Holography and thermodynamics of a non-conformal quantum field theory

Author: Adrià Olives-Salmerón, aolivesa80@alumnes.ub.edu
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Yago Bea, yagobea@ub.edu
(Dated: June 9, 2025)

Abstract: In this work, we study the thermodynamics of a non-conformal QFT using holography, specifically the AdS/CFT correspondence. The dual description consists of five-dimensional AdS spacetime with Einstein gravity coupled to a scalar field that breaks conformal symmetry, in order to model non-conformal systems inspired by QCD. By numerically solving the Einstein-Klein-Gordon equations, we construct black brane solutions and extract the temperature and entropy associated with thermal states in the dual QFT. The $s/T^3(T)$ plot exhibits the expected behaviour of a CFT at both high and low temperatures, but shows non-conformal behaviour in the intermediate region.
Keywords: holography, AdS/CFT correspondence, quantum field theories, quantum chromodynamics, black branes, equation of state.
SDGs: 4. Quality Education. 9. Industry, Innovation and Infrastructure.

I. INTRODUCTION

Performing calculations at low energies in quantum chromodynamics (QCD) is difficult due to the strong coupling constant. The anti-de Sitter/conformal field theory (AdS/CFT) correspondence provides a framework that enables useful insights into QCD. It maps the problem to a five-dimensional gravity theory that approximates the equation of state (EOS) of a quantum field theory (QFT) by using black hole solutions in that spacetime.

In this work, we focus on obtaining the EOS $s = s(T)$ of a non-conformal QFT—motivated by the non-conformal nature of QCD—at zero chemical potential μ. To do so, we construct different black hole solutions, each of which is dual to a thermal state in the dual theory.

This paper is organized as follows. In section II, we present the theoretical foundations behind this study, from the holographic principle to the EOS. In section III, we describe the main calculation details and computational methods. In section IV, we analyse, discuss, and compare our results with theoretical expectations. In section V, we summarize our conclusions.

II. THEORETICAL FRAMEWORK

A. The holographic principle and the AdS/CFT correspondence

The holographic principle states that the degrees of freedom and the dynamics within a volume V of a quantum gravity theory can be encoded on its boundary ∂V, in terms of an effective theory—a theory that is applicable only under specific conditions [1]—whose precise form may be unknown [2].

One of the most important realizations of the holographic principle is the gauge/gravity duality, originally motivated by string theory. This duality conjectures that a QFT with gauge symmetry—the effective theory—is dual—bijective map—to a theory of quantum gravity defined in the bulk [2].

The most prominent example of a gauge/gravity duality is the AdS/CFT correspondence, also known as the Maldacena conjecture [3]. This duality, characterized by a high degree of symmetry, relates two supersymmetric and conformal theories. The canonical example of the AdS/CFT correspondence is the duality between $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory, a (3 + 1)-dimensional conformal QFT, and a ten-dimensional type IIB closed superstring theory on $\mathrm{AdS}_5 \times S^5$. It should be emphasized that the correspondence remains a conjecture, although there is evidence in its favour [2, 4].

B. Large N and strong coupling limits

Focusing on the duality between $\mathcal{N} = 4$ SYM and type IIB closed superstring theory on $\mathrm{AdS}_5 \times S^5$, we can consider two limits that simplify the treatment [2].

1. Large N or 't Hooft limit: $N \to \infty$, where N is the rank of the special unitary group $SU(N)$—the gauge group—keeping the 't Hooft coupling of the QFT, $\lambda = g^2_{\mathrm{YM}} N$, fixed.
2. Strong coupling limit: $\lambda \to \infty$.

One can show that now our problem has been reduced to the duality between strongly coupled $\mathcal{N} = 4$ SYM and type IIB supergravity on weakly curved $\mathrm{AdS}_5 \times S^5$, i.e., classical gravity rather than quantum gravity [2].

C. Dimensional reduction and consistent truncation

The treatment can be further simplified by working in a lower-dimensional effective theory through dimensional reduction and consistent truncation.

By compactifying the five dimensions of the 5-sphere S^5, we obtain a five-dimensional Einstein gravity theory on AdS_5 coupled to additional fields. This procedure is known as dimensional reduction. Furthermore, to avoid working with a large number of fields, we can consistently set most of them to zero and retain only a few. This procedure is known as consistent truncation [2, 5].

The essential idea is that any solution of the five-dimensional effective theory with extra fields corresponds to a solution of the full ten-dimensional theory.

D. Extension of AdS/CFT: bottom-up models

In the previous subsections, we explained the so-called top-down approach: starting from a well-defined theory in the context of duality, such as string theory—which is dual to a QFT—and ending up with a reduced effective theory.

An alternative is the bottom-up approach. The idea is to start with an appropriate effective theory on the gravity side and conjecture that it has a dual QFT, in the same sense as in top-down models. The main advantage of these models is that one can impose specific properties that the QFT is expected to exhibit, whereas the main disadvantage is that one does not actually know whether a dual QFT exists [2, 4].

E. Non-conformal QFT and gravity

Let us now particularize the above discussion to this study. QCD is a non-conformal $SU(3)$ Yang-Mills theory without supersymmetry, in contrast to $\mathcal{N}=4$ SYM [4]. Hence, if we aim to study certain aspects of non-conformal theories using holography, it is necessary to break conformal invariance on the gravity side. Modifying the bulk theory so that it captures this essential feature that boundary theory has, motivates the use of a bottom-up approach.

In the top-down approach one finds that strongly coupled $\mathcal{N}=4$ SYM is dual to type IIB supergravity on weakly curved $AdS_5 \times S^5$. By performing a dimensional reduction, we work in a five-dimensional Einstein gravity theory on AdS_5 with additional fields. Let us now examine consistent truncation.

Because we want to continue describing a theory of gravity, we retain the gravity field, which is the metric tensor $g_{\mu\nu}$. Moreover, we also retain one more field, a scalar field ϕ, which allows us to break the conformal invariance and thus get closer to QCD. This scalar field has an associated scalar potential $V(\phi)$, which must be chosen—here is the bottom-up approach. Our potential is based on the one presented in [6] but with a different convention chosen to be consistent with [7]. It is defined in terms of the superpotential

$$W(\phi) := -\frac{3}{2L} - \frac{1}{8L}\phi^2 - \frac{1}{64L\phi_M^2}\phi^4 + \frac{1}{64L\phi_Q}\phi^6, \quad (1)$$

which leads to the scalar potential—see Fig. 1

$$V(\phi) := -\frac{16}{3}W(\phi)^2 + 8W'(\phi)^2, \quad (2)$$

where L is the radius of curvature of the asymptotic AdS geometry and ϕ_M and ϕ_Q are constants. In this study we focus on the model with $L=1$, $\phi_M=1$, and $\phi_Q=10$.

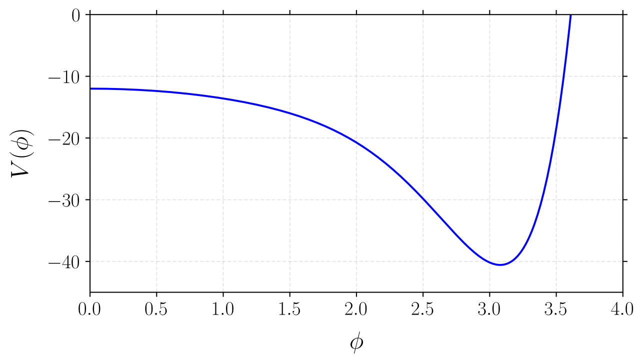

Figure 1: Scalar potential $V(\phi)$ defined in Eq. (2) for the model parameters $L=1$, $\phi_M=1$, and $\phi_Q=10$.

Our approach is then based on a bottom-up model where we have left a scalar field to capture qualitative features inspired by QCD—although the dual theory is not intended to reproduce QCD—with the scalar potential chosen according to the Eq. (2). Therefore, we have the duality between AdS_5 spacetime with two additional fields—metric and scalar field—dual to a non-conformal QFT.

We conclude this subsection by justifying how to obtain the temperature and the entropy of a thermal state on the QFT side, and why a thermal state corresponds to a black brane solution on the gravity side.

When the gauge theory has a finite temperature, the thermal states are computed as follows. It can be shown that the gauge theory at zero temperature is dual to an AdS_5 spacetime, while at non-zero finite temperature it is dual to an AdS_5 black hole. If this black hole is the simplest one in AdS_5 spacetime—the so called Schwarzschild-AdS_5 black hole ($SAdS_5$)—the dual QFT is $\mathcal{N}=4$ SYM. Moreover, if a perturbation is added via a field, one obtains more exotic AdS black holes, breaking symmetries. The temperature and entropy of the QFT are then related to the Hawking temperature and Bekenstein–Hawking entropy, i.e., the quantities evaluated at the event horizon of the black hole, $r=r_H$ [5].

Thermal states are characterized by their spacetime-translation invariance, i.e., invariance under translations in the $\vec{x}$ and t directions. It is therefore natural to require that the dual gravity theory also has these symmetries, and this is what fixes the planar topology of the black hole, commonly referred to as a black brane.

F. Lagrangian density, metric and scalar field ansatz, equations of motion, and thermodynamics

We follow [7] throughout this work. On the gravitational side, we use geometrized-like units, setting $c = k_{\rm B} = 1$ and $G = \frac{1}{8\pi}$. Since we also fix the scale $L = 1$, all quantities become dimensionless within this unit system—see Appendix A for further discussion. Let us now develop our effective five-dimensional gravity model coupled to a scalar field, defined by the following Lagrangian density

$$\mathcal{L} = \frac{1}{2\kappa^2}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right], \tag{3}$$

where $\kappa^2 = \frac{8\pi G}{c^4} = 1$ is related to Newton's gravitational constant and R is the Ricci scalar. This Lagrangian can be directly motivated from Einstein-Hilbert and Klein-Gordon Lagrangians, by generalizing the mass term in the latter through a non-trivial potential $V(\phi)$ instead of $\frac{1}{2}m^2\phi^2$. When the scalar field is constant the potential $V(\phi)$ becomes a constant as well, effectively acting as a negative cosmological constant and yielding an SAdS_5 geometry.

We now propose an ansatz for the metric and scalar field. Using $(t, \vec{x})$ as spacetime coordinates and r as the holographic coordinate, the symmetries of thermal states discussed in the previous subsection imply that all functions must not depend on $(t, \vec{x})$ coordinates. Therefore, $\phi = \phi(r)$, and the most general form for an AdS_5 black brane geometry consistent with these symmetries is

$$ds^2 = e^{2A(r)}\left[-h(r)dt^2 + d\vec{x}^2\right] + \frac{e^{2B(r)}}{h(r)}dr^2, \tag{4}$$

where $A(r)$ and $B(r)$ are functions of the holographic coordinate, and $h(r)$ is called the blackening factor, as it vanishes at the horizon, $h(r_{\rm H}) = 0$.

Minimizing the action defined by Lagrangian (3), and inserting $\phi = \phi(r)$ and (4), one obtains the equations of motion (EOMs)—see Appendix B. Fixing the gauge we can set $B(r) = 0$. It yields to the second-order ordinary differential equations (ODEs)

$$\begin{aligned} A'' + \frac{1}{6}\phi'^2 &= 0 \\ h'' + 4A'h' &= 0 \\ \phi'' + \left(4A' + \frac{h'}{h}\right)\phi' - \frac{1}{h}\frac{dV}{d\phi} &= 0, \end{aligned} \tag{5}$$

and the first-order ODE—namely, the zero-energy constraint

$$h(24A'^2 - \phi'^2) + 6A'h' + 2V(\phi) = 0, \tag{6}$$

for $\phi(r)$, $A(r)$, and $h(r)$.

We conclude this subsection by presenting the explicit expressions for the temperature and entropy of a thermal state on the QFT side. We need to compute the temperature T and entropy density s at the event horizon of the black brane, r_H. This is a standard procedure within the AdS/CFT framework and it can be found in most AdS/CFT textbooks, e.g., in [2, p. 80]. However, it is important to note that in general each solution exhibits different asymptotic behaviour as $r \to \infty$ and, more importantly, the behaviour of $\phi(r)$ in this limit is characterized by ϕ_A, which introduces an energy scale in the dual QFT—see Appendix C. Therefore, to construct the EOS we must compare all results within the framework of the same QFT, i.e., within the same energy scale. This requirement implies that the asymptotic metric and ϕ_A values must be the same for all solutions. Since this is generally not true, it is necessary to make a rescaling of coordinates $(\tilde{t}, \tilde{\vec{x}}, \tilde{r})$ requiring the metric to be asymptotically AdS_5 and the dual QFT to share the common energy scale, $\tilde{\phi}_A$. The expressions are—see Appendix D

$$T = \frac{e^{\tilde{A}(\tilde{r})}}{4\pi}\frac{d\tilde{h}}{d\tilde{r}}\bigg|_{\tilde{r}=\tilde{r}_{\rm H}}, \qquad s = \frac{2\pi}{\kappa^2}e^{3\tilde{A}(\tilde{r})}\bigg|_{\tilde{r}=\tilde{r}_{\rm H}}, \tag{7}$$

where all quantities are expressed in this new set of rescaled coordinates such that the metric asymptotically approaches canonical AdS_5 and the QFT energy scale is given by $\tilde{\phi}_A$—see section III C for the expressions we used.

G. IR and UV limits

The holographic coordinate r encodes the energy scale of the boundary theory. It can be demonstrated that large values of r correspond to high energies—ultraviolet (UV)—while small values, in particular those near the black brane horizon, correspond to low energies—infrared (IR)— [5].

To ensure an asymptotically AdS_5 geometry as $r \to \infty$, the scalar field must vanish in this limit, as shown in Appendix C. Therefore, in terms of the scalar field, the UV limit corresponds to $\phi \to 0$, while near the black brane horizon the scalar field approaches a finite value $\phi(r_{\rm H}) =: \phi_H$, which may or may not be close to the minimum of the potential. Since $r_{\rm H}$ is a finite value this corresponds to the IR limit.

The full solution can then be understood as the result of integrating the EOMs from a value $\phi_{\rm H}$—associated with the IR region—to the UV point at $\phi = 0$. The thermodynamic quantities T and s of each black brane solution, which depend on the functions $A(r)$ and $h(r)$, are then determined by the scalar field profile—particularly by $\phi_{\rm H}$—and the shape of the potential $V(\phi)$. To obtain different black brane solutions, we integrate the EOMs for different values of $\phi_{\rm H}$, perturbing the SAdS_5 geometry.

III. CALCULATION

A. Initial conditions

We have three functions—$\phi(r)$, $A(r)$, and $h(r)$—governed by second-order ODEs. This would normally require six initial conditions, which we choose to specify at the horizon $r_{\rm H} = 1$. However, the constraint relates the functions and their derivatives, reducing the number of independent initial conditions from six to five, and imposing regularity of the solution near the horizon introduces an additional condition, leaving only four independent initial conditions.

The initial conditions are set as follows. By definition, $\phi(r_{\rm H}) = \phi_{\rm H}$, with $\phi_{\rm H} \in (\phi_{\rm min}, \phi_{\rm max})$, where $\phi_{\rm min} = 0$ and $\phi_{\rm max} \approx 3.08$ correspond to the values of the scalar field at the maximum and minimum of the potential (2), respectively. The value $A(r_{\rm H}) = 0$ is fixed by rescaling t and $\vec{x}$. By the black brane ansatz, $h(r_{\rm H}) = 0$. Finally, we set $h'(r_{\rm H}) = 1/L$ by rescaling t.

To ensure regularity and avoid divergences, the functions must admit a Taylor expansion near the horizon. The initial conditions used are then evaluated at $r = r_{\rm H} + \varepsilon$, with $\varepsilon = 0.001$. Hence, $\phi_0 = \phi_{\rm H}$, $A_0 = 0$, $h_0 = 0$, and $h_1 = 1/L$, while the remaining coefficients are determined from the EOMs and the constraint using `Series` and `Solve` in Mathematica. Thus, the input of initial conditions must be

$$X(r_{\rm H}+\varepsilon) = \sum_{k=0}^{\infty} X_k \varepsilon^k, \; X'(r_{\rm H}+\varepsilon) = \sum_{k=1}^{\infty} k X_k \varepsilon^{k-1}, \quad (8)$$

for each $X \in \{\phi, A, h\}$, where the coefficients $\{X_i\}$ are determined from the expansion. Note that we now recover the six initial conditions originally expected, since the constraint and the regularity condition are now implicitly present.

B. Numerical integration and parameters

We integrate the EOMs (5) for each value of $\phi_{\rm H}$ using `NDSolve` in Mathematica, with the initial conditions given in (8), and the functions $\phi(r)$, $A(r)$, and $h(r)$ over the interval $r \in [r_{\rm H} + \varepsilon, r_{\rm lim}]$. Appendix E illustrates numerical solutions corresponding to different values of $\phi_{\rm H}$.

Below is a summary of all relevant parameters: $r_{\rm H} = 1$, $L = 1$, $\phi_{\rm M} = 1$, $\phi_{\rm Q} = 10$, $\kappa = 1$, $\phi_0 = \phi_{\rm H} \in (\phi_{\rm min}, \phi_{\rm max})$, $A_0 = 0$, $h_0 = 0$, $h_1 = 1/L$, $\varepsilon = 0.001$, and $\delta = 0.001$—to ensure that the interval for $\phi_{\rm H}$ is open. It is also important to note that the sampling of $\phi_{\rm H}$ values is non-uniform, following a Gaussian-like distribution to better resolve the transition region. The upper limit of integration is set to $r_{\rm lim} = 10$. The expansions for the initial conditions are performed up to third order around $r = r_{\rm H}$. As discussed in subsection III C, $\nu = 1$, and $r_{\rm AdS}$ is defined as the value of the holographic coordinate such that $|h'(r_{\rm AdS})| < 10^{-10}$.

C. UV normalization

As explained at the end of subsection II F, although the expressions for temperature and entropy only require the IR behaviour of the functions, we must integrate the EOMs up to the UV. This is because normalization of the AdS_5 spacetime to its canonical form is necessary so that all thermal states obtained correspond to the same QFT, meaning they have the same energy scale $\tilde{\phi}_A$. Hence, we must express our metric so that it matches the canonical AdS_5 form with $\tilde{\phi}_A$. The coordinate transformation given in (C22), derived in Appendix C, ensures this condition. Expressing Eqs. (7) using this change of coordinates—see Appendix F—yields the following formulas for T and s, expressed so that they do not depend on the energy scale chosen,

$$\frac{T}{\tilde{\phi}_A^{1/\nu}} = \frac{1}{4\pi} \frac{1}{L\phi_A^{1/\nu}\sqrt{h_0^{\rm far}}}, \qquad \frac{s}{\tilde{\phi}_A^{3/\nu}} = \frac{2\pi}{\kappa^2}\frac{1}{\phi_A^{3/\nu}}, \quad (9)$$

where $h_0^{\rm far}$ and ϕ_A are defined in the UV expansions (C12) and (C15), respectively, and ν is defined in (C8). In our case, $\nu = 1$, as derived in Appendix C. We extract these values by evaluating $h(r)$ and $\phi(r)$ in the limit $r \to \infty$ and solving the equations for them. In practice, we define a finite cutoff $r_{\rm AdS}$ that ensures that asymptotic AdS_5 behaviour has been reached. One option is to demand that $|h'(r_{\rm AdS})|$ be less than an arbitrarily small number, e.g., 10^{-10}, in view of $h(r)$ UV behaviour. Substituting into the expansions (C12) and (C15) yields

$$\phi_A = \phi(r_{\rm AdS}) e^{\nu A(r_{\rm AdS})}, \qquad h_0^{\rm far} = h(r_{\rm AdS}). \quad (10)$$

IV. EQUATION OF STATE

Conditions of thermodynamic stability imply that the specific—here understood as per unit volume—heat capacity at constant chemical potential, defined by

$$c_\mu := T\left(\frac{\partial s}{\partial T}\right)_\mu, \quad (11)$$

must be positive. As shown in Fig. 2, there is a region of instability and hence the Maxwell construction must be performed. As a result, the function $s(T)$ exhibits a discontinuity, i.e., a discontinuity in the first derivative of the free energy with respect to temperature, namely a first-order phase transition.

In the conformal case, thermodynamics is fixed by conformal symmetry, and it is well known that $s \propto T^3$, i.e., s/T^3 remains constant [2, 5]. As shown in Fig. 3, conformal behaviour is observed at both high and low temperatures. These asymptotic limits correspond to values of $\phi_{\rm H} = \phi_{\rm min} = 0$—at high temperatures—and $\phi_{\rm H} = \phi_{\rm max} \approx 3.08$—at low temperature—which are precisely the values of ϕ such that $V(\phi)$

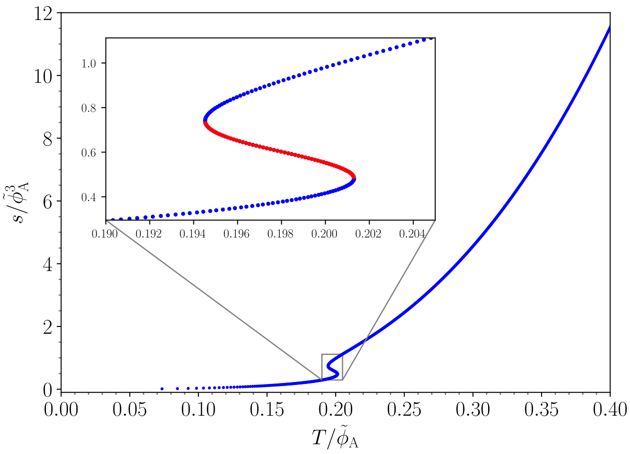

Figure 2: Equation of state $s(T)$ of the non-conformal quantum field theory obtained using holography.

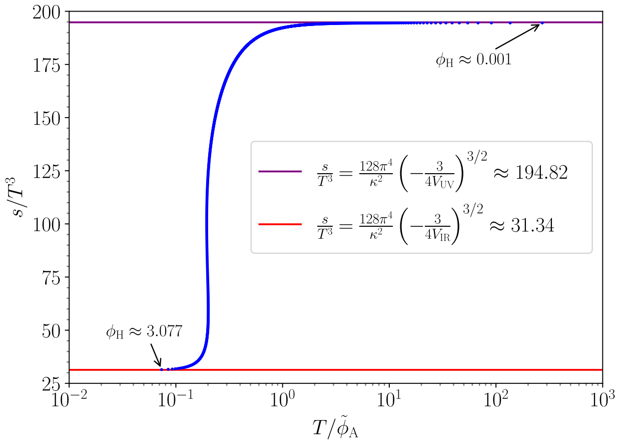

Figure 3: Entropy density over T^3, as a function of temperature. The two extreme values of ϕ_H and the analytically expected asymptotic behaviour—see Eq. (H12)—are indicated.

attains the maximum $V(\phi_{\min}) = -12$ and the minimum $V(\phi_{\max}) \approx -40.57$, respectively—see Fig. 1. According to the proposition proven in Appendix G—which states that the scalar field remains constant, $\phi(r) = \phi_H$, if and only if ϕ_H is an extremum of $V(\phi)$—we conclude that in both cases $\phi(r) = \phi_H$ for all r. Therefore, as shown in Appendix H, the corresponding solutions are $SAdS_5$ black branes. We can thus use Eq. (H12) to compare the numeric trend with the analytic values, $s/T^3 \approx 31.34$ and $s/T^3 \approx 194.82$.

In contrast, the intermediate region exhibits non-conformal behaviour. The corresponding values of ϕ_H are neither a maximum nor a minimum of $V(\phi)$, so, by the above proposition, the scalar field has a non-trivial profile, $\phi(r) \neq \phi_H$, caused by the $dV/d\phi$ term in (5). This affects the metric functions $A(r)$ and $h(r)$, leading to deformations of the $SAdS_5$ geometry—more exotic black brane solutions. Since these functions determine the thermodynamic behaviour of the dual QFT—see Eqs. (9)—their deformation gives rise to richer thermodynamic phenomena, such as the first-order phase transitions observed in this model, in contrast with $\mathcal{N} = 4$ SYM theory, which features only a plasma phase and no phase transitions [5]. The presence of a first-order phase transition in our model highlights the departure from conformality, confirming the effectiveness of the scalar potential chosen in generating non-conformal behaviour.

These results align with subsection II G: $\phi_H \to 0$ corresponds to the UV—high temperatures—while $\phi_H \neq 0$ corresponds to the IR—low temperatures.

V. CONCLUSIONS

We obtain the EOS of a non-conformal QFT via holography. By introducing a scalar field, we deform the $SAdS_5$ black brane solutions by tuning different values of ϕ_H. Each configuration corresponds to a different thermal state characterized by a specific temperature and entropy. The scalar potential used in this model gives rise to a first-order phase transition between two conformal regimes—each characterized by a distinct effective cosmological constant—while conformal symmetry is broken in the intermediate region.

In future work, we aim to include a Maxwell field to study the system at non-zero chemical potential $\mu \neq 0$, and to explore alternative scalar potentials that could lead to different kinds of phase transitions.

Acknowledgments

I would like to express my deepest gratitude to my advisor, Dr. Yago Bea, for giving me the opportunity to carry out this work, as well as for his trust and guidance. I also wish to thank my friends and family for their encouragement and support throughout these years.

---

[1] J. D. Wells, Springer (2012).
[2] M. Ammon, J. Erdmenger, Cambridge University Press (2015).
[3] J. M. Maldacena, Adv. Theor. Math. Phys., **2**, 231-252 (1998).
[4] B. Zwiebach, Cambridge University Press (2004).
[5] M. Natsuume, Springer (2015).
[6] Y. Bea *et al.*, Phys. Rev. D **104**, L121903 (2021).
[7] O. DeWolfe, S. S. Gubser and C. Rosen, Phys. Rev. D **83**, 086005 (2011).
[8] S. S. Gubser *et al.*, Phys. Rev. Lett. **101**, 131601 (2008).

Holografia i termodinàmica d'una teoria quàntica de camps no conforme

Author: Adrià Olives-Salmerón, aolivesa80@alumnes.ub.edu
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Yago Bea, yagobea@ub.edu
(Dated: June 9, 2025)

Resum: En aquest treball, estudiem la termodinàmica d'una QFT no conforme mitjançant l'holografia, en particular la correspondència AdS/CFT. La descripció dual consisteix en un espaitemps de AdS de cinc dimensions amb gravetat d'Einstein acoblada a un camp escalar que trenca la simetria conforme, amb l'objectiu de modelar sistemes no conformes inspirats per la QCD. Resolent numèricament les equacions d'Einstein-Klein-Gordon, construïm solucions de branes negres per obtenir la temperatura i l'entropia associada als estats tèrmics de la QFT dual. La representació $s/T^3(T)$ mostra que, a altes i baixes temperatures, s'obté el comportament característic d'una CFT, mentre que en la regió intermèdia trobem un comportament no conforme.
Paraules clau: holografia, correspondència AdS/CFT, teories quàntiques de camps, cromodinàmica quàntica, branes negres, equació d'estat.
ODSs: 4. Educació de qualitat i 9. Indústria, innovació, infraestructures.

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures	X		

El contingut d'aquest TFG, del grau universitari de Física i centrat en el desenvolupament de tècniques de recerca avançades, es relaciona amb l'ODS 4. Educació de qualitat, i en particular amb la fita 4.4, ja que contribueix a la formació universitària en metodologies científiques de gran rellevància. A més a més, també es pot vincular amb l'ODS 9. Indústria, innovació i infraestructures, i en particular amb la fita 9.4, atès que aquest àmbit de la física teòrica contribueix a una millor comprensió fonamental de les lleis de la natura, imprescindible per impulsar el progrés industrial i tecnològic de manera sostenible.

Appendix A: Units and dimensions

Natural unit systems are systems in which certain physical constants are set to unity. As a result, some physical dimensions become equivalent and can be expressed in terms of a reduced set of base dimensions.

1. Gravitational side

The standard choice of natural units in classical gravitation is

$$c = G = k_{\mathrm{B}} = 1, \tag{A1}$$

where c is the speed of light, G is Newton's gravitational constant, and k_{B} is Boltzmann's constant. This unit system is known as the geometrized unit system.

In this work, however, we adopt the convention

$$c = k_{\mathrm{B}} = 1, \quad G = \frac{1}{8\pi}, \tag{A2}$$

which we refer to as a "reduced geometrized units". In ordinary units, all quantities relevant to our analysis have dimensions that can be expressed as powers of length L, mass M, time T, and absolute temperature Θ. By contrast, in the reduced geometrized units, all such quantities share a common dimension, e.g., length L. That is,

$$\mathsf{L} = \mathsf{M} = \mathsf{T} = \Theta = \mathsf{E}. \tag{A3}$$

It is important to note that we can not set $\hbar = 1$ in classical gravity, since quantum effects are neglected. In the limit $\hbar \to 0$, the uncertainty principle becomes trivial, and assigning $\hbar$ a finite value would be inconsistent with the classical regime.

In addition to setting certain fundamental constants to be pure numbers, we also fix a dimensionless length scale by choosing the AdS curvature radius to be $L = 1$. As a result, all base units collapse into dimensionless derived unit,

$$\mathsf{L} = \mathsf{M} = \mathsf{T} = \Theta = \mathsf{E} = 1. \tag{A4}$$

Hence, all physical quantities on the gravitational side are dimensionless—represented by pure numbers—in this unit system.

2. Quantum field theory side

The standard choice of natural units in QFT is

$$\hbar = c = k_{\mathrm{B}} = 1. \tag{A5}$$

In this unit system,

$$\mathsf{L} = \mathsf{M}^{-1} = \mathsf{T} = \Theta^{-1} = \mathsf{E}^{-1}. \tag{A6}$$

Therefore, on the QFT side, we work in energy units.

Appendix B: Derivation of the equations of motion

In the next, we derive the system of ODEs (5) and (6).

In view of the Lagrangian (3), and the ansatz $\phi = \phi(r)$ and (4), the action is

$$\begin{aligned} S &= \int d^5x\sqrt{-g}\;\mathcal{L} \\ &= \int dt d\vec{x} \int dr \sqrt{-g}\frac{1}{2\kappa^2}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right] \\ &= \frac{V_4}{2\kappa^2}\int dr\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]. \end{aligned} \tag{B1}$$

Since a constant factor in the action does not alter the EOMs, we can derive them for the holographic coordinate action/Lagrangian defined as

$$S_r := \int dr\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right] =: \int dr \mathcal{L}_r. \tag{B2}$$

Since we are going to use Euler-Lagrange equations it is important to include the determinant of the metric tensor matrix in the Lagrangian. To construct our Lagrangian we need the following ingredients.

- The value of $\sqrt{-g}$.

$$\begin{aligned} \sqrt{-g} &= \sqrt{-\left(-h(r)e^{2A(r)}\right)\left(e^{2A(r)}\right)^3\left(\frac{e^{2B(r)}}{h(r)}\right)} \\ &= e^{4A(r)+B(r)} \end{aligned} \tag{B3}$$

- The Ricci scalar R. The only non-vanishing Christoffel symbols are the next thirteen

$$\begin{aligned} \Gamma^t_{tr} &= \Gamma^t_{rt} = A' + \frac{h'}{2h}, \\ \Gamma^i_{ir} &= \Gamma^i_{ri} = A', \\ \Gamma^r_{tt} &= \frac{1}{2}he^{2(A-B)}\left(h' + 2A'h\right), \\ \Gamma^r_{ii} &= -A'he^{2(A-B)}, \\ \Gamma^r_{rr} &= B' + \frac{h'}{2h}, \end{aligned}$$

where we use i for x, y, z coordinates and we omit the function dependencies for simplicity. Therefore,

$$\begin{aligned} R = e^{-2B}&\left(-8A''h - h'' - 20A'^2h\right. \\ &\left.+8A'B'h - 9A'h' + B'h'\right). \end{aligned} \tag{B4}$$

- The scalar product $(\partial\phi)^2$.

$$\begin{aligned} (\partial\phi)^2 &= \partial_\mu\phi\partial^\mu\phi \\ &= g^{\mu\nu}\partial_\mu\phi(r)\partial_\nu\phi(r) \\ &= g^{rr}\partial_r\phi\partial_r\phi \\ &= he^{-2B}\phi'^2. \end{aligned} \tag{B5}$$

Introducing (B3)-(B5) in the $\mathcal{L}_r$ yields

$$\mathcal{L}_r = e^{4A-B}\Big[-8A''h - h'' - 20A'^2h + 8A'B'h - 9A'h' + B'h' - \frac{1}{2}h\phi'^2 - e^{2B}V(\phi)\Big]. \tag{B6}$$

Now we need to use the Euler-Lagrange equations with second-order derivatives, namely

$$\frac{d^2}{dr^2}\left(\frac{\partial \mathcal{L}_r}{\partial f''}\right) - \frac{d}{dr}\left(\frac{\partial \mathcal{L}_r}{\partial f'}\right) + \frac{\partial \mathcal{L}_r}{\partial f} = 0, \tag{B7}$$

with $f \in \{\phi, A, h\}$.

For ϕ straightforwardly leads to the ODE

$$\phi'' + \left(4A' - B' + \frac{h'}{h}\right)\phi' - \frac{e^{2B}}{h}\frac{dV}{d\phi} = 0. \tag{B8}$$

For B also straightforwardly leads to the ODE

$$h(24A'^2 - \phi'^2) + 6A'h' + 2e^{2B}V(\phi) = 0, \tag{B9}$$

that is, the constraint.

For h straightforwardly leads to the ODE

$$A'' - A'B' + \frac{1}{6}\phi'^2 = 0. \tag{B10}$$

Finally, for A leads to the ODE

$$-24\left(A'' + 2A'^2 - A'B' + \frac{1}{12}\phi'^2\right)h - 3h'' - 3\left(8A' - B'\right)h' - 4e^{2B}V(\phi) = 0,$$

and using (B9) and (B10) one finds that

$$h'' + (4A' - B')h' = 0. \tag{B11}$$

Appendix C: UV expansions of ϕ, A, and h, and UV normalization

Our metric for a generic choice of scaled-coordinates—defined with respect to the original ones by an affine transformation—is

$$d\tilde{s}^2 = e^{2\tilde{A}(\tilde{r})}\left[-\tilde{h}(\tilde{r})d\tilde{t}^2 + d\tilde{\vec{x}}^2\right] + \frac{1}{\tilde{h}(\tilde{r})}d\tilde{r}^2. \tag{C1}$$

AdS_5 metric in Poincaré patch coordinates [2] is

$$ds^2 = \frac{\rho^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{\rho^2}d\rho^2, \tag{C2}$$

where $\rho \in [0,\infty)$, but let us define a new coordinate r such that $\frac{\rho}{L} = e^{\frac{r}{L}}$, hence $r \in (-\infty,\infty)$. Then,

$$ds^2 = e^{2\frac{r}{L}}\left(-dt^2 + d\vec{x}^2\right) + dr^2. \tag{C3}$$

Demanding our metric to be asymptotically—when $\tilde{r} \to \infty$—AdS_5 in the canonical form, i.e., without scaling factors, we obtain by comparison between (C1) and (C3)

$$\tilde{A}(\tilde{r} \to \infty) := \lim_{\tilde{r}\to\infty}\tilde{A}(\tilde{r}) = \frac{\tilde{r}}{L}, \tag{C4}$$

and

$$\tilde{h}(\tilde{r} \to \infty) := \lim_{\tilde{r}\to\infty}\tilde{h}(\tilde{r}) = 1, \tag{C5}$$

and hence $\tilde{A}'(\tilde{r} \to \infty) = \frac{1}{L}$, $\tilde{A}''(\tilde{r} \to \infty) = 0$, and $\tilde{h}'(\tilde{r} \to \infty) = \tilde{h}''(\tilde{r} \to \infty) = 0$.

If $\phi(r) = 0$ or $\phi(r) = \phi_H = \text{const.}$, then

$$\tilde{V} = -12\tilde{A}'^2\tilde{h} = -12/L^2$$

and the trend of $\tilde{\phi}$ is trivially—note that if the coordinates are not scaled, this value could be different.

Let us study the case where $\phi(r)$ is not trivial. Plugging everything into the EOMs (5) and constraint (6) yields different information:

1. $\tilde{\phi}'(\tilde{r} \to \infty) = 0$, implying that $\tilde{\phi}(\tilde{r} \to \infty) = \text{const.}$,
2. $\tilde{V}'(\tilde{\phi}(\tilde{r} \to \infty)) = 0$, i.e., the potential $\tilde{V}(\phi)$ has no $\tilde{\phi}$ term, and has an extremum at this value,
3. $\tilde{V}(\tilde{\phi}(\tilde{r} \to \infty)) = -\frac{12}{L^2} = \text{const.}$, which if $\phi \in (\phi_{\min}, \phi_{\max})$ as in our case, that implies $\tilde{\phi}(\tilde{r} \to \infty) = 0$.

By performing an iterative process, we can obtain an explicit dependence for $\tilde{\phi}$. Let us introduce new parameters. The potential can be expanded as a Taylor series around $\tilde{\phi} = 0$ in the UV limit. We know that there is no linear term, and also we know from section II F that the second-order term is just the mass term of Klein-Gordon Lagrangian. Setting $\phi(r) = \tilde{\phi}(\tilde{r})$—scalar field is invariant under a change of coordinates—we obtain

$$\tilde{V}(\tilde{\phi}) = -\frac{12}{L^2} + \frac{1}{2}m_\phi^2\tilde{\phi}^2 + \mathcal{O}(\tilde{\phi}^3). \tag{C6}$$

The dimension of the operator $\mathcal{O}_\phi$, which is dual to ϕ, is Δ_ϕ and it is defined as

$$m_\phi^2 L^2 =: \Delta_\phi(\Delta_\phi - 4). \tag{C7}$$

Here we consider the case where $\Delta_\phi \in (2,4)$ [8]. We also define

$$\nu := 4 - \Delta_\phi. \tag{C8}$$

Introducing again all the above information into the third EOM but computing it for $\tilde{\phi}(\tilde{r} \to \infty)$ relaxing the conditions of the null limits, we find

$$\tilde{\phi}''(\tilde{r} \to \infty) + \frac{4}{L}\tilde{\phi}'(\tilde{r} \to \infty) - \frac{\nu(\nu-4)}{L^2}\tilde{\phi}(\tilde{r} \to \infty) = 0,$$

that yields

$$\tilde{\phi}(\tilde{r} \to \infty) = \tilde{\phi}_A e^{-\frac{\nu}{L}\tilde{r}} + \tilde{\phi}_B e^{-\frac{(4-\nu)}{L}\tilde{r}}.$$

For our potential (2) we have that $\Delta_\phi = 4 - \nu = 3$, because comparing the second-order coefficient of (2) with (C6) yields $m_\phi^2 = -3/L^2$, that implies $\Delta_\phi = 3$. Hence, the second term of $\tilde{\phi}(\tilde{r} \to \infty)$ can be neglected, consequently

$$\tilde{\phi}(\tilde{r} \to \infty) = \tilde{\phi}_A e^{-\frac{\nu}{L}\tilde{r}}. \tag{C9}$$

Therefore,

$$\therefore \quad \begin{aligned} \tilde{\phi}(\tilde{r} \to \infty) &= \tilde{\phi}_A e^{-\frac{\nu}{L}\tilde{r}}, \\ \tilde{A}(\tilde{r} \to \infty) &= \frac{\tilde{r}}{L}, \\ \tilde{h}(\tilde{r} \to \infty) &= 1. \end{aligned} \tag{C10}$$

This asymptotic behaviour corresponds to canonical AdS_5.

We are now in a position to derive the asymptotic expansions of the functions ϕ, A, and h in the UV limit using another coordinates that asymptotically do not match with canonical AdS_5—instead, match with scaled AdS_5. However, we must first establish that $\phi(r \to \infty) = 0$, and consequently, $\phi'(r \to \infty) = \phi''(r \to \infty) = 0$, and that $V(\phi(r \to \infty)) = -\frac{12}{L^2}$, as proven before. Inserting this information into the EOMs (5) and (6) yields further constraints on the UV behaviour of the fields, which will guide the structure of their asymptotic expansions. In what follows, we adopt the constants so that they match those used in [?].

From the first equation in (5), we obtain trivially

$$A(r) = \frac{A_{-1}^{\text{far}}}{L} r + A_0^{\text{far}} =: \alpha(r), \tag{C11}$$

straightforwardly from the second one,

$$h(r) = h_0^{\text{far}} + h_4^{\text{far}} e^{-4\alpha(r)}, \tag{C12}$$

the third one simply verifies that $V'(\phi) = 0$, and for the constraint (6), we obtain the relation

$$V(\phi(r \to \infty)) = -\frac{12}{L^2} {A_{-1}^{\text{far}}}^2 h_0^{\text{far}} \stackrel{!}{=} -\frac{12}{L^2},$$

that imposing the value of $V(\phi(r \to \infty))$, yields the following relation

$$A_{-1}^{\text{far}} = \frac{1}{\sqrt{h_0^{\text{far}}}}. \tag{C13}$$

The negative sign has been discarded, as it leads to a behaviour of $A(r)$ that is inconsistent with the form required by Eq. (C4).

Before computing the UV expansion of the scalar field, let us follow the same argument as in the canonical case. The potential can be expanded as a Taylor series around $\phi = 0$ in the UV limit. We know that there is no linear term, and also we know from section II F that the second-order term is just the mass term of Klein-Gordon Lagrangian. Therefore,

$$V(\phi) = -\frac{12}{L^2} + \frac{1}{2} m_\phi^2 \phi^2 + \mathcal{O}(\phi^3). \tag{C14}$$

This finally allows us to determine the UV expansion of the scalar field. Plugging all of the above into the third ODE in (5), in order to solve it for ϕ, and neglecting the exponential terms—since $r \to \infty$ and $A_{-1}^{\text{far}} > 0$, as implied by its particular form in (C4)—yields

$$\phi'' + \left(4\frac{A_{-1}^{\text{far}}}{L}\right)\phi' - {A_{-1}^{\text{far}}}^2 \frac{\nu(\nu-4)}{L^2}\phi = 0,$$

whose solution is

$$\begin{aligned} \phi(r) &= C_1 e^{-\nu \frac{A_{-1}^{\text{far}}}{L} r} + C_2 e^{-(4-\nu)\frac{A_{-1}^{\text{far}}}{L} r} \\ &= C_1 e^{-\nu(\alpha(r) - A_0^{\text{far}})} + C_2 e^{-(4-\nu)(\alpha(r) - A_0^{\text{far}})} \\ &= C_1 e^{\nu A_0^{\text{far}}} e^{-\nu\alpha(r)} + C_2 e^{(4-\nu)A_0^{\text{far}}} e^{-(4-\nu)\alpha(r)} \end{aligned}$$

and redefining the constants

$$\phi(r) = \phi_A e^{-\nu\alpha(r)} + \phi_B e^{-(4-\nu)\alpha(r)}. \tag{C15}$$

Therefore,

$$\therefore \quad \begin{aligned} \phi(r \to \infty) &= \phi_A e^{-\nu\alpha(r)} + \phi_B e^{-(4-\nu)\alpha(r)}, \\ A(r \to \infty) &= \frac{A_{-1}^{\text{far}}}{L} r + A_0^{\text{far}} =: \alpha(r), \\ h(r \to \infty) &= h_0^{\text{far}} + h_4^{\text{far}} e^{-4\alpha(r)}. \end{aligned} \tag{C16}$$

This asymptotic behaviour corresponds to scaled AdS_5.

Finally, using (C10) and (C16), we can establish the relation between both coordinate systems in the UV limit. It is important to remark that although we compare both coordinates in the UV limit it must be true for all values of the holographic coordinate since we are only scaling coordinates using an affine transformation.

Recalling that $\tilde{\phi}(\tilde{r} \to \infty) = \phi(r \to \infty)$ and $\Delta_\phi \sim 4 \Leftrightarrow \nu \ll 1$, we have

$$\frac{\tilde{r}}{L} = \alpha(r) - \ln\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu} = \frac{A_{-1}^{\text{far}}}{L} r + A_0^{\text{far}} - \ln\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu}. \tag{C17}$$

It then immediately follows that

$$\tilde{A}(\tilde{r}) = A(r) - \ln\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu}. \tag{C18}$$

Setting $d\tilde{s}^2 = ds^2$,

$$\vec{\tilde{x}} = \left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu} \vec{x}, \tag{C19}$$

and

$$\tilde{t} = \left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu} \sqrt{h_0^{\text{far}}}\, t. \tag{C20}$$

Moreover, by direct comparison,

$$\tilde{h}(\tilde{r}) = \frac{1}{h_0^{\text{far}}} h(r). \tag{C21}$$

In summary,

$$\begin{aligned}
\tilde{t} &= \left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu} \sqrt{h_0^{\text{far}}}\, t,\\
\tilde{\vec{x}} &= \left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu} \vec{x},\\
\frac{\tilde{r}}{L} &= \alpha(r) - \ln\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu} = \frac{A_{-1}^{\text{far}}}{L} r + A_0^{\text{far}} - \ln\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu},\\
\tilde{A}(\tilde{r}) &= A(r) - \ln\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu},\\
\tilde{h}(\tilde{r}) &= \frac{1}{h_0^{\text{far}}} h(r).
\end{aligned} \tag{C22}$$

Appendix D: Derivation of the equations for T and s

In this Appendix, we derive Eqs. (7) starting from [2, p. 82, Eq. (2.159)]. If the metric is

$$ds^2 = -f(r)dt^2 + \cdots + \frac{1}{g(r)} dr^2, \tag{D1}$$

then the temperature at the horizon is

$$T_{\text{H}} = \frac{\sqrt{f'(r_{\text{H}})g'(r_{\text{H}})}}{4\pi}. \tag{D2}$$

In view of our metric,

$$d\tilde{s}^2 = e^{2\tilde{A}(\tilde{r})}\left[-\tilde{h}(\tilde{r})d\tilde{t}^2 + d\tilde{\vec{x}}^2\right] + \frac{1}{\tilde{h}(\tilde{r})} d\tilde{r}^2, \tag{D3}$$

we have $f(\tilde{r}) = \tilde{h}(\tilde{r})e^{2\tilde{A}(\tilde{r})}$ and $g(\tilde{r}) = \tilde{h}(\tilde{r})$, hence $f'(\tilde{r}) = \left(\tilde{h}'(\tilde{r}) + 2\tilde{A}'(\tilde{r})\tilde{h}(\tilde{r})\right) e^{2\tilde{A}(\tilde{r})}$ and $g'(\tilde{r}) = \tilde{h}'(\tilde{r})$. Then, the expression for the temperature is

$$\begin{aligned}
T_{\text{H}} &= \frac{\sqrt{\tilde{h}'(\tilde{r}_{\text{H}})e^{2\tilde{A}(\tilde{r}_{\text{H}})}\tilde{h}'(\tilde{r}_{\text{H}})}}{4\pi}\\
&= \frac{\tilde{h}'(\tilde{r}_{\text{H}})e^{\tilde{A}(\tilde{r}_{\text{H}})}}{4\pi}\\
&= \frac{e^{\tilde{A}(\tilde{r})}}{4\pi}\frac{d\tilde{h}}{d\tilde{r}}\bigg|_{\tilde{r}=\tilde{r}_{\text{H}}}.
\end{aligned} \tag{D4}$$

For the entropy density, we have the Bekenstein-Hawking equation [2, p. 84, Eq. (2.182)],

$$S = \frac{A}{4G} = \frac{2\pi A}{\kappa^2}, \tag{D5}$$

where A is the area of the black hole horizon. Computing it,

$$\begin{aligned}
A &= \int_{\tilde{r}=\tilde{r}_{\text{H}}} d\tilde{x}d\tilde{y}d\tilde{z}\sqrt{\tilde{g}_{\text{subspace}}}\\
&= \int_{\tilde{r}=\tilde{r}_{\text{H}}} d\tilde{x}d\tilde{y}d\tilde{z}\; e^{3\tilde{A}(\tilde{r})}\\
&= e^{3\tilde{A}(\tilde{r})}\int_{\tilde{r}=\tilde{r}_{\text{H}}} d\tilde{x}d\tilde{y}d\tilde{z}\\
&= e^{3\tilde{A}(r_{\text{H}})}\tilde{V}_3,
\end{aligned} \tag{D6}$$

where the determinant is computed as in (B3) but in this subspace, yields

$$s = \frac{2\pi}{\kappa^2} e^{3\tilde{A}(\tilde{r})}\bigg|_{\tilde{r}=\tilde{r}_{\text{H}}}. \tag{D7}$$

Appendix E: Examples of numerical solutions for different values of ϕ_{H}

We show the numerical profiles of $\phi(r)$, $A(r)$, and $h(r)$ for some values of ϕ_{H}, from the horizon $r = 1$ to the UV $r > 1$, where the geometry approaches AdS_5.

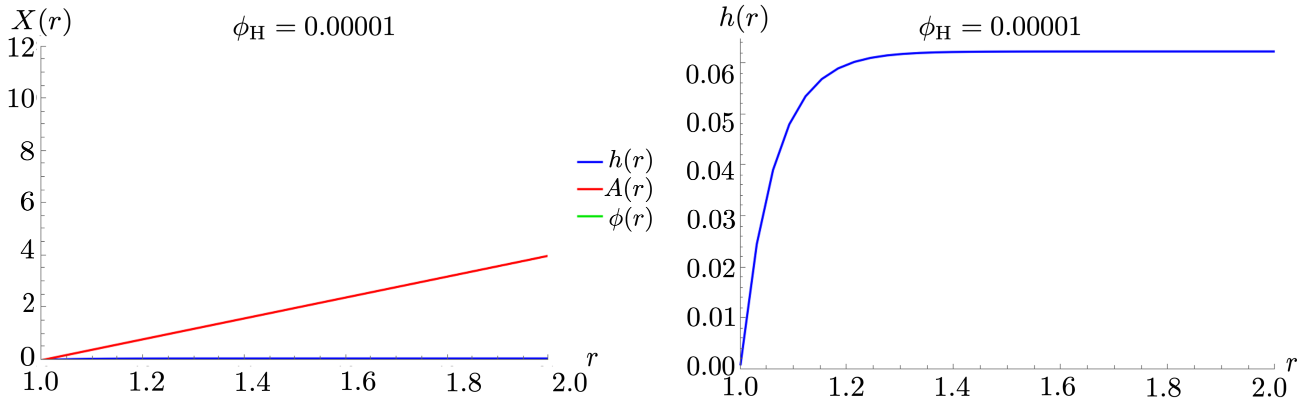

Figure 4: Numerical solution for $\phi(r)$, $A(r)$, and $h(r)$ with horizon value $\phi_{\rm H} = 0.00001$.

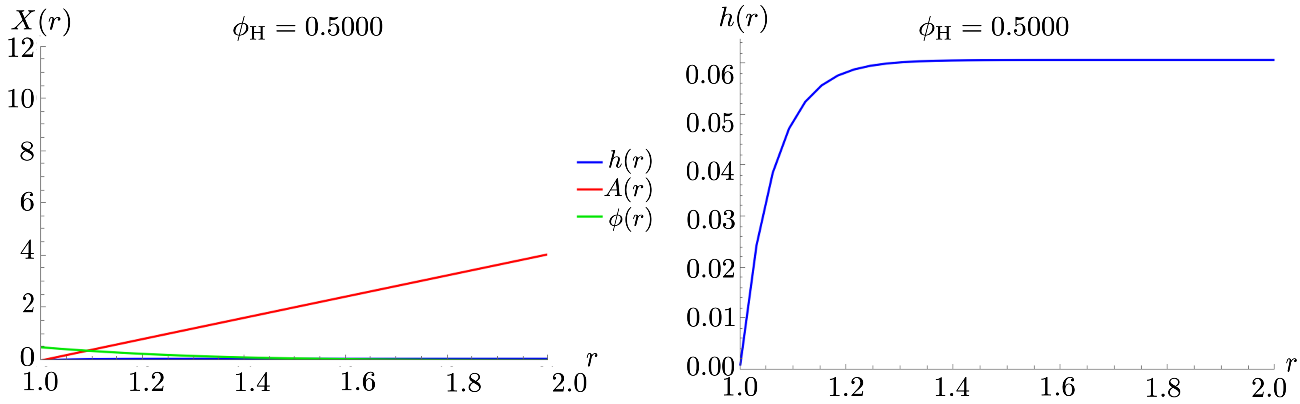

Figure 5: Numerical solution for $\phi(r)$, $A(r)$, and $h(r)$ with horizon value $\phi_{\rm H} = 0.50000$.

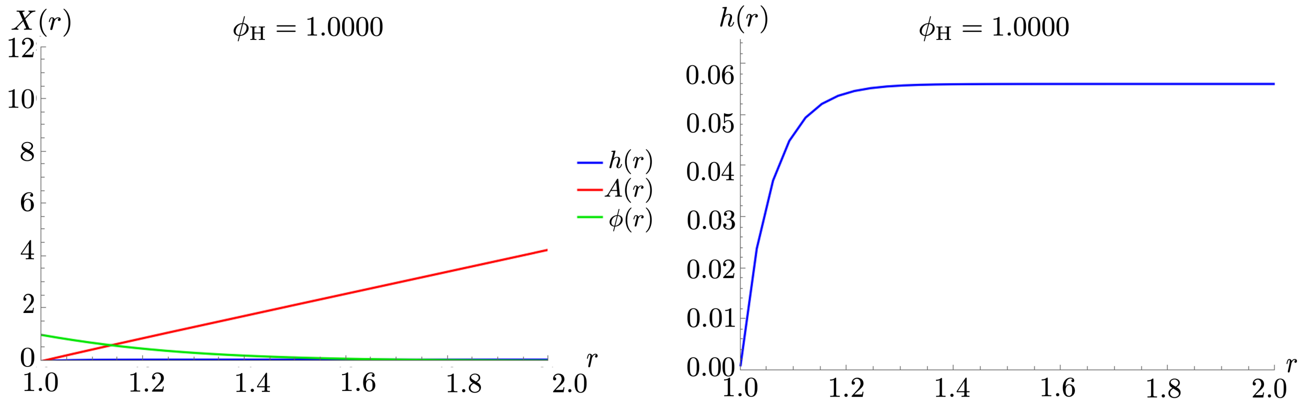

Figure 6: Numerical solution for $\phi(r)$, $A(r)$, and $h(r)$ with horizon value $\phi_{\rm H} = 1.00000$.

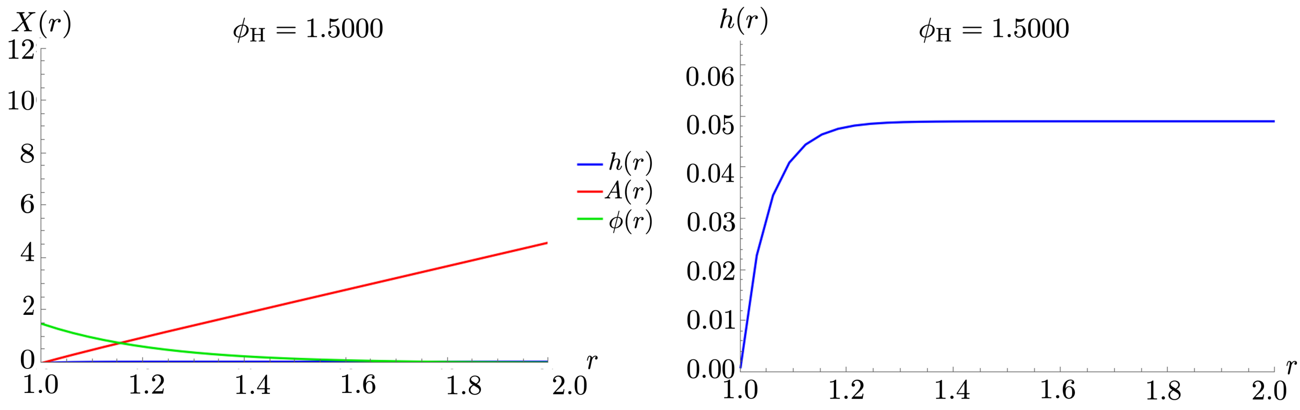

Figure 7: Numerical solution for $\phi(r)$, $A(r)$, and $h(r)$ with horizon value $\phi_H = 1.50000$.

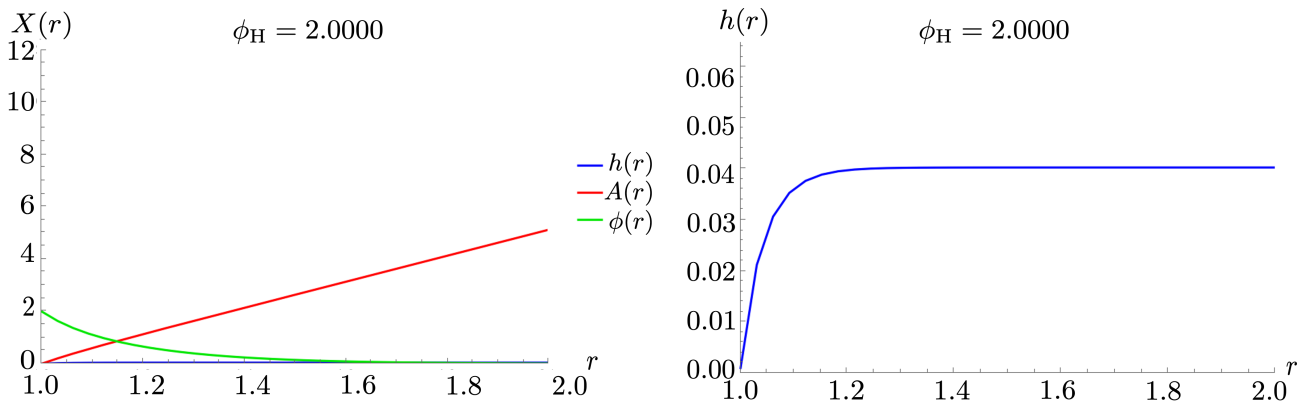

Figure 8: Numerical solution for $\phi(r)$, $A(r)$, and $h(r)$ with horizon value $\phi_H = 2.00000$.

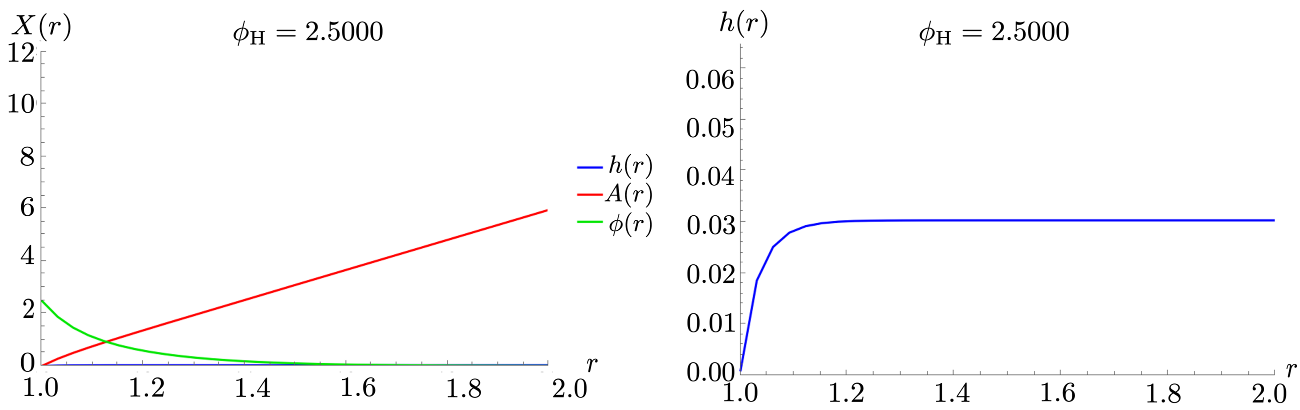

Figure 9: Numerical solution for $\phi(r)$, $A(r)$, and $h(r)$ with horizon value $\phi_H = 2.50000$.

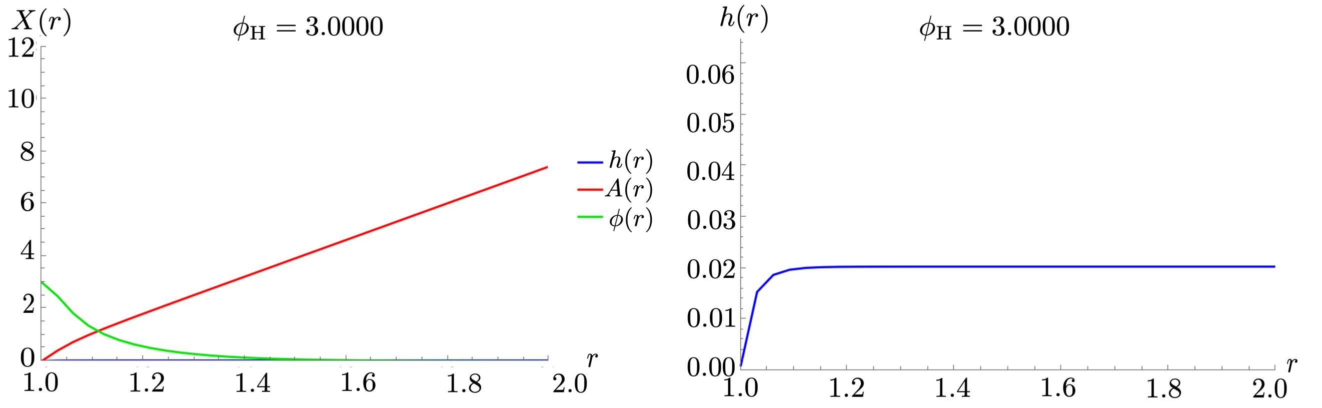

Figure 10: Numerical solution for $\phi(r)$, $A(r)$, and $h(r)$ with horizon value $\phi_H = 3.00000$.

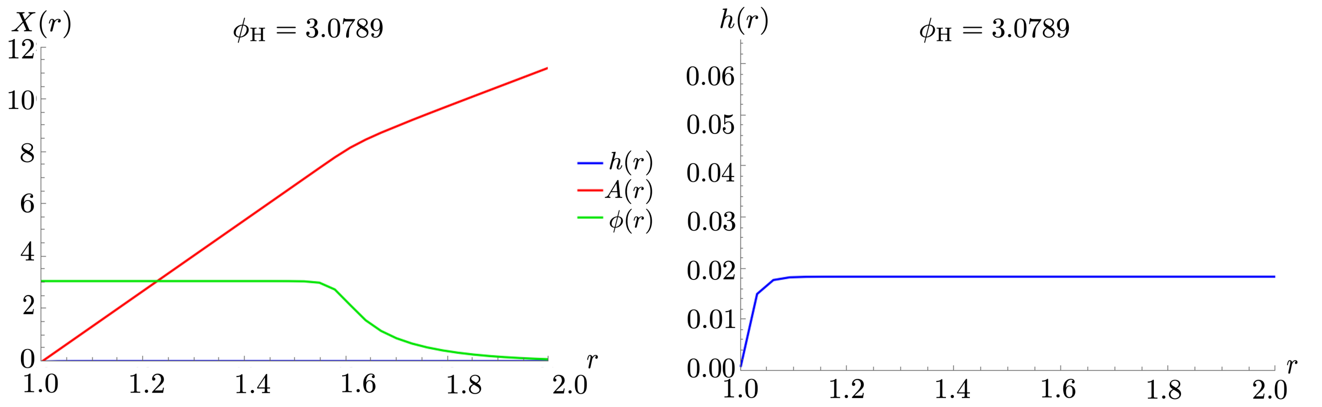

Figure 11: Numerical solution for $\phi(r)$, $A(r)$, and $h(r)$ with horizon value $\phi_H = 3.07891$.

Appendix F: Expressions for T and s in terms of UV quantities

This appendix is devoted to obtaining expressions for T and s in scaled coordinates, i.e., coordinates in which the metric is not asymptotically AdS_5 in its canonical form—a more general scenario that allows us to use these formulas even when solutions are scaled by arbitrary constants. This procedure is referred to as scaling, as it effectively normalizes the asymptotic form of AdS_5 to its canonical form.

The coordinate transformation is given in (C22). Let us start with the temperature T,

$$\begin{aligned} T &= \frac{e^{\tilde{A}(\tilde{r})}}{4\pi}\frac{d\tilde{h}}{d\tilde{r}}\bigg|_{\tilde{r}=\tilde{r}_{\rm H}} \\ &= \frac{e^{A(r)-\ln\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu}}}{4\pi}\frac{1}{h_0^{\rm far}}\frac{dh}{dr}\frac{dr}{d\tilde{r}}\bigg|_{r=r_{\rm H}} \\ &= \frac{1}{4\pi}\frac{1}{\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu}\sqrt{h_0^{\rm far}}}e^{A(r)}h'(r)\bigg|_{r=r_{\rm H}}, \end{aligned}$$

and plugging the initial conditions $A(r_{\rm H}) = 0$ and $h'(r_{\rm H}) = 1/L$ we obtain

$$\frac{T}{\tilde{\phi}_A^{1/\nu}} = \frac{1}{4\pi}\frac{1}{L\phi_A^{1/\nu}\sqrt{h_0^{\rm far}}}. \tag{F1}$$

Finally the entropy density s,

$$\begin{aligned} s &= \frac{2\pi}{\kappa^2}e^{3\tilde{A}(\tilde{r})}\bigg|_{\tilde{r}=\tilde{r}_{\rm H}} \\ &= \frac{2\pi}{\kappa^2}e^{3A(r)-3\ln\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{1/\nu}}\bigg|_{r=r_{\rm H}} \\ &= \frac{2\pi}{\kappa^2}e^{3A(r)}\left(\frac{\phi_A}{\tilde{\phi}_A}\right)^{-3/\nu}\bigg|_{r=r_{\rm H}}, \end{aligned}$$

and applying the initial condition $A(r_{\rm H}) = 0$ yields

$$\frac{s}{\tilde{\phi}_A^{3/\nu}} = \frac{2\pi}{\kappa^2}\frac{1}{\phi_A^{3/\nu}}. \tag{F2}$$

Appendix G: $\phi_{\rm H} = \phi_{\rm min}$ or $\phi_{\rm max} \Leftrightarrow \phi(r) = \phi_{\rm H}$

In this appendix, we prove an important result: if the scalar field at the horizon is located at either a maximum or a minimum of the potential, then it remains constant along the holographic direction and vice versa. The intuitive idea behind this is as follows. When a potential energy reaches an extremum, the associated force vanishes, i.e., $dV/d\phi = 0$. Furthermore, if the field starts at "rest"—namely $\phi'(r_{\rm H}) = 0$—there is no source to perturb its initial position and it remains stationary.

Proposition 1. *$\phi_{\rm H}$ is an extremum of $V(\phi)$ if and only if $\phi(r) = \phi_{\rm H}$ for all values of r.*

Proof. ($\Leftarrow$) Suppose that $\phi(r) = \phi_{\rm H}$ is constant for all r. Then, $\phi'(r) = \phi''(r) = 0$. Substituting into the scalar field EOM,

$$\phi'' + \left(4A' + \frac{h'}{h}\right)\phi' - \frac{1}{h}\frac{dV}{d\phi} = 0,$$

we find

$$-\frac{1}{h(r)}\frac{dV}{d\phi}\bigg|_{\phi_{\rm H}} = 0,$$

which implies that $\frac{dV}{d\phi}\Big|_{\phi_{\rm H}} = 0$. Therefore, $\phi_{\rm H}$ is an extremum of the potential.

($\Rightarrow$) Now suppose that $\phi_{\rm H}$ is an extremum of $V(\phi)$, i.e., $\frac{dV}{d\phi}\Big|_{\phi_{\rm H}} = 0$. Let us consider the EOM in the $r \to r_{\rm H}$ limit:

$$\begin{aligned} &\phi''(r \to r_{\rm H}) + \left(4A'(r \to r_{\rm H}) + \frac{h'(r \to r_{\rm H})}{h(r \to r_{\rm H})}\right)\phi'(r \to r_{\rm H}) \\ &\quad - \frac{1}{h(r \to r_{\rm H})}\frac{dV}{d\phi}\bigg|_{(r\to r_{\rm H})} = 0. \end{aligned}$$

The coefficient of the first-order derivative term diverges, hence $\phi'(r \to r_{\rm H}) = 0$ to ensure a regular solution. The potential term vanishes although the $1/h$ term diverges, because it decreases faster than $|dV/d\phi|$ increases because $\phi'(r \to r_{\rm H}) = 0$, and regular condition implies

$$\begin{aligned} \phi(r) &= \phi(r_{\rm H}) + \phi'(r_{\rm H})(r - r_{\rm H}) + \mathcal{O}\left((r - r_{\rm H})^2\right) \approx \phi_{\rm H} \\ h(r) &= h(r_{\rm H}) + h'(r_{\rm H})(r - r_{\rm H}) + \mathcal{O}\left((r - r_{\rm H})^2\right) \\ &\approx \frac{1}{L}(r - r_{\rm H}). \end{aligned}$$

Therefore, $\phi''(r \to r_{\rm H}) = C$, but $C = 0$ because $\phi'(r \to r_{\rm H}) = 0$.

We now proceed by mathematical induction to show that all higher derivatives vanish at the horizon, implying $\phi(r) = \phi_{\rm H}$ for all r.

Base case ($k = 0$): We have already shown that $\phi^{(1)}(r_{\rm H}) = \phi'(r_{\rm H}) = 0$.

Induction step: Assuming that $\phi^{(k)}(r_{\rm H}) = 0$ for all $1 \le k \le n$. Let us prove that $\phi^{(k+1)}(r_{\rm H}) = 0$. By the inductive hypothesis,

$$\begin{aligned} \phi(r) &= \phi_{\rm H} + \sum_{n=k+1}^{\infty}\phi_n(r - r_{\rm H})^n \\ &= \phi_{\rm H} + \phi_{k+1}(r - r_{\rm H})^{k+1} + \mathcal{O}\left((r - r_{\rm H})^{k+2}\right) \\ &= \phi_{\rm H} + \mathcal{O}\left((r - r_{\rm H})^{k+1}\right). \end{aligned}$$

Therefore,

$$\phi'(r) = (k+1)\phi_{k+1}(r - r_{\rm H})^k + \mathcal{O}\left((r - r_{\rm H})^{k+1}\right),$$

and

$$\phi''(r) = (k+1)k\phi_{k+1}(r-r_{\rm H})^{k-1} + \mathcal{O}\left((r-r_{\rm H})^k\right).$$

Since $h(r)$ has a simple zero, $h(r) = \mathcal{O}(r-r_{\rm H})$, therefore $h'(r) = h_1 + \mathcal{O}(r-r_{\rm H})$. Moreover, $1/h(r) = \mathcal{O}\left((r-r_{\rm H})^{-1}\right)$, and $A'(r) = A_1 + \mathcal{O}(r-r_{\rm H})$. Finally,

$$\begin{aligned}\frac{dV}{d\phi} &= \left.\frac{dV}{d\phi}\right|_{\phi_{\rm H}} + \left.\frac{d^2V}{d\phi^2}\right|_{\phi_{\rm H}} (\phi(r)-\phi_{\rm H}) + \dots \\ &= \mathcal{O}\left((r-r_{\rm H})^{k+1}\right),\end{aligned}$$

since the first term is zero and $\phi_{\rm H}$ cancels with the corresponding term in $\phi(r)$. Plugging everything into the EOM, one finds that:

- the second derivative term, has terms of order $(r-r_{\rm H})^{k-1}$ and higher,
- the first derivative term, also has terms of order $(r-r_{\rm H})^{k-1}$ and higher,
- and the potential term, has terms of order $(r-r_{\rm H})^k$ and higher.

Equating terms of order $(r-r_{\rm H})^{k-1}$ terms, one obtains:

$$(k+1)k\phi_{k+1} + (\text{const.})(k+1)\phi_{k+1} = 0,$$

it follows that $\phi^{(k+1)} \propto \phi_{k+1} = 0$.

Therefore, by induction, all derivatives of ϕ vanish at $r = r_{\rm H}$, and thus $\phi(r) = \phi_{\rm H}$ for all r. □

Appendix H: Analytic solution for the case $\phi(r) = \phi_{\rm H}$

As shown in Appendix G, if $\phi_{\rm H} = \phi_{\min}$ or $\phi_{\max}$ then $\phi(r) = \phi_{\rm H}$ remains constant, and the system admits an analytic solution, the SAdS$_5$. Therefore, the EOMs are

$$A'' = 0 \tag{H1}$$

$$h'' + 4A'h' = 0. \tag{H2}$$

Additionally, the constraint becomes

$$24A'^2h + 6A'h' + 2V = 0. \tag{H3}$$

Integrating the system yields

$$\begin{aligned}A(r) &= b + ar, \\ h(r) &= c + de^{-4ar},\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

From the constraint (H3), we derive a condition that, together with the initial conditions—see III A—allows us to fix the solution:

$$\begin{aligned}24a^2\left(c + de^{-4ar}\right) + 6a\left(-4ade^{-4ar}\right) + 2V &= 0 \\ 24a^2c + 24a^2de^{-4ar} - 24a^2de^{-4ar} + 2V &= 0 \\ 24a^2c + 2V &= 0 \\ c &= -\frac{V}{12a^2}.\end{aligned}$$

Thus,

$$A(r) = ar + b, \tag{H4}$$

$$h(r) = -\frac{V}{12a^2} + de^{-4ar}, \tag{H5}$$

where $a, b, d \in \mathbb{R}$.

Let us now determine the integration constants, starting with the blackening function $h(r)$:

$$\begin{cases} h'(r_{\rm H}) = -\dfrac{V}{3a} \overset{!}{=} \dfrac{1}{L} \Rightarrow a = -\dfrac{1}{3}LV \\ h(r_{\rm H}) = -\dfrac{V}{12a^2} + de^{-4ar_{\rm H}} \overset{!}{=} 0 \Rightarrow d = \dfrac{3}{4L^2V}e^{-\frac{4}{3}LVr_{\rm H}}. \end{cases}$$

For $A(r)$, one has

$$A(r_{\rm H}) = ar_{\rm H} + b \overset{!}{=} 0 \Rightarrow b = -ar_{\rm H}.$$

Therefore,

$$A(r) = -\frac{1}{3}LV(r-r_{\rm H}), \tag{H6}$$

$$h(r) = -\frac{3}{4L^2V}\left(1 - e^{\frac{4}{3}LV(r-r_{\rm H})}\right). \tag{H7}$$

It should be noted that to recover an AdS solution in the UV limit, namely (C16)

$$A(r\to\infty) = \frac{1}{L\sqrt{h_0^{\rm far}}}r + A_0^{\rm far},$$

$$h(r\to\infty) = h_0^{\rm far} + h_4^{\rm far}e^{-\frac{4A_{-1}^{\rm far}}{L}r - 4A_0^{\rm far}},$$

it must be true that $V < 0$. This is consistent with the fact that the potential plays the role of a negative cosmological constant in AdS spacetime.

By comparison, the analytic expressions for the UV coefficients for $\phi(r) = \phi_{\rm H}$ case are:

$$A_{-1}^{\rm far} = -\frac{1}{3}L^2V, \tag{H8}$$

$$A_0^{\rm far} = \frac{1}{3}LVr_{\rm H}, \tag{H9}$$

$$h_0^{\rm far} = -\frac{3}{4L^2V}, \tag{H10}$$

$$h_4^{\rm far} = \frac{3}{4L^2V}. \tag{H11}$$

We can now compute s/T^3 in this analytic case—$\phi(r) = \phi_{\rm H}$—using Eqs. (F1) and (F2):

$$\begin{aligned}\frac{s}{T^3} &= \frac{2\pi}{\kappa^2}\frac{1}{\phi_A^{3/\nu}}\left[\frac{1}{4\pi}\frac{1}{L\phi_A^{1/\nu}\sqrt{h_0^{\rm far}}}\right]^{-3} \\ &= \frac{128\pi^4L^3}{\kappa^2}\left(h_0^{\rm far}\right)^{3/2} \\ &= \frac{128\pi^4L^3}{\kappa^2}\left(-\frac{3}{4L^2V}\right)^{3/2} \\ &= \frac{128\pi^4}{\kappa^2}\left(-\frac{3}{4V}\right)^{3/2}.\end{aligned} \tag{H12}$$