Regular Black Holes

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Abstract: General Relativity (GR) predicts that black holes have singularities in their interiors. How they get resolved is a fundamental problem in theoretical physics. Regular black holes (RBHs) are a class of spacetimes that achieve this by replacing the singularities in their interiors with regular cores. In this work, we first present the general properties of RBHs and explain why they are difficult to obtain as solutions to actual physical theories. Then, we review a recently proposed mechanism for obtaining RBHs as solutions to GR coupled to infinite towers of higher-curvature corrections in $D \geq 5$ spacetime dimensions. As a new result in the literature, we construct a regular black hole with four horizons in $D = 8$ for one of these theories. This represents the first example of this class with more than two horizons.
Keywords: General Relativity, Black Holes, Spacetime Singularities, Curvature, Event Horizon.
SDGs: SDG 4 - Quality Education.

I. INTRODUCTION

Black holes represent some of the most fundamental and mysterious objects in physics. A black hole is characterized by the presence of an event horizon. This is a spacetime boundary which causally disconnects its interior from its exterior: no object, not even light, which goes through an event horizon can ever go back (at least not classically). General relativity (GR) predicts that the generic result of the gravitational collapse of ordinary matter is a black hole which hides a spacetime singularity in its interior [1]. A singularity is a region of spacetime in which the gravitational field would become infinitely intense and the very notions of space and time would cease to make sense. As a consequence, spacetime singularities are expected to be unphysical. Their resolution within a theory of gravity beyond GR is one of the most important challenges in theoretical physics.

One possibility is that black holes might not actually contain singularities; instead, they could be regular objects. Regular black holes (RBHs) are characterized by the presence of two or more horizons and the absence of curvature singularities. For them, the near-singularity region is replaced by a regular core where all physical magnitudes remain finite. However, RBHs turn out to be remarkably difficult to obtain as solutions to sensible generalizations of GR.

RBHs have a long history, the first proposals dating back to the 60's of the previous century. These involved *ad hoc* metrics, specifically customized to avoid singularities but which did not satisfy any equations of motion. Notable examples of this type include the models proposed by Bardeen, Hayward, Dynmnikova and Sakharov [2–5]. Consider for instance the Hayward black hole metric. This is a static and spherically symmetric (SSS) spacetime with metric

$$ds^2 = -N(r)^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{(D-2)} \qquad (1)$$

where

$$N(r) = 1\,, \quad f(r) = 1 - \frac{mr^2}{r^{(D-1)} + \alpha m}\,. \qquad (2)$$

Naturally, this reduces to the D-dimensional Schwarzschild spacetime when the parameter α vanishes, as well as in the asymptotic region. On the other hand, the singularity is replaced by a regular core. One finds

$$f(r) \overset{r\to\infty}{=} 1 - \frac{m}{r^{D-3}} + \cdots\,, \quad f(r) \overset{r\to 0}{=} 1 - \frac{r^2}{\alpha} + \cdots \qquad (3)$$

where it is apparent that the interior region is replaced by a regular de Sitter core.

More recently, RBHs have been obtained as solutions of GR coupled to non-linear electrodynamics—see *e.g.*, [6]. Unfortunately, these require highly unusual and poorly motivated Lagrangians, the solutions are non-generic, and they require a certain degree of fine tuning between the theory parameters and the physical properties of the solutions.

On more general grounds, singularities are expected to be resolved within a complete quantum theory of gravity. Quantum gravity corrections to GR typically arise in the form of infinite towers of higher-curvature corrections to the Einstein-Hilbert action, weighted by some fundamental scale(s). For example, such kind of towers are predicted by string theory, where the scale is related to the fundamental string tension. Unfortunately, capturing the effects of such towers within a top-down setup is very challenging in general. Recently, a bottom-up approach to capture these effects has been proposed in the context of spherically symmetric black holes [7]. Notably, this leads to a generic resolution of the Schwarzschild black hole singularity in $D \geq 5$ spacetime dimensions. In this TFG, we review solutions of RBHs already discussed in the literature and introduce some new cases using this approach.

The rest of the document is organized as follows: in section II we present the general properties of RBHs.

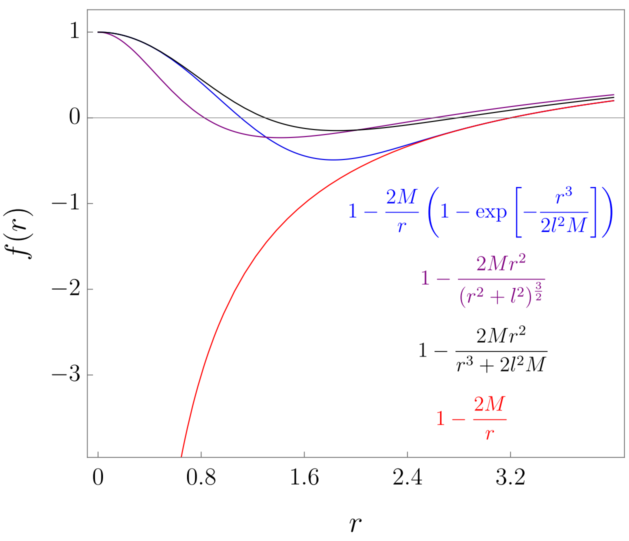

FIG. 1: The metric function $f(r)$ is represented for various black hole models in $D = 4$. We plot the Dymnikova (blue line), Bardeen (purple line) and Hayward (black line) solutions, all of which are regular. We also plot the Schwarzschild solution (red line). In all models, the mass M and characteristic length l are set to identical values. In all cases there are two horizons and the metric behaves as (5) near $r = 0$.

In section III we discuss the gravitational effective action. Section IV introduces Quasi-topological gravities, the framework in which we will work through the rest of the TFG. In Subsection IV.A we explain the construction of RBH solutions for those theories. This includes the previously known cases with two horizons, and a new solution with four horizons in $D = 8$.

II. GENERAL PROPERTIES OF REGULAR BLACK HOLES

RBHs replace the usual spacetime singularity by a regular core. By doing so, they eliminate the point where GR predicts infinite curvature and replace it with a region where these quantities take finite values [8, 9].

This class of black holes has finite curvature invariants everywhere, therefore avoiding the infinities of these scalars associated with classical models. Any divergence in curvature scalars indicates the presence of a true spacetime singularity. For example, consider the Kretschmann invariant, defined as the contraction of two Riemann tensors, $K = R_{abcd}R^{abcd}$. The Kretschmann invariant for the Schwarzschild and the previously introduced Hayward metrics takes the following values close to $r = 0$

$$K_{\text{Schwarzschild}} \overset{r\to 0}{\sim} r^{-2(D-1)}, \quad K_{\text{Hayward}} \overset{r\to 0}{\sim} \alpha^{-2} \qquad (4)$$

for an arbitrary D spacetime dimensions.

Generically, for static and spherically symmetric RBHs characterized by a single metric function, the behavior of $f(r)$ near $r = 0$ in Schwarzschild coordinates reads

$$f = 1 - \mathcal{O}(r^2)\,, \qquad (5)$$

which ensures the smoothness and regularity of the solution. At long distances, however, the metrics are usually built such that they approach the Schwarzschild one. In Fig. 1 we plot various proposed metrics describing RBHs.

RBHs have an outer event horizon, beyond which nothing can escape, but they also possess an inner horizon, where the metric is static again. The inner horizon separates the outside part of the event horizon interior from the region with the regular core at the center. Inner horizons are known to be potentially unstable under small perturbations, giving rise to the so-called “mass-inflation instabilities” [10, 11]. These would involve an infinite blueshift of even tiny amounts of energy which crossed it, leading to a breakdown of spacetime and, effectively, to a spacetime singularity. However, in the absence of dynamical models in which RBHs arise as actual solutions, it is difficult to address the existence and implications of such putative instabilities. In fact, recent results suggest that mass-inflation instabilities can be avoided by considering inner-extremal RBHs [12], namely, RBHs for which the inner horizon has a vanishing surface gravity.

As mentioned earlier, finding RBHs as *bona fide* solutions to gravitational theories has been an outstanding challenge since the first *ad hoc* models were proposed. While quantum gravity models are expected to resolve spacetime singularities, finding explicit mechanisms which achieve this feature has remained completely out of reach, both due to conceptual and technical reasons, which we briefly comment on next.

III. QUANTUM GRAVITY AND GRAVITATIONAL EFFECTIVE ACTION

GR is an incomplete theory. On the one hand, it predicts the existence of singularities and, on the other, it is a classical theory, namely, it does not respect the principles of quantum mechanics.

Several candidates for a quantum theory of gravity have been proposed in the literature. A prototypical example is string theory, which at low energies predicts GR coupled to various matter fields (in the form of a supergravity theory) as well as a series of corrections. These corrections involve contractions of the Riemann tensor and the rest of fields and can be organized in a series expansion of increasing higher-curvature terms. Restricting the discussion to the gravitational sector, the corrections would read $\alpha R^2 + \beta R^2_{abcd} + \cdots$, where the couplings $(\alpha, \beta, ...)$ would be controlled by some fundamental scale, such as the string tension [13]. In general, it is extremely difficult to fully account for all these corrections and their effects and the existing studies confine their analysis to the study of the leading perturbative corrections.

Alternatively, we can pursue a bottom-up approach. This involves using the gravitational effective action, which includes all diffeomorphism-invariant corrections to the classical action up to field redefinitions [14]. In this method, we expand the action as a series of terms, again organized in powers of the curvature, where the couplings are a priori unconstrained. One feature of this method is that we can choose different equivalent bases of invariants, as long as the couplings are considered perturbative corrections to Einstein gravity. A particularly useful basis is provided by Quasi-topological (QT) gravities, which we will discuss next [15].

IV. QUASI-TOPOLOGICAL GRAVITY BLACK HOLES

Let us consider a general theory of gravity in D dimensions, involving arbitrary contractions of the Riemann tensor and the metric. The Lagrangian can be written as $\mathcal{L}(g^{ab}, R_{cdef})$. The field equations for this theory can be expressed as

$$P_a{}^{cde} R_{bcde} - \frac{1}{2} g_{ab} \mathcal{L} - 2\nabla^c \nabla^d P_{abcd} = 0\,, \tag{6}$$

where $P^{abcd} \equiv \partial\mathcal{L}/\partial R_{abcd}$. Consequently, the equations of motion generally contain up to fourth-order derivatives of g_{ab}, with the contributions involving more than two derivatives arising from the last term. In Lovelock theories [16], however, this term is absent, meaning that

$$\nabla^a P_{abcd} = 0\,, \qquad \text{(Lovelock)} \tag{7}$$

so the field equations remain second-order for any metric.

QT gravities [17–20] are a more general class of high-curvature theories of gravity. Their defining property is that their field equations involve up to two derivatives when restricted to a general SSS metric (1), namely,

$$\nabla^a P_{abcd}|_{\text{SSS}} = 0\,, \qquad \text{(Quasi-topological)}. \tag{8}$$

Hence, Lovelock gravities are particular examples of QT gravities. However, the latter exist at all curvature orders for any dimension $D \geq 5$, as opposed to Lovelock theories, which only exist for $n \leq \lfloor (D-1)/2 \rfloor$.

The action of these QT gravities is constructed by introducing an arbitrary number of terms into the Einstein-Hilbert action, and it takes the following form

$$I = \frac{1}{16\pi G} \int \mathrm{d}^D x \sqrt{|g|} \left[R + \sum_{n=2}^{n_{\max}} \alpha_n \mathcal{Z}_n \right], \tag{9}$$

where R is the Ricci scalar, α_n represent arbitrary coupling constants with dimensions of length$^{2(n-1)}$ and $\mathcal{Z}_n$ are the quasi-toplogical densitites of order n in curvature. These are constructed from contractions of the Riemann tensor and the metric. At each curvature order there exist several possible QT densities, but all of them contribute in the same way to the equations of motion of SSS spacetimes, so it suffices to choose a single representative at each order. Representatives of the densities corresponding to the first five curvature orders can be found in the Appendix.

Remarkably, there exists a recursive formula which allows one to construct arbitrarily high order QT densities taking the first five as seeds [21]. This reads

$$\mathcal{Z}_{n+5} = \frac{3(n+3)\mathcal{Z}_1 \mathcal{Z}_{n+4}}{D(D-1)(n+1)} - \frac{3(n+4)\mathcal{Z}_2 \mathcal{Z}_{n+3}}{D(D-1)n} \tag{10}$$

$$+ \frac{(n+3)(n+4)\mathcal{Z}_3 \mathcal{Z}_{n+2}}{D(D-1)n(n+1)}\,. \tag{11}$$

Applying the ansatz presented in (1) to the action (9), it is possible to show that the full equations of motion of the theory get drastically simplified and reduce to[22]:

$$\frac{\mathrm{d}N}{\mathrm{d}r} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}[r^{D-1} h(\psi)] = 0\,, \tag{12}$$

where :

$$h(\psi) \equiv \psi + \sum_{n=2}^{n_{\max}} \alpha_n \psi^n, \quad \psi \equiv \frac{1 - f(r)}{r^2}\,. \tag{13}$$

Consequently we will have that $N(r) = 1$ and that $f(r)$ is determined by the algebraic equation

$$h(\psi) = \frac{m}{r^{D-1}}\,, \tag{14}$$

where m is an integration constant, proportional to the ADM mass. The simplicity of these equations allows us to study QT black hole solutions beyond the perturbative regime, which we do next.

A. Regular black holes

In this section we will explain how RBHs arise as solutions to QT gravities when an infinite number of terms is included in the action [7]. In the case of $n_{\max} = N$, when a finite number of terms are added to the action, the black hole core is dominated by the highest order contribution and the metric function around $r = 0$ behaves as follows:

$$f = 1 - \left(\frac{m}{\alpha_N}\right)^{1/N} r^{2-(D-1)/N} + \ldots \tag{15}$$

For finite N, a curvature singularity exists at $r = 0$, but as N becomes larger the singularity weakens. Remarkably, when N tends to infinity, the singularity completely disappears, resulting in a de Sitter core and finite curvature invariants everywhere. This phenomenon takes place whenever the coupling constants of the theory satisfy certain mild and qualitative conditions. For instance, a set of sufficient conditions reads

$$\alpha_n \geq 0 \;\forall n, \quad \lim_{n\to\infty} (\alpha_n)^{1/n} = C > 0\,. \tag{16}$$

The conditions essentially arise out of the necessity of inverting (14) for all values of r, which requires $h(\psi)$ to

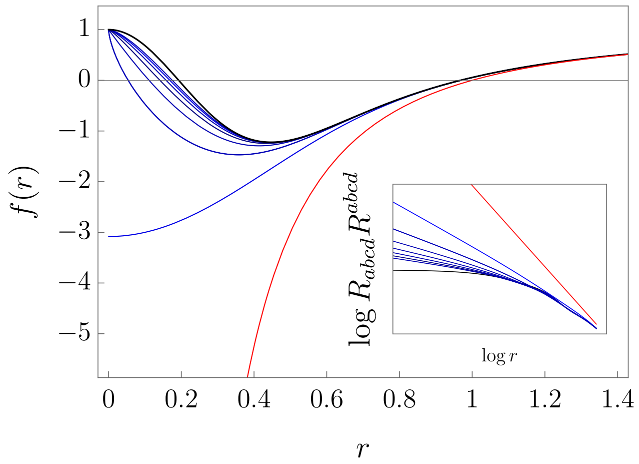

FIG. 2: The metric function $f(r)$ presented in (17) is shown for a regular black hole with two horizons in $D = 5$ with $m = 1$, $\alpha = 0.03$. The black line represents the regular solution, while the blue ones represent the solution for $n_{\max} = 2, 3, 4, 5, 6, 7, 9$. The red line corresponds to the Schwarzschild black hole. The inset plot represents the Kretschmann invariant of the solutions in a log-log scale. Observe that it is only when $n_{\max} \to \infty$ that K remains finite at $r = 0$.

be a monotonic function whose range covers all positive values.

As opposed to all known previous mechanisms, RBHs arise in this context as the unique SSS solutions of the corresponding theories and for generic values of the gravitational couplings. In fact, it has been shown that QT theories satisfy a Birkhoff theorem, which means that their RBHs are in fact their most general spherically symmetric solutions and that they describe the exterior of any spherical matter distribution [21].

1. *Regular black holes with two horizons*

The first class of regular black holes that we explore are those characterized by two horizons. They can be constructed using (13) when the sum goes to infinity.

Following the work in [7], we now examine a solution of a regular black hole when we consider $\alpha_n = n\alpha^{n-1}$. We obtain it by evaluating the sum in (13) and solving (14). The metric function is

$$f(r) = 1 - \frac{2mr^2}{2\alpha m + r^{D-1} + \sqrt{r^{2(D-1)} + 4\alpha m r^{D-1}}} \,. \quad (17)$$

As shown in Fig. 2, the Schwarzschild solution diverges at $r = 0$, while the solution for the regular black hole and the case with N finite (provided $N > 2$) tend to 1. We observe that both the Schwarzschild solution and the solution for finite N exhibit a curvature singularity at the origin, while in the regular one the curvature invariant tends towards a constant value, as illustrated in the inset plot.

2. *Regular black holes with four horizons*

We will now examine a variant of RBH solutions with four horizons in $D = 8$, using the same method we used for two horizons, and confirm that these solutions are regular. This is the first time a RBH with four horizons has been presented using the purely gravitational models introduced earlier in (9) and the first RBH with more than two horizons embedded in an actual gravitational theory. In order to obtain four horizons, we extend the sum in (13) to $n_{\max} = 4$, yielding the following expression:

$$\psi + \alpha_2\psi^2 + \alpha_3\psi^3 + \alpha_4\psi^4 = \frac{m}{r^7} \,. \quad (18)$$

In order to determine the range values of the parameters, we use that $f(r_h) = 0$ at the horizons. Therefore, we solve (18) with $\psi = 1/r^2$ and look for theories for which this has four positive roots. We find an instance for the following values: $m = 0.108$, $\alpha_2 = -1$, $\alpha_3 = \frac{2}{5}$, $\alpha_4 = 0.001$, but note that additional solutions with four horizons can be found by continuously varying the above values of the mass and the rest of parameters.

With these choices we obtain a black hole with four horizons. However, it is not regular because its behavior near $r = 0$ is not like $f = 1 - \mathcal{O}(r^2)$. In order to construct a regular solution, we need to add to (18) a sum from $n = 5$ to infinity. We can do so by using Hayward-type couplings, namely, $\alpha_n = \alpha^{n-1}$ with $\alpha = 0.00023$. Consequently, we would need to solve

$$\psi + \alpha_2\psi^2 + \alpha_3\psi^3 + \alpha_4\psi^4 + \frac{\alpha\psi^5}{1 - \alpha\psi} = \frac{m}{r^7} \quad (19)$$

in order to find $f(r)$, which cannot be done explicitly. We plot it in Fig. 3. Near $r = 0$, it behaves as

$$f(r) \overset{r\to 0}{=} 1 - \frac{100000}{23} r^2 + \mathcal{O}(r^9) \,,$$

and it has four positive roots corresponding to the four horizons. It also behaves asymptotically like a Schwarzschild black hole,

$$f(r) \overset{r\to\infty}{=} 1 - \frac{27}{250} r^{-5} + \mathcal{O}(r^{-12}) \,.$$

V. CONCLUSIONS

In this TFG, we have introduced RBHs and reviewed a proposed mechanism to obtain them as generic solutions to purely gravitational theories which involve correcting GR by infinite towers of higher-curvature corrections of the Quasi-topological class. We have then applied this method to construct a RBH solution with two horizons in general D as well as another one with four horizons in $D = 8$, which is the first of this kind. Naturally, it would be interesting to perform a full exploration of the general conditions under which multiple-horizon black holes may

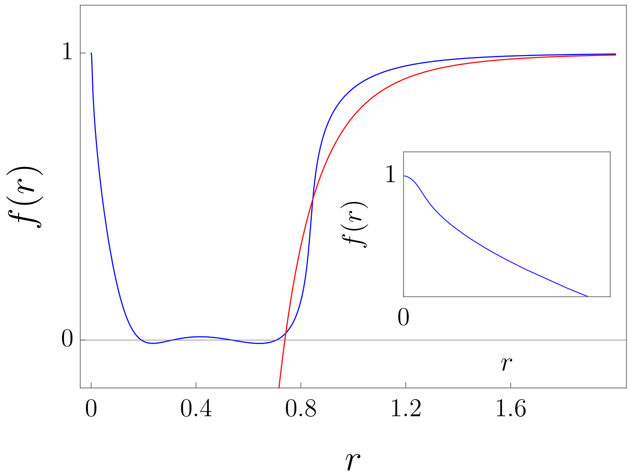

FIG. 3: The $f(r)$ is represented for a RBH with four horizons in $D = 8$, utilizing the previously specified parameter values (blue line). We also plot the Schwarzschild black hole of the same mass (red line). The inset plot provides a zoomed-in view of the function near $r = 0$.

exist in each spacetime dimension while maintaining the solutions' regularity.

It is known that black holes are formed from the collapse of matter according to GR, but in this TFG we have not considered how this takes place for RBHs within QT theories. Such analysis was performed in [21] in the case of two-horizon RBHs and it would be interesting to repeat it in the multiple-horizon case, where new phenomena may arise due to the expected intricate structure of the corresponding effective potentials for collapsing shells or stars.

Finally, we have stated several times that QT gravities are only defined in dimensions larger than four. From a mathematical perspective, this is because QT densities do not exist in $D = 4$. It would be interesting to investigate if a similar mechanism to resolve singularities could be found in $D = 4$ using a different class of higher-curvature modifications to Einstein gravity.

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[22] See *e.g.*, appendix A of [7]

Forats Negres Regulars

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Resum: La Relativitat General (RG) prediu que els forats negres tenen singularitats en els seus interiors. La seva resolució presenta un problema fonamental en la física teòrica. Els forats negres regulars són una classe d'espaitemps que ho aconsegueixen substituint les singularitats dels seus interiors per nuclis regulars. En aquest document primer es presenten les propietats generals dels forats negres regulars i s'explica per què és difícil obtenir-los com a solucions de teories físiques. Després, revisem un mecanisme proposat recentment per a obtenir aquest tipus de forats negres com a solucions de RG amb una torre infinita de correccions d'alta curvatura en $D \geq 5$ dimensions espaitemporals. Com a resultat nou en la literatura, hem construït un forat negre regular amb quatre horitzons en $D = 8$ per a una d'aquestes teories. Aquest representa el primer exemple d'aquesta classe amb més de dos horitzons.

Paraules clau: Relativitat General, Forats Negres, Singularitats de l'Espaitemps, Curvatura, Horitzó d'Esdeveniments.

ODSs: ODS 4 - Educació de Qualitat.

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

El contingut d'aquest TFG es pot relacionar amb l'ODS 4, d'educació de qualitat, ja que contribueix a l'eduació universitària en el camp de física fonamental.

Appendix A: Quasi-topological densities

Let $\mathcal{Z}_n$ denote the densities constructed from n-th order curvature invariants. W_{abcd} denotes the Weyl curvature tensor and Z_{ab} the traceless part of the Ricci curvature tensor. The first five densities read:

$$
\begin{aligned}
\mathcal{Z}_{(1)} &= R\,,\\
\mathcal{Z}_{(2)} &= \frac{1}{(D-2)}\left[\frac{W^{abcd}W_{abcd}}{D-3} - \frac{4Z_{ab}Z^{ab}}{D-2}\right] + \frac{\mathcal{Z}_{(1)}^2}{D(D-1)}\,,\\
\mathcal{Z}_{(3)} &= \frac{24}{(D-2)(D-3)}\left[\frac{W_{ac}{}^{bd}Z_b^aZ_d^c}{(D-2)^2} - \frac{W_{acde}W^{bcde}Z_b^a}{(D-2)(D-4)} + \frac{2(D-3)Z_b^aZ_c^bZ_a^c}{3(D-2)^3} + \frac{(2D-3)W^{ab}{}_{cd}W^{cd}{}_{ef}W^{ef}{}_{ab}}{12(D((D-9)D+26)-22)}\right]\\
&+ \frac{3\mathcal{Z}_{(1)}\mathcal{Z}_{(2)}}{D(D-1)} - \frac{2\mathcal{Z}_{(1)}^3}{D^2(D-1)^2}\,,\\
\mathcal{Z}_{(4)} &= \frac{96}{(D-2)^2(D-3)}\left[\frac{(D-1)(W_{abcd}W^{abcd})^2}{8D(D-2)^2(D-3)} - \frac{(2D-3)Z_e^fZ_f^eW_{abcd}W^{abcd}}{4(D-1)(D-2)^2} - \frac{2W_{abcd}W^{cefg}W^d{}_{efg}Z^{ab}}{D(D-3)(D-4)}\right.\\
&\left.- \frac{4Z_{ac}Z_{de}W^{bdce}Z_a^b}{(D-2)^2(D-4)} + \frac{(D^2-3D+3)(Z_a^bZ_b^a)^2}{D(D-1)(D-2)^3} - \frac{Z_a^bZ_b^cZ_c^dZ_d^a}{(D-2)^3} + \frac{(2D-1)W_{abcd}W^{aecf}Z^{bd}Z_{ef}}{D(D-2)(D-3)}\right]\\
&+ \frac{4\mathcal{Z}_{(1)}\mathcal{Z}_{(3)} - 3\mathcal{Z}_{(2)}^2}{D(D-1)}\,,\\
\mathcal{Z}_{(5)} &= \frac{960(D-1)}{(D-2)^4(D-3)^2}\left[\frac{(D-2)W_{ghij}W^{ghij}W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab}}{40D(D^3-9D^2+26D-22)} + \frac{4(D-3)Z_a^bZ_b^cZ_c^dZ_d^eZ_e^a}{5(D-1)(D-2)^2(D-4)}\right.\\
&- \frac{(3D-1)W^{ghij}W_{ghij}W_{acde}W^{bcde}Z_b^a}{10D(D-1)^2(D-4)} - \frac{4(D-3)(D^2-2D+2)Z_a^bZ_b^aZ_c^dZ_d^eZ_e^c}{5D(D-1)^2(D-2)^2(D-4)}\\
&- \frac{(D-3)(3D-1)(D^2+2D-4)W^{ghij}W_{ghij}Z_c^dZ_d^eZ_e^c}{10D(D-1)^2(D+1)(D-2)^2(D-4)} + \frac{(5D^2-7D+6)Z_g^hZ_h^gW_{abcd}Z^{ac}Z^{bd}}{10D(D-1)^2(D-2)})\\
&+ \frac{(D-2)(D-3)(15D^5-184D^4+527D^3-800D^2+472D-88)W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab}Z_g^hZ_h^g}{40D(D-1)^2(D-4)(D^5-15D^4+91D^3-277D^2+418D-242)}\\
&- \frac{2(3D-1)Z^{ab}W_{abcd}Z^{ef}W_e{}^c{}_f{}^gZ_g^d}{D(D^2-1)(D-4)} - \frac{Z_a^bZ_b^cZ_{cd}Z_{ef}W^{eafd}}{(D-1)(D-2)} + \frac{(D-3)W_{acde}W^{bcde}Z_b^aZ_c^dZ_d^c}{5D(D-1)^2(D-4)}\\
&\left.- \frac{(D-2)(D-3)(3D-2)Z_b^aZ_c^bW_{daef}W^{efgh}W_{gh}{}^{dc}}{4(D-1)^2(D-4)(D^2-6D+11)} + \frac{W_{ghij}W^{ghij}Z^{ac}Z^{bd}W_{abcd}}{20D(D-1)^2}\right]\\
&+ \frac{5\mathcal{Z}_{(1)}\mathcal{Z}_{(4)} - 2\mathcal{Z}_{(2)}\mathcal{Z}_{(3)}}{D(D-1)} + \frac{6\mathcal{Z}_{(1)}\mathcal{Z}_{(2)}^2 - 8\mathcal{Z}_{(1)}^2\mathcal{Z}_{(3)}}{D^2(D-1)^2}\,.
\end{aligned}
$$