On the physical and mathematical properties of coherent states

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Abstract: In this work, coherent states and their properties have been studied. It has been shown that the electric field of a coherent state resembles the classical solution of Maxwell equations. The phase of a coherent state has been studied, showing that in the asymptotic limit of average photon numbers, a coherent state has a well-defined phase. Finally, the squeezed state of the vacuum has been studied, showing it is related to the Lie group SU(1,1), both for 1 and 2 modes of oscillation, and has a smaller uncertainty in one of the quadratures than a coherent state.

Keywords: Quantum, coherent,phase,operator,squeeze,uncertainty.
SDGs: This work is related to the following ODS: **4.3**, **7.4**, **13.3**

I. INTRODUCTION

Coherent states were introduced by Glauber in [1] who showed that they provide an adequate means for a quantum description of coherent laser light beams. It turns out that a coherent state is a quantum superposition of states with different number of photons, which has no classical counterpart whatsoever. Yet, the expectation value of the electric field in a coherent state is the expression of the classical monochromatic wave solution of the Maxwell equations. Moreover, for a large number of the photon mean value, the coherent state (i.e., the laser beam) has a well defined phase, as will be discussed below following the work of [2]. Squeezed states are also superpositions of states with different photon numbers. These states have less uncertainty in a quadrature than a coherent state and are related to the group of dilations, either by one mode or by two, as will be discussed following [4], [6] and [3]. Throughout this work, we will restrict to the study of one mode of oscillation, unless the latter part of the work. We will work in natural units $\hbar = 1$

II. GLAUBER COHERENT STATES

Coherent states may be defined as the eigenstates of the annihilation operator of the quantum oscillator, which means

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle . \tag{1}$$

Let $\hat{q}$ and $\hat{p}$ be operators such that $[\hat{q}, \hat{p}] = i$. These operators are the so-called quadratures of the optical phase space. The quadrature operators are defined by:

$$\begin{aligned} \hat{q} &= \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \\ \hat{p} &= i\left(\frac{\hat{a}^\dagger - \hat{a}}{\sqrt{2}}\right). \end{aligned} \tag{2}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators for the quadratures $\hat{q}$ and $\hat{p}$.

Now we define the coherent states by

$$|\alpha\rangle = \hat{D}(\alpha) |0\rangle , \tag{3}$$

where $\hat{D}(\alpha)$ is the displacement operator in the optical phase space

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}. \tag{4}$$

The coherent states are defined to be displaced forms of the ground state $|0\rangle$ by an amount $\alpha \in \mathbb{C}$

To find the form of $|\alpha\rangle$ we use the Baker-Campbell-Hausdorff formula to express $\hat{D}(\alpha)$ as a product of exponential operators

$$\hat{D}(\alpha) = e^{\frac{-1}{2}|\alpha^2|} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}, \tag{5}$$

since $[\hat{a}, \hat{a}^\dagger] = 1$.

As $\hat{a} |0\rangle = 0$, equation (3) can be expressed as

$$|\alpha\rangle = \hat{D}(\alpha) |0\rangle = e^{\frac{-1}{2}|\alpha^2|} e^{\alpha\hat{a}^\dagger} |0\rangle \tag{6}$$

$$= e^{\frac{-1}{2}|\alpha^2|} \sum_n \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle , \tag{7}$$

given that $|n\rangle = (n!)^{-1/2}(\hat{a}^\dagger)^n |0\rangle$.

Now the expectation value for the number of photons for a coherent state

$$\langle \hat{N} \rangle = \langle\alpha| \hat{a}^\dagger\hat{a} |\alpha\rangle = \langle\alpha| \alpha^*\alpha |\alpha\rangle = |\alpha|^2$$
$$\langle \hat{N}^2 \rangle = \langle\alpha| \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} |\alpha\rangle = |\alpha|^2(\langle\alpha| \hat{a}^\dagger\hat{a} |\alpha\rangle + 1) = |\alpha|^4 + |\alpha|^2,$$

and

$$\Delta\hat{N} = \sqrt{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2} = |\alpha|.$$

From these expressions, one can extract that the probability of finding n photons in a coherent state $|\langle n|\alpha\rangle|^2$ follows a Poisson distribution with variance equal to $\langle \hat{N} \rangle$

$$| \langle n|\alpha\rangle |^2 = \frac{|\alpha|^2}{n!} e^{-|\alpha|^2} = \frac{\langle \hat{N}\rangle^n}{n!} e^{-\langle \hat{N}\rangle}. \quad (8)$$

.

Since the number states $|n\rangle$ evolve with the Hamiltonian $\hat{H} = \omega \hat{N}$ as $|n\rangle_t = e^{-iwnt} |n\rangle$, the time evolution in the Schrödinger picture for a coherent state will be

$$|\alpha\rangle_t = e^{-\frac{1}{2}|\alpha^2|} \sum_n \frac{(\alpha e^{-iwt})^n}{\sqrt{n!}} |n\rangle = |\alpha(t)\rangle , \quad (9)$$

where we have defined $\alpha(t) \equiv \alpha e^{-iwt}$ Now we can see that the fluctuations $\Delta \hat{N}$ are not time dependent.

$$\Delta \hat{N}^2(t) = |\alpha(t)|^2 = |\alpha|^2 = \Delta \hat{N}^2. \quad (10)$$

A. ELECTRIC FIELD OF A COHERENT STATE

The expression for the one mode contribution to the electric field using quantum field formalism is

$$\vec{E}(\vec{x}) = i\sqrt{\frac{\omega}{2V}} \left(\vec{e}_{\vec{k}} e^{i\vec{k}\vec{x}} \hat{a} - \vec{e}^{\,*}_{\vec{k}} e^{-i\vec{k}\vec{x}} \hat{a}^\dagger \right), \quad (11)$$

where $\vec{e}_{\vec{k}}$ is the polarization vector, perpendicular to the mode of oscillation, i.e. $\vec{k} \cdot \vec{e}_{\vec{k}} = 0$.

It can be seen that the expectation value for the electric field for any photon number state $|n\rangle$

$$\langle n|\vec{E}(\vec{x})|n\rangle = 0.$$

However, averaging using a coherent state, the electric field not only doesn't vanish, but becomes a plane wave.

$$\langle \alpha(t)|\vec{E}(\vec{x})|\alpha(t)\rangle = \langle \alpha(t)|\, i\sqrt{\frac{\omega}{2V}} \left(\vec{e}_{\vec{k}} e^{i\vec{k}\vec{x}} \hat{a} - h.c. \right) |\alpha(t)\rangle$$

Using our definition (1) and using basic complex algebra since $\alpha(t) = |\alpha| e^{i\phi} e^{-i\omega t}$

$$\begin{aligned} \langle \alpha(t)|\vec{E}(\vec{x})|\alpha(t)\rangle &= i\sqrt{\frac{\omega}{2V}} \left(\vec{e}_{\vec{k}} e^{i\vec{k}\vec{x}} \alpha(t) - c.c. \right) \\ &= \sqrt{\frac{2\omega}{V}} |\alpha| \sin(\vec{k}\vec{x} - \omega t + \phi) \vec{e}_{\vec{k}}. \end{aligned} \quad (12)$$

It can also be seen, as $\vec{e}_{\vec{k}} \cdot \vec{e}^{\,*}_{\vec{k}} = 1$

$$\langle \alpha(t)|\vec{E}^2(\vec{x})|\alpha(t)\rangle = \frac{\omega}{2V} \left((1 + 4|\alpha|^2 \sin^2(\vec{k}\vec{x} - \omega t + \phi) \right), \quad (13)$$

which leads to a constant and time independent uncertainty

$$(\Delta \vec{E})^2 = \frac{\omega}{2V}. \quad (14)$$

From these results, one extracts that coherent states resembles the classical solution for Maxwell equations.

III. PHASE OPERATOR

Since we have characterized the coherent states as displaced forms of the ground state, and we have related the amplitude of that displacement to the expected number of photons, we now try to get insight on the phase of that displacement on the optical phase space.

A. Number and phase operators

Following [2], we consider the operators defined as

$$(\widehat{e^{i\phi}})^\dagger = \hat{a}^\dagger (\hat{N}+1)^{-1/2}, \;\; \widehat{e^{i\phi}} = (\hat{N}+1)^{-1/2} \hat{a}. \quad (15)$$

It should be noted that these operators do not correspond to an exponential of some operator, but are a definition which will help us to later define operators which are indeed hermitian.

The operator $\widehat{e^{i\phi}}$ is the conjugate operator for the number operator, as it can be seen that

$$\widehat{e^{i\phi}} \hat{N} (\widehat{e^{i\phi}})^\dagger = \hat{N} + 1. \quad (16)$$

This is shown by

$$\widehat{e^{i\phi}} \hat{N} (\widehat{e^{i\phi}})^\dagger = (\hat{N}+1)^{-1/2} \hat{a}\hat{a}^\dagger \hat{a}\hat{a}^\dagger (\hat{N}+1)^{-1/2}, \quad (17)$$

which is

$$\widehat{e^{i\phi}} \hat{N} (\widehat{e^{i\phi}})^\dagger = (\hat{N}+1)^{-1/2} (\hat{N}+1)^2 (\hat{N}+1)^{-1/2} = \hat{N} + 1. \quad (18)$$

In analogy with the momentum and position operators, the operator $\widehat{e^{i\phi}}$ displaces by a discrete unit the number operator, as the latter has a discrete spectrum.

These operators satisfy the relation $\widehat{e^{i\phi}} (\widehat{e^{i\phi}})^\dagger = \mathbb{I}$ but are not unitary, as $(\widehat{e^{i\phi}})^\dagger \widehat{e^{i\phi}} \neq \mathbb{I}$. This is shown using that $(\widehat{e^{i\phi}})^\dagger$ acts on the number states as the raising operator without the prefactor

$$(\widehat{e^{-\phi}})^\dagger |n\rangle = |n+1\rangle , \;\; \widehat{e^{i\phi}} |n\rangle = |n-1\rangle . \quad (19)$$

Then, the operators can be written in this form

$$\begin{aligned} (\widehat{e^{i\phi}})^\dagger &= |1\rangle \langle 0| + |2\rangle \langle 1| + |3\rangle \langle 2| + ... \\ \widehat{e^{i\phi}} &= |0\rangle \langle 1| + |1\rangle \langle 2| + |2\rangle \langle 3| + ... \end{aligned} \quad (20)$$

So the product

$$(\widehat{e^{i\phi}})^\dagger \widehat{e^{i\phi}} = |1\rangle \langle 1| + |2\rangle \langle 2| + ... |n\rangle \langle n| = \mathbb{I} - |0\rangle \langle 0| \quad (21)$$

is clearly not equal to the identity.

Equation (21) shows that the existence of a state $|\psi\rangle$ such that $\hat{a}|\psi\rangle = 0$, the so-called ground state, is what makes the operator $\widehat{e^{i\phi}}$ not unitary. Were these operators unitary, then the displacement on the number operator would have been given by a hermitian phase operator $\hat{\phi}$.

Nevertheless, we try to find an eigenvector for the operator $\widehat{e^{i\phi}}$ with an eigenvalue $e^{i\theta}$, that is

$$\widehat{e^{i\phi}}\left|\theta\right\rangle = e^{i\theta}\left|\theta\right\rangle.$$

As the operator $\widehat{e^{i\phi}}$ acts on the number states like the lowering operator, the state $|\theta\rangle$ might be defined by

$$\left|\theta\right\rangle = \frac{1}{\sqrt{2\pi}}\sum_n e^{in\theta}\left|n\right\rangle. \tag{22}$$

Now applying the phase operator to our state we find

$$\begin{aligned} \widehat{e^{i\phi}}\left|\theta\right\rangle &= \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty} e^{in\theta}\left|n-1\right\rangle \\ &= \frac{1}{\sqrt{2\pi}}e^{i\theta}\sum_m e^{im\theta}\left|m\right\rangle = e^{i\theta}\left|\theta\right\rangle. \end{aligned} \tag{23}$$

B. Phase on a coherent state

Although the phase operator has been shown not to correspond to a hermitian operator, applied to a coherent state with large number of photons, i.e. $|\alpha|^2 \to \infty$, the contribution of the ground state becomes ignorable.

Overlapping $|\theta\rangle$ with $|\alpha\rangle$

$$\langle\theta|\alpha\rangle = \frac{1}{\sqrt{2\pi}}\sum_n e^{-|\alpha|^2/2}\frac{(\alpha e^{-i\theta})^n}{\sqrt{n!}}. \tag{24}$$

Now since $\alpha = |\alpha|e^{i\delta}$

$$\langle\theta|\alpha\rangle = \sum_n \frac{1}{\sqrt{2\pi}}\sqrt{e^{-|\alpha|^2}\frac{(\alpha)^{2n}}{n!}}e^{in(\delta-\theta)}. \tag{25}$$

We can now identify the term inside the square root as a Poisson distribution of variance $|\alpha|$, so the probability distribution $|\langle\theta|\alpha\rangle|^2$ follows a Poisson distribution.

For very large values of its variance, a Poisson distribution can be approximated as a Gaussian distribution. With some work, equation (25) can be expressed as

$$\langle\theta|\alpha\rangle \simeq \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi|\alpha|^2)^{1/4}}e^{i|\alpha|^2(\delta-\theta)}\sqrt{4\pi|\alpha|^2}e^{-|\alpha|^2(\delta-\theta)^2}. \tag{26}$$

The probability distribution $|\langle\theta|\alpha\rangle|^2$ then becomes

$$|\langle\theta|\alpha\rangle|^2 \simeq \sqrt{\frac{2|\alpha|^2}{\pi}}e^{-2|\alpha|^2(\theta-\delta)^2}, \tag{27}$$

which is a Gaussian distribution with variance $\Delta\theta = \frac{1}{2|\alpha|}$. Notice the product of the uncertainties

$$\Delta N\Delta\theta = |\alpha|\frac{1}{2|\alpha|} = \frac{1}{2}$$

satisfies the Heisenberg uncertainty principle with the equality sign.

In the limit $|\alpha|^2 >> 1$ the dispersion in phase decreases and the probability density becomes a Dirac delta $\delta(\theta-\delta)$, thus the phase of a coherent state with large number of photons becomes asymptotically well-defined.

This can be understood as that in the limit $|\alpha|^2 \to \infty$ the projection of a coherent state on the ground state tends to 0

$$\langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \to 0. \tag{28}$$

Now recalling the product (21), applying it to a coherent state, with the limit $|\alpha| >> 1$:

$$(\widehat{e^{i\phi}})^\dagger\widehat{e^{i\phi}}\left|\alpha\right\rangle = \mathbb{I}\left|\alpha\right\rangle - \left|0\right\rangle\left\langle 0|\alpha\right\rangle \simeq \mathbb{I}\left|\alpha\right\rangle. \tag{29}$$

So the product, for the limit of $|\alpha| >> 1$ behaves like the identity when applied on a coherent state, thus asymptotically solving our problems with the definition (15). On physical grounds, we are satisfied, but mathematically, this is not formal, as the operators $\widehat{e^{i\phi}}$ and $(\widehat{e^{i\phi}})^\dagger$ remain not to be unitary, so we introduce the cosine operator

$$\widehat{\cos(\phi)} = \frac{1}{2}\left(\widehat{e^{i\phi}} + (\widehat{e^{i\phi}})^\dagger\right), \tag{30}$$

which is clearly hermitian. This operator has eigenvectors $|\cos(\theta)\rangle$ with eigenvalues $\cos(\theta)$.

The operator $\widehat{\cos(\phi)}$ acts on the number states as

$$\widehat{\cos(\phi)}\left|n\right\rangle = \frac{1}{2}\left(\left|n+1\right\rangle + \left|n-1\right\rangle\right). \tag{31}$$

Now the uncertainty for $\widehat{\cos(\phi)}$ operator acting on the coherent states can be calculated as well.

$$\Delta\widehat{\cos(\phi)} = \sqrt{\langle\alpha|\widehat{\cos(\phi)}^2|\alpha\rangle - \langle\alpha|\widehat{\cos(\phi)}|\alpha\rangle^2} \tag{32}$$

Using equation (31) and following [5], it is found that in the limit $|\alpha|^2 >> 1$

$$\Delta\widehat{\cos(\phi)} = \left(1 - \frac{1}{8|\alpha|^2} + ...\right)\frac{\sin(\theta)}{2|\alpha|} \simeq \frac{\sin(\theta)}{2|\alpha|}. \tag{33}$$

This shows that in this limit, the uncertainty product becomes

$$\Delta\widehat{\cos(\phi)}\Delta\hat{N} \simeq \frac{\sin(\theta)}{2|\alpha|}|\alpha| = \frac{\sin(\theta)}{2}, \tag{34}$$

which gives us a similar result than the one obtained with our less formal procedure.

IV. SQUEEZED STATES

Coherent states are states that satisfy the uncertainty relation for the quadratures with the equality sign, as is widely known, that is: $\Delta\hat{p} = \Delta\hat{q} = \frac{1}{2}$. But in the Heisenberg uncertainty principle there are no constraints to the uncertainty $\Delta\hat{p}$ being less than $\frac{1}{2}$, this is $\Delta\hat{p} < \frac{1}{2}$ as long as the uncertainty of the other quadrature satisfies $\Delta\hat{p}\Delta\hat{q} = \frac{1}{4}$. The states that satisfy this property are the squeezed states

A. Squeezed states and SU(1,1)

Consider the scaling transformation on a general wavefunction

$$\psi(x) \longrightarrow \tilde{\psi}(x) = \sqrt{\lambda}\psi(\lambda x). \tag{35}$$

This transformation preserves the normalization

$$\int dx|\tilde{\psi}(x)|^2 = \int dx\lambda|\psi(\lambda x)|^2 = \int du|\psi(u)|^2.$$

Now let $\lambda = 1 + \delta_\lambda$ such that δ_λ is an infinitesimal parameter. Then the transformation yields

$$\tilde{\psi}(x) = \left(1 + \frac{\delta_\lambda}{2}\right)\psi(x + \delta_\lambda x). \tag{36}$$

Expanding to and only considering first order terms in δ_λ

$$\begin{aligned}\tilde{\psi}(x) &= \left(1 + \frac{\delta_\lambda}{2}\right)\left(\psi(x) + \delta_\lambda x\frac{d}{dx}(\psi(x)\right). \\ &= \psi(x) + i\delta_\lambda\left(-\frac{i}{2} + x(-i\frac{d}{dx})\right)\psi(x),\end{aligned} \tag{37}$$

and using the definition of the momentum operator and $[\hat{X}, \hat{P}] = -i$

$$\tilde{\psi}(x) = \left(\mathbb{I} + i\delta_\lambda\left(\frac{1}{2}[\hat{X}, \hat{P}] + \hat{X}\hat{P}\right)\right)\psi(x), \tag{38}$$

which can be expressed as

$$\tilde{\psi}(x) = \left(\mathbb{I} + i\delta_\lambda\hat{G} + O(\delta_\lambda^2)\right)\psi(x) \tag{39}$$

with $\hat{G} = \frac{\hat{X}\hat{P}+\hat{P}\hat{X}}{2}$.
This shows that the scaling transformation is generated by the operator $\hat{G}$

$$\hat{G} = \frac{\hat{a}^2 - (\hat{a}^\dagger)^2}{2i}.$$

Now, equation (39) can be expressed as

$$\tilde{\psi}(x) = e^{i\lambda\hat{G}}\psi(x) = \hat{S}(\lambda)\psi(x) = e^{\frac{1}{2}\lambda(\hat{a}^2-(\hat{a}^\dagger)^2)}\psi(x). \tag{40}$$

In analogy with the angular momentum and the rotations, the operator $\hat{G}$ generates the scaling transformations on the Hilbert space of the wavefunctions.
This operator $e^{i\lambda\hat{G}} = \hat{S}(\lambda)$, applied to the creation and destruction operators, acts as

$$\begin{pmatrix}\hat{S}\hat{a}\hat{S}^\dagger \\ \hat{S}\hat{a}^\dagger\hat{S}^\dagger\end{pmatrix} = \begin{pmatrix}\cosh(\lambda) & \sinh(\lambda) \\ \sinh(\lambda) & \cosh(\lambda)\end{pmatrix}\begin{pmatrix}\hat{a} \\ \hat{a}^\dagger\end{pmatrix} \tag{41}$$

where it turns out that the matrix is an element of the SU(1,1) group, detailed in the appendix.

Equation (41) can be proved using the decomposition of the exponential operators, detailed in the appendix.
Now consider the operators

$$\hat{K}_0 = \hat{a}^\dagger\hat{a} + \frac{1}{2},\ \ \hat{K}_+ = \frac{1}{2}(\hat{a}^\dagger)^2,\ \ \hat{K}_- = -\frac{1}{2}\hat{a}^2. \tag{42}$$

These operators have the same commutation rules of the SU(1,1) generators

$$[\hat{K}_+, \hat{K}_-] = \hat{K}_0,\ \ [\hat{K}_0, \hat{K}_-] = -2\hat{K}_-,\ \ [\hat{K}_0, \hat{K}_+] = 2\hat{K}_+ \tag{43}$$

The operators $\hat{K}_0, \hat{K}_+\hat{K}_-$ are a representation of the SU(1,1) Lie algebra.
The operator $\hat{S}(\lambda)$ is the exponential of the difference of $\hat{K}_-$ and $\hat{K}_+$, i.e.

$$\hat{S}(\lambda) = e^{\lambda(\hat{K}_- - \hat{K}_+)}. \tag{44}$$

Perelomov, in [3] took advantage of this property and derived

$$\begin{aligned}\hat{S}(\lambda) = \frac{1}{\sqrt{\cosh(\lambda)}} \times e^{-(\hat{a}^\dagger)^2\tanh(\lambda)} \times \\ e^{-(\hat{a}^\dagger\hat{a}\ln(\cosh\lambda)} \times e^{\hat{a}^2\tanh(\lambda)}.\end{aligned} \tag{45}$$

Applying this operator on the ground state, we define the squeezed vacuum

$$|s\rangle = \hat{S}(\lambda)|0\rangle. \tag{46}$$

Now, with the help of equation (45) we obtain the form of the squeezed vacuum on Fock's basis

$$|s\rangle = \frac{1}{\sqrt{\cosh(\lambda)}}\sum_n(\tanh(\lambda))^n\frac{1}{\sqrt{(2n)!}}|2n\rangle. \tag{47}$$

B. Properties of the squeezed vacuum

Now that we have derived an expression for the squeezed vacuum, we can have a look at its properties. The uncertainty for the quadratures will be determined by

$$(\Delta\hat{q})^2 = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2. \tag{48}$$

Now we need to evaluate the term

$$\langle s|\hat{a}|s\rangle = \langle 0|\hat{S}^\dagger(\lambda)\hat{a}\hat{S}(\lambda)|0\rangle. \tag{49}$$

Following a procedure similar to that used to derive equation (41), we find

$$\langle 0|\hat{S}^\dagger(\lambda)\hat{a}\hat{S}(\lambda)|0\rangle = \langle 0|\left(\hat{a}\cosh(\lambda) - \hat{a}^\dagger\sinh(\lambda)\right)|0\rangle. \tag{50}$$

So the contribution to the uncertainty of the terms $\langle s|\hat{a}|s\rangle$ and $\langle s|\hat{a}^\dagger|s\rangle$ will be equal to 0. By consequence, the calculation of uncertainty of the quadrature will be reduced to

$$(\Delta\hat{q})^2 = \langle\hat{q}^2\rangle = \langle s|\left(\frac{(\hat{a}^\dagger)^2 + \hat{a}^2 + 2\hat{N} + 1}{2}\right)|s\rangle. \tag{51}$$

Using the unitarity of the squeeze operator

$$\langle s|(\hat{a}^\dagger)^2|s\rangle = \langle 0|\hat{S}^\dagger(\lambda)\hat{a}^\dagger\hat{S}(\lambda)\hat{S}^\dagger(\lambda)\hat{a}^\dagger\hat{S}(\lambda)|0\rangle\,. \quad (52)$$

Using (50), equation (52) becomes

$$\langle s|(\hat{a}^\dagger)^2|s\rangle = \sinh(\lambda)\cosh(\lambda). \quad (53)$$

Similarly

$$\langle s|\hat{a}^2|s\rangle = \sinh(\lambda)\cosh(\lambda). \quad (54)$$

Now using a parallel reasoning to the one used in (52)

$$\langle s|\hat{N}|s\rangle = \langle 0|\hat{S}^\dagger(\lambda)\hat{a}^\dagger\hat{S}(\lambda)\hat{S}^\dagger(\lambda)\hat{a}\hat{S}(\lambda)|0\rangle\,, \quad (55)$$

in which only the term $\langle 0|\sinh^2(\lambda)\hat{a}\hat{a}^\dagger|0\rangle$ survives. These results give

$$(\Delta\hat{q})^2 = \frac{1}{4}e^{-2\lambda} \quad (56)$$

This shows that the uncertainty on the quadrature becomes squeezed by a factor $e^{-\lambda}$ with respect to the uncertainty for a coherent state.
By an almost identical derivation, the $(\Delta p)^2$

$$(\Delta\hat{p})^2 = \frac{1}{4}e^{2\lambda} \quad (57)$$

Now for the conjugate quadrature, the uncertainty is enlarged by a factor e^{λ} with respect to the uncertainty for a coherent state in order to preserve the Heisenberg uncertainty principle.
It should be noted that for the squeezed vacuum the uncertainty relation is satisfied with the equality sign as well $\Delta\hat{q}\Delta\hat{p} = \frac{1}{4}$.
The parameter λ which is taken to be real, can appear in the literature to be a complex number, although this changes nothing in squeezing terms, as the squeezing depends on its amplitude $|\lambda|$.

C. Two mode squeezed states

Throughout the entirety of this work we have only considered one mode quantum states, but for this section will be worth considering a two mode state. The two mode squeezing operator is defined as:

$$\hat{S}(\zeta) = e^{\frac{1}{2}(\zeta\hat{a}\hat{b} - \zeta\hat{a}^\dagger\hat{b}^\dagger)}. \quad (58)$$

With $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$ being the annihilation and creation operators for two different modes.
Now considering the operators defined by

$$\begin{aligned}\hat{K}_+ &= \hat{a}^\dagger\hat{b}^\dagger, \quad \hat{K}_- = \hat{a}\hat{b}\\ \hat{K}_0 &= (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 1).\end{aligned} \quad (59)$$

These operators are also a representation of the SU(1,1) algebra. Following a similar reasoning as that for the state of one mode

$$\begin{aligned}\hat{S}(\zeta) = \frac{1}{\cosh(\zeta)} \times e^{-\hat{a}^\dagger\hat{b}^\dagger\tanh(\zeta)} \times\\ e^{-(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})\ln(\cosh(\zeta))} \times e^{(\hat{a}\hat{b}\tanh(\zeta))}.\end{aligned} \quad (60)$$

The two mode squeeze operator in the form of (60) applied to the two mode vacuum gives us the expression shown in [6]

$$|s\rangle = \frac{1}{\cosh(\zeta)}\sum_n (\tanh\zeta)^n e^{in\theta}\,|n\rangle_a \otimes |n\rangle_b\,. \quad (61)$$

V. CONCLUSIONS

Coherent states are quantum superpositions of different number states, and although they do not have any classical analogy, coherent states have an electric field analog to a plane electromagnetic wave. We have studied the phase and the cosine operators and related them to coherent states by showing that have an asymptotically well-defined phase in the limit for large expected number of photons.
Moreover, we have studied states with smaller uncertainties on a quadrature than the coherent states, the squeezed states, and shown that these states have a relation with the Lie Group SU(1,1), either for one mode or two modes of oscillation.

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Propietats físiques i matemàtiques dels estats coeherents.

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Resum: En aquest treball s'han estudiat els estats coherents i les seves propietats. S'ha mostrat que el camp elèctric d'un estat coherent és una ona electromagnètica plana, resemblant així el resultat clàssic. S'ha estudiat la fase d'un estat coherent, mostrant que en el límit asimptòtic de nombres mitjans de fotons, un estat coherent té una fase ben definida. Finalment s'ha estudiat l'estat squeezed del buit. S'ha mostrat que té una relació amb el grup de Lie SU(1,1), tant com per 1 i 2 modes d'oscil·lació, i que té una incertesa menor en les quadratures que un estat coherent.
Paraules clau: Quàntica, coherent, fase, operador, estrényer, incertesa.
ODSs: Aquest TFG està relacionat amb els següents Objectius de Desenvolupament Sostenible (SDGs): **4.3**, **7.4**, **13.3**

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	X
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible	X	16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

VII. APPENDIX

A. The Baker-Campbell-Hausdorff formula

The B-C-H formula is

$$\exp \hat{X} \exp \hat{Y} = \exp(\hat{Y} + [\hat{X}, \hat{Y}] + \frac{1}{2!}[\hat{X}, [\hat{X}, \hat{Y}]] + ...) \tag{62}$$

The decomposition of the exponential operators reads as

$$e^{\hat{X}} \hat{O} e^{-\hat{X}} = \hat{O} + [\hat{X}, \hat{O}] + \frac{1}{2!}[\hat{X}, [\hat{X}, \hat{O}]] + ... \tag{63}$$

B. SU(1,1) group

The group SU(1,1) has for elements the matrices M that satisfy

$$M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{64}$$

that is, the matrices M that have the form

$$M = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} \tag{65}$$

with $\det(M) = 1$, which means $|u|^2 - |v|^2 = 1$

This group has for generators

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; \Lambda_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \;\; \Lambda_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{66}$$

Now forming lineal combinations we get the operators

$$\Lambda_+ = \frac{i\Lambda_2 + \Lambda_1}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \;\; \Lambda_- = \frac{i\Lambda_2 - \Lambda_1}{2} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

with their commutation rules

$$[\Lambda_+, \Lambda_]] = \Lambda_0, \;\; [\Lambda_0, \Lambda_+] = 2\Lambda_+, \;\; [\Lambda_+, \Lambda_-] = -2\Lambda_- \tag{67}$$

C. Characterization of the eigenvalues and eigenvectors for $\widehat{\cos(\phi)}$

Following Susskind and Glogower, we expand $|\cos(\theta)\rangle$ in the number representation i.e $|\cos(\theta)\rangle = \sum_n C_n |n\rangle$ Now the eigenvalue equation reads as

$$2\widehat{\cos(\phi)} |\cos(\theta)\rangle = \sum_n (C_n |n+1\rangle + C_n |n-1\rangle) = 2\lambda \sum_n C_n |n\rangle \tag{68}$$

This gives us the recursive relation

$$C_1 = 2\lambda C_0, \quad C_n + C_{n+2} = 2\lambda C_{n+1} \tag{69}$$

The general solution for the second equation can be proposed to be a power series

$$C_n = A\rho^n + B\rho^{-n} \tag{70}$$

Since the operator $\widehat{\cos(\phi)}$ is hermitian, the eigenvalue λ must be real. Following the first equation $\rho + \frac{1}{\rho} = 2\lambda$, and imposing $\lambda \in \mathbb{R}$, we find

$$\mathrm{Im}\left(\rho + \frac{1}{\rho}\right) = \sin(\theta)\left(|\rho| - \frac{1}{|\rho|}\right) = 0 \Rightarrow |\rho|^2 = 1 \tag{71}$$

Notice how the condition $|\rho|^2 = 1$ grants us that the coefficient C_n stays bounded and does not diverge. Now let $\rho = e^{i\theta}$. Then the eigenvalue $\lambda = \cos(\theta)$, and inserting this into the recursion relation we get

$$|\cos(\theta)\rangle = \sum_n \sin(n+1)\theta \, |n\rangle \tag{72}$$