Study of bounded confidence models of opinion dynamics on networks

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Abstract: Using the Deffuant model of opinion dynamics, in which the agents in a network can only influence each other if the discrepancy in their opinions is less than the confidence bound parameter, we observe the formation of clusters during the dynamic. In this process there is a fraction of agents that stay isolated and don't change their opinion or they form really small clusters that barely change their opinion from the start and stay in the extreme opinions. This work is centered on studying this fraction of isolated agents, how the probability changes for different topologies, dynamics and stages.
Keywords: Opinion dynamics, bounded confidence, Erdős-Rényi networks, phase transition, percolation, isolated nodes
SDGs: Quality education

I. INTRODUCTION

Arriving at a consensus is key to design urban policies and for taking climate action by all agents (citizens, enterprises and governments). This study about the factors that influence the reaching of strong consensus gives insight in the processes of participation and communication that reinforce the acceptance of sustainable projects.

Recently there is an interest in the application of statistical physics into opinion dynamics in complex networks. The first models on opinion dynamics used binary opinions (”in favor”=1 and ”against”=0). These simple models fail to show the complexity of human behavior; to get over this limitation it's considered a continuous opinion where each agent of the network can be at any point in the spectrum of opinions [1].

These models introduce the concept of bounded confidence; this makes it so that the agents in the network can only influence each other if the difference in their opinion is less than a threshold d. This is to show things like confirmation bias or homophily and simulates the resistance to dialogue with people with an opinion really different. The Deffuant model [1] proposes that each agent starts with an opinion $x_i \in [0,1]$ and for each interaction a couple of connected nodes is chosen. If they satisfy the confidence bound condition, they interact and influence each other to have similar opinions; this is repeated until the network arrives at a stationary configuration.

A particular effect we observe is the formation of ”wings”, groups of agents that stay isolated or low influenced by the majority. The topology of the network changes the result; in a well-mixed case, there is a fast and uniform convergence and isolation is minimal, but in a more realistic network, the results show a higher probability of isolation. This can be interpreted as a radicalization or lack of integration. This study is centered around the probability of isolation in Erdös-Rényi random graphs for different times: the initial condition, the early times of the dynamic and the final configuration. We focus on the dependency with the parameter that characterizes the network (the average connectivity k) and the dynamic.

II. MODEL

Let $G(N,p)$ be an Erdős–Rényi graph with N nodes and probability p of having an edge between two nodes, so that the average connections per node are $k = p(N-1)$, and the number of neighbors per node, i.e. its degree is $\deg(i) \sim Bin(N-1,p)$ [3].

Assign to each node i an opinion $x_i \in [0,1]$ following a uniform distribution and fix a threshold $d \in [0,1]$. We define a node as ”isolated” if **all** its neighbors j satisfy $|x_i - x_j| \geq d$

For each iteration, a random edge is chosen; the two nodes i and j, if their opinions satisfy $|x_i - x_j| < d$ the interaction will be:

$$\begin{aligned} x_i(n+1) &= x_i(n) + \mu(x_j(n) - x_i(n)) \\ x_j(n+1) &= x_j(n) + \mu(x_i(n) - x_j(n)) \end{aligned} \tag{1}$$

The parameter μ is the speed at which the agents influence each other. For $\mu = 0$, there is no influence, and for $\mu = 0.5$, the agents end up with the same opinion after one interaction. After all the interactions the system arrives at a consensus where the opinions don't change anymore. This consensus in the simulation is defined as the moment when the change in opinions for a certain number of interactions ($10N$) is less than a fixed tolerance (10^{-6}). For the complete mixing case, the number of clusters formed at the end is the integer part of $1/2d$ [2]. All simulations in this work are done on an Erdős–Rényi graph with 900 nodes, $d = 0.3$, and $\mu \in [0.001, 0.5]$. Averages are performed over 1000 samples, except for section C, where the simulation is done with 10000 nodes over different values of d. In fig 1 we see how the majority falls into consensus while an isolated minority creates these ”wings” by barely changing their opinion.

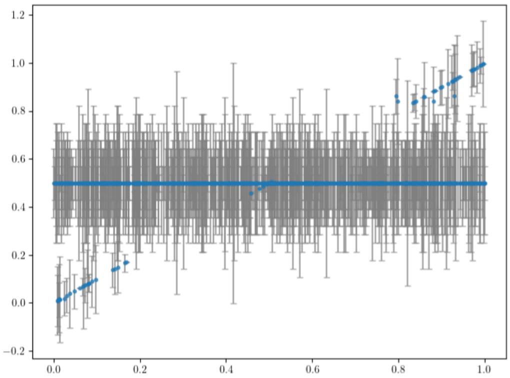

FIG. 1: Final opinion (vertical axis) vs initial opinion (horizontal axis) for average connectivity 4, $d = 0.3$ and 900 nodes. The bars show the number of connections of each node.

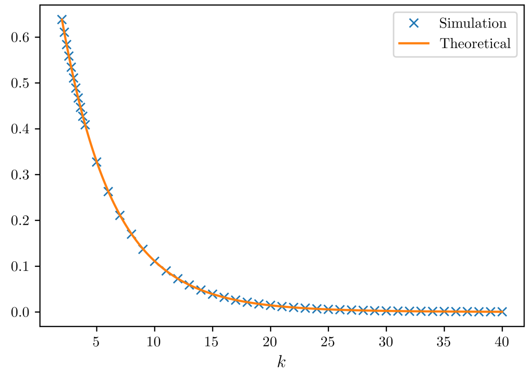

FIG. 2: Probability of isolation for the initial configuration as a function of the average connectivity for $d = 0.3$

III. PROBABILITY OF ISOLATION

A. Initial isolation

For a node with opinion x the probability of a neighbor being inside the confidence parameter d, if we assume a uniform distribution of opinions, is:

$$f(x) = \begin{cases} x+d, & 0 \le x < d \\ 2d, & d \le x \le 1-d \\ 1-x+d, & 1-d < x \le 1 \end{cases} \tag{2}$$

Conditioning on the value $x_i = x$ of the node's opinion and the number of neighbors m, we have

$$P(i \text{ isolated} \mid x, m) = [P(|x_j - x| \ge d)]^m = [1 - f(x)]^m \tag{3}$$

Since $\deg(i) \sim Bin(N-1, p)$, the probability of a node i being isolated is:

$$P_{iso} = \sum_{m=0}^{N-1} \binom{N-1}{m} p^m (1-p)^{N-1-m} \int_0^1 (1 - f(x))^m dx \tag{4}$$

Exchanging the sum and integral and using the binomial theorem, the following compact form is obtained

$$P_{iso} = \int_0^1 [1 - pf(x)]^{N-1} \tag{5}$$

When $N \to \infty$ with fixed k (recall $k = p(N-1)$)

$$[1 - pf(x)]^{N-1} \approx e^{-kf(x)} \tag{6}$$

and

$$P_{iso} = \int_0^1 e^{-kf(x)} dx = 2\frac{e^{-kd}(1 - e^{-kd})}{k} + (1 - 2d)e^{-2kd} \tag{7}$$

This expression summarizes as a function of k and d the probability that a node is isolated at the start of the dynamic (fig 2). In general if the initial distribution is $\rho(x)$ the probability of having k' neighbors connected ($|x_j - x| < d$) given a node with k neighbors is:

$$P(k'|k, x, \rho(x)) = \binom{k}{k'} (a(x))^{k'} (1 - a(x))^{k-k'} \tag{8}$$

Where $a(x)$ is the probability of a node with opinion x being connected to one of its neighbors.

$$a(x) = \int_{max(0,x-d)}^{min(1,x+d)} \rho(y) dy$$

For a uniform distribution:

$$P(k'|k, x, \rho(x)) = \binom{k}{k'} (f(x))^{k'} (1 - f(x))^{k-k'} \tag{9}$$

and for $k' = 0$ we recover equation (3). For a Gaussian distribution centered at 0.5 and normalized in [0,1]:

$$a(x) = \frac{1}{Z}\left[\Phi\left(\frac{min(1, x+d) - 0.5}{\sigma}\right) - \Phi\left(\frac{max(0, x-d) - 0.5}{\sigma}\right)\right]$$

Where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du$$

and

$$Z = \Phi\left(\frac{0.5}{\sigma}\right) - \Phi\left(\frac{-0.5}{\sigma}\right)$$

B. Phase transition

Erdős-Rényi networks show a second order percolation transition in $k = 1$. The order parameter is S, the fraction of nodes that belong to the biggest connected component in the graph. So $S = 0$ for $k < 1$ and $S \sim (k-1)^\beta$ for $k > 1$ with $\beta = 1$. The mean size of the rest of the components $<s> \sim |k-1|^{-\gamma}$, with $\gamma = 1$, is the susceptibility of this transition [3]. In our case for the initial distribution of opinions and the confidence bound we observe that the parameter d defines a temporal network [4] with an effective connectivity in the graph k_{eff} so the phase transition is at $k_{eff} = 1$.

$$k_{eff} = kP(|x_i - x_j| \le d)$$

$$P(|x_i - x_j| \leq d) = \int_0^1 \int_0^1 1_{|x_i - x_j| \leq d} dx_i dx_j = 2d - d^2$$

So the condition for the critical point is:

$$d = 1 - \sqrt{1 - \frac{1}{k}}$$

In fig 3 we observe the percolation transition where the theory predicts $k_{eff} = 1$.

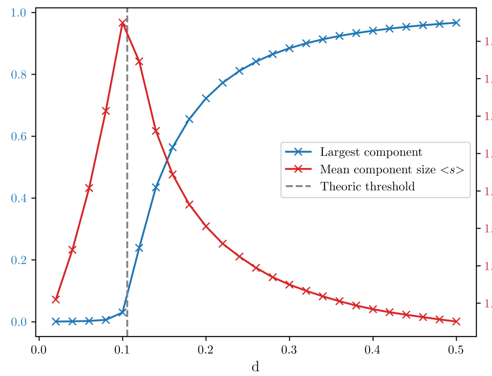

FIG. 3: Size of the largest connected component as a function of d and the mean component size excluding the largest. The vertical line indicates where the theory predicts the phase transition. Simulation with $k = 5$

C. Early times

We use simulations where we take the list of edges in the network and only make one interaction for each edge in a random order. Thus, each node will interact with all of its neighbors once. After all the interactions, we look at how many nodes were left isolated. We look at the isolation in two different intervals of opinions, the bulk ($x \in [d, 1-d]$) and the tails ($x \in [0, d] \cup [1-d, 1]$). Simulations are done for different values of μ and k.

Let's look first at the isolation in the tails. For the initial configuration we had the probability of isolation to be the isolation in the tails $2e^{-kd}(1 - e^{-kd})/k$ plus the isolation in the bulk $(1 - 2d)e^{-2kd}$. Looking at the isolation in the tails for a small μ, we expect to find a similar isolation to the initial one because for a small number of iterations and small μ it is less probable to have significant changes in opinions. And that is exactly what we find in fig 4. For big values of μ (~ 0.5) we observe that the theoretical curve doesn't fit as well but it still is a good approximation. For $k < 5$ the dependency on μ accounts for changes around 1% and for bigger k around 0.4% as we can see in fig 6.

Now let's look at the bulk region. Following the same reasoning as before we expect the isolation in the bulk to follow something similar to $(1-2d)e^{-kd}$. However, what we see is that the isolation has decreased very quickly with these few iterations compared to the initial isolation

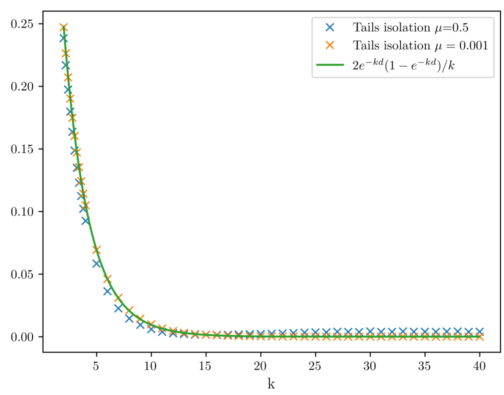

FIG. 4: Probability of isolation in the tails for early times dynamics

(fig 5). Again, the dependency on μ is a small contribution that goes to 0 very fast when k increases. As we

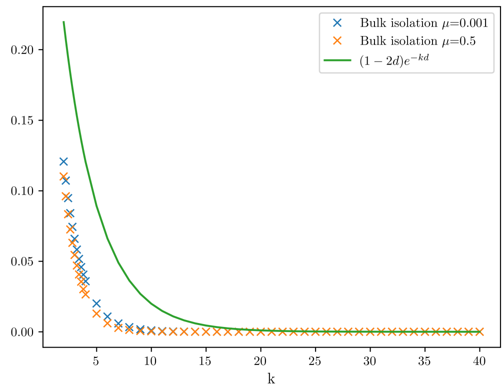

FIG. 5: Probability of isolation in the bulk for early times dynamics

can see the isolation is dependent on k and has a weak contribution depending on μ. In the tails for small k the probability decreases with μ , but when k increases the tendency changes, for small values of μ ($\lesssim 0.2$) the isolation is constant in μ and for larger values of μ the isolation increases (see fig 6).

D. Final configuration

When we let the system evolve with random interactions, we arrive at a point where there are no changes in opinions due to cluster formation with the same opinion; in our case, all the simulations finished in 1 big cluster with opinion 0.5 and a small portion of nodes left in the tails. There are two ways in which we looked at the isolation. The first is the same way we saw before: for each node, we check if all its neighbors are at a greater distance than d. The other way is to check all the nodes that didn't fall into the main cluster. This second way is

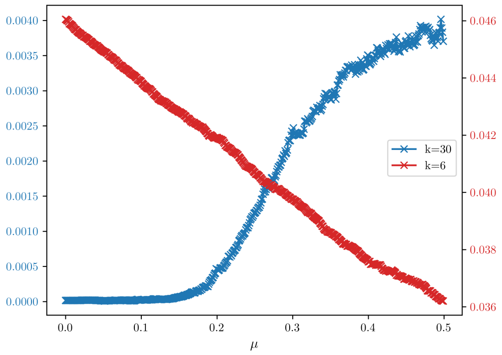

FIG. 6: Probability of isolation in the tails for early times dynamics as a function of μ

not isolation by definition, but may be useful to study the lack of integration and the formation of extreme opinions in society. The idea is that for example a node ends with 3 neighbors only, with the same opinion, and an opinion very close to their initial, then we consider these nodes to not have participated in the dynamic because they started and ended at the same point and only interacted within a small bubble.

First, we take a look at completely isolated nodes. We observe something similar to before, the dependency with k follows $P_{iso} \approx e^{-kd}/k$ (this is an ansatz, and works better for small μ) (fig 7). With μ, again, there is a clear dependency which is small compared to the changes in k ($\lesssim 1\%$) (fig 8). Lastly, we consider the second way to see

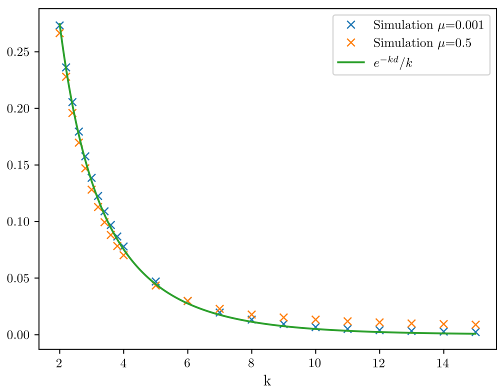

FIG. 7: Probability of isolation for the final configuration as a function of k

the fraction of nodes that don't fall in the central cluster. Once more the difference in magnitude of the effects between k and μ are similar, in this case μ has a bigger contribution ($\sim 2\%$) than before and it is notable when k increases because the probability is close to zero and a 2% change is relevant. The probability as a function of k follows a tendency that looks like the previous cases, but doesn't quite fit with any of the functions proposed, the only observation is that for $k \leq 4$ the probability behaves

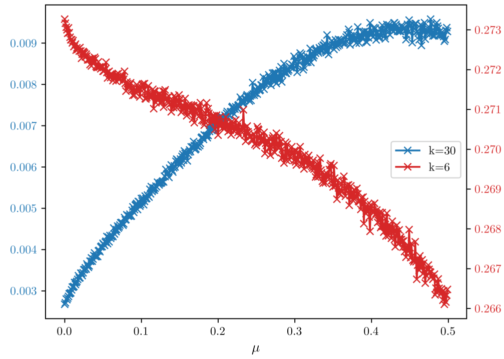

FIG. 8: Probability of isolation for the final configuration as a function of μ

like (fig 9)

$$P = \frac{A}{ln(k)} - B \ ; \ 2 \leq k \leq 4 \tag{10}$$

where A and B are constants that are different for every value of μ. If we plot $A(\mu)$ and $B(\mu)$ it looks linear with a lot of dispersion ($R^2 \approx 0.7$) (fig 10), this is because the probability in function of μ has a high variance (fig 12).

$$\begin{aligned} A(\mu) &\approx 0.50 - 0.05\mu \\ B(\mu) &\approx 0.26 - 0.04\mu \end{aligned} \tag{11}$$

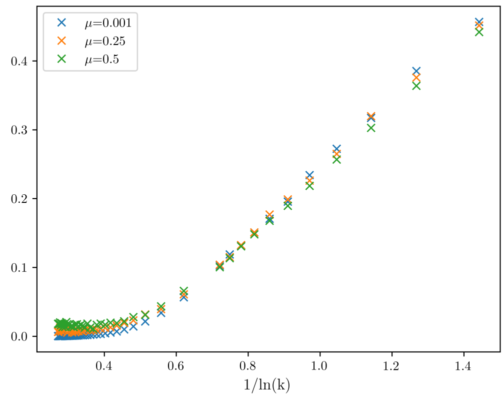

FIG. 9: Probability of isolation for the final configuration as a function of $1/ln(k)$

IV. CONCLUSIONS

The probability of isolation is governed mainly by k, while the effects of μ are a small contribution that is more noticeable when k increases and the probability drops. μ it's a parameter whose principal effect is just to change the time of convergence, it doesn't change the macroscopic result. The probability of isolation at the final

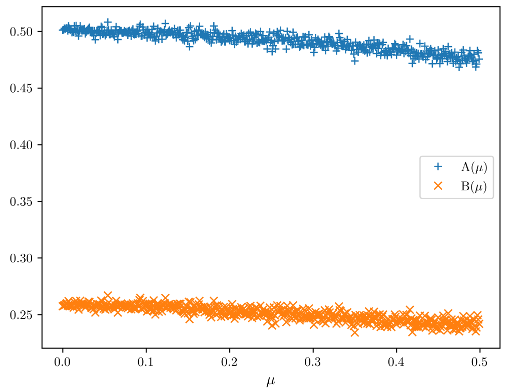

FIG. 10: Coeficients A and B as a function of μ

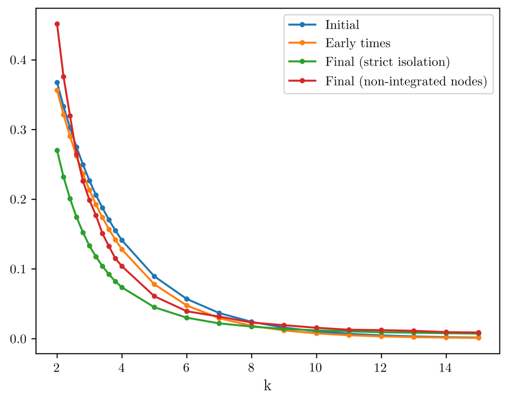

FIG. 11: Probability of isolation for different times and $\mu = 0.25$

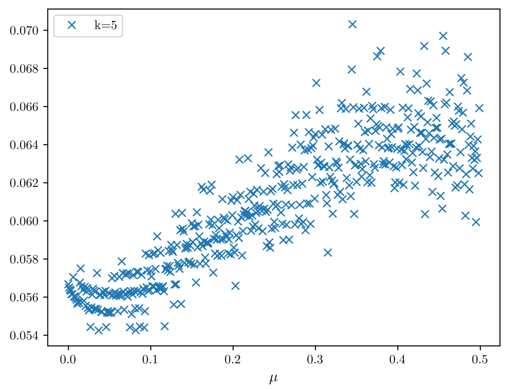

FIG. 12: Probability of isolation (non-integrated nodes)

consensus follows the empirical law $P = e^{-kd}/k$. The small effects of μ are to decrease the isolation for small k when we increase μ and around $k = 15$ the tendency flips and increases the probability with μ. This indicates that for well-connected networks, the sudden changes due to a big μ make a few nodes that don't have time to interact isolated. We also observe that most of the isolation happens in the tails, while in the bulk decreases very quickly, for high values of k go to zero.

For the final configuration, the nodes that don't fall into the consensus decrease quickly with k and for $2 \leq k \leq 4$ follow equation (10). With A, B depending linearly on μ (equation (11)), but with a lot of dispersion ($R^2 \approx 0.7$). This shows how after $k = 4$ the probability saturates close to 0. Due to the confidence bound condition there is a temporal network with an effective connectivity k_{eff} and the phase transition we see for $k = 1$ is at the initial configuration there is a threshold $k_{eff} = 1$ (with $k_{eff} = k(2d - d^2)$) that shows the same phase transition as an Erdös-Rényi graph.

Acknowledgments

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Estudi de models de confiança acotada de dinamica d'opinions en xarxes

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Resum: Utilitzant el model de Deffaunt de dinàmica d'opinió, en què els agents d'una xarxa només poden influir-se mútuament si la discrepància en les seves opinions és inferior al paràmetre del llindar de confiança, s'observa la formació de clústers durant la dinàmica. En aquest procés, hi ha una fracció d'agents que es manté aïllada i no canvia la seva opinió, o bé forma clústers molt petits que gairebé no modifiquen la seva opinió des del principi i es mantenen en posicions extremes. Aquest treball se centra a estudiar aquesta fracció d'agents aïllats i com varia la probabilitat segons diferents topologies, dinàmiques i etapes. **Paraules clau:** Dinàmica d'opinions, confiança acotada, xarxes Erdős-Rényi, transició de fase, percolació, nodes aïllats
ODSs: Educació de qualitat

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	X
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat		13. Acció climàtica	X
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			