Keynes' Principle of Effective Demand: A Statistical Mechanics Approach

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Abstract: This paper constructs a model of the macroeconomy using Ensemble Theory. This approach, although more complex, is far more rewarding since more rigorous results are obtained. In particular, this model successfully proves Keynes' Principle of Effective demand. This is, raising demand of goods in the economy boosts its Gross Domestic Product (GDP). Moreover, the model is calibrated with Spanish data from 2019 and 2020 to further enhance its robustness. A more broad aim of this paper is to expand the Economics toolkit by incorporating Statistical Mechanics concepts, in order to get Economics closer to the analysis of natural sciences.
Keywords: Statistical Physics, Computational simulation, Stochastic model.
SDGs: 4, 8, 9, 10

I. INTRODUCTION

The study of economics occupies a unique position among the sciences: it has been historically considered a social science, shaped by human behavior and institutional contexts rather than by immutable natural laws. As such, it has never settled on a single paradigm. The most mythical debate in the context of Macroeconomics was open as early as 1776 with Adam Smith, who exposed the pivotal role supply has over GDP [8]. He was the first Classical, whose position is well summarized by Say's law: *"Supply creates its own demand"* [7], ; this is, supply will create demand, and thus, is the driver of GDP. The antagonist position was led by Keynes and its Principle of Effective Demand [3], which states that actually, it is demand which checks supply, and thus, Gross Domestic Product (GDP). This lack of consensus has often been cited as evidence that economics cannot attain the same scientific rigor found in physics or chemistry.
Yet, beginning in the nineteenth century, scholars endeavored to bridge this gap through mathematical formalism and analogies to physical systems. William Stanley Jevons', pioneered this approach by treating equilibrium as a calculable state akin to mechanical balance and by applying differential equations to marginal utility [2]. This thesis aims to contribute to Jevons' plan, and set economic science closer to a natural science. Previous research [6] has proved that incorporating more complex mathematical modeling techniques can be used to find new relationships between economic variables. Continuing with the trend, this thesis will apply ensemble theory to give arguments that support Keynes' Principle of Effective Demand.
Statistical Mechanics is very useful discipline to apply to macroeconomic analysis because it allows to describe systems (be them physical or social) that are composed by a great amount of elements, such that it is impossible to analytically study all of its interactions. Consider, for example one mole of an ideal gas. Since the system has a number of particles of the order of $N \sim 10^{23}$, no theory can describe the dynamics of all particles. However, statistical mechanics provides a framework that describes macroscopic properties of such system. This is, the ideal gas macro-state can be described by $pV = nRT$, with p its pressure, V its volume, R a constant and T its temperature.
An economy may not be much different than this gas. It is composed by a great amount of agents (citizens, companies, public agencies, etc.) for which it is impossible to model all interactions. However, decades of research have shown that there exists aggregate variables that can be related and characterize an economy.

II. FOUNDATIONS OF ENSEMBLE THEORY

As shown on [5], the microcanonical ensemble provides a good theoretical background to hold ensemble theory. From it, we borrow its main postulates:

- **Postulate 1 (The Equiprobability Postulate):** All microstates compatible with the same macrostate in an isolated system are equally probable.
- **Postulate 2:** The number of microstates corresponding to two systems in thermodynamical equilibrium and isolated with the rest of the universe (denoted by Ω), is maximum with respect any of the variables of any of the two systems.

A. The canonical ensemble

We will use the canonical ensemble to set the grounds of the grand canonical ensemble, which will be the building block of the model. The natural variables of the canonical ensemble are (T, N, V), temperature, number of particles and volume, respectively. The central result of this ensemble is the canonical partition function, which acts as the generating function of the macro-state of the system:

$$Z(T,N,V) \equiv \sum_{E_r} \Omega(E_r) e^{-\beta E_r} = \sum_r e^{-\beta E_r}, \qquad (1)$$

with r denoting possible microstates of the system, E_r the possible energy values, and $\beta \equiv 1/k_B T$, being $k_B \approx 1.38 \cdot 10^{-23}$ J/K the Boltzmann constant, and T the temperature.

B. The grand canonical ensemble

In this ensemble, the macro state of the system is characterized by variables μ, V and T, chemical potential, volume and temperature, respectively. Its main result is the grand canonical partition function:

$$Q \equiv \sum_{N_s=0}^{\infty} \sum_{E_r} \Omega(E_r, N_s) e^{-\beta E_r - \alpha N_s} = \sum_{N_s=0}^{\infty} \sum_r e^{-\beta E_r - \alpha N_s}, \qquad (2)$$

with N_s and E_r possible values of energy and number of particles the system can have, and $\alpha \equiv -\mu/k_B T$. The grand canonical partition function can be written in terms of the canonical partition function:

$$\begin{aligned} Q(z,V,T) &= \sum_{N_s=0}^{\infty} \sum_{E_r} \Omega(E_r, N_s)\, e^{-\beta E_r}\, z^{N_s} \\ &= \sum_{N_s=0}^{\infty} Z(N_s, V, T)\, z^{N_s}, \end{aligned} \qquad (3)$$

where $z \equiv e^{-\alpha} = e^{\beta\mu}$ is the fugacity of the system.

III. THE MACROECONOMIC MODEL

To model the macroeconomy, we follow the usual strategy of defining aggregate supply and aggregate demand function, and study the equilibrium case, where markets clear (supply equals demand). This is justified because when supply and demand are not balanced, there are incentives for economic agents to return to the market equilibrium, and thus this will be what we will most likely observe. This principle is analogous to Le Chatelier's principle [4], physical systems tend to be in equilibrium by contrasting external shocks.

A. Aggregate supply

Assume the economy is composed by $K \in \mathbb{N}$ sectors and, for simplicity, consider the only input sectors use to produce is labor. Each sector differentiates itself because it has different productivities $c_1 < c_2 < \cdots < c_K$, defined as the infinitesimal increase in production (y_k) of each sector $1 \leq k \leq K$ given by an infinitesimal increase in labor employed (n_k) $c_k \equiv dy_k/dn_k$, which are constant. The number of employed workers in the economy is constrained by $N \equiv \sum_{k=1}^{K} n_k$. The total number of workers, L is exogenously given. Therefore, unemployment in this model is $U = L - N$. Finally, aggregate supply, this is, the total production of this economy is given by:

$$Y = \sum_{k=1}^{K} y_k = \sum_{k=1}^{K} c_k n_k. \qquad (4)$$

B. Aggregate demand (AD)

Aggregate demand is the sum of the amount of goods and services economic agents would like to consume. In demand models of the macroeconomy, it is divided into demand of goods for consumption of households, for investment by each sector, for the government and for exporting (net of imports). However, since the goal of this thesis is to analyze the effect of aggregate demand on GDP, it is not necessary to provide a complex structure to aggregate demand. Therefore, in the model, we take aggregate demand as exogenously given, $D \in \mathbb{R}$.

C. Equilibrium in the model

In reality, GDP is constrained by aggregate demand, because sectors seek for producing only the quantity to be sold. Demand is subject to many fluctuations since it is the aggregation of a great number of decisions. Consequently, GDP also fluctuates stochastically. In this model we assume that, in the short term, D is constant on average, exogenous, and stochasticity is concentrated on the supply function. The equilibrium condition is:

$$\langle Y \rangle = D, \qquad (5)$$

with this condition, and using postulates 1 and 2, it can be proved that sectorial GDP in this economy is exponentially distributed.

At this point, we have a system of N particles and K energy levels with energy c_k per particle and arbitrary degeneration. Moreover, n_k is the occupation number of each level, and average energy D is fixed. This system follows the Maxwell-Boltzmann statistics. Nonetheless, it is subject to constraints, $N = \sum_{k=1}^{K} n_k$ and $D = \sum_{k=1}^{K} c_k n_k$ [10].

Now, to get the output probability distribution, we need to obtain the labor productivity distribution, since labor is the only input in production for this model. This is a combinatorial problem equivalent to distributing N indistinguishable balls into K boxes. The combinatorial number that represents allocation vector $n = (n_1, ..., n_k)$ is :

$$W_n = \frac{N!}{\prod_{k=1}^{K} n_k!}. \qquad (6)$$

Since the number of all possible ways to allocate N different balls into K different boxes is K^N, the probability of obtaining a particular allocation vector $n = (n_1, ..., n_k)$ is:

$$P(n) = \frac{W_n}{K^N} = \frac{1}{K^N}\frac{N!}{\prod_{k=1}^K n_k!}. \tag{7}$$

To find the labor allocation in equilibrium, Postulate 2 states that the equilibrium labor allocation will be the one that maximizes $P(n)$ under macroconstraints of the economy, that is:

$$\begin{aligned} \max_{\{n_i\}_{i=1,\dots,K}} \quad & \tfrac{1}{K^N}\tfrac{N!}{\prod_{k=1}^K n_k!} \\ \text{s.t.} \quad & D = \textstyle\sum_{k=1}^K d_k n_k \\ & N = \textstyle\sum_{k=1}^K n_k. \end{aligned} \tag{8}$$

To make calculations easier, we can maximize $\log P(n)$ instead of $P(n)$. Using the Stirling approximation formula (valid for large arguments):

$$\begin{aligned} \log P(n) &= \log N! - N\log K - \textstyle\sum_{k=1}^K \log n_k! \\ &\approx N\log N - N - N\log K + N - \textstyle\sum_{k=1}^K n_k \log n_k \\ &= \textstyle\sum_{k=1}^K n_k(\log N - \log n_k - \log K) \\ &= \textstyle\sum_{k=1}^K n_k \log p_k - \sum_{k=1}^K n_k \log K. \end{aligned} \tag{9}$$

Ignoring constants, as it is a maximization problem, (11) is equivalent to:

$$\begin{aligned} \max_{\{n_i\}_{i=1,\dots,K}} \quad & S = -\textstyle\sum_{k=1}^K p_k \log p_k \\ \text{s.t.} \quad & D = \textstyle\sum_{k=1}^K d_k n_k \\ & N = \textstyle\sum_{k=1}^K n_k, \end{aligned} \tag{10}$$

with $p_k \equiv n_k/N$. We can now solve (12), setting up the Lagrangian:

$$L = -\sum_{k=1}^K \frac{n_k}{N}\log\frac{n_k}{N} + \alpha\left[N - \sum_{k=1}^K n_k\right] + \beta\left[D - \sum_{k=1}^K d_k n_k\right]. \tag{11}$$

The First Order Conditions are, for $k = 1, .., K$

$$[n_k] : \log\frac{n_k}{N} = -1 - \alpha N - \beta N d_k. \tag{12}$$

This are equivalent to

$$\frac{n_k}{N} = \exp[-1 - \alpha N - \beta N d_k]. \tag{13}$$

Since n_k/N sums up to one, we obtain:

$$\frac{n_k}{N} = \frac{e^{-\beta N d_k}}{\sum_{k=1}^K e^{-\beta N d_k}}. \tag{14}$$

This is the Boltzmann distribution. Now, since n_k can be interpreted as the number of cases where sectorial output takes value Y_k, the following result can be reinterpreted as the probability of Y_k:

$$g(Y) = \frac{e^{-\beta Y_i}}{\sum_i e^{-\beta Y_i}}. \tag{15}$$

Because of (17), this economy can be studied with the canonical ensemble, since sectorial output, as energy in a physical system, is exponentially distributed. Thus, the canonical partition function for the economy is:

$$Z = \sum_i e^{-\beta Y_i}. \tag{16}$$

However, the summation over the allocation of workers into productivity-organized workplaces is difficult to compute. It is easier to work our calculation using the grand canonical partition function.

$$\Phi = \sum_{N=0}^{\infty} z^N Z_N, \tag{17}$$

where $z^N = e^{\beta\mu}$ the fugacity of the system. In ensemble theory, μ is the chemical potential, and it measures the marginal contribution in terms of energy of an additional particle to the system.
It is also assumed that the number of workers at a firm with productivity c_k is constrained by f_k, the number of potential jobs sites with productivity c_j ($n_k \in \{0, 1, ..., f_k\}$). f_k can be described by a Markov model, and Appendix A shows that, in equilibrium, it follows a power law distribution $f(c_k) \sim c_k^{-\alpha}$ with $\alpha > 1$.
Using the canonical partition function (18), the definition of fugacity and constraint $N = \sum_{k=1}^K n_k$, a functional expression of the grand canonical partition function arises:

$$\Phi = \prod_{j=1}^K [1 + e^{\beta(\mu - c_j)} + ... + e^{f_j \beta(\mu - c_j)}]. \tag{18}$$

(20) acts as a generator of macro-state properties. In this case, its derivative with respect to μ provides the expected value of employed agents $\langle N\rangle$:

$$\begin{aligned} \frac{1}{\beta}\left(\frac{\partial}{\partial\mu}\log\Phi\right) &= \frac{1}{\beta}\left(\frac{\partial}{\partial\mu}\log\left(\sum_{N=0}^{\infty} e^{\beta\mu N} Z_N\right)\right) \\ &= \frac{1}{\beta}\left(\frac{\beta\sum_{N=0}^{\infty} N e^{\beta\mu N} Z_N}{\sum_{N=0}^{\infty} e^{\beta\mu N} Z_N}\right) \\ &= \sum_{N=0}^{\infty} N\, p(N) = \langle N\rangle. \end{aligned} \tag{19}$$

Therefore, using (20):

$$\begin{aligned} \langle N\rangle &= \frac{1}{\beta}\left[\frac{\partial}{\partial\mu}\log\Phi\right] \\ &= \frac{1}{\beta}\sum_{j=1}^K \frac{\partial}{\partial\mu}\log\left(1 + e^{\beta(\mu - c_j)} + \cdots + e^{f_j\beta(\mu - c_j)}\right) \\ &= \sum_{j=1}^K \left[\frac{e^{-(f_j-1)\beta(\mu-c_j)} + 2e^{-(f_j-2)\beta(\mu-c_j)} + \cdots + f_j}{e^{-f_j\beta(\mu-c_j)} + e^{-(f_j-1)\beta(\mu-c_j)} + \cdots + 1}\right]. \end{aligned} \tag{20}$$

This equation also provides the expected value of the number of workers employed on each sector:

$$\langle n_j \rangle = \left[\frac{e^{-(f_j-1)\beta(\mu-c_j)} + 2e^{-(f_j-2)\beta(\mu-c_j)} + \cdots + f_j}{e^{-f_j\beta(\mu-c_j)} + e^{-(f_j-1)\beta(\mu-c_j)} + \cdots + 1} \right]. \tag{21}$$

Equation (23) determines the distribution of workers across job sites with different levels of productivity. This distribution depends on the aggregate demand level via $\beta < 0$. From (13), $\beta = \frac{\partial S}{\partial D}$. Because of (16), we can associate lower values of beta to higher values of aggregate demand.

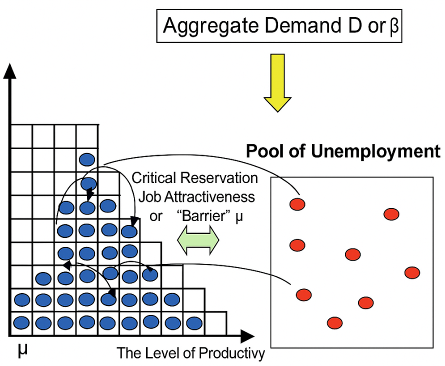

FIG. 1: Stochasticity in the model, extracted from [9]

Figure 1 shows the model mechanics. This economy features employed and unemployed agents. There is a pool of unemployed, regulated by μ, which plays a similar role to the reservation wage in standard models. When μ is high, the unemployed worker is "choosy", and vice versa. When aggregate demand is high ($\beta = -\infty$), $\langle n_k \rangle = 0$ for $c_k < \mu$. In contrast, when aggregate demand is low ($\beta = 0$), $\langle n_k \rangle = f_k/2$ for all values of c_k. This is, when demand allows, no workers take jobs whose productivity is lower than μ. The number of employed workers evolves stochastically depending on aggregate demand and μ, with expected value (22). Workers stochastically move back and forth from the unemployment pool and between different productivity levels, with expected value of the employed workers in each sector given by (23).

IV. KEYNES' PRINCIPLE OF EFFECTIVE DEMAND

Consider a simple calibration of the model, assuming that the level of productivity is $c_1 = 1, ..., c_{200} = 200$. μ is set to 25 and the labor force is assumed to be $L = 630$. The number of potential jobs is $f_j = 10$ for $c_1, ..., c_{50}$, while it declines for the other productivity levels according to $f_j \sim 1/c_j^2$. This economy can be simulated using Matlab with different levels of aggregate demand (see figure 2):

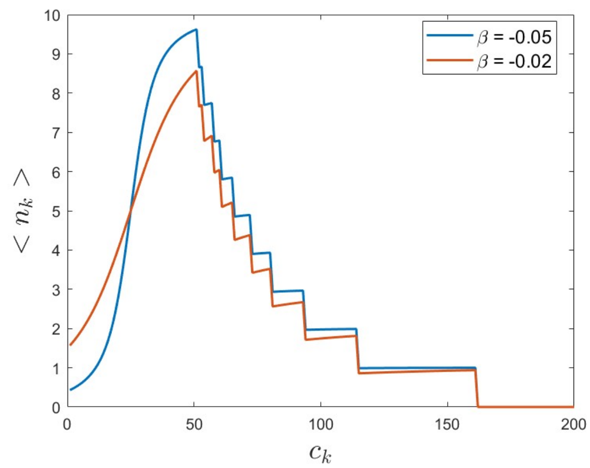

FIG. 2: Average number of employees (thousands) per sector for a simulation with $c_1 = 1, ..., c_{200} = 200$, $f_j = 10$ for $c_1, ..., c_{50}$, and $f_j \sim 1/c_j^2$ for the rest, $\mu = 25$ and $L = 630$.

The red line corresponds to a case with low aggregate demand ($\beta = -0.02$), while the blue line is associated to high aggregate demand ($\beta = -0.05$). In both cases, n_k increases up to $j = 50$ and then declines from $j = 51 - 200$. In the model, workers strive to obtain better jobs offered by firms with higher productivity. This is why the number of workers n_k increases as the level of productivity rises in the relatively low productivity region. Remember that, in this region, f_j is constant (virtually, no ceiling). The number of workers n_k turns out to be a decreasing function of productivity c_k in the high productivity region because the number of potentially available jobs f_j declines as c_j rises.
When aggregate demand D increases, the distribution shifts such that more workers are employed at high productivity jobs. Moreover, the number of employed workers N, which corresponds to the area below the curve, increases. Specifically, $N_{\beta=-0.05} = 618$ and $N_{\beta=-0.02} = 582$. Thus, when aggregate demand raises, employment raises accordingly, and workers occupy more productive jobs. This implies that GDP raises, exactly as Keynes's Principle of Effective Demand states.

V. EMPIRICAL EVIDENCE

We ought to test if the model can successfully replicate empirical data. For the case study, Spanish data for productivity and employment was taken from [1] for 2019 and 2020. These two years were selected strategically, since 2020 showed a very low aggregate demand relative to 2019 because of the COVID-19 crisis. This will allow us to test the model in a wide range of aggregate demand values for more robustness. We plot the data and calibrate the model with reasonable parameters. Take $L = 37.000$ thousands of workers and 35 sectors with c_k according to the labor productivity brought by the data (appendix B). Figures 3 and 4 show the fit of the model:

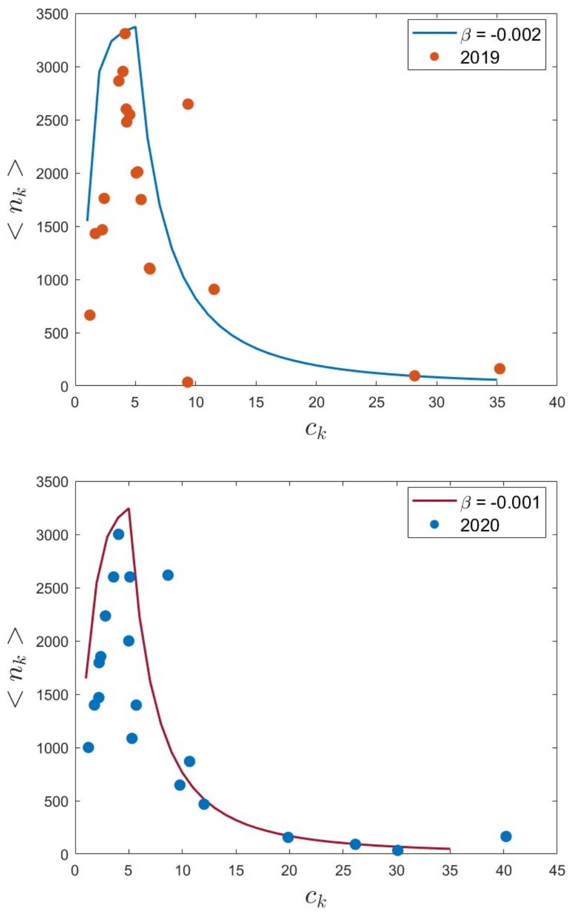

FIG. 3: Average number of employees (thousands) per sector for 2019 and 2020. The scatterplot is real data for Spain, while the lines are the calibration of the model.

Calculating the Mean Absolute Percentage Errors, it is found that $\text{MAPE}_{2019} = 4.6\%$ and $\text{MAPE}_{2020} = 3.2\%$. Therefore, on average, the model's sectoral employment predictions miss the observed Spanish values by less than 5%, which is an acceptable prediction given the stochastic nature of statistical mechanics. We find that the model fits well the data for 2019 with $\beta_{2019} = -0.002$, and $\beta_{2020} = -0.001$ for 2020. This is what it was expected, lower aggregate demand because of the crisis.

VI. CONCLUSIONS

As it was also shown in [6], this thesis also stresses the necessity of the economics toolkit to expand beyond calculus. In this case, the adoption of Ensemble Theory has provided new and rich conclusions not foreseen by classic economic analysis.

- **At a theoretical level**, this new model has succesfully provided a framework that proves Keynes' Principle of Effective Demand. Increasing demand pushes employment and provides a more productive labor distribution, which boosts GDP.

- **At an empirical level**, The model is able to replicate spanish data for different levels of aggregate demand, so we are confident that the model adapts to the current economic reality for Spain.

Acknowledgments

I would like to express my deepest gratitude to Dr. Montero for his unwavering support and patience.
To my family, I am very thankful for their support throughout this tough years at the university.
Finally, to my fiancée, who encouraged me to fight for my dreams and supported me while dealing with all the difficulties university has brought. I hope you can forever enlighten me like you have been doing these years.

[1] Instituto Nacional de Estadística. (2025). *National Statistics Institute of Spain.* Retrieved June 12, 2025, from https://www.ine.es
[2] Jevons, W. S. (1871). *The Theory of Political Economy.* London: Macmillan.
[3] Keynes, J. M. (1936). *The General Theory of Employment, Interest and Money.* London: Macmillan.
[4] Le Châtelier, H. (1884). Sur un énoncé général des lois des équilibres chimiques. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 99, 786–789.
[5] Ortín, J., & Sancho, J. M. (2021). *Curso de Física Estadística.* Barcelona: UB editions.
[6] Rabanal Cajal, P. (2025). *Uncertainty and Policy Ineffectiveness: A Macroeconomic Model with Markov Chains.* Final Degree Project, Degree in Mathematics. UB.
[7] Say, J.-B. (1803). *Traité d'économie politique, ou simple exposition de la manière dont se forment, se distribuent et se consomment les richesses.* Paris: Deterville.
[8] Smith, A. (1776). *An Inquiry into the Nature and Causes of the Wealth of Nations.* London: W. Strahan.
[9] Yoshikawa, H. (2015). *Stochastic macro-equilibrium: A micro-foundation for the Keynesian economics. Journal of Economic Interaction and Coordination*, 10(1), 31–56.
[10] Yoshikawa, H. (2022). *Reconstruction of Macroeconomics: Methods of Statistical Physics, and Keynes' Principle of Effective Demand.*

El principi de la demanda efectiva de Keynes: una perspectiva des de la mecànica estadística

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Resum: Aquest article construeix un model macroeconòmic partint de la teoria de col·lectivitats. Aquest enfocament, tot i ser més complex, resulta molt més enriquidor, ja que permet obtenir resultats més rigorosos. En particular, l'anàlisi proporciona un model que demostra el principi de demanda efectiva de Keynes, que exposa que un augment de la demanda de béns i serveis a l'economia impulsa el seu Producte Interior Brut (PIB). A continuació, el model es calibra amb dades espanyoles dels anys 2019 i 2020 per reforçar-ne la robustesa. Un objectiu més ampli d'aquest treball és ampliar les tècniques d'anàlisi de l'economia mitjançant la incorporació de conceptes de la mecànica estadística, amb l'objectiu d'apropar l'economia a l'anàlisi pròpi de les ciències naturals.
Paraules clau: Física estadística, simulació computacional, model estocàstic.
ODSs: Aquest TFG està relacionat amb els Objectius de Desenvolupament Sostenible (SDGs)

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	X
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic	X	17. Aliança pels objectius	
9. Indústria, innovació, infraestructures	X		

Appendix A: Dynamics of Potential Job Creation/Destruction

In this model, the dynamics of the potential jobs follow a discrete state jump Markov process [6].
Over a short time interval dt, the productivity level c of a given job site may increase by one unit with probability $w_+(c)\,dt$, or decrease by one unit with probability $w_-(c)\,dt$. Without loss of generality, we assume the step size is one. The functions $w_+(c)$ and $w_-(c)$ represent the transition rates for the processes $c \to c+1$ and $c \to c-1$, respectively, and both depend on the current level of c.

These changes in productivity can be due to a wide variety of factors. They may reflect technical progress or innovation. In other cases, productivity varies due to changes in demand conditions faced by firms. For instance, when demand for a firm's products drops, productivity may decline as a result of labor hoarding (see Fay & Medoff, 1985). Such dynamics are modeled through the transition rates $w_+(c)$ and $w_-(c)$ within a Markovian framework.

Additionally, we assume new job sites are created with productivity level $c = 1$ at a rate $p\,dt$. Conversely, a job site with productivity $c = 1$ may disappear if c drops to zero, with an exit probability given by $w_-(1)\,dt$.

Under this framework, the evolution of the average number of job sites with productivity c at time t, denoted by $f(c,t)$, follows the master equation:

$$\frac{\partial f(c,t)}{\partial t} = w_+(c-1)\,f(c-1,t) + w_-(c+1)\,f(c+1,t) - w_+(c)\,f(c,t) - w_-(c)\,f(c,t) + p\,\delta_{c,1} \tag{A1}$$

We now consider the stationary regime of Eq. (A1), assuming that $\partial f(c,t)/\partial t = 0$. Under this condition, the steady-state distribution $f(c)$ can be obtained by applying the boundary condition $w_-(1)\,f(1) = p$, which leads to:

$$f(c) = f(1)\prod_{k=1}^{c-1}\frac{w_+(c-k)}{w_-(c-k+1)}. \tag{A2}$$

To proceed, we impose a simplifying assumption on the transition rates $w_+(c)$ and $w_-(c)$: we suppose that the likelihood of upward or downward productivity changes depends on the current productivity level. In particular, we assume that the higher the productivity c, the more likely it is to experience a change. This leads us to define:

$$w_+(c) = a_+\,c^{\alpha}, \quad w_-(c) = a_-\,c^{\alpha},$$

where a_+, a_- are positive constants, and $\alpha > 1$. Under these assumptions, Eq. (A2) simplifies to:

$$f(c) = \frac{f(1)}{1 - f(1)/C(\alpha)}\cdot\frac{(1 - f(1)/C(\alpha))^c}{c^{\alpha}} \simeq c^{-\alpha}e^{-c/c^*}, \tag{A3}$$

where $C(\alpha)$ is a normalizing constant and c^* is a characteristic scale for the exponential cutoff. To ensure consistency with Eq. (A3), we make use of the relation $a_+/a_- = 1 - f(1)/C(\alpha)$. The approximation in Eq. (A3) is justified under the condition $n(1)/C(\alpha) \ll 1$, which ensures that the exponential cutoff becomes relevant only as c approaches a threshold value c^*.

Nevertheless, since c^* tends to be sufficiently large in practice, the exponential decay becomes negligible over a broad range of c, allowing the distribution $n(c)$ to exhibit a power-law behavior $f(c) \propto c^{-\alpha}$ across a wide interval of productivity levels. Therefore, under this plausible assumption, the stationary distribution of job sites, f_j, follows a power-law.

Appendix B: Data files

Data from figures is extracted from [1], and it is used to test the model in figure 3.

2019		
ACTIVITY	LABOR PRODUCTIVITY	EMPLOYED (Thousands)
Agriculture, livestock, hunting and related services	11,52	906,90
Extractive industries	9,33	34,80
Manufacturing industry	9,36	2646,60
Electricity, gas, steam and air conditioning supply	28,16	94,70
Water supply, sewerage, waste management and remediation activities	5,46	1750,30
Construction	6,16	1415,30
Wholesale and retail trade; repair of motor vehicles and motorcycles	4,14	3308,50
Transportation and storage	6,21	1097,60
Accommodation and food service activities	4,24	2600,00
Information and communication	5,06	2000,00
Financial and insurance activities	5,20	2010,00
Real estate activities	35,23	161,10
Professional, scientific and technical activities	3,96	2953,50
Administrative and support service activities	3,62	2864,00
Public administration and defence; compulsory social security	2,25	1466,30
Education	1,66	1430,80
Human health and social work activities	2,41	1761,00
Arts, entertainment and recreation	4,52	2254,90
Other services	4,27	2478,50
Activities of households as employers of domestic personnel; activities of households producing goods and services for own use	1,21	664,40

2020		
ACTIVITY	LABOR PRODUCTIVITY	EMPLOYED (Thousands)
Agriculture, livestock, hunting and related services	10,68	869,00
Extractive industries	8,11	34,30
Manufacturing industry	8,66	2616,80
Electricity, gas, steam and air conditioning supply	26,16	91,30
Water supply, sewerage, waste management and remediation activities	5,88	156,10
Construction	5,70	1397,50
Wholesale and retail trade; repair of motor vehicles and motorcycles	4,05	3000,10
Transportation and storage	5,30	1085,30
Accommodation and food service activities	2,23	1796,30
Information and communication	4,77	646,90
Financial and insurance activities	5,53	467,70
Real estate activities	40,22	164,70
Professional, scientific and technical activities	3,59	2600,00
Administrative and support service activities	2,82	2234,00
Public administration and defence; compulsory social security	2,20	1468,50
Education	1,79	1398,60
Human health and social work activities	2,38	1852,80
Arts, entertainment and recreation	5,10	2600,00
Other services	5,00	2000,00
Activities of households as employers of domestic personnel; activities of households producing goods and services for own use	1,23	1000,00

FIG. 4: Spanish data for sectorial labor productivtiy and employed workers for 2019 and 2020, taken from [1].

Appendix C: Matlab codes

This section contains two codes made with Matlab. Figure 5a) contains the code that generated the simulation in figure 2, and figure 5b), the simulations and plots of data shown in figure 3.

```matlab
% TFG Simulation: Number of employed workers with respect to labor productivity.


% 1) Parameters & data
mu   = 25;                           % chemical potential
c_k  = 1:200;                        % productivity vector
K    = numel(c_k);                   % number of sectors
L    = 630;                          % workforce size
betas = [-0.05, -0.02];              % two levels of aggregate demand

% 2) Upper bounds per sector
f = zeros(1, K);
f(1:50)   = 10;
f(51:end) = floor(26010 ./ (c_k(51:end).^2));

% 3) Single figure
figure; hold on;

% 4) Calculations and plots
for i = 1:length(betas)
    beta = betas(i);
    mean_n = zeros(1, K);
    for j = 1:K
        fj = f(j);
        n = 0:fj;
        w = exp(beta * (mu - c_k(j)) * n);
        numer = sum(n .* w);
        denom = sum(w);
        mean_n(j) = numer / denom;
    end
    plot(c_k, mean_n, 'LineWidth', 1.5, 'DisplayName', ['\beta = ', num2str(beta)]);
end

% 5) Graph details
xlabel('$c_k$', 'Interpreter','latex', 'FontSize', 18);
ylabel('$ <n_k>$', 'Interpreter','latex', 'FontSize', 18);
legend('Location','best','FontSize',12);
grid off;
```

```matlab
% TFG Simulation: Number of employed workers with respect to labor
% productivity for Spain in 2019 and 2020.

% 1) Parameters & data
mu   = 1.1;                          % chemical potential
c_k  = 1:35;                         % productivity vector
K    = numel(c_k);                   % number of sectors
L    = 37000000;                     % workforce size
betas = [-0.002, -0.001];            % two levels of aggregate demand

% 2) Upper bounds per sector
f = zeros(1, K);
f(1:5)   = 3500;
f(6:end) = floor(87500 ./ (c_k(6:end).^2));

% 3) Single figure
figure; hold on;

% 4) Loop over beta values
for i = 1:length(betas)
    beta = betas(i);
    mean_n = zeros(1, K);
    for j = 1:K
        fj = f(j);
        n = 0:fj;
        w = exp(beta * (mu - c_k(j)) * n);
        numer = sum(n .* w);
        denom = sum(w);
        mean_n(j) = numer / denom;
    end
    plot(c_k, mean_n, 'LineWidth', 1.5, 'DisplayName', ['\beta = ', num2str(beta)]);
end

% 5) Labels and legend
xlabel('Productivity levels c_j');
ylabel('Mean sectoral employment \langle n_j \rangle');
legend show;
grid on;

% Productivity and employment (thousands) for 2019
prod_2019 = [11.524,  9.332, 28.167,  5.463,  9.363,  6.160,  4.137,  6.211, ...
              4.236,  5.063,  5.195, 35.228,  3.963,  3.622,  2.246,  1.665, ...
              2.406,  4.522,  4.271,  1.214];
emp_2019  = [906.9,   34.8,   94.7, 1750.3, 2646.6, 1105.3, 3308.5, 1097.6, ...
             2600.0, 2000.0, 2010.0, 161.1, 2953.5, 2864.0, 1466.3, 1430.8, ...
             1761.0, 2549.0, 2478.5,  664.4];

% Productivity and employment for 2020
prod_2020 = [10.677,  30.112, 26.156,  19.877,  8.663,  5.697,  4.051,  5.301, ...
              2.230,  9.772,  12.028, 40.223,  3.586,  2.821,  2.203,  1.793, ...
              2.382,  5.100,  5.000,  1.232];
emp_2020  = [869.0,   34.3,   91.3,  156.1, 2616.8, 1397.5, 3000.1, 1085.3, ...
             1796.0,  646.9,  467.7, 164.7, 2600.0, 2234.0, 1468.5, 1398.6, ...
             1852.8, 2600.0, 2000.0  1000.0];
% 2) Create a single figure and hold for multiple series
figure('Color','w'); hold on;

%3) Plot 2019 data as filled blue circles
 scatter(prod_2019, emp_2019, 80, 'filled', ...
     'MarkerFaceColor',[0.2 0.6 1], 'DisplayName','2019');

% 4) Plot 2020 data as orange squares
scatter(prod_2020, emp_2020, 80, 's', ...
   'MarkerEdgeColor',[1 0.4 0.2], 'LineWidth',1.5, 'DisplayName','2020');

% 5) Labels, title, legend, grid
%xlabel('c_k','FontSize',12);
xlabel('$c_k$', 'Interpreter','latex', 'FontSize', 18);
ylabel('$ <n_k>$', 'Interpreter','latex', 'FontSize', 18);

legend('Location','best','FontSize',12);
grid off;
```

FIG. 5: **FIG. 5a) (left) and 5b) (right):** Matlab code for the simulations in figures 2 and 3.