it is said the contrary, the step size used is $h = 10^{-2}$.

Without loss of generality, the system is initially set in the stable state $y_i = 0$ and $\dot{y_i} = 0$. Imposing random initial displacements only would change the time it takes to the system to first start slipping. The movement of the driving plate causes energy to flow into the system increasing the elastic potential stored in it in a uniform way throughout the blocks. When the net force acting on a certain block overcomes static friction, it begins to slip. While the block is moving, the forces applied on the rest of the blocks change, so it may trigger the slipping of other blocks.

It is called an *event* (related to earthquakes) what goes on from when a block starts slipping to when all blocks remain stuck. Events goes off in such a short time and the driving plate moves at such slow velocity that during an event the driving plate is considered as stopped. Then, two times can be defined: the *system time*, measured in v^{-1} units and where events happen suddenly in zero time and the *event time*, in dimensional units as said before. After each event, I compute the time at which next event starts, called as *interverent time.* Then, the program advances the system to this time's state, and it computes its temporal evolution in *event time* until it goes off again.

Because the velocity is a continuous variable, it is necessary to establish a criterion for when a stuck block begins to move or when a moving block sticks, in order to determine the beginning and end of an event. Above has been said that a block starts to slip when the stress on it exceeds the static friction β_i, so that stuck block will move at the next time step. Conversely, it will remain stuck. On the other hand, a slipping block will become stuck if at the next time step its velocity is negative or it is decelerating with a velocity smaller than a certain threshold v_0 and a stress on it smaller than the static friction β_i. In this paper, $v_0 = 10^{-6}$. This convention is the same as [4].

III. SIMULATION RESULTS

A. One block

For a single block, $\beta = 1$, I find a periodic behaviour. As there are not interacting blocks, the stress pattern is repeated: when the stress on it overcomes β it slips. The system is fully deterministic, so I recover the elastic rebound theory and it does not exhibit complexity at all.

B. Bifurcation diagram with two blocks

A slightly more complex system includes two blocks, which has now interactions between them. It must be solved eq. (4) with $\beta_1 = 1$ for one block and $\beta_2 = 1 + R$ for the other. This model is fairly simple to describe the behaviour of a single fault but it is a good approximation of the dynamics of two coupled large segments of a fault. It allows to relate the spatial inhomogeneity of an active zone, *i.e.* the factor R, to the degree of predictability of the earthquakes originated by it [2]. Fig. 2 shows the period of successive events as varied the factor R for a fixed $\alpha = 0.9$, $\mu = 1.0$, $v = 0.01$ and a step size $h = 10^{-4}$. At $R = 0$, the behaviour is periodic (not appreciable in figure) and both blocks slip synchronized. So there is only a single frequency, which indeed depends on the particular initial condition chosen [2]. As R is increased, the system drives to chaos via period-doubling bifurcations. Also, inside chaos appear some regular windows of periods such as 1, 3, 4 and 5 that also undergo period-doubling. I find a similar behaviour when plotting distance between blocks at each global stick.

C. More blocks

Now more blocks are added up to five. In Fig. 3, I plotted the displacement evolution of each block from its initial position. As can be seen, there are some phases of great events in which all blocks are involved and others in which there are only small displacements. These small events occur before a large event takes place.

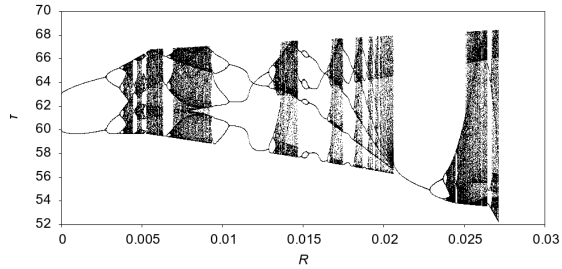

FIG. 2: Bifurcation diagram of the period of sucessive earthquakes in adimensional units depending on parameter R for $\alpha = 0.9$, $\mu = 1.0$, $v = 0.01$ and a step size $h = 10^{-4}$.

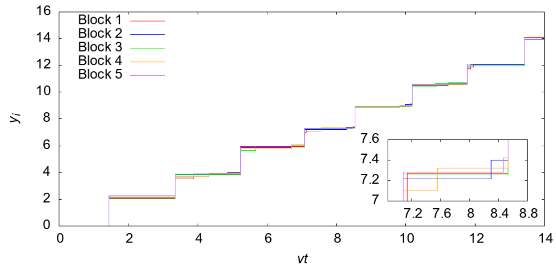

FIG. 3: Displacement evoluction in v^{-1} adimensional time units of five blocks for $\alpha = 9.0$, $\mu = 3.0$ and $R = 1.0$.

D. Many more blocks and statistics

Now, I examine a system with many more blocks in order to obtain a power-law behaviour like the Guttenberg-Ritcher law in real earthquakes and observe how its exponents change as varying some parameters. First, I examine the evolution of the average stress per block, as depicted in Fig. 4 for a system with $N = 400$, $\alpha = 9.0$, $\mu = 3.0$ and $R = 1.0$. Initially, the system goes through transients until it reaches a *critical state*, in which earthquakes of a large range of magnitudes can take place, only limited by the number of blocks. Thus, the system drives itself into that state and it will no longer leave it. From then on, the average stress per block fluctuates around a constant value. That behaviour is an example of SOC. In the case of Fig. 4, transients arrive more or less to time 16 which corresponds to around 2,500 events. In the following simulations, only are taken into account events from 10,000.

In total, 10^6 events are simulated of a system with $N = 400$ blocks. Similar results are obtained for fewer blocks, like 200. A power-law behaviour gives a distribution that goes as $\propto M^{-b}$ for moment. For recurrence time, an exponential distribution $\propto exp(-c\tau/\langle\tau\rangle)$ emerges.

First, the effect of varying R is examined, as shown in Fig. 5. Increasing R means raising the disorder, so for

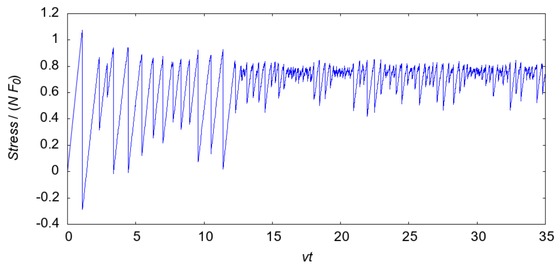

FIG. 4: Average stress per block evolution in v^{-1} adimensional time units for a system with $N = 400$, $\alpha = 9.0$, $\mu = 3.0$ and $R = 1.0$.

$R = 0$ the system is periodic and all blocks move synchronized, resulting in a unique value of magnitude. When R is increased but still a lower value, such as 0.5 and 1.0, the distribution exhibits a pronounced peak at a given large magnitude, whereas the GR is satisfied for small events. It means that smaller events exhibit a *self-similar critical* behaviour while large events exhibit an *off-critical* or characteristic behaviour. However, this peak is reduced while increasing R until at some value it disappears and all events then satisfy the GR law. So, for small R most of the blocks slip at similar stress thresholds, so they usually slip simultaneously or nearly producing a sharp peak in a characteristic magnitude. At increasing disorder, slipping conditions for each block are much more different leading to fewer characteristic events and therefore that peak gradually disappears. After that, all events follow the power law. The transition occurs for R between 1.0 and 2.0. In addition, the absolute value of the exponent of the power law for those events that satisfy it is greater as R is smaller.

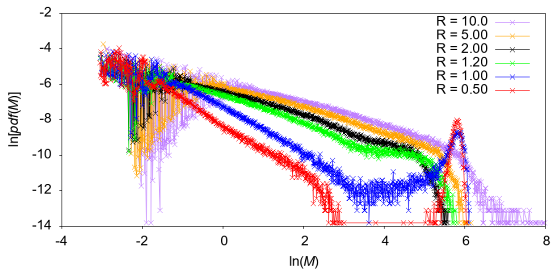

FIG. 5: Natural logarithm of the distribution of moments in natural logarithm scale for 10^6 events with $N = 400$, $\alpha = 9.0$ and $\mu = 3.0$, for different values of R.

As regards μ parameter, lower values imply more dissipation because Φ_i drops slower with velocity. Thus, blocks tend to stop more rapidly than when using larger μ values and, in general, they cannot activate other blocks to slip. On the contrary, upper values imply less dissipation, so there is more energy available to share with the rest of the blocks and that makes more blocks slip. Therefore, bigger events with more blocks involved take place. However, because the system is slippery blocks are usually not stressed at their maximum capacity and a wide range of magnitudes are obtained for big events. An exponent of $b = (-0.688 \pm 0.004)$ is obtained for $\mu = 10.0$ from a linear least-squares regression, with a lineal correlation coefficient $r^2 = 0.98$. For μ values a bit smaller, friction decays slower, so more energy is dissipated and therefore, less supplied to the rest of the blocks. Then, fewer blocks are forced to slip among them and more energy remains in the system. At some point, those blocks with bigger static friction get full stressed and slip releasing so much energy and triggering other blocks motion leading to a characteristic magnitude. The magnitude of that peak really doesn't depend on the size of the system, *i.e.* is really characteristic to the system. In that case, the exponent for $\mu = 2.0$ obtained from a linear least-squares regression is $b = (-1.404 \pm 0.010)$, with $r^2 = 0.91$. Finally, for μ very small like 0.5 friction decays very slowly. Thus, blocks stop quickly and total displacements are smaller. By the way magnitude is measured, this means smaller magnitudes and there are no large events. In some papers, they refer to it as *creeping-like* behaviour. Now, the exponent for $\mu = 2.0$ obtained from a linear least-squares regression is $b = (-0.651 \pm 0.004)$, with $r^2 = 0.99$. On the other hand, it can be seen that GR law is better satisfied at mid μ values between 1.0 and 3.0. Also, these values have a greater slope so it means that moment distribution decreases faster as moments is increased.

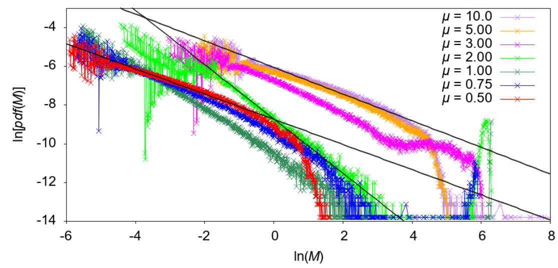

FIG. 6: Natural logarithm of the distribution of moments in natural logarithm scale for 10^6 events with $N = 400$, $\alpha = 9.0$ and $R = 1.1$, for different values of μ. Moreover, linear regression for $\mu = 0.5$, 2.0 and 10.0 is plotted.

As regards α, a higher value means blocks more tightly bounded, and energy can be shared more easily among them. Therefore, blocks are more influenced by others and they behave in a more coherent way. When a block slips, it is easier to trigger other blocks movement. By this reason, bigger α values exhibit a peak or nearly at large magnitudes, as can be seen in Fig. 7. Big events tend to be of a characteristic magnitude, while small events follow GR law. For small α, blocks are more independent and there is not a characteristic magnitude. In addition, as slipping blocks are less bounded less energy is transmitted so they can slip a long distance then, implying larger magnitudes.

In the following, I focus on the recurrence time, defined as the time (in system time) between successive events.

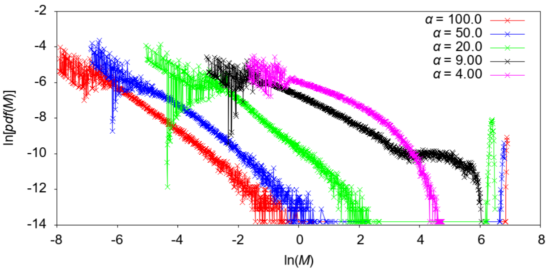

FIG. 7: Natural logarithm of the distribution of moment in natural logarithm scale for 10^6 events with $N = 400$, $\mu = 3.0$ and $R = 1.1$, for different values of α.

As shown in Fig. 8, it appears to follow an exponential law, except at short recurrence times for events with $\alpha > 20$. This behaviour implies that earthquakes are, in most cases, statistically independent, and can be effectively described by a Poisson process. Consequently, it is a memoryless process, meaning that the occurrence of an event doesn't provide information about when the next will occur. For $\alpha > 20$, the system becomes highly bounded, and events with short recurrence times are not fully independent.

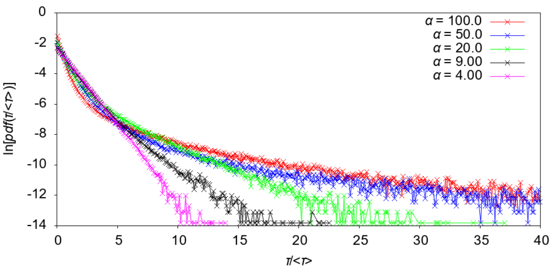

FIG. 8: Natural logarithm of the distribution of recurrence time for 10^6 events with $N = 400$, $\mu = 3.0$ and $R = 1.1$, for different values of α.

IV. CONCLUSIONS

- Using the well-known Burridge-Knopoff model, it was possible to satisfactorily reproduce earthquake dynamics in a mean-field approximation, particularly by using a velocity-weakening friction and heterogeneous static friction for each block. For a single block, the system exhibits deterministic dynamics consistent with the elastic rebound theory. When more blocks are added and coupled through interactions, the system develops complex dynamics. Specifically, with two blocks, a period-doubling route to chaos appears, along with intermediate periodic windows of periods 1, 3, 4, and 5.

- Considering a large number of blocks, it has been shown that Burridge-Knopoff is effective in reproducing a power-law behaviour in the magnitude distribution, as expected for real earthquakes. Moreover, for certain values of the parameter μ, which controls the decay of dynamic friction, the moment distribution exhibits a peak at a characteristic magnitude. I found a similar effect for large values of stiffness between blocks. Statistical analysis also revealed an exponential distribution of recurrence time, indicating that they are governed by a Poisson process. This suggest that events are statistically independent, regardless of the previous event.

- A drawback of using mean-field approximation in the present approach is the need to assign a different static friction threshold to each block in order to introduce disorder. This adds another parameter to take into account. In this work, this parameter is denoted as R. As a result, the outcomes for different values of μ and α depend on the specific choice of it.

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Model de Burridge-Knopoff en camp mig per entendre els terratrèmols

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Resum: En aquest article, s'estudien les allaus d'un model de blocs i molles de camp mitjà per simular la dinàmica dels terratrèmols. Així doncs, s'implementa un model en *Fortran90* que resol mitjançant un Runge-Kutta de 4t ordre les equacions de moviment. L'estudi comença amb l'anàlisi d'un sol bloc, observant-se un comportament determinista consistent amb la teoria del rebot elàstic. A continuació, la introducció d'interaccions porta a l'aparició de dinàmiques complexes, observant-se per dos blocs una bifurcació *period-doubling* a mesura que augmenta l'heterogeneïtat. A causa de l'existència de complexitat, es duu a terme una anàlisi estadística dels terratrèmols en un sistema de 400 blocs. Aleshores, s'observa que la distribució de magnituds, relacionada amb el logaritme de l'energia alliberada, presenta invariància d'escala, consistent amb un comportament de llei de potències. En canvi, el temps de recurrència entre terratrèmols segueix una distribució exponencial, característica d'un procés de Poisson, suggerint independència estadística entre terratrèmols.
Paraules clau: Criticalitat, camp mig, llei de potències, caos, integració amb RK4.
ODSs: 3. Salut i benestar, 4. Educació de qualitat.

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar	X	12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

El contingut d'aquest TFG, part d'un grau universitari de Física, es relaciona amb l'ODS 4, i en particular amb la fita 4.4, ja que contribueix a l'educació a nivell universitari. També es pot relacionar amb l'ODS 3 perquè pretèn aprofundir en el coneixement dels terratrèmols per millorar la prevenció i així disminuir les possibles afectacions a la població.