On quantum teleportation with partial entanglement

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Abstract: Quantum teleportation cannot be achieved with certainty when using a partially entangled resource; however, full-fidelity teleportation can be realized probabilistically, by allowing the protocol to fail sometimes. We present two strategies for conclusive teleportation based on generalized measurements: one involves a filtering operation applied by the receiver, while the other uses non-orthogonal state discrimination performed by the sender. Both methods are optimal in that they succeed with the maximum possible probability. Additionally, we study this maximal probability in the general d-dimensional case.
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I. INTRODUCTION

In quantum information theory, quantum teleportation [1] refers to the process of recreating an unknown quantum state at a remote location by sending only classical information. This is achieved through a shared, maximally entangled pair between the sender and the receiver. The sender, Alice, has an unknown qubit $|\psi\rangle$ she wants to send to the receiver, Bob:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \tag{1}$$

The coefficients α and β are unknown and satisfy $|\alpha|^2 + |\beta|^2 = 1$. The qubit is expressed in the computational basis: $\{|0\rangle = \binom{1}{0},\, |1\rangle = \binom{0}{1}\}$.

At the same time, Alice and Bob share one of the Bell basis states $\Big\{|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|00\rangle_{AB} + |11\rangle_{AB}\big),\, |\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|00\rangle_{AB} - |11\rangle_{AB}\big),\, |\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|01\rangle_{AB} + |10\rangle_{AB}\big),\, |\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|01\rangle_{AB} - |10\rangle_{AB}\big)\Big\}$. Subscripts A and B refer to Alice's half and Bob's half, respectively. To teleport the qubit $|\psi\rangle$ from Alice to Bob, it is combined with the shared pair. The three-qubit system becomes:

$$\begin{aligned}
|\Psi\rangle_{CAB} &= |\psi\rangle_C\,|\Phi^+\rangle_{AB}\\
&= \begin{pmatrix}\alpha\\ \beta\end{pmatrix}_C \frac{1}{\sqrt{2}}\big(|00\rangle_{AB} + |11\rangle_{AB}\big)\\
&= \frac{1}{2}\Bigg[|\Phi^+\rangle_{CA}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}_B + |\Phi^-\rangle_{CA}\begin{pmatrix}\alpha\\ -\beta\end{pmatrix}_B\\
&\quad + |\Psi^+\rangle_{CA}\begin{pmatrix}\beta\\ \alpha\end{pmatrix}_B + |\Psi^-\rangle_{CA}\begin{pmatrix}-\beta\\ \alpha\end{pmatrix}_B\Bigg]
\end{aligned} \tag{2}$$

Here, the subscript C indicates the system of the original qubit to be teleported. This system belongs to Alice.

This state can be rearranged as a superposition of Bell states, as shown in equation (2). From this form, it is easy to see that if Alice measures her two-qubit system CA in the Bell basis $\{|\Phi^+\rangle_{CA},\, |\Phi^-\rangle_{CA},\, |\Psi^+\rangle_{CA},\, |\Psi^-\rangle_{CA}\}$, each outcome occurs with equal probability, and Bob's half of the Bell pair then collapses to a rotated version of $|\psi\rangle$: $\{\binom{\alpha}{\beta}_B, \binom{\alpha}{-\beta}_B, \binom{\beta}{\alpha}_B, \binom{-\beta}{\alpha}_B\}$.

Alice sends the result of her measurement to Bob as two classical bits. Using this information, Bob knows which rotation to apply to his qubit via quantum gates $\{I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\, Z = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),\, X = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\, XZ = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\}$. Finally, he obtains the original qubit $|\psi\rangle_B = \binom{\alpha}{\beta}_B$ on his system, successfully completing the teleportation.

This protocol only works if Alice and Bob share a maximally entangled state. If, instead, they share a partially entangled state, Bob's final state will not match the original qubit. Let Alice and Bob share a general entangled state:

$$|\chi\rangle_{AB} = a\,|00\rangle_{AB} + b\,|11\rangle_{AB} \tag{3}$$

This represents the most general entangled state (up to a local change of basis) according to the Schmidt decomposition, where a and b are both real and positive coefficients satisfying $a^2 + b^2 = 1$. Without loss of generality, we assume $a \geq b > 0$. The special case $a = b$ corresponds to maximal entanglement. Similarly to before, the new three-qubit system then becomes:

$$\begin{aligned}
|\Omega\rangle_{CAB} &= |\psi\rangle_C\,|\chi\rangle_{AB}\\
&= \begin{pmatrix}\alpha\\ \beta\end{pmatrix}_C \big(a\,|00\rangle_{AB} + b\,|11\rangle_{AB}\big)\\
&= \frac{1}{\sqrt{2}}\Bigg[|\Phi^+\rangle_{CA}\begin{pmatrix}a\alpha\\ b\beta\end{pmatrix}_B + |\Phi^-\rangle_{CA}\begin{pmatrix}a\alpha\\ -b\beta\end{pmatrix}_B\\
&\quad + |\Psi^+\rangle_{CA}\begin{pmatrix}a\beta\\ b\alpha\end{pmatrix}_B + |\Psi^-\rangle_{CA}\begin{pmatrix}-a\beta\\ b\alpha\end{pmatrix}_B\Bigg]
\end{aligned} \tag{4}$$

This state can also be rewritten in terms of the Bell basis states. However, in this case, Bob's state is no longer a simple rotation of the original state $|\psi\rangle$. Alice's measurement in the Bell basis yields the outcomes $|\Phi^\pm\rangle$ each with probability $\frac{1}{2}\left(a^2|\alpha|^2 + b^2|\beta|^2\right)$, and the outcomes

$|\Psi^{\pm}\rangle$ each with probability $\frac{1}{2}\left(a^2|\beta|^2+b^2|\alpha|^2\right)$. Depending on the outcome of Alice's measurement, Bob's qubit is projected into one of the following unnormalized states $|\phi_i\rangle \in \left\{\binom{a\alpha}{b\beta}, \binom{a\alpha}{-b\beta}, \binom{a\beta}{b\alpha}, \binom{-a\beta}{b\alpha}\right\}$. Their norms relate to the probabilities for Alice via $p_i = \frac{1}{2}\,\||\phi_i\rangle\|^2$.

Even after applying quantum gates $\{I,\, Z,\, X,\, XZ\}$, these states generally do not coincide with $|\psi\rangle$, indicating imperfect teleportation unless entanglement is maximal. To achieve full fidelity, Bob can use an ancillary qubit and a unitary transformation to convert his state $|\phi\rangle$ to a rotation of $|\psi\rangle$, as presented by Wan-Li Li *et al.* [2].

II. PROBABILISTIC TELEPORTATION

After Alice measures in the Bell basis, Bob's state collapses to one of the unnormalized states $|\phi_i\rangle$ with probability $p_i = \frac{1}{2}\,\||\phi_i\rangle\|^2$. He knows which state he has received based on the information sent by Alice. To recover the original state, Bob wants to transform $|\phi_i\rangle$ into the original $|\psi\rangle = \binom{\alpha}{\beta}$, however, this transformation cannot be unitary while universal for all $\binom{\alpha}{\beta}$. To apply such a non-unitary operation, he appends an ancilla qubit in a known state $|0\rangle_a$ and performs a unitary transformation on the extended space, followed by a measurement. This comes at the cost of introducing a finite probability of failure. Depending on the measurement outcome, the procedure either succeeds, yielding the desired rotated state, or fails, resulting in a complete loss of information. Crucially, Bob is always able to determine whether the recovery was successful. From the combined initial state:

$$\frac{1}{\mathcal{N}}\,|0\rangle_a|\phi\rangle = \frac{1}{\mathcal{N}}\,|0\rangle_a\left(a\kappa\,|0\rangle + b\eta\,|1\rangle\,\right) \tag{5}$$

Where $\mathcal{N}^2 = \||\phi\rangle\|^2 = |a\kappa|^2 + |b\eta|^2$ is a normalization constant and $(\kappa,\eta) = (\alpha,\pm\beta)$ or $(\pm\beta,\alpha)$ represent all possible states $\{|\phi_i\rangle\}$. The unitary operator U acts as follows:

$$\begin{aligned} &U\left(\frac{1}{\mathcal{N}}\,|0\rangle_a\left(a\kappa\,|0\rangle + b\eta\,|1\rangle\,\right)\right) \\ &= \frac{1}{\mathcal{N}}\left[|0\rangle_a\left(\frac{b}{a}a\kappa\,|0\rangle + b\eta\,|1\rangle\right) + |1\rangle_a\sqrt{1-\frac{b^2}{a^2}}a\kappa\,|1\rangle\right] \\ &= \frac{1}{\mathcal{N}}\left[|0\rangle_a\left(b\kappa\,|0\rangle + b\eta\,|1\rangle\,\right) + |1\rangle_a\sqrt{a^2-b^2}\kappa\,|1\rangle\right] \end{aligned} \tag{6}$$

The term proportional to $|0\rangle_a$, indicating success, selectively dampens the $|0\rangle$ component by a factor of $\frac{b}{a}$ while leaving the $|1\rangle$ component intact. It follows that the state corresponding to $|0\rangle_a$ is, up to normalization, a rotation of the original state $|\psi\rangle$. For example, the unnormalized state $|\phi\rangle = \binom{a\alpha}{b\beta}$ is mapped to $b\,|\psi\rangle = \binom{b\alpha}{b\beta}$ upon successful recovery. The factor $\sqrt{1-\frac{b^2}{a^2}}$ in the term proportional to $|1\rangle_a$, indicating failure, arises from the need to preserve unitarity, ensuring normalization after modifying $a\kappa\,|0\rangle$ to $b\kappa\,|0\rangle$.

The probabilities for each outcome are $\frac{1}{\mathcal{N}^2}b^2$ for success and $\frac{1}{\mathcal{N}^2}\left(a^2-b^2\right)|\kappa|^2$ for failure. Note that they both depend on κ and η (recall $\mathcal{N}^2 = |a\kappa|^2+|b\eta|^2$); still, they add up to 1. This dependence disappears when averaging over all states $|\phi_i\rangle$ with probabilities $p_i = \frac{1}{2}\mathcal{N}_i^2$. The probabilities become $2b^2$ for success, transforming the state to a rotation of $|\psi\rangle$: $\left\{\binom{\alpha}{\beta}, \binom{\alpha}{-\beta}, \binom{\beta}{\alpha}, \binom{-\beta}{\alpha}\right\}$; and a^2-b^2 for failure, with no information about $|\psi\rangle$.

This unitary transformation can be explicitly written as the following matrix:

$$U = \begin{pmatrix} \frac{b}{a} & 0 & 0 & -\sqrt{1-\frac{b^2}{a^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{1-\frac{b^2}{a^2}} & 0 & 0 & \frac{b}{a} \end{pmatrix} \tag{7}$$

in basis $\left\{\,|0\rangle_a|0\rangle, |0\rangle_a|1\rangle, |1\rangle_a|0\rangle, |1\rangle_a|1\rangle\,\right\}$.

Physically, this transformation can be interpreted as a time evolution operator $U(t)$ with evolution parameter t such that $\cos t = \frac{b}{a}$ for some $t \in \left[0, \frac{\pi}{2}\right]$. Hence, we write:

$$U(t) = \begin{pmatrix} \cos t & 0 & 0 & -\sin t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin t & 0 & 0 & \cos t \end{pmatrix} \tag{8}$$

with a Hamiltonian H satisfying $U(t) = e^{-iHt}$:

$$H = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

The evolution combined with the measurement of the ancilla qubit can be described by a Positive Operator-Valued Measure (POVM) with Kraus operators:

$$K_0 = \begin{pmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{pmatrix} \quad K_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{1-\frac{b^2}{a^2}} & 0 \end{pmatrix} \tag{10}$$

which correspond to the first two columns of U. This leads to the decomposition:

$$U\big(\,|0\rangle_a|\phi\rangle\,\big) = |0\rangle_a K_0\,|\phi\rangle + |1\rangle_a K_1\,|\phi\rangle \tag{11}$$

where K_0 and K_1 act only on $|\phi\rangle$. The POVM elements are given by $A_i = K_i^{\dagger}K_i$:

$$A_0 = \begin{pmatrix} \frac{b^2}{a^2} & 0 \\ 0 & 1 \end{pmatrix} \quad A_1 = \begin{pmatrix} 1-\frac{b^2}{a^2} & 0 \\ 0 & 0 \end{pmatrix} \tag{12}$$

with non-negative eigenvalues and summing up to the identity operator: $A_0 + A_1 = I$.

III. POVM: GENERALIZED MEASUREMENTS

A POVM represents the most general type of measurement on a quantum state and is therefore also known as a generalized measurement. It can be understood as the effective action on a subsystem arising from a projective measurement performed on a larger, entangled system. Formally, a POVM is a set of positive semi-definite Hermitian operators $A_i = A_i^\dagger$ that satisfy the completeness relation: $\sum_i A_i = I$, where I is the identity operator. Positive semi-definiteness ensures that each operator's eigenvalues are non-negative, corresponding to real probabilities, while the completeness relation guarantees that the probabilities of all possible outcomes sum to 1.

Each POVM element A_i represents a different outcome, and can be decomposed as $A_i = K_i^\dagger K_i$, where K_i are the Kraus operators associated with the measurement outcomes. This decomposition is not unique, since for any unitary operator W, the Kraus operators $M_i = WK_i$ lead to the same POVM elements: $M_i^\dagger M_i = K_i^\dagger W^\dagger W K_i = K_i^\dagger K_i = A_i$.

As stated by Neumark's theorem, any POVM can be implemented as a projective measurement on an extended Hilbert space. This is achieved by introducing an ancillary system (ancilla) and performing a suitable unitary transformation on the joint system, as previously discussed. The dimension of the ancillary space must be at least equal to the number of POVM elements, since the outcomes correspond to projective measurements in an orthonormal basis of the ancilla. More generally, given a POVM $\{A_i\}$ with Kraus operators $\{K_i\}$, one can construct a unitary operator U whose first columns correspond to the K_i:

$$U = \left(\begin{array}{c|c} \cdots & \\ \hline K_i & \\ \hline \cdots & \end{array} \right) \tag{13}$$

with the remaining columns chosen to ensure that U is unitary $U^\dagger U = I$. This construction is not unique but does not affect the resulting POVM. This operator will take an initial state $|0\rangle_a |\psi\rangle$ to a superposition, assigning an ancillary state $|i\rangle_a$ to each outcome $K_i |\psi\rangle$:

$$U\big(|0\rangle_a |\psi\rangle \big) = \sum_i |i\rangle_a K_i |\psi\rangle \tag{14}$$

where each operator K_i acts only on $|\psi\rangle$. Measuring the ancilla in the orthonormal basis $\{|i\rangle_a\}$ will result in a collapse to the corresponding $K_i |\psi\rangle$ with probability $\| K_i |\psi\rangle \|^2$, thereby realizing the generalized measurement.

IV. CONCLUSIVE TELEPORTATION WITH UNAMBIGUOUS STATE DISCRIMINATION

The use of generalized measurements enables the unambiguous discrimination of non-orthogonal quantum states, at the expense of introducing a finite probability of inconclusive outcomes. This approach underlies an alternative teleportation protocol using partially entangled resources, introduced by Gilles Brassard *et al.* [3] as conclusive teleportation. In this process, Alice measures in a non-orthogonal basis, then sends the information to Bob, who applies the quantum gates as in the original protocol. This contrasts with the prior method in which Bob was responsible for the more complex state transformation.

Consider the three-qubit system from equation (4):

$$\begin{aligned} |\Omega\rangle_{CAB} &= |\psi\rangle_C |\chi\rangle_{AB} \\ &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_C \big(a\,|00\rangle_{AB} + b\,|11\rangle_{AB} \big) \\ &= \frac{1}{2} \Bigg[\big(a\,|00\rangle_{CA} + b\,|11\rangle_{CA} \big) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_B \\ &\quad + \big(a\,|00\rangle_{CA} - b\,|11\rangle_{CA} \big) \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}_B \\ &\quad + \big(b\,|01\rangle_{CA} + a\,|10\rangle_{CA} \big) \begin{pmatrix} \beta \\ \alpha \end{pmatrix}_B \\ &\quad + \big(b\,|01\rangle_{CA} - a\,|10\rangle_{CA} \big) \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}_B \Bigg] \end{aligned} \tag{15}$$

Now interpreted as a superposition of rotations of $|\psi\rangle$. We can name $|\Phi_1\rangle = a\,|00\rangle + b\,|11\rangle$, $|\Phi_2\rangle = a\,|00\rangle - b\,|11\rangle$, $|\Psi_1\rangle = b\,|01\rangle + a\,|10\rangle$, and $|\Psi_2\rangle = b\,|01\rangle - a\,|10\rangle$, paralleling the Bell basis states. Alice aims to measure over $\big\{|\Phi_1\rangle_{CA}, |\Phi_2\rangle_{CA}, |\Psi_1\rangle_{CA}, |\Psi_2\rangle_{CA}\big\}$. Although the states $\big\{ |\Phi_i\rangle \big\}$ and $\big\{ |\Psi_i\rangle \big\}$ are not mutually orthogonal within their respective sets: $\langle \Phi_i | \Phi_j \rangle \neq \delta_{ij}$, $\langle \Psi_i | \Psi_j \rangle \neq \delta_{ij}$, the subspaces they occupy are orthogonal to each other: $\langle \Phi_i | \Psi_j \rangle = 0$. Consequently, Alice can first perform a projective measurement to determine whether her two-qubit state lies in the subspace spanned by $\big\{|00\rangle, |11\rangle\big\}$ or in the subspace spanned by $\big\{|01\rangle, |10\rangle\big\}$, effectively distinguishing between the $|\Phi_i\rangle$ and $|\Psi_i\rangle$ states. Each outcome occurs with equal probability. This can be implemented by measuring a degenerate observable projecting onto these two subspaces, for example, the parity operator $\hat{\mathcal{P}} = |00\rangle\langle 00| + |11\rangle\langle 11| - |01\rangle\langle 01| - |10\rangle\langle 10|$. Once the subspace is identified, Alice distinguishes between two non-orthogonal states by performing a POVM for non-orthogonal state discrimination conditioned on the first outcome.

In general, for an ensemble of two non-orthogonal states $|\psi_1\rangle$, $|\psi_2\rangle$ with equal *a priori* probabilities and overlap $\langle \psi_1 | \psi_2 \rangle \neq 0$, the unambiguous state discrimination (USD) POVM is [4]:

$$\begin{aligned} A_1 &= \frac{1}{1 + |\langle \psi_1 | \psi_2 \rangle|} |\psi_2^\perp\rangle \langle \psi_2^\perp| \\ A_2 &= \frac{1}{1 + |\langle \psi_1 | \psi_2 \rangle|} |\psi_1^\perp\rangle \langle \psi_1^\perp| \\ A_0 &= I - A_1 - A_2 \end{aligned} \tag{16}$$

Where $|\psi_1^\perp\rangle$ and $|\psi_2^\perp\rangle$ are states orthogonal to $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. These operators satisfy $A_1 + A_2 + A_0 =$

I, and have the properties $A_1 |\psi_2\rangle = 0$ and $A_2 |\psi_1\rangle = 0$. This implies that if the measurement outcome corresponds to A_1, the state cannot be $|\psi_2\rangle$, and vice versa. Since the ensemble contains only the two states, an outcome A_1 corresponds to conclusively identifying $|\psi_1\rangle$ and an outcome A_2 to conclusively identifying $|\psi_2\rangle$, while an outcome A_0 indicates a failed discrimination. The prefactor $\frac{1}{1+|\langle\psi_1|\psi_2\rangle|}$ maximizes the probability of success (A_1 or A_2) while ensuring that A_0 is positive semi-definite. This probability can be found to be $1 - |\langle\psi_1|\psi_2\rangle|$.

Suppose Alice's measurement yields the parity eigenvalue $+1$, meaning her state falls in the subspace spanned by $|00\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|11\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. The two states to discriminate are $|\Phi_1\rangle = a|00\rangle + b|11\rangle = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$ and $|\Phi_2\rangle = a|00\rangle - b|11\rangle = \left(\begin{smallmatrix}a\\-b\end{smallmatrix}\right)$, with factor $1 + |\langle\Phi_1|\Phi_2\rangle| = 1 + \left|\big(a\langle 00| + b\langle 11|\big)\big(a|00\rangle - b|11\rangle\big)\right| = 1 + a^2 - b^2 = 2a^2$ (remember $a^2 + b^2 = 1$ and $a \geq b$). We find orthogonal states $|\Phi_1^{\perp}\rangle = \left(\begin{smallmatrix}b\\-a\end{smallmatrix}\right)$ and $|\Phi_2^{\perp}\rangle = \left(\begin{smallmatrix}b\\a\end{smallmatrix}\right)$, since a and b are real. The discrimination POVM becomes:

$$\begin{aligned} A_1^{\Phi} &= \frac{1}{2a^2} |\Phi_2^{\perp}\rangle\langle\Phi_2^{\perp}| = \frac{1}{2a^2}\begin{pmatrix} b^2 & ab \\ ab & a^2 \end{pmatrix} \\ A_2^{\Phi} &= \frac{1}{2a^2} |\Phi_1^{\perp}\rangle\langle\Phi_1^{\perp}| = \frac{1}{2a^2}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \\ A_0^{\Phi} &= I - A_1^{\Phi} - A_2^{\Phi} = \begin{pmatrix} 1-\frac{b^2}{a^2} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{17}$$

These are all proportional to projectors, i.e., have rank one. The probability of success is $1 - |\langle\Phi_1|\Phi_2\rangle| = 1 - (a^2 - b^2) = 2b^2$.

To implement this POVM following Neumark's theorem, a simple choice for the corresponding Kraus operators is $K_i^{\Phi} = \sqrt{A_i^{\Phi}}$:

$$\begin{aligned} K_1^{\Phi} &= \frac{1}{\sqrt{2}a} |\Phi_2^{\perp}\rangle\langle\Phi_2^{\perp}| = \frac{1}{\sqrt{2}a}\begin{pmatrix} b^2 & ab \\ ab & a^2 \end{pmatrix} \\ K_2^{\Phi} &= \frac{1}{\sqrt{2}a} |\Phi_1^{\perp}\rangle\langle\Phi_1^{\perp}| = \frac{1}{\sqrt{2}a}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \\ K_0^{\Phi} &= \sqrt{I - A_1^{\Phi} - A_2^{\Phi}} = \begin{pmatrix} \sqrt{1-\frac{b^2}{a^2}} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{18}$$

A corresponding unitary transformation acts as follows:

$$\begin{aligned} U(|0\rangle_a |\Phi\rangle) &= \sum_i |i\rangle_a K_i^{\Phi} |\Phi\rangle \\ &= |1\rangle_a K_1^{\Phi}|\Phi\rangle + |2\rangle_a K_2^{\Phi}|\Phi\rangle + |0\rangle_a K_0^{\Phi}|\Phi\rangle \end{aligned} \tag{19}$$

A projective measurement over $\{|0\rangle_a, |1\rangle_a, |2\rangle_a\}$ collapses the state to $\{K_0^{\Phi}|\Phi\rangle, K_1^{\Phi}|\Phi\rangle, K_2^{\Phi}|\Phi\rangle\}$, with the outcome $|0\rangle_a$ indicating failure. If $|\Phi\rangle = |\Phi_1\rangle$, then:

$$\begin{aligned} U(|0\rangle_a |\Phi_1\rangle) &= |1\rangle_a K_1^{\Phi}|\Phi_1\rangle + |0\rangle_a K_0^{\Phi}|\Phi_1\rangle \\ &= |1\rangle_a \sqrt{2}b\,|\Phi_2^{\perp}\rangle + |0\rangle_a \sqrt{a^2-b^2}\,|00\rangle \end{aligned} \tag{20}$$

where outcome $|2\rangle_a$ occurs with 0 probability. Similarly, for $|\Phi\rangle = |\Phi_2\rangle$:

$$\begin{aligned} U(|0\rangle_a |\Phi_2\rangle) &= |2\rangle_a K_2^{\Phi}|\Phi_2\rangle + |0\rangle_a K_0^{\Phi}|\Phi_2\rangle \\ &= |2\rangle_a \sqrt{2}b\,|\Phi_1^{\perp}\rangle + |0\rangle_a \sqrt{a^2-b^2}\,|00\rangle \end{aligned} \tag{21}$$

with 0 probability for outcome $|1\rangle_a$. In both cases, the probability of success is $(\sqrt{2}b)^2 = 2b^2$.

If instead, Alice measures the parity eigenvalue -1, the POVM elements for the subspace spanned by $|01\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|10\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, discriminating between $|\Psi_1\rangle = b|01\rangle + a|10\rangle = \left(\begin{smallmatrix}b\\a\end{smallmatrix}\right)$ and $|\Psi_2\rangle = b|01\rangle - a|10\rangle = \left(\begin{smallmatrix}b\\-a\end{smallmatrix}\right)$, with factor $1 + |\langle\Psi_1|\Psi_2\rangle| = 2a^2$ are:

$$\begin{aligned} A_1^{\Psi} &= \frac{1}{2a^2} |\Psi_2^{\perp}\rangle\langle\Psi_2^{\perp}| = \frac{1}{2a^2}\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} \\ A_2^{\Psi} &= \frac{1}{2a^2} |\Psi_1^{\perp}\rangle\langle\Psi_1^{\perp}| = \frac{1}{2a^2}\begin{pmatrix} a^2 & -ab \\ -ab & b^2 \end{pmatrix} \\ A_0^{\Psi} &= I - A_1^{\Psi} - A_2^{\Psi} = \begin{pmatrix} 0 & 0 \\ 0 & 1-\frac{b^2}{a^2} \end{pmatrix} \end{aligned} \tag{22}$$

where $|\Psi_1^{\perp}\rangle = \left(\begin{smallmatrix}a\\-b\end{smallmatrix}\right)$ and $|\Psi_2^{\perp}\rangle = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$. With the choice $K_i^{\Psi} = \sqrt{A_i^{\Psi}}$, the unitary transformation acts as:

$$\begin{aligned} U(|0\rangle_a |\Psi_1\rangle) &= |1\rangle_a K_1^{\Psi}|\Psi_1\rangle + |0\rangle_a K_0^{\Psi}|\Psi_1\rangle \\ &= |1\rangle_a \sqrt{2}b\,|\Psi_2^{\perp}\rangle + |0\rangle_a \sqrt{a^2-b^2}\,|10\rangle \\ U(|0\rangle_a |\Psi_2\rangle) &= |2\rangle_a K_2^{\Psi}|\Psi_2\rangle + |0\rangle_a K_0^{\Psi}|\Psi_2\rangle \\ &= |2\rangle_a \sqrt{2}b\,|\Psi_1^{\perp}\rangle - |0\rangle_a \sqrt{a^2-b^2}\,|10\rangle \end{aligned} \tag{23}$$

Yielding the same probability for success.

V. PROBABILITY OF SUCCESSFUL TELEPORTATION IN HIGHER DIMENSIONS

Note that the probability of successful teleportation for both schemes is given by $2b^2$. In actuality, this value represents the maximum achievable probability for successful conclusive teleportation with a partially entangled state $|\chi\rangle = a|00\rangle + b|11\rangle$. Generalizing this result to a d-dimensional Hilbert space, consider the partially entangled state with Schmidt decomposition:

$$|\chi\rangle = \sum_{i=0}^{d-1} a_i\,|ii\rangle \tag{24}$$

with real, non-negative coefficients a_i, normalized such that $\sum_i a_i^2 = 1$. The maximum probability of successful teleportation in this case is given by $p_{\max} = d\min_i\{a_i^2\}$, as demonstrated by W. Son *et al.* [5]. For the two-dimensional case, where $a \geq b$, this reduces to $d\min_i\{a_i^2\} = 2b^2$. Note that if any a_i is zero, then the probability drops to 0, indicating that teleportation always fails in the absence of entanglement across all basis states.

This upper limit for success can be understood by considering that perfect teleportation of an unknown quantum state requires a maximally entangled resource, such as the generalized Bell state $|\Phi^+\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |ii\rangle$. To achieve teleportation with certainty, Alice and Bob can apply a POVM to convert their partially entangled state $|\chi\rangle$ into the maximally entangled state $|\Phi^+\rangle$ with finite probability, and with the outcome revealing whether the conversion has failed. If successful, they can proceed as in the original protocol.

This process is known as entanglement concentration, in particular, for a single copy. As stated by Vidal's theorem [6], the maximal probability of success for such a transformation is $d \min_i\{a_i^2\}$, matching the teleportation upper bound. It can be realized via a filtering operation that rescales all coefficients a_i to the smallest one, similar to the one on equation (6). It acts as follows:

$$U(|0\rangle_a |\chi\rangle) = |0\rangle_a F |\chi\rangle + |1\rangle_a \sqrt{I - F^2} |\chi\rangle \qquad (25)$$

Where I is the $d \times d$ identity operator and F is a Kraus-like operator that acts on any of the two sides of the entangled pair (say, Alice's side) defined by:

$$F |i\rangle = \frac{\min_j\{a_j\}}{a_i} |i\rangle \qquad (26)$$

As such, $F|\chi\rangle = \sqrt{d}\min_i\{a_i\} |\Phi^+\rangle$, converting $|\chi\rangle$ into a maximally entangled state, up to normalization:

$$\begin{aligned} F|\chi\rangle &= \sum_{i=0}^{d-1} \frac{\min_j\{a_j\}}{a_i} a_i |ii\rangle = \min_j\{a_j\} \sum_{i=0}^{d-1} |ii\rangle \\ &= \min_j\{a_j\} \sum_{i=0}^{d-1} \frac{\sqrt{d}}{\sqrt{d}} |ii\rangle = \sqrt{d} \min_j\{a_j\} |\Phi^+\rangle \end{aligned} \qquad (27)$$

Since F is Hermitian and all its eigenvalues lie in the interval $[0, 1]$, the operator $I - F^2$ has non-negative eigenvalues. This ensures that $\{F^2,\ I - F^2\}$ form a valid POVM. It also ensures that $\sqrt{I - F^2}$ is well defined and that U is norm-preserving. The probability of success is given by $\| F |\chi\rangle \|^2 = d \min_i\{a_i^2\}$, since $|\Phi^+\rangle$ is normalized, establishing an upper limit for the probability of perfect teleportation with a partially entangled resource.

VI. CONCLUSIONS

In this paper, we have demonstrated that perfect teleportation cannot be achieved when the sender, Alice, and the receiver, Bob, share only a partially entangled pair. With such a resource, the best one can accomplish is probabilistic teleportation, providing full fidelity upon success in exchange for a finite probability of failure.

We have explored two methods for attaining conclusive teleportation with partial entanglement. The main difference between the two methods lies in which party, Alice or Bob, bears the additional complexity beyond the original protocol. Both approaches rely on Positive Operator-Valued Measures (POVM) or generalized measurements. We have provided a general overview of POVMs and their implementation with an ancillary system and a unitary transformation, followed by a projective measurement on the ancilla, following Neumark's theorem.

In the first method, Alice measures in the Bell basis as in the standard scheme, and Bob applies a filtering POVM to recover the original qubit from his altered state. In contrast, in the second method, Alice performs a non-orthogonal measurement on her two-qubit system and, if successful, sends the information to Bob, leaving him with a rotation of the original state along with the knowledge to rotate it back.

Finally, we have shown that both protocols yield the same maximal probability of success, as demonstrated by a generalization to the d-dimensional case. We have illustrated this upper bound using a strategy of entanglement concentration prior to any measurement on either side.

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Sobre la teleportació quàntica amb entrellaçament parcial.

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Resum: La teleportació quàntica no es pot aconseguir amb certesa quan s'utilitza un recurs parcialment entrellaçat; tanmateix, es pot realitzar una teleportació amb fidelitat completa de manera probabilística, permetent que el protocol falli en algunes ocasions. Presentem dues estratègies per a la teleportació conclusiva basades en mesures generalitzades: una consisteix en una operació de filtratge aplicada pel receptor, mentre que l'altra utilitza la discriminació d'estats no ortogonals realitzada per l'emissor. Ambdós mètodes són òptims, ja que tenen èxit amb la màxima probabilitat possible. A més, estudiem aquesta probabilitat màxima en el cas general de dimensió d.
Paraules clau: Teleportació quàntica, entrellaçament parcial, mesures generalitzades, POVM
ODSs: 4.4, 9.5, 9.8

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures	X		

El contingut d'aquest TFG, part d'un grau universitari de Física, es relaciona amb l'ODS 4, i en particular amb la fita 4.4, ja que contribueix a l'educació a nivell universitari. També es pot relacionar amb l'ODS 9, fita 9.5, perquè aporta a la investigació científica, i 9.8, perquè tracta sobre un protocol d'informació i comunicació.