The unique conformally and duality-invariant theory of non-linear electrodynamics

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Abstract: We study extensions of Maxwell electrodynamics which respect Lorentz invariance, gauge symmetry, conformal symmetry and are invariant under electromagnetic duality rotations. We carefully derive and show that there exists a unique one-parameter family of theories that preserves all these properties. It receives the name of ModMax and we examine how Coulomb's law is modified within this set of theories and the appearance of birefringence in vacuum. **Keywords:** Non-linear electrodynamics, duality invariance, conformal invariance.
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I. INTRODUCTION

From a profound perspective, Maxwell's electrodynamics represents the simplest example of a Lorentz and gauge invariant theory for a four-vector field A_μ. In terms of its field strength $F_{\mu\nu}$, defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{1}$$

Maxwell's Lagrangian is given by

$$\mathcal{L}_{\text{Max}} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{2}$$

where the overall factor has been set to $\frac{1}{4}$ for the sake of convenience. As a matter of fact, we will be using throughout the document natural units in which $c = \mu_0 = \varepsilon_0 = 1$ and the sign convention for the Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

However, as we have learnt in the last century in theoretical physics, symmetries are the guiding principle towards the finding of the fundamental laws that rule the universe. In this context, one may argue whether there are other Lorentz and gauge invariant theories of electrodynamics beyond Maxwell's theory. It is a straightforward exercise to note that *any* theory for a four-vector A_μ which is built by Lorentz scalars formed by contractions of its field strength $F_{\mu\nu}$ will meet such requirements. Since the subsequent theories will provide equations of motion which are no longer linear in the derivatives of $F_{\mu\nu}$, these are called *theories of non-linear electrodynamics.*

However, as it turns out, Maxwell's theory possesses two additional symmetries: it is conformal and is invariant under electromagnetic duality rotations (we will review these transformations later). As a consequence, it is a natural question to explore the existence of non-linear electrodynamics beyond Maxwell which share the same symmetries as Maxwell: Lorentz and gauge invariance, conformal symmetry, and duality invariance. The purpose of this article is to show that there is a one-parameter generalization of Maxwell's theory that preserves these symmetries. The resulting family of theories is called *ModMax* and we will show how this theory predicts a modification of Coulomb's law and the phenomenon of birefringence in vacuum.

II. THEORIES OF NON-LINEAR ELECTRODYNAMICS

Define the following invariants:

$$s \equiv \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2}\left(B^2 - E^2\right) \tag{3}$$

$$p \equiv \tfrac{1}{4} F_{\mu\nu} (\star F)^{\mu\nu} = \vec{E} \cdot \vec{B} \tag{4}$$

where $\star$ stands for the Hodge dual transformation:

$$(\star F)^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{5}$$

Our first goal will be to show that any theory of non-linear electrodynamics built out of Lorentz scalars formed through the contraction of field strengths F_{ab} is a function of the invariants s and p.

To this end, consider a Lorentz scalar $\mathcal{I}$ constructed with N field strengths, so that we may write schematically $\mathcal{I} \sim F^N$. We will now see how if we form a scalar with this quantity we end up with something that depends on s and p. To show it one may compute the eigenvalues of the field strength $F_a^{\ b}$ and see that these are given by:

$$\lambda_1 = \pm f(s,p)\,, \quad \lambda_2 = \pm g(s,p)\,. \tag{6}$$

Where $f(s,p)$ and $g(s,p)$ are functions which depend only on the invariants s and p. This calculation does not add any insight and these two functions are not very elegant, so we are not going to write their explicit form. Since any invariant will be expressed in terms of these eigenvalues, we conclude that $\mathcal{I}$ must be solely a function of s and p.

If we notice that out of the four eigenvalues, two are of opposite sign and the other two are also opposite in sign, then we can also conclude that an odd contraction of the electromagnetic tensor with itself will result in zero. That it to say that $Tr(F^{2k+1}) = 0$, where $k \in \mathbb{Z}$ since:

$$Tr((F \cdot \eta)^{2k+1}) = \lambda_1^{2k+1} - \lambda_1^{2k+1} + \lambda_2^{2k+1} - \lambda_2^{2k+1} = 0 \tag{7}$$

The next thing we could think is that several different contractions between F and $\star F$ are possible. However, the identity

$$F_a{}^b F_b{}^c - (\star F)_a{}^b (\star F)_b{}^c = 2s \mathbb{1}_a{}^c \tag{8}$$

guarantees that it doesn't matter how we contract the field strength and its dual with each other, it will always result in something that depends only on s and p. Finally, noting that:

$$F_a^b (\star F_b^c) = p \delta_a^c \,, \tag{9}$$

we conclude that any Lorentz-invariant Lagrangian built out from contractions of the field strength is a function of s and p.

A. Field equations

Let us start by reviewing the equations of motion of Maxwell's electrodynamics. These are given by:

$$\partial_\mu F^{\mu\nu} = 0 \tag{10}$$

$$\partial_\mu (\star F)^{\mu\nu} = 0 \tag{11}$$

The first equation comes from the extremisation of the Maxwell Lagrangian $\mathcal{L}_{\text{Max}} = s$, while the second equation is a consequence of the definition of F_{ab} and is called the Bianchi identity. While the Bianchi identity will remain unmodified for all theories on non-linear electrodynamics, this will no longer be the case for the first equation above.

Indeed, in a given theory of non-linear electrodynamics $\mathcal{L}$, the equations of motion are given by:

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} \right) = 0 \,, \quad \partial_\mu (\star F)^{\mu\nu} = 0 \,. \tag{12}$$

We want to write the equations of motion in terms of the invariants s and p. For that, using the chain rule:

$$\frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} = \frac{\partial \mathcal{L}}{\partial s} \frac{\partial s}{\partial F_{\alpha\beta}} + \frac{\partial \mathcal{L}}{\partial p} \frac{\partial p}{\partial F_{\alpha\beta}} \tag{13}$$

Now we only need to compute two more derivatives:

$$\begin{aligned} \tfrac{\partial s}{\partial F_{\alpha\beta}} &= \tfrac{1}{4} \eta^{\mu\epsilon} \eta^{\nu\delta} (\delta_\mu^{\ \alpha} \delta_\nu^{\ \beta} F_{\epsilon\delta} + F_{\mu\nu} \delta_\epsilon^{\ \alpha} \delta_\delta^{\ \beta}) \\ &= \tfrac{1}{2} (F^{\alpha\beta}) \end{aligned} \tag{14}$$

$$\begin{aligned} \tfrac{\partial p}{\partial F_{\alpha\beta}} &= \tfrac{1}{8} (\delta^\alpha_{\ \mu} \delta^\beta_{\ \nu} \epsilon^{\mu\nu\epsilon\delta} F_{\epsilon\delta} + F_{\mu\nu} \epsilon^{\mu\nu\epsilon\delta} \delta_\epsilon^{\ \alpha} \delta_\delta^{\ \beta}) \\ &= \tfrac{1}{2} (\star F)^{\alpha\beta} \end{aligned} \tag{15}$$

With these two derivatives we now may rewrite (12) and (13) into the following form:

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial s} F^{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial p} (\star F)^{\alpha\beta} \right) = 0 \tag{16}$$

This last equation together with the Bianchi identity form our new field equations.

III. ELECTROMAGNETIC DUALITY

Inspired by the form of the equations of motion for a general theory of non-linear electrodynamics, define a new contravariant tensor $H^{\mu\nu} \equiv -2 \star \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}$. This tensor in Maxwell's theory is:

$$\begin{aligned} H^{ab} = -2 \star \tfrac{\partial \mathcal{L}}{\partial F_{ab}} &= -2 \star \left(-\tfrac{1}{4} (2 \ \delta^a_{\ \mu} \delta^b_{\ \nu} \eta^{\mu\epsilon} \eta^{\nu\alpha} F_{\epsilon\alpha}) \right) = \\ &= \star F^{ab} \end{aligned} \tag{17}$$

Consider the following SO(2) rotation producing the new fields [7] H' and F':

$$\begin{pmatrix} F' \\ H' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix} \tag{18}$$

In order for this transformation to be a symmetry, the relation between H' and F' must be formally the same as the relation between H and F. In the case of Maxwell's theory, we note that:

$$\begin{aligned} &\star F' = H' \to \\ \to\ &\cos(\alpha) \star F - \sin(\alpha) \star H = \sin(\alpha) F + \cos(\alpha) H \to \\ &\to\ \sin(\alpha)(F + \star H) = \cos(\alpha)(\star F - H) \end{aligned} \tag{19}$$

If equation (19) is to hold for all α then:

$$\star F - H = 0 \tag{20}$$

If we apply the Hodge operator to (20) and we remember that $\star\star = -\mathbb{1}$ we can see that imposing (20) also means that $F + \star H = 0$, which makes (19) hold for all α. Equation (20) proves exactly what we were looking for: SO(2) duality rotations leave, in Maxwell's equations, the functional form of H invariant. This is a peculiarity of Maxwell electrodynamics and this property does not hold in general.

Therefore, we now aim to find more general theories of non-linear electrodynamics satisfying this property of *duality invariance*, which amounts to the requirement that $H' = \mathcal{F}(F')$ if $H = \mathcal{F}(F)$, where $\mathcal{F}$ is a certain function of the field strength $F_{\mu\nu}$.

Let us now derive an equation that allows us to compute such Lagrangians. To do that it suffices to make our Lagrangian invariant under infinitesimal SO(2) duality transformations. From (18) if we differentiate with respect to α and take the limit as $\alpha \to 0$ then:

$$\delta H' = \delta\alpha F \tag{21}$$

$$\delta F' = -\delta\alpha H \tag{22}$$

Now if we take into account that $H' = H'(F)$ then the variation of H' is:

$$\delta H'_{ab} = \frac{\partial H_{ab}}{\partial F_{cd}} \delta F_{cd} \tag{23}$$

and if we now use (21) and (22) we have our result:

$$-\frac{\partial H_{ab}}{\partial F_{cd}} \delta\alpha H_{cd} = \delta\alpha F_{ab} \tag{24}$$

Now if we use the definition of the H tensor $H_{ab} = -2 \star \frac{\partial \mathcal{L}}{\partial F^{ab}}$ then (24) transforms into:

$$
\begin{aligned}
F_{ab} &= -2 \left(\star \frac{\partial^2 \mathcal{L}}{\partial F^{ab} \partial F_{cd}} \right) H_{cd} \\
&= 4 \tfrac{1}{4} \epsilon_{abgh} \frac{\partial^2 \mathcal{L}}{\partial F_{gh} \partial F_{cd}} \epsilon_{cdlm} \frac{\partial \mathcal{L}}{\partial F_{lm}} \qquad (25)
\end{aligned}
$$

If we now contract both sides of (25) with $\frac{1}{2}\epsilon^{rsab}$ then:

$$
\tfrac{1}{2}\epsilon^{rsab} F_{ab} = \tfrac{1}{2}\epsilon^{rsab} \epsilon_{abgh} \frac{\partial^2 \mathcal{L}}{\partial F_{gh} \partial F_{cd}} \epsilon_{cdlm} \frac{\partial \mathcal{L}}{\partial F_{lm}} = \qquad (26)
$$

$$
= \delta^{rs}_{gh} \frac{\partial^2 \mathcal{L}}{\partial F_{gh} \partial F_{cd}} \epsilon_{cdlm} \frac{\partial \mathcal{L}}{\partial F_{lm}} \qquad (27)
$$

where in (27) we have used the following property of the levi-civita symbol:

$$
\epsilon^{abcd} \epsilon_{cdlm} = 2 \delta^{ab}_{lm} \qquad (28)
$$

the delta in (28) represents the generalised Kronecker delta. If we compute the contraction of the right-hand side of (27) we get:

$$
\delta^{rs}_{gh} \frac{\partial^2 \mathcal{L}}{\partial F_{gh} \partial F_{cd}} \epsilon_{cdlm} \frac{\partial \mathcal{L}}{\partial F_{lm}} = \qquad (29)
$$

$$
= 2 \frac{\partial^2 \mathcal{L}}{\partial F_{rs} \partial F_{cd}} \epsilon_{cdlm} \frac{\partial \mathcal{L}}{\partial F_{lm}} \qquad (30)
$$

then (27) finally transforms into:

$$
\frac{1}{2}\epsilon^{rsab} F_{ab} = 2 \frac{\partial^2 \mathcal{L}}{\partial F_{rs} \partial F_{cd}} \epsilon_{cdlm} \frac{\partial \mathcal{L}}{\partial F_{lm}} \qquad (31)
$$

We now notice that the right-hand side looks like a chain rule:

$$
2 \frac{\partial^2 \mathcal{L}}{\partial F_{rs} \partial F_{cd}} \epsilon_{cdlm} \frac{\partial \mathcal{L}}{\partial F_{lm}} = \epsilon_{cdlm} \frac{\partial}{\partial F_{rs}} \left(\frac{\partial \mathcal{L}}{\partial F_{cd}} \frac{\partial \mathcal{L}}{\partial F_{lm}} \right) \qquad (32)
$$

and therefore (31) turns into:

$$
\frac{1}{2}\epsilon^{rsab} F_{ab} = \epsilon_{cdlm} \frac{\partial}{\partial F_{rs}} \left(\frac{\partial \mathcal{L}}{\partial F_{cd}} \frac{\partial \mathcal{L}}{\partial F_{lm}} \right) \qquad (33)
$$

Now if we integrate both sides with respect to F and set all integration constants to zero we get:

$$
\frac{1}{4}\epsilon^{rsab} F_{ab} F_{rs} = \frac{\partial \mathcal{L}}{\partial F_{cd}} \frac{\partial \mathcal{L}}{\partial F_{lm}} \epsilon_{cdlm} \qquad (34)
$$

which is functional relation to find Lagrangians invariant under duality rotations. Now finally, if we use our definitions of s and p we can make this equation explicitly depend on s and p, after using the chain rule on (34) and using (14) and (15) we get:

$$
1 = -\mathcal{L}_p^2 - \frac{2s}{p} \mathcal{L}_p \mathcal{L}_s + \mathcal{L}_s^2 \qquad (35)
$$

where $\mathcal{L}_p \equiv \frac{\partial \mathcal{L}}{\partial p}$ and $\mathcal{L}_s \equiv \frac{\partial \mathcal{L}}{\partial s}$. This is one of the equations we will use from now on to find the, possibly multiple, Lagrangians of our new theory.

IV. CONFORMAL SYMMETRY

The conformal symmetry consists of the invariance of the action under the following coordinate transformations:

$$
x'^{\mu} = \Omega x^{\mu} \qquad (36)
$$

Where Ω is a constant. if the action is to remain invariant under this transformation then:

$$
S'[\mathcal{L}'] = S[\mathcal{L}] \qquad (37)
$$

which implies:

$$
\int d^4x' \mathcal{L}' = \int d^4x \mathcal{L} \qquad (38)
$$

Under these transformations our tensor field transforms as follows:

$$
F'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} F_{\alpha\beta} = \Omega^{-2} F_{\mu\nu} \qquad (39)
$$

Taking into account (39) and remembering the definition of s and p (38) implies:

$$
\begin{aligned}
\int d^4x \mathcal{L}(s,p) = \int d^4x' \mathcal{L}(s',p') = \int d^4x \left|\tfrac{\partial x'}{\partial x}\right| \mathcal{L}(s',p') = \\
= \int d^4x \; \Omega^{-4} \mathcal{L}(\Omega^4 s, \Omega^4 p) \qquad (40)
\end{aligned}
$$

where in (40) we use the Jacobian determinant to transform the volume element.

If (40) is to hold for all s and p then the Lagrangian must be a homogeneous function of degree one in order to cancel out the Ω^{-4} term. There is a theorem called "Euler's homogeneous function theorem" which guarantees that all homogeneous functions of degree one must satisfy, in our case, the following differential equation:

$$
s \frac{\partial \mathcal{L}}{\partial s} + p \frac{\partial \mathcal{L}}{\partial p} = \mathcal{L}(s,p) \qquad (41)
$$

With this final result we have two non-linear partial differential equations, namely (41) and (35), to compute Lagrangians which follow: Conformal symmetry, Lorentz invariance, Gauge symmetry and electromagnetic duality.

V. MODMAX LAGRANGIAN

We now perform the following change of variables in equations (41) and (35):

$$
\alpha \equiv s; \; \beta \equiv \sqrt{s^2 + p^2} \qquad (42)
$$

$$
q \equiv \tfrac{1}{2}(\alpha + \beta); \; l \equiv \tfrac{1}{2}(\alpha - \beta) \qquad (43)
$$

after this change they change to an easier form:

$$
\mathcal{L}_q \mathcal{L}_l = 1 \qquad (44)
$$

$$
q\mathcal{L}_q + l\mathcal{L}_l = \mathcal{L}(q,l) \qquad (45)
$$

Euler's theorem guarantees that all functions obeying (41) and (45) must be homogeneous. The most general homogeneous function is:

$$\mathcal{L}(q,l) = qg(l/q) \tag{46}$$

where $g(l/q)$ is an arbitrary function. If we now use this piece of information and introduce it in (44) we get:

$$(g(u) - g'(u)u)\, g'(u) = 1 \tag{47}$$

where $u \equiv l/q$. To solve this non-linear differential equation we choose as an Ansatz the most general function possible:

$$g(u) \equiv \sum_{i=0}^{\infty} a_i u^i \tag{48}$$

After distributing the product of functions we get:

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u^{i+k-1} k\,(a_i a_k - i a_i a_k) = 1 \tag{49}$$

If this is to be 1 for all u then it's obvious that, as long as $k \neq 0$, then:

$$a_i a_k\,(1-i) = 0\,, \quad \text{for } k \geq 2-i\,. \tag{50}$$

We now fix $a_0 = 1/c$ where c is a constant. if we do that then by (50), $a_{k\geq 2} = 0$. Which means our solution is linear. To find a_1 we simply introduce our results in (49) and we get that $a_1 = c$. Then finally after undoing every change of variable and noticing that $\left(\frac{1}{2c} + \frac{c}{2}\right)^2 - \left(\frac{1}{2c} - \frac{c}{2}\right)^2 = 1$, we are led with a unique family of duality- and conformally invariant theories of non-linear electrodynamics which take the form:

$$\mathcal{L}(s,p) = \cosh(\gamma)s + \sinh(\gamma)\sqrt{s^2+p^2}\,, \tag{51}$$

where γ is a dimensionless parameter. This family of theories is called ModMax [1] and reduces to Maxwell electrodynamics in the limit $\gamma = 0$.

VI. MODIFIED COULOMB'S LAW AND BIREFRINGENCE IN THE VACUUM

Now we want to see how the field equations change with this new Lagrangian. Introducing the ModMax Lagrangian into (16) we get:

$$\cosh\gamma\partial_\mu F^{\mu\beta} + \sinh\gamma\partial_\mu\left(\frac{sF^{\mu\beta}+p(\star F)^{\mu\beta}}{\sqrt{s^2+p^2}}\right) = 0 \tag{52}$$

$$\partial_\mu(\star F)^{\mu\nu} = 0 \tag{53}$$

We can also think about the easiest way to couple (51) with a current, which is to introduce into the Lagrangian a $-J^\mu A_\mu$ term to transform (52) into:

$$\cosh\gamma\partial_\mu F^{\mu\beta} + \sinh\gamma\partial_\mu\left(\frac{sF^{\mu\beta}+p(\star F)^{\mu\beta}}{\sqrt{s^2+p^2}}\right) = J^\beta \tag{54}$$

To see how Coulomb's law changes let us consider $A^\mu = \left(\phi(r), \vec{0}\right)$. With this four potential it can be shown that $p = 0$ and our equation (54) changes to a familiar form:

$$e^{sgn(s)\gamma}\partial_\mu F^{\mu\beta} = J^\beta \tag{55}$$

It can also be shown, by calculating the contraction with $p = 0$, that for a charge at rest $s < 0$. These equations are the dynamic Maxwell equations but just with a different "effective" charge which depends on γ. We know the solution of Maxwell equations when the current is a point charge at rest, and since the functional form of the equation is the same then we can guarantee that:

$$\phi(r) = \frac{e^\gamma q}{4\pi r} \tag{56}$$

On the other hand we know that Coulomb's law is very precise, that means that γ must be very small. The PVLAS experiment, that can be found in [2], has set a lower bound which is $-3 \cdot 10^{-22} \leq \gamma \leq 0$.

Another important aspect of every theory in physics is that it must be causal. In section (2.1) of [3] they explain the constrains of every Lagrangian, in the context of non-linear electrodynamics, in order for it to be obey causality.

Lagrangians which obey causality must be convex down functions in terms of s and p. That is to say $\mathcal{L}_{ss} < 0$ and $\mathcal{L}_{pp} < 0$. It is important to notice that in many articles they define s and p with opposite signs. If we did that the Lagrangian should be convex up instead. Using this piece of information it is easy to see that γ must be negative.

Before studying birefringence, which arises in vacuum under this Lagrangian when a strong background magnetic field is present, we first define what birefringence is: It is a phenomenon predicted by Maxwell's equations, but only when light passes through a material with specific properties. It consists of a single light beam being split into two distinct beams. However in the vacuum Maxwell's equations predict no splitting, even under strong background magnetic fields.

Let us define $F^{\mu\nu} = f^{\mu\nu} + F_B^{\mu\nu}$, where $F_B^{\mu\nu}$ is the background field and $f^{\mu\nu}$ is the small perturbation of an electromagnetic wave going through the vacuum. We now proceed to make a Taylor expansion of (52) up to second order in $F^{\mu\nu}$ and around $F_B^{\mu\nu}$. After computing the first and second derivatives we arrive to:

$$\mathcal{L} = \frac{1}{2}G_B^{\mu\nu} f_{\mu\nu} + \frac{1}{8}Q_B^{\mu\nu\alpha\beta} f_{\mu\nu} f_{\alpha\beta} + \frac{1}{4}c_1 f_{\mu\nu} f^{\mu\nu} + \frac{1}{4}c_2 f_{\mu\nu}(\star f)^{\mu\nu} \tag{57}$$

Where we defined the following background fields:

$$G_B^{\mu\nu} = c_1 F_B^{\mu\nu} + c_2(\star F)_B^{\mu\nu} \tag{58}$$

$$\begin{aligned} Q_B^{\mu\nu\alpha\beta} &= d_1 F_B^{\mu\nu} F_B^{\alpha\beta} + d_3(\star F)^{\alpha\beta} F^{\mu\nu} + d_3 F^{\alpha\beta}(\star F)^{\mu\nu} \\ &+ d_2(\star F)^{\mu\nu}(\star F)^{\alpha\beta} \end{aligned} \tag{59}$$

while the constants $(c_1, c_2, d_1, d_2, d_3)$ are defined as follows:

$$c_1 \equiv \frac{\partial \mathcal{L}}{\partial s}\Big|_B ; c_2 \equiv \frac{\partial \mathcal{L}}{\partial p}\Big|_B$$
$$d_1 \equiv \frac{\partial^2 \mathcal{L}}{\partial s^2}\Big|_B ; d_2 \equiv \frac{\partial^2 \mathcal{L}}{\partial p^2}\Big|_B$$
$$d_3 \equiv \frac{\partial^2 \mathcal{L}}{\partial s \partial p}\Big|_B$$

Where the "B" subscript means that the derivative is evaluated around the background field. We now write our new set of equations:

$$\partial_\mu \left(G_B^{\mu\nu} + \tfrac{1}{2} Q^{\mu\nu\alpha\beta} f_{\alpha\beta} + c_1 f^{\mu\nu} \right) = 0 \tag{60}$$
$$\partial_\mu (\star f)^{\mu\nu} = 0 \tag{61}$$

Here it is interesting to check that if we set $c_1 = 1$ and the other constants to zero we end up with Maxwell's equations again.

Assuming a constant magnetic field as background, then:

$$\partial_\mu \left(\tfrac{1}{2} Q^{\mu\nu\alpha\beta} f_{\alpha\beta} + c_1 f^{\mu\nu} \right) = 0 \tag{62}$$
$$\partial_\mu (\star f)^{\mu\nu} = 0 \tag{63}$$

It can be proved that under these circumstances $p_B = \frac{1}{4}((\star F)^{\mu\nu})_B (F_{\mu\nu})_B = 0$ which means that $d_3 \propto p = 0$. To continue we also make use of the fact that $F^{\alpha\beta} f_{\alpha\beta} = 2\ \vec{B} \cdot \vec{b}\ ;\ (\star F)^{\alpha\beta} f_{\alpha\beta} = -2\ \vec{B} \cdot \vec{e}$ where capital letter means background field and lowercase letter means the perturbative field.

With all this information now we may write (60) in a more familiar way:

$$(\nu = 0) \quad \nabla \cdot \vec{e} + \tfrac{d_2}{c_1} \nabla(\vec{B} \cdot \vec{e}) \cdot \vec{B} = 0 \tag{64}$$
$$(\nu = 1,2,3) \quad - d_1 \vec{B} \times \nabla(\vec{B} \cdot \vec{b}) + d_2 \vec{B} \partial_t (\vec{B} \cdot \vec{e}) = = c_1(-\partial_t \vec{e} + \nabla \times \vec{b}) \tag{65}$$

and (61):

$$\nabla \times \vec{e} + \partial_t \vec{b} = 0,\ \nabla \cdot \vec{b} = 0 \tag{66}$$

Now with these equations we use the planar wave Ansatz:

$$\vec{e} \equiv \vec{e}_0 e^{i(k \cdot x - \omega t)};\ \vec{b} \equiv \vec{b}_0 e^{i(k \cdot x - \omega t)} \tag{67}$$

where $\vec{e}_0$ and $\vec{b}_0$ are constant amplitudes. After introducing this Ansatz into (64)-(66) we are left with a set of linear equations whose determinant must be zero to avoid trivial solutions. The condition for the determinant to be zero is the following:

$$\omega_1^2 = k^2 \left(1 + \tfrac{d_1}{c_1} (\vec{B} \times \hat{k})^2 \right) \tag{68}$$
$$\omega_2^2 = k^2 \left(1 - \tfrac{d_2}{-c_1 + d_2 B^2} (\vec{B} \times \hat{k})^2 \right) \tag{69}$$

If we apply this solution to the ModMax Lagrangian, then $c_1 = \cosh\gamma + \sinh\gamma$ and $d_2 = \frac{\sinh\gamma}{|s|} = \frac{2\sinh\gamma}{\vec{B}^2}$. The other constants, namely c_2, d_1 and d_3 are zero. This is a consequence of $p = 0$ in our specific case. Therefore:

$$\omega_1^2 = k^2 \tag{70}$$
$$\omega_2^2 = k^2 \left(\cos^2\beta + e^{2\gamma} \sin^2\beta \right) \tag{71}$$

Where $\cos\beta = \frac{\hat{k} \cdot \vec{B}}{|\vec{B}|}$. As a consequence, we observe that there are two modes that propagate with different velocities and in different directions. Indeed, while there is always one mode that propagates at the speed of light (see (70)), the second mode (71) is subluminal, for $\gamma < 0$, whenever the direction of propagation is not parallel to the uniform magnetic field. This is precisely the phenomenon of birefringence in vacuum, by which the velocity of propagation depends on the direction of polarization of the incoming wave.

Conclusions

We have identified all theories of non-linear electrodynamics which are gauge-, Lorentz, conformal and duality-invariant. These form a one-parameter family of theories — containing Maxwell's electrodynamics — that are known as ModMax. In particular, we have explicitly shown that there are no additional theories of non-linear electrodynamics beyond this class which share the same symmetries as Maxwell theory. Afterwards, we embarked on the study of this novel class of electrodynamics. We have focused our efforts on the examination of the subsequent Coulomb's law and the phenomenon of birefringence in vacuum. In this regard, we were able to show that Coulomb's law is modified and electromagnetic waves in the vacuum suffer from Birefringence.

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[7] To ease notation, we avoid here the inclusion of indices, as no confusion may arise.

La única teoria d'electrodinàmica no lineal que és conforme i invariable per dualitat

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Resum: Estudiem extensions de l'electrodinàmica de Maxwell que respecten la invariància de Lorentz, la simetria de gauge, la simetria conforme i que són invariants sota rotacions de dualitat electromagnètica. Derivem acuradament i mostrem que existeix una família única de teories amb un sol paràmetre que preserva totes aquestes propietats. Aquesta rep el nom de ModMax, i examinem com es modifica la llei de Coulomb dins d'aquest conjunt de teories i l'aparició de la birefringència al buit. **Paraules clau:** Electrodinàmica no lineal, Invariància sota la dualitat, Invariància conforme.
ODSs: Educació de qualitat, innovació.

El contingut d'aquest TFG està relacionat amb l'educació de qualitat i amb la innovació. El primer punt es relaciona amb el treball, ja que aquest TFG aprofundeix en aspectes de l'assignatura d'electrodinàmica que no es poden veure per manca de temps (les seves simetries, per exemple). El segon punt és clar, ja que mostrem el fenomen de la birrefringència en el buit, que és força exòtic i innovador.