Stabilizer codes and absolutely maximally entangled states for mixed-dimensional systems

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A major difficulty in quantum computation is the ability to implement fault tolerant computations, protecting information against undesired interactions with the environment. The theory of stabiliser codes has been developed over recent years which protects information when storing or applying computations in Hilbert spaces where the local dimension is fixed, i.e. in Hilbert spaces of the form $(\mathbb{C}^D)^{\otimes n}$. If D is a prime power then one can consider stabiliser codes over finite fields [KKKS06], which allows a deeper mathematical structure to be used to develop stabiliser codes. However, there is no practical reason that the subsystems should be required to have the same local dimension and in this work, we introduce a stabiliser formalism for mixed dimension Hilbert spaces, i.e. of the form $\mathbb{C}^{D_1} \otimes \cdots \otimes \mathbb{C}^{D_n}$. We redefine entanglement measures for these Hilbert spaces and follow [HESG18] to define absolutely maximally entangled states as states which maximize this entanglement measure, and give an example of such a state on a mixed dimension Hilbert space.

Keywords: Stabilizer codes, Entanglement measures, Absolute Maximal Entanglement, Mixed dimensional systems

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1 Introduction

Ever since Claude Shannon introduced his mathematical theory for communication on fundamentally noisy channels[Sha48], error correcting codes have been studied for classical channels to trade off bandwidth for uncertainty. From early single-error-correcting Hamming codes[Ham50] and multi-error-correcting Golay codes[Gol49] to high-tolerance polynomial-based Reed-Solomon codes[RS60] for storage applications and efficiently decoded convolutional codes[Vit03] for real-time communication applications, to the Turbo codes supplanting them approaching the theoretical limit using iterative decoding processes,[BGT93] classical error correction made leaps and bounds that informed the development of quantum error correction methods[HDB07, Got96, Pra20, LXW08, AGK+07, FGG07, WHB13], including the correspondence of classical additive codes and quantum stabilizer codes[Got97], and the study of block code weight enumerators[SL97, Rai02b, Rai02a] leading to non-existence results[Rai02c] for both. The inherent need to correct for imperfect quantum processes as well as natural decoherence of information is as intrinsic a motivation for quantum error correction as Johnson-Nyquist thermal noise is to classical error correction, as the cost of taking advantage of superposition and entanglement properties which enable quantum computing theoretical algorithmic advantages[Gro96]. Throughout many branches of quantum sciences, entanglement shows up as a valuable resource in application, algorithm, and analysis. Absolutely Maximally Entangled (AME) states, then, enable various applications even moreso by taking this notion to the extreme, such as quantum secret sharing[SWGW25, HCL+12], open-destination teleportation[HC13], and holographic quantum error correction[PYHP15]. Where first understandings of maximal entanglement yield Bell states and GHZ states, which perfectly determine correlations on remaining subsystems when measuring one qubit or qudit, absolutely maximally entangled states guarantee perfect correlations between all choices of bipartition for the systems involved. Due to A. J. Scott, it is known there is in fact a direct correspondence between requiring all these correlations on such states, to maximally distance separable (MDS) stabilizer codes[Sco04], meaning it is at once both possible to use non-existence results of one to show non-existence of the other, as well as the explicit constructions for one to directly construct the other. In particular, all known results so far cover only specific instances or classes of Hilbert spaces, and there are even fewer published results generalizing to systems consisting of a heterogeneous mix of qudits. In this work, we explore this relation between AME states and MDS codes on mixed-dimensional systems, generalize to mixed dimensionality both the notions of (maximal) distance for error-correcting codes and a class of multipartite entanglement measures based on the subsystem linear entropy, and provide an unconventional stabilizer construction for an explicit MDS code, corresponding to an absolutely maximally entangled state over a mix of qubits and qutrits.

2 Background

2.1 Generalizations of Pauli operators

The Pauli operators combined form an orthonormal, Hermitian, traceless, and local-error basis for operators on systems of arbitrary numbers of qubits when combined with the identity and pairwise commute up to a phase, which are five properties helpful for different respective reasons:

- Their orthonormality and completeness as a basis means any operator can be uniquely decomposed as a linear combination of them, which allows substitution of treating only local-Pauli operators in place of treatment of arbitrary operators.
- Their Hermitivity means they directly correspond to physical observables and do not require additional ancillas or basis transformations to measure.
- The tracelessness of individual Paulis means all operators describable as a tensor product of at least one are traceless as well, leaving only the identity as non-traceless in the basis of operators.
- The ability to compose a local-error basis for arbitrary numbers of qubits is tied to simple recursive analyses.
- The proportionality of E_1E_2 to E_2E_1 for all errors in the basis is tied to simple analyses of products of basis elements involving commutation relations.

To extend this to qudits of prime local dimensions, there are two main families of matrices due to Murray Gell-Mann and James Joseph Sylvester which manage to preserve most of the above properties of niceness, sacrificing one in exchange for arbitrary dimensionality.

The generalized Gell-Mann matrices are orthonormal, traceless aside from identity, and Hermitian, which is useful in expressions involving error transposes. Their constructions can be conceptualized in three families:

X-like matrices: For each pair of off-diagonal entries $e_{jk}, e_{kj}, j \neq k$, there is a basis matrix with those two entries set to 1 and all other entries set to 0.

Y-like matrices: For each pair of off-diagonal entries $e_{jk}, e_{kj}, j \neq k$, there is also a basis matrix with those two entries set to $+i$ and $-i$ so that it is Hermitian, and all other entries set to 0.

Z-like matrices: For each $k \leq d$, there is a basis matrix with all entries $e_{jj} = 1, j \leq k$, $e_{k+1,k+1} = -k$ set to make the matrix traceless, and a normalization factor of $\sqrt{\frac{2}{k(k+1)}}$, where for $k = d$ this is instead the identity for d dimensions.

The other family of generalizations is the Weyl-Heisenberg matrices, which are orthonormal, traceless, and form a multiplicative group of order d for the additive basis with only two generating elements, which is useful in expressions involving exchanges of error order. Their constructions are in general thus:

Shift operator: like the Pauli X operator, the shift operator takes $|j\rangle$ to $|j+1 \mod d\rangle$.

Clock operator: like the Pauli Z operator, the clock operator takes $|j\rangle$ to $\omega^j |j\rangle$, where $\omega = e^{\frac{2\pi i}{d}}$ is the d-th root of unity.

Since both of these constructions led to d^2 orthonormal matrices, they do indeed form complete bases for all $d \times d$ matrices.

2.2 Quantum Error Correcting Codes

A quantum code embedded in a Hilbert space is a subspace of k dimensions, which allows for the encoding of k orthonormal logical states, also referred to sometimes as codewords as in classical coding theory. The use of a proper subspace instead of using the k-dimensional Hilbert space directly means some physical errors can take a system into the code's orthogonal space. Any error that does so, is said to be **detectable**. Formally, an error E is detectable if and only if for any two logical states $|i\rangle, |j\rangle$ of an orthonormal basis spanning the code $\mathcal{Q}$,

$$\langle j| E |i\rangle = C(E)\delta_{ij} \tag{1}$$

with $C(E)$ a constant depending on E. This allows a **syndrome** measurement to be made without destroying the logical state. A syndrome measurement is a combination of measurements which partition the full physical Hilbert space into subspaces of states which yield the same measured outcomes, one of which contains all of the code's logical states and only those logical states. Then, along with the defined subspaces, there are operators which act to move between those subspaces, which can be applied depending on the syndrome readout to send the physical state into the code space.

Additionally, if we make the assumption that under a local-error model it is exponentially unlikely to have errors occurring with higher weight, where weight is defined as the number of parties the overall error acts nontrivially upon, we can heuristically choose to send a state back into the code space using the lowest weight operators possible, probabilistically favouring it over higher-weight operators. This informs an approach of separating into a set of errors we make ourselves resistant to, and a set of errors we accept remaining vulnerable to. Formally, a set of errors $\mathcal{E}$ is said to be **correctable** by $\mathcal{Q}$ if and only if for all $E_1, E_2 \in \mathcal{E}$,

$$\langle j| E_2^\dagger E_1 |i\rangle = C(E)\delta_{ij} \tag{2}$$

Additionally, a code is said to be **pure** if $C(E) = 1$ only for $E = I$, and $C(E) = 0$ otherwise.

A $((n, K, d))_D$ quantum-error-correcting code is defined as a a K-dimensional subspace of $(\mathbb{C}^{\mathbb{D}})^{\otimes n}$ spanned by an orthonormal logical basis $\{|j_L\rangle \,|j = 0, \ldots, K-1\}$ such that

$$\langle j_L| E |i_L\rangle = C(E)\delta_{ij} \tag{3}$$

for all $E \in \mathcal{E}$ for some error basis $\mathcal{E}$, that spans all operators over $(\mathbb{C}^{\mathbb{D}})^{\otimes n}$ with weight less than d. An $((n, 1, d))_D$ code is required to be pure by convention.

$((n, K, d))_D$ Quantum-error-correcting codes obey the **quantum Singleton bound**[Jos58]:

$$K \leq D^{n-2d+2} \tag{4}$$

and codes which saturate this bound are called **Maximally Distance Separable** (MDS) codes. Quantum Maximally Distance Separable codes have been proven to be pure in [Rai99], and each correctable error corresponds to exactly one unique syndrome readout[KKÖ15].

2.3 Stabilizer codes

One notable class of codes conducive to error correction is constructed with respect to its **stabilizers**: for a set $\mathcal{S}$ of unitary operators over a Hilbert space $\mathbb{H}$, define the corresponding **stabilizer code** $\mathcal{Q}_\mathcal{S}$ as the joint $+1$ eigenspace of all of $\mathcal{S}$. That is:

$$\mathcal{Q}_\mathcal{S} := \{v \in \mathbb{H} | Uv = v, \forall U \in \mathcal{S}\} \tag{5}$$

Because the defining condition for any two operators $U_1, U_2 \in \mathcal{S}$, $U_1 U_2 v = U_1(U_2 v) = U_1 v = v$ for all v, so $\mathcal{S}$ can be extended to include $U_1 U_2$ without removing any vector from $\mathcal{Q}_\mathcal{S}$. Also, because the definition is a joint eigenspace, no vector can be added to $\mathcal{Q}_\mathcal{S}$ by extending $\mathcal{S}$. Then, we can extend $\mathcal{S}$ to the multiplicative group generated by its elements without changing $\mathcal{Q}_\mathcal{S}$, and it suffices to define the stabilizer group by a set of linearly independent generators.

On the other hand, because $\mathcal{Q}_{\mathcal{S}}$ is defined as nothing less and nothing more than the whole subspace stabilized by $\mathcal{S}$, we have

$$\langle v|\, U\, |v\rangle = 1, \forall U \in \mathcal{S} \tag{6}$$

and can define

$$\mathcal{Q}_{\mathcal{S}}^{\perp} := \{|u\rangle \in \mathbb{H} | \langle u|v\rangle = 0, \forall\, |v\rangle \in \mathcal{Q}_{\mathcal{S}}\} \tag{7}$$

for which

$$\forall v \in \mathcal{Q}_{\mathcal{S}}^{\perp}, \exists U \in \mathcal{S}, \text{s.t. } \langle v|\, U\, |v\rangle \neq 1 \tag{8}$$

That is, any error that projects off of the code space by definition requires it to have a nonzero probability in an observable outcome different from $+1$ for some stabilizer in $\mathcal{S}$, and it is safe to measure all of them for any state in the code space, without disturbing the code state.

Then, a set of errors that do not commute with all generators of $\mathcal{S}$ is detectable. For each distinct syndrome, a correction can be made that maps a coset of logically equivalent states into the set of codewords.

Taken together, this suggests measuring each element of the stabilizer in some way constitutes a projector for the code space, a **code projector**. To see this, let $P_{\mathcal{S}}$ be the unweighted average over $\mathcal{S}$:

$$P_{\mathcal{S}} := \frac{1}{|\mathcal{S}|} \sum_{M \in \mathcal{S}} M \tag{9}$$

Theorem 1. *The dimension of $\mathcal{Q}_{\mathcal{S}}$ is*

$$\frac{1}{|\mathcal{S}|} \sum_{M \in \mathcal{S}} \operatorname{tr}(M). \tag{10}$$

Proof. Observe that

$$P_{\mathcal{S}}^2 = \frac{1}{|\mathcal{S}|} \sum_{N \in \mathcal{S}} N \frac{1}{|\mathcal{S}|} \sum_{M \in \mathcal{S}} M = \sum_{N, M \in \mathcal{S}} \frac{1}{|\mathcal{S}|^2} NM = \frac{1}{|\mathcal{S}|} \sum_{M \in \mathcal{S}} M = P_{\mathcal{S}} \tag{11}$$

and that

$$\frac{1}{|\mathcal{S}|} \sum_{M \in \mathcal{S}} M \tag{12}$$

fixes any element of $\mathcal{Q}_{\mathcal{S}}$.

Since $P_{\mathcal{S}}^2 = P_{\mathcal{S}}$, the eigenvalues of $P_{\mathcal{S}}$ are zero and one. The image of $P_{\mathcal{S}}$ is its eigenspace of eigenvalue one, which is also $\mathcal{Q}_{\mathcal{S}}$. Thus, $P_{\mathcal{S}}$ is the projector onto the subspace $\mathcal{Q}_{\mathcal{S}}$. Since the eigenvalues of $P_{\mathcal{S}}$ are zero and one, the dimension of the eigenspace of eigenvalue one is equal to the sum of eigenvalues, which is $\operatorname{tr}(P_{\mathcal{S}})$.

$$\dim(\mathcal{Q}_{\mathcal{S}}) = \operatorname{tr}(P_{\mathcal{S}}) = \frac{1}{|\mathcal{S}|} \sum_{M \in \mathcal{S}} \operatorname{tr}(M) \tag{13}$$

□

2.4 Entanglement measures

Entanglement is a quantum property defined abstractly oppositely to the property of systems that can be represented as several independent systems together. There is agreement that a **separable** state which can be written entirely as a tensor product is minimally entangled, and a system which has maximal correlation between subsystems is maximally entangled, but the choice of entanglement measures for values in-between have some degree of flexibility.

From [BDSW96] a postulate is established that entanglement measures that capture this genuinely quantum property, are to be defined such that they are non-increasing on average under **local operations and classical communication** (LOCC): for a system $\mathbb{H}^A \otimes \mathbb{H}^B$, entanglement between systems A and B do not increase by operators of form $U_A(\langle\psi|_B U_B |\psi\rangle_B) \otimes I_B$, $I_A \otimes U_B(\langle\psi|_A U_A |\psi_A\rangle)$, acting on one side of the bipartition dependent on the classically communicated results of any measurement (including identity) of the other.

To choose an entanglement measure befitting our purpose, we consider that the properties of maximal and minimal entanglement can be thought of in terms of what happens under partial trace:

A minimally entangled pure state of independent systems retains purity in any subsystem S when tracing out its complement, S^C. A maximally entangled pure state becomes maximally mixed in any subsystem S when tracing out a complement S^C of higher dimension. Then, we can use the starting point of linearized subsystem purity, $\operatorname{tr} \rho_S^2$, where $\rho_S = \operatorname{tr}_{S^C} |\psi\rangle\langle\psi|$ is the reduced density operator for the subsystem after partially tracing out S^C. As we are free to define the range of our measure aside from vanishing for separable states, we'll arbitrarily choose [0, 1]. To normalize to a minimum point of 0 from the maximum purity of 1 we change to 1 - $\operatorname{tr} \rho_S^2$, and to normalize to a maximum point of 1 from the minimum purity of $\frac{1}{D_S}$ we multiply by a factor of $\frac{D_S - 1}{D_S}$, where D_S is the dimension of the smaller of the two partitions. Finally, since we are interested in a relevant overall measure for entanglement over the entire system, we wish to align a notion of maximal overall entanglement with reaching maximal entanglement in any choice of bipartition. That means we wish to impose the above on every choice of S, so after taking a normalized sum and putting it all together:

$$Q(\psi) \equiv \frac{1}{2^n} \sum_{S \subseteq \{1,\ldots,n\}} \frac{D_S}{D_S - 1} \left(1 - \operatorname{tr}(\operatorname{tr}_{S^C} |\psi\rangle\langle\psi|)^2\right) \tag{14}$$

Where the normalization factor $\frac{1}{2^n}$ is due to the size of the power set of a set of n parties.

To prepare for a correspondence with the notion of distance, it will also be of use to consider dividing up this normalized sum into a family of entanglement measures due to Scott[Sco04]. For the set of all bipartitions $\{(S, S^C) | |S| = m, m < n\}$, originally defined for a uniform system of n qudits of dimension D:

$$Q_m(\psi) = \frac{1}{\binom{n}{m}} \sum_{|S|=m} \frac{D_S}{D_S - 1} \left(1 - \mathrm{tr}(\mathrm{tr}_{S^C} |\psi\rangle\langle\psi|)^2\right) \tag{15}$$

$$= \frac{m!(n-m)!}{n!} \sum_{|S|=m} \frac{D^m}{D^m - 1} \left(1 - \mathrm{tr}(\mathrm{tr}_{S^C} |\psi\rangle\langle\psi|)^2\right) \tag{16}$$

$$= \frac{D^m}{D^m - 1} \left(1 - \frac{m!(n-m)!}{n!} \sum_{|S|=m} \mathrm{tr}(\mathrm{tr}_{S^C} |\psi\rangle\langle\psi|)^2\right) \tag{17}$$

Where the reduced final form is the original given by Scott, more obviously starring the average linear subsystem entropy.

2.5 Absolutely Maximally Entangled States

With the choice of multipartite entanglement measures defined above, we can define a class of **absolutely maximally entangled** (AME) states, which are pure states that saturate the limit of $Q(\psi) = 1$, as well as a class of m-**uniform** states, which are states over qudits of equal dimension that saturate the limit of $Q_m(\psi) = 1$. An absolutely maximally entangled state over qudits of equal dimension is then also m-uniform for all $m < \lfloor \frac{n}{2} \rfloor$.

Interestingly thanks to Scott[Sco04, Proposition 3], there is a known bijective relation between AME states and QMDS stabilizer codes on uniform-dimension systems which we replicate below, in order to extend to mixed-dimensional systems in Theorem 6

Theorem 2. *$Q_m(\psi) = 1 \iff |\psi\rangle$ is a pure $((n, 1, m+1))_D$ quantum-error-correcting code.*

Proof. Let us consider an element of the orthonormal basis of displacement operators $\mathcal{D}(\mu, \nu)$ with support S, $\mathrm{wt}[\mathcal{D}(\mu, \nu)] \equiv |S| = k \leq m$, such that $\mathcal{D}(\mu, \nu) = I_{S^C} \otimes \mathcal{D}_E$, where $\mathcal{D}_E = \mathrm{tr}_{S^C} \mathcal{D}(\mu, \nu)$ is the local error operator reduced to only where it acts nontrivially, and I_{S^C} is the identity on S^C and has dimension D^{n-k}. This basis is chosen such that it is traceless over any non-empty subset of its support. Note also that for an m-uniform state, by definition, $\mathrm{tr}_{S^C} |\psi\rangle\langle\psi| = D^{-k} I$.

$$\langle\psi| \mathcal{D}(\mu, \nu) |\psi\rangle = \mathrm{tr}[|\psi\rangle\langle\psi| \mathcal{D}(\mu, \nu)] \tag{18}$$

$$= \mathrm{tr}_S[\mathrm{tr}_{S^C}[|\psi\rangle\langle\psi| \mathcal{D}(\mu, \nu)]] \tag{19}$$

$$= \mathrm{tr}_S[\mathrm{tr}_{S^C}[|\psi\rangle\langle\psi| (\mathcal{D}_E \otimes I_{S^C})]] \tag{20}$$

$$= D^{-k} \mathrm{tr}_S[I_S \mathcal{D}_E] \tag{21}$$

$$= \delta_{\mu\mathbf{0}} \delta_{\nu\mathbf{0}} \tag{22}$$

Which is indeed the quantum-error-correcting code condition for the set of errors with weight m or less.

Since the basis chosen is an orthonormal basis for all operators of dimensions $D^n \times D^n$, we can decompose $|\psi\rangle\langle\psi|$ in it using coefficients $c_{\mu\nu} = \langle\psi| \mathcal{D}(\mu, \nu) |\psi\rangle$:

$$D^n |\psi\rangle\langle\psi| = I + \sum_{1 \leq \mathrm{wt}[\mathcal{D}(\mu,\nu)] \leq m} c_{\mu\nu} \mathcal{D}(\mu, \nu) + \sum_{m+1 \leq \mathrm{wt}[\mathcal{D}(\mu,\nu)] \leq n} c_{\mu\nu} \mathcal{D}(\mu, \nu) \tag{23}$$

Where we have chosen the first sum such that all of its coefficients are zero due to the quantum-error-correcting code condition. Then, tracing down to m or fewer parties is traceless for everything in the second sum, because a local displacement operator must be traced over, and all local displacement operators are themselves traceless. So, we are left with only

$$\mathrm{tr}_{S^C} |\psi\rangle\langle\psi| = D^{-k} I \tag{24}$$

This holding for all choices of S with $k = m$ is the condition of m-uniformity, and yields $Q_m(\psi) = 1$.

□

2.6 Shadow inequalities

Due to Rains[Rai02c], it is known that for all positive semi-definite Hermitian operators M and N on parties $(1, ..., n)$ of dimensions $(d_1, d_2, \ldots, d_n)$ and any choice of subset $T \subseteq \{1, ..., n\}$,

$$\sum_{S\subseteq\{1,...,n\}} (-1)^{|S\cap T|} \mathrm{tr}_S \left[\mathrm{tr}_{S^C}(M)\, \mathrm{tr}_{S^C}(N)\right] \geq 0 \tag{25}$$

In particular, choosing $M = N = \rho = |\psi\rangle\langle\psi|$ reduces this to

$$\sum_{S\subseteq\{1,...,n\}} (-1)^{|S\cap T|} \mathrm{tr}_S \left[\rho_S^2\right] \geq 0 \tag{26}$$

Which is a constraint on subsystem purities with an exponential number of terms. While this is true in general, there are simplifications which can be made for uniform-dimensional cases.

Consider a hypothetical AME state on a system of n qudits of equal dimension D. By the definition of AME states, reduction to any bipartition $\{S, S^C\}$ yields a density operator of rank $\min\{|S|, \left|S^C\right|\}$ with equal eigenvalues, which results in a partial trace square of

$$\mathrm{tr}_S[\mathrm{tr}_{S^C}[\rho]^2] = \frac{1}{D^{\min\{|S|,|S^C|\}}} \tag{27}$$

Then, we can organize the terms in this sum by size, as all choices of $|S| = m$ yield the same contribution magnitude, though not necessarily sign, and the sum is symmetrical between choices of size $|n/2 - m|$. Consider the terms with $|S| = m, m+1 \leq n-m-1$. There are $\binom{n}{m}$ ways to choose S, each of which yields a term with magnitude $\frac{1}{D^m}$. There are $\binom{n}{m+1}$ terms for the next sizes of subsets, each with magnitude $\frac{1}{D^{m+1}}$. There are $(n-m)/(m+1)$ as many terms of size $|S| = m + 1$ as there are of size $|S| = m$, after canceling out parts of factorials. The magnitude of each $|S| = m + 1$ has a magnitude $\frac{1}{D}$ smaller than each $|S| = m$. The total of the contribution magnitudes of a level is greatest when its ratio to the previous level is not less than 1, and its ratio to the next level is not more than 1.

That means the comparison we care about is $(n - m)/D(m + 1)$ and 1, or equivalently $n - m$ and $D(m + 1)$. Setting them equal gives a threshold of

$$n - m = D(m + 1) \iff n = (D + 1)m + D \iff m = (n - D)/(D + 1) \tag{28}$$

For $D = 2$, that means $n = 4$ gives $m = \lceil 2/3 \rceil = 1$ yields the most dominant level of contributions. Choosing $T = \{1, 2, 3, 4\}$ to be the full system makes the signs of all contributions only dependent on m, $(-1)^{|S \cap T|} = (-1)^m$, which maximizes negative contributions from the dominant level. Indeed, choosing $T = \{1, 2, 3, 4\}$ gives that a four-qubit AME state is forbidden with a sum of -1/2.

Choosing $T = \{1, 2, 3, 4\}$ also yields a negative sum for systems on $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, and $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, but not $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. There are no other ways to choose T such that a lower sum is yielded in any of the above systems, and indeed we will see it is possible to construct an AME state for $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, once we establish some extensions for our machinery into mixed dimensionality.

3 Extension to mixed dimensions

3.1 Absolutely Maximal Entanglement in mixed dimensions

For a system of parties with mixed dimensions $D_1, D_2, D_3, \ldots D_n$, we extend the notion of absolutely maximally entangled states in the same sense as Huber et al.[HESG18], that tracing out a subsystem S^C of dimension $D_{S^C} = \prod_{i \in S^C} D_i$ equal to or larger than that of its complement $D_S = \prod_{i \in S} D_i$ always leaves it maximally mixed. Equivalently to $D_S \leq D_{S^C}$, since $D_S D_{S^C} = \prod D_i$, the condition can be written:

$$\forall S \subset \{1, 2, 3 \ldots n\}, D_S \leq \sqrt{\prod D_i}, \operatorname{tr}_{S^C} |\psi\rangle\langle\psi| = \frac{1}{D_S} I \tag{29}$$

3.2 Entanglement measure

For general systems of mixed-dimensional parties, choosing the same number of parties does not always yield the same dimension of subsystem. Worse, some such choices may yield the smaller subsystem and some the larger. It makes more sense for our purposes then, to define a family of entanglement measures analogously based on subsystem dimension instead. Let

$$Q_M(\psi) := \frac{1}{f} \sum_{\dim S = M} \frac{M}{M-1} (1 - \operatorname{tr} \rho_S^2) \tag{30}$$

Where f is the number of ways to choose any number of parties with dimensions factorizing M.

To justify this is indeed a entanglement measure, we now show three essential properties.

Define $|\psi\rangle$ to be an M**-separable** state if $|\psi\rangle$ is a product state on any bipartition of dimensions M, $\left(\prod D_i\right)/M$. That is,

$$\forall S \text{ s.t. } |S| = M, \exists \, |\psi_A\rangle_S \in \mathbb{H}_S, |\psi_B\rangle_{S^C} \in \mathbb{H}_{S^C}, |\psi\rangle = |\psi_A\rangle_S \otimes |\psi_B\rangle_{S^C} \tag{31}$$

Lemma 3 (Vanishing for product states)**.** $Q_M(|\psi\rangle) = 0$ *if and only if* $|\psi\rangle$ *is* M*-separable.*

Proof. For an M-separable state $|\psi\rangle$, any choice of bipartition of dimensions M, $\left(\prod D_i\right)/M$ yields a pure state on each subsystem when partially tracing out the other, leading to every term in the measure evaluating to 0.

For any choice of S, the Schmidt decomposition of an arbitrary $|\psi\rangle$ can be written

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |\psi_i\rangle_S |\phi_i\rangle_{S^C} \tag{32}$$

with some integer n, $\sum_{i=1}^{n} |\lambda_i|^2 = 1$, some orthonormal states $\{|\psi_i\rangle_S\}$ over subsystem S, and some orthonormal states $\{|\phi_i\rangle_{S^C}\}$ over its complement subsystem S^C,

$$1 - \operatorname{tr}\left(\rho_S^2\right) = 1 - \sum_{i=1}^{n} |\lambda_i|^4 \tag{33}$$

With the quadratic normalization over eigenvalues, this equation only equals 0 if there is only one non-zero eigenvalue, $\{\lambda_i\} = \{1\}$, meaning the decomposition is a product state, $|\psi\rangle = |\psi_A\rangle_S \otimes |\psi_B\rangle_{S^C}$.
This measure being a weighted average of non-negative numbers, as the trace of any state squared cannot exceed 1, means

$$Q_M(\psi) = 0 \implies (1 - \operatorname{tr}\rho_S^2) = 0, \forall S \text{ s.t.} \dim S = M \tag{34}$$

thereby requiring $|\psi\rangle$ to be a product state for any choice of S with dimension M. □

Lemma 4. *The proposed entanglement measure is 1 if and only if $|\psi\rangle$ is maximally entangled for any choice of S with dimension M.*

Proof. The minimal purity for a density operator on a subsystem of dimension M is for a mixed state of M equal eigenvalues, $|\lambda_i|^2 = \frac{1}{M}$, which has

$$\operatorname{tr}\left(\rho^2\right) = \sum_{i=1}^{M} |\lambda_i|^4 = \frac{1}{M} \tag{35}$$

which means the maximal value for $(1 - \operatorname{tr}\{\rho_S^2\})$ is $\frac{M-1}{M}$, and normalizing with the prefactor means the maximal value for each term of the sum is 1.

If $|\psi\rangle$ is maximally entangled for any choice of S with dimension M, $\operatorname{tr}_{S^C} |\psi\rangle\langle\psi|$ is a maximally mixed state $\mathbb{I}/M$ and satisfies $Q_M(\psi) = 1$.
Additionally, as this is the maximal value for an sum,

$$Q_M(\psi) = 1 \implies \frac{M}{M-1}(1 - \operatorname{tr}\rho_S^2) = 1, \forall S \text{ s.t.} \dim S = M \tag{36}$$

thereby requiring $\rho_S = \operatorname{tr}_{S^C} |\psi\rangle\langle\psi|$ to be a maximally mixed state $\mathbf{I}/M$ for any choice of S with dimension M. □

Lemma 5. *The proposed entanglement measure is non-increasing on average under LOCC.*

Proof. Due to [Życ03] it is established that the linear entropy is non-increasing under LOCC. As $Q_M(\psi)$ is a linear sum of non-increasing functions, it is also non-increasing. □

Then, we can also define M-**uniform states** as states $|\psi\rangle$ which saturate the bound $Q_M(\psi) = 1$. Note that while m-uniform states on uniform-dimensional systems imply $m-1$ uniformity, the nature of mixed-dimensional systems mean M-uniformity does *not* imply $M-1$-uniformity. Instead, a state which is M-uniform is also M/D_i-uniform, for all $D_i \in (D_1, \ldots D_n)$ which factor M.

Finally, note that while these measures relax some conditions for the states corresponded to by states which fulfill their maximal and minimum values, it is possible to write the full average (14) as

$$Q(\psi) = \sum_{M=0}^{\prod D_i} \frac{1}{|\{S \in \{1, \ldots n\} | \dim(S) = M\}|} Q_M(\psi) \tag{37}$$

which does recover the properties of bijection between $Q(\psi) = 0$ and $|\psi\rangle\langle\psi|$ being fully separable and between $Q(\psi) = 1$ and $|\psi\rangle\langle\psi|$ being absolutely maximally entangled.

3.3 Quantum-error-correcting codes

Similar to entanglement measures, while it is still possible to preserve the same notion of distance, it may prove more useful to define a metric based on the minimum dimensions affected for an uncaught error rather than the number of parties. We define the **dimensional weight** as

$$\mathrm{dimwt}(E) = \prod_{i \in \mathrm{supp}(E)} D_i \tag{38}$$

Let $\mathcal{Q}$ be a K-dimensional subspace of $\mathbb{H} = \mathbb{C}^{D_1} \otimes \ldots \mathbb{C}^{D_n}$ spanned by the orthonormal basis $\{|j_L\rangle \,| j = 0, \ldots, K-1\}$, and $\mathcal{E}$ an orthonormal basis spanning the space of operators over $\mathbb{H}$, in which all elements are traceless over any subsets of its support. We define $\mathcal{Q}$ to be a $(((D_1, \ldots D_n), K, \Delta))$ quantum-error-correcting code if

$$\langle j_L| E |i_L\rangle = C(E)\delta_{ij} \tag{39}$$

for all $E \in \mathcal{E}$ for an error basis $\mathcal{E}$ spanning all operators over $\mathbb{H}$ with $\mathrm{dimwt}(E) < \Delta$ and $0 \leq i, j \leq K-1$. The corresponding correction capability is that a $(((D_1, \ldots D_n), K, \Delta))$ QECC can detect and recover all errors acting on $< \sqrt{\Delta}$ dimensions.

3.4 QECC-AME bijection

Theorem 6. *$Q_M(\psi) = 1 \iff |\psi\rangle$ is a pure $(((D_1, \ldots D_n), 1, M+1))$ quantum-error-correcting code.*

Proof. From the definition above, it is now plain to see we can rearrange the decomposition of a $(((D_1, \ldots D_n), 1, M+1))$ QECC $|\psi\rangle\langle\psi|$ in basis $\mathcal{E} = \{E_i\}$:

$$\left(\prod D_i\right) |\psi\rangle\langle\psi| = I + \sum_{1 \leq \mathrm{dimwt}(E) \leq M} c_E E \; + \sum_{M+1 \leq \mathrm{dimwt}(E) \leq \dim(\mathbb{H})} c_E E \tag{40}$$

where $c_E = \langle\psi| E |\psi\rangle$ is zero in the first sum. Tracing down to M or fewer dimensions requires tracing over a non-empty subset of the support of any operator in the second sum, which is always traceless. Then,

$$\rho_S = \mathrm{tr}_{SC} |\psi\rangle\langle\psi| = D^{-M} I_S \tag{41}$$

whenever $\prod_{i\in S} D^i = M$. Thus $|\psi\rangle$ is M-uniform and $Q_M(\psi) = 1$.

To show the converse, consider an element E_i of the orthonormal basis $\mathcal{E}$, with $\text{dimwt}(E) = D_S \leq M$, such that $E_i = I_{S^C} \otimes E_{iS}$, where $S = \text{supp}(E)$, $D_S = \prod_{i\in S} D_i$, $E_{iS} = \text{tr}_{S^C} E$ is the nontrivial portion of E, and I_{S^C} is the identity on S^C and has dimension $\prod_{i\notin S} D_i$. Note also that for an M-uniform state, by definition, $\text{tr}_{S^C} |\psi\rangle\langle\psi| = D_S I$.

$$\begin{aligned}
\langle\psi| E_i |\psi\rangle &= \text{tr}[|\psi\rangle\langle\psi| E_i] && (42)\\
&= \text{tr}_S[\text{tr}_{S^C}[|\psi\rangle\langle\psi| E_i]] && (43)\\
&= \text{tr}_S[\text{tr}_{S^C}[|\psi\rangle\langle\psi| E_{iS} \otimes I_{S^C}]] && (44)\\
&= D_S \,\text{tr}_S[I_S E_i] && (45)\\
&= c(E_i) && (46)
\end{aligned}$$

Where $c(E_i)$ is 1 if and only if $E_i = I$ and 0 otherwise, as all other basis elements are traceless. This fulfills the condition for a pure quantum-error-correcting code for the set of errors with dimensional weight M or less. □

□

4 Examples

Consider the mixed system $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, the only 4-party mix of qubits and qutrits for which absolutely maximally entangled states can exist.

4.1 Numerical example

In [HESG18], Huber et al gave an example of an absolutely maximally entangled state on this system found numerically by means of semidefinite programming.

$$\begin{aligned}
|\psi_H\rangle = \alpha(&- |0011\rangle + |0022\rangle + |0102\rangle + |0120\rangle - |0201\rangle - |0210\rangle + \\
&|1012\rangle - |1021\rangle + |1101\rangle - |1110\rangle - |1202\rangle + |1220\rangle) \\
+\beta(&- |0012\rangle + |0021\rangle - |0101\rangle + |0110\rangle + |0202\rangle - |0220\rangle \\
&- |1011\rangle + |1022\rangle + |1102\rangle + |1120\rangle - |1201\rangle - |1210\rangle)
\end{aligned} \quad (47)$$

where two sets of coefficients were specified given by

$$\alpha = \frac{1}{6}\sqrt{\frac{3}{2} \pm \frac{\sqrt{65}}{6}},\ \beta = \frac{1}{54\alpha} = \frac{1}{6}\sqrt{\frac{3}{2} \mp \frac{\sqrt{65}}{6}} \quad (48)$$

corresponding to constraints

$$12(\alpha^2 + \beta^2) = 1,\ 54\alpha\beta = 1 \quad (49)$$

This was achieved with a semi-definite program [Hub17] iteratively alternating maximizing the expectation value of a density operator ρ with respect to a fixed vector $\left|\psi^{(i)}\right\rangle$, and setting $\left|\psi^{(i+1)}\right\rangle$ to the eigenvector corresponding to the maximal eigenvalue of ρ, subject to maximally mixed subsystem constraints and standard quantum normalization constraints, until convergence.

4.2 Stabilizer construction example

We give an example AME state on the same system motivated from a stabilizer code construction, prove that the code spanned by $|\psi\rangle$ can detect all errors of dimensional weight at most 6, which is the largest error within the threshold $\sqrt{\prod D_i} = \sqrt{54}$, and apply Theorem 6 to prove that $|\psi\rangle$ is an absolutely maximally entangled state.

Let

$$|\psi\rangle = \frac{1}{\sqrt{12}}(|0\rangle(|022\rangle + |201\rangle + |120\rangle + |011\rangle + |102\rangle + |210\rangle)) \\ + |1\rangle(-|101\rangle + |110\rangle - |012\rangle + |202\rangle - |220\rangle + |021\rangle)) \tag{50}$$

Firstly, we prove that the subspace spanned by $|\psi\rangle$ is actually a stabilizer code.
Let

$$U_0 = \mathbf{I} \otimes \begin{pmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \tag{51}$$

and observe that $U_0|\psi\rangle = |\psi\rangle$.

To be able to state the other stabilizers of $|\psi\rangle$ we define an orthogonal set for $\mathbb{H}$,

$$\begin{array}{llll} |\phi_1\rangle = |0\rangle|022\rangle & |\phi_7\rangle = -|1\rangle|110\rangle & |\phi_{13}\rangle = |1\rangle|022\rangle & |\phi_{19}\rangle = |0\rangle|110\rangle \\ |\phi_2\rangle = |0\rangle|210\rangle & |\phi_8\rangle = |1\rangle|101\rangle & |\phi_{14}\rangle = |1\rangle|210\rangle & |\phi_{20}\rangle = -|0\rangle|101\rangle \\ |\phi_3\rangle = |0\rangle|102\rangle & |\phi_9\rangle = -|1\rangle|021\rangle & |\phi_{15}\rangle = |1\rangle|102\rangle & |\phi_{21}\rangle = |0\rangle|021\rangle \\ |\phi_4\rangle = |0\rangle|011\rangle & |\phi_{10}\rangle = |1\rangle|220\rangle & |\phi_{16}\rangle = |1\rangle|011\rangle & |\phi_{22}\rangle = -|0\rangle|220\rangle \\ |\phi_5\rangle = |0\rangle|120\rangle & |\phi_{11}\rangle = -|1\rangle|202\rangle & |\phi_{17}\rangle = |1\rangle|120\rangle & |\phi_{23}\rangle = |0\rangle|202\rangle \\ |\phi_6\rangle = |0\rangle|201\rangle & |\phi_{12}\rangle = |1\rangle|012\rangle & |\phi_{18}\rangle = |1\rangle|201\rangle & |\phi_{24}\rangle = -|0\rangle|012\rangle\,. \end{array} \tag{52}$$

To define U_1 we use the notation $1_2 \mapsto 2_3$ to mean that $|1\rangle$ on the second system gets mapped to $|2\rangle$ on the third system, etc. Using this notation

$$U_1 = Z \otimes (1_3 \mapsto 1_2 \mapsto 1_4 \mapsto 2_3 \mapsto 2_2 \mapsto 2_4 \mapsto)(0_3 \mapsto 0_2 \mapsto 0_4 \mapsto). \tag{53}$$

As a permutation of the orthogonal set $|\phi_j\rangle$ this is the permutation (on the indices)

$$(1\ 2\ 3\ 4\ 5\ 6\ )(7\ 8\ 9\ 10\ 11\ 12)(13\ 14^-\ 15\ 16^-\ 17\ 18^-)(19\ 20^-\ 21\ 22^-\ 23\ 24^-). \tag{54}$$

where 14^- indicates $-|\phi_{14}\rangle$.

Note that U_1 fixes ψ since

$$|\psi\rangle = \frac{1}{\sqrt{12}}\sum_{j=1}^{12}|\phi_j\rangle\,. \tag{55}$$

Using the same notation, we define

$$U_2 = ZX \otimes (0_2 2_3 2_4 \mapsto 0_2 2_3 1_4 \mapsto -0_2 1_3 1_4 \mapsto 0_2 1_3 2_4 \mapsto) \\ (1_2 0_3 2_4 \mapsto 2_2 0_3 2_4 \mapsto -2_2 0_3 1_4 \mapsto 1_2 0_3 1_4 \mapsto)(1_2 2_3 0_4 \mapsto 1_2 1_3 0_4 \mapsto -2_2 1_3 0_4 \mapsto 2_2 2_3 0_4 \mapsto) \tag{56}$$

and identity on the remaining elements in the computational basis.

As a permutation of the orthogonal set $|\phi_j\rangle$ this is the permutation (again on the indices)

$$(1\ 9\ 4\ 12)(2\ 10\ 5\ 7)(3\ 11\ 6\ 8)(13\ 21\ 16\ 24)(14\ 22\ 17\ 19)(15\ 23\ 18\ 20). \tag{57}$$

Hence, U_2 also fixes $|\psi\rangle$.

Since U_0 is a local operator whose components are diagonal matrices it commutes with both U_1 and U_2. Furthermore,

$$U_1U_2 = U_2U_1 = (1\ 10\ 6\ 9\ 5\ 8\ 4\ 7\ 3\ 12\ 2\ 11)(13\ 22^-\ 18\ 21^-\ 17\ 20^-\ 16\ 19^-\ 15\ 24^-\ 14\ 23^-) \tag{58}$$

and U_2 acts as the identity on the remaining kets in the computational basis.

We can then define an abelian (commutative) subgroup

$$\mathcal{S} = \langle U_0, U_1, U_2 \rangle \tag{59}$$

of linear operators on $\mathbb{H}$.

Our next step is to prove that the subspace $Q(\mathcal{S})$ of states which are stabilized by $\mathcal{S}$ is the one-dimensional subspace spanned by $|\psi\rangle$. To do this, we calculate the trace of all the elements of S and apply Theorem 1.

Since U_0 has order 4, U_1 has order 6, U_2 has order 4 and $\mathcal{S}$ is Abelian, we have that $|\mathcal{S}| = 96$.

The 48 operators with non-zero trace are listed in the following table. Note that, since U_1 is the traceless Z operator on the qubit system, all other operators in $\mathcal{S}$ will have trace zero.

	tr		tr		tr		tr
$\mathbf{I}$	54	U_0	$2(2i-1)^3$	U_2	30	U_0U_2	$-4i-2$
U_0^2	-2	U_0^3	$2(-2i-1)^3$	$U_0^2U_2$	-26	$U_0^3U_2$	$4i-2$
U_1^2	6	$U_0U_1^2$	$-4i-2$	$U_1^2U_2$	6	$U_0U_1^2U_2$	$-4i-2$
$U_0^2U_1^2$	-2	$U_0^3U_1^2$	$4i-2$	$U_0^2U_1^2U_2$	-2	$U_0^3U_1^2U_2$	$4i-2$
U_1^4	6	$U_0U_1^4$	$-4i-2$	$U_1^4U_2$	6	$U_0U_1^4U_2$	$-4i-2$
$U_0^2U_1^4$	-2	$U_0^3U_1^4$	$4i-2$	$U_0^2U_1^4U_2$	-2	$U_0^3U_1^4U_2$	$4i-2$
U_2^2	30	$U_0U_2^2$	$-4i-2$	U_2^3	30	$U_0U_2^3$	$-4i-2$
$U_0^2U_2^2$	-26	$U_0^3U_2^2$	$4i-2$	$U_0^2U_2^3$	-26	$U_0^3U_2^3$	$4i-2$
$U_1^2U_2^2$	6	$U_0U_1^2U_2^2$	$-4i-2$	$U_1^2U_2^3$	6	$U_0U_1^2U_2^3$	$-4i-2$
$U_0^2U_1^2U_2^2$	-2	$U_0^3U_1^2U_2^2$	$4i-2$	$U_0^2U_1^2U_2^3$	-2	$U_0^3U_1^2U_2^3$	$4i-2$
$U_1^4U_2^2$	6	$U_0U_1^4U_2^2$	$-4i-2$	$U_1^4U_2^3$	6	$U_0U_1^4U_2^3$	$-4i-2$
$U_0^2U_1^4U_2^2$	-2	$U_0^3U_1^4U_2^2$	$4i-2$	$U_0^2U_1^4U_2^3$	-2	$U_0^3U_1^4U_2^3$	$4i-2$

By Theorem 1, the dimension of $Q(\mathcal{S})$ is (summing the sums of the four columns)

$$(72 + 24 + 24 - 24)/96 = 1 \tag{60}$$

since

$$2(2i-1)^3 + 2(-2i-1)^3 = 44. \tag{61}$$

Since we already observed that $|\psi\rangle$ is stabilized by the elements of $\mathcal{S}$, we conclude that

$$Q(\mathcal{S}) = \langle |\psi\rangle \rangle, \tag{62}$$

where

$$|\psi\rangle = \frac{1}{\sqrt{12}}(|0\rangle(|022\rangle + |201\rangle + |120\rangle + |011\rangle + |102\rangle + |210\rangle) \\ + |1\rangle(-|101\rangle + |110\rangle - |012\rangle + |202\rangle - |220\rangle + |021\rangle)) \tag{63}$$

It remains to prove that

$$\langle\psi| E |\psi\rangle = 0 \tag{64}$$

for all operators E of dimensional weight less than $\sqrt{54}$.

By arguments of symmetry, it suffices to consider

$$E \in \{\sigma_1 \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}, \mathbf{I} \otimes \sigma_2 \otimes \mathbf{I} \otimes \mathbf{I}, \sigma_1 \otimes \sigma_2 \otimes \mathbf{I} \otimes \mathbf{I}\}, \tag{65}$$

where σ_1 is a Pauli operator on the qubit system and σ_2 is a Weyl-Heisenberg type generalized Pauli operator on the qutrit system.

One can readily see that for $\sigma_1 \in \{X, Z, XZ\}$, we have

$$\langle\psi| X \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = 0, \tag{66}$$

$$\langle\psi| Z \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = \frac{1}{12}(6-6) = 0 \tag{67}$$

$$\langle\psi| XZ \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = 0. \tag{68}$$

For all $a, b \in \mathbb{Z}/3\mathbb{Z}$, $a \neq 0$,

$$\langle\psi| \mathbf{I} \otimes X(a)Z(b) \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = 0, \tag{69}$$

and

$$\langle\psi| \mathbf{I} \otimes Z(b) \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = \frac{1}{12}(2(1+\eta+\eta^2) + 2(1+\eta+\eta^2)) = 0, \tag{70}$$

since $\eta = e^{2\pi i/3}$.

For the dimensional weight 6 operators, we have

$$\begin{aligned}\langle\psi| X \otimes X(1)Z(b) \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = \frac{\eta^{2b}}{12}(((\langle 1220| + \langle 1202|)(-|1220\rangle + |1202\rangle) \\ +(-\langle 0201| + \langle 0210|)(|0201\rangle + |0210\rangle)) = 0, \quad (71) \\ \langle\psi| X \otimes X(2)Z(b) \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = \frac{\eta^{b}}{12}(((\langle 0102| - \langle 0120|)(|0102\rangle + |0120\rangle) \\ +(\langle 1101| + \langle 1110|)(-|1101\rangle + |1110\rangle)) = 0, \quad (72) \\ \langle\psi| XZ \otimes X(1)Z(b) \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = \frac{\eta^{2b}}{12}(-((\langle 1220| + \langle 1202|)(-|1220\rangle + |1202\rangle) \\ +(-\langle 0201| + \langle 0210|)(|0201\rangle + |0210\rangle)) = 0, \quad (73) \\ \langle\psi| XZ \otimes X(2)Z(b) \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = \frac{\eta^{b}}{12}(((\langle 0102| - \langle 0120|)(|0102\rangle + |0120\rangle) \\ -(\langle 1101| + \langle 1110|)(-|1101\rangle + |1110\rangle)) = 0. \quad (74)\end{aligned}$$

For $a \neq 0$

$$\langle\psi| Z \otimes X(a)Z(b) \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = 0, \tag{75}$$

and

$$\langle\psi| Z \otimes Z(b) \otimes \mathbf{I} \otimes \mathbf{I} |\psi\rangle = \frac{1}{12}(2(1+\eta+\eta^2) - 2(1+\eta+\eta^2)) = 0. \tag{76}$$

Thus, $Q(\mathcal{S})$ is a $(((2,3,3,3),1,\sqrt{54}))$ quantum error correcting code and by Theorem 6, $|\psi\rangle$ is an absolutely maximally entangled state.

4.3 Higher dimensional codes over $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

One might consider subgroups of $\mathcal{S}$ of index two, in the hope that these might provide higher dimensional codes which can also detect some non-trivial class of errors. However, in each case we find a two-dimensional undetectable error.

name	generators	basis for $Q(\mathcal{S}_i)$	undetectable error
$\mathcal{S}_0$	U_0^2, U_1, U_2	$\|\psi\rangle, \|0000\rangle$	$Z \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}$
$\mathcal{S}_1$	U_0, U_1^2, U_2	$\|\psi\rangle, \frac{1}{\sqrt{12}} \sum_{j=13}^{24} \|\phi_j\rangle$	$ZX \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}$
$\mathcal{S}_2$	U_0, U_1, U_2^2	$\frac{1}{\sqrt{6}} \sum_{j=1}^{6} \|\phi_j\rangle, \frac{1}{\sqrt{6}} \sum_{j=7}^{12} \|\phi_j\rangle$	$Z \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}$

4.4 Other AME states over $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

While the example state Huber et al. gave (47) is appreciably different than that which we detailed above, showing Theorem 6 is fulfilled actually weakens the constraints not to require the right-hand side of (49). Because of this, actually (α, β) can take on any pair of values in a circle, 10 arbitrary choices of which we have numerically verified. Of note are the choices of (1, 0) or (0, 1), which select only the α coefficient half of the state or the β coefficient half of the state respectively, and end up with two simpler 12-term AME states which are also orthogonal to each other.

Then, we redefine $|\psi_H\rangle = \alpha |\psi_{H\alpha}\rangle + \beta |\psi_{H\beta}\rangle$, where

$$\begin{aligned} |\psi_{H\alpha}\rangle = &- |0011\rangle + |0022\rangle + |0102\rangle + |0120\rangle - |0201\rangle - |0210\rangle \\ &+ |1012\rangle - |1021\rangle + |1101\rangle - |1110\rangle - |1202\rangle + |1220\rangle \end{aligned} \tag{77}$$

and similarly

$$\begin{aligned} |\psi_{H\beta}\rangle = &- |0012\rangle + |0021\rangle - |0101\rangle + |0110\rangle + |0202\rangle - |0220\rangle \\ &- |1011\rangle + |1022\rangle + |1102\rangle + |1120\rangle - |1201\rangle - |1210\rangle \end{aligned} \tag{78}$$

The same principle applies to our example, and actually $|\psi_\beta\rangle := \sum_{j=13}^{24} |\phi_j\rangle$ constitutes another example of an AME state over the same system, although it is not stabilized by the exact same stabilizers, instead corresponding to $\langle U_0, -U_1, -U_2\rangle$. This can be thought of as a change of logical basis on the qubit system and its operators, resulting in a mapping of $Z \to -Z$ and $X \to X$, and any other change of basis will do, but are linearly dependent on these. Similar to $|\psi_{H\alpha}\rangle$ and $|\psi_{H\beta}\rangle$, we now refer to our first example as $|\psi_\alpha\rangle := \sum_{j=1}^{12} |\phi_j\rangle$.

5 Discussion

5.1 Use of nonlocal stabilizers

Of notable concern is our use of the "skew-permutation" U_1 and nonlocal but in-place U_2 operators, which deviate from the uniform-dimensional norm of using Weyl-Heisenberg Pauli extensions to construct local errors for the generators of the stabilizer group. However for a mixed system of D_a- and D_b-dimensional qudits, with D_a, D_b co-prime, it is only possible to have two such local stabilizers commute if they independently commute over the qudits of each dimension. This is because a pair of Weyl-Heisenberg operators of dimension D commute up to a phase of $e^{\frac{2\pi ij}{D}}$, for some integer j. With a single qubit, this would restrict the qubit part of such local operators to only be the identity, and codes constructed from such operators would fail to detect any errors on the qubit.

During this work, the potential existence of some local error basis was not formally ruled out using Gell-Mann matrices for the stabilizers corresponding to AME states on $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, but even if a construction were found, due to significantly non-trivial commutation relations it is believed it would be much less generalizable to higher dimensions, as the complexity of generalized Gell-Mann matrix products scales in general with only embeddings of $SU(2)$ to look to for hope in simplification, to the authors' knowledge.

5.2 Further constructions of similar type

It is conjectured further codes and therefore AME states may be found over other mixed-dimensional systems by generalizing the notions of U_0, U_1, and U_2:

U_0 is a diagonal operator which selects a subset $\{|\psi_j\rangle\}$ of the computational basis of size ab and discards the remaining $\prod d_i - ab$.

U_2 organizes the selected subset into a cycles of b states, while acting as identity on the remaining $\prod d_i - ab$.

U_1 permutes between a qudits of equal dimension, connecting between subsets of c cycles with order ac, where c divides b.

5.3 Orthogonality of AME states

By noticing that the Huber state can be written in terms of $|\phi_j\rangle$, we actually see

$$\begin{aligned} |\psi_{H\alpha}\rangle = &- |0011\rangle + |0022\rangle + |0102\rangle + |0120\rangle - |0201\rangle - |0210\rangle \\ &+ |1012\rangle - |1021\rangle + |1101\rangle - |1110\rangle - |1202\rangle + |1220\rangle \quad (79) \\ = &- |\phi_4\rangle + |\phi_1\rangle + |\phi_3\rangle + |\phi_5\rangle - |\phi_6\rangle - |\phi_2\rangle + \sum_{j=7}^{12} |\phi_j\rangle \quad (80) \end{aligned}$$

and similarly

$$\begin{aligned} |\psi_{H\beta}\rangle = &- |0012\rangle + |0021\rangle - |0101\rangle + |0110\rangle + |0202\rangle - |0220\rangle \\ &- |1011\rangle + |1022\rangle + |1102\rangle + |1120\rangle - |1201\rangle - |1210\rangle \quad (81) \\ = &\sum_{j=19}^{24} |\phi_j\rangle - |\phi_{16}\rangle + |\phi_{13}\rangle + |\phi_{15}\rangle + |\phi_{17}\rangle - |\phi_{18}\rangle - |\phi_{14}\rangle \quad (82) \end{aligned}$$

Then, $\frac{1}{\sqrt{2}}(|\psi_{H\alpha}\rangle - |\psi_{H\beta}\rangle)$ is orthogonal to our examples $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$.

5.4 Application

While quantum computing device manufacturers typically produce symmetrical qudits (ignoring connectivity) intending for uniform-dimensional systems, both manufacturing defects and operational drift yield different error rates on each physical qudit as well as the two-qudit operations. Using error-correcting-codes on mixed-dimensional systems allow quantum compilers to take better advantage of calibration data to be hardware-aware in error rate optimization heuristics by assigning lower-dimension logical requirements to higher-error physical components. In particular, while both 4-qutrit and now 3-qutrit-1-qubit systems are known to have absolutely maximally entangled states and their corresponding stabilizer codes, it is also known due to shadow inequalities that 4-qubit systems do not have them.

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A Shadow inequality counterexamples

For a system on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^p \otimes \mathbb{C}^q$, a choice of $T = \{1, 2, 3, 4\}$ yields the terms:

$$\begin{aligned}
|S| = 0 : &\qquad +1 \\
|S| = 1 : &\qquad -1/2 - 1/2 - 1/p - 1/q = -1 - (p+q)/pq \\
|S| = 2 : &\qquad 1/4 * 2 + 1/2p * 4 = +1/2 + 2/p \\
|S| = 3 : &\qquad -1/2 - 1/2 - 1/p - 1/q = -1 - (p+q)/pq \\
|S| = 4 : &\qquad +1
\end{aligned}$$

Then the condition is:

$$\begin{aligned}
2 + 2(-1 - (p+q)/pq)) + 1/2 + 2/p &\geq 0 \\
1/2 - 2(p+q)/pq + 2/p &\geq 0 \\
(pq/2 - 2p - 2q + 2q)/pq &\geq 0 \\
pq/2 - 2p &\geq 0 \\
q &\geq 4
\end{aligned}$$

B Code used

The code used to set up linear algebra and verify stabilizer candidates over the course of this work, including code to generate matrix representations based on our mapping representations of nonlocal stabilizers, is available at https://github.com/Saphius1a/stabiliser-verification under Creative Commons license CC0 1.0 Universal.