Émilie du Châtelet and Newton on the spheroid's problem: from geometry to calculus

Author: Adrià Vilanova Martínez
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Enric Pérez Canals

Abstract: This paper analyzes and compares the resolution made by Newton in the *Principia* and Du Châtelet in the commentary of her *Principes* of the problem of finding the attraction exerted by a spheroid on a corpuscle on its axis. We find her solution to be simpler and more complete.
Keywords: History of Physics, Mechanics, Celestial Mechanics, Physics in the XVIII century.
SDGs: 4. Quality education. 5. Gender equality.

I. INTRODUCTION

Isaac Newton's (1643–1727) *Philosophiæ Naturalis Principia Mathematica* (1687) is a well-known book in the scientific community, which marks a milestone in mechanics. After Newton's death, physicist and mathematician Émilie du Châtelet (1706–1749) wrote a translation and commentary in French, being published posthumously as *Principes mathématiques de la philosophie naturelle* (1759). The commentary features a section with analytic solutions to some of the problems that appear in the *Principia* and others that go even further.

In this paper, belonging to the realm of the history of physics, I study the problem regarding the attraction of a corpuscle by a spheroid[1] when placed on its axis, as solved by Newton in his *Principia* and Du Châtelet in her commentary, and compare them with the goal of highlighting her contributions. We will see that Du Châtelet's mathematical methods allow a simpler and clearer explanation of the physics behind the problem.

In my Mathematics bachelor's thesis, I already carried out a comparative analysis of another problem of the *Principia* [1]. While developing it, I read that Du Châtelet also developed solutions for problems that Newton did not solve [2], motivating the study of this problem, since Newton does not give a complete proof.

This paper starts with a brief contextualization, followed by a discussion of the resolution of two problems by both authors: the attraction exerted by a circle on a corpuscle, and that exerted by a spheroid—the former is a prerequisite for solving the latter. Finally, we compare them and present some conclusions.

The main bibliography used to analyze the problems includes I. Bernard Cohen and Anne Whitman's commentary and translation of the *Principia* [3],[2] a facsimile of the first edition of the *Principes* [4],[3] and an original manuscript of the *Principes* [5].

[1] A spheroid is the result of rotating an ellipse about one of its principal axes. The spheroid's axis is this axis of revolution.
[2] The reason for choosing it is that their translation has been kept as close as possible to the original but uses modern notation to make it more readable—e.g., "*PF quad.*" is translated as "PF^2".
[3] All its quotes are my own translations to English.

II. CONTEXTUALIZATION

The publication of the *Principia* is regarded as a key turning point in the development of physics. To understand its importance, we must first introduce the state of physics and scientific thinking that existed before it, and upon which Newton and Du Châtelet built.

For nearly two millennia, the prevailing system of the universe was the one proposed by Aristotle (IV century BCE). In it, the Earth was at the center, and the rest of the celestial objects moved in a composition of perfect circles. Gravity inside the terrestrial sphere was explained by objects naturally seeking their proper place.

However, these long-held assumptions were challenged by many natural philosophers, such as Nicolaus Copernicus (1473–1543), who introduced the heliocentric model of the solar system. Galileo Galilei (1564–1642) provided evidence of it using his telescope, and also questioned Aristotle's ideas about motion through some studies [6].

Tycho Brahe (1546–1601) made some meticulous and extensive naked-eye astronomical measurements of the positions of planets. Johannes Kepler (1571–1630) later used them to determine that planets moved in elliptical orbits and proposed his laws.

René Descartes (1596–1650) proposed his "*Mécanisme des tourbillons*" ("Vortex Theory"), one of the most influential theories of Newton's time, especially in France. It attempted to explain the movement of celestial objects by physical vortices in an ether [7, pp. 41–71].

It was in this context that Newton introduced his *Principia*. With it, Newton created the foundations of classical mechanics with his three laws and the universal law of gravitation [3, p. 128]. One of the major breakthroughs was that, as opposed to Descartes, Newton did not describe the philosophical cause behind motion. Instead, he described motion based on a purely mathematical construct, without having to deal with controversial philosophical topics such as attraction [3, pp. 150–155].

The *Principia* consists of three books. The first one, titled "*De motu corporum*" ("the motion of bodies"), gives his method of first and last ratios—a mathematical tool that lets him calculate limits of proportions of magnitudes—and treats the movement of bodies in the absence of friction by constructing the aforementioned mathematical construct. Later, in Book III, "*De Mundi Systemate*" ("The System of the World"), he presents some rules ("*regulæ*") to reason, whereby he establishes a way to proceed in natural philosophy. He presents some empirical evidence of the physical world ("*phenomena"*), verifies the mathematical construct with them, and presents the universal law of gravitation [3, pp. 195–198].

As will be shown in this paper, Newton constructed the mathematical model using Euclidean geometry. He did it "because the Ancients for making things certain admitted nothing into Geometry before it was demonstrated synthetically [i.e., with Euclidean geometry], [...] [and so] that the System of the Heavens might be founded upon good geometry" [3, p. 141].

The *Principia* was first published in 1687, with a second edition published in 1713 and a third one in 1726.

In 1744, Du Châtelet embarked on a project to translate Newton's *Principia* into French. In 1745, she had already finished the translation and started writing the commentary, which she finished in 1749. Just before she died that same year, she sent the manuscripts to the *Bibliothèque du Roi* so they were not lost and nobody doubted her authorship. Alexis Clairaut (1713–1765), who had already been reviewing her work, edited the book and published it in 1759.

The publication consists of two volumes. The second one includes the commentary, which has two parts: the "*Exposition abregée du Systême du Monde*" and the "*Solution analytique des principaux problêmes qui concernent le Systême du Monde*". In the latter, she gives the solutions via Leibniz's calculus. We will see that this simplifies and clarifies the results.

Until hers, the only translation available was an English one written by Andrew Motte (1696–1734). The French translation—including the commentary—was much needed to spread Newtonianism to France, where people resisted it in favor of Descartes. It is also important since French was the common language of Europe [4, vol. 1, p. ix]. Thus, a French translation was more accessible to European readers, many of whom did not understand English nor Latin. It remains the only translation ever made into French, and the one of reference for Francophones [8].

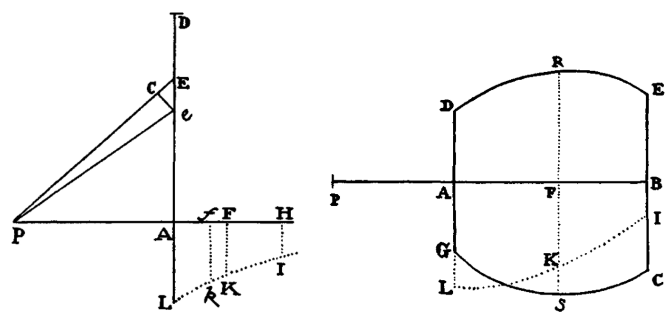
According to Zinsser, only dozens of people were capable of fully understanding calculus [8]. Thus, just translating the work itself was an arduous challenge due to the *Principia*'s complexity, but writing a commentary and expanding on it was even harder. Such was Émilie's scientific maturity that Voltaire's (1694–1778) *Elémens de la Philosophie de Neuton* (1738)—another work that helped spread Newtonianism in France—was written thanks to the essential help given by her [9, pp. 521-522].

III. COMPARATIVE ANALYSIS OF THE SPHEROID'S PROBLEM

To identify a part of Du Châtelet's commentary containing contributions that went further than Newton's, I first wrote a summary of a commentary on the different sections of the *Principia* [3, pp. 128–198]. Afterwards, I worked with the table of contents of her commentary [4, vol. 2, pp. 287–297] to gain a similar overview.

I then compared both summaries and proposed two research lines: studying their descriptions of the precession of the equinoxes, or studying how both approached the problem regarding the attraction of a corpuscle by a spheroid when placed on its axis. After analyzing in more detail how both authors work on these topics, I discovered that the latter shows more clearly her contributions, and therefore, I focused my research on that topic.

A. Newton's development



(a) Diagram for proposition 90. (b) Diagram for proposition 91.

FIG. 1. Diagrams from the *Principia* [3, pp. 614–615] that accompany the propositions we discuss.

1. Attraction of a circle (proposition 90)

In proposition 90 [3, pp. 613–615], a method is given to calculate the attraction of a circle with center A and radius AD to the corpuscle P situated in a line PA perpendicular to the circle and passing through its center (see FIG. 1a).

Given any point E in the circle, draw segment $PF = PE$ in PA. In an auxiliary plane, draw segment FK perpendicular to PA to be as the force by which the single point E attracts corpuscle P. Then, Newton proposes that the "force of attraction will be as the area $AHIL$ multiplied by the altitude AP."

To prove this, he starts saying that the force by which point E attracts the body P toward A—the component of the force perpendicular to the circle—is as $\frac{AP \times FK}{PE}$. Though he does not make it explicit, the parallel component can be ignored because it cancels out with the one exerted by the opposite point of the circle. We can imagine he used this implicitly since he previously gives similar arguments (e.g., proposition 70).

Newton then constructs the "minimally small line Ee," a ring "described with center A and radius AE," and draws C in PE such that $PC = Pe$. We assume the ring is the minimally small area enclosed by circumferences with radii Ae and AE since he says its area is $AE \times Ee$.

Then, "the force by which the whole ring attracts body P toward A is as the ring and $\frac{AP \times FK}{PE}$ jointly" since ultimately—when Ee is diminished infinitely—all points in the ring attract the body with the same force.

Afterwards he states that "PE is to AE as Ee to CE." This is true because PEA and eEC are similar triangles since $\widehat{ECe}$ becomes ultimately a right angle. This implies that the rectangle $AE \times Ee$ "is equal to [...] $PE \times CE$." "Or $PE \times Ff$" since $CE = PE - Pe = PF - Pf = Ff$.

Thus, the force of the ring will be as $Ff \times AP \times FK$. The proof comes to an end because "the sum of forces by which all the rings in the circle [...] attract body P toward A is as the whole area $AHIKL$ multiplied by AP." Here he is implicitly using lemma 2 on the method of first and last ratios, which in this case states that if parallelograms with base fF and height FK covering AH have their bases diminished and their number augmented infinitely, their sum will be as the area $AHIKL$ [1, pp. 46–47].

In corollary 1, Newton says that "if the forces of the parts decrease in the squared ratio of the distances, that is, if FK is as $\frac{1}{PF^2}$ (and thus the area $AHIKL$ is as $\frac{1}{PA} - \frac{1}{PH}$), the attraction [...] will be [...] as $\frac{AH}{PH}$."

The calculation of the quadrature of area $AHIKL$ is not specified. Following an argument given by Cohen and Whitman, we can assume he derived it from his calculus in *Tractatus de Quadratura Curvarum* [3, pp. 125–126].

2. *Attraction of a solid of revolution (proposition 91) and a spheroid in particular (corollaries 2 and 3)*

In proposition 91 [3, pp. 615–618], he gives a method to "find the attraction of a corpuscle placed in the axis of a round solid, to each of whose individual points there tend equal centripetal forces decreasing in any ratio of the distances."

The method begins by sectioning the solid of revolution DECG into circles perpendicular to its axis AB (see FIG. 1b). Then, construct an auxiliary plane passing through the axis, and draw a curve LKI on it such that for each sectioned circle RFS, the distance FK from the curve to the axis is proportional to the force exerted by the circle on corpuscle P calculated in the preceding proposition. Finally, Newton states that the total attraction will be as the area under the curve, that is, $LABI$.

Jacquier and Lesseur say that the corollary of lemma 4 on the method of first and last ratios justifies the result [10, vol. 1, p. 395]. Since the individual segments FK (multiplied by the minimally small increments in the axis) are as the partial attractions of the small sections of the solid, the total area will be as the total attraction.

In corollary 2, this method is applied to the case of an oblate spheroid when the corpuscle is on its exterior.

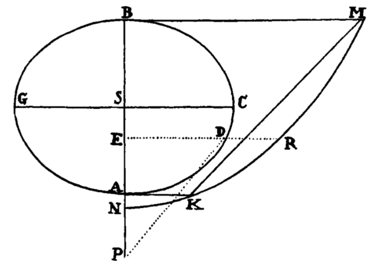


FIG. 2. Diagram accompanying cor. 2 of prop. 91 [3, p. 616].

Given a spheroid with center S, and greatest semidiameter SC, Newton constructs a conic $NKRM$ "whose ordinate ER [...] is always equal to the length of the line PD, which is drawn to the point D in which the ordinate cuts the spheroid" (see FIG. 2).

Then, he states that "the force by which the spheroid attracts the body P will be to the force by which a sphere described with diameter AB attracts the same body as $\frac{AS \times CS^2 - PS \times KMRK}{PS^2 + CS^2 - AS^2}$ to $\frac{AS^3}{3PS^2}$," but doesn't prove it.

In the section on spherical bodies, Newton says he could calculate the law of attraction on other bodies, "but to treat these in particular cases is not essential to my design." This might explain the lack of detail. Some proofs for this expression have been given by other authors: [10, 11] develop proofs similar to what Newton would have done, and [12] proves it with modern calculus.

Corollary 3 treats the case of P being inside the spheroid. Newton states that "the attraction will be as its distance from the center," and proves it by showing geometrically that the forces exerted by the outer shells cancel out, so only the inner spheroid exerts a net force.

B. Du Chatelet's development

The proofs given by Du Châtelet are very detailed. Here we only show the relevant steps, including additional information that is helpful in understanding them.

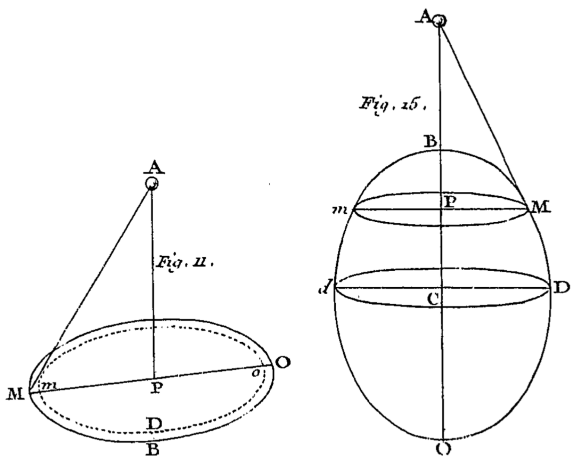
1. *Attraction of a circle (article 22)*

In article 22 [4, vol. 2, pp. 168–169], the circle MBO is also sectioned into rings (see FIG. 3a). She takes a particle M in the ring, which is supposed to attract the body as AM^n, and only considers the component of the attraction exerted by it towards the center of the circle:

$$\frac{AP}{AM} \times AM^n = AP \times AM^{n-1}, \tag{1}$$

since she explicitly mentions that the other component will be canceled out by the opposite particle in the circle.

Émilie then lets $AP = a$, $PM = x$, $AM = \sqrt{a^2 + x^2}$, $Mm = dx$. Since the area of ring $MmoD$ is $\frac{cx\,dx}{r}$—where $\frac{c}{r}$ is the ratio of the circumference to its radius [13], i.e., 2π—and "all particles [in the ring] attract in the same



(a) Diagram for article 22. (b) Diagram for article 37.

FIG. 3. Diagrams from the *Principes* [4, vol. 2, p. 108] which accompany the articles we discuss.

way," the attraction of the whole ring is:

$$\frac{acx\,dx}{r} \times (a^2+x^2)^{\frac{n-1}{2}}. \tag{2}$$

Finally, Du Châtelet integrates (2) over the whole circle to find the total attraction:

$$\frac{c}{r(n+1)}(AP \times AM^{n+1} - AP^{n+2}). \tag{3}$$

2. *Attraction of a spheroid (article 37)*

In article 37 [4, vol. 2, pp. 178–183], Du Châtelet finds "the attraction that a spheroid BMO [oblate/prolate] exerts on a corpuscle A on its axis, supposing that its parts attract in inverse proportion to the square of the distance."

She starts defining (see FIG. 3b) $AB = f$, $BC = a =$ semiaxis of the spheroid, $PB = x$, $PM = y$, and $CD = b =$ radius at the equator. Afterwards, she uses property $y = \frac{b}{a}\sqrt{2ax - x^2}$ that stems from the implicit equation of the ellipse, Pythagoras' theorem ($AM = \sqrt{AP^2 + PM^2}$), and $AP = f + x$ to find the expression of the distance:

$$AM = \sqrt{f^2 + 2\left(f + \frac{b^2}{a}\right)x + \left(1 - \frac{b^2}{a^2}\right)x^2}. \tag{4}$$

Then, Du Châtelet uses (3) with $n = -2$ to obtain that the attraction of a circle section of the spheroid is $\frac{c}{r}(1 - \frac{AP}{AM})$. Thus, integrating it over the whole spheroid, she obtains the total attraction:

$$\frac{c}{r}\int_0^{2a}\left(dx - \frac{(f+x)\,dx}{\sqrt{f^2 + 2\left(f + \frac{b^2}{a}\right)x + \left(1 - \frac{b^2}{a^2}\right)x^2}}\right). \tag{5}$$

The integral of the second term will depend on whether $b > a$ (oblate spheroid) or $b < a$ (prolate spheroid).

In the first case, after defining $\frac{g^2}{a^2} = \frac{b^2}{a^2} - 1$, $h = f + \frac{b^2}{a}$ and the change of variables $u = \frac{ha^2}{g^2}$, it becomes:

$$\frac{c}{r}\int\left(-du - \frac{\frac{a}{g}u\,du}{\sqrt{\frac{a^2f^2}{g^2} + \frac{h^2a^4}{g^4} - u^2}}\right) + \frac{c}{r}\int\frac{\left(\frac{ha^3}{g^3} + \frac{af}{g}\right)du}{\sqrt{\frac{a^2f^2}{g^2} + \frac{h^2a^4}{g^4} - u^2}}. \tag{6}$$

The first integral can be calculated directly, while the second one "can be reduced to an arc of a circle." After integrating it, Du Châtelet gives the total attraction:

$$\frac{c}{r}\left(2a + \frac{2a^3}{g^2} - \left(\frac{ha^3}{g^3} + \frac{af}{g}\right) \times \arcsin\frac{ha^2}{\sqrt{a^2f^2g^2 + h^2a^4}} + \left(\frac{ha^3}{g^3} + \frac{af}{g}\right) \times \arcsin\frac{ha^2 - 2ag^2}{\sqrt{a^2f^2g^2 + h^2a^4}}\right). \tag{7}$$

In the second case, Du Châtelet proceeds similarly, but $\frac{g^2}{a^2}$ is defined so it is still positive: $\frac{g^2}{a^2} = 1 - \frac{b^2}{a^2}$. The total attraction in that case is:

$$\frac{c}{r}\left(-\frac{2ab^2}{g^2} - \left(\frac{af}{g} - \frac{ha^3}{g^3}\right) \times \log\frac{-\frac{ha^2}{g^2} + \frac{af}{g}}{\sqrt{\frac{a^2f^2}{g^2} - \frac{h^2a^4}{g^4}}} + \left(\frac{af}{g} - \frac{ha^3}{g^3}\right) \times \log\frac{-\frac{ha^2}{g^2} - 2a + \frac{a}{g}(f+2a)}{\sqrt{\frac{a^2f^2}{g^2} - \frac{h^2a^4}{g^4}}}\right). \tag{8}$$

The proofs have been verified, allowing us to discover many typographical errors in the first published edition. These are not present in Du Châtelet's manuscript. Additionally, in the manuscript, two minor mistakes have been found in the last expression of the proof. A list of *errata* has been compiled in annexes A and B.

Clairaut changed some letters in the published version from the manuscript, possibly causing him some confusion. This would explain part of the errors.

C. Comparison

As we have seen, both authors first solve the attraction of a circle, and then use it to calculate the attraction of a spheroid. Regarding solving the attraction of a circle, at a general level, both authors also follow similar paths. Nevertheless, Du Châtelet explicitly states why she ignores the parallel component of the force, while Newton does not and uses it implicitly.

The main difference is in the methods used: Newton uses Euclidean geometry, a synthetic approach, and his method of first and last ratios, while Du Châtelet uses analytic geometry, algebra, and Leibniz's calculus.

Except for the spheroid, Newton gives methods to find the attraction instead of expressions. However, all of them depend on being able to calculate the area under a certain curve constructed *ad hoc*, for which he does not give a method. On the other hand, Du Châtelet does not need to introduce auxiliary geometrical constructs, giving the solutions as closed algebraic expressions.

Thus, calculus is a helpful abstraction to replace Newton's *ad hoc* geometric constructions. It also lets Du Châtelet solve the problem of the spheroid without having to deal first with a generic solid of revolution. Furthermore, recalling the hypothesis that Newton might be taking some expressions for quadratures from his results in *Tractatus de Quadratura Curvarum*, one could argue that, in the end, calculus—whether Leibnizian or Newtonian—is the right tool to approach these problems.

Regarding the attraction of a spheroid, Du Châtelet gives a detailed and extensive proof that can easily be followed. Newton, however, presents the result without a proof, and only for an oblate spheroid. He only gives a detailed proof for the case that the corpuscle is inside the spheroid, which Du Châtelet does not address. Nevertheless, using Newton's argument that the forces exerted by the outer shells cancel out, one can easily obtain the same result by using Du Châtelet's expressions (7) or (8) with the inner spheroid. This shows the versatility of her solutions since the same expression can be used to give the solution for both Newton's corollaries.

IV. CONCLUSIONS

- We have provided additional details to facilitate following both developments, as well as putting them into context via the ideas given by several historians.
- We can confirm that Du Châtelet solved a problem whose proof is not provided by Newton in its entirety.
- The comparison shows that hers is clearer, with a shorter path to resolution. Her mathematical methods give a broader physical vision of the problem, helping us understand it better. In particular, we have discussed why calculus is well-suited for its resolution.
- We also discovered typographical errors in the *Principes*, attributing them to the editor. Yet, the historical preface says "Clairaut also had the calculations reviewed by a third party [...] so that it is morally impossible that an error of inattention could have slipped into this work" [4, vol. 1, p. ix], thus highlighting the importance of consulting primary sources such as manuscripts.
- Some of the arguments and methods used by both authors are still taught in the Physics degree. This work is important in contextualizing them historically.
- Some topics that have not been studied due to time constraints include previous work on rebuilding the proofs of the *Principia* using calculus, and the section "*De la figure de la terre*" of the commentary, where Du Châtelet gives more results on spheroids.
- In conclusion, this paper highlights the importance of Du Châtelet's solution to the problem studied and how she contributed to it further than Newton did.

ACKNOWLEDGMENTS

I want to wholeheartedly thank my advisor, Enric, for being very supportive and for the enlightening conversations we have had while researching this topic. I would also like to extend my gratitude to my family and friends—their support has been essential to me.

[1] A. Vilanova Martínez, *Émilie du Châtelet (1706–1749) i la seva visió dels Principia Mathematica de Newton*, Bachelor's thesis, Facultat de Matemàtiques i Estadística, Universitat Politècnica de Catalunya (2025).
[2] M. Touzery, Émilie du Châtelet, un passeur scientifique au XVIIIe siècle: D'Euclide à Leibniz, La Revue pour l'histoire du CNRS (2008).
[3] I. Newton, I. B. Cohen, A. Whitman, and J. Budenz, *The Principia: The Authoritative Translation and Guide: Mathematical Principles of Natural Philosophy* (University of California Press, 1999).
[4] É. du Châtelet, *Principes mathématiques de la philosophie naturelle* (Éditions Jacques Gabay, 1990).
[5] É. du Châtelet, « Principes mathématiques de la philosophie naturelle, par M. NEWTON, traduits en français par Madame la marquise DU CHASTELLET, avec un comentaire sur les propositions qui ont raport au sistème du monde. » Français 12268, Département des Manuscrits, Bibliothèque nationale de France, Paris (1749).
[6] D. Tur Otero, *On a disputed argument in Galileo's Discorsi*, Bachelor's thesis, Facultat de Física, Universitat de Barcelona (2024).
[7] C. Pask, *Magnificent Principia* (Prom. Books, 2013).
[8] J. P. Zinsser, Translating Newton's 'Principia': The Marquise du Châtelet's revisions and additions for a French audience, Notes and Records of the Royal Society of London **55**, 227 (2001).
[9] W. Johnson and S. Chandrasekar, Voltaire's contribution to the spread of Newtonianism—II. Élémens de la philosophie de Neuton: The elements of the philosophy of sir Isaac Newton, International Journal of Mechanical Sciences **32**, 521 (1990).
[10] I. Newton, F. Jacquier, and T. Leseur, *Philosophiae naturalis principia mathematica. Perpetuis commentariis illustrata*, Vol. 1 (A. et J. M. Duncan, 1822) pp. 395–400.
[11] A. N. Kriloff, On Sir Isaac Newton's formula for the attraction of a spheroid on a point of its axis, Monthly Notices of the Royal Astronomical Society **85**, 571 (1925).
[12] S. Chandrasekhar, *Newton's principia for the common reader* (Clarendon Press, 1995) pp. 313–316.
[13] F. Cajori, *A History of Mathematical Notations*, Vol. 2 (The Open Court Publishing Company, 1929) p. 9.

Émilie du Châtelet i els *Principia* de Newton: de la geometria al càlcul

Author: Adrià Vilanova Martínez
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Enric Pérez Canals

Resum: Aquest article analitza i compara la resolució feta per Newton als *Principia* i Du Châtelet al comentari dels seus *Principes* del problema de trobar l'atracció exercida per un esferoide en un corpuscle que es troba al seu eix. Trobem que la solució de Du Châtelet és més simple i completa.
Paraules clau: Història de la Física, Mecànica, Mecànica celeste, Física al segle XVIII.
ODSs: Aquest TFG està relacionat amb els Objectius de Desenvolupament Sostenible (SDGs).

OBJECTIUS DE DESENVOLUPAMENT SOSTENIBLE (ODSS O SDGS)

1. Fi de les desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere	X	14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

Appendix A: *Errata* of the published *Principes* [4]

Page 169, line 11: *instead of* $AmoD$, *read* $MmoD$.

Page 180, line 8: in the first term of the square root, *instead of* $\frac{aa}{ff}$, *read* $\frac{aaff}{gg}$.

Page 180, line 11: in the last term, *instead of* $-\frac{ha^2}{gg}$, *read* $-\frac{fa^2}{gg}$.

Page 181, line 3: in the second term, *instead of* $-\frac{haa}{gg}$, *read* $-\frac{faa}{gg}$.

Page 182, line 2: multiply the entire expression by $\frac{c}{r}$.

Page 182, line 6: in the first term of the logarithm, *instead of* $-\frac{ha^2g}{gg}$, *read* $-\frac{ha^2}{gg}$.

Page 182, line 12: inside the logarithm, *instead of* $-\frac{haa}{gg}-\frac{af}{g}$, *read* $-\frac{haa}{gg}+\frac{af}{g}$.

Page 183, line 1: inside the logarithm, *instead of* $\overline{f-2a}$, *read* $\overline{f+2a}$.

Page 183, line 2: inside the logarithm, *instead of* $-\frac{ha^2}{gg}+\frac{af}{g}$, *read* $\frac{ha^2}{gg}-\frac{af}{g}$.

Appendix B: *Errata* of manuscript *Fr 12268* [5]

Page 66, line 4: inside the logarithm, *instead of* $-2d$, *read* $-\frac{hd^2}{g^2}-2d$.

Page 66, line 5: in the first term, *instead of* a^2d^2, *read* $\frac{d^2a}{c^2}$.