Theoretical study of the mixing of the π^0-η-η'-axion system

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Abstract: The mass term of the Chiral Lagrangian at leading order mixes the strangeness-zero states. In order to perform calculations of physical quantities such as decay rates and scattering amplitudes involving the neutral π^0, η and η' mesons, the mass matrix of the Lagrangian needs to be diagonalized. In this work, we find the relation between the Lagrangian and physical states in terms of mixing angles as well as the expressions for the corresponding physical meson masses. We also incorporate mixing with the axion, thus preparing the Lagrangian for applications in axion phenomenology.
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I. INTRODUCTION

The π^0, η and η' are neutral pseudoscalar mesons that arise from the spontaneous breaking of chiral symmetry in Quantum Chromodynamics (QCD). Although ideally these states would be pure flavor eigenstates, isospin-breaking effects, due to the mass difference between the up and down quarks, introduce a small but significant mixing among them. Moreover, since the π^0, η, and η' all possess the same quantum numbers ($J^{PC} = 0^{-+}$), such mixing is not forbidden by symmetry principles and is naturally accommodated within the theory. As a result, the physical meson states are linear combinations of the original flavor states. Understanding this π^0-η-η' mixing is crucial since it has notable phenomenological consequences, particularly in processes such as radiative decays and electromagnetic transitions.

Beyond the meson sector, a key unresolved issue in the Standard Model (SM) is the so-called strong CP problem, which refers to the absence of CP violation in strong interactions despite the presence of a term in the QCD Lagrangian that would naturally induce it. To address this issue, Peccei and Quinn (PQ) proposed a dynamical mechanism involving the introduction of a new global U(1) symmetry, which is spontaneously broken. This mechanism predicts the existence of a hypothetical pseudoscalar particle known as the axion [1].

In addition to addressing the unnatural smallness of the QCD θ-angle, axions, and more generally, axion-like particles (ALPs), have emerged as compelling candidates for dark matter and mediators of new physics beyond the SM. Their weak coupling to SM particles and their light masses allow them to evade current experimental constraints while giving rise to rich phenomenology.

Motivated by these considerations, in this work, we present a theoretical analysis of the mixing within the π^0-η-η' system. We will find the relation between the original and physical states in terms of mixing angles and the expressions for the physical meson masses. In addition, we extend our analysis of the π^0–η–η' system to include mixing with the axion. This inclusion not only enriches the theoretical framework but also prepares the effective Lagrangian for applications in axion phenomenology.

This work is structured as follows. In Section II, we outline the theoretical framework, presenting the leading-order chiral perturbation theory (ChPT) Lagrangian and deriving the π^0–η–η' mixing. In Section III, we extend the formalism by incorporating the axion into the mesonic sector, thus making the framework suitable for axion phenomenology. Finally, in Section IV, we summarize our main results and present our conclusions.

II. THEORETICAL FRAMEWORK

A. Chiral Perturbation Theory and mass mixing matrix

Chiral Perturbation Theory (ChPT) is the effective low-energy theory that systematically describes the interactions of the lightest mesons originating from the spontaneous breaking of chiral symmetry in QCD [2]. In the present analysis, we adopt the ChPT Lagrangian at leading-order (LO), which reads:

$$\begin{aligned}\mathcal{L}^{\chi\text{PT, LO}} &= \frac{f_\pi^2}{4}\text{Tr}[\partial_\mu U^\dagger \partial^\mu U] \\ &+ \frac{f_\pi^2}{4}\text{Tr}[2B_0(\mathcal{M}U + \mathcal{M}U^\dagger)] - \frac{1}{2}m_0^2\eta_0^2\,,\end{aligned} \tag{1}$$

where $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ is the quark mass matrix, $U = \exp\left(\frac{i\sqrt{2}\Phi}{f_\pi}\right)$, Φ is the matrix containing the pseudoscalar meson fields (see Appendix A), f_π is the pion decay constant and B_0 is a low energy constant related to the quark condensate.

The quadratic mass term in Eq. (1) mixes strangeness-zero chiral mesons fields π_3, η_8 and η_0. We can express these quadratic Lagrangian terms as $\mathcal{L}^{\chi\text{PT, LO}} \supset -\frac{1}{2}\phi^T M^2 \phi$ with $\phi = (\pi_3, \eta_8, \eta_0)$, and where the mass matrix is given by:

$$M^2 = \begin{pmatrix} m^2_{\pi_3} & m^2_{\pi_3\eta_8} & m^2_{\pi_3\eta_0} \\ m^2_{\pi_3\eta_8} & m^2_{\eta_8} & m^2_{\eta_8\eta_0} \\ m^2_{\pi_3\eta_0} & m^2_{\eta_8\eta_0} & m^2_{\eta_0} \end{pmatrix}, \tag{2}$$

where the explicit expressions for the constituent elements of this matrix are detailed in Appendix A. In order to obtain the physical meson states π^0, η and η', the mass mixing matrix given by Eq. (2) must be diagonalized.

B. η-η' mixing

To elucidate the mixing structure in a simplified setting, we first consider the isospin limit, where the mass of quark up and is the same as the quark down. In this case, the off-diagonal mass terms $m^2_{\pi_3\eta_8}$ and $m^2_{\pi_3\eta_0}$ vanish, and the mass matrix in Eq. (2) simplifies such that only the η_8 and η_0 fields mix. The physical η and η' states are then obtained by performing a rotation of the η_8-η_0 subsystem via:

$$\begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix} = \begin{pmatrix} \cos\theta_{\eta\eta'} & \sin\theta_{\eta\eta'} \\ -\sin\theta_{\eta\eta'} & \cos\theta_{\eta\eta'} \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix}, \tag{3}$$

where $\theta_{\eta\eta'}$ is the mixing angle characterizing the rotation between the flavor and mass eigenstates. If we do the corresponding diagonalization of 2×2 submatrix we obtain the expressions for the physical meson masses,

$$m^2_{\eta,\eta'} = \frac{1}{2}\left[(m^2_{\eta_8} + m^2_{\eta_0}) \mp \sqrt{(m^2_{\eta_8} - m^2_{\eta_0})^2 + 4m^4_{\eta_8\eta_0}}\right], \tag{4}$$

and the mixing angle obeys the equation

$$\tan 2\theta_{\eta\eta'} = \frac{2m^2_{\eta_8\eta_0}}{m^2_{\eta_0} - m^2_{\eta_8}}. \tag{5}$$

C. π^0-η-η' mixing

Having established the physical masses of the η and η' mesons and the associated mixing angle $\theta_{\eta\eta'}$, we now extend the analysis to incorporate isospin symmetry breaking. This inclusion introduces mixing among the π^0, η, and η' states. The transformation between the symmetry eigenstates (π_3, η_8, η_0) and the physical mass eigenstates (π^0, η, η') is expressed through the following rotation [3]:

$$\begin{pmatrix} \pi_3 \\ \eta_8 \\ \eta_0 \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon_{\pi\eta} & -\epsilon_{\pi\eta'} \\ \epsilon_{\pi\eta} c\theta_{\eta\eta'} + \epsilon_{\pi\eta'} s\theta_{\eta\eta'} & c\theta_{\eta\eta'} & s\theta_{\eta\eta'} \\ \epsilon_{\pi\eta'} c\theta_{\eta\eta'} - \epsilon_{\pi\eta} s\theta_{\eta\eta'} & -s\theta_{\eta\eta'} & c\theta_{\eta\eta'} \end{pmatrix} \begin{pmatrix} \pi^0 \\ \eta \\ \eta' \end{pmatrix}, \tag{6}$$

with $s\theta_{\eta\eta'} = \sin\theta_{\eta\eta'}$ and $c\theta_{\eta\eta'} = \cos\theta_{\eta\eta'}$.

To systematically diagonalize the mass matrix, we proceed by treating the mixing in subsystems. The full rotation matrix can be factorized as: $R = R_1(\theta_{\eta\eta'})R_2(\epsilon_{\pi\eta})R_3(\epsilon_{\pi\eta'})$, where:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\theta_{\eta\eta'} & s\theta_{\eta\eta'} \\ 0 & -s\theta_{\eta\eta'} & c\theta_{\eta\eta'} \end{pmatrix} \begin{pmatrix} 1 & -\epsilon_{\pi\eta} & 0 \\ \epsilon_{\pi\eta} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\epsilon_{\pi\eta'} \\ 0 & 1 & 0 \\ \epsilon_{\pi\eta'} & 0 & 1 \end{pmatrix}. \tag{7}$$

Here, we assume that the mixing angles $\epsilon_{\pi\eta}$ and $\epsilon_{\pi\eta'}$ are small, allowing us to apply a Taylor expansion for the sine and cosine functions, retaining only leading-order terms.

The first subsystem η-η' has already been diagonalized in the previous section. Therefore, we now focus on diagonalizing the π^0-η and π^0-η' mixings. The resulting expressions for the small mixing angles are:

$$\epsilon_{\pi\eta} = \frac{m^2_{\pi_3\eta_0}\sin\theta_{\eta\eta'} - m^2_{\pi_3\eta_8}\cos\theta_{\eta\eta'}}{m^2_{\eta_1} - m^2_{\pi_3}}, \tag{8}$$

$$\epsilon_{\pi\eta'} = \frac{m^2_{\pi_3\eta_8}\sin\theta_{\eta\eta'} + m^2_{\pi_3\eta_0}\cos\theta_{\eta\eta'}}{m^2_{\pi_1} - m^2_{\eta'_1}}, \tag{9}$$

where $m^2_{\eta_1}$ and $m^2_{\eta'_1}$ refers to the mass eigenvalues of the diagonalized $\eta - \eta'$ subsystem, Eq. (4). Finally, the new expressions for the physical meson masses, accounting for the full mixing structure, are:

$$m^2_{\eta,\pi_1} = \frac{1}{2}\Big[m^2_{\eta_1} + m^2_{\pi_3} \pm \big[(m^2_{\eta_1} - m^2_{\pi_3})^2 + 4(m^2_{\pi_3\eta_8} c\theta_{\eta\eta'} - m^2_{\pi_3\eta_0} s\theta_{\eta\eta'})^2\big]^{\frac{1}{2}}\Big], \tag{10}$$

$$m^2_{\pi,\eta'} = \frac{1}{2}\Big[m^2_{\eta'_1} + m^2_{\pi_1} \mp \big[(m^2_{\eta'_1} - m^2_{\pi_1})^2 + 2(m^4_{\pi_3\eta_0} + m^4_{\pi_3\eta_8}) + 2(m^4_{\pi_3\eta_0} - m^4_{\pi_3\eta_8})c2\theta_{\eta\eta'} + 4m^2_{\pi_3\eta_0} m^2_{\pi_3\eta_8} s2\theta_{\eta\eta'}\big]^{\frac{1}{2}}\Big]. \tag{11}$$

In Eq. (10), the upper (plus) sign corresponds to the physical η mass, m^2_η, while the lower (minus) sign corresponds to $m^2_{\pi_1}$, which is an intermediate pion mass eigenvalue after the $R_2(\epsilon_{\pi\eta})$ rotation. In Eq. (11), the upper (minus) sign yields the final physical pion mass, m^2_π, and the lower (plus) sign gives the physical η' mass $m^2_{\eta'}$.

III. INCLUSION OF THE AXION IN THE NEUTRAL PSEUDOSCALAR MESON SECTOR

A. Axion-meson mixing

In the context of low-energy QCD, axions can couple to the gluon field-strength topological term via the anomaly, leading to interactions with mesons through mixing. As such, their inclusion in chiral perturbation theory (ChPT) naturally extends the framework to accommodate potential signatures of new physics in mesonic processes. In this section, we incorporate an axion into the mesonic sector in order to study its impact on neutral meson and to make the formalism suitable for axion phenomenology.

From now on, we will adopt the leading order ALP-$\mathcal{X}PT$ Lagrangian [4, 5]:

$$\mathcal{L}_{\rm ALP}^{\mathcal{X}\rm PT,\,LO} = \frac{1}{2}\partial_\mu a\partial^\mu a - \frac{1}{2}M_{\not{PQ}}^2 a^2 - \frac{1}{2}m_0^2\left(\eta_0 - \frac{Q_G}{\sqrt{6}}\frac{f_\pi}{f_a}a\right)^2 + \frac{f_\pi^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger\partial^\mu U] + \frac{f_\pi^2}{4}\mathrm{Tr}[2B_0(M_q(a)U + M_q(a)U^\dagger)]\,, \tag{12}$$

where $M_q(a)$ is the ALP-dependent quark mass matrix (Appendix A) and $M_{\not{PQ}}$ represents a bare PQ-breaking contribution to the ALP mass. In this framework, the mass-squared matrix that governs the mixing among the neutral pseudoscalar mesons and the axion is given by:

$$M^2 = \begin{pmatrix} m_{\pi_3}^2 & m_{\pi_3\eta_8}^2 & m_{\pi_3\eta_0}^2 & m_{a\pi_3}^2 \\ m_{\pi_3\eta_8}^2 & m_{\eta_8}^2 & m_{\eta_8\eta_0}^2 & m_{a\eta_8}^2 \\ m_{\pi_3\eta_0}^2 & m_{\eta_8\eta_0}^2 & m_{\eta_0}^2 & m_{a\eta_0}^2 \\ m_{a\pi_3}^2 & m_{a\eta_8}^2 & m_{a\eta_0}^2 & m_{aa}^2 \end{pmatrix}. \tag{13}$$

The flavor eigenstates π_3, η_8, η_0, and a can be related to the physical mass eigenstates π^0, η, η', and $a^{\rm phys}$ through a unitary rotation:

$$\begin{pmatrix}\pi_3\\ \eta_8\\ \eta_0\\ a\end{pmatrix} = \begin{pmatrix} \theta_{\pi_3\pi_3} & \theta_{\pi_3\eta_8} & \theta_{\pi_3\eta_0} & \theta_{\pi_3 a} \\ \theta_{\eta_8\pi_3} & \theta_{\eta_8\eta_8} & \theta_{\eta_8\eta_0} & \theta_{\eta_8 a} \\ \theta_{\eta_0\pi_3} & \theta_{\eta_0\eta_8} & \theta_{\eta_0\eta_0} & \theta_{\eta_0 a} \\ \theta_{a\pi_3} & \theta_{a\eta_8} & \theta_{a\eta_0} & \theta_{aa} \end{pmatrix} \begin{pmatrix}\pi^0\\ \eta\\ \eta'\\ a^{\rm Phys}\end{pmatrix}. \tag{14}$$

In the limit where $M_{\not{PQ}} = 0$, which corresponds to the case of the QCD axion, the mass of the physical axion $m_{a^{\rm phys}}$ receives a contribution induced by its mixing with the meson sector. This mass can be expressed as [5]:

$$m_{a^{\rm phys}}^2 = \frac{\det(M^2)}{\det(M_<^2)}\,, \tag{15}$$

where $\det(M_<^2)$ is the determinant of the 3×3 submatrix obtained from M^2 by removing the last row and last column. This expression arises from the standard method for extracting the eigenvalue associated with a small perturbative coupling. At leading order in the expansion parameter f_π/f_a, this result provides a predictive expression for the axion mass generated by meson-axion mixing, which is:

$$m_{a^{\rm phys}}^2 = \frac{B_0 m_u m_d m_s Q^2}{(m_u m_d + m_u m_s + m_d m_s + \frac{6B_0 m_u m_d m_s}{m_0^2})}\frac{f_\pi^2}{f_a^2}\,, \tag{16}$$

where $Q = Q_u + Q_d + Q_s + Q_G$.

On the other hand, the mixing parameters θ_{ij} can be obtained by solving the eigenvalue problem for the mass matrix M^2,

$$(M^2 - m_{a^{\rm phys}}^2\mathbb{I})\begin{pmatrix}\theta_{\pi_3 a}\\ \theta_{\eta_8 a}\\ \theta_{\eta_0 a}\\ \theta_{aa}\end{pmatrix} = 0, \tag{17}$$

which implies a system of four equations. The axion-meson mixing coefficients have been computed with the help of *Mathematica* (a symbolic computation software) and their expressions are given by:

$$\theta_{a\pi_3}^{\rm (PQ)} = -\frac{f_\pi}{f_a}\frac{1}{1+\epsilon}\left[\frac{Q_u m_u - Q_d m_d}{m_u + m_d} + \frac{m_u - m_d}{m_u + m_d}\frac{Q_G}{2} + \epsilon\frac{Q_u - Q_d}{2}\right], \tag{18}$$

$$\theta_{a\eta_8}^{\rm (PQ)} = -\frac{f_\pi}{f_a}\frac{\sqrt{3}}{2}\frac{1}{1+\epsilon}\left[\frac{3Q_s + Q_G}{3} - \epsilon\frac{(Q_u + Q_d + 2Q_G/3) + \frac{2B_0 m_s}{m_0^2}(Q_u + Q_d - 2Q_s)}{1 + \frac{6B_0 m_s}{m_0^2}}\right], \tag{19}$$

$$\theta_{a\eta_8}^{\rm (PQ)} = -\frac{f_\pi}{f_a}\frac{1}{\sqrt{6}}\frac{1}{1+\epsilon}\left[Q_G + \epsilon\frac{Q_G - \frac{6B_0 m_s}{m_0^2}(Q_u + Q_d + Q_s)}{1 + \frac{6B_0 m_s}{m_0^2}}\right], \tag{20}$$

$$\theta_{aa}^{\rm (PQ)} = 1\,, \tag{21}$$

where $\epsilon = \frac{m_u m_d}{m_s(m_u + m_d)}\left(1 + \frac{6B_0 m_s}{m_0^2}\right)$.

B. Numerical Estimation of the QCD Axion Mass

In this section, we provide a quantitative estimation of the QCD axion mass, based on phenomenological inputs from low-energy QCD and chiral perturbation theory.

We begin by determining the parameter B_0 from the pion mass. At leading order in chiral perturbation theory, the pion mass is expressed as $m_\pi^2 = B_0(m_u + m_d)$, which leads to:

$$B_0 = \frac{m_\pi^2}{m_u + m_d} \approx 2.6 \text{ GeV}, \tag{22}$$

where we have used the experimental values $m_\pi \approx 135$ MeV, $m_u \approx 2.2$ MeV, and $m_d \approx 4.7$ [6]. The anomaly-induced mass parameter m_0^2 can be approximated using the relation:

$$m_0^2 \approx m_\eta^2 + m_{\eta'}^2 - B_0(m_u + m_d + 2m_s), \tag{23}$$

which yields $m_0 \approx 850$ MeV when employing the experimental masses m_η and $m_{\eta'}$. The anomaly coefficient $Q = Q_u + Q_d + Q_s + Q_G$ depends on the specific axion model under consideration. For instance, in the KSVZ model (hadronic axion), $Q \approx Q_G$, while in the DFSZ model (where axions couple to quarks and leptons), Q depends on the Peccei-Quinn charge assignments. For estimation purposes, we adopt $Q \sim 1$. Substituting these parameters into the expression for the physical axion mass and simplifying, we obtain:

$$m_{a^{\text{phys}}} \approx 5.8\ \mu\text{eV} \left(\frac{10^{12}\ \text{GeV}}{f_a}\right). \tag{24}$$

In summary, the QCD axion mass is estimated to be approximately 5.8 μeV for a Peccei-Quinn scale of $f_a = 10^{12}$ GeV.

C. ALPs-meson mixing

In the previous subsection we have estimated the QCD axion mass. We now consider the case where the PQ symmetry is explicitly broken, corresponding to a finite $M_{\not{PQ}}$ mass (when the ALP is no longer the QCD axion). Assuming $f_a \gg f_\pi$ and neglecting terms of $O(\theta^2_{a\phi})$, where $\phi = \pi_3, \eta_8, \eta_0$, the mass-squared of the physical ALP is:

$$m_a^2 = \left(m_a^{(\text{PQ})}\right)^2 + M^2_{\not{PQ}}, \tag{25}$$

where $\left(m_a^{(\text{PQ})}\right)^2$ is the QCD axion mass, Eq. (16). Also we can obtain the new ALP meson mixing angles, which are given by:

$$\theta_{a\pi_3} = \theta^{(\text{PQ})}_{a\pi_3}\left(1 + \frac{M^2_{\not{PQ}}}{m_\pi^2 - m_a^2}\right), \tag{26}$$

$$\begin{aligned}\theta_{a\eta_8} = \theta^{(\text{PQ})}_{a\eta_8}&\left(1 + \cos^2\theta_{\eta\eta'}\frac{M^2_{\not{PQ}}}{m_\eta^2 - m_a^2} + \sin^2\theta_{\eta\eta'}\frac{M^2_{\not{PQ}}}{m_{\eta'}^2 - m_a^2}\right)\\ &+ \theta^{(\text{PQ})}_{a\eta_0}\cdot\frac{\sin 2\theta_{\eta\eta'}}{2}\left(\frac{M^2_{\not{PQ}}}{m_{\eta'}^2 - m_a^2} - \frac{M^2_{\not{PQ}}}{m_\eta^2 - m_a^2}\right),\end{aligned} \tag{27}$$

$$\begin{aligned}\theta_{a\eta_0} = \theta^{(\text{PQ})}_{a\eta_0}&\left(1 + \sin^2\theta_{\eta\eta'}\frac{M^2_{\not{PQ}}}{m_\eta^2 - m_a^2} + \cos^2\theta_{\eta\eta'}\frac{M^2_{\not{PQ}}}{m_{\eta'}^2 - m_a^2}\right)\\ &+ \theta^{(\text{PQ})}_{a\eta_8}\cdot\frac{\sin 2\theta_{\eta\eta'}}{2}\left(\frac{M^2_{\not{PQ}}}{m_{\eta'}^2 - m_a^2} - \frac{M^2_{\not{PQ}}}{m_\eta^2 - m_a^2}\right).\end{aligned} \tag{28}$$

In deriving these expressions, we have neglected mixing between the π^0-η and the π^0-η'. We have also assumed small mixing angles, allowing a first order Taylor expansion: $\tan(\theta_{a\phi}) \approx \theta_{a\phi}$.

Note that these expressions exhibit pole singularities: the mixing angles diverge when the ALP mass approaches the mass of any of the pseudoscalar mesons. However, if we do not use the small-angle approximation, these singularities are regularized and the mixing angles remain finite. The qualitative behavior remains the same; the divergence is simply replaced by a large but finite enhancement. We can plot the mixing angle $\theta_{a\pi}$ as a function of the ALP mass to illustrate this behavior (FIG. 1):

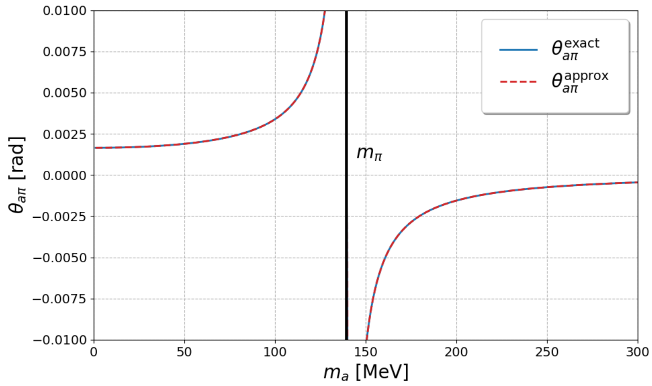

FIG. 1: Comparison of the mixing angle between the exact and approximate solutions. The red dashed curve represents the approximate solution for the mixing angle, while the blue curve shows the exact result. The black vertical line indicates the specific pion mass value at which the mixing diverges.

As can be seen in FIG. 1, when the ALP mass approaches the pion mass, a divergence occurs. This divergence arises from the pole structure in Eq. (26). Although this is not clearly visible at this scale (corresponding to small mixing angles), the blue curve (exact solution) saturates at an angle of $\pm\frac{\pi}{4}$, whereas the red curve (approximate solution) diverges to $\pm\infty$. It is important to emphasize that for the exact values $m_a = m_\pi$, $m_a = m_\eta$ and $m_a = m'_\eta$ the physical ALP state cannot exist. This is because the mathematical conditions required for consistent diagonalization lead to a vanishing mixing term, as discussed in Appendix B.

D. Phenomenological examples

With the diagonalization of the π^0-η-η'-axion system complete, and the physical masses and mixing angles established, the Lagrangian is prepared for phenomenological analysis. As an illustrative application, in this section we will calculate the amplitude and corresponding branching ratio for the decay process $\eta^{(\prime)} \to \pi^0\pi^0 a$.

From the Eq. (12), we can obtain the corresponding leading-order amplitudes. To achieve this, we expand the matrix U and calculate the trace $\text{Tr}[M_q(a)U + M_q(a)U^\dagger]$. Finally, we identify the coefficient of interest, which corresponds to a term containing one η meson, two π^0 mesons, and one axion (a). In this manner, we obtain the Eq. (29) which compactly encodes the amplitudes for both decay processes:

$$\mathcal{A}(\eta^{(\prime)} \to \pi^0\pi^0 a) = \frac{m_\pi^2}{f_\pi^2}\left[C_{\eta^{(\prime)}}\frac{m_u A_u + mdA_d}{m_u + m_d} + O(\epsilon_{\pi\eta})\right], \tag{29}$$

where we have defined:

$$C_\eta = \frac{\cos\theta_{\eta\eta'}}{\sqrt{3}} - \frac{\sin\theta_{\eta\eta'}}{\sqrt{3/2}}\,, \tag{30}$$

$$C_{\eta'} = \frac{\cos\theta_{\eta\eta'}}{\sqrt{3/2}} + \frac{\sin\theta_{\eta\eta'}}{\sqrt{3}}\,, \tag{31}$$

$$A_u = \frac{f_\pi}{f_a}Q_u + \theta_{a\pi} + \frac{\theta_{a\eta_8}}{\sqrt{3}} + \frac{\theta_{a\eta_0}}{\sqrt{3/2}}\,, \tag{32}$$

$$A_d = \frac{f_\pi}{f_a}Q_d - \theta_{a\pi} + \frac{\theta_{a\eta_8}}{\sqrt{3}} + \frac{\theta_{a\eta_0}}{\sqrt{3/2}}\,. \tag{33}$$

With our results for the leading order amplitudes we can now extract the branching ratios for the single ALP decays $\eta \to \pi^0\pi^0 a$ and $\eta' \to \pi^0\pi^0 a$ whit the process described in the Appendix C. Plotting the branching ratios as a function of the ALP mass we obtain the FIG. 2:

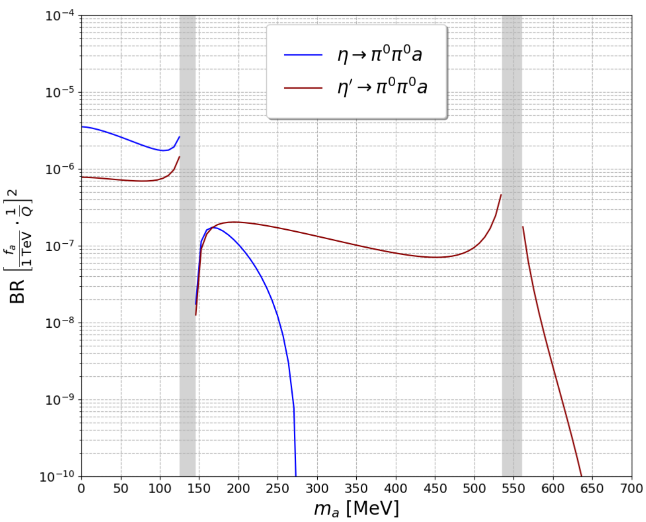

FIG. 2: (Branching ratios)$\times(f_a/Q)^2$ for $\eta \to \pi^0\pi^0 a$ (blue line) and $\eta' \to \pi^0\pi^0 a$ (red line), as a function of the ALP mass, m_a. We have considered gluon-dominance. Since our small mixing approximations are not valid when $m_a \approx m_\pi$ and $m_a \approx m_\eta$ we have masked out this regions in grey.

IV. CONCLUSIONS

Our work provides a comprehensive theoretical analysis of neutral pseudoscalar meson mixing within Chiral Perturbation Theory, extended to include the axion. We began by outlining the theoretical framework of leading-order ChPT and deriving the mixing relations for the π^0 η, and η' mesons. We systematically diagonalized the mass matrix, first considering the η-η' mixing in the isospin limit and then incorporating isospin symmetry breaking to account for the full π^0-η-η' mixing. This allowed us to derive expressions for the mixing angles and the physical meson masses.

A significant part of this work was dedicated to incorporating the axion into the neutral pseudoscalar meson sector. We analyzed both the QCD axion limit, estimating its mass around $5.8\,\mu$eV for $f_a = 10^{12}$ GeV and the more general case of axion-like particles (ALPs) with explicit Peccei-Quinn symmetry breaking. For the PQ-breaking scenario, we obtained expressions for the ALP mass and its mixing angles with the mesons, finding resonant enhancements when the ALP mass approaches those of the pseudoscalar mesons.

Finally, we demonstrated the phenomenological utility of our extended theoretical framework by calculating the amplitudes and branching ratios for the decay processes $\eta \to \pi^0\pi^0 a$ and $\eta' \to \pi^0\pi^0 a$. These calculations provide concrete examples of how our derived mixing angles and masses can be used to predict observable signatures in experiments, offering avenues for probing new physics beyond the Standard Model through axion phenomenology. The presented analysis lays the groundwork for further investigations into axion-meson interactions and their implications for particle physics and cosmology.

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Estudi teòric de la barreja del sistema π^0-η-η'-axió

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Resum: El terme de massa del Lagrangià Quiral a primer ordre (Leading Order) barreja els estats amb estranyesa zero. No obstant això, per realitzar càlculs de quantitats físiques, com ara taxes de decaïment i amplituds de dispersió que involucren els mesons neutres π^0, η i η', la matriu de masses del Lagrangià ha de ser diagonalitzada. En el present treball, trobarem la relació entre els estats del Lagrangià i els estats físics en termes d'angles de mescla, així com les expressions per a les masses dels mesons físics. Així mateix, hem inclòs la barreja amb l'axió, preparant el Lagrangià per a aplicacions en fenomenologia d'axions.
Paraules clau: Axió i ALPs, Lagrangià Quiral, simetria de trencament d'isospí
ODSs: Aquest TFG està vinculat amb l'Objectiu de Desenvolupament Sostenible número 4, i més concretament amb la fita 4.4, ja que contribueix a la millora de l'educació en l'àmbit universitari.

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

Appendix A: Notation and definitions

Throughout this work, certain mathematical expressions have been used without explicit definition, under the assumption that the reader is already familiar with them. This choice was made primarily because some of these expressions are lengthy and not central to the main discussion. Nevertheless, for completeness and clarity, they are provided in this section.

The matrix containing the pseudoscalar mesons fields, Φ, is defined as:

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi_3 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi_3 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 \end{pmatrix} \tag{A1}$$

The elements of the mass matrix of Eq. (2) and Eq. (13) are defined as:

$$m^2_{\pi_3} = B_0(m_u + m_d)\,, \tag{A2}$$

$$m^2_{\pi_3\eta_8} = \frac{B_0}{\sqrt{3}}(m_u - m_d)\,, \tag{A3}$$

$$m^2_{\pi_3\eta_0} = \sqrt{\frac{2}{3}}B_0(m_u - m_d)\,, \tag{A4}$$

$$m^2_{\eta_8} = \frac{B_0}{3}(m_u + m_d + 4m_s)\,, \tag{A5}$$

$$m^2_{\eta_8\eta_0} = \sqrt{\frac{2}{3}}B_0(m_u + m_d - 2m_s)\,, \tag{A6}$$

$$m^2_{\eta_0} = m_0^2 + \frac{2}{3}B_0(m_u + m_d + m_s)\,, \tag{A7}$$

$$m^2_{a\pi_3} = \frac{f_\pi}{f_a}B_0(m_uQ_u - m_dQ_d), \tag{A8}$$

$$m^2_{a\eta_8} = \frac{f_\pi}{f_a}\frac{B_0}{\sqrt{3}}(m_uQ_u + m_dQ_d - 2m_sQ_s)\,, \tag{A9}$$

$$m^2_{a\eta_0} = \frac{f_\pi}{f_a}\sqrt{\frac{2}{3}}B_0(m_uQ_u + m_dQ_d + m_sQ_s) - \frac{f_\pi}{f_a}\frac{m_0^2 Q_G}{\sqrt{6}}\,, \tag{A10}$$

$$m^2_a = \frac{f_\pi^2}{f_a^2}B_0(m_uQ_u^2 + m_dQ_d^2 + m_sQ_s^2) + \frac{f_\pi^2}{f_a^2}\frac{m_0^2 Q_G}{6} + M^2_{\not{P}Q}\,. \tag{A11}$$

The ALP-dependent quark mass matrix, $M_q(a)$ is given by:

$$M_q(a) = \begin{pmatrix} m_u e^{iQ_u a/f_a} & & \\ & m_d e^{iQ_d a/f_a} & \\ & & m_s e^{iQ_s a/f_a} \end{pmatrix}\,. \tag{A12}$$

Appendix B: ALP with pseudoscalar mesons mass

Let us consider the case where the axion-like particle (ALP) can mix with the pion. The mass matrix is given by:

$$M^2 = \begin{pmatrix} m^2_{\pi_3} & m^2_{\pi_3 a} \\ m^2_{\pi_3 a} & m^2_a \end{pmatrix}\,. \tag{B1}$$

After diagonalizing this matrix, we obtain the physical masses:

$$m^2_\pi = \frac{1}{2}\left[(m^2_{\pi_3} + m^2_a) - \sqrt{(m^2_{\pi_3} - m^2_a)^2 + 4m^4_{\pi_3 a}}\right]\,, \tag{B2}$$

$$m^2_{a^{phys}} = \frac{1}{2}\left[(m^2_{\pi_3} + m^2_a) + \sqrt{(m^2_{\pi_3} - m^2_a)^2 + 4m^4_{\pi_3 a}}\right] . \tag{B3}$$

Now, suppose that the physical masses of the pion and the ALP are equal. In that case, we must have:

$$m^2_{a^{phys}} - m^2_\pi = 0 = \sqrt{(m^2_{\pi_3} - m^2_a)^2 + 4m^4_{\pi_3 a}} . \tag{B4}$$

This implies two conditions must be simultaneously satisfied:

$$(m^2_{\pi_3} - m^2_a)^2 = 0 \quad \text{and} \quad m^2_{\pi_3 a} = 0 . \tag{B5}$$

However, these conditions also imply that there is no mixing between the ALP and the pion. Therefore, we conclude that an ALP with a physical mass equal to that of the pion cannot exist, as it would require the absence of any mixing (contradicting the assumption of mixing in the first place).

Appendix C: Formal Derivation of Branching Ratios from Decay Amplitudes

The theoretical determination of branching ratios for particle decay processes is a fundamental aspect of particle physics, connecting theoretical models with experimental observables. This section details the methodology employed to calculate the branching ratio for the decay process $\eta^{(\prime)} \to \pi^0\pi^0 a$.

The foundation of this calculation is the partial decay rate (Γ), which quantifies the probability per unit time for a particle to decay into a specific final state. For a three-body decay such as $\eta^{(\prime)} \to \pi^0\pi^0 a$, the partial decay rate is given by the phase space integral of the squared decay amplitude. Incorporating the identical nature of the two π^0 particles in the final state, which introduces a symmetry factor of $1/2!$, the partial decay rate is expressed as [4]:

$$\Gamma(\eta^{(\prime)} \to \pi^0\pi^0 a) = \frac{1}{2!}\frac{1}{(2\pi)^3}\frac{1}{32m^3_{\eta^{(\prime)}}}\int_{s_{\min}}^{s_{\max}} ds \int_{t_{\min}(s)}^{t_{\max}(s)} dt\, |\mathcal{A}(s,t,u)|^2 , \tag{C1}$$

where $m_{\eta^{(\prime)}}$ represents the mass of the decaying η or η' meson, $\mathcal{A}(s,t,u)$ is the Lorentz-invariant decay amplitude. It is a function of the Mandelstam variables s, t, and u, which describe the squared invariant masses of different particle combinations in the decay.

The boundaries for the integration variables s and t are determined by kinematic constraints to ensure physical decay products. The integration limits for t are dependent on s and are given by:

$$t_{\max/\min}(s) = \frac{1}{2}\left[m^2_{\eta^{(\prime)}} + m^2_a + 2m^2_{\pi^0} - s \pm \frac{\lambda^{1/2}(s, m^2_{\eta^{(\prime)}}, m^2_a)\lambda^{1/2}(s, m^2_{\pi^0}, m^2_{\pi^0})}{s}\right] , \tag{C2}$$

where m_a is the axion mass, m_{π^0} is the neutral pion mass, and $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ is the Källen function, which frequently appears in phase space calculations.

The allowed range for the Mandelstam variable s is bounded by the threshold for creating two pions and the maximum energy available for the system containing the axion and one pion:

$$s_{\min} = (2m_{\pi^0})^2 , \quad s_{\max} = (m_{\eta^{(\prime)}} - m_a)^2 . \tag{C3}$$

Finally, the branching ratio (BR) for a specific decay mode is defined as the proportion of decays that proceed via that particular channel. It is calculated by normalizing the partial decay rate by the total decay rate (Γ_{Total}) of the parent particle:

$$BR(\eta^{(\prime)} \to \pi^0\pi^0 a) = \frac{\Gamma(\eta^{(\prime)} \to \pi^0\pi^0 a)}{\Gamma_{\text{Total}}} . \tag{C4}$$

The total decay rate, Γ_{Total}, is the sum of all possible decay modes of the $\eta^{(\prime)}$ meson and is typically obtained from experimental measurements.