# Optimal partition of geometric complex networks

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**Abstract:** We introduce a method to find a low sparsity partition and an estimate $h$ of the Cheeger constant of complex networks by exploiting the geometric properties that many networks exhibit. We generate synthetic networks from the $\mathbb{S}^1/\mathbb{H}^2$ model and obtain estimates for $h$ that are between one and three orders of magnitude lower than the average sparsity over a large number of random partitions, $\langle s \rangle$, and decrease with network size. We then select seven real networks, infer an embedding into the hyperbolic disk and obtain estimates for $h$ that are all lower than $\langle s \rangle$, but only three of them are at least one order of magnitude below. In conclusion, the geometric method provides better results than random in all cases and, if the network exhibits an underlying metric space, it provides estimates that are orders of magnitude lower than random and decrease with network size. **Keywords:** Complex networks. **SDGs:** 4.

## I. INTRODUCTION

Complex networks have become increasingly popular as a way to organize and analyze large and interconnected data sets, with plenty of applications in diverse fields such as communication networks, computer science, social sciences, economics, biology, etc.

A fundamental problem that arises when analyzing such networks is reducing them to smaller, simpler and easier to study networks, that is, finding a partition of the network into two subsets such that both subsets are similar in size and there are as little links between them as possible. In order to balance these two conditions, we shall define the sparsity of a partition.

Let $G$ be a complex network on $N$ nodes and we consider a partition of $G$: two disjoint subsets $A$ and $B$ on $N_A$ and $N_B$ nodes, respectively, such that every node of $G$ is in either $A$ or $B$ (or in other words, such that $N = N_A + B_N$). If we denote the links between $A$ and $B$ as $E_{AB}$, then the sparsity of the partition is defined as:

$$s_{AB} = \frac{E_{AB}}{\min\{N_A, N_B\}}, \tag{1}$$

and we can then define the Cheeger constant of the network $G$ as:

$$h_G = \min_{A,B}\{s_{AB}\}. \tag{2}$$

We say a partition $A$, $B$ of a network $G$ is optimal if it satisfies $s_{AB} = h_G$. However, finding optimal partitions in general requires exploring every possible partition of the network and calculating their sparsity in order to determine the Cheeger constant of the network. This is an NP-hard problem and, as such, it is not practically feasible in large networks.

The present project aims to show a polynomial-time ($\mathcal{O}(N^2)$) method to estimate the Cheeger constant of a network and to provide a partition which satisfies that estimate by exploiting the geometric properties that many real complex networks exhibit.

### A. The $\mathbb{S}^1/\mathbb{H}^2$ model

As it was shown in [1], many real complex networks can be mapped into low-dimensional metric spaces with hyperbolic geometry, the $\mathbb{S}^1/\mathbb{H}^2$ model. In the $\mathbb{S}^1$ model, this can be done by assigning two hidden variables to each node in the network; a hidden degree $\kappa$, which quantifies its popularity, and an angle $\theta \in [0, 2\pi)$ which determines its position in the similarity space: a circumference of radius $R = \frac{N}{2\pi}$. Nodes that are similar are assigned similar angles and, as such, the angular distance between nodes gives us an idea of the likelihood of their connection. In the $\mathbb{S}^1$ model, the hidden degree distribution $\rho(\kappa)$ is given by a power law:

$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma}, \quad \kappa > \kappa_0, \tag{3}$$

where $\gamma > 2$ is a parameter and $\kappa_0$ is given by:

$$\kappa_0 = \frac{\gamma - 2}{\gamma - 1}\langle k \rangle . \tag{4}$$

The connection probability between nodes $i$ and $j$, with hidden degrees $\kappa_i$, $\kappa_j$ and angles $\theta_i$, $\theta_j$, respectively, is then given by:

$$p_{ij} = \frac{1}{1 + \left(\frac{R\Delta\theta_{ij}}{\mu\kappa_i\kappa_j}\right)^{\beta}}, \tag{5}$$

where $\beta > 1$ is a parameter (the inverse temperature) that controls the level of clustering of the network, $\Delta\theta_{ij}$ is the (shortest) angular distance between nodes $i$ and $j$, and $\mu$ controls the average degree of the network $\langle k \rangle$ as follows:

$$\Delta\theta_{ij} = \pi - |\pi - |\theta_i - \theta_j|| \tag{6}$$

$$\mu = \frac{\beta}{2\pi\langle k \rangle}\sin\left(\frac{\pi}{\beta}\right). \tag{7}$$

The $\mathbb{H}^2$ model instead maps the complex network to the hyperbolic plane. A point $(r, \theta)$ in the hyperbolic disk is assigned to each node, which combines the popularity and similarity dimensions that we have discussed in the $\mathbb{S}^1$ model. Hyperbolic distance between two nodes is then an indicator of the likelihood of their connection.

It can be shown that the $\mathbb{S}^1$ and the $\mathbb{H}^2$ models are quasi-isomorphic and we can use them indistinctly, hence we call it the $\mathbb{S}^1/\mathbb{H}^2$ model. This model can be generalized to higher dimensions, the $\mathbb{S}^D/\mathbb{H}^{D+1}$ model, but the one-dimensional model is simpler and it is enough to capture the geometric properties of many real complex networks, so we shall only use the one-dimensional model in this project.

## II. METHOD TO ESTIMATE THE CHEEGER CONSTANT

We shall now exploit the hidden geometry of a given complex network in order to find a low sparsity partition and an estimate of its Cheeger constant.

Nodes that are close in the similarity space of a complex network described by the $\mathbb{S}^1$ model have a small angular distance $\Delta\theta_{ij}$ and, as seen in Eq. (5), they are much more likely to be connected than further away nodes with a larger angular distance. Taking this into consideration, we can partition the similarity space into two continuous regions, $A = \{i \in \mathit{nodes} \mid 0 \leq \theta_i < x\}$ and $B = \{i \in \mathit{nodes} \mid x \leq \theta_i < 2\pi\}$ where $0 < x < 2\pi$ is what we call the partition angle. We note that this partition of the similarity space also defines a partition of the complex network itself and, as we have discussed, it is reasonable to think that partitions of this type provide a good estimate of the Cheeger constant of the network, as long as the network can be described by the $\mathbb{S}^1$ model. The method we propose is as follows.

1. We infer an embedding of the complex network into the hyperbolic disk using the program Mercator (see [4]) and, for each node $i$, we obtain its position in the similarity space, its angle $\theta_i$.

2. We choose a step for the partition angle, $x_{step}$, and a step for the initial angle, $\theta_{step}$, as well as an arbitrary starting initial angle $\theta_0$ (for simplicity, we shall use 0 as our initial angle throughout this project). We set the initial angle $\theta = \theta_0$. Note that the lower the step values, the more geometric partitions that will be calculated, and we will obtain a better Cheeger constant estimate at the cost of extra computing time.

3. Find the node with the closest angle to $\theta$, node $i$, such that its angle $\theta_i$ satisfies $\theta \leq \theta_i$ (this is to ensure the partition has at least one node in subset $A$). Set the partition angle $x$ to $x = \theta_i + x_{step}$.

4. Partition the complex network into subsets $A = \{i \in \mathit{nodes} \mid \theta \leq \theta_i < x, \mod 2\pi\}$ and $B = \{i \in \mathit{nodes} \mid x \leq \theta_i < 2\pi + \theta, \mod 2\pi\}$.

5. Calculate the sparsity $s_{AB}$ of the partition made in step 4 using Eq. (1). Set $s_{min} = s_{AB}$ if $s_{AB} < s_{min}$, or if $s_{min}$ had not yet been set.

6. Set the partition angle to $x \to x + x_{step}$. Repeat steps 4-6 while there is at least one node in subset $B$.

7. Set the initial angle to $\theta \to \theta + \theta_{step}$. Repeat steps 3-7 while $\theta < \theta_0 + 2\pi$.

8. Our estimate for the Cheeger constant of the complex network is $s_{min}$.

It was shown in [4] that the computational complexity of Mercator scales as $\mathcal{O}(N^2)$ and, since it is the slowest part of the geometric method, it also scales as $\mathcal{O}(N^2)$.

We have written a program that implements the aforementioned method, and we shall test the method by applying it to different scenarios and comparing its Cheeger constant estimate, $h$, with the average sparsity of a large number of random partitions of the network, $\langle s\rangle$. In order to do so, we shall first generate synthetic networks using the $\mathbb{S}^1/\mathbb{H}^2$ model for different values of $N$, $\beta$ and $\gamma$. In theory we expect the geometric method to provide considerably better results than random partitions: that $h$ is orders of magnitude lower than $\langle s\rangle$.

Afterwards, we shall perform the same comparison but instead using real complex networks: we shall map the networks to their underlying metric space using Mercator, then apply our method on them to obtain an estimate of the Cheeger constant, $h$, and compare it to the random average sparsity, $\langle s\rangle$. If real complex networks can be well described by the $\mathbb{S}^1/\mathbb{H}^2$ model, then we also expect $h$ to be orders of magnitude lower than $\langle s\rangle$.

## III. RESULTS FOR SYNTHETIC COMPLEX NETWORKS

We have used the SD-model program found in [3] to generate synthetic complex networks from the $\mathbb{S}^1$ model. For each $\gamma \in \{2.5, 3.5\}$ and for each $\beta \in \{1.0001, 1.25, 1.5, 1.75, 2, 3\}$ we have generated complex networks of sizes $N \in \{1000, 3000, 5000, 7000, 10000\}$, all of them with average degree $\langle k\rangle = 10$. We shall analyze these networks in order to validate whether synthetic networks behave as expected.

Since synthetic networks from the $\mathbb{S}^1$ model have a homogeneous angular distribution of nodes, we expect that the sparsity of geometric partitions decreases as the partition angle $x$ approaches $\pi$ (both subsets are of similar size), since then we have $N_A \sim N_B \sim N/2$, and the denominator of Eq. (1) takes its greatest possible value. This was true for every synthetic network we analyzed, and we can see an example in Fig. 1. Since networks are discrete rather than continuous, we observe some fluctuations with respect to the expected behavior caused by inhomogeneities in the angular distribution of nodes.

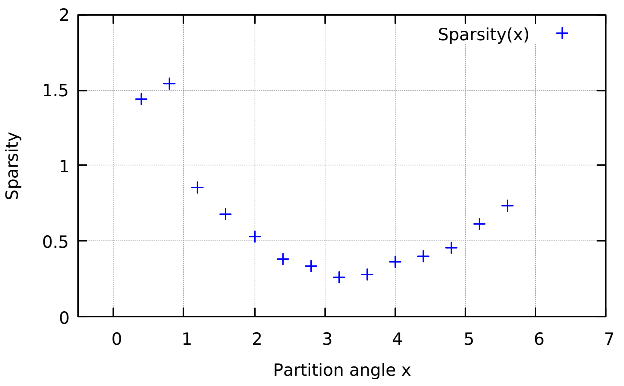


FIG. 1: Sparsity as a function of the partition angle $x$ for multiple geometric partitions of a synthetic $\mathbb{S}^1$ network with parameters $\gamma = 2.5$, $\beta = 2$, $N = 7000$ and $\langle k \rangle = 10$. We have chosen steps $x_{step} = \theta_{step} = 0.4$. The partitions shown correspond to the initial angle for which we obtained the partition with the lowest sparsity, $\theta = 2.4$. As expected, sparsity decreases as the partition angle approaches $\pi$.

On average, we expect random partitions to be of size $N_A, N_B \sim N/2$ and to have $E_{AB} \sim \langle k \rangle N/2$ links between each subset, so the sparsity of an average random partition would be, using Eq. (1); $s_{rand} \sim \frac{\langle k \rangle N/2}{N/2} = \langle k \rangle$, which does not depend on the size $N$ of the network. This also proved to be true for every network we analyzed except for some small fluctuations caused once again by the discrete nature of complex networks. An example of this behavior can be seen in Fig. 2.

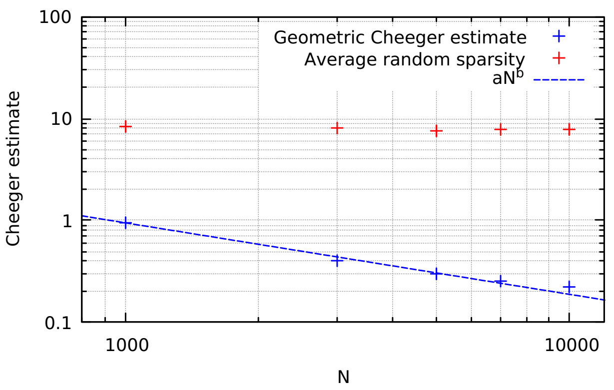


FIG. 2: Comparison between the average random sparsity and the geometric Cheeger constant estimate of synthetic $\mathbb{S}^1$ networks as functions of their size $N$. We have used parameters $\gamma = 2.5$, $\beta = 2.5$ and $\langle k \rangle = 10$ to generate all the networks shown. We have chosen steps $x_{step} = \theta_{step} = 0.4$ for the geometric partitions, and performed 1000 random partitions to calculate the average random sparsity of each network. We have also represented a power law fit $aN^b$ of the geometric Cheeger estimates as a dashed line, with $a = 120 \pm 37$ and $b = -0.70 \pm 0.04$. As expected, the average random sparsity $\langle s \rangle$ is constant with $N$ and of order $\langle s \rangle \sim \langle k \rangle = 10$, and the geometric Cheeger estimate decreases as a power law with $N$.

It was shown in [2] Appendix 4 that, for complex networks with a homogeneous hidden degree distribution, the Cheeger constant scales with the size $N$ of the network as $h_G \simeq c_1 N^{1-\beta} + c_2 N^{-1}$, and so in the thermodynamic limit $N \to \infty$, the Cheeger constant decays to zero (since $\beta > 1$). Because of this, we expect that networks from the $\mathbb{S}^1$ model, which have a power law hidden degree distribution given by Eq. (3), exhibit similar behavior and their Cheeger constant estimates decay to zero for increasingly larger networks following a power law trend, and that they decrease faster for larger values of $\beta$. Once again, this was true for all the networks we analyzed, and we can see an example in Fig. 2. In Figs. 3-4, we can observe that the geometric Cheeger constant estimate does indeed decrease as $\beta$ increases and, by comparing both figures, we can tell that the same is true for $\gamma$. We can also see that, in all cases, the geometric Cheeger estimate as a function of $N$ can be well approximated as a power law of the form $aN^b$. The results of the power law fits can be found in Tables I,II of Appendix A. As we expected, the exponent $b$ decreases as $\beta$ increases: the geometric Cheeger estimate decays faster with the size of the network $N$ for larger values of the inverse temperature $\beta$.

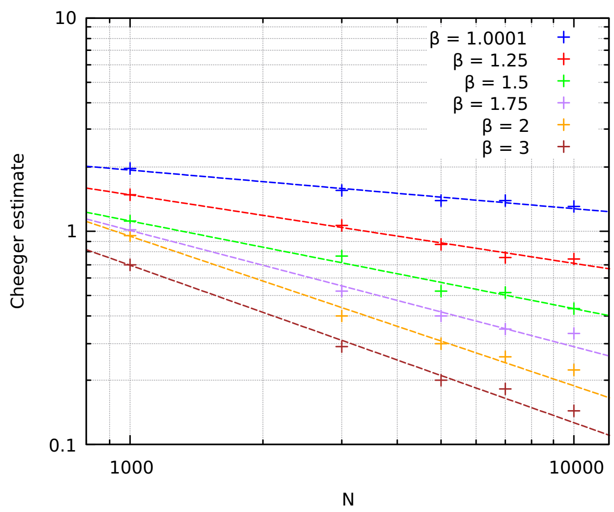


FIG. 3: Geometric estimate of the Cheeger constant as a function of $N$ for synthetic $\mathbb{S}^1$ networks with parameters $\gamma = 2.5$, $\langle k \rangle = 10$ and different sizes and values of the inverse temperature $\beta$. We have chosen steps $x_{step} = \theta_{step} = 0.4$. The dashed lines represent power law fits ($aN^b$) of the geometric Cheeger estimate as a function of N for each value of $\beta$.

Lastly and as we have already discussed, we expect to obtain a much better estimate of the Cheeger constant through our method than the sparsity of an average random partition, even more so for larger networks and larger values of $\beta$. Figs. 2-4 show that this is the case: the average random sparsity is of order $\langle s \rangle \sim \langle k \rangle = 10$ in all cases, whereas the geometric estimates of the Cheeger constant are all at least one order of magnitude below, with increasingly better results for larger values of $N$, $\beta$

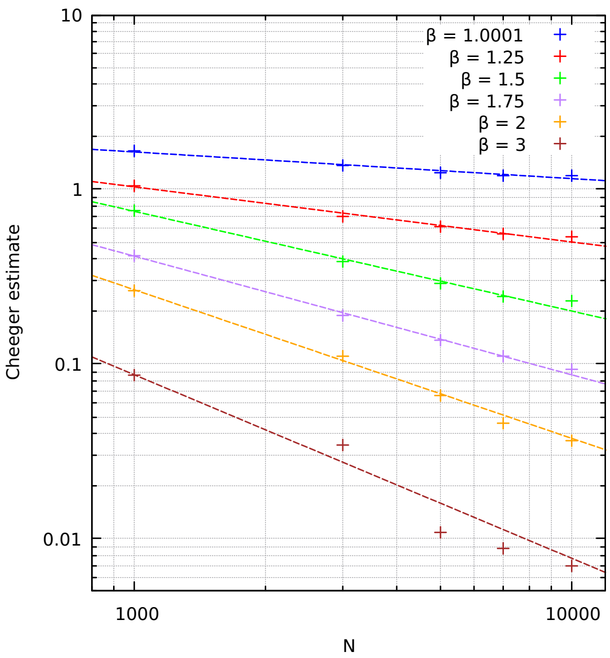


FIG. 4: Geometric estimate of the Cheeger constant as a function of $N$ for synthetic $\mathbb{S}^1$ networks with parameters $\gamma = 3.5$, $\langle k \rangle = 10$ and different sizes and values of the inverse temperature $\beta$. We have chosen steps $x_{step} = \theta_{step} = 0.4$. The dashed lines represent power law fits ($aN^b$) of the geometric Cheeger estimate as a function of N for each value of $\beta$.

and $\gamma$; with the best results being three orders of magnitude below the average random sparsity.

## IV. RESULTS FOR REAL COMPLEX NETWORKS

After validating that the geometric method to estimate the Cheeger constant works as expected for synthetic $\mathbb{S}^1$ model networks, we selected seven real complex networks [6–14] in order to test our method on them. We first used the Mercator program [4] to infer an embedding of each network into the hyperbolic disk (see Fig. 5 for two examples), and then we calculated the average random sparsity and the geometric Cheeger estimate of every network. Since the aforementioned networks have different average degrees $\langle k \rangle$, in Fig. 6 we represented the average random sparsity and the geometric Cheeger estimate divided by $\langle k \rangle$, so that we can compare the results of different networks.

As it can be seen in Fig. 6, it is also true for real networks that the average random sparsity is of order $\langle s \rangle \sim \langle k \rangle$, independent of the size $N$ of the network.

We also observe in Fig. 6 that the geometric Cheeger estimate is lower than the average random sparsity $\langle s \rangle$ in all cases and, while some estimates are one, two or even three orders of magnitude lower than $\langle s \rangle$, our method does not provide estimates of at least one order of magnitud below $\langle s \rangle$ in all cases like it did with synthetic complex networks. This is possibly because some real networks cannot be properly described by the $\mathbb{S}^1/\mathbb{H}^2$ model, or because they have different parameters $\gamma$ and $\beta$, thus making them harder to compare directly.

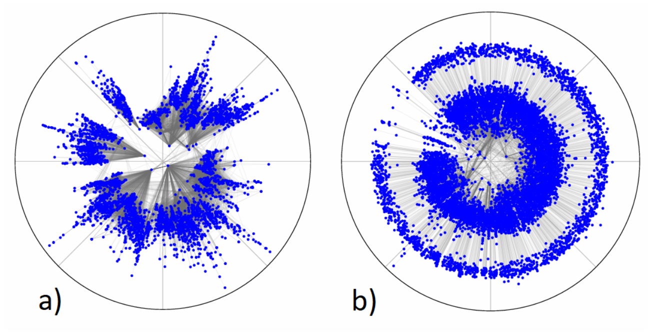


FIG. 5: Embeddings of two real networks into the hyperbolic disk inferred using Mercator. a) Network [10] of size $N = 4039$. Nodes represent Facebook users and edges represent friendships between users. We inferred $\beta = 2.34134$. b) Network [12] of size $N = 10680$. Nodes represent users of the Pretty Good Privacy algorithm and edges represent trust relationships between users. We inferred $\beta = 1.94417$.

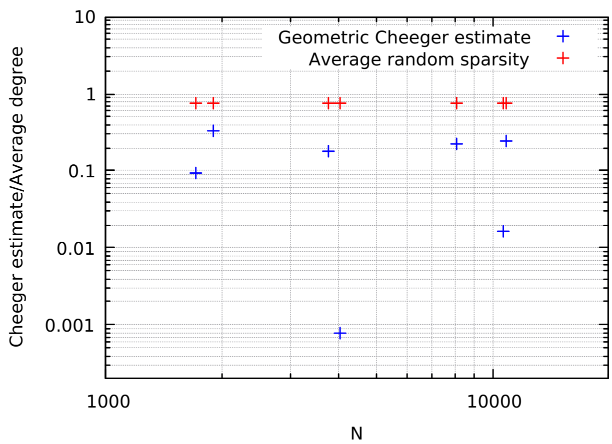


FIG. 6: Comparison between the average random sparsity and the geometric Cheeger constant estimate, both of them divided by the average degree $\langle k \rangle$ of the network, for real networks as functions of their size $N$. We have chosen steps $x_{step} = \theta_{step} = 0.1$ for the geometric partitions, and have performed 10000 random partitions to calculate the average random sparsity of each network.

## V. CONCLUSIONS

We have introduced a method to find a low sparsity partition and an estimate of the Cheeger constant of a complex network by taking advantage of its hidden geometry.

We first tested the method on synthetic networks: we generated synthetic networks from the $\mathbb{S}^1/\mathbb{H}^2$ model using [3] with various sizes and parameters and we verified all the expected results. We saw that the sparsity as a function of the partition angle decreases as the partition angle approaches $\pi$; that is, as both subsets get closer in size. We then observed that the average random sparsity $\langle s \rangle$ is of order $\langle s \rangle \sim \langle k \rangle$ and is independent of the size of the network $N$. Furthermore, we noticed that the geometric estimate of the Cheeger constant decreases with the size of the network $N$, and with parameters $\gamma$ and $\beta$. We concluded our analysis of synthetic networks by pointing out that geometric estimates of the Cheeger constant are at least about one order of magnitude lower than the average random sparsity in all cases, with lower results (up to three orders of magnitude lower) as $N$, $\beta$ and $\gamma$ increase. In view of these results, synthetic $\mathbb{S}^1/\mathbb{H}^2$ model networks behave as we expected and geometric partitions have a much lower sparsity than random partitions, even more so the larger the network and parameters $\gamma$ and $\beta$ are. Consequently, ours is a good method (at least much better than random partitions) to estimate the Cheeger constant and to find a partition that satisfies that estimate for complex networks that are well described by the $\mathbb{S}^1/\mathbb{H}^2$ model.

Then we tested the geometric method on real complex networks. We selected seven real networks [6–14] of different sizes and inferred an embedding into the hyperbolic disk using Mercator [4] for every network. We calculated the average random sparsity $\langle s \rangle$ of each network and, just like we found out for synthetic networks, real networks also have behave as $\langle s \rangle \sim \langle k \rangle$, independent of the size $N$ of the network. We then applied the geometric partition method to obtain an estimate of the Cheeger constant of each network and compared it to $\langle s \rangle$. While the geometric estimate of the Cheeger constant was lower than $\langle s \rangle$ on all cases, it was not at least one order of magnitude lower on each network unlike in the synthetic network case. This is likely because some real networks are not well described by the $\mathbb{S}^1/\mathbb{H}^2$ model, and so the geometric method is unable find a low sparsity partition. Regardless, we have obtained results one order of magnitude lower than $\langle s \rangle$ on three of the seven networks we have analyzed, and even three orders of magnitude lower on one of them. As such, our method gives generally positive results (at least better than random partitions) for real networks as well as synthetic ones, especially if they are well described by the $\mathbb{S}^1/\mathbb{H}^2$ model.

While it was not the goal of this project, a possible next step could be to compare the estimates of the Cheeger constant $h$ provided by the geometric method with other general methods to estimate $h$ of complex networks.

**Acknowledgments**

I would like to thank my advisor, Dr. Marián Boguñá, for guiding me through this project and solving the issues I came across. I would also like to thank my parents for their support throughout the development of this project.

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# Partició òptima de xarxes complexes geomètriques

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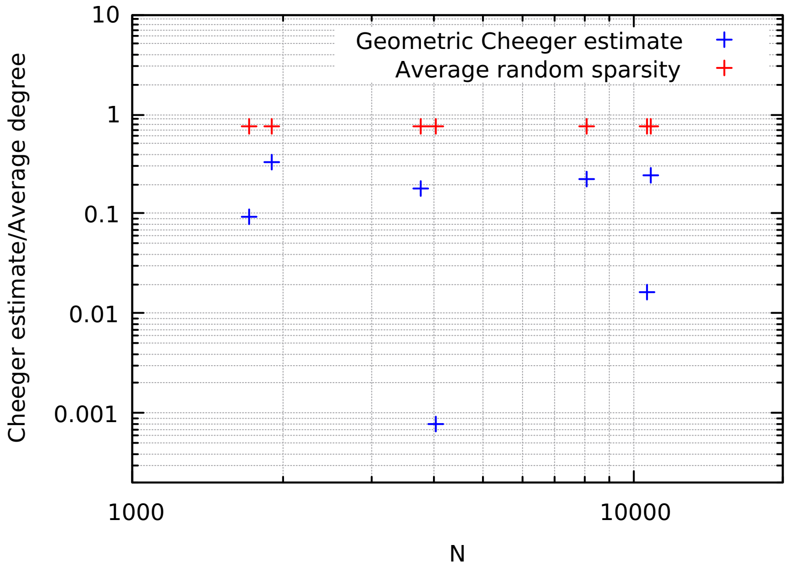
**Resum:** Introduïm un mètode per a trobar particions amb *sparsity* petita i una estimació de la constant de Cheeger $h$ de xarxes complexes tot aprofitant les propietats geomètriques que presenten moltes xarxes. Generem xarxes sintètiques a partir del model $\mathbb{S}^1/\mathbb{H}^2$ i obtenim estimacions de $h$ que es troben entre un i tres ordres de magnitud per sota de la *sparsity* mitjana de moltes particions aleatòries, $\langle s \rangle$, i que decreixen amb el tamany de la xarxa. Després seleccionem set xarxes complexes reals, inferim un *embedding* al disc hiperbòlic i obtenim estimacions de $h$ que són totes més petites que $\langle s \rangle$, però només tres d'elles es troben un o més ordres de magnitud per sota. En conclusió, el mètode geomètric proporciona resultats millors que els aleatoris en tots els casos i, si la xarxa presenta un espai mètric ocult, les estimacions són ordres de magnitud millors que les aleatòries, i decreixen amb el tamany de la xarxa. **Paraules clau:** Xarxes complexes. **ODSs:** Aquest TFG està relacionat amb els Objectius de Desenvolupament Sostenible (SDGs)

**Objectius de Desenvolupament Sostenible (ODSs o SDGs)**

| | | | |
|---|---|---|---|
| 1. Fi de la pobresa | | 10. Reducció de les desigualtats | |
| 2. Fam zero | | 11. Ciutats i comunitats sostenibles | |
| 3. Salut i benestar | | 12. Consum i producció responsables | |
| 4. Educació de qualitat | X | 13. Acció climàtica | |
| 5. Igualtat de gènere | | 14. Vida submarina | |
| 6. Aigua neta i sanejament | | 15. Vida terrestre | |
| 7. Energia neta i sostenible | | 16. Pau, justícia i institucions sòlides | |
| 8. Treball digne i creixement econòmic | | 17. Aliança pels objectius | |
| 9. Indústria, innovació, infraestructures | | | |

El contingut d'aquest TFG, part d'un grau universitari de Física, es relaciona amb l'ODS 4, i en particular amb la fita 4.4, ja que contribueix a l'educació a nivell universitari.

**GRAPHICAL ABSTRACT**



### Appendix A: Results of the power law fits

TABLE I: Results of the power law fits $aN^b$ of Fig. 3.

| $\beta$ | $a$ | $b$ |
|---|---|---|
| 1.0001 | $6.7 \pm 0.8$ | $-0.181 \pm 0.015$ |
| 1.25 | $13.6 \pm 1.8$ | $-0.321 \pm 0.016$ |
| 1.5 | $19 \pm 5$ | $-0.41 \pm 0.03$ |
| 1.75 | $43 \pm 10$ | $-0.55 \pm 0.03$ |
| 2 | $120 \pm 40$ | $-0.70 \pm 0.04$ |
| 3 | $113 \pm 30$ | $-0.74 \pm 0.04$ |

TABLE II: Results of the power law fits $aN^b$ of Fig. 4.

| $\beta$ | $a$ | $b$ |
|---|---|---|
| 1.0001 | $4.7 \pm 0.5$ | $-0.153 \pm 0.013$ |
| 1.25 | $9.1 \pm 1.5$ | $-0.32 \pm 0.02$ |
| 1.5 | $38 \pm 8$ | $-0.57 \pm 0.03$ |
| 1.75 | $45 \pm 4$ | $-0.679 \pm 0.013$ |
| 2 | $94 \pm 17$ | $-0.85 \pm 0.03$ |
| 3 | $122 \pm 99$ | $-1.05 \pm 0.11$ |

### Appendix B: Real network description

[6]: Size $N = 1707$. Nodes represent places and names from the King James Bible and edges represent co-occurrence in verses. We inferred $\beta = 3.15$.

[7]: Size $N = 1893$. Nodes represent users of an online social network at the University of California, Irvine, and edges represent private messages between users. We inferred $\beta = 0.621955$.

[8]: Size $N = 3775$. Nodes represent users of a Bitcoin trading platform called Bitcoin Alpha and edges represent trust ratings between users. We inferred $\beta = 1.09999$.

[10]: Size $N = 4039$. Nodes represent Facebook users and edges represent friendships between users. We inferred $\beta = 2.34134$.

[11]: Size $N = 8114$. Nodes represent genes of Drosophilas and edges represent genetic interactions. We inferred $\beta = 1.07889$.

[12]: Size $N = 10680$. Nodes represent users of the Pretty Good Privacy algorithm and edges represent trust relationships between users. We inferred $\beta = 1.94417$.

[13]: Size $N = 10879$. Nodes represent hosts in the Gnutella network topology and edges represent connections between hosts. We inferred $\beta = 0.732776$.