UNIVERSITAT DE BARCELONA

# Periodic boundary points for transcendental Fatou components

Anna Jové Campabadal

UNIVERSITAT DE BARCELONA

# PERIODIC BOUNDARY POINTS
## FOR TRANSCENDENTAL FATOU COMPONENTS

by

**Anna Jové Campabadal**

PhD Dissertation

Advisor: Núria Fagella Rabionet

Doctoral Program of Mathematics and Computer Science
Barcelona, September 2025

# Universitat de Barcelona

## Programa de doctorat en Matemàtiques

Tesi presentada per Anna Jové Campabadal per optar al grau de
Doctora en Matemàtiques

Directora: Núria Fagella Rabionet

Facultat de Matemàtiques i Informàtica
Setembre de 2025

Certifico que la següent tesi ha estat realitzada per Anna Jové Campabadal sota la meva direcció.

Barcelona, Setembre de 2025
Núria Fagella Rabionet

# Acknowledgements

Abans que res, et vull donar les gràcies a tu, Núria, per haver-me donat la oportunitat de fer el doctorat amb tu. No tinc paraules per expressar tot l'agraïment que sento per tot el que has fet per mi. Per la teva paciència, per donar-me la llibertat de treballar en problemes que m'agraden, per guiar-me, per aconsellar-me, per donar-me totes les facilitats possible per fer la tesi. Espero poder fer matemàtiques amb tu molt anys! I també et dono les gràcies a tu, Xavi, pel teu suport incondicional, per estar sempre al meu costat, per escoltar-me i ajudar-me quan ho he necessitat. Sou tots dos una font d'inspiració per a mi.

Gràcies a tot el grup HOLODYN: Gustavo, Kostya, Jordi, Leti, Robert (i també a l'Àlex, a l'Igsyl i al Robert Cardona). Toni, et trobo a faltar! Per tota la teoria ergòdica que vaig aprendre amb vosaltres, i que és el fonament d'aquesta tesi. Gustavo, thank you for all your interest in prime ends and boundary dynamics since you landed in Barcelona.

I have been very lucky to have attended so many conferences and schools abroad. Thank you to all of you, who shared mathematics, adventures and coffee and cake with me. David, Leti and Vasso, thank you for always taking care of me and being like older siblings for me. And to Anna Benini and Jasmin Raissy, for always looking after me. Special thanks to Nikolai Prochorov for sharing all the PhD journey with me. Discussions with Oleg Ivrii and Lasse Rempe, and their inspiring questions, motivated some of the chapters of this thesis. I also indebted with Phil Rippon and Gwyneth Stallard, for all the interesting discussions and all the oportunities that were given to me. I also want to thank all the Polish group, not only for all the oportunities you gave me and all your interest in my work, but also for taking care of me during my stay in Warsaw. Especially Anna Zdunik and Łukasz Pawelec, for giving me the amazing opportunity to learn from you and working together. *Dziękuję*!

Gràcies a tot el departament de matemàtiques de la UB. En especial, al Miquel Bosch, per guiar-me, i a la Carme Cascante i l'Albert Clop, per la docència compartida, el vostre suport i les facilitats que m'heu donat. A la Clara, per haver fet que Introducció al Càlcul Diferencial es convertís en una Introducció a la Docència a ritme accelerat i el començament d'una amistat. A tots els companys de despatx, i a la resta de companys de doctorat, al Jose i el nostre *Chaotic Seminar*, a la Mar. Al Jordi i a l'Enric per solucionar-me sempre tots els problemes logístics que he tingut i fer que mai m'hagués de preocupar de res. A la Irene, la meva veïna al despatx. A tots els alumnes que he tingut, que heu deixat una petjada immensa en mi.

Gràcies a les meves amigues del col·le (Cristina, Júlia, Mariama) i als meus amics del grau (a tots, però en especial al David, per haver viscut tota l'aventura de grau, màster i doctorat l'un al costat de l'altre). Al TRI 4.40 i al Coetus Triatló Lleida, en especial a l'Eva i al Pablo. Al papa i a la mama, al Joan, a la padrina Fina, a la tieta Carme i al tiet Alfonso. A tots vosaltres (i als que segur que em deixo), gràcies per tot!

The work presented in this dissertation has been developed with the financial support of the following grants from the Spanish Agencia Estatal de Investigación (AEI) .

- FPI PRE2021-097372.
- PID2020-118281GB-C32.
- PID2023-147252NBI00.

# Summary

This thesis fits within the field of Complex Dynamics, which deals with the study of discrete dynamical systems generated by the iteration of holomorphic maps.

More precisely, let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental function, either entire or meromorphic, and consider the sequence of iterates $\{f^n(z)\}_n$, for all $z \in \mathbb{C}$. Then, the complex plane is divided into two totally invariant sets: the *Fatou set* $\mathcal{F}(f)$, the set of points $z \in \mathbb{C}$ such that $\{f^n\}_{n\in\mathbb{N}}$ is well-defined and forms a normal family in some neighbourhood of $z$; and the *Julia set* $\mathcal{J}(f)$, its complement, where the dynamics is chaotic. The Fatou set is open and consists in general of infinitely many components, which are called *Fatou components*, and are either periodic, preperiodic or wandering.

One of the basic results of this field, already established by Fatou and Julia for rational maps, is that

$$\mathcal{J}(f) = \overline{\{\text{repelling periodic points of } f\}}.$$

This was generalized by Baker [Bak68, Thm. 1] for entire maps, and by Baker, Kotus and Lü [BKL91, Thm. 1], for meromorphic transcendental functions.

Observe that, if $U$ is a $p$-periodic Fatou component, then $\{f^n(\partial U)\}_{n=0}^{p-1}$ is a closed invariant subset of the Julia set. Hence, one shall ask the following question.

**Question.** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be a periodic Fatou component. Are periodic points dense on $\partial U$?*

Note that although periodic points are dense in the Julia set, *a priori* they could accumulate on $\partial U$ from the complement of $\overline{U}$, without being in $\partial U$. For instance, if $U$ is a rotation domain with locally connected boundary, then there are no periodic points in $\partial U$ at all. Nevertheless, F. Przytycki and A. Zdunik showed that, for rational maps, rotation domains (i.e. Siegel disks and Herman rings) are the only possible exception for which periodic boundary points are not dense. Namely, they gave a positive answer to the question above for basins (attracting or parabolic) of rational maps.

**Theorem. (Przytycki-Zdunik,** [PZ94]**)** *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map, and let $U$ be an attracting or parabolic basin for $f$. Then, periodic points are dense on $\partial U$.*

The seminal work of F. Przytycki and A. Zdunik [PZ94] unveils that the answer to the previous elementary question is far from being straightforward. Indeed, it requires a deep understanding of the boundaries of such Fatou components (which may be not even locally connected), combining tools from dynamics, measure theory and conformal analysis. In the particular case of simply connected attracting basins, the proof relies strongly on the measure-theoretical properties of $f|_{\partial U}$ and Lyapunov exponents, previously developed in [Prz85, Prz86,

Prz93], as well as precise estimates on the distortion of Riemann maps and finite Blaschke products on the unit circle, and conformal Pesin theory.

For unbounded Fatou components of transcendental maps, the situation is even more delicate, due to the presence of the essential singularity, and most of the previous techniques do not apply. In fact, due to the lack of compactness of $\partial U$, the following is also an open question.

**Question.** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be an unbounded periodic Fatou component. Does $\partial U$ contain at least one periodic point?*

In view of the previous questions, and the work developed in [DG87, BW91, BD99, BF01, Bar08, BK07, BFJK17, RS18, BFJK19] to understand the boundaries of transcendental Fatou components, the following conjecture arises, which is a major open problem in transcendental dynamics.

**Conjecture.** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be a $p$-periodic simply connected Fatou component, such that $f^p|_U$ is not univalent. Then,*

(a) *there exists a periodic point on $\partial U$.*

(b) *Morever, if $U$ is an attracting or parabolic basin, or a doubly parabolic Baker domain, then periodic points are dense on $\partial U$.*

This thesis has to be understood as significant progress towards the proof of the previous conjecture. Indeed, we prove existence and density of periodic boundary points under mild assumptions on the postsingular set, together with additional results on boundary dynamics, concerning escaping points and accessibility. To this end, we develope new techniques, such as distortion estimates for inner functions and Pesin theory for transcendental maps.

The work presented in this thesis corresponds to the following articles and preprints (each of them corresponding to a chapter, numbered accordingly), preceded by a chapter devoted to the main tool in understanding simply connected Fatou components: the associated inner functions.

1. N. Fagella and A. Jové, *A model for boundary dynamics of Baker domains*, Math. Z. **303** (2023), n. 4, Paper n. 95, 36.

2. A. Jové and N. Fagella, *Boundary dynamics in unbounded Fatou components*, Trans. Amer. Math. Soc. **378** (2025), 2321-2362.

3. A. Jové, *Periodic boundary points for simply connected Fatou components of transcendental maps*, Math. Ann. (2025). Published online.

4. A. Jové, *Pesin theory for transcendental maps and applications*, Preprint (2024), available at arxiv:2410.19703. Submitted.

5. A. Jové, *Boundaries of hyperbolic and simply parabolic Baker domains*, Preprint (2024), available at arxiv:2410.19726. Submitted.

# Resum

Aquesta tesi s'emmarca en el camp de la Dinàmica Complexa, que estudia els sistemes dinàmics discrets generats per la iteració de funcions holomorfes.

De manera més precisa, sigui $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ una funció transcendent, entera o meromorfa, i considerem la successió d'iterats $\{f^n(z)\}_n$, per tot $z \in \mathbb{C}$. Llavors el pla complex es divideix en dos conjunts totalment invariants: el *conjunt de Fatou* $\mathcal{F}(f)$, el conjunt de punts $z \in \mathbb{C}$ tals que $\{f^n\}_{n\in\mathbb{N}}$ està ben definit i forma una família normal en algun entorn de $z$; i el *conjunt de Julia* $\mathcal{J}(f)$, el seu complement, on la dinàmica és caòtica. El conjunt de Fatou és obert i en general té infinites components connexes, anomenades *components de Fatou*, i són periòdiques, pre-periòdiques, o errants.

Un dels resultats bàsics en Dinàmica Complexa (demostrat per Fatou i Julia per a funcions racionals) és que

$$\mathcal{J}(f) = \overline{\{\text{punts periòdics repulsors de } f\}}.$$

Aquest resultat va ser generalitzat per Baker [Bak68, Thm. 1] per funcions enteres, i per Baker, Kotus i Lü [BKL91, Thm. 1], per funcions meromorfes transcendents.

Observem que, si $U$ és una component de Fatou $p$-periòdica, llavors $\{f^n(\partial U)\}_{n=0}^{p-1}$ és un subconjunt tancat invariant del conjunt de Julia. Aleshores, la següent pregunta apareix de manera natural.

**Pregunta.** *Sigui $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ una funció meromorfa, i sigui $U$ una component de Fatou periòdica. Els punts periòdics són densos a $\partial U$?*

Notem que, tot i que els punts periòdics són densos al conjunt de Julia, *a priori* es podrien acumular a $\partial U$ des del complement de $\overline{U}$, sense estar a $\partial U$. Per exemple, si $U$ és un domini de rotació amb frontera localment connexa, llavors no hi ha cap punt periòdic a $\partial U$. Tot i això, F. Przytycki i A. Zdunik van demostrar que, per funcions racionals, els dominis de rotació (i.e. discs de Siegel i anells de Herman) són les úniques excepcions per les quals els punts periòdics no són densos a la frontera. En particular, van donar una resposta positiva a la pregunta anterior per conques d'atracció o parabòliques de funcions racionals.

**Teorem. (Przytycki-Zdunik,** [PZ94]) *Sigui $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ una funció racional, i sigui $U$ una conca atractora o parabòlica per $f$. Aleshores els punts periòdics són densos a $\partial U$.*

El treball de F. Przytycki and A. Zdunik [PZ94] ja ens mostra que la resposta a una pregunta tan elemental és lluny de ser senzilla. En efecte, és necessari un estudi exhaustiu de les fronteres de tals components de Fatou (que poden no ser localment connexes), combinant eines de dinàmica, teoria de la mesura i anàlisi conforme. En el cas particular de conques

d'atracció simplement connexes, la demostració es basa en les propietats de $f|_{\partial U}$ des del punt de vista de la teoria de la mesura i els exponents de Lyapunov, desenvolupada prèviament a [Prz85, Prz86, Prz93], així com estimacions precises de la distorsió de l'aplicació de Riemann i els productes de Blaschke finits al cercle unitat, i teoria de Pesin conforme.

Per a components de Fatou no acotades de funcions transcendents, la situació és encara més delicada, degut a la presència de la singularitat essencial, i la majoria de les tècniques anteriors no es poden aplicar. A més a més, com que $\partial U$ no és compacta, la següent també és una pregunta oberta.

**Pregunta.** *Sigui* $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ *una funció meromorfa, sigui* $U$ *una component de Fatou invariant no acotada. Hi ha algun punt periòdic a* $\partial U$*?*

En vista de les preguntes anteriors, i el treball previ desenvolupat a [DG87, BW91, BD99, BF01, Bar08, BK07, BFJK17, RS18, BFJK19] per entendre les fronteres de components de Fatou transcendents, sorgeix de manera natural la següent conjectura, que és un gran problema obert en dinàmica transcendent.

**Conjectura.** *Sigui* $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ *una funció meromorfa, i sigui* $U$ *una component de Fatou p-periòdica simplement connexa, tal que* $f^p|_U$ *no és univalent. Aleshores,*

(a) *existeix un punt periòdic a* $\partial U$*.*

(b) *A més a més, si* $U$ *és una conca atractora o parabòlica, or un domini de Baker doblement parabòlic, llavors els punts periòdics són densos a* $\partial U$*.*

Aquesta tesi s'ha d'entendre com a un progrés significatiu en la demostració de la conjectura anterior. En efecte, demostrem l'existència i densitat de punts periòdics a la frontera de components de Fatou sota hipòtesis molt febles en el conjunt postsingular, juntament amb resultats addicionals en relació a la dinàmica a la frontera, punts d'escapament i accessibilitat. Durant la tesi s'han demostrat noves tècniques, tals com estimacions en la distorsió de funcions internes i teoria de Pesin per a funcions transcendents.

# Contents

# Introduction

Consider a transcendental function $f\colon \mathbb{C} \to \mathbb{C}$ (entire or meromorphic), and denote by $\{f^n\}_{n\in\mathbb{N}}$ its iterates, which generate a discrete dynamical system in $\mathbb{C}$. Then, the complex plane is divided into two totally invariant sets: the *Fatou set* $\mathcal{F}(f)$, defined to be the set of points $z \in \mathbb{C}$ such that $\{f^n\}_{n\in\mathbb{N}}$ is well-defined and forms a normal family in some neighbourhood of $z$; and the *Julia set* $\mathcal{J}(f)$, its complement. Another dynamically relevant set is the *escaping set* $\mathcal{I}(f)$, where points converge to infinity, the essential singularity of the function. For background on the iteration of entire and meromorphic functions see e.g. [Ber93].

On the one hand, the Fatou set is open and consists typically of infinitely many connected components, called *Fatou components*. Due to the invariance of the Fatou and the Julia sets under $f$, Fatou components map among themselves and hence are periodic, preperiodic or wandering. On the other hand, periodic points are dense in the Julia set, and the map $f\colon \mathcal{J}(f) \to \mathcal{J}(f)$ is chaotic in the sense of Devaney [Dev89]. Even though dynamics in Fatou components are well-understood, the dynamics of $f|_{\mathcal{J}(f)}$ is much more intricate, specially in the case of transcendental functions, for which the essential singularity plays a significant role and adds complexity to the system.

In this thesis we focus on boundaries of Fatou components of transcendental maps, regarded as subsets of the Julia set for which the dynamics is more understandable (at least *a priori* it should be somehow related with the well-understood interior dynamics of the Fatou component). More precisely, let $U$ be an invariant Fatou component (i.e. $f(U) \subset U$). Then, $\partial U$ is a forward invariant subset of the Julia set. Thus, our goal is to understand the function $f\colon \partial U \to \partial U$, and to discuss existence (and density) of periodic points for such map. As we will see, such questions are intimately related with the topology of such boundaries, becoming a matter of interest throughout the thesis.

The case of rational maps (and polynomials) is better understood. Indeed, for polynomials, the boundary of any bounded periodic Fatou component which is not a Siegel disk is locally connected [RY08, RY22]. Hence, $f|_{\partial U}$ is topologically conjugate to $z \mapsto z^d$ on the unit circle $\partial\mathbb{D}$, providing a good description for the boundary dynamics. In particular, this implies that periodic points are dense of the boundary of such Fatou components.

For rational maps, periodic points are also dense on the boundary of attracting and parabolic basins [PZ94]. For transcendental maps, even though local connectivity is proven for certain Fatou components [BFJK25], the situation is wilder and less understood, and complicated boundaries arise naturally. In fact, for a large class of unbounded Fatou components of transcendental entire maps, their boundary is always non-locally connected [BW91, BD99, Bar08]. However, successful results have been obtained in some cases, in which it is possible to relate the boundary dynamics with the internal dynamics in the Fatou component.

Let us restrict to simply connected invariant Fatou components, so that the Riemann map can be used as a uniformization for the internal dynamics. More precisely, let $U$ be a invariant Fatou component of $f$ and let $\varphi\colon \mathbb{D} \to U$ be a Riemann map. Then,

$$g\colon \mathbb{D} \longrightarrow \mathbb{D}, \qquad g := \varphi^{-1} \circ f \circ \varphi$$

is an analytic self-map of $\mathbb{D}$, and $f|_U$ and $g|_{\mathbb{D}}$ are conformally conjugate by $\varphi$.

The study of holomorphic self-maps of $\mathbb{D}$ is a good approach to analyze the dynamics of $f|_U$. Indeed, the Denjoy-Wolff Theorem asserts that, whenever a holomorphic self-map $g$ of $\mathbb{D}$ is not conjugate to a rotation, all orbits converge to the same point $p \in \overline{\mathbb{D}}$ (the *Denjoy-Wolff point* of $g$). From this celebrated result, the classification theorem of invariant Fatou components of entire maps can be deduced, which was proved earlier by Fatou [Fat20] using different techniques. More precisely, a simply connected invariant Fatou component is either a *Siegel disk* (when it is conjugate to an irrational rotation), an *attracting basin* (when all orbits converge to the same point in $U$) or a *parabolic basin* or a *Baker domain* (when all orbits converge to the same point in $\partial U$). The difference between the last two possibilities comes from the nature of the convergence point: for Baker domains it is the essential singularity, so $f$ is not defined at it; whereas for parabolic basins, it is a fixed point of multiplier 1.

One may ask if the previous conjugacy with a holomorphic self-map of $\mathbb{D}$ can be used to describe the dynamics of $f$ in the boundary of $U$. First, from the fact that $f(\partial U) \subset \partial U$, it can be deduced that $g$ is an *inner function*, i.e. an analytic self-map of $\partial\mathbb{D}$ such that the radial limits belong to $\partial\mathbb{D}$ for almost every point in $\partial\mathbb{D}$. Hence, a boundary extension

$$g^*\colon \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$$

can be defined using radial limits almost everywhere with respect to the Lebesgue measure, and it induces a dynamical system defined almost everywhere on $\partial\mathbb{D}$. One may expect *a priori* that $f|_{\partial U}$ and $g^*|_{\partial\mathbb{D}}$ share dynamical properties. Nevertheless, this cannot be assumed, since the Riemann map may not extend continuously to the boundary. In fact, this is the usual case for unbounded Fatou components of transcendental entire functions [BW91, BD99, Bar08]. Therefore, $\varphi$ may not be a conjugacy on $\partial\mathbb{D}$ and properties of $g^*|_{\partial\mathbb{D}}$ do not transfer to $f|_{\partial U}$ in general, although some connections can be established, as we show next.

First, Devaney and Goldberg studied the exponential family $\lambda e^z$ with $0 < \lambda < \frac{1}{e}$, [DG87], whose Fatou set is connected and consists of a totally invariant attracting basin $U$. From the explicit computation of the inner function, accesses to infinity were characterized, and the boundary of $U$, which is precisely the Julia set, was shown to be organized in curves of escaping points and their endpoints, the latter being the only accessible points from $U$. Such results were generalized to a larger family of functions having a totally invariant attracting basin [Bar07, BK07].

On the basis of this successful example, inner functions have been used systematically to understand the dynamics on the boundary of Fatou components. On the one hand, results of [BW91, BD99, Bar08, BFJK17] describe the topology of the boundary of unbounded Fatou components of entire functions and their accesses to infinity. On the other hand, the revealing work in [DM91], further developed in [RS18, BFJK19], describe their ergodic properties. However, many questions are still unanswered. For instance, concerning periodic points, it is not known whether there exists a single periodic point on the boundary of an unbounded Fatou component of a transcendental map, and under which conditions these periodic points are dense. Interesting questions also arise concerning the accessibility of boundary points and their dynamics.

One precise type of periodic Fatou components, exclusive of transcendental functions, is of special interest: *Baker domains*. Baker domains are (invariant) Fatou components in which iterates converge locally uniformly to infinity, the essential singularity of the map. Maps possessing Baker domains are not hyperbolic, nor of bounded type (i.e. the set of singularities of the inverse branches of the function is unbounded [EL92]). In contrast with the other periodic Fatou components, in which the dynamics around the convergence point can be conjugate to some predetermined normal form, three different asymptotics are possible for Baker domains (see Thm. II.3.5). This leads to a further classification according to their internal dynamics into *doubly parabolic*, *hyperbolic* and *simply parabolic* Baker domains, which also present different boundary properties.

Even though all orbits in a Baker domain tend to infinity, it is still unknown whether a single escaping point always exists in $\partial U$. For hyperbolic and simply parabolic univalent Baker domains this question was answered affirmatively by Rippon and Stallard [RS18], who showed that the set of boundary escaping points has full harmonic measure with respect to the Baker domain. This result was generalized to finite degree Baker domains [BFJK19, Thm. A]. On the contrary, for doubly parabolic Baker domains of finite degree the set of escaping boundary points is known to have zero harmonic measure [BFJK19, Thm. B]. The question is still unanswered in the general case.

Hence, the following conjecture is an open problem in transcendental dynamics, which is supported by the previously mentioned works.

**Conjecture.** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be an invariant simply connected Fatou component, such that $f|_U$ is not univalent. Then,*

(a) *there exists an escaping point on $\partial U$.*

(b) *there exists a periodic point on $\partial U$.*

(c) *Morever, if $U$ is an attracting or parabolic basin, or a doubly parabolic Baker domain, then periodic points are dense on $\partial U$.*

The goal of this thesis is to make progress towards the proof of the previous conjecture, specially the statements concerning periodic boundary points. Let us note that the nature of the problem is different from the ones considered previously, and hence new tools and ideas should be provided. On the one hand, the topological tools used in [BW91, BD99, Bar08] are not sufficient to tackle dynamical questions. On the other hand, the measure-theoretical approach developed in [DM91, RS18, BFJK19] is rather crude to deal with the qualitative (and measure-zero) sets described in the conjecture. However, a combination of both approaches, together with a deeper understanding on the associated inner function and the boundary behaviour of the Riemann map is what leads us to the results presented in this thesis.

Next we summarize our main results, together with the tools and techniques we use to prove them.

### A model for boundary dynamics of Baker domains

One first approach to tackle the previous conjecture is to study an explicit example, which serves as a toy model to understand the boundary dynamics of Baker domains. Indeed, we present a detailed analysis of the dynamics of the transcendental entire function $f(z) = z+e^{-z}$, which possesses countably many doubly parabolic Baker domains of degree two. A good understanding of this model throws some light about the correspondence between the inner function and the boundary map, in the explicit way we pursue. In our work, other interesting properties of both the inner function and the boundary of the Baker domain arise, which hold for a wider family of functions, as we discuss later on.

The function we consider, $f(z) = z + e^{-z}$, is one of the few known explicit examples having doubly parabolic Baker domains of finite degree, and was studied previously in [BD99, FH06, BFJK19]. However, many aspects concerning boundary dynamics are still unexplored, and are the object of our study.

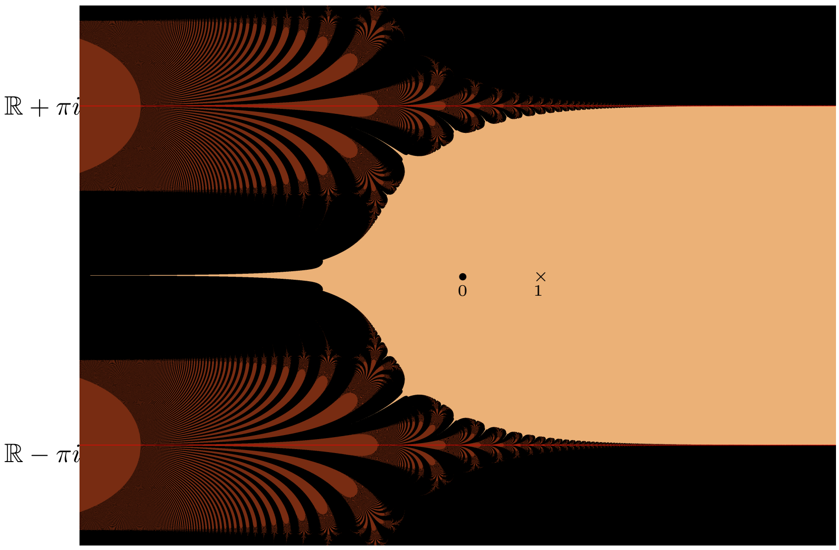



Figure 1: Dynamical plane for $f(z) = z + e^{-z}$. In red, the Julia set of $f$. In beige, the Baker domain contained in the strip $\{-\pi < \text{Im } z < \pi\}$. In black, the rest of the Fatou set of $f$. The only critical point on the strip (0) is also marked, as well as the corresponding critical value (1).

First, Baker and Domínguez [BD99, Thm. 5.1] and Fagella and Henriksen [FH06, Example 3] proved, using different arguments, the existence of a doubly parabolic Baker domain $U_k$ of degree two in each strip $S_k := \{(2k-1)\pi \leq \text{Im } z \leq (2k+1)\pi\}$, for all $k \in \mathbb{Z}$. Since the dynamics in all of them are the same, we consider only the Baker domain $U := U_0$ in the strip $S := S_0 = \{-\pi \leq \text{Im } z \leq \pi\}$ (see Figure 1). In [BD99, Thm. 5.2], the associated inner function is computed explicitly.

The topology of $\partial U$ is addressed in [BD99, Section 6], where it is deduced that $\partial U$ is non-locally connected and preimages of infinity by the Riemann map $\varphi\colon \mathbb{D} \to U$ (in the sense of radial limits) are dense in the unit circle. Going one step further, they proved that the impression of the prime end corresponding to 1 is precisely $\partial S \cup \{\infty\}$. Using this, accesses to infinity from $U$ were explicitly characterized in terms of the inner function.

Finally, in [BFJK19, Example 1.2], they describe some dynamical sets in $\partial U$ in terms of measure, as an application of a general theorem [BFJK19, Thm. B]. More precisely, they show that almost every point with respect to the harmonic measure has a dense orbit in $\partial U$. Therefore, the escaping points in $\partial U$ have zero harmonic measure. Moreover, they conjectured that all escaping points in $\partial U$ are non-accessible from $U$ and accessible repelling periodic points are dense in $\partial U$. We prove both conjectures, and additional results concerning boundary dynamics.

**Theorem 1.** *Let $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z + e^{-z}$, and let $U$ be an invariant Baker domain for $f$. Then, the following holds.*

1.A (The boundary of $U$) *Every escaping point in $\partial U$ can be connected to $\infty$ by a unique curve of escaping points in $\partial U$ (a* dynamic ray*). Moreover, $\partial U$ is the closure of such dynamic rays.*

1.B (Landing and non-landing dynamic rays) *There exist uncountably many dynamic rays which land at a finite endpoint, and there exists uncountably many dynamic rays which do not land. The accumulation set (on the Riemann sphere) of such a non-landing ray is an indecomposable continuum which contains the ray itself.*

1.C (Accessible points) *Escaping points on $\partial U$ are non-accessible from $U$, while points on $\partial U$ having a bounded orbit are all accessible from $U$.*

1.D (Periodic points) *Periodic points are dense on $\partial U$.*

### Boundary dynamics in unbounded Fatou components of entire maps

Once the previous example is understood, one should ask which of the previous properties rely on specific features of the map considered, and which are susceptible to be generalized for a larger class of Fatou components. At this moment, it is important to note that we analyze the dynamics in a semilocal way (i.e. restricted to a Fatou component and its boundary), so we only need to control the function in some neighbourhood of the Fatou component we consider.

On the one hand, note that the crucial step in the proof of Theorem 1 is to give an accurate topological description of the boundary of $U$ in terms of dynamic rays. This topological structure is used as a foundation to prove the remaining statements. In this sense, for transcedental entire maps, one should observe that a similar structure can be deduced from [BW91, BD99, Bar08] in terms of accesses to infinity. This is true not only for doubly parabolic Baker domains but for all Fatou components for which $f|_{\partial U}$ is ergodic with respect to the harmonic measure $\omega_U$ (i.e. Siegel disks, attracting and parabolic basins, and doubly parabolic Baker domains, see Thm. II.5.4). It is worth noting the interplay between the ergodic properties of $f|_{\partial U}$ and the topological properties of $\partial U$, as established in Theorem 2, as well as the connections with the set of singularities of the associated inner function.

On the other hand, consider the set $SV(f)$ of *singular values* of $f$ (i.e. the singularities of $f^{-1}$: critical and asymptotic values and accumulation thereof, see Sect. I.4), and the *postsingular set* of $f$, defined as

$$P(f) \coloneqq \overline{\bigcup_{s\in SV(f)} \bigcup_{n\geq 0} f^n(s)}.$$

It is a transversal argument in complex dynamics (rational and transcendental) that orbits of singular values govern the dynamics. In particular, singular values not lying on the Julia set translate to tamer dynamics (as happens for $f(z) = z + e^{-z}$, whose singular values are all critical values and lie inside the Baker domains, allowing an accurate study of the boundary dynamics). However, standard assumptions on the singular or the postsingular set, such as bounded type or hyperbolicity, are too restrictive to deal with Baker domains. As we will see, here and in the sequel, we require certain control on $P(f)$ in a neighbourhood of $\partial U$ – assumptions which are weak enough to allow the existence of Baker domains, but sufficient to prove the results concerning boundary dynamics we are interested in.

**Theorem 2.** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g = \varphi \circ f \circ \varphi^{-1}$ be the associated inner function.*

2.A (The boundary of $U$) *If $f|_{\partial U}$ is ergodic, $\partial U$ is the disjoint union of cluster sets $Cl(\varphi, \cdot)$ of $\varphi$ in $\mathbb{C}$, i.e.*

$$\partial U = \bigsqcup_{\xi\in\partial\mathbb{D}} Cl(\varphi, \xi) \cap \mathbb{C},$$

*where*

$$Cl(\varphi, \xi) \coloneqq \left\{ w \in \widehat{\mathbb{C}} \colon \text{ there exists } \{z_n\}_n \subset \mathbb{D} \text{ with } z_n \to \xi \text{ and } \varphi(z_n) \to w \right\}.$$

*Moreover, either $Cl(\varphi, \xi) \cap \mathbb{C}$ is empty, or has at most two connected components. If $Cl(\varphi, \xi) \cap \mathbb{C}$ is disconnected, then $\varphi^*(\xi) = \infty$.*

2.B (Periodic points in Siegel disks) *If $U$ is a Siegel disk, there are no periodic points on $\partial U$.*

*Moreover, assume there exists a simply connected domain $\Omega$ and a domain $V$ such that $\overline{V} \subset U$, $\overline{U} \subset \Omega$, and $P(f) \cap \Omega \subset V$.*

2.C (Singularities of the associated inner functions) *The set of singularities of $g$ (points at which $g$ cannot be extended analytically) has zero Lebesgue measure in $\partial\mathbb{D}$. Moreover, if $\xi \in \partial\mathbb{D}$ is a singularity of $g$, then $\varphi^*(\xi) = \infty$.*

2.D (Boundary dynamics) *Periodic points in $\partial U$ are accessible from $U$. Moreover, if $f|_{\partial U}$ is recurrent, then both periodic and escaping points are dense on $\partial U$.*

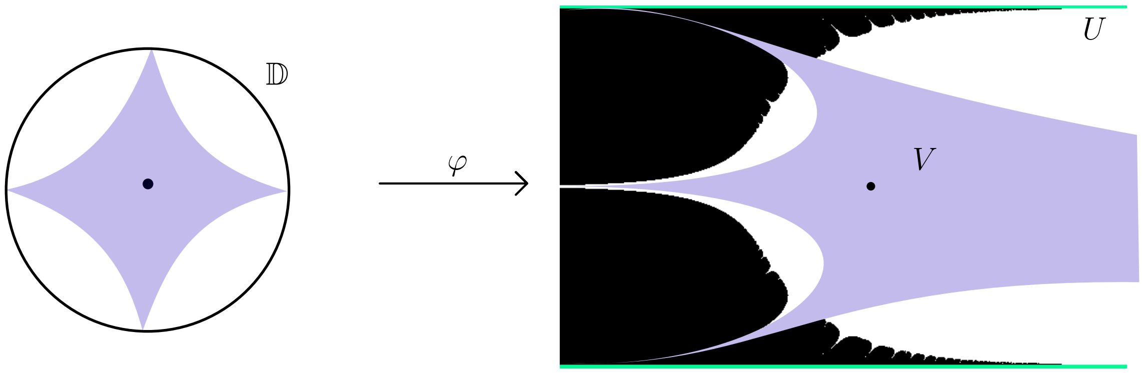



Figure 2: Schematic representation of the assumption in Theorem 2 concerning the postsingular set. The Fatou component of the right is a Baker domain of $z + e^{-z}$.

**Periodic boundary points through recurrence**

From now on, let us focus on the particular question of density of periodic boundary points. For rational maps, F. Przytycki and A. Zdunik [PZ94] showed that periodic points are dense on the boundaries of attracting and parabolic basins; for transcendental maps this question is essentially unexplored. Recall that although periodic points are dense in the Julia set, *a priori* they could accumulate on $\partial U$ only from the complement of $\overline{U}$, without being in $\partial U$.

First note that, unlike the other properties addressed in Theorem 1 and Theorem 2, which are of topological nature, density of periodic boundary points depends essentially on the recurrence of the boundary map.

Indeed, the basic idea to find periodic boundary points is to find an inverse branch of $f^n$ which sends a disk $D(x, r)$, $x \in \partial U$, inside itself, and hence having a fixed point, which is periodic for $f$. The difficulty of the proof is showing that this point is actually on $\partial U$ (note that inverse branches may not leave the Fatou component invariant). In [PZ94], this is solved by analyzing carefully the inner function, which is always a centered finite Blaschke product. Then, if the boundary map is recurrent, this argument can be applied densely along the boundary (as long as inverse branches are well-defined, and one can control that the resulting periodic point is in $\partial U$), showing that periodic boundary points are dense.

Therefore, to study density of periodic boundary points, there is no need to restrict to unbounded Fatou components of transcendental entire maps, and one shall work with periodic Fatou components of meromorphic functions, under the assumption that they are simply connected and the boundary map is recurrent.

Following the approach of F. Przytycki and A. Zdunik [PZ94] for basins of rationals maps, we prove the following. (Recall that, given a simply connected domain $U$, $C \subset U$ is a *crosscut* if $C$ is a Jordan arc such that $\overline{C} = C \cup \{a, b\}$, with $a, b \in \partial U$, $a \neq b$; any of the two connected components of $U \smallsetminus C$ is a *crosscut neighbourhood*.)

**Theorem 3.** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be a periodic simply connected Fatou component for $f$. Assume the following conditions are satisfied.*

- *$U$ is either an attracting basin, a parabolic basin, or a doubly parabolic Baker domain, with $f|_{\partial U}$ recurrent with respect to the harmonic measure $\omega_U$.*
- *There exists $x \in \partial U$ and $r > 0$ such that $P(f) \cap D(x, r) = \emptyset$.*
- *There exists a crosscut neighbourhood $N \subset U$ with $P(f) \cap N = \emptyset$.*

*Then, periodic points are dense on $\partial U$.*

Observe that, although the boundary map of a Siegel disk $U$ is recurrent, it always holds $\partial U \subset P(f)$, so they never satisfy the assumptions of Theorem 3.

As we will see in Chapter 3, the proof of this theorem is inspired in [PZ94], but the extension of the arguments to transcendental maps is far from being straightforward. Indeed, their proof, which combines tools from dynamics, measure theory and conformal analysis, relies strongly on the measure-theoretical properties of $f|_{\partial U}$ and Lyapunov exponents, previously developed in [Prz85, Prz86, Prz93], as well as precise estimates on the distortion of Riemann maps and finite Blaschke products on the unit circle, and conformal Pesin theory. In the transcendental case, most of these tools fail to be applicable, fundamentally due to the lack of compactness of the phase space.

In Chapter 3 we will see how these difficulties are overcome (essentially, by a more careful study of the dynamics, a clever use of hyperbolic metric, and a deeper study of the associated inner function), and we will justify the hypotheses we need in Theorem 3.

Finally, note that Theorem 3 holds for doubly parabolic Baker domains, as long as $f|_{\partial U}$ is recurrent (see Thm. II.5.4 for criteria to determine when it is the case). As mentioned before, it turns out that boundary dynamics depends essentially on the ergodic properties of the boundary map, more than on the particular type of Fatou component we are dealing with. Hence, a proof meant for attracting or parabolic basins should also work for doubly parabolic Baker domains.

### Pesin theory for transcendental maps

One can argue that the hypothesis of Theorem 3 on the postsingular set $P(f)$ is quite strong (in particular, $P(f)$ is assumed to be nowhere dense on $\partial U$). Observe that F.

Przytycki and A. Zdunik [PZ94] showed density of periodic points on $\partial U$ even if $\partial U \subset P(f)$. Even though rational maps are simpler than transcendental maps (for instance, rational maps have only finitely many critical values, while transcendental maps may have uncountably many singular values), their result indicates that, under some control on singular values (but not on their orbit– the postsingular set), it should be possible to prove density of periodic boundary points.

First of all, let us point out that the seminal paper of F. Przytycki and A. Zdunik [PZ94] should be viewed as the culmination of a deep study of the measure-theoretical properties of $f|_{\partial U}$ and Lyapunov exponents, done in [Prz85, Prz86, Prz93] for a rational basin $U$. In particular, a conformal version of Pesin theory (originally created in the context of smooth dynamical systems generated by diffeomorphisms) is developed in order to prove the existence of certain well-defined iterated inverse branches around almost every point on $\partial U$, which is the basis to construct periodic boundary points.

The hypothesis in Theorem 3 of the existence of $x \in \partial U$ and $r > 0$ such that $P(f) \cap D(x,r) = \emptyset$ implies that *all* iterated inverse branches are well-defined (and conformal) in $D(x,r)$. As one deduces from the proof of [PZ94] and Theorem 3, such a strong control is not required: one only needs that *certain* iterated inverse branches are well-defined in $D(x,r)$, the ones used to construct the periodic boundary points. The role of conformal Pesin theory is to guarantee that *these* iterated inverse branches are well-defined, for boundaries of attracting basins of rational maps.

Note that the cornerstones from which the rational Pesin theory is built (namely, compact phase space, finitely many critical values, and existence of ergodic invariant probability measures) no longer hold in general for boundary maps of transcendental Fatou components. However, taking advantage of the invariant measures on the boundaries of Fatou components given by Doering and Mañé [DM91], and under some mild assumptions on the geometric distribution of singular values and the order of growth of the function, we are able to overcome the difficulties arising from the lack of compactness, the infinite degree and the presence of infinitely many singular values. Our techniques include refined estimates on harmonic measure and the construction of an appropriate conformal metric. In this manner, we can develop Pesin theory on the boundary of some transcendental Fatou components in a quite successful way.

One of our results (in a very simplified version) reads as follows.

**Theorem 4.** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be a simply connected attracting basin for $f$, with attracting fixed point $p \in U$. Let $\omega_U$ be the harmonic measure in $\partial U$ with base point $p$. Assume $\log|f'| \in L^1(\omega_U)$ with $\int_{\partial U} \log|f'|\, d\omega_U > 0$. Let us suppose also that there are finitely many singular values on $\partial U$.*
*Then, for every countable collection of measurable sets $\{A_k\}_k \subset \partial U$ with $\omega_U(A_k) > 0$, $k \in \mathbb{N}$, and for $\omega_U$-almost every $x_0 \in \partial U$, there exists a backward orbit $\{x_n\}_n \subset \partial U$ and $r > 0$ such that*

- *there exists a sequence $n_k \to \infty$ such that $x_{n_k} \in A_k$;*
- *the inverse branch $F_n$ sending $x_0$ to $x_n$ is well-defined in $D(x_0, r)$;*

- diam $F_n(D(x_0, r)) \to 0$, *as* $n \to \infty$.

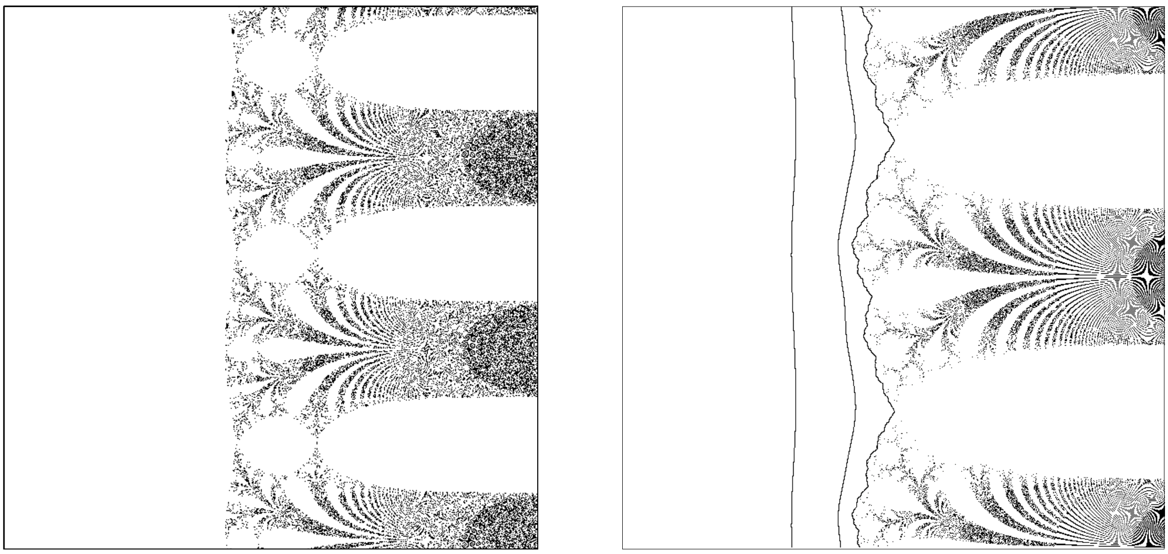
In Chapter 4, one finds variations of the previous theorem in order to deal with related situations (attracting basins with infinitely many singular values on their boundary, parabolic basins and doubly parabolic Baker domains, and centered inner functions, among others). We also show that hypotheses of Theorem 4 are enough to show density of periodic boundary points. Conditions which imply the integrability of $\log |f'|$ are also provided.

### Boundaries of hyperbolic and simply parabolic Baker domains

At this point, one shall ask which of the previous properties hold for hyperbolic and simply parabolic Baker domains, or, more in general, what can be said about the boundaries of such Baker domains.

Let us note that the previous techniques do not apply (since they essentially rely on ergodic properties of $g^*|_{\partial\mathbb{D}}$ which are no longer satisfied here). As a consequence, such Baker domains remain somehow unexplored, except for the results in [RS18] and [BFJK19, Thm. A] (see Thm. II.5.4), which establish the measure of the escaping set in $\partial U$, under certain conditions on the associated inner function. Additionally, several particular examples have been studied in the literature, which we analyze next.

Some of the well-known examples of such Baker domains confirm that hyperbolic and simply parabolic Baker domains may exhibit a completely different boundary behaviour than doubly parabolic Baker domains (see Fig. 3).



$f(z) = 2 - \log 2 + 2z - e^z$ $\qquad$ $f(z) = z + 2\pi i\alpha + e^z$

Figure 3: Examples of univalent Baker domains of hyperbolic and simply parabolic type, respectively (in both cases, the Baker domain is the Fatou component which contains a left half-plane, and its boundary is a Jordan curve in $\widehat{\mathbb{C}}$). Compare with Section II.5.4.

For instance, for the simply parabolic Baker domain of the function

$$f(z) = z + 2\pi i\alpha + e^z,$$

for appropriate $\alpha \in [0,1] \smallsetminus \mathbb{Q}$, studied in [BW91, Thm. 4] (see also Ex. II.5.12), all points in the boundary, which is a Jordan curve, converge to infinity under iteration.

Similarly, the hyperbolic Baker domain of the function

$$f(z) = 2 - \log 2 + 2z - e^z$$

has a unique fixed point in the boundary, and any other point is escaping [Ber93] (see also Ex. II.5.13). See Figure 3.

One may argue that the 'pathologies' observed in the previous Baker domains are not attributable to the fact they are hyperbolic or simply parabolic, but to univalency. Indeed, the associated inner function is a Möbius transformation, which has either none or a single periodic point (which is fixed).

When looking into examples of non-univalent hyperbolic or simply parabolic Baker domains [Rip06, Bar08, BZ12], one notices that the situation is completely opposite: there are plenty of periodic points for the associated inner function, and periodic points may even happen to be dense on $\partial U$, as for the Baker domain of the function

$$f(z) = 2z - 3 + e^z,$$

studied in [Bar08, Ex. 3.6] (in this case, the fact that periodic points are dense on $\partial U$ is deduced straightaway from the equalities $\mathcal{F}(f) = U$ and $\mathcal{J}(f) = \partial U$).

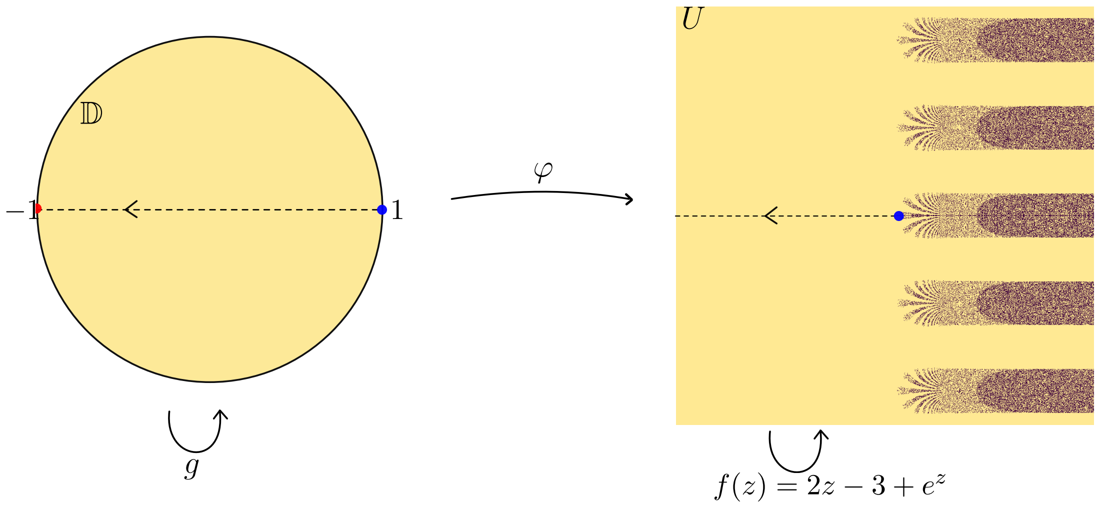



Figure 4: Dynamical plane of $f(z) = 2z - 3 + e^z$, with the hyperbolic Baker domain $U$ (yellow) of infinite degree [Bar08, Ex. 3.6]. The Riemann map $\varphi\colon \mathbb{D} \to U$ is depicted, together with the inner function. Note that $-1$ is the Denjoy-Wolff point. Since $\mathcal{J}(f) = \partial U$, periodic points are dense on $\partial U$.

With this in mind we prove the following.

**Theorem 5.** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be a simply connected Baker domain, of hyperbolic or simply parabolic type. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g = \varphi \circ f \circ \varphi^{-1}$ be the associated inner function. Then, the following holds.*

5.A (Ergodic properties of $f|_{\partial U}$) *$f|_{\partial U}$ is non-ergodic and non-recurrent with respect to the harmonic measure $\omega_U$.*

5.B (Density of periodic points) *Assume $f$ is entire, $f|_U$ is not univalent and $\mathcal{J}(g) \neq \partial\mathbb{D}$. Then, periodic points are not dense on $\partial U$.*

5.C (Existence of periodic points) *Assume $f|_U$ is not univalent. Then, there exists a crosscut neighbourhood $N_\xi$ of $\xi \in \mathcal{J}(g)$ such that $\overline{\varphi(N_\xi)} \cap P(f) = \emptyset$. Then, there exist infinitely many periodic points on $\partial U$, of arbitrarily large period.*

We note that the techniques and tools used and developed up to now become obsolete in this new context, since they rely on the ergodicity and recurrence of the boundary map, which no longer hold. Instead, the main tool now is the 'topological recurrence' on certain subsets of $\partial\mathbb{D}$ for the radial extension of the inner function. Let us note that the complexity of the proofs increases substantially.

**Selected tools and final remarks**

As noted above, the study of boundary dynamics (and, more precisely, the construction of periodic boundary points) is based on the interplay of different techniques coming from topology, ergodic theory, and conformal mapping, among others. However, in some cases, the existent tools are not enough to tackle our problems, and new ideas should be provided. In particular, we developed two tools, presented next and used intensively in this thesis, which are of independent interest, and which we believe to be potentially useful when dealing with other boundary problems. These are a control of the distortion of iterated inverse branches of inner functions at points on the unit circle (Theorem A), and the Carathéodory set (Definition B). Both concepts rely on the notion of crosscut neighbourhood and the Carathéodory's compactification of a simply connected domain, for which we refer to Section II.4.

First observe that one of the difficulties of working in the transcendental setting is the emergence of inner functions of infinite degree, and dealing with such a map on the unit circle (which may not extend continuously at any point). In contrast, in the case of a rational attracting basin considered in [PZ94], $g$ is a finite Blaschke product, which can be chosen to satisfy $g(0) = 0$. We shall view $g$ as a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, extended by Schwarz reflection. Then, its critical values (which are finitely many) are compactly contained in $\mathbb{D}$ (and, by reflection, in $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$) and their orbits converge uniformly to 0 (or to $\infty$), which are attracting fixed points. Hence, inverse branches of $g$ are well-defined for all points in $\partial\mathbb{D}$. Moreover, precise estimates on the behaviour of such inverse branches are given in [PZ94, Lemma 2].

In the general situation we consider (including Baker domains of infinite degree), $g$ is no longer a finite Blaschke product, and may not have an attracting fixed point in $\mathbb{D}$. However, having some mild control on the postsingular values of $g$ allows us to control inverse branches for the associated inner function $g$ at $\lambda$-almost every point in $\mathbb{D}$, even if

$g$ has infinite degree. Indeed, we consider the maximal meromorphic extension of $g$:

$$g\colon \widehat{\mathbb{C}} \smallsetminus E(g) \to \widehat{\mathbb{C}},$$

where $E(g) \subset \partial\mathbb{D}$ denotes the set of singularities of $g$. In this situation, we prove the following result concerning inner functions (not necessarily associated with Fatou components), which is of independent interest.

**Theorem A. (Iterated inverse branches at boundary points)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function, such that $g^*|_{\partial\mathbb{D}}$ is recurrent. Assume that there exists a crosscut neighbourhood $N \subset \mathbb{D}$ without postsingular values. Then, for $\lambda$-almost every $\xi \in \partial\mathbb{D}$, there exists $\rho_0 := \rho_0(\xi) > 0$ such that all branches $G_n$ of $g^{-n}$ are well-defined in $D(\xi, \rho_0)$. In particular, the set $E(g)$ of singularities of $g$ has $\lambda$-measure zero.*
*In addition, for all $0 < \alpha < \frac{\pi}{2}$, there exists $\rho_1 < \rho_0$ such that, for all $n \geq 0$, all branches $G_n$ of $g^{-n}$ are well-defined in $D(\xi, \rho_1)$ and, if $R_\xi$ denotes the radial segment at $\xi$, then the curve $G_n(R_\xi)$ tends to $G_n(\xi)$ non-tangetially with angle at most $\alpha$.*

Note that $\alpha$ does not depend on $n$, nor on the chosen inverse branch. Apart from giving a precise characterization of inverse branches, Theorem A also describes measure-theoretically the set of singularities, improving the results in [EFJS19, ERS20]. Compare also with the situation for one component inner functions (a more restrictive class of inner functions) described in [IU23, Part III], as well with the results in [IU24].

Moreover, in order to study the boundary dynamics on Baker domains, one needs a way to characterise whether or not points on the boundary have the same dynamic behaviour as points in the interior of the Baker domain. One possible approach, following the ideas in [BEF$^+$24] to describe the boundary dynamics of wandering domains, is to define the *Denjoy-Wolff set* of $f|_U$ as the set of points $x \in \partial U$ such that $f^n(x) \to \infty$ (i.e. $\textrm{dist}_{\widehat{\mathbb{C}}}(f^n(x), \infty) \to 0$, see [BEF$^+$24, Sect. 9]).

However, the main limitation of the Denjoy-Wolff set is that it does not capture in which direction boundary orbits converge to infinity. Indeed, for points inside the Baker domain, the convergence takes place through the same access to infinity (known as the *dynamical access*, see [BFJK17]). Thus, we introduce the notion of *Carathéodory set* as the set of points in $\partial U$ which converge to the image under $\varphi^*$ of the Denjoy-Wolff point with respect to the Carathéodory topology of $\partial U$ (or, morally, the points on $\partial U$ which converge to $\infty$ through the dynamical access).

**Definition B. (Carathéodory set)** Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Baker domain. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g = \varphi^{-1} \circ f \circ \varphi$ be the inner function associated with $(f, U)$ by $\varphi$. We say that $x \in \partial U$ is in the *Carathéodory set* if, for any crosscut neighbourhood $N \subset \mathbb{D}$ at the Denjoy-Wolff point $p \in \partial\mathbb{D}$, there exists $k_0 \geq 0$ such that, for all $k \geq k_0$,

$$f^k(x) \in \overline{\varphi(N)}.$$

Several properties of this set are explored throughout the thesis.

**Remarks.** At this point, we shall make some additional remarks, in order to clarify and contextualize our results. First, one should observe that the class of (transcendental) meromorphic functions is not closed under composition: iterates $f^n$ of a transcendental meromorphic function $f$ have, in general, countably many essential singularities, so they are no longer meromorphic functions of the plane. Hence, we consider functions in class $\mathbb{K}$, the smallest class of functions which includes transcendental meromorphic functions and which is closed under composition. Formally, $f \in \mathbb{K}$ if there exists a compact countable set $E(f) \subset \widehat{\mathbb{C}}$ such that

$$f \colon \widehat{\mathbb{C}} \setminus E(f) \to \widehat{\mathbb{C}}$$

is meromorphic in $\widehat{\mathbb{C}} \setminus E(f)$ but in no larger set. The theory of Fatou and Julia of iteration of rational maps was extended to class $\mathbb{K}$ by Bolsch, and Baker, Domínguez and Herring [Bol96, Bol97, Bol99, BDH01, BDH04, Dom10]. Although more sophisticated tools are needed for the proofs, the main features of iteration theory extend successfully to class $\mathbb{K}$ (see Sect. II.1). In particular, if $f \in \mathbb{K}$, then for any $k \geq 1$, $f^k \in \mathbb{K}$ and $\mathcal{F}(f) = \mathcal{F}(f^k)$. This allows us to reduce the study of $k$-periodic Fatou components to the study of the invariant ones, just replacing $f$ by $f^k$.

Second, we deal with periodic Fatou components which are simply connected. Recall that periodic Fatou components of entire maps are always simply connected [Bak84]. It is well-known that Fatou components of meromorphic functions (and for functions in class $\mathbb{K}$) are either simply connected, doubly connected or infinitely connected. However, there are plenty of examples of functions and classes of functions whose Fatou components are simply connected. For example, if $f$ is an entire function, then its Newton's method

$$N_f \colon \mathbb{C} \to \widehat{\mathbb{C}}, \qquad N_f(z) := z - \frac{f(z)}{f'(z)}$$

is a meromorphic function, whose Fatou components are simply connected [FJT08, FJT11, BFJK14, BFJK18].

Third, in this introduction, theorems are presented in a simplified and incomplete form, but hopefully more readable, for convenience of the reader. Theorems in their complete form can be found in the corresponding chapters inside the thesis.

**Structure of the thesis.** The first chapter is devoted to some preliminary definitions and results, which are used intensively later on. This includes abstract ergodic theory, planar topology, and distortion estimates for conformal maps, among others. In the second chapter, we deal with Fatou components and associated inner functions, in a quite extensive way, summarizing the state-of-the-art in this area, and providing some new results and tools (namely Theorem A and Definition B).

The remaining chapters, numbered with arabic numbers, correspond each to one of the papers or preprints on which this thesis is based (as indicated in page vi), including the complete statement and proof of the corresponding theorem (the one numbered with the same number).

# Chapter I

## Basic background and tools

Before diving in the flied of complex dynamics, we discuss here some wide-ranging topics which arise as valuable tools in our proofs. This includes ergodic theory, planar topology, and distortion estimates for conformal maps, among others.

## I.1 Abstract Ergodic Theory

*Ergodic Theory* is the branch of dynamical systems devoted to study measurable transformations. In contrast with holomorphic dynamics, where a large degree of regularity of the function is assumed, the theory presented here applies to transformations which are only assumed to be measurable. As anticipated in the introduction, we will deal often with measurable transformations, which are only approachable with the tools given by Ergodic Theory.

We note that, although many books and monographs on Ergodic Theory restrict themselves to measure-preserving transformations on probability spaces, the theory presented here applies to general transformations on infinite-measure spaces, unless otherwise specified. Several of the measures considered later on are not probability measures, so we certainly need a theory that applies in this more general context.

### I.1.1 Background on Measure Theory

In the sequel, let $(X, \mathcal{A})$ be a measurable space, and let $(X, \mathcal{A}, \mu)$ be a measure space (i.e. a measurable space endowed with a measure). If $\mu(X) = 1$, we say that $(X, \mathcal{A}, \mu)$ is a *probability space*. Note that any measure space with finite measure (i.e. $\mu(X) < \infty$) can be turned into a probability space by reescaling the measure. We say that the measure $\mu$ is *σ-finite* if there exists a countable collection of measurable sets $\{A_n\}_n \subset \mathcal{A}$ such that $X = \bigcup_n A_n$ and $\mu(A_n) < \infty$.

Recall that, given $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ measurable spaces, we say that $T\colon X_1 \to X_2$ is *measurable* if $T^{-1}(A) \in \mathcal{A}_1$, for every $A \in \mathcal{A}_2$.

**Definition I.1.1. (Properties of measurable maps)** Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$

be measure spaces, and let $T\colon X_1 \to X_2$ be measurable. Then, $T$ is:

- *non-singular*, if, for every $A \in \mathcal{A}_2$, it holds $\mu_1(T^{-1}(A)) = 0$ if and only if $\mu_2(A) = 0$;
- *measure-preserving*, if, for every $A \in \mathcal{A}_2$, it holds $\mu_1(T^{-1}(A)) = \mu_2(A)$.

Mostly, we will use the measure space $(\partial\mathbb{D}, \mathcal{B}(\partial\mathbb{D}), \lambda)$, where $\mathcal{B}(\partial\mathbb{D})$ denotes the Borel $\sigma$-algebra of $\partial\mathbb{D}$, and $\lambda$, its normalized Lebesgue measure.

**Definition I.1.2. (Lebesgue density)** Given a Borel set $A \in \mathcal{B}(\partial\mathbb{D})$, the *Lebesgue density* of $A$ at $\xi \in \partial\mathbb{D}$ is defined as

$$d_\xi(A) := \lim_{\rho\to 0} \frac{\lambda(A \cap D(\xi,\rho))}{\lambda(D(\xi,\rho))}.$$

A point $\xi \in \partial\mathbb{D}$ is called a *Lebesgue density point* for $A$ if $d_\xi(A) = 1$.

**Proposition I.1.3. (Almost every point is a Lebesgue density point,** [Rud87, p. 138]**)** *Given a Borel set $A \in \mathcal{B}(\partial\mathbb{D})$, with $\lambda(A) > 0$, then $\lambda$-almost every point in $A$ is a Lebesgue density point for $A$.*

We will make use of the following well-known results.

**Lemma I.1.4. (First Borel-Cantelli lemma,** [Bog07, 1.12.89]**)** *Let $(X, \mathcal{A}, \mu)$ be a probability space, let $\{A_n\}_n \subset \mathcal{A}$, and let*

$$B := \{x \in X \colon x \in A_n \text{ for infinitely many } n\text{'s}\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.$$

*Then, if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, it holds $\mu(B) = 0$.*

### I.1.2 Basic notions of Ergodic Theory

Now, let us turn to the case when only one measure space $(X, \mathcal{A}, \mu)$ is considered, and $T$ is a measurable transformation mapping $X$ into itself: the setting of Ergodic Theory. We present here the basic concepts used to describe these measure-theoretical dynamical systems, as well as fundamental results on the area, such as the Poincaré Recurrence Theorem, or the Birkhoff Ergodic Theorem. For more details, see e.g. [Aar97, PU10, Haw21].

**Definition I.1.5. (Ergodic properties of measurable maps)** Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $T\colon X \to X$ be measurable. Then, $T$ is:

- $\mu$ is $T$- *invariant* if $T$ is measure-preserving;
- *recurrent*, if for every $A \in \mathcal{A}$ and $\mu$-almost every $x \in A$, there exists a sequence $n_k \to \infty$ such that $T^{n_k}(x) \in A$;
- *ergodic*, if $T$ is non-singular and for every $A \in \mathcal{A}$ with $T^{-1}(A) = A$, it holds $\mu(A) = 0$ or $\mu(X \smallsetminus A) = 0$.

Clearly, invariance implies non-singularity. Moreover, the following holds true.

**Theorem I.1.6. (Poincaré Recurrence Theorem,** [Haw21, Thm. 2.12]**)** *Let* $(X, \mathcal{A}, \mu)$ *be a measure space, and let* $T\colon X \to X$ *be a measurable transformation. Assume* $\mu(X) < \infty$*, and* $T$ *is* $\mu$*-preserving. Then,* $T$ *is recurrent with respect to* $\mu$*.*

**Theorem I.1.7. (Almost every orbit is dense,** [Aar97, Prop. 1.2.2]**)** *Let* $(X, \mathcal{A}, \mu)$ *be a measure space, and let* $T\colon X \to X$ *be non-singular. Then, the following are equivalent.*

(a) $T$ *is ergodic and recurrent.*

(b) *For every* $A \in \mathcal{A}$ *with* $\mu(A) > 0$*, we have that for* $\mu$*-almost every* $x \in X$*, there exists a sequence* $n_k \to \infty$ *such that* $T^{n_k}(x) \in A$*.*

Note that, if the space $X$ is endowed with a topology whose open sets are measurable and have positive measure, then statement (b) implies that $\mu$-almost every orbit is dense in $X$.

In holomorphic dynamics, it is possible to replace the function by an iterate of it, since the dynamics remain essentially the same. Thus, we are interested in knowing which ergodic properties remain under taking iterates of the function.

**Lemma I.1.8. (Ergodic properties for $T^k$)** *Let* $(X, \mathcal{A}, \mu)$ *be a measure space, and let* $T\colon X \to X$ *be non-singular. Let* $k$ *be a positive integer. Then,*

(a) $T$ *is recurrent if and only if* $T^k$ *is recurrent.*

(b) *If* $T^k$ *is ergodic, so is* $T$*. The converse is not true in general.*

*Proof.* (a) It is clear that $T^k$ recurrent implies $T$ recurrent. We shall see the converse. To do so, consider $A \in \mathcal{A}$ with $\mu(A) > 0$. Since $T$ is assumed to be recurrent, for $\mu$-almost every $x \in A$ there exists a sequence $n_j \to \infty$ such that $T^{n_j}(x) \in A$. For such $x$, consider the following subsequences

$$\left\{T^{kn}(x)\right\}_n, \left\{T^{kn+1}(x)\right\}_n, \dots, \left\{T^{2kn-1}(x)\right\}_n.$$

At least one of them, say $\left\{T^{kn+l}(x)\right\}_n$, contains infinitely many $T^{n_j}(x)$'s. Choose $n$ and $k$ so that

$$y \coloneqq T^{kn+l}(x) \in A.$$

Then, $y$ is a point in $A$ whose orbit returns to $A$ infinitely many times under $T^k$. We claim that such points have full measure in $A$. Assume that, on the contrary, there exists $B \subset A$ with $\mu(B) > 0$ such that, for all $x \in B$, $T^{nk}(x) \in A$, only for finitely many $n$'s. Applying the same procedure as before to $B$ we can find a point in $B$ whose orbit returns to $B$, and hence to $A$, infinitely many times under $T^k$, which is a contradiction. Hence, $T^k$ is recurrent.

(b) Let $A \in \mathcal{A}$ be such that $T^{-1}(A) = A$. Then, $T^{-k}(A) = A$, and, since $T^k$ is assumed to be ergodic, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. Thus, $T$ is ergodic.

To see that the converse is not true in general, consider the space $\mathbb{Z}$ endowed with the counting measure $\mu$, i.e. given $X \subset \mathbb{Z}$, $\mu(X)$ is the number of elements of $X$. Then, the translation $T \coloneqq x \mapsto x+1$ is ergodic, since there are no proper $T$-invariant subsets of $X$. However, $T^2 = x \mapsto x+2$ is not ergodic, since $2\mathbb{Z}$ is invariant, and $\mu(2\mathbb{Z}) > 0$ and $\mu(\mathbb{Z} \setminus 2\mathbb{Z}) > 0$.

$\square$

Finally, we state Birkhoff Ergodic Theorem, for measure-preserving transformations in probability spaces.

**Theorem I.1.9. (Birkhoff Ergodic Theorem,** [KH95, Sect. 4.1]**)** *Let $(X, \mathcal{A}, \mu)$ be a probability space together with a measure-preserving transformation $T\colon X \to X$, and let $\varphi \in L^1(\mu)$. Then,*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x))$$

*exists for $\mu$-almost every $x \in X$. If $T$ is an automorphism, the equality*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x)) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^{-k}(x))$$

*holds $\mu$-almost everywhere.*
*Finally, if $T$ is ergodic with respect to $\mu$, then for $\mu$-almost every $x \in X$ it holds*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x)) = \int_X \varphi d\mu.$$

## I.2 Basic planar topology

We provide now some standard notions and results of planar topology.

By a *simple arc*, or a *Jordan arc*, we mean a set homeomorphic to the closed interval $[0,1]$. By a *closed simple curve*, or a *Jordan curve*, we mean a set homeomorphic to a circle. Recall the well-known Jordan Curve Theorem.

**Theorem I.2.1. (Jordan Curve Theorem)** *Let $\gamma$ be a simple closed curve in $\widehat{\mathbb{C}}$. Then, $\gamma$ separates $\widehat{\mathbb{C}}$ into precisely two connected components.*

By a *domain* $U \subset \mathbb{C}$, we mean a connected open set. A domain $U$ is *simply connected* if every closed curve in $U$ is homotopic to a point in $U$. We shall use the following criterion to characterize when a domain is simply connected.

**Theorem I.2.2. (Criterion for simple connectivity,** [Bea91, Prop. 5.1.3]**)** *Let $U$ be a domain in $\widehat{\mathbb{C}}$. Then, $U$ is simply connected if, and only if, $\widehat{\mathbb{C}} \setminus U$ is connected.*

Given a domain $U \subset \mathbb{C}$, one shall be interested in studying which points in $\partial U$ which can be reached from $U$ by curves; the so-called *accessible points*. To define them, we need the following preliminary definition.

**Definition I.2.3. (Landing set of a curve)** Given a curve $\gamma\colon [0,1) \to \widehat{\mathbb{C}}$, we consider its *landing set*

$$L(\gamma) := \left\{ v \in \widehat{\mathbb{C}}\colon \text{ there exists } \{t_n\}_n \subset [0,1)\,, t_n \to 1 \text{ such that } \gamma(t_n) \to v \right\}.$$

By definition, $L(\gamma)$ is a connected, compact subset of $\widehat{\mathbb{C}}$. We say that $\gamma$ *lands* at $v \in \widehat{\mathbb{C}}$ if $L(\gamma) = \{v\}$, or, equivalently, if

$$\lim_{t\to 1^-} \gamma(t) = v.$$

Accessible points are defined as follows.

**Definition I.2.4. (Accessible point)** Given a domain $U \subset \mathbb{C}$, a point $p \in \widehat{\partial} U$ is *accessible* from $U$ if there exists a curve $\gamma \subset U$ landing at $p$.

There may exist different curves landing at the same boundary point which are not (homotopically) equivalent. This leads to the following definition.

**Definition I.2.5. (Access)** Let $U \subset \widehat{\mathbb{C}}$ be a simply connected domain. Given $z_0 \in U$ and $p \in \widehat{\partial} U$, a homotopy class (with fixed endpoints) of curves $\gamma\colon [0,1] \to \widehat{\mathbb{C}}$ such that $\gamma([0,1)) \subset U$, $\gamma(0) = z_0$ and $\gamma(1) = p$ is called an *access* from $U$ to $p$.

## I.3 Distortion estimates for conformal maps

The following results provide explicit bounds for the distortion of conformal maps.

**Theorem I.3.1. (De Branges,** [dB85]**)** *Let $\varphi\colon \mathbb{D} \to \mathbb{C}$ be univalent, with $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then,*

$$\varphi(z) = z + \sum_{n\geq 2} a_n z^n,$$

*with $|a_n| \leq n$, for $n \geq 2$.*

**Corollary I.3.2. (Distortion estimates for univalent maps)** *Let $\varphi\colon D(z_0, r_0) \to \mathbb{C}$ be univalent, and let $r \in (0, r_0)$. Then, there exists $C := C(r, r_0)$, with $C(r, r_0) \to 0$ as $\frac{r}{r_0} \to 0$, such that, for all $z \in D(z_0, r)$,*

$$|\varphi(z) - L(z)| \leq C\, |\varphi'(z_0)|\, |z - z_0|\,,$$

*where $L$ stands for the liner map $L(z) := \varphi(z_0) + \varphi'(z_0)(z - z_0)$.*
*In particular, if $\varphi$ additionally satisfies $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then, for all $r \in (0,1)$, there exists $C := C(r)$, with $C(r) \to 0$ as $r \to 0$, for all $z \in D(0, r)$,*

$$\left| \frac{\varphi(z)}{z} - 1 \right| \leq C.$$

Note that, in both cases, $C$ does not depend on the univalent map considered.

*Proof.* Let us start by proving the particular case of $\varphi\colon \mathbb{D} \to \mathbb{C}$, satisfying $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then, by Theorem I.3.1, for all $z \in \mathbb{D}$, it holds

$$\frac{\varphi(z)}{z} = 1 + \sum_{n\geq 2} a_n z^{n-1},$$

with $|a_n| \leq n$, for $n \geq 2$. Hence, for $r \in (0,1)$ and $z \in D(0,r)$, it holds

$$\left|\frac{\varphi(z)}{z} - 1\right| = \left|\sum_{n\geq 2} a_n z^{n-1}\right| \leq \sum_{n\geq 2} |a_n|\, r^{n-1} \leq \sum_{n\geq 2} n r^{n-1} =: C(r).$$

Note that the last power series converges for $r < 1$, and $C(r) \to 0$ as $r \to 0$, as desired.

Now, consider any univalent map $\varphi\colon \mathbb{D} \to \mathbb{C}$, and let $\psi\colon \mathbb{D} \to \mathbb{C}$ be defined as

$$\psi(w) \coloneqq \frac{\varphi(z_0 + r_0 w) - \varphi(z_0)}{r_0 \varphi'(z_0)}.$$

Note that $\psi$ is univalent, and satisfies $\psi(0) = 0$ and $\psi'(0) = 1$. Let $r < r_0$ and $\rho \coloneqq \frac{r}{r_0} < 1$. Hence, there exists $C \coloneqq C(\rho)$, such that, for $w \in D(0,\rho)$,

$$\left|\frac{\psi(w)}{w} - 1\right| \leq C.$$

Letting $z = z_0 + r_0 w$, we get that, for $z \in D(z_0, r)$,

$$\frac{|\varphi(z) - (\varphi(z_0) + \varphi'(z_0)(z - z_0))|}{|z - z_0|\, |\varphi'(z_0)|} \leq C,$$

as desired. □

**Theorem I.3.3. (Koebe's distortion estimates,** [BFJK20, p. 639]**)** *Let $z \in \mathbb{C}$, $r > 0$, and let $\varphi\colon D(z,r) \to \mathbb{C}$ be a univalent map. Then,*

$$D\left(\varphi(z), \frac{1}{4} \cdot |\varphi'(z)| \cdot r\right) \subset \varphi(D(z,r)).$$

*Moreover, for all $\lambda \in (0,1)$ and $z \in \overline{D(x, \lambda r)}$, it holds*

$$|\varphi'(x)| \cdot \frac{1-\lambda}{(1+\lambda)^3} \leq |\varphi'(z)| \leq |\varphi'(x)| \cdot \frac{1+\lambda}{(1-\lambda)^3},$$

$$\varphi(D(x,\lambda r)) \subset D\left(\varphi(x), r \cdot |\varphi'(x)| \cdot \frac{1+\lambda}{(1-\lambda)^3}\right).$$

## I.4 Regular and singular values for holomorphic maps

It is well-known that singular values play a central role in holomorphic dynamics (see e.g. [Ber93, Sect. 4.3] and references therein). Hence, as expected, throughout the thesis we will make an extensive use of the concepts of regular and singular values. Although these definitions are quite standard in the context of entire or meromorphic maps (i.e. with one single essential singularity), we believe it is useful to give definitions in the rather general context of functions of class $\mathbb{K}$ or inner functions, which is the setting we are going to work in.

We consider the following class of meromorphic functions, denoted by $\mathbb{M}$, consisting of functions

$$f\colon \widehat{\mathbb{C}} \smallsetminus E(f) \longrightarrow \widehat{\mathbb{C}},$$

where $\Omega(f) \coloneqq \widehat{\mathbb{C}} \smallsetminus E(f)$ is the largest set where $f$ is meromorphic, and, for all $z \in E(f)$, the cluster set $Cl(f, z)$ of $f$ at $z$ is $\widehat{\mathbb{C}}$, that is

$$Cl(f,z) = \left\{ w \in \widehat{\mathbb{C}}\colon \text{ there exists } \{z_n\}_n \subset \Omega(f),\, z_n \to z,\, f(z_n) \to w \right\} = \widehat{\mathbb{C}}.$$

If $E(f) = \emptyset$, then $f$ is rational and we make the further assumption that $f$ is non-constant. Note that $\Omega(f)$ is open, and $E(f)$ has empty interior. Indeed, if $z$ is an interior point for $E$, there does not exist any sequence in $\Omega(f)$ converging to $z$, and hence $Cl(f, z)$ is empty, a contradiction.

In this general setting, regular and singular values, and critical and asymptotic values, are defined as follows. Appropriate charts have to be used when dealing with $\infty$.

**Definition I.4.1. (Regular and singular values)** Given a value $v \in \widehat{\mathbb{C}}$, we say that $v$ is a *regular* value for $f$ if there exists $r \coloneqq r(v) > 0$ such that all branches $F_1$ of $f^{-1}$ are well-defined (and, hence, conformal) in $D(v, r)$. Otherwise we say that $v$ is a *singular* value for $f$.

The set of singular values of $f$ is denoted by $SV(f)$. Note that $SV(f)$ is closed by definition, and it is the smallest set for which

$$f\colon \widehat{\mathbb{C}} \smallsetminus (E(f) \cup f^{-1}(SV(f))) \longrightarrow \widehat{\mathbb{C}} \smallsetminus SV(f)$$

is a covering map.

**Definition I.4.2. (Critical and asymptotic values)** Given a value $v \in \widehat{\mathbb{C}}$, we say that $v$ is a *critical value* if there exists $z \in \Omega$ such that $f'(z) = 0$ and $f(z) = v$. We say that $z$ is a *critical point.*
We say that $v$ is an *asymptotic value* if there exists a curve $\gamma\colon\ [0, 1) \to \Omega$ such that $\gamma(t) \to \partial\Omega$ and $f(\gamma(t)) \to v$, as $t \to 1$. We say that the curve $\gamma$ is an *asymptotic path.*

The set of critical values of $f$ is denoted by $CV(f)$, while $AV(f)$ stands for the set of asymptotic values. Note that we do not assume, in general, that $\gamma(t)$ lands at a definite point $\partial\Omega$ as $t \to 1$. However, this is the case for both inner functions and functions in class $\mathbb{K}$ (Lemma II.3.13). We say that $v$ is an asymptotic value corresponding to $x \in \partial\Omega$ if $v = \lim_{t\to 1} f(\gamma(t))$, where $\gamma$ is a curve such that $\gamma(t) \to x$ as $t \to 1$. Note that an asymptotic value may correspond to more than one point in $\partial\Omega$.

The following lemma makes explicit the relation between regular and singular values, and critical and asymptotic values, in the sense of Iversen [Ive14, BE95]. We refer to [BE95] for a proof of it (although the proof in [BE95] is done for meromorphic functions, the argument is local, and hence works for $f \in \mathbb{M}$ as well).

**Lemma I.4.3. (Characterization of singular values)** *Let $f \in \mathbb{M}$. Then,*

$$SV(f) = \overline{CV(f) \cup AV(f)}.$$

# Chapter II

# Fatou components and associated inner functions

This chapter is devoted to study the main tool when dealing with invariant Fatou components: the associated inner function.

As anticipated in the introduction, for simply connected invariant Fatou components, the Riemann map can be used as a uniformization. Indeed, let $U$ be a invariant Fatou component of $f$ and let $\varphi$ be a Riemann map from the open unit disk $\mathbb{D}$ onto $U$. Then,

$$g\colon \mathbb{D} \longrightarrow \mathbb{D}, \qquad g \coloneqq \varphi^{-1} \circ f \circ \varphi$$

is an analytic self-map of $\mathbb{D}$, and $f|_U$ and $g|_{\mathbb{D}}$ are conformally conjugate by $\varphi$. Therefore, the study of holomorphic self-maps of $\mathbb{D}$ is a good approach to analyze the dynamics of $f|_U$ and, as we will see, to describe $f|_{\partial U}$, even though neither $\varphi$ nor $g$ extend continuously to $\overline{\mathbb{D}}$ in general.

We will make this precise next (Sect. II.1), together with the description of the dynamics of inner functions (in the unit disk and on the boundary, Sect. II.3), the boundary behaviour of the Riemann map (Sect. II.4), and its transference to the dynamical plane (Sect. II.5).

Most of the results presented in this section are well-known (or are not difficult consequences of well-known statements, whose proofs we include for completeness). However, the set of tools used in the literature when working with inner functions is far-reaching, and includes several rather different topics (conformal mapping and Riemann maps, harmonic measure and estimates, prime ends and accessibility, among others), spread out among several references. Hence, for convenience of the reader, this chapter of the thesis aims to be an up-to-date and self-contained explanation of the method of associating inner functions with Fatou components.

As mentioned in the introduction, two new tools have been developed, used systematically in this thesis, and with the potentiality of being useful to deal with other boundary problems. First, we study when inverse branches are well-defined around points in the unit circle, and we estimate how the radius is distorted under such inverse branches (Theorem A). Second, for Baker domains, we introduce the notion of Carathéodory set as the set of points in the boundary with the same asymptotic behaviour as points in the interior of the Baker domain, with respect to the Carathéodory topology of $\partial U$ (Definition B). Morally,it is the set of points on $\partial U$ which converge to the essential singularity through the dynamical access and, as we will see, plays an important role in the boundary dynamics.

## II.1 Iteration in class $\mathbb{K}$ and associated inner functions

Consider $f \in \mathbb{K}$, i.e.

$$f\colon \widehat{\mathbb{C}} \smallsetminus E(f) \to \widehat{\mathbb{C}},$$

where $\Omega(f) := \widehat{\mathbb{C}} \smallsetminus E(f)$ is the largest set where $f$ is meromorphic and $E(f)$ is the set of singularities of $f$, which is assumed to be closed and countable. Note that $\Omega(f)$ is open.

**Notation.** Having fixed a function $f \in \mathbb{K}$, we denote $\Omega(f)$ and $E(f)$ simply by $\Omega$ and $E$, respectively. Given a domain $U \subset \Omega$, we denote by $\partial U$ the boundary of $U$ in $\Omega$, and we keep the notation $\widehat{\partial} U$ for the boundary with respect to $\widehat{\mathbb{C}}$.

The dynamics of such functions has been studied in [Bol96, Bol97, Bol99, BDH01, BDH04, Dom10, DMdOS22]. The standard theory of Fatou and Julia for rational or entire functions extends successfully to this more general setting. The Fatou set $\mathcal{F}(f)$ is defined as the largest open set in which $\{f^n\}_n$ is well defined and normal, and the Julia set $\mathcal{J}(f)$, as its complement in $\widehat{\mathbb{C}}$. We need the following properties.

**Theorem II.1.1. (Properties of Fatou and Julia sets,** [BDH01, Thm. A]**)** *Let $f \in \mathbb{K}$. Then,*

(a) *$\mathcal{F}(f)$ is completely invariant in the sense that $z \in \mathcal{F}(f)$ if and only if $f(z) \in \mathcal{F}(f)$;*

(b) *for every positive integer $k$, $f^k \in \mathbb{K}$, $\mathcal{F}(f^k) = \mathcal{F}(f)$ and $\mathcal{J}(f^k) = \mathcal{J}(f)$;*

(c) *$\mathcal{J}(f)$ is perfect;*

(d) *repelling periodic points are dense in $\mathcal{J}(f)$.*

By (a), Fatou components (i.e. connected components of $\mathcal{F}(f)$) are mapped among themselves, and hence classified into periodic, preperiodic or wandering. By (b), the study of periodic Fatou components reduces to the invariant ones, i.e. those for which $f(U) \subset U$. Those Fatou components are classified into attracting basins, parabolic basins, Siegel disks, Herman rings and Baker domains [BDH01, Thm. C]. A *Baker domain* is, by definition, a periodic Fatou component $U$ of period $k \geq 1$ for which there exists $z_0 \in \widehat{\partial} U$ such that $f^{nk}(z) \to z_0$, for all $z \in U$ as $n \to \infty$, but $f^k$ is not meromorphic at $z_0$. In such case, $z_0$ is accessible from $U$ [BDH01, p. 658].

**Theorem II.1.2. (Connectivity of Fatou components,** [Bol99]**)** *Let $f \in \mathbb{K}$, and let $U$ be a periodic Fatou component of $f$. Then, the connectivity of $U$ is 1, 2, or $\infty$.*

In the sequel, we focus on simply connected periodic Fatou components, which we assume to be invariant. There are plenty of examples of functions and classes of functions whose Fatou components are simply connected. For instance, periodic Fatou components of entire maps are always simply connected [Bak84]. Moreover, if $f$ is an entire function, then its Newton's method

$$N_f\colon \mathbb{C} \to \widehat{\mathbb{C}}, \quad N_f(z) \coloneqq z - \frac{f(z)}{f'(z)}$$

is a meromorphic function, whose Fatou components are simply connected [FJT08, FJT11, BFJK14, BFJK18].

The main tool when working with a simply connected invariant Fatou component is the Riemann map, and the conjugacy that it induces between the original function in the Fatou component and an inner function of the unit disk $\mathbb{D}$. Before proceeding, let us recall the definition of inner function.

**Definition II.1.3. (Inner function)** A holomorphic self-map of the unit disk $g\colon \mathbb{D} \to \mathbb{D}$ is an *inner function* if, for $\lambda$-almost every point $\xi \in \partial\mathbb{D}$,

$$g^*(\xi) \coloneqq \lim_{t\to 1^-} g(t\xi) \in \partial\mathbb{D}.$$

If, in addition, $g(0) = 0$, we say that $g$ is a *centered inner function*.

Then, let $f \in \mathbb{K}$ and let $U$ be a simply connected invariant Fatou component for $f$, which we assume to be simply connected. Consider $\varphi\colon \mathbb{D} \to U$ to be a Riemann map. Then, $f\colon U \to U$ is conjugate by $\varphi$ to a holomorphic map $g\colon \mathbb{D} \to \mathbb{D}$, such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & U \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ \mathbb{D} & \xrightarrow{\ g\ } & \mathbb{D} \end{array}$$

commutes.

For entire or meromorphic functions, where the unique essential singularity lies at $\infty$, it is well-known that $g$ is an inner function (see e.g. [EFJS19, Sect. 2.3], or [ERS20, Prop. 1.1]). The same holds for functions in class $\mathbb{K}$ (see Prop. II.5.3). We say that $g$ is an *inner function associated with* $(f, U)$. Note that two inner functions associated with the same $(f, U)$ are conformally conjugate and hence have the same dynamical behaviour.

Since $U$ is unbounded, $f|_U$ need not be a proper self-map of $U$, and thus $f|_U$ has infinite degree. In this case, the associated inner function $g$ has also infinite degree, and must have at least one singularity on the boundary of the unit disk.

**Definition II.1.4. (Singularity of an inner function)** Let $g$ be an inner function. A point $\xi \in \partial\mathbb{D}$ is called a *singularity* of $g$ if $g$ cannot be continued analytically to a neighbourhood of $\xi$. Denote the set of all singularities of $g$ by $E(g)$.

Throughout this paper we assume that an inner function $g$ is always continued to $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by the reflection principle, and to $\partial\mathbb{D} \setminus E(g)$ by analytic continuation. In other words, $g$ is considered as a meromorphic function

$$g \colon \widehat{\mathbb{C}} \setminus E(g) \longrightarrow \widehat{\mathbb{C}}.$$

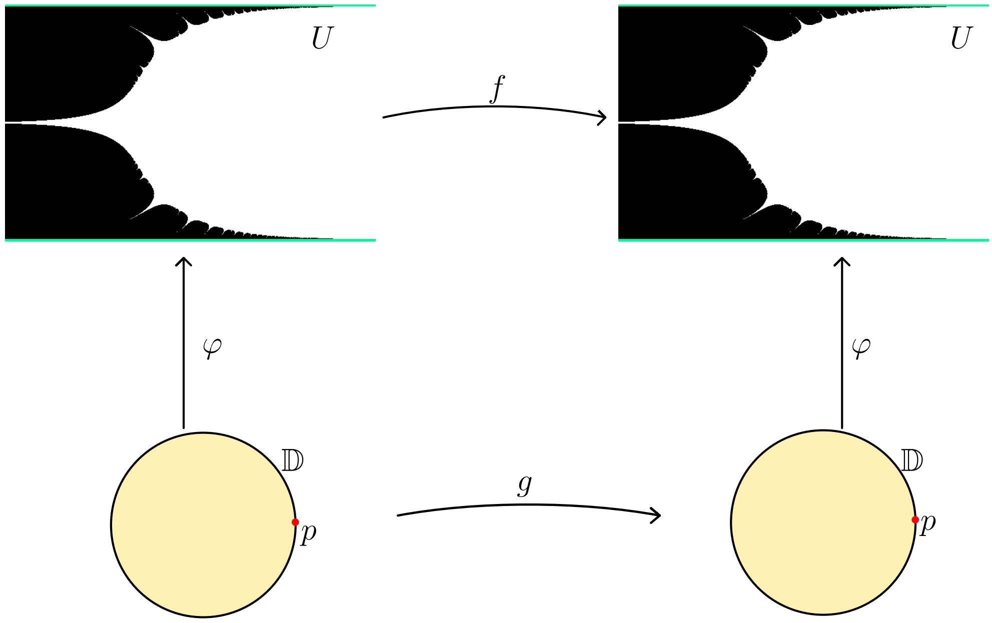



Figure II.1: This diagram shows the construction of the inner function. Here, we have the dynamical plane of $f(z) = z + e^{-z}$, with one of its invariant Baker domains $U$ (in white), compare with Chapter 1. In the Baker domain, iterates converge to $\infty$ under iteration, while in the unit disk, they converge to the Denjoy-Wolff point $p \in \partial\mathbb{D}$. The inner function associated with this particular domain was computed explicitly in [BD99, Thm. 5.2]: $g(z) = \frac{z^2+3}{1+3z^2}$.

## II.2 Boundary extension of meromorphic maps $h \colon \mathbb{D} \to \widehat{\mathbb{C}}$

Throughout the thesis, we shall make an intensive use of the following concepts, which describe ways one may approach a boundary point $\xi \in \partial\mathbb{D}$.

In the sequel, we denote the (Euclidean) disk of radius $\rho > 0$ centered at $\xi \in \partial\mathbb{D}$ by $D(\xi, \rho)$. We also consider the radial segment at $\xi$ of length $\rho > 0$,

$$R_\rho(\xi) \coloneqq \{r\xi \colon r \in (1-\rho, 1)\}.$$

**Definition II.2.1. (Crosscut neighbourhoods and Stolz angles)** Let $\xi \in \partial\mathbb{D}$.

- A *crosscut* $C$ is an open Jordan arc $C \subset \mathbb{D}$ such that $\overline{C} = C \cup \{a, b\}$, with $a, b \in \partial\mathbb{D}$. If $a = b$, we say that $C$ is *degenerate*; otherwise it is *non-degenerate*.

- A *crosscut neighbourhood* of $\xi \in \partial\mathbb{D}$ is an open set $N \subset \mathbb{D}$ such that $\xi \in \partial N$, and $C \coloneqq \partial N \cap \mathbb{D}$ is a non-degenerate crosscut. We usually write $N_\xi$ or $N_C$, to stress the dependence on the point $\xi$ or on the crosscut $C$. Note that for a crosscut neighbourhood $N$, $\partial\mathbb{D} \cap \overline{N}$ is a non-trivial arc.

- Given $\xi \in \partial\mathbb{D}$, a *Stolz angle*[1] at $\xi$ is a set of the form

$$\Delta_{\alpha,\rho} = \{z \in \mathbb{D} \colon\ |\text{Arg}\ \xi - \text{Arg}\ (\xi - z)| < \alpha, |z| > 1 - \rho\}\,.$$

- We say that $\gamma$ *lands non-tangentially* at $\xi \in \partial\mathbb{D}$ if $\gamma$ lands at $\xi$, and there exists a Stolz angle $\Delta_{\alpha,\rho}$ at $\xi$ with $\gamma \subset \Delta_{\alpha,\rho}$.

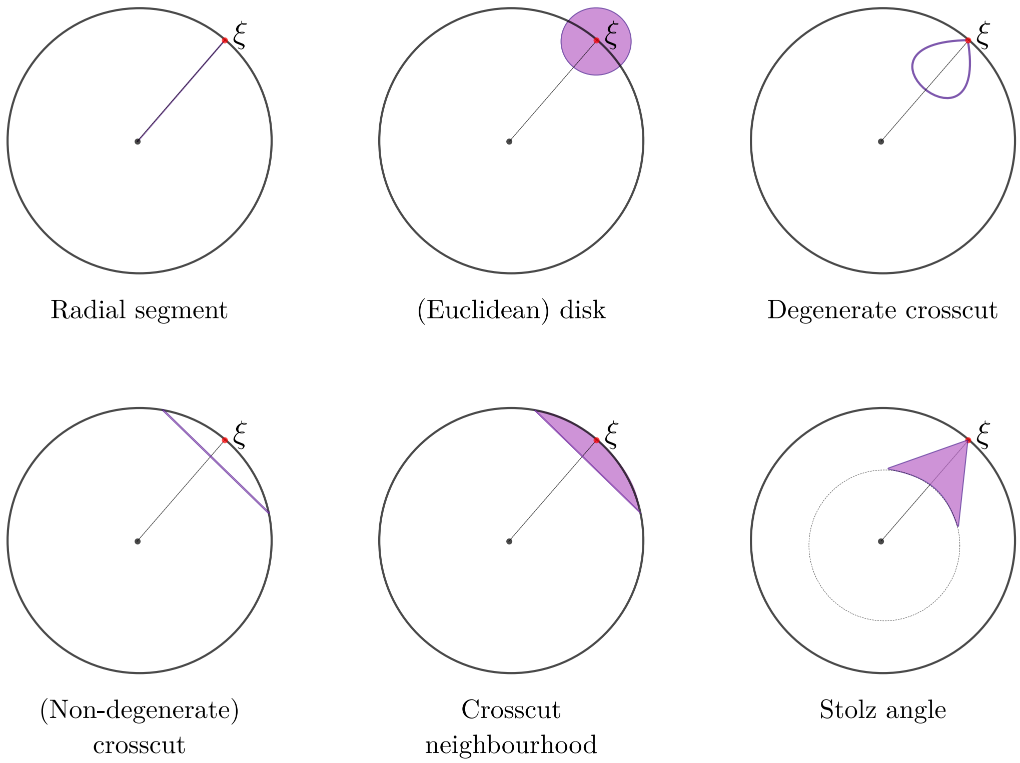



Figure II.2: Different sets related to $\xi \in \partial\mathbb{D}$.

Some times it is more convenient to work in the upper half-plane $\mathbb{H}$ rather than in the unit disk $\mathbb{D}$. The previous concepts can be defined analogously for points in $\partial\mathbb{H}$. In particular, the specific formulas for both the radial segment and Stolz angles at a point $x \in \mathbb{R}$ are

$$R_\rho^{\mathbb{H}}(x) \coloneqq \{z \in \mathbb{H} \colon \text{Im}\ w < \rho, \text{Re}\ w = x\}\,;$$

$$\Delta_{\alpha,\rho}^{\mathbb{H}}(x) \coloneqq \left\{z \in \mathbb{H} \colon \text{Im}\ w < \rho, \frac{|\text{Re}\ w - x|}{\text{Im}\ w} < \tan\alpha\right\}.$$

A more flexible notion of radial segment and Stolz angle will be needed for our purposes.

[1]Note that the usual defintion of Stolz angle is

$$\Delta = \{z \in \mathbb{D} \colon\ |\text{Arg}\ \xi - \text{Arg}\ (\xi - z)| < \alpha, |\xi - z| < \rho\}\,.$$

However, both definitions are equivalent for our purposes, and the stated one is slightly more convenient in our setting.

**Definition II.2.2. (Generalized radial arc and Stolz angle)** Let $p \in \overline{\mathbb{D}}$ and let $\xi \in \partial\mathbb{D}$, $\xi \neq p$. Let $\rho > 0$ and $0 < \alpha < \pi/2$.

- If $p \in \mathbb{D}$, consider the Möbius transformation $M\colon \mathbb{D} \to \mathbb{D}$, $M(z) = \dfrac{p-z}{1-\overline{p}z}$. Then, the *(generalized) radial segment* $R_\rho(\xi, p)$ of length $\rho$ at $\xi$ is defined as the preimage under $M$ of the radial segment $R_\rho(M(\xi))$. Analogously, the *(generalized) Stolz angle* $\Delta_{\alpha,\rho}(\xi, p)$ of angle $\alpha$ and length $\rho$ is the preimage under $M$ of the Stolz angle $\Delta_{\alpha,\rho}(M(\xi))$. That is,
$$R_\rho(\xi, p) \coloneqq M^{-1}(R_\rho(M(\xi))),$$
$$\Delta_{\alpha,\rho}(\xi, p) \coloneqq M^{-1}(\Delta_{\alpha,\rho}(M(\xi))).$$
- If $p \in \partial\mathbb{D}$, consider the Möbius transformation $M\colon \mathbb{D} \to \mathbb{H}$, $M(z) = i\dfrac{p+z}{p-z}$. Then, the *(generalized) radial segment* and *Stolz angle* at $\xi$ are defined as the preimages of the corresponding radial segment and Stolz angle at $M(\xi) \in \mathbb{R}$. That is,
$$R_\rho(\xi, p) \coloneqq M^{-1}(R^{\mathbb{H}}_\rho(M(\xi)))$$
$$\Delta_{\alpha,\rho}(\xi, p) \coloneqq M^{-1}(\Delta^{\mathbb{H}}_{\alpha,\rho}(M(\xi))).$$

See Figures II.3 and II.4.

Observe that $R_\rho(\xi) = R_\rho(\xi, 0)$, and $\Delta_{\alpha,\rho}(\xi) = \Delta_{\alpha,\rho}(\xi, 0)$. Note also that $R_\rho(\xi, p)$ is a curve landing non-tangentially at $\xi \in \partial\mathbb{D}$, while $\Delta_{\alpha,\rho}(\xi, p)$ is an angular neighbourhood of $\xi$, since Möbius transformations are conformal, and hence angle-preserving.

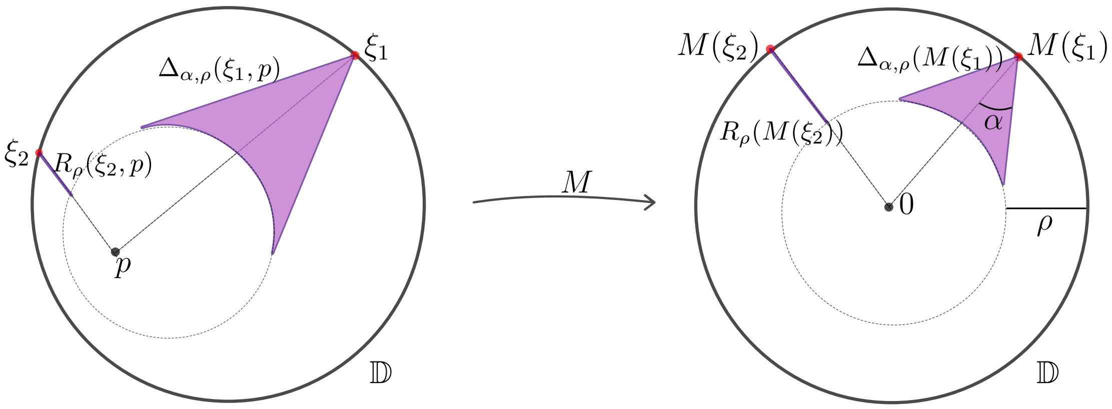



Figure II.3: Radial arc and angular neighbourhood with respect to $p \in \partial\mathbb{D}$.

We are interested in the boundary behaviour of meromorphic maps $h\colon \mathbb{D} \to \widehat{\mathbb{C}}$. Since $h$ may not extend continuously to $\partial\mathbb{D}$, the concepts of radial and angular limit are a keystone on studying the boundary behavior of $h$.

**Definition II.2.3. (Radial and angular limit)** Let $h\colon \mathbb{D} \to \widehat{\mathbb{C}}$ be a meromorphic map, and let $\xi \in \partial\mathbb{D}$. We say that $h$ has *radial limit* at $\xi$ if the limit
$$h^*(\xi) \coloneqq \lim_{t \to 1^-} h(t\xi)$$

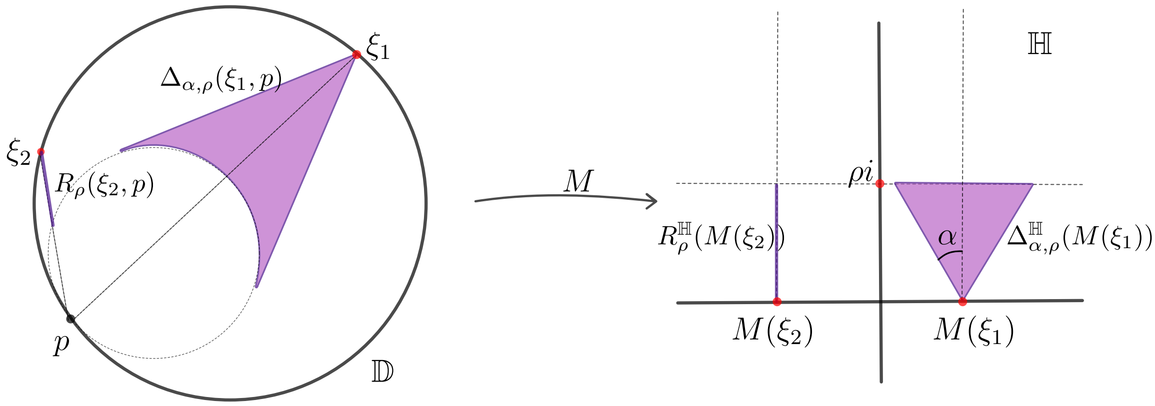



Figure II.4: Radial arc and angular neighbourhood with respect to $p \in \partial\mathbb{D}$.

exists. We say that $h$ has *angular limit* at $\xi$ if, for any Stolz angle $\Delta$ at $\xi$, the limit

$$\lim_{z\to\xi, z\in\Delta} h(z)$$

exists.

Note that, whenever we write $h^*(\xi) = p$ we are assuming implicitly that the radial limit exists, and equals $p$. The map

$$h^*\colon \partial\mathbb{D} \to \widehat{\mathbb{C}}$$

is called the *radial extension* of $h$ (defined wherever the radial limit exists).

**Theorem II.2.4. (Radial extensions are measurable,** [Pom92, Prop. 6.5]**)** *Let $h\colon \mathbb{D} \to \widehat{\mathbb{C}}$ be continuous. Then, the points $\xi \in \partial\mathbb{D}$ where the radial limit $h^*$ exists form a Borel set, and if $A \subset \widehat{\mathbb{C}}$ is a Borel set, then*

$$(h^*)^{-1}(A) := \{\xi \in \partial\mathbb{D} \colon h^*(\xi) \in A\} \subset \partial\mathbb{D}$$

*is also a Borel set.*

For maps $h\colon \mathbb{D} \to \widehat{\mathbb{C}}$ omitting three values in $\widehat{\mathbb{C}}$, the following well-known theorem of Lehto and Virtanen relates radial and angular limits.

**Theorem II.2.5. (Lehto-Virtanen,** [Pom92, Sect. 4.1]**)** *Let $h\colon \mathbb{D} \to \widehat{\mathbb{C}}$ be a meromorphic map omitting three values in $\widehat{\mathbb{C}}$. Let $\gamma$ be a curve in $\mathbb{D}$ landing at $\xi \in \partial\mathbb{D}$. If $h(\gamma)$ lands at a point $p \in \mathbb{C}$, then $h$ has angular limit at $\xi$ equal to $p$. In particular, radial and angular limits are the same.*

**Remark II.2.6. (Limit on generalized radial arcs and Stolz angles)** Note that, in particular, the Lehto-Virtanen Theorem justifies that, for meromorphic maps omitting three values, it is equivalent to take the limit along the radial segment, than along any generalized radial arc. Likewise, the angular limit can be computed along generalized Stolz angles.

## II.3 Dynamics of inner functions

This section contains the fundamental results concerning iteration of inner functions both in the unit disk $\mathbb{D}$ and on the unit circle $\partial\mathbb{D}$, as well as the proof of Theorem A.

### II.3.1 Internal dynamics of self-maps of the unit disk

The asymptotic behaviour of the iterates of a holomorphic self-map of the unit disk is essentially determined by the Denjoy-Wolff theorem. Moreover, results by Schwarz, Wolff and Cowen give a more precise description of the dynamics. We collect their work in this section are valid, which is valid for any holomorphic self-map of $\mathbb{D}$, not necessarily an inner function.

**Theorem II.3.1. (Denjoy-Wolff,** [Mil06, Thm. 5.2]**)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be holomorphic, which is not the identity nor an elliptic Möbius transformation. Then, there exists a point $p \in \overline{\mathbb{D}}$, the Denjoy-Wolff point of $g$, such that for all $z \in \mathbb{D}$, $g^n(z) \to p$.*

Hence, holomorphic self-maps of $\mathbb{D}$ are classified into two types: the elliptic ones, for which $p \in \mathbb{D}$, and the non-elliptic ones, with $p \in \partial\mathbb{D}$. In the first case, the Schwarz lemma describes the dynamics precisely.

**Theorem II.3.2. (Schwarz lemma,** [Mil06, Lemma 1.2]**)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be holomorphic, with $g(0) = 0$. Then, for all $z \in \mathbb{D}$, $|g(z)| \leq |z|$, and $|g'(0)| \leq 1$.*

An analogous result was obtained by Wolff for non-elliptic self-maps of $\mathbb{D}$.

**Theorem II.3.3. (Wolff lemma,** [Wol26]**)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be holomorphic, with Denjoy-Wolff point $p \in \partial\mathbb{D}$. Let $D \subset \mathbb{D}$ be an open disk tangent to $\partial\mathbb{D}$ at $p$. Then, $g(D) \subset D$. In particular, $g^*(p) = p$.*

Another equivalent way of stating Wolff lemma is that, for any holomorphic function $h\colon \mathbb{H} \to \mathbb{H}$ with Denjoy-Wolff point $\infty$ and any upper half-plane $H$, $h(H) \subset H$ (see also [Bar08, Lemma 2.33]).

Note that, in the elliptic case, $g$ is holomorphic in a neighbourhood of the Denjoy-Wolff point $p \in \mathbb{D}$, which is fixed and it is either attracting (if $|g'(p)| \in (0, 1)$) or superattracting (if $g'(p) = 0$). In the former case, $g$ is conjugate to $z \mapsto |g'(p)|\, z$ in a neighourhood of $p$ (by Koenigs Theorem, see e.g. [Mil06, Chap. 8]). In the latter case, the dynamics are conjugate to those of $z \mapsto z^d$, where $d$ stands for the local degree of $g$ at $p$ (by Böttcher Theorem, see e.g. [Mil06, Chap. 9]).

An analogous result for the non-elliptic case is given by the following result of Cowen, which leads to a classification of non-elliptic self-maps of $\mathbb{D}$ in terms of the dynamics near the Denjoy-Wolff point.

**Definition II.3.4. (Absorbing domains and fundamental sets)** Let $U$ be a domain in $\mathbb{C}$ and let $f\colon U \to U$ be a holomorphic map. A domain $V \subset U$ is said to be an *absorbing domain* for $f$ in $U$ if $f(V) \subset V$ and for every compact set $K \subset U$ there exists $n \geq 0$ such that $f^n(K) \subset V$. If, additionally, $V$ is simply connected and $f|_V$ is univalent, $V$ is said to be a *fundamental set* for $f$ in $U$.

**Theorem II.3.5. (Cowen's classification of self-maps of $\mathbb{D}$,** [Cow81]) *Let $g$ be a holomorphic self-map of $\mathbb{D}$ with Denjoy-Wolff point $p \in \partial\mathbb{D}$. Then, there exists a fundamental set $V$ for $g$ in $\mathbb{D}$.*
*Moreover, given a fundamental set $V$, there exists a domain $\Omega$ equal to $\mathbb{C}$ or $\mathbb{H} = \{\operatorname{Im} z > 0\}$, a holomorphic map $\psi\colon \mathbb{D} \to \Omega$, and a Möbius transformation $T\colon \Omega \to \Omega$, such that:*

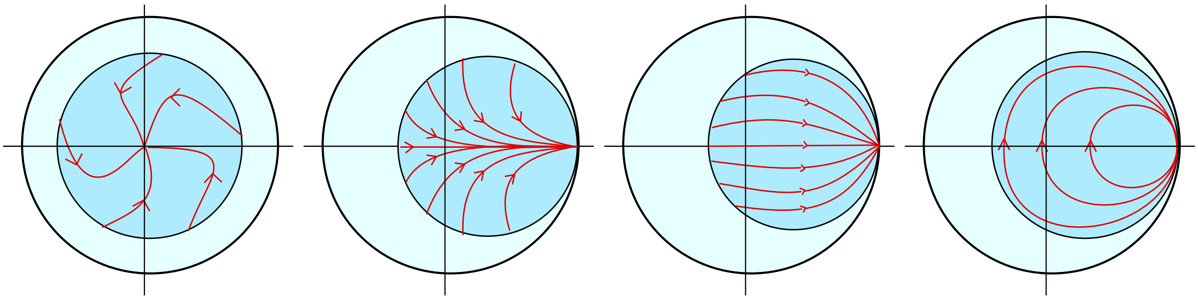
(a) *$\psi(V)$ is a fundamental set for $T$ in $\Omega$,*

(b) *$\psi \circ g = T \circ \psi$ in $\mathbb{D}$,*

(c) *$\psi$ is univalent in $V$.*

*Moreover, $T$ and $\Omega$ depend only on the map $g$, not on the fundamental set $V$. In fact (up to a conjugacy of $T$ by a Möbius transformation preserving $\Omega$), one of the following cases holds:*

- $\Omega = \mathbb{C}$, $T = \mathrm{id}_{\mathbb{C}} + 1$ (doubly parabolic type),
- $\Omega = \mathbb{H}$, $T = \lambda \mathrm{id}_{\mathbb{H}}$, *for some* $\lambda > 1$ (hyperbolic type),
- $\Omega = \mathbb{H}$, $T = \mathrm{id}_{\mathbb{H}} \pm 1$ (simply parabolic type).



(a) Elliptic (b) Doubly parabolic (c) Hyperbolic (d) Simply parabolic

Figure II.5: The different types of convergence to the Denjoy-Wolff point.

Finally, note that if $g$ is a self-map of $\mathbb{D}$, so is $g^k$, for all $k \geq 1$. The type in Cowen's classification is preserved by taking iterates.

**Lemma II.3.6. (Cowen's classification for $g^k$)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ holomorphic, and let $k$ be a positive integer. Then, $g$ is elliptic (resp. doubly parabolic, hyperbolic, simply parabolic) if and only if so is $g^k$.*

*Proof.* It is clear that $g$ is of elliptic type if and only if so is $g^k$. Now, assume that $p \in \partial\mathbb{D}$ is the Denjoy-Wolff point of $g$, and choose a fundamental set $V$ for $g$ in $\mathbb{D}$. Then, $V$ is a fundamental set for $g^k$ in $\mathbb{D}$. It follows that $g|_V$ is conformally conjugate to $T_1\colon \Omega_1 \to \Omega_1$, and $g^k|_V$ is conformally conjugate to $T_2\colon \Omega_2 \to \Omega_2$. Hence, $T_1\colon \Omega_1 \to \Omega_1$ and $T_2\colon \Omega_2 \to \Omega_2$ are conformally conjugate. Since $T$ and $\Omega$ are unique up to conformal conjugacy, and do not depend on the choice of the fundamental set, it follows that $g$ and $g^k$ are of the same type in Cowen's classification. □

### II.3.2 Dynamics of inner functions on the unit circle

Once the dynamics of inner functions inside the unit disk are well-understood, let us examine the dynamics induced by such functions on the unit circle. Note that inner functions may not extend continuously to any point on $\partial\mathbb{D}$, so it is *a priori* unclear how to proceed.

We find two different approaches in the literature. On the one hand, Aaronson [Aar78] and Doering and Mañé [DM91] studied the measure-theoretical dynamical system given by the radial extension $g^*\colon \partial\mathbb{D} \to \partial\mathbb{D}$, from an ergodic point of view. On the other hand, Baker and Domínguez [BD99] and Bargmann [Bar08] define and study Fatou and Julia sets of inner functions, with a more topological point of view. We present both theories next, which together provide a detailed description of the dynamics.

#### The measure-theoretical approach of Aaronson, Doering and Mañé

Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function, and consider the dynamical system induced by its the radial extension
$$g^*\colon \partial\mathbb{D} \to \partial\mathbb{D}.$$
Recall that if $g$ is an inner function, so is $g^k$ [BD99, Lemma 4], so the equality
$$(g^n)^*(\xi) = (g^*)^n(\xi)$$
holds for all $n \geq 0$ $\lambda$-almost everywhere. Moreover, the radial extension $g^*$ is measurable (Thm. II.2.4), and hence analyzable from the point of view of ergodic theory. The following is a recollection of ergodic properties of $g^*$, with precise references.

**Theorem II.3.7. (Ergodic properties of $g^*$)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function with Denjoy-Wolff point $p \in \overline{\mathbb{D}}$. The following holds.*

(a) *$g^*$ is non-singular. In particular, for $\lambda$-almost every $\xi \in \partial\mathbb{D}$, its infinite orbit under $g^*$, $\{(g^n)^*(\xi)\}_n$, is well-defined.*

(b) *$g^*$ is ergodic if and only if $g$ is elliptic or doubly parabolic.*

(c) *If $g^*$ is recurrent, then it is ergodic. In this case, for every $A \in \mathcal{B}(\mathbb{D})$ with $\lambda(A) > 0$, we have that for $\lambda$-almost every $\xi \in \partial\mathbb{D}$, there exists a sequence $n_k \to \infty$ such that $(g^{n_k})^*(\xi) \in A$. In particular, $\lambda$-almost every $\xi \in \partial\mathbb{D}$, $\{(g^n)^*(\xi)\}_n$ is dense in $\partial\mathbb{D}$.*

(d) *If $g$ is an elliptic inner function, then $g^*$ is recurrent.*

(e) *The radial extension of a doubly parabolic inner function is not recurrent in general. However, if $g$ is doubly parabolic and the Denjoy-Wolff point $p$ is not a singularity for $g$, then $g^*$ is recurrent. Moreover, if $g$ is doubly parabolic and there exists $z \in \mathbb{D}$ and $r > 1$ such that*

$$\operatorname{dist}_{\mathbb{D}}(g^{n+1}(z), g^n(z)) \leq \frac{1}{n} + O\left(\frac{1}{n^r}\right),$$

*as $n \to \infty$, then $g^*$ is recurrent.*

(f) *If $g$ is hyperbolic or simply parabolic, then $g^*$ is non-ergodic and non-recurrent. Moreover, $\lambda$-almost every $\xi \in \partial\mathbb{D}$ converges to the Denjoy-Wolff point under the iteration of $g^*$.*

(g) *Let $k$ be a positive integer. Then, $g^k$ is an inner function. Moreover, $g^*$ is ergodic (resp. recurrent) if and only if $(g^k)^*$ is ergodic (resp. recurrent).*

*Proof.* (a) The proof that $g^*$ is non-singular can be found in [Aar97, Prop. 6.1.1]. We claim that this already implies that, for $\lambda$-almost every $\xi \in \partial\mathbb{D}$, its infinite orbit $\{(g^n)^*(\xi)\}_n$ is well-defined. We shall prove it by induction. First, it is clear that the set

$$\{\xi \in \partial\mathbb{D} \colon g^*(\xi) \text{ is well-defined}\}$$

has full measure, and, since $g^*$ is non-singular, $g^*(\partial\mathbb{D}) = 1$. Now, assume that the set

$$\left\{\xi \in \partial\mathbb{D} \colon \{(g^n)^*(\xi)\}_{n=0}^{k-1} \text{ is well-defined}\right\}$$

has full measure, and $\lambda((g^{k-1})^*(\partial\mathbb{D})) = 1$. Then, the set

$$(g^{k-1})^*(\partial\mathbb{D}) \cap \{\xi \in \partial\mathbb{D} \colon (g^n)^*(\xi) \text{ is well-defined}\}$$

has also full measure, proving that the orbit $\{g^*(\xi)\}_{n=0}^k$ is well-defined for $\lambda$-almost every $\xi \in \partial\mathbb{D}$, as desired.

(b) It follows from combining [DM91, Thm. G] with [Bon97, Thm. 1.4].

(c) See [DM91, Thm. E, F], as well as [Aar97, Thm. 6.1.7].

(d) The first statement is [DM91, Corol. 1] (see also [Aar97, Thm. 6.1.8]). The second statement follows from applying Theorem I.1.7, and applying that open sets in $\partial\mathbb{D}$ have positive measure. See also [BEF$^+$24, Sect. 8.3].

(e) An example of a doubly parabolic inner function whose boundary map is not recurrent is given in [BFJK19, Example 1.3]. Conditions which imply recurrence are found in [BFJK19, Thm. B] and [BFJK19, Thm. E], respectively.

(f) See [DM91, Thm. 3.1, Thm. 4.1, Corol. 4.3], combined with [Bar08, Lemma 2.6].

(g) Finally, the proof that $g^k$ is an inner function can be found in [BD99, Lemma 4]. Since ergodicity depends only on the type in Cowen's classification II.3.5, which is invariant under talking iterates (Lemma II.3.6), $g^*$ is ergodic if and only if $(g^k)^*$ is ergodic. Recurrence is always preserved under taking iterates (see Lemma I.1.8).

□

The next theorem deals with the existence of invariant measures.

**Theorem II.3.8. (Invariant measures for $g^*|_{\partial\mathbb{D}}$,** [DM91, Thm. A, C]**)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function.*

(i) *If $g$ is elliptic, assume $0$ is the Denjoy-Wolff point of $g$. Then, the Lebesgue measure $\lambda$ is invariant under $g^*$.*

(ii) *If $g$ is doubly parabolic, assume $1$ is the Denjoy-Wolff point of $g$. Then, the $\sigma$-finite measure*
$$\lambda_{\mathbb{R}}(A) := \int_A \frac{1}{|w-1|^2} d\lambda(w), \qquad A \in \mathcal{B}(\partial\mathbb{D}),$$
*is invariant under $g^*$.*

Note that the measure $\lambda_{\mathbb{R}}$ is the push-forward of the Lebesgue measure on $\mathbb{R}$ (up to multiplication by a constant) under any Möbius map transforming the upper half plane to the unit disk, and sending $\infty$ to 1.

We need the following result, which is extracted from the proof of [DM91, Thm. 3.1].

**Theorem II.3.9.** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be a hyperbolic or simply parabolic inner function. Then, there exists $h\colon \mathbb{D} \to \mathbb{D}$ and a Möbius transformation $T\colon \mathbb{D} \to \mathbb{D}$ such that $h \circ g = T \circ h$. Moreover, $h = \lim_n h^n$, where $h_n = T_n \circ g^n$, $T_n\colon \mathbb{D} \to \mathbb{D}$ Möbius, and $h_n(0) = 0$ for all $n$.*

It follows from [Fer23] that $h$ is inner (and $h(0) = 0$, and hence preserves the Lebesgue measure on $\partial\mathbb{D}$); moreover, $h$ is a Möbius transformation if and only if $g$ is univalent, and otherwise $h$ has infinite degree. Note that $T$ cannot have fixed points, and thus is hyperbolic or simply parabolic (doubly parabolic inner functions are never univalent).

Applying the Lehto-Virtanen Theorem II.2.5, one deduces that $h^*$ and $h^* \circ g^*$ exists $\lambda$-almost everywhere, and
$$h^* \circ g^* = T \circ h^*,$$
where defined.

### The topological approach of Baker, Domínguez and Bargmann

Another approach to describe the dynamics of inner functions in $\partial\mathbb{D}$ is developed in [BD99, Bar08], where instead of considering the induced measure-theoretical dynamical system as in [Aar78, DM91] it is considered the holomorphic dynamical system given by the maximal meromorphic extension of $g$, and the normality of the sequences of iterates is studied.

More precisely, assume $g$ is not a Möbius transformation. Then, the *Fatou set* $\mathcal{F}(g)$ is the set of all points $z \in \widehat{\mathbb{C}}$ for which there exists an open neighbourhood $U \subset \widehat{\mathbb{C}}$ of $z$ such that $\{g^n|_U\}_n$ is well-defined and normal. The *Julia set* $\mathcal{J}(g)$ is the complement of $\mathcal{F}(g)$ in $\widehat{\mathbb{C}}$. In view of the Denjoy-Wolff Theorem, it is clear that the Fatou set is precisely the set of points for which iterates converge locally uniformly to the Denjoy-Wolff point.

It follows from Montel's theorem that $\mathcal{J}(g) \subset \partial\mathbb{D}$. Note also that, if $g$ has finite degree, then it is a Blaschke product, and thus a rational map. In this case, the definition of the Fatou and Julia sets agrees with the usual one for rational functions. Moreover, the following holds.

**Lemma II.3.10. (Properties of Fatou and Julia sets of inner functions,** [BD99, Lemma 8], [Bar08, Thm. 2.34] **)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function. Then,*

(a) *$g(\mathcal{F}(g)) \subset \mathcal{F}(g)$;*

(b) *if $g$ is non-Möbius, then $\mathcal{J}(g)$ is a perfect set;*

(c) *if $g$ is non-rational, then $\mathcal{J}(g) = \overline{\bigcup_n E(g^n)}$.*

It is well-known that, if $g$ is elliptic or doubly parabolic, then $\mathcal{J}(g) = \partial\mathbb{D}$ [Bar08, Thm. 2.24]. For non-Möbius hyperbolic and simply parabolic inner functions, it may happen $\mathcal{J}(g) = \partial\mathbb{D}$, as well as $\mathcal{J}(g) \neq \partial\mathbb{D}$, see [Bar08, Sect. 2.5] (both situations can happen also for inner functions associated with non-univalent Baker domains of entire function; see [Bar08, Ex. 3.6], where $\mathcal{J}(g) = \partial\mathbb{D}$; and [BZ12], where $\mathcal{J}(g) \neq \partial\mathbb{D}$).

We need the following additional properties.

**Lemma II.3.11. (Characterization of singularities of inner functions)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function. Then, $\xi \in E(g)$ if and only if, for any crosscut neighbourhood $N_\xi$ of $\xi$,*

$$\overline{g(N_\xi)} = \overline{\mathbb{D}}.$$

*Moreover, for every $\eta \in \partial\mathbb{D}$ and every neighbourhood $U$ of $\xi$, there exists $\zeta \in U \cap \partial\mathbb{D}$ such that $g^*(\zeta) = \eta$.*

*Proof.* The first statement can be found in [Gar07, Thm. II.6.6], while the second is proven in [BD99, Lemma 5 and Corollary]. □

**Lemma II.3.12. (Iterated preimages are dense in the Julia set)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be a non-Möbius inner function, and let $\zeta \in \partial\mathbb{D}$. Then,*

$$\overline{\bigcup_{n\geq 0} \{\xi \in \partial\mathbb{D} \colon (g^n)^*(\xi) = \zeta\}} \supset \mathcal{J}(g).$$

*Proof.* In the case when $g$ has finite degree, it follows from the standard theory of Fatou and Julia sets. In the case when $g$ has infinite degree, there exists at least one singularity of $g$, and

$$\mathcal{J}(g) = \overline{\bigcup_{n\geq 0} E(g^n)},$$

by Lemma II.3.10(c). By Lemma II.3.11, each singularity is approximated by radial preimages of every point in $\partial\mathbb{D}$, and the lemma follows. □

### II.3.3 Iterated inverse branches on $\partial\mathbb{D}$ and distortion

Finally, we study iteration on $\partial\mathbb{D}$ from a different point of view: iterating backwards. Indeed, we analyze when iterated inverse branches are well-defined (and hence, conformal), and the quantify the distortion exercised on the radial segment, as stated in Theorem A. To do so, we study carefully the singular values for inner functions, as indicated in Section I.4, and combine this results with estimates of conformal maps, given in Section I.3.

#### Singular values for inner functions

Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function, considered as its maximal meromorphic extension

$$g\colon \widehat{\mathbb{C}} \smallsetminus E(g) \to \widehat{\mathbb{C}},$$

as before. We shall consider regular and singular values for $g$ as introduced in Section I.4. We observe that for any inner function $g$ and any $z \in E(g)$, the cluster set (as defined in Section I.4) is $\widehat{\mathbb{C}}$. Indeed, if $z \in E(g)$, then Lemma II.3.11 and Schwarz reflection imply that $Cl(g, z) = \widehat{\mathbb{C}}$ (compare also with [BD99]). Thus, $g \in \mathbb{M}$, and all the previous description of singular values applies.

Note that, since $\mathbb{D}$ and $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ are totally invariant, there cannot be critical points nor critical values in $\partial\mathbb{D}$. Moreover, by symmetry, if $v \in \mathbb{D}$ is regular (resp. singular), then so is $1/\overline{v}$. Hence, we consider

$$SV(g, \overline{\mathbb{D}}) = \left\{ v \in \overline{\mathbb{D}}\colon v \text{ is singular} \right\}.$$

We start by proving that asymptotic paths actually land at points in $E(g)$.

**Lemma II.3.13. (Asymptotic paths land)** *Let $v \in \overline{\mathbb{D}}$ be an asymptotic value for $g$, and let $\gamma\colon [0, 1) \to \widehat{\mathbb{C}} \smallsetminus E(g)$ be an asymptotic path for $v$. Then, there exists a singularity $\xi \in E(g) \subset \partial\mathbb{D}$ such that $\gamma(t) \to \xi$, as $t \to 1$.*

*Proof.* Assume, on the contrary, that the landing set $L(\gamma)$ of the asymptotic path $\gamma$ is a continuum in $E(g)$. Then, $L(\gamma)$ is a closed non-degenerate interval in the unit circle.

On the one hand, for $\lambda$-almost every point $\xi$ in $L(\gamma)$, the radial limit $g^*(\xi)$ exists. Let us denote the radial segment at $\xi$ by $R_\xi$.

On the other hand, without loss of generality, we can assume $\gamma\colon [0, 1) \to \overline{\mathbb{D}} \smallsetminus E(g)$. Then, since $L(\gamma) \subset E(g)$ and $\gamma \subset \overline{\mathbb{D}} \smallsetminus E(g)$, for every point $\xi \in L(\gamma)$ (except at the endpoints), there exists a sequence $\{\xi_n\} \subset \gamma \cap R_\xi$ with $\xi_n \to \xi$. Then, $g(\xi_n) \to v$, implying that the radial limit $g^*(\xi)$ equals $v$. This contradicts the fact that radial limits are different almost everywhere (Thm. 4.8.4). □

Next we prove that singular values in $\partial\mathbb{D}$ correspond to accumulation points of singular values in $\mathbb{D}$.

**Proposition II.3.14. (Singular values on $\partial\mathbb{D}$)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function, and let $\xi \in \partial\mathbb{D}$. The following are equivalent.*

(a) *There exists a crosscut $C$, with crosscut neighbourhood $N_C$ and $\xi \in \partial N_C$ such that $SV(g) \cap N_C = \emptyset$.*

(b) *$v$ is regular, i.e. there exists $\rho := \rho(\xi) > 0$ such that all inverse branches $G_1$ of $g$ are well-defined in $D(\xi, \rho)$.*

*Proof.* The implication (b)⇒(a) is trivial. Let us prove (a)⇒(b). Without loss of generality, we can assume that there are no singular values in $\partial\mathbb{D} \cap \overline{N_C}$. Moreover, note that (a) implies that all inverse branches $G_1$ are well-defined (and conformal) in $N_C$. The assumption that there are no singular values in $\partial\mathbb{D} \cap \overline{N_C}$ implies that $G_1$ is holomorphic in $\overline{N_C} \cap \mathbb{D}$. We shall show that $G_1$ can be extended across $\partial\mathbb{D}$ by Schwarz reflection (see Fig. II.6).

To this end, let $\varphi\colon \mathbb{D} \to N_C$ be a Riemann map. Note that it extends homeomorphically to $\partial N_C$; we shall denote this extension again by $\varphi$. Then,

$$G_1 \circ \varphi\colon \mathbb{D} \to G_1(N_C)$$

is a Riemann map for the simply connected domain $G_1(N_C)$, where $G_1$ is any branch of $g^{-1}$.

Consider the radial extension

$$G_1^*\colon \partial N_C \to \partial G_1(N_C),$$

defined as

$$G_1^*(x) = (G_1 \circ \varphi)^*(\varphi^{-1}(x)),$$

for $x \in \partial N_C$. Note that, since $G_1|_{N_C}$ is bounded, the radial extension is well-defined almost everywhere in $\partial N_C$. Since we assume that $G_1$ is holomorphic in $\overline{N_C} \cap \mathbb{D}$, it follows that $G_1(\overline{N_C} \cap \mathbb{D}) \subset \mathbb{D}$. Indeed, assume $z \in \overline{N_C} \cap \mathbb{D}$ and $G_1(z) \in \partial\mathbb{D}$. Since $G_1$ is conformal, it would map points in $\mathbb{D}$ to points in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, a contradiction.

Moreover, modifying slightly the crosscut if needed, we can assume that, for the two endpoints $\{\xi_1, \xi_2\} = \partial N_C \cap \partial\mathbb{D}$, the limit

$$G_1^*(\xi_i) = \lim_{z \to \xi_i, z \in N_C \cap \mathbb{D}} G_1(z)$$

exists, for $i = 1, 2$. This is possible since the radial extension is well-defined almost everywhere. Note that, since $g$ is assumed to be an inner function, $G_1^*(\xi_i) \in \partial\mathbb{D}$ (otherwise, there would exist a point in $\mathbb{D}$ mapped by $g$ to $\partial\mathbb{D}$, a contradiction). Hence, $G_1(\partial N_C \cap \mathbb{D})$ is a crosscut in $\mathbb{D}$; we shall denote this crosscut by $C'$. On the other hand, note that, for $x \in \partial N_C \cap \partial\mathbb{D}$, it holds

$$Cl(G_1, x) := \left\{ w \in \overline{\mathbb{D}} \colon \text{ there exists } \{x_n\}_n \subset N_C \text{ with } x_n \to x \text{ and } G_1(x_n) \to w \right\} \subset \partial\mathbb{D},$$

since $g(\mathbb{D}) \subset \mathbb{D}$. Hence,

$$\partial G_1(N_C) \subset C' \cup \partial\mathbb{D}.$$

Therefore, $\partial G_1(N_C)$ is locally connected (and, in fact, a Jordan curve), so $G_1$ extends homeomorphically to $\partial N_C$, and $G_1(\partial N_C \cap \partial\mathbb{D}) \subset \partial\mathbb{D}$.

Thus, by Schwarz reflection, we can extend holomorphically $G_1$ to

$$N_C \cup (\partial N_C \cap \partial\mathbb{D}) \cup \left\{ z \in \widehat{\mathbb{C}} \colon \frac{1}{\overline{z}} \in N_C \right\},$$

(see Fig. II.6). In particular, there exists $\rho \coloneqq \rho(\xi) > 0$ such that all inverse branches of $g$ are well-defined in $D(\xi, \rho)$, as desired.

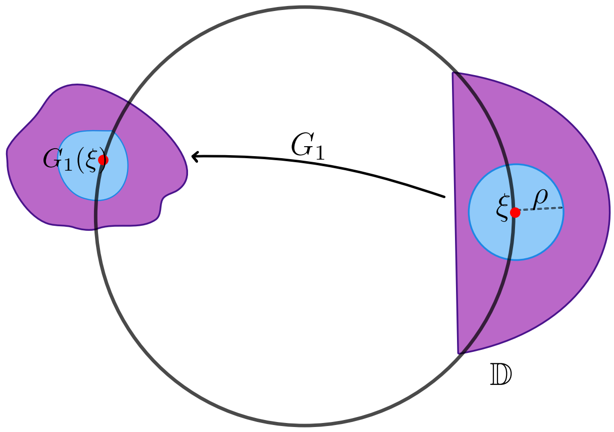



Figure II.6: Whenever an inverse branch $G_1$ is well-defined in a crosscut neighbourhood, it can be extended across the unit circle by Schwarz reflection.

□

It follows from Proposition II.3.14 that a value $v \in \partial\mathbb{D}$ is singular for $g$ if and only if it is accumulated by singular values in $\mathbb{D}$, i.e. $v \in \overline{SV(g) \cap \mathbb{D}}$. Clearly, for finite Blaschke products, all values $v \in \partial\mathbb{D}$ are regular, and the same is true if $SV(g) \cap \mathbb{D}$ is compactly contained in $\mathbb{D}$. Moreover,

**Corollary II.3.15. (Non-singular Denjoy-Wolff point)** *Let $g \colon \mathbb{D} \to \mathbb{D}$ be an inner function with Denjoy-Wolff point $p \in \partial\mathbb{D}$. If $p \notin \overline{SV(g)}$, then $p$ is not a singularity for $g$.*

*Proof.* If $p \notin \overline{SV(g)}$, then there exists a crosscut neighbourhood $N_p$ such that $p \in \partial N_p$ and $SV(g) \cap N_p = \emptyset$. Since the Denjoy-Wolff point is radially fixed (Thm. II.3.3), there exists a curve $\gamma \subset N_p$ landing at $p$, such that $g(\gamma) \subset N_p$ also lands at $p$. Consider $G_1$ the inverse branch of $g^{-1}$ defined in $N_p$ such that $G_1(g(\gamma)) = \gamma$. By Proposition II.3.14, $G_1$ extends conformally to $D(p, \rho)$ for some $\rho > 0$, and $G_1(p) = p$. Then, $D_1 \coloneqq G_1(D(p, \rho))$ is a negihbourhood of $p$, and

$$g \colon D_1 \to D(p, \rho)$$

conformally. Therefore, by Lemma II.3.11, $p$ is not a singularity for $g$, as desired. □

**Remark II.3.16. (At most one asymptotic value per singularity)** By the Lehto-Virtanen Theorem II.2.5, given a singularity $\xi \in E(g)$, there exists at most one asymptotic value $v \in \overline{\mathbb{D}}$ corresponding to $\xi$. Indeed, if $v$ is an asymptotic value corresponding to the singularity $\xi$, there exists a curve landing at $\xi$ whose image lands at $v$. By Lehto-Virtanen Theorem, $g^*(\xi) = v$. Since radial limits, if they exist, are unique, there cannot be more asymptotic values corresponding to $\xi$. Hence,

$$\#E(g) \geq \#AV(g) \cap \overline{\mathbb{D}}.$$

This differs from when a meromorphic function $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is considered, where the essential singularity (infinity) can have infinitely many asymptotic values corresponding to it.

### Distortion of inverse branches

Next, we analyse the distortion induced by inverse branches near $\partial\mathbb{D}$, and how we can control the preimages of radial limits in terms of Stolz angles.

**Proposition II.3.17. (Control of radial limits in terms of Stolz angles)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function with Denjoy-Wolff point $p \in \overline{\mathbb{D}}$. Let $\xi \in \partial\mathbb{D}$, $\xi \neq p$. Assume there exists $\rho_0 > 0$ such that $D(\xi, \rho_0) \cap SV(g) \neq 0$. Then, for all $0 < \alpha < \frac{\pi}{2}$, there exists $\rho_1 := \rho_1(\alpha, \rho_0) < \rho_0$ such that all branches $G_1$ of $g^{-1}$ are well-defined in $D(\xi, \rho_1)$ and, for all $\rho < \rho_1$,*

$$G_1(R_\rho(\xi, p)) \subset \Delta_{\alpha,\rho}(G_1(\xi), p),$$

*where $R_\rho(\cdot, p)$ and $\Delta_{\alpha,\rho}(\cdot, p)$ stand for the generalized radial segment and Stolz angle with respect to $p$* (Def. II.2.2).

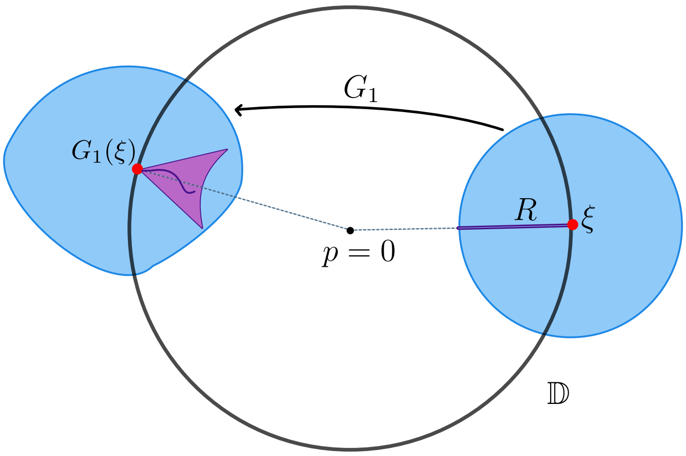



Figure II.7: Whenever an inverse branch $G_1$ is well-defined at a boundary point $\xi \in \partial\mathbb{D}$, $G_1$ sends radial segments into angular neighbourhoods of a given openning. In the figure, $p = 0 \in \mathbb{D}$.

Note that $\rho_1$ depends only on $\rho_0$ and $\alpha$, but not on the point $\xi \in \partial\mathbb{D}$, nor on the inverse branch $G_1$.

*Proof.* Note that, since $D(\xi, \rho_0) \cap SV(g) \neq \emptyset$, all branches $G_1$ of $g^{-1}$ are well-defined in $D(\xi, \rho_0)$. We shall distinguish two cases.

- Assume first that $g$ is elliptic, so $p \in \mathbb{D}$. According to Definition II.2.2, it is enough to consider $g\colon \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$ and prove

$$G_1(R_\rho(\xi, 0)) \subset \Delta_{\alpha,\rho}(G_1(\xi), 0).$$

By Schwarz lemma II.3.2, $|G_1(z)| \geq |z|$ for $z \in D(\xi, \rho_0) \cap \mathbb{D}$. It is left to see that, for $z \in R_\rho(\xi, 0)$,

$$|\text{Arg } G_1(\xi) - \text{Arg } (G_1(\xi) - G_1(z))| < \alpha.$$

To do so, consider the linear map

$$L_1(z) := G_1(\xi) + G_1'(\xi)(z - \xi).$$

Note that $|L_1(z) - G_1(\xi)| = |G_1'(\xi)|\,|z - \xi|$. Moreover, by Corollary I.3.2, there exists $\rho_1 < \rho_0$ and a constant $C(\rho_1) > 0$ such that

$$|G_1(z) - L_1(z)| \leq C(\rho_1)\,|z - \xi|\,|G_1'(\xi)|\,,$$

for all $z \in D(\xi, \rho_1)$. That is, the point $G_1(z)$ belongs to the disk of center $L_1(z)$ and radius $C(\rho_1)\,|z - \xi|\,|G_1'(\xi)|$ (see Fig. II.8).

Since $C(\rho_1) \to 0$ as $\rho_1 \to 0$, we have $\frac{C(\rho_1)}{1-C(\rho_1)} \to 0$ as $\rho_1 \to 0$. Without loss of generality, we assume $C(\rho_1)$ satisfies

$$\frac{C(\rho_1)}{1 - C(\rho_1)} < \tan\alpha.$$

Let

$$\beta := |\text{Arg } G_1(\xi) - \text{Arg } (G_1(\xi) - G_1(z))|\,.$$

We claim that, if $z \in R_\rho(\xi, 0)$, then $\text{Arg } L_1(z) = \text{Arg } (G_1(\xi))$. Indeed, $L_1$ is the affine map associated to $G_1$, which is an inverse branch of $g$. The map $G_1$ is conformal in $D(\xi, \rho_0)$, and hence angle preserving, and $G_1(D(\xi, \rho_0) \cap \partial\mathbb{D}) \subset \partial\mathbb{D}$. From this, it follows that, if $\text{Arg } z = \text{Arg } \xi$, then $\text{Arg } L_1(z) = \text{Arg } (G_1(\xi))$, i.e. if $z$ lies on the radial segment at $\xi$, then $L_1(z)$ lies on the radial segment at $G_1(\xi)$.

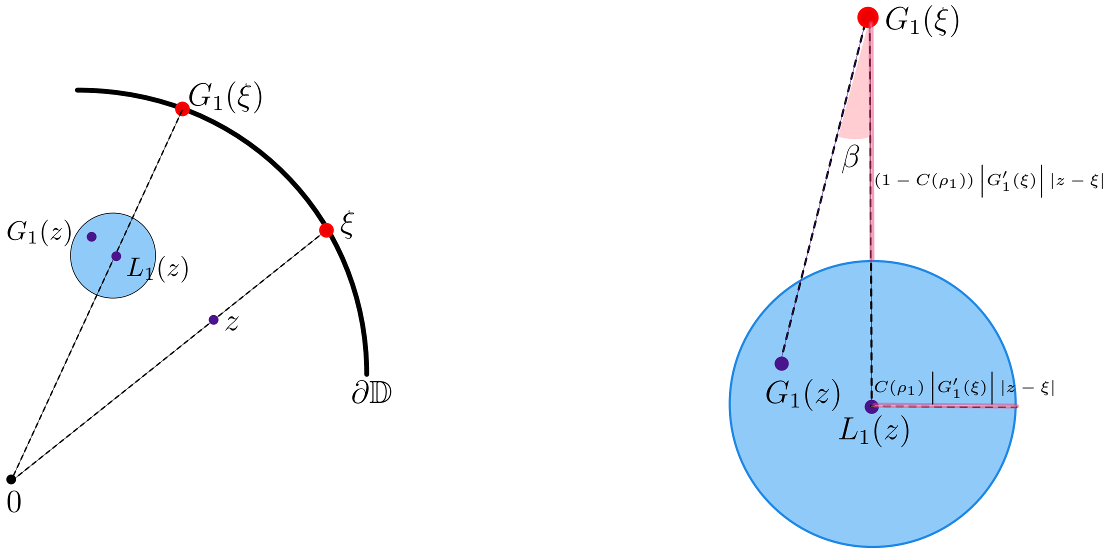
Then, noting that $G_1(z)$ belongs to the disk of center $L_1(z)$ and radius $C(\rho_1)\,|z - \xi|\,|G_1'(\xi)|$, it follows

$$\tan\beta \leq \frac{C(\rho_1)\,|G_1'(\xi)|\,|z - \xi|}{(1 - C(\rho_1))\,|G_1'(\xi)|\,|z - \xi|} = \frac{C(\rho_1)}{1 - C(\rho_1)} \leq \tan\alpha,$$

as desired. See also Figure II.8.

- Assume $g$ is non-elliptic, so $p \in \partial\mathbb{D}$. Note that $G_1(\xi) \neq p$, since $p$ is the Denjoy-Wolff point and, hence, it is radially fixed (Thm. II.3.3).

Now, consider $h\colon \mathbb{H} \to \mathbb{H}$, $h := M \circ g \circ M^{-1}$, where $M\colon \mathbb{D} \to \mathbb{H}$, $M(z) = i\dfrac{p + z}{p - z}$. Then, there exists $\widetilde{\rho}_0$ such that $D(M(\xi), \widetilde{\rho}_0) \cap SV(h) = \emptyset$, and consider $H_1$ the




(a) (b)

Figure II.8: Image (a) shows how the inverse branch $G_1$ acts near $\xi$, and (b) provides more detail on it. Indeed, the point $G_1(z)$ lies on the disk $D(L_1(z), C(\rho_1)\,|G_1'(\xi)|\,|z-\xi|)$. The angle $\beta$ captures the opening of the vector $G_1(z)-G_1(\xi)$ with respect to the vector $L_1(z)-G_1(\xi)$. Hence, $\tan\beta$ is bounded above by the maximal distance between $G_1(z)$ and $L_1(z)$ divided by the minimal distance between $G_1(\xi)$ and $G_1(z)$.

branch of $h^{-1}$ corresponding to $G_1$, well-defined in $D(M(\xi), \widetilde{\rho}_0)$. It is enough to prove that there exists $\widetilde{\rho}_1 < \widetilde{\rho}_0$ such that, for all $\rho < \widetilde{\rho}_1$,

$$H_1(R_\rho^{\mathbb{H}}(M(\xi))) \subset \Delta_{\alpha,\rho}^{\mathbb{H}}(H_1(M(\xi))).$$

First note that, by Wolff lemma II.3.3, if $\text{Im } w < \rho$, then $\text{Im } H_1(w) < \rho$. Now, consider the linear map

$$L_1(w) := H_1(M(\xi)) + H_1'(M(\xi))(w - M(\xi)).$$

Note that $|L_1(w) - H_1(\xi)| = |H_1'(M(\xi))|\,|w - M(\xi)|$. Moreover, by Corollary I.3.2, there exists $\widetilde{\rho}_1 < \widetilde{\rho}_0$ and a constant $C(\widetilde{\rho}_1) > 0$ such that

$$|H_1(w) - L_1(w)| \leq C(\widetilde{\rho}_1)\,|w - M(\xi)|\,|H_1'(M(\xi))|\,,$$

for all $w \in D(\xi, \widetilde{\rho}_1)$. We assume, without loss of generality,

$$\frac{C(\widetilde{\rho}_1)}{1 - C(\widetilde{\rho}_1)} < \tan\alpha.$$

Since $h(\mathbb{H}) \subset \mathbb{H}$, and $H_1$ is a branch of $h^{-1}$, conformal where defined, then if $w \in R_\rho^{\mathbb{H}}(M(\xi))$, then $\text{Re } L_1(w) = H_1(M(\xi))$, i.e. if $w$ lies on the radial segment at $M(\xi)$, then $L_1(w)$ lies on the radial segment at $L_1(M(\xi))$.

We claim that, for $w \in R_\rho^{\mathbb{H}}(M(\xi))$, it holds

$$\frac{|\text{Re } H_1(w) - H_1(M(\xi))|}{\text{Im } H_1(w)} < \tan\alpha.$$

Indeed,

$$|\text{Re } H_1(w) - H_1(M(\xi))| = |\text{Re } H_1(w) - \text{Re } L_1(w)| \leq C(\widetilde{\rho}_1)\, |w - M(\xi)|\, |H_1'(M(\xi))|\, ,$$

$$\text{Im } H_1(w) = \text{Im } (H_1(w) - H_1(M(\xi))) \geq (1 - C(\widetilde{\rho}_1))\, |H_1'(M(\xi))|\, |w - M(\xi)|\, ,$$

as desired. See also Figure II.9.

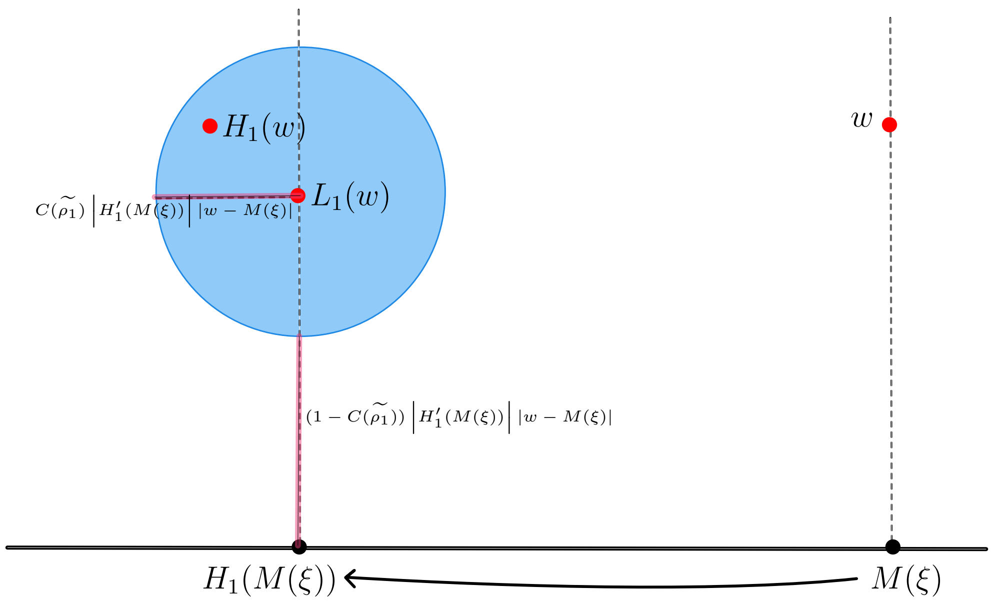



Figure II.9: The point $H_1(w)$ lies on the disk $D(L_1(w), C(\widetilde{\rho}_1)\, |H_1'(\xi)|\, |z - \xi|)$, and this gives the estimates on its real and imaginary part.

□

### Distortion of iterated inverse branches

Consider the *postsingular set*

$$P(g) := \overline{\bigcup_{v \in SV(g)} \bigcup_{n \geq 0} g^n(v)}.$$

The following theorem is now straightforward (and it is the precise version of Theorem A stated at the beginning of the section).

**Theorem II.3.18. (Iterated inverse branches at boundary points I)** *Let* $g \colon \mathbb{D} \to \mathbb{D}$ *be an inner function. Assume there exists* $\xi \in \partial\mathbb{D}$ *and a crosscut neighbourhood* $N_\xi$ *of* $\xi$ *such that* $P(g) \cap N_\xi = \emptyset$*. Then, there exists* $\rho_0 > 0$ *such that all branches* $G_n$ *of* $g^{-n}$ *are well-defined in* $D(\xi, \rho_0)$*, and, for all* $\rho < \rho_0$*,*

$$G_n(R_\rho(\xi), p) \subset \Delta_{\alpha,\rho}(G_n(\xi), p),$$

*where* $R_\rho(\cdot, p)$ *and* $\Delta_{\alpha,\rho}(\cdot, p)$ *stand for the radial segment and the Stolz angle with respect to* $p$ (Def. II.2.2).

If, in addtion, $g^*|_{\partial\mathbb{D}}$ is recurrent, the existence of one crosscut neighbourhood without postsingular values already implies that iterated inverse branches are well-defined $\lambda$-almost everywhere, and singularities have zero $\lambda$-measure. This is the content of the following theorem.

**Theorem II.3.19. (Iterated inverse branches at boundary points II)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function, such that $g^*|_{\partial\mathbb{D}}$ is recurrent. Assume there exists $\zeta \in \partial\mathbb{D}$ and a crosscut neighbourhood $N_\zeta$ of $\zeta$ such that $P(g) \cap N_\zeta = \emptyset$. Then, for $\lambda$-almost every $\xi \in \partial\mathbb{D}$, there exists $\rho_0 := \rho_0(\xi) > 0$ such that all branches $G_n$ of $g^{-n}$ are well-defined in $D(\xi, \rho_0)$. In particular, the set $E(g)$ of singularities of $g$ has $\lambda$-measure zero.*
*In addition, for all $0 < \alpha < \frac{\pi}{2}$, there exists $\rho_1 < \rho_0$ such that, for all $n \geq 0$, all branches $G_n$ of $g^{-n}$ are well-defined in $D(\xi, \rho_1)$ and, for all $\rho < \rho_1$,*

$$G_n(R_\rho(\xi), p) \subset \Delta_{\alpha,\rho}(G_n(\xi), p),$$

*where $R_\rho(\cdot, p)$ and $\Delta_{\alpha,\rho}(\cdot, p)$ stand for the radial segment and the Stolz angle with respect to $p$* (Def. II.2.2).

*Proof.* By Proposition II.3.14, the existence of a crosscut neighbourhood $N_\zeta$ of $\zeta \in \partial\mathbb{D}$ such that $P(g) \cap N_\zeta = \emptyset$, implies the existence of $\rho_\zeta > 0$ such that all branches $G_n$ of $g^{-n}$ are well-defined in $D(\zeta, \rho_\zeta)$.

We have to see that, for $\lambda$-almost every $\xi \in \partial\mathbb{D}$, there exists $\rho_\xi > 0$ such that all branches $G_n$ of $g^{-n}$ are well-defined in $D(\xi, \rho_\xi)$. Since we are assuming $g^*$ to be recurrent, for $\lambda$-almost every $\xi \in \partial\mathbb{D}$, $\{(g^n)^*(\xi)\}_n$ is dense in $\partial\mathbb{D}$ (Thm. II.3.7(c)). Therefore, there exists $n_0 := n_0(\xi)$ such that $(g^{n_0})^*(\xi) \in D(\zeta, \rho_\zeta)$. This already implies the existence of $\rho_\xi > 0$ such that all inverse branches $G_n$ of $g^{-n}$ are well-defined in $D(\xi, \rho_\xi)$.

Next we prove that $\lambda(E(g)) = 0$. To do so, let

$$K := \left\{\xi \in \partial\mathbb{D} \colon \exists \rho > 0 \text{ such that all branches } G_n \text{ of } g^{-n} \text{ are well-defined in } D(\xi, \rho)\right\}.$$

Note that points in $g^{-1}(K)$ do not belong to $E(g)$. Indeed, if $\zeta \in g^{-1}(K)$, then there exists a neighbourhood of $\zeta$ which is mapped conformally to $D(g(\zeta), \rho)$, and hence $\zeta$ cannot be a singularity. Therefore, it is enough to prove that $\lambda(g^{-1}(K)) = 1$. This follows from the fact that $\lambda(K) = 1$ and that $g^*$ is non-singular (Thm. II.3.7(a)).

Finally, the control of the image of radial segments by inverse branches in terms of angular neighbourhoods follows from Proposition II.3.17. Indeed, note that the estimates obtained therein do not depend on the inverse branches considered, but only on the radius of the disk where they are defined. □

**Remark II.3.20.** Observe that a sufficient condition so that hypotheses of Theorem II.3.18 are satisfied is that singular values are compactly contained in $\mathbb{D}$. Indeed, it is enough to show that, if singular values of $g|_\mathbb{D}$ are compactly contained in $\mathbb{D}$, then there exists $\zeta \in \partial\mathbb{D}$ and a crosscut neighbourhood $N_\zeta$ of $\zeta$ such that $P(g) \cap N_\zeta = \emptyset$.

Assume first that $g$ is elliptic, with Denjoy-Wolff point $p \in \mathbb{D}$. After conjugating by a Möbius transformation, we assume $p = 0$. Now, consider an Euclidean disk $D(0, r)$, with $r \in (0,1)$ big enough so that $SV(g) \subset D(0,r)$. By Schwarz Lemma II.3.2, $D(0,r)$ is forward invariant under $g$, so $P(g) \subset D(0,r)$. This implies that, for all $\zeta \in \partial\mathbb{D}$, we can find a crosscut neighbourhood of $\zeta$ disjoint from the postsingular set.

If $g$ is doubly parabolic, the procedure is analogous, with the difference that we should work with a holodisk tangent to the Denjoy-Wolff point instead of an Euclidean disk, and we apply Wolff Lemma II.3.3 instead of Schwarz Lemma. Note that we can find a crosscut neighbourhood disjoint from the postsingular set for any $\zeta \in \partial\mathbb{D}$, except for the Denjoy-Wolff point.

## II.4 Boundary behaviour of Riemann maps

In order to transfer the previous results on the iteration of inner functions on the unit circle $\partial\mathbb{D}$, we need a deep understanding of the Riemann map $\varphi\colon \mathbb{D} \to U$. We collect here the results concerning the boundary behaviour of Riemann maps needed in this thesis, and refer to [Pom92] for a wider exposition on the topic. In the sequel we assume $U \subsetneq \mathbb{C}$; this can be achieved without loss of generality by postcomposing $\varphi$ by a Möbius transformation.

Let us start with a classical result by Beurling [Pom92, Thm. 9.19].

**Theorem II.4.1. (Existence of radial limits)** *Let $\varphi\colon \mathbb{D} \to U \subsetneq \mathbb{C}$ be a Riemann map. Then, for all $\xi \in \partial\mathbb{D}$ apart from a set of logarithmic capacity zero, the radial limit $\varphi^*$ exists and its finite.*

We shall not discuss here the concept of logarithmic capacity (for which we refer to [Pom92, Chap. 9]), but keep in mind the following properties. First, the notion of logarithmic capacity is usually defined for compact subsets of the plane, but the notion extends to Borel sets [Pom92, Thm. 9.12]; in particular, a Borel set has logarithmic capacity zero if it does not contain any compact set of positive capacity. Recall that sets of logarithmic capacity zero are extremely thin: they cannot contain non-degenerate continua, and its Hausdorff dimension is zero [Pom92, Thm. 10.1.3]. Moreover, the union of countably many sets of capacity zero has capacity zero [Pom92, Corol. 9.13], and, if $\varphi$ is a Möbius transformation and $E$ has capacity zero, then $\varphi(E)$ has also capacity zero.

In particular, since sets of logarithmic capacity zero have Lebesgue measure zero, radial limits exist and are different $\lambda$-almost everywhere – this is also known as the Fatou, Riesz and Riesz Theorem (see e.g. [Mil06, Thm. 17.4]). Moreover, the following is true.

**Theorem II.4.2.** ([Pom92, Corol. 2.19]) *Let $\varphi\colon \mathbb{D} \to \mathbb{C}$ be a homeomorphism. Then, there are at most countably many points $a \in \widehat{\mathbb{C}}$ such that $\varphi^*(\xi_j) = a$ for three distinct points $\xi_j \in \partial\mathbb{D}$.*

### II.4.1 Prime ends and cluster sets

The definition of cluster set and radial cluster set reads as follows.

**Definition II.4.3. (Cluster sets and radial cluster sets)** Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map and let $\xi \in \partial\mathbb{D}$.

- The *radial cluster set* $Cl_R(\varphi, \xi)$ of $\varphi$ at $\xi$ is defined as the set of values $w \in \widehat{\mathbb{C}}$ for which there is an increasing sequence $\{t_n\}_n \subset (0,1)$ such that $t_n \to 1$ and $\varphi(t_n\xi) \to w$, as $n \to \infty$.
- The *cluster set* $Cl(\varphi, \xi)$ of $\varphi$ at $\xi$ is the set of values $w \in \widehat{\mathbb{C}}$ for which there is a sequence $\{z_n\}_n \subset \mathbb{D}$ such that $z_n \to \xi$ and $\varphi(z_n) \to w$, as $n \to \infty$.
- If $U$ is unbounded, we define the *cluster set in* $\mathbb{C}$ as

$$Cl_{\mathbb{C}}(\varphi, \xi) \coloneqq Cl(\varphi, \xi) \cap \mathbb{C}.$$

Observe that every point in $\partial U$ must belong to the cluster set of some $\xi \in \partial U$. In some sense, the previous concepts replace the notion of image under $\varphi$ for points in $\partial\mathbb{D}$, and allow us to describe the topology of $\partial U$. Cluster sets (and radial cluster sets) are, by definition, non-empty compact subsets of $\widehat{\mathbb{C}}$. However, cluster sets in $\mathbb{C}$ may be empty.

Prime ends give a more geometrical approach to the same concepts, and are defined as follows. Consider a simply connected domain $U$, and fix a basepoint $z_0 \in U$. In the same spirit as in Definition II.2.1, we say that $C$ is a *(non-degenerate)crosscut* in $U$ if $C$ is an open Jordan arc in $U$ such that $\overline{C} = C \cup \{a, b\}$, with $a, b \in \partial U$; we allow $a = b$. If $C$ is a crosscut of $U$ and $z_0 \notin C$, then $U \smallsetminus D$ has exactly one component which does not contain $z_0$; let us denote this component by $N_C$. We say that $N_C$ is a *crosscut neighbourhood* in $U$ associated to $C$.

A *null-chain* in $U$ is a sequence of crosscuts $\{C_n\}_n \subset U$ with disjoint closures, such that the corresponding crosscut neighbourhoods are nested, i.e. $N_{C_{n+1}} \subset N_{C_n}$ for $n \geq 0$; and the spherical diameter of $C_n$ tends to zero as $n \to \infty$. We say that two null-chains $\{C_n\}_n$ and $\{C'_n\}_n$ are equivalent if, for every sufficiently large $m$, there exists $n$ such that $N_{C_n} \subset N_{C'_m}$ and $N_{C'_n} \subset N_{C_m}$. This defines an equivalence relation between null-chains. The equivalence classes are called the *prime ends* of $U$. The impression of a prime end $P$ is defined as

$$I(P) \coloneqq \bigcap_{n \geq 0} \overline{N_{C_n}} \subset \partial U.$$

If $U = \mathbb{D}$ (or any set with locally connected boundary) the impression of every prime end is a single point. In general, a Riemann map $\varphi\colon \mathbb{D} \to U$ gives a bijection between points in $\partial\mathbb{D}$ and prime ends of $U$ (Carathéodory's Theorem, [Pom92, Thm. 2.15]). We denote by $P(\varphi, \xi)$ the prime end in $U$ corresponding to $\xi \in \partial\mathbb{D}$.

Given a prime end $P$, we say that $w \in \widehat{\partial} U$ is a *principal point* of $P$, if $P$ can be represented by a null-chain $\{C_n\}_n$ satisfying that, for all $r > 0$, there exists $n_0$ such that the crosscuts $C_n$ are contained in the disk $D(w, r)$ for $n \geq n_0$. Let $\Pi(P)$ denote the set of all principal points of $P$.

The following theorem gives explicitly the relation between cluster sets and prime ends, and between radial cluster sets and principal points.

**Theorem II.4.4. (Prime ends and cluster sets,** [Pom92, Thm. 2.16]**)** *Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $\xi \in \partial\mathbb{D}$. Then,*

$$I(P(\varphi, \xi)) = Cl(\varphi, \xi), \text{ and } \Pi(P(\varphi, \xi)) = Cl_R(\varphi, \xi).$$

### II.4.2 Accessing the boundary of a simply connected domain

A classical result about Riemann maps is the following.

**Theorem II.4.5. (Lindelöf Theorem,** [CG93, Thm. I.2.2]**)** *Let $\gamma\colon [0, 1) \to U$ be a curve which lands at a point $p \in \widehat{\partial} U$. Then, the curve $\varphi^{-1}(\gamma)$ in $\mathbb{D}$ lands at some point $\xi \in \partial\mathbb{D}$. Moreover, $\varphi$ has the radial limit at $\xi$ equal to $p$. In particular, curves that land at different points in $\widehat{\partial} U$ correspond to curves which land at different points of $\partial\mathbb{D}$.*

Accessible points (and accesses) are in bijection with points in $\partial\mathbb{D}$ for which $\varphi^*$ exists, as it is shown in the following well-known theorem [Pom92, p. 35, Ex. 5]. For a complete proof, see [BFJK17].

**Theorem II.4.6. (Correspondence Theorem)** *Let $U \subset \widehat{\mathbb{C}}$ be a simply connected domain, $\varphi\colon \mathbb{D} \to U$ a Riemann map, and let $p \in \widehat{\partial} U$. Then, there is a one-to-one correspondence between accesses from $U$ to $p$ and the points $\xi \in \partial\mathbb{D}$ such that $\varphi^*(\xi) = p$. The correspondence is given as follows.*

(a) *If $\mathcal{A}$ is an access to $p \in \partial U$, then there is a point $\xi \in \partial\mathbb{D}$ with $\varphi^*(\xi) = p$. Moreover, different accesses correspond to different points in $\partial\mathbb{D}$.*

(b) *If, at a point $\xi \in \partial\mathbb{D}$, the radial limit $\varphi^*$ exists and it is equal to $p \in \partial U$, then there exists an access $\mathcal{A}$ to $p$. Moreover, for every curve $\eta \subset \mathbb{D}$ landing at $\xi$, if $\varphi(\eta)$ lands at some point $q \in \widehat{\mathbb{C}}$, then $p = q$ and $\varphi(\eta) \in \mathcal{A}$.*

### II.4.3 Separating the boundary of simply connected domains

Next, we state the following theorem, which exploits the possibility of separating sets in $\partial U$ with arcs contained in $U$.

**Theorem II.4.7. (Separation of simply connected domains,** [CP02, Prop. 2]**)** *Let $U \subset \widehat{\mathbb{C}}$ be a simply connected domain, and let $E \subset \widehat{\partial} U$ be a continuum. Let $w_1, w_2$ be points in different connected components of $\widehat{\partial} U \setminus E$. Then, there exists a Jordan arc $\gamma \subset U$ with $\widehat{\gamma} \setminus \gamma \subset E$ such that $\gamma \cup E$ separates $w_1$ and $w_2$ in $\widehat{\mathbb{C}}$.*

As a consequence of the previous theorem, we describe under which conditions cluster sets are disconnected when restricted to $\mathbb{C}$. Indeed, the next proposition gives a precise characterization of disconnected cluster sets. In particular, if a radial limit achieves a finite value, the corresponding cluster set is connected in $\mathbb{C}$.

**Proposition II.4.8. (Disconnected cluster sets)** *Let $f$ be a transcendental entire function and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map. Let $\xi \in \partial\mathbb{D}$ be such that $Cl_{\mathbb{C}}(\varphi, \xi)$ is contained in more than one component of $\partial U$. Then, $\varphi^*(\xi) = \infty$, and $Cl_{\mathbb{C}}(\varphi, \xi)$ is contained in exactly two components of $\partial U$.*

*Proof.* Consider $\Sigma_1, \Sigma_2$ connected components of $\partial U$, such that both intersect $Cl_{\mathbb{C}}(\varphi, \xi)$. Set $w_1 \in \Sigma_1 \cap Cl_{\mathbb{C}}(\varphi, \xi)$, $w_2 \in \Sigma_2 \cap Cl_{\mathbb{C}}(\varphi, \xi)$. Now, apply Theorem II.4.7, with $E = \{\infty\}$ and $w_1, w_2$ chosen before, which lie on different connected components of $\widehat{\partial} U \smallsetminus \{\infty\} = \partial U$. Hence, there exists a simple arc $\gamma \subset U$, such that $\widehat{\gamma} \smallsetminus \gamma = \{\infty\}$ and $\widehat{\gamma}$ separates $w_1$ and $w_2$ in $\widehat{\mathbb{C}}$.

It remains to see that $\varphi^{-1}(\gamma)$ has one endpoint at $\xi$, and then the Correspondence Theorem II.4.6 would imply that $\varphi^*(\xi) = \infty$. See Figure II.10.

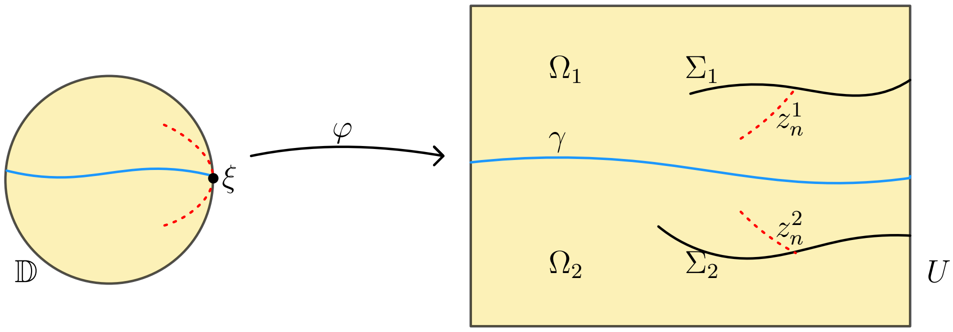



Figure II.10: Diagram of the setup of the second part of the proof of Lemma II.4.8, when it is shown that, if $Cl_{\mathbb{C}}(\varphi, \xi)$ is disconnected, then $\varphi^*(\xi) = \infty$.

Since $\widehat{\gamma}$ is a closed simple curve in $\widehat{\mathbb{C}}$, by the Jordan Curve Theorem I.2.1, $\widehat{\mathbb{C}} \smallsetminus \widehat{\gamma}$ has exactly two connected components, say $\Omega_1$ and $\Omega_2$, with $\Sigma_i \subset \Omega_i \subset \mathbb{C}$, for $i = 1, 2$. Moreover, since $\widehat{\gamma} \subset U \cup \{\infty\}$, each $U_i := \Omega_i \cap U$ is non-empty and connected. Hence, there exists a sequence of points $\{z_n^i\}_n$ in $U_i$ converging to $w_i$. Since $w_i \in Cl_{\mathbb{C}}(\varphi, \xi)$, we can assume that the sequences $\{z_n^i\}_n$ have been chosen so that $\{\varphi^{-1}(z_n^i)\}_n$ both converge to $\xi \in \partial\mathbb{D}$.

Now, consider a null-chain $\{C_n\}_n$ in $\mathbb{D}$, giving the sequence of crosscuts neighbourhoods $\{N_n\}_n$ converging to $\xi$, and such that $\{\varphi(C_n)\}_n$ gives a null-chain in $U$. For the existence of such null-chain, we refer to [Mil06, Lemma 17.9]. For all $n \geq 0$, there is $m_n$ such that $z_{m_n}^i \in \varphi(N_n)$, for $i = 1, 2$. Hence, for all $n \geq 0$, there exists $z_n \in \gamma \cap \varphi(N_n)$.

Observe that, by the Correspondence Theorem II.4.6, $\varphi^{-1}(\gamma)$ lands at two different points $\xi_1, \xi_2 \in \partial\mathbb{D}$. Hence, for every null-chain in $\mathbb{D}$ not corresponding to $\xi_1$ nor $\xi_2$, $\varphi^{-1}(\gamma)$ intersects only a finite number of crosscut neighbourhoods of it. Since $\varphi^{-1}(\gamma)$ intersects every $N_n$, it follows that either $\xi_1 = \xi$, or $\xi_2 = \xi$, so $\varphi^{-1}(\gamma)$ lands at $\xi$, as desired.

Next, we shall prove that $Cl_{\mathbb{C}}(\varphi, \xi)$ is contained in exactly of two connected components of $\partial U$. Assume, on the contrary, that there exists $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ connected components of $\partial U$ which intersect $Cl_{\mathbb{C}}(\varphi, \xi)$. By the previous argument, there exists a simple arc $\gamma \subset U$, separating $\mathbb{C}$ into two connected components, $\Omega_1$ and $\Omega_2$, with $\Sigma_i \subset \Omega_i$, for $i = 1, 2$. Since $\widehat{\gamma} \subset U \cup \{\infty\}$, $\Sigma_3$ is either contained in $\Omega_1$ or in $\Omega_2$. Without loss of generality, assume $\Sigma_3 \subset \Omega_1$. Now, let us consider $U_1 := U \cap \Omega_1$, which is connected, simply connected, and $\Sigma_1$ and $\Sigma_3$ are different connected components of $\partial U_1$. Hence, there exists $\gamma' \subset U_1$, separating $\Sigma_1$ and $\Sigma_3$. Both curves $\gamma$ and $\gamma'$ are disjoint, $\widehat{\gamma} \cup \widehat{\gamma'} = \{\infty\}$, and $\varphi^{-1}(\gamma(t)) \to \xi$ and $\varphi^{-1}(\gamma(t)) \to \xi$, as $t \to +\infty$. Hence, $\mathbb{C} \smallsetminus (\gamma \cup \gamma')$ consists precisely of three connected components, each of them containing exactly one $\Sigma_i$. See Figure II.11.

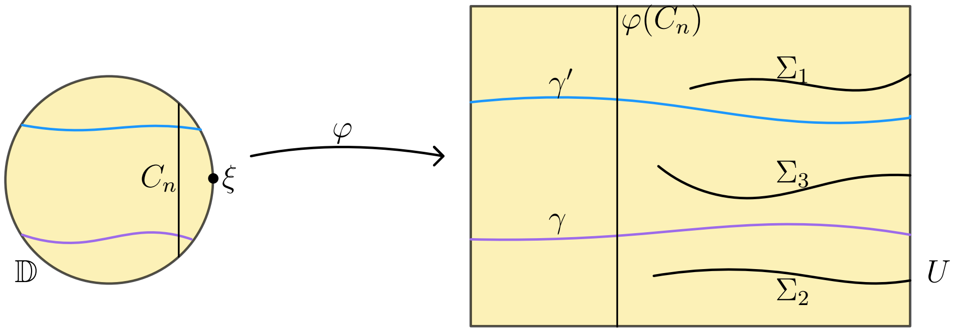



Figure II.11: Diagram of the setup of the second part of the proof of Lemma II.4.8, when it is shown that $Cl_{\mathbb{C}}(\varphi, \xi)$ cannot have more that two connected components.

We want to see that $\gamma$ and $\gamma'$ define different accesses to $\infty$ in $U$, leading to a contradiction with the Correspondence Theorem II.4.6 (indeed, if $\gamma$ and $\gamma'$ define different accesses to $\infty$ in $U$, then $\varphi^{-1}(\gamma)$ and $\varphi^{-1}(\gamma')$ cannot land at the same point $\xi \in \partial\mathbb{D}$).

To do so, we fix a crosscut $\varphi(C_n)$ of the null-chain $\{\varphi(C_n)\}_n$ defined above. Since both $\varphi^{-1}(\gamma(t))$ and $\varphi^{-1}(\gamma'(t))$ converge to $\xi$, as $t \to \infty$, there exist $t_\gamma$, $t_{\gamma'}$ satisfying that

$$\gamma\left(t_\gamma\right), \gamma'\left(t_{\gamma'}\right) \in \varphi(C_n),$$

$$\gamma\left([t_\gamma, +\infty)\right) \cup \gamma'\left([t_{\gamma'}, +\infty)\right) \subset \varphi(N_n)$$

Denote by $\eta$ the connected arc in $\varphi(N_n)$ satisfying that

$$\widetilde{\gamma} := \eta \cup \gamma\left([t_\gamma, +\infty)\right) \cup \gamma'\left([t_{\gamma'}, +\infty)\right)$$

is a simple arc in $U$, and $\widehat{\widetilde{\gamma}}$ is a closed simple curve in $U \cup \{\infty\}$. Moreover, since

$$\widetilde{\gamma}, \Sigma_1, \Sigma_2, \Sigma_3 \subset \overline{\varphi(N_n)},$$

it follows that $\widetilde{\gamma}$ separates exactly one $\Sigma_i$ from the others. Hence, $\gamma\left([t_\gamma, +\infty)\right)$ and $\gamma'\left([t_{\gamma'}, +\infty)\right)$ define different accesses to $\infty$, although both $\varphi^{-1}(\gamma\left([t_\gamma, +\infty)\right))$ and $\varphi^{-1}(\gamma'\left([t_{\gamma'}, +\infty)\right))$ land at $\xi$, contradicting the Correspondence Theorem. This finishes the proof of the proposition. □

### II.4.4 Harmonic measure

Let $U \subset \widehat{\mathbb{C}}$ be a hyperbolic simply connected domain (i.e. $U$ omits at least three points), and let $\varphi\colon \mathbb{D} \to U$ be a Riemann map. We are concerned with the extension of $\varphi$ to the unit circle $\partial\mathbb{D}$ given in terms of radial limits

$$\varphi^*(\xi) \coloneqq \lim_{t\to 1^-} \varphi(t\xi),$$

which exist $\lambda$-almost everywhere. The radial extension of its Riemann map $\varphi^*\colon \partial\mathbb{D} \to \widehat{\partial} U$ is used to define a measure in $\widehat{\partial} U$, the *harmonic measure*, in terms of the push-forward of the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$.

**Definition II.4.9. (Harmonic measure)** Let $U \subset \widehat{\mathbb{C}}$ be a hyperbolic simply connected domain, $z \in U$, and let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, such that $\varphi(0) = z \in U$. Let $(\partial\mathbb{D}, \mathcal{B}, \lambda)$ be the measure space on $\partial\mathbb{D}$ defined by $\mathcal{B}$, the Borel $\sigma$-algebra of $\partial\mathbb{D}$, and $\lambda$, its normalized Lebesgue measure. Consider the measurable space $(\widehat{\mathbb{C}}, \mathcal{B}(\widehat{\mathbb{C}}))$, where $\mathcal{B}(\widehat{\mathbb{C}})$ is the Borel $\sigma$-algebra of $\widehat{\mathbb{C}}$. Then, given $B \in \mathcal{B}(\widehat{\mathbb{C}})$, the *harmonic measure at $z$ relative to $U$* of the set $B$ is defined as

$$\omega_U(z, B) \coloneqq \lambda((\varphi^*)^{-1}(B)).$$

We refer to [GM05, Pom92] for equivalent definitions and further properties of the harmonic measure.

Let $U \subset \widehat{\mathbb{C}}$ be a hyperbolic simply connected domain. We need the following simple facts.

- Let $B \in \mathcal{B}(\widehat{\mathbb{C}})$. If there exists $z_0 \in U$ such that $\omega_U(z_0, B) = 0$ (resp. $\omega_U(z_0, B) = 1$), then $\omega_U(z, B) = 0$ (resp. $\omega_U(z, B) = 1$) for all $z \in U$. In this case, we say that the set $B$ has *zero* (resp. *full*) *harmonic measure relative to* $U$, and we write $\omega_U(B) = 0$ (resp. $\omega_U(B) = 1$).

- $\operatorname{supp} \omega_U = \widehat{\partial} U$. That is, for all $x \in \widehat{\partial} U$ and $r > 0$, $\omega_U(D(x, r)) > 0$.

## II.5 Transference to the dynamical plane: dynamics of $f|_{\partial U}$

Let us see now how the previous results transfer to the dynamical plane. First, it is clear that $f|_U$ is conformally conjugate to $g|_{\mathbb{D}}$. This implies that inner functions associated with Siegel disks or attracting basins are elliptic (Möbius or non-Möbius, respectively), while inner functions associated with parabolic basins or Baker domains are non-elliptic. More precisely, dynamics inside a Baker domain can be eventually conjugate to $\mathrm{id}_{\mathbb{C}} + 1$, or to $\lambda \mathrm{id}_{\mathbb{H}}$, $\lambda > 1$, or to $\mathrm{id}_{\mathbb{H}} + 1$. (being all types possible, [Kö99]); whereas inner functions associated with parabolic basins are always of doubly parabolic type (this can be shown using Fatou coordinates,see e.g. [Mil06, Sect. 10]).

The transference of the boundary dynamics is much more intricate, and the topic of this section. Before proceeding, note that, if $U$ is an invariant Baker domain, then all points in $U$ escape to an essential singularity, say $\infty$, under iteration. Thus, $U$ is clearly unbounded in $\Omega(f)$ and infinity is accessible from it. Indeed, given any point $z \in U$ and a curve joining $z$ and $f(z)$ within $U$, then the curve $\gamma \coloneqq \bigcup_{n\geq 0} f^n(\gamma)$ is unbounded and lands at infinity, defining an access which is called the *dynamical access to infinity.*

### II.5.1 Singular values

As usual, singular values (defined in Sect. I.4) play a distinguished role in the dynamics. It follows from [Bol97, Thm. 1.2] that $Cl(f, z) = \widehat{\mathbb{C}}$, for every $z \in E(f)$ (see also [BDH01, p. 651]). Thus, $\mathbb{K} \subset \mathbb{M}$, and the discussion in Section I.4 applies.

The dynamics of $f|_U$ is controlled by the singular values in $U$, and

$$SV(g) \subset \varphi^{-1}(SV(f) \cap U), \quad P(g) \subset \varphi^{-1}(P(f) \cap U).$$

It is well-known that $SV(f) \cap U \neq \emptyset$ whenever $U$ is an attracting or parabolic basin, or a doubly parabolic Baker domain. If $SV(f)$ is compactly contained in $U$, the behaviour of the associated inner function $g$ is well understood, in the following sense.

**Proposition II.5.1. (Singular values compactly contained)** *Let $f \in \mathbb{K}$, let $U$ be an invariant simply connected Fatou component for $f$, and $g$ its associated inner function. Assume $SV(f)$ is compactly contained in $U$. Then, the following hold.*

(a) *If $U$ is an attracting basin, then, for all $\xi \in \partial\mathbb{D}$, there exists a crosscut neighbourhood $N_\xi$ of $\xi$, such that $N_\xi \cap P(g, \mathbb{D}) = \emptyset$.*

(b) *If $U$ is either a parabolic basin or a Baker domain, then the Denjoy-Wolff point $p \in \partial\mathbb{D}$ of $g$ is not a singularity for $g$. Moreover, for all $\xi \in \partial\mathbb{D}$, $\xi \neq p$, there exists a crosscut neighbourhood $N_\xi$ of $\xi \in \partial\mathbb{D}$, such that $N_\xi \cap P(g, \mathbb{D}) = \emptyset$.*

*Proof.* (a) Let $z_0 \in U$ be the attracting fixed point, and consider $g$ to be the inner function associated by a Riemann map $\varphi$, with $\varphi(0) = z_0$. Then, there exists $r \in (0, 1)$ big enough so that $SV(g) \subset D(0, r)$. By Schwarz lemma II.3.2, $D(0, r)$ is forward invariant, so $P(g, \mathbb{D}) \subset D(0, r)$, and (a) follows trivially.

(b) By Corollary II.3.15, it is enough to find a crosscut neighbourhood $N_p$ of $p$ such that $N_p \cap SV(g) = \emptyset$, and this is immediate from the hypothesis. The second statement follows for applying the same argument as in (a), using a tangent disk at the Denjoy-Wolff point and Wolff Lemma II.3.3.

□

### II.5.2 Ergodic properties of the boundary map $f\colon \partial U \to \partial U$

Let $\varphi\colon \mathbb{D} \to U$ be the Riemann map. Then, the radial extension

$$\varphi^*\colon \partial\mathbb{D} \to \widehat{\partial} U$$

is well-defined $\lambda$-almost everywhere, and $\widehat{\partial} U$ admits a harmonic measure $\omega_U$, which stands for the push-forward of the normalized Lebesgue measure $\lambda$ of $\partial\mathbb{D}$. The ergodic properties of $f|_{\partial U}$ will be derived from the ergodic properties of $g^*|_{\partial\mathbb{D}}$, where $g$ is the inner function associated with $(f, U)$.

We start by proving that $g$ is actually an inner function. To that end, consider the following subsets of $\partial\mathbb{D}$.

$$\Theta_E := \{\xi \in \partial\mathbb{D} \colon \varphi^*(\xi) \in E(f)\}$$
$$\Theta_\Omega := \{\xi \in \partial\mathbb{D} \colon \varphi^*(\xi) \in \Omega(f)\}$$

Note that, since $E(f)$ is countable, $\lambda(\Theta_E) = 0$, so $\lambda(\Theta_\Omega) = 1$. Moreover, the conjugacy $f \circ \varphi = \varphi \circ g$ extends for the radial extensions wherever it makes sense, as it is shown in the following lemma.

**Lemma II.5.2. (Radial limits commute)** *Let $\xi \in \Theta_\Omega$, then $g^*(\xi)$ and $\varphi^*(g^*(\xi))$ are well-defined, and*

$$f(\varphi^*(\xi)) = \varphi^*(g^*(\xi)).$$

*Proof.* Let $R_\xi(t) = t\xi$, with $t \in [0, 1)$. By assumption, $\varphi(R_\xi(t)) \to \varphi^*(\xi) =: w \in \partial U$, as $t \to 1^-$ (recall that $\partial U$ denotes the boundary of $U$ taken in $\Omega$). Since $f$ is continuous at $w$ and $f \circ \varphi = \varphi \circ g$, for all $0 < t < 1$,

$$\varphi(g(R_\xi(t))) = f(\varphi(R_\xi(t))) \to f(w) \in \widehat{\partial} U,$$

as $t \to 1^-$. We claim that this already implies that $\gamma(t) := g(R_\xi(t))$ lands at some $\zeta \in \partial\mathbb{D}$. Indeed, consider

$$L_{\gamma,1} := \left\{ z \in \overline{\mathbb{D}} \colon \text{ there exists } t_n \to 1^- \text{ such that } \gamma(t_n) \to z \right\},$$

which is a non-empty, compact, connected set contained in $\partial\mathbb{D}$, since points in $\mathbb{D}$ are mapped to $U$ by $\varphi$. If $L_{\gamma,1}$ is a non-degenerate arc $I$, with $\lambda(I) > 0$, for $\lambda$-almost every $\zeta \in I$, $\varphi^*(\zeta) = f(w)$, which is a contradiction with Theorem 4.8.4. Hence, $L_{\gamma,1} = \zeta \in \partial\mathbb{D}$.

Finally, the Lehto-Virtanen Theorem II.2.5 implies that $g^*(\xi) = \zeta$ and $\varphi^*(g^*(\xi)) = f(w)$, as desired. □

**Proposition II.5.3. ($g$ is inner)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Fatou component of $f$. Then, the associated map $g \colon \mathbb{D} \to \mathbb{D}$ is an inner function.*

*Proof.* Since $g$ is a self-map of the unit disk, its radial extension $g^*$ is well-defined $\lambda$-almost everywhere. We have to see that $|g^*(\xi)| = 1$, for $\lambda$-almost every $\xi \in \partial\mathbb{D}$.

Assume, on the contrary, that there exists $A \subset \partial\mathbb{D}$ with $\lambda(A) > 0$ and $|g^*(\xi)| < 1$, for all $\xi \in A$. We can assume, without loss of generality, that $\varphi^*(\xi) \in \partial U \subset \mathcal{J}(f)$, for all $\xi \in A$. By Lemma II.5.2, for all $\xi \in A$, $\varphi^*(\xi) \in \mathcal{J}(f) \smallsetminus E(f)$, and

$$f(\varphi^*(\xi)) = \varphi^*(g^*(\xi)) \in U \subset \mathcal{F}(f)$$

which is a contradiction with the total invariance of the Fatou set. □

We note that, since $\omega_U(E(f)) = 0$, for every Borel set $B \subset \widehat{\mathbb{C}}$ and $z \in U$, we have

$$\omega_U(z, B) = \omega_U(z, B \cap \Omega(f)).$$

With these tools at hand, and those developed in the previous sections, we can now prove ergodic properties like ergodicity and recurrence for the boundary map of Fatou components of maps in class $\mathbb{K}$, generalizing the results of Doering and Mañé [DM91] for rational maps.

**Theorem II.5.4. (Ergodic properties of the boundary map)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Fatou component for $f$. Let $g$ be an inner function associated with $(f, U)$. Then, the following are satisfied.*

(i) *If $U$ is either an attracting basin, a parabolic basin, or a Siegel disk, then $g^*|_{\partial\mathbb{D}}$ is ergodic and recurrent with respect to the Lebesgue measure $\lambda$.*

(ii) *If $U$ is a doubly parabolic Baker domain, $g^*|_{\partial\mathbb{D}}$ is ergodic with respect to $\lambda$. In addition, assume one of the following conditions is satisfied.*

  (a) *$f|_U$ has finite degree.*

  (b) *$SV(f) \cap U$ is compactly contained in $U$.*

  (c) *The Denjoy-Wolff point of $g$ is not a singularity for $g$.*

  (d) *There exists $z \in U$ and $r > 1$ such that*

$$\text{dist}_U(f^{n+1}(z), f^n(z)) \leq \frac{1}{n} + O\left(\frac{1}{n^r}\right),$$

  *as $n \to \infty$, where $\text{dist}_U$ denotes the hyperbolic distance in $U$.*

  *Then, $g^*|_{\partial\mathbb{D}}$ is recurrent with respect to $\lambda$.*

(iii) *If $g^*|_{\partial\mathbb{D}}$ is ergodic (resp. recurrent) with respect to $\lambda$, so is $f|_{\partial U}$ with respect to $\omega_U$.*

  *If $g^*|_{\partial\mathbb{D}}$ is recurrent with respect to $\lambda$, then for $\omega_U$-almost every point $x \in \partial U$, $\{f^n(x)\}_n$ is dense in $\partial U$. In particular, escaping points have zero harmonic measure.*

(iv) *If $U$ is a hyperbolic or simply parabolic Baker domain of finite degree, then $\mathcal{I}(f) \cap \partial U$ (the set of escaping points in $\partial U$) has full harmonic measure.*

(v) *Let $k$ be a positive integer. Then, the inner function associated with $(f, U)$ has the same ergodic properties than the inner function associated with $(f^k, U)$.*

*Proof.* (i) The associated inner function to these Fatou components is either elliptic or doubly parabolic, so $g^*$ is ergodic (Thm. II.3.7(b)).

Recurrence for the inner function associated with a Siegel disk or an attracting basin follows from the fact that these inner functions are always elliptic (Thm. II.3.7(d)). Recurrence for parabolic basins follows from [DM91, Thm. 6.1] (although it is stated for rational maps, the proof only uses the local behaviour around the parabolic fixed point, so it is valid for $f \in \mathbb{K}$).

(ii) Ergodicity of doubly parabolic Fatou components follows from Theorem II.3.7(b). For conditions (a)-(c) it is straightforward. To see that condition (d) implies recurrence, note that

$$\text{dist}_U(f^{n+1}(z), f^n(z)) = \text{dist}_{\mathbb{D}}(g^{n+1}(\varphi^{-1}(z)), g^n(\varphi^{-1}(z))),$$

and apply again Theorem II.3.7(e).

(iii) We start with ergodicity. Let $A \subset \partial U$ be measurable. If $A = f^{-1}(A)$, then

$$(\varphi^*)^{-1}(A) = (\varphi^*)^{-1}(f^{-1}(A)) = (g^*)^{-1}((\varphi^*)^{-1}(A)).$$

If $g^*$ is ergodic, then $\lambda((\varphi^*)^{-1}(A)) = 0$ or $\lambda((\varphi^*)^{-1}(A)) = 1$. Then, $\omega_U(A) = 0$ or $\omega_U(A) = 1$, and $f|_{\partial U}$ is ergodic.

For the recurrence, assume $A \subset \partial U$ is measurable, and consider $(\varphi^*)^{-1}(A)$. Then, for $\lambda$-almost every $\xi \in (\varphi^*)^{-1}(A)$, there exists $n_k \to \infty$ such that $(g^{n_k})^*(\xi) \in (\varphi^*)^{-1}(A)$. Since $A \subset \partial U \subset \Omega$, Lemma II.5.2 applies, and

$$\varphi^*((g^{n_k})^*(\xi)) = f^{n_k}(\varphi^*(\xi)) \in A,$$

proving recurrence for $f|_{\partial U}$. Since supp $\omega_U = \widehat{\partial} U$, it follows from Theorem I.1.7 that $\omega_U$-almost every orbit is dense.

(iv) This is the content of [RS18], [BFJK19, Thm. A].

(v) It follows from Theorem II.3.7(g).

□

The following result concern the existence of invariant measures for $f|_{\partial U}$, built as the push-forward measures of those invariants for the radial extension of the associated inner function (Thm. II.3.8).

**Corollary II.5.5. (Invariant measures for $f|_{\partial U}$)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Fatou component for $f$.*

(i) *If $U$ is an attracting basin or a Siegel disk with fixed point $p \in U$, the harmonic measure $\omega_U(p, \cdot)$ is invariant under $f$.*

(ii) *If $U$ is a parabolic basin or a doubly parabolic Baker domain, with convergence point $p \in \widehat{\partial} U$. Then, the push-forward of the measure*

$$\lambda_{\mathbb{R}}(A) = \int_A \frac{1}{|w-1|^2} d\lambda(w), \quad A \in \mathcal{B}(\partial \mathbb{D}),$$

*under the Riemann map $\varphi\colon \mathbb{D} \to U$, $\varphi^*(1) = p$, i.e.*

$$\mu := (\varphi^*)_* \lambda_{\mathbb{R}},$$

*is invariant under $f$. The support of $\mu$ is $\widehat{\partial} U$.*

### II.5.3 Entire functions and accesses to infinity

Next, we deal with the particular case that $U$ is an invariant Fatou component of an entire function $f$. In this setting, accesses to infinity play an important role in order to describe the dynamics and the topology of boundaries of Fatou components, since infinity is the unique essential singularity of the map, and has no finite preimage. This has been already exploited in [BW91, BD99, Bar08] (see also [BFJK17]), as we show next.

We use the following notation.

$$\Theta_\infty = \Theta_E = \{\xi \in \partial\mathbb{D} \colon \varphi^*(\xi) = \infty\}$$

$$\partial\mathbb{D} \smallsetminus \Theta_\infty = \{\xi \in \partial\mathbb{D} \colon Cl_R(\varphi, \xi) \neq \{\infty\}\}$$

$$\Theta_\mathbb{C} = \Theta_\Omega = \{\xi \in \partial\mathbb{D} \colon \varphi^*(\xi) \in \mathbb{C}\}$$

The following holds.

**Theorem II.5.6. (Boundaries of unbounded Fatou components)** *Let $f$ be a transcendental entire function, and let $U$ be an unbounded invariant Fatou component. Consider $\varphi \colon \mathbb{D} \to U$ to be a Riemann map. The following holds.*

(a) (All cluster sets contain infinity, [BW91]) *If $U$ is ergodic, then $\infty \in Cl(\varphi, \xi)$, for all $\xi \in \partial\mathbb{D}$.*

(b) (Accesses to infinity are dense, [BD99, Bar08]) *If $U$ is ergodic and $\infty$ is accessible from $U$, then $\Theta_\infty$ is dense in $\partial\mathbb{D}$.*

(c) (Accesses to infinity and Julia sets, [BD99, Bar08]) *If $f|_U$ is non-univalent and $\infty$ is accessible from $U$, then $\mathcal{J}(g) \subset \overline{\Theta_\infty}$.*

**Theorem II.5.7. (Accesses to boundary points,** [Bar08, Thm. 3.8]**)** *Let $f \colon \mathbb{C} \to \mathbb{C}$ be an entire function, and let $U$ be a Baker domain. Let $\varphi \colon \mathbb{D} \to U$ be a Riemann map. Then, the map*

$$\Theta_\mathbb{C} \to AP(U), \quad \xi \mapsto \varphi^*(\xi),$$

*is a bijection.*

Finally, we give a precise description of how radial limits and cluster sets are mapped under $f$, which can be interpreted as a stronger version of Lemma II.5.2 in the case where the iterated map is entire.

**Lemma II.5.8. (Radial limits and cluster sets for the associated inner function of entire functions)** *Let $f$ be an entire function, and let $U$ be an invariant Fatou component for $f$. Consider $\varphi \colon \mathbb{D} \to U$ a Riemann map, and $g := \varphi^{-1} \circ f \circ \varphi$ an associated inner function. Let $\xi \in \partial\mathbb{D}$. Then, the following holds.*

(a) (Radial limit for the associated inner function) *If $\varphi^*(\xi)$ is well-defined and not equal to $\infty$, then $g^*(\xi)$ and $\varphi^*(g^*(\xi))$ are well-defined and*

$$f(\varphi^*(\xi)) = \varphi^*(g^*(\xi)).$$

(b) (Action of $f$ on cluster sets) *If $\xi \in \partial\mathbb{D}$ is not a singularity for $g$, then*

$$f(Cl_{\mathbb{C}}(\varphi, \xi)) \subset Cl_{\mathbb{C}}(\varphi, g(\xi)).$$

(c) (Action of $f$ on radial cluster sets) *Assume $\xi \in \partial\mathbb{D} \smallsetminus \Theta_\infty$ and $g^*(\xi)$ exists. Then, $g^*(\xi)$ belongs to $\partial\mathbb{D} \smallsetminus \Theta_\infty$, and*

$$f(Cl_R(\varphi, \xi) \cap \mathbb{C}) \subset Cl_R(\varphi, g^*(\xi)) \cap \mathbb{C}.$$

(d) (Backwards invariance of $\Theta_\infty$) *If $\xi \in \Theta_\infty$, then for all $\zeta \in \partial\mathbb{D}$ with $g^*(\zeta) = \xi$, it holds $\zeta \in \Theta_\infty$.*

*Proof.* (a) This is Lemma II.5.2 in this particular setting.

(b) Let $z \in Cl_{\mathbb{C}}(\varphi, \xi)$. Then, there exists a sequence $\{z_n\}_n \subset \mathbb{D}$ such that $z_n \to \xi$ and $\varphi(z_n) \to z$, as $n \to \infty$. Since $f$ is continuous at $z$, and $f$ and $g$ are conjugate by $\varphi$, we have

$$\varphi(g(z_n)) = f(\varphi(z_n)) \to f(z),$$

as $n \to \infty$. Since $\xi$ is not a singularity of $g$, the sequence $\{g(z_n)\}_n \subset \mathbb{D}$ approaches $g(\xi)$, as $n \to \infty$. Hence, $f(z) \in Cl_{\mathbb{C}}(\varphi, g(\xi))$, as desired.

(c) Let $\xi \in \partial\mathbb{D} \smallsetminus \Theta_\infty$. Assume first $\varphi^*(\xi)$ exists, so $\varphi^*(\xi) = Cl_R(\varphi, \xi) \in \mathbb{C}$. Then, by (a), $g^*(\xi)$ and $\varphi^*(g^*(\xi))$ are well-defined and

$$f(Cl_R(\varphi, \xi)) = f(\varphi^*(\xi)) = \varphi^*(g^*(\xi)) = Cl_R(\varphi, g^*(\xi)) \in \partial U.$$

Hence, $g^*(\xi) \in \partial\mathbb{D} \smallsetminus \Theta_\infty$.

Assume now that $Cl_R(\varphi, \xi)$ is a non-degenerate continuum in $\widehat{\mathbb{C}}$. Since critical points are discrete in $\mathbb{C}$, we can find $z \in Cl_R(\varphi, \xi) \cap \mathbb{C}$ which is not a critical point. Hence, there exists $r > 0$ small enough so that $f|_{D(z,r)}$ is a homeomorphism onto its image. On the other hand, since $z \in Cl_R(\varphi, \xi)$, it is a principal point (see Thm. II.4.4), so we can find a null-chain $\{C_n\}_n \subset D(z, r)$.

We claim that $\{f(C_n)\}_n \subset f(D(z, r))$ is a null-chain. We have to check that $f(C_n)$ is a crosscut for all $n \geq 0$, that different crosscuts have disjoint closures, that the corresponding crosscut neighbourhoods are nested, and that its spherical diameter tends to zero as $n \to \infty$.

First, it is clear that $f(C_n)$ is a crosscut for all $n \geq 0$, since $f|_{D(z,r)}$ is a homeomorphism, and $f(U) \subset U$ and $f(\partial U) \subset \partial U$. From the fact that $f|_{D(z,r)}$ is a homeomorphism and the original crosscuts $\{C_n\}_n$ have disjoint closures, one deduces that the crosscuts $\{f(C_n)\}_n$ also have disjoint closures. It is also clear that the diameter of the crosscuts $\{f(C_n)\}_n$ tends to zero.

We must still see that the crosscut neighbourhoods corresponding to the crosscuts $\{f(C_n)\}_n$ are nested. To do so, consider $R_\xi$ to be the radial segment at $\xi$. Since $g^*(\xi)$

exists, the curve $g(R_\xi)$ lands at $g^*(\xi)$. This implies that, for any crosscut $D$ at $\xi$, if its image is again a crosscut (which it is, because $f$ acts locally as a homeomorphism in the dynamical plane), it is a crosscut at $g^*(\xi)$. Therefore, $\{\varphi^{-1}(f(C_n))\}_n$ is a null-chain in $\mathbb{D}$, corresponding to $g^*(\xi) \in \partial\mathbb{D}$, and $\{f(C_n)\}_n$ is a null-chain in $U$.

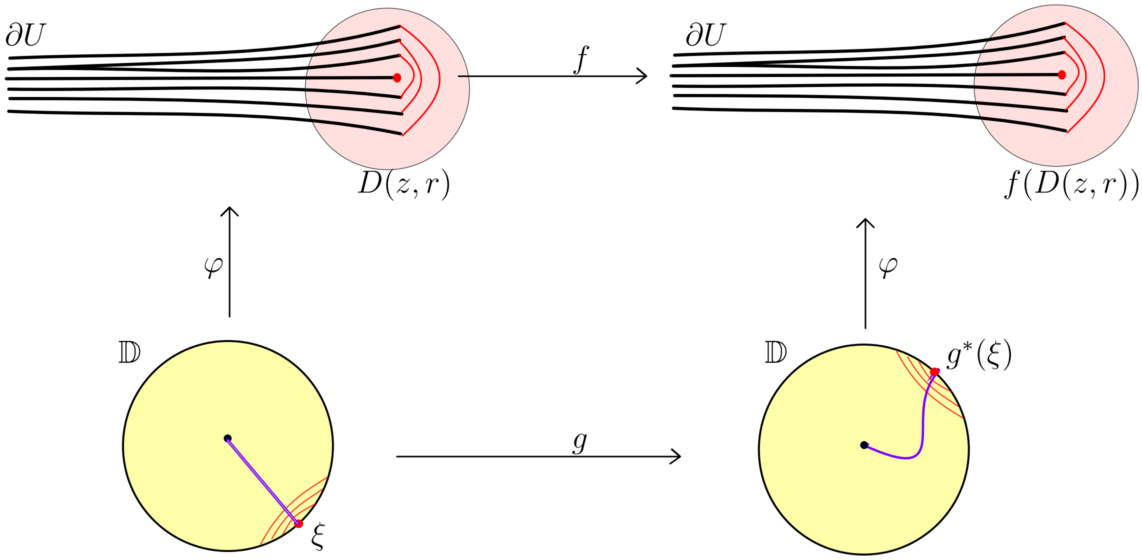



Figure II.12: Set-up of the proof of Lemma II.5.2(c). On the one hand, since $f$ acts homeomorphically on a neighbourhood of $z$, the image of a crosscut near $z$ is a crosscut near $f(z)$. On the other hand, the fact that $g^*(\xi)$ exists allows us to prove that the corresponding crosscut neighbourhoods are nested.

We claim that $g^*(\xi) \in \partial\mathbb{D} \setminus \Theta_\infty$. Indeed, $f(z)$ is a principal point in the prime end of $g^*(\xi)$. Then, by Theorem II.4.4, $f(z) \in Cl_R(\varphi, g^*(\xi))$, and hence $g^*(\xi) \in \partial\mathbb{D} \setminus \Theta_\infty$.

Finally, it is left to see that

$$f(Cl_R(\varphi, \xi) \cap \mathbb{C}) \subset Cl_R(\varphi, g^*(\xi)) \cap \mathbb{C}.$$

From the previous construction, we have that, for all $z \in Cl_R(\varphi, \xi) \cap \mathbb{C}$ which is not a critical point,

$$f(z) \in Cl_R(\varphi, g^*(\xi)).$$

Since $Cl_R(\varphi, \xi)$ is closed and critical points are discrete, if $z \in Cl_R(\varphi, \xi) \cap \mathbb{C}$ is a critical point, we can approximate it by a sequence $\{z_n\}_n$ of non-critical points in $Cl_R(\varphi, \xi) \cap \mathbb{C}$. Since $f$ is continuous, $f(z_n) \to f(z)$, and $f(z_n) \in Cl_R(\varphi, g^*(\xi))$. Then, $f(z) \in Cl_R(\varphi, g^*(\xi))$, because $Cl_R(\varphi, g^*(\xi))$ is closed. Thus,

$$f(Cl_R(\varphi, \xi) \cap \mathbb{C}) \subset Cl_R(\varphi, g^*(\xi)) \cap \mathbb{C},$$

as desired.

(d) Let $\zeta \in \partial\mathbb{D}$ such that $g^*(\zeta) = \xi$, and assume, on the contrary, that $e\zeta \in \partial\mathbb{D} \setminus \Theta_\infty$. Then, there exists $z \in Cl_R(\varphi, \zeta)$, $z \neq \infty$. By (c), $f(z) \in Cl_R(\varphi, g^*(\zeta)) = Cl_R(\varphi, \xi)$, $f(z) \neq \infty$. Hence, $\xi \in \partial\mathbb{D} \setminus \Theta_\infty$, a contradiction.

□

**Remark II.5.9.** The statements in Lemma II.5.8 deserve a few comments.

- In (a), one has to assume that $\varphi^*(\xi) \neq \infty$, otherwise $f(\varphi^*(\xi))$ is not defined. Moreover, the existence of $g^*(\xi)$ does not imply that $\varphi^*(\xi)$ exists, as shown by Baker domains of $f(z) = z + e^{-z}$ (compare with Chap. 1).

- In (b), the assumption of $\xi$ not being a singularity for $g$ is crucial. Indeed, if $\xi$ is a singularity for $g$, $Cl(g, \xi) = \overline{\mathbb{D}}$ [Gar07, Thm. II.6.6].

  We also note that, under the same assumptions, we cannot expect $f(Cl_{\mathbb{C}}(\varphi, \xi)) = Cl_{\mathbb{C}}(\varphi, g(\xi))$, due to the possible existence of omitted values in $\partial U$. For the same reason, one cannot expect, in general, equality in (c).

- Concerning (d), note that $\Theta_\infty$ is not always forward invariant. Compare with the example of the exponential basin considered in [DG87], where $-1 \in \Theta_\infty$ but $g^*(-1) = 0 \notin \Theta_\infty$. Even though, $-1$ is a singularity for $g$, $\Theta_\infty$ is not always forward invariant even at points which are not singularities. Indeed, the inner function $g$ associated to the parabolic basin of $f(z) = ze^{-z}$ is a Blaschke product of degree 2, which can be chosen to have the Denjoy-Wolff point at 1 and $g(-1) = 1$. Then, $g$ satisfies $-1 \in \Theta_\infty$ and $1 = g(-1) \in \partial\mathbb{D} \setminus \Theta_\infty$ (compare [BD99], also Chap. 1).

### II.5.4 The Carathéodory set

Let $U$ be an invariant Baker domain. Following the approach taken in [BEF+24] to describe the boundary dynamics of wandering domains, one can define the *Denjoy-Wolff set* of $f|_U$ as the set of points $x \in \partial U$ such that $f^n(x) \to \infty$ (i.e. $\text{dist}_{\widehat{\mathbb{C}}}(f^n(x), \infty) \to 0$, see [BEF+24, Sect. 9]). As stated in Theorem II.5.4, if $U$ is a hyperbolic or simply parabolic Baker domain such that the Denjoy-Wolff point of the associated inner function is not a singularity (in particular, if $f|_U$ is univalent or has finite degree), then the Denjoy-Wolff set has full harmonic measure. For doubly parabolic Baker domains such that the Denjoy-Wolff point of the associated inner function is not a singularity (in particular, if $f|_U$ has finite degree), the Denjoy-Wolff set has zero harmonic measure.

However, the main limitation of the Denjoy-Wolff set is that it does not capture in which direction boundary orbits converge to infinity. Indeed, for points inside the Baker domain, the convergence takes place through the same access to infinity (the *dynamical access*). Thus, we introduce the notion of *Carathéodory set* as the set of points in $\partial U$ which converge to the image under $\varphi^*$ of the Denjoy-Wolff point with respect to the Carathéodory topology of $\partial U$ (or, morally, the points in $\partial U$ which converge to $\infty$ through the dynamical access).

Let us recall that the Carathéodory's compactification of $U$ endows $\overline{U}$ with a topology such that $\varphi\colon \mathbb{D} \to U$ extends to $\overline{\mathbb{D}}$ as a homeomorphism, and a sequence $\{z_n\}_n \subset U$, $z_n \to \partial U$, is convergent if there exists $\xi \in \partial\mathbb{D}$ such that for every crosscut neighbourhood $N \subset \mathbb{D}$ of $\xi$, there exists $n_0$ such that for all $n \geq n_0$, $z_n \in \varphi(N)$.

**Definition B. (Carathéodory set)** Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Baker domain. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g = \varphi^{-1} \circ f \circ \varphi$ be the inner function associated with $(f, U)$ by $\varphi$. We say that $x \in \partial U$ is in the *Carathéodory set* if, for any crosscut neighbourhood $N \subset \mathbb{D}$ at the Denjoy-Wolff point $p \in \partial\mathbb{D}$, there exists $k_0 \geq 0$ such that, for all $k \geq k_0$,

$$f^k(x) \in \overline{\varphi(N)}.$$

By Theorem II.3.7, the Carathéodory set has full harmonic measure (and, in particular, it is dense in $\partial U$) for simply connected Baker domains, of hyperbolic or simply parabolic type. Moreover the escaping points constructed in [RS18], [BFJK19, Thm. A] are also in the Carathéodory set (and since they are escaping, they are also in the Denjoy-Wolff set). However, points in the Carathéodory set may fail to converge to infinity in general, if the cluster set of the Denjoy-Wolff point is non-degenerate.

Let us describe the Carathéodory set in some illustrative examples. Let us start with doubly parabolic Baker domains of entire functions. Theorem 2 will give us that

$$\partial U = \bigsqcup_{\xi \in \partial\mathbb{D}} Cl(\varphi, \xi) \cap \mathbb{C}.$$

In particular, this implies that the images under $\varphi$ of disjoint crosscut neighbourhoods in $\mathbb{D}$ have disjoint closures in $\overline{U}$. Note that the topology of the boundary of a general simply connected domain may be more complicated and the previous property need not be satisfied. In particular, we have that the Carathéodory set of such Baker domains consists precisely of points on $\partial U$ which belong to $Cl(\varphi, \xi)$, for $\xi \in \partial\mathbb{D}$ with $(g^*)^n(\xi) \to p$. The results in [DM91, Thm. F], [BFJK19, Thm. C] imply that in many cases the Carathéodory set has harmonic measure zero; we do next a deeper analysis.

Let us start with doubly parabolic Baker domains for which the Denjoy-Wolff point $p$ of the associated inner function $g$ is not a singularity. In this case, $p$ is a parabolic fixed point with two petals, and it is (weakly) repelling when restricted to $\partial\mathbb{D}$. The only points on $\partial\mathbb{D}$ that converge to $p$ under iteration are its (radial) iterated preimages. Thus, the Carathéodory set is

$$\bigcup_{n \geq 0} \bigcup_{(g^*)^n(\xi) = p} Cl(\varphi, \xi) \cap \mathbb{C}.$$

In the following example, we see that this is compatible with having a curve of escaping points in the cluster set of every $\xi \in \partial\mathbb{D}$ (note that such points converge to $\infty$, but the convergence does not take place through the dynamical access).

**Example II.5.10. (Doubly parabolic Baker domain, finite degree,** [BD99, Sect. 5, 6], [FH06, Ex. 3], Chapter 1) The function

$$f(z) = z + e^{-z}$$

has a doubly parabolic Baker domain $U_k$ of degree 2 in each strip $S_k := \{(2k-1)\pi \leq \text{Im}z \leq (2k+1)\pi\}$. Due to the $2\pi i$-periodicity of the function, it suffices to study $U_0$. It is easy to see that $\mathbb{R} \subset U_0$ and

$$L^{\pm} := \{z \in \mathbb{C}\colon \text{Im } z = \pm\pi i\} \subset \partial U_0.$$

Fix the Riemann map $\varphi\colon \mathbb{D} \to U_0$ (chosen as in [BD99], see also Chapter 1). Then, the associated inner function can be computed explicitly as

$$g\colon \mathbb{D} \to \mathbb{D}, \qquad g(z) = \frac{3z^2+1}{3+z^2}.$$

Note that the Denjoy-Wolff point is 1, and one can prove that $Cl(\varphi, 1) = L^{\pm} \cup \{\infty\}$.

Thus, the Carathéodory set of $U_0$ is

$$\bigcup_{n\geq 0} \bigcup_{g^n(\xi)=1} Cl(\varphi, \xi) \cap \mathbb{C} = \bigcup_{n\geq 0} (f^{-n}(L^{\pm})).$$

Note that all points in the Carathéodory set are non-accessible from $U$.

Moreover, one can show that, for every $\xi \in \partial\mathbb{D}$, $g^n(\xi) \neq 1$ for all $n \geq 0$, $Cl(\varphi, \xi) \cap \mathbb{C}$ consists of a curve of escaping points, landing at infinity from one end, and at a finite endpoint in the plane from the other end, or accumulating along itself, giving rise to an indecomposable countinua. In any case, this shows the existence of plenty of escaping points which are not in the Carathéodory set. See Figure II.13.

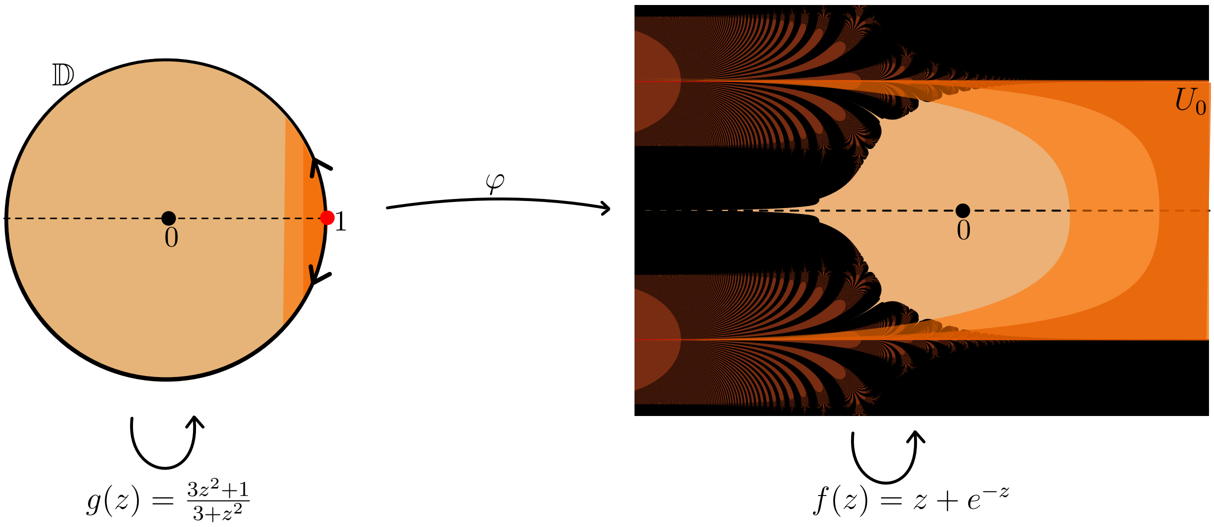



Figure II.13: Dynamical plane of $f(z) = z+e^{-z}$, with the doubly parabolic Baker domain $U_0$ (orange). The Riemann map $\varphi\colon \mathbb{D} \to U_0$ is depicted, together with the inner function. Note that 1 is the Denjoy-Wolff point, and it is repelling when restricted to $\partial\mathbb{D}$. Crosscut neighbourhoods at the Denjoy-Wolff point are indicated, as well as their image in the dynamical plane. By definition, the Carathéodory set consists of those points on $\partial U$ whose orbit eventually enters the image of every crosscut neighbourhood of 1. By the dynamics of $g$, one deduces that the Carathéodory set is $Cl(\varphi, 1) \cap \mathbb{C}$ and its iterated preimages.

Next we deal with doubly parabolic Baker domains for which the Denjoy-Wolff point $p$ of the associated inner function $g$ is a singularity. In this case, the Carathéodory set may be larger, as shown in the following example.

**Example II.5.11. (Doubly parabolic Baker domain, infinite degree,** [Evd16], [BFJK19, Ex. 1.5]**)** The function

$$f(z) = z + 1 + e^{-z},$$

known as Fatou's function, has a completely invariant Baker domain $U$, which contains a right half-plane, and $\mathcal{J}(f) = \partial U$.

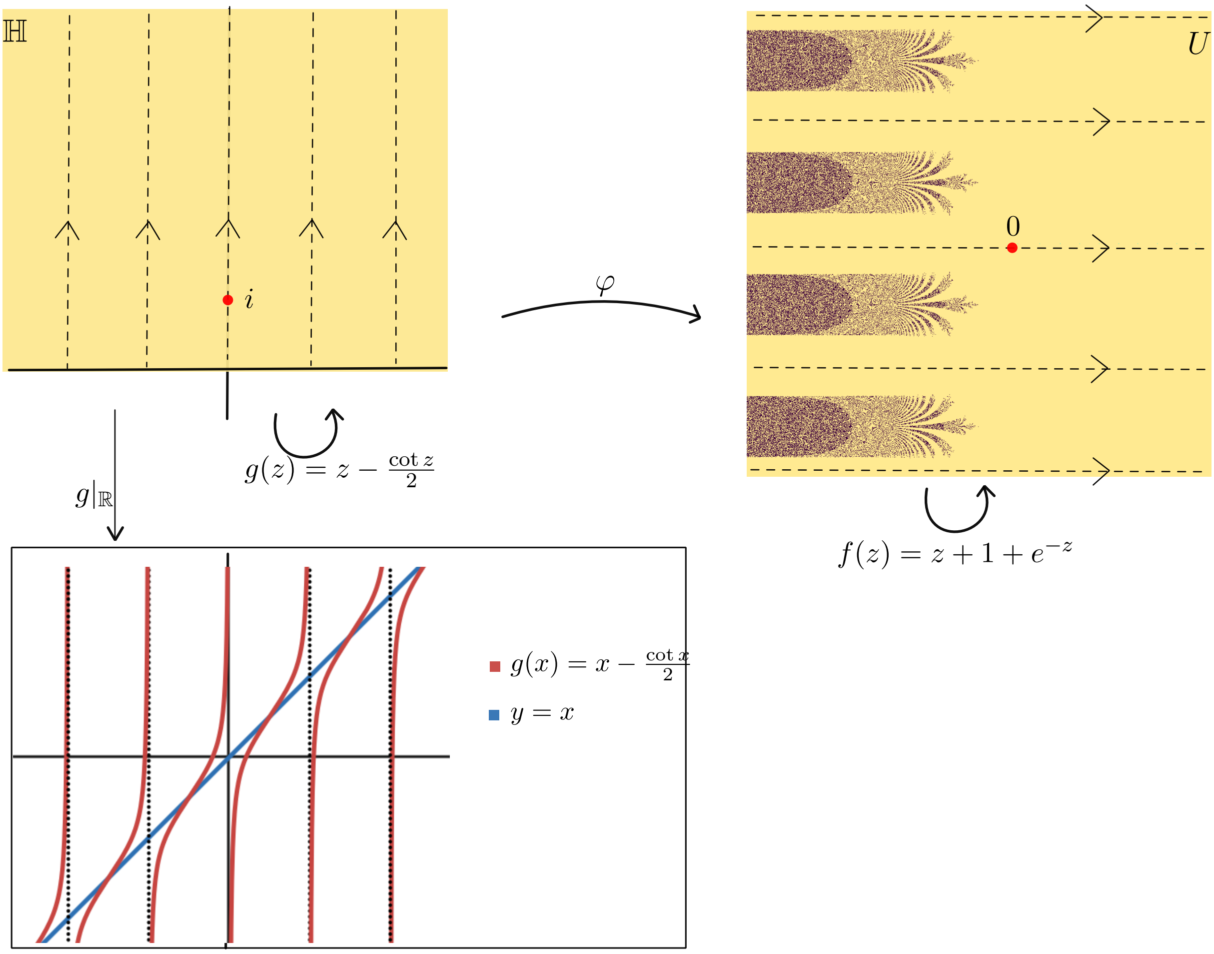



Figure II.14: Dynamical plane of $f(z) = z + 1 + e^{-z}$, with the doubly parabolic Baker domain $U$ (yellow), of infinite degree. The Riemann map $\varphi\colon \mathbb{H} \to U$ is depicted, together with the inner function, and the graphic of the inner function restricted to the real line, $g|_{\mathbb{R}}$. Note that $\infty$ is the Denjoy-Wolff point, and it is a singularity of $g$. It is easy to see that there exists points in $\mathbb{R}$ (which are not poles or prepoles) converging to $\infty$ under iteration; this points correspond to escaping endpoints in the dynamical plane, and their hairs, and they form the Carathéodory set of $U$.

It follows from [BFJK19, Thm. D] that $\omega_U$-almost every point has a dense orbit, however we claim that the Carathéodory set is non-empty, and in fact contains accessible points. Indeed, it corresponds to the hairs whose endpoint (and thus the whole hair) escapes to $\infty$. Such endpoints (which have harmonic measure zero) have been studied in [Evd16].

Moreover, the associated inner function can be computed explicitly as

$$g\colon \mathbb{H} \to \mathbb{H}, \qquad g(z) = z - \frac{\cot z}{2},$$

for a suitable Riemann map $\varphi\colon \mathbb{H} \to U$ [ERS20, Thm. 1.9]. Escaping endpoints correspond to points in $\mathbb{R}$ which converge to the Denjoy-Wolff point under iteration of $g$. See Figure II.14.

For hyperbolic and simply parabolic Baker domains, the situation is fundamentally diferent. Indeed, when the Denjoy-Wolff point of $g$ is not a singularity, it attracts points in the unit circle (from both sides if $g$ is hyperbolic, or from one side if $g$ is simply parabolic). Moreover, the Carathéodory set has always full measure (Thm. II.3.7).

If we restrict ourselves to univalent Baker domains, the situation is even more tamer. The inner function (considered in the upper half-plane) is a Möbius transformation $M\colon \mathbb{H} \to \mathbb{H}$, and can be taken to be $z \mapsto \lambda z$, $\lambda > 0$ (hyperbolic), or $z \mapsto z \pm 1$ (simply parabolic). Note that, in the first case, there is a single point in $\mathbb{R}$ which does not converge to $\infty$ (0, which is fixed), and none in the second case. For univalent Baker domains whose boundary is a Jordan curve, this implies that the Carathéodory set at most omits one point on $\partial U$. We refer to the following examples.

**Example II.5.12. (Univalent Baker domain, simply parabolic type,** [Her85, p. 609], [BW91, Thm. 4], [BF01, Sect. 5.3]**)** The function

$$f(z) = z + 2\pi i\alpha + e^z,$$

for appropriate $\alpha \in [0,1] \smallsetminus \mathbb{Q}$, has a univalent Baker domain $U$, of simply parabolic type, contained in a left half-plane. One can choose $\alpha$ so that $\widehat{\partial} U$ is a Jordan curve. In particular, the Carathéodory set is the whole boundary of the Baker domain.

**Example II.5.13. (Univalent Baker domain, hyperbolic type,** [Ber95a], [BF01, Sect. 5.1]**)** The function

$$f(z) = 2 - \log 2 + 2z - e^z$$

has a univalent Baker domain $U$, of hyperbolic type, contained in a left half-plane, and $\widehat{\partial} U$ is a Jordan curve. Then, Carathéodory set is the whole boundary of the Baker domain, except one point (the fixed point that corresponds to the repelling fixed point of the inner function under the Riemann map).

Let us note that other examples of univalent Baker domains of hyperbolic type, such as the ones considered in [BF01, Sect. 5.2] satisfy that the Carathéodory set is the whole boundary, since the cluster set of the repelling fixed point of the inner function is $\{\infty\}$.

Finally, we consider the following example of a non-univalent hyperbolic Baker domain, due to Bargmann [Bar08, Ex. 3.6], which exhibits a richer boundary dynamics. Indeed, although the Carathéodory set has full harmonic measure, periodic and bungee points are dense on $\partial U$.

**Example II.5.14. (Hyperbolic Baker domain, infinite degree,** [Bar08, Ex. 3.6]**)** The function

$$f(z) = 2z - 3 + e^z$$

has a completely invariant Baker domain $U$, and $\mathcal{F}(f) = U$ and $\mathcal{J}(f) = \partial U$. Indeed, $\exp \circ F = F \circ \exp$, with

$$F(z) = z^2 e^{z-3}.$$

It is easy to see that this latter function has a completely invariant super-attracting basin $V$ with fixed point 0, and $\mathcal{F}(F) = V$. Thus, $\mathcal{J}(F) = \partial V$, and by a result of Barański [Bar07], $\mathcal{J}(F)$ consists of disjoint curves (hairs) of escaping points, landing at infinity from one end and to a finite endpoint from the other end, which is the only point in the hair accessible from $V$.

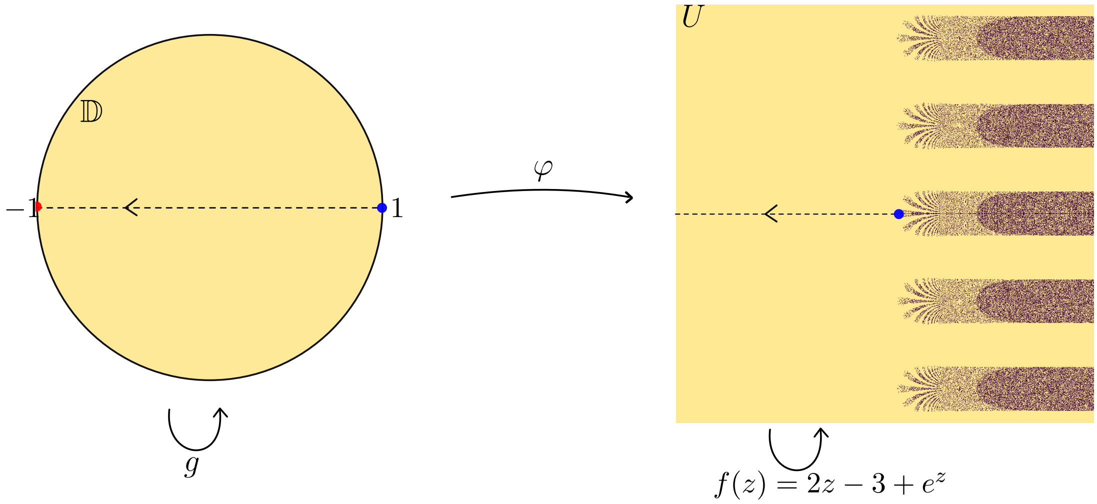



Figure II.15: Dynamical plane of $f(z) = 2z - 3 + e^z$, with the hyperbolic Baker domain $U$ (yellow) of infinite degree. The Riemann map $\varphi\colon \mathbb{D} \to U$ is depicted, together with the inner function. Note that $-1$ is the Denjoy-Wolff point. The boundary $\partial U$ is a Cantor bouquet, and the Carathéodory set consists of those endpoints whose orbit converges to $\infty$ with $|\mathrm{Im} f^n(x)| \to \infty$ and the corresponding hairs. Such endpoints have full harmonic measure.

This implies that $\mathcal{J}(f) = \partial U$, and $\mathcal{J}(f)$ consists also of hairs of escaping points, contained in the right half-plane. The Carathéodory set consists of those hairs whose endpoint $x$ (and thus the whole hair) converges to infinity with $|\mathrm{Im} f^n(x)| \to \infty$. Note that the set of such endpoints have full harmonic measure.

Thus, observe that the Carathéodory set is dense, but non-Carathéodory set is also dense. Indeed, repelling periodic points are dense on $\partial U$ (and accessible from $U$), as well as points whose orbit is dense on $\partial U$. Note also the existence of escaping points which are not in the Carathéodory set. We shall remark the poor regularity of the associated inner function $g$: its Denjoy-Wolff point is a singularity (in fact, the only singularity of $g$), and $\mathcal{J}(g) = \partial\mathbb{D}$.

Part of our work concerning the Carathéodory set will be to show that, under certain conditions, the non-Carathéodory set of a hyperbolic or simply parabolic Baker domain is non-empty (and in fact contains accessible points), despite having zero harmonic measure. See Chapter 5.

# Chapter 1

# A model for boundary dynamics of Baker domains

We consider the transcendental entire function $f(z) = z + e^{-z}$, which has a doubly parabolic Baker domain $U$ of degree two. It is known from general results that the dynamics on the boundary is ergodic and recurrent and that the set of points in $\partial U$ whose orbit escapes to infinity has zero harmonic measure (Thm. II.5.4). Following the approach of [BFJK19], we aim to give an explicit description of the sets of full and zero harmonic measure which appear as a result of the general ergodic theorems.

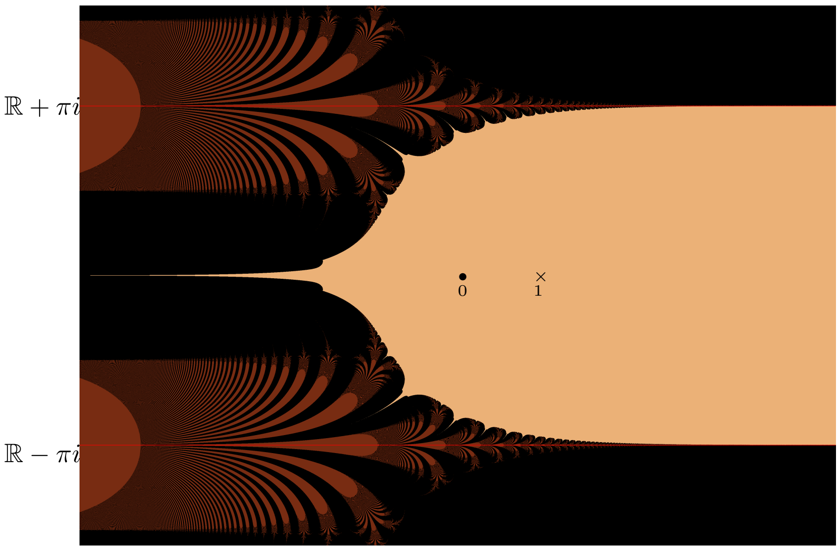



Figure 1.1: Dynamical plane for $f(z) = z + e^{-z}$. In red, the Julia set of $f$. In beige, the Baker domain contained in the strip $\{-\pi < \text{Im } z < \pi\}$. In black, the rest of the Fatou set of $f$. The only critical point on the strip (0) is also marked, as well as the corresponding critical value (1).

For this model we show that stronger results hold, namely that the escaping set is non-empty, and it is organized in curves encoded by some symbolic dynamics, whose closure is precisely $\partial U$. We also prove that nevertheless, all escaping points in $\partial U$ are non-accessible from $U$, as opposed to points in $\partial U$ having a bounded orbit, which are all accessible. Moreover, repelling periodic points are shown to be dense in $\partial U$, answering a question posted in [BFJK19]. None of these features are known to occur for a general doubly parabolic Baker domain.

First we will describe the escaping set in $\partial U$. Recall that it is known to have zero harmonic measure so, a priori, it is unknown whether it is non-empty. We prove that escaping points do exist in $\partial U$ and are organized in curves (known as *dynamic rays* or *hairs*) encoded by some symbolic dynamics, as it is not uncommon for transcendental entire functions. All escaping points are proved to belong to such curves, while non-escaping points are in their accumulation sets. This leads to the following description of the boundary of $U$.

**Theorem 1.A. (The boundary of $U$)** *Every escaping point in $\partial U$ can be connected to $\infty$ by a unique curve of escaping points in $\partial U$. Moreover, $\partial U$ is the closure of such curves.*

The existence of these dynamic rays follows from general results of [RRRS11] applied to $h(w) = we^{-w}$, semiconjugate to $f$ by $w = e^{-z}$. From these general results it is deduced that $h$, and therefore $f$, are *criniferous functions*, i.e. that all points in the escaping set can be connected to infinity by a curve of escaping points: the dynamic ray. Criniferous functions were introduced in [BR20], and further studied in [PS22]. Nevertheless, in order to have a better control on the geometry of the dynamic rays and their relation with the boundary of $U$, we choose to prove Theorem 1.A with an explicit construction, which gives us additionally a parametrization and certain continuity properties.

Other remarkable properties are observed, such as that all points in $\partial U$ escape to $\infty$ in a different 'direction' than that of the dynamical access. This connects with the fact that, for the inner function, there is no escaping point (in the sense that there are no boundary orbits converging to the Denjoy-Wolff point, apart from the preimages of itself). Moreover, escaping orbits in $\partial U$ converge to $\infty$ exponentially fast, while points in $U$ do so in a slower fashion, being the map close to the identity.

Next, we study the landing properties of the dynamic rays mentioned above. More precisely, we prove the following.

**Theorem 1.B. (Landing and non-landing dynamic rays)** *There exist uncountably many dynamic rays which land at a finite end-point, and there exists uncountably many dynamic rays which do not land. The accumulation set (on the Riemann sphere) of such a non-landing ray is an indecomposable continuum which contains the ray itself.*

This contrasts with the exponential maps $\lambda e^z$, with $0 < \lambda < \frac{1}{e}$, where all dynamic rays land, due to hyperbolicity.

On the other hand, indecomposable continua were shown to exist in the Julia set of some non-hyperbolic exponential maps $E_\kappa(z) = e^z + \kappa$, for some values of $\kappa$, first in [Dev93] and later on [DJ02, DJR05], although not as the accumulation set of a dynamic ray. It was shown by Rempe [Rem03, Rem07] that indecomposable continua appear as the accumulation set of a dynamic ray in exponential maps $E_\kappa$, for some values of

$\kappa$. More precisely, he proves that if the singular $\kappa$ is on a dynamic ray, then there exist uncountably many dynamic rays whose accumulation set is an indecomposable continuum. However, for the exponential maps $E_\kappa$, if the singular value is on a dynamic ray, then $\mathcal{J}(E_\kappa) = \mathbb{C}$ or the Fatou set consists of Siegel disks and preimages of them (see e.g. [Dev94]). Contrastingly, we find these indecomposable continua in the boundary of a Baker domain (and in the boundary of the projected parabolic basin, see Sect. 1.1).

We also address the problem of relating the previous sets, of escaping and non-escaping points, with the set of accessible boundary points from $U$. Again, symbolic dynamics play an important role, in this case to connect the dynamics in the unit circle with the behaviour in $\partial U$.

**Theorem 1.C. (Accessible points)** *Escaping points in $\partial U$ are non-accessible from $U$, while points in $\partial U$ having a bounded orbit are all accessible from $U$.*

Finally, we study periodic points in $\partial U$. We show that $g|_{\partial \mathbb{D}}$ is conjugate to the doubling map (see Sect. 1.4), so periodic points for $g$ are dense in $\partial \mathbb{D}$. Moreover, Theorem 1.C asserts that periodic points in $\partial U$, if they exist, are accessible. Both things suggest that periodic points might be dense in $\partial U$, which is indeed proven in the following theorem.

**Theorem 1.D. (Periodic points)** *Periodic points are dense in $\partial U$.*

We observe that first statement in Theorem 1.C corresponds to the first part of the conjecture in [BFJK19], while the second statement together with Theorem 1.D provide a positive answer to the second part.

**Structure of the chapter.** In Section 1.1 one finds the auxiliary results about the dynamics of $f(z) = z + e^{-z}$, which are used recurrently in the following sections. For completeness, a sketch of the general dynamics of $f$ is included, summarizing the ideas of [BD99] and [FH06]. Section 1.2 is devoted to studying the escaping set and its organization in dynamic rays, proving Theorem 1.A. The landing properties of such rays are discussed in Section 1.3. The remaining theorems are proved in Sections 1.4 and 1.5.

**Notation.** The horizontal strips of width $2\pi$ are denoted by

$$S_k := \{(2k-1)\pi \leq \text{Im } z \leq (2k+1)\pi\}.$$

We denote by $U_k$ the unique Baker domain contained in the strip $S_k$. We shall denote by $S$ the central strip $S_0$, and by $U$ its Baker domain, just to lighten the notation.

## 1.1 Basic properties of the dynamics of $f$

In this section we gather some of the properties of the function $f(z) = z + e^{-z}$, as well as its dynamics, which are used recurrently during the proofs of the main theorems. First, we include a quick description of the general dynamics of $f$, summarizing the ideas of [BD99, FH06]. From there, it will be deduced that only the study of $f$ on the strip $S := \{z \in \mathbb{C} \colon |\text{Im } z| \leq \pi\}$ is needed.

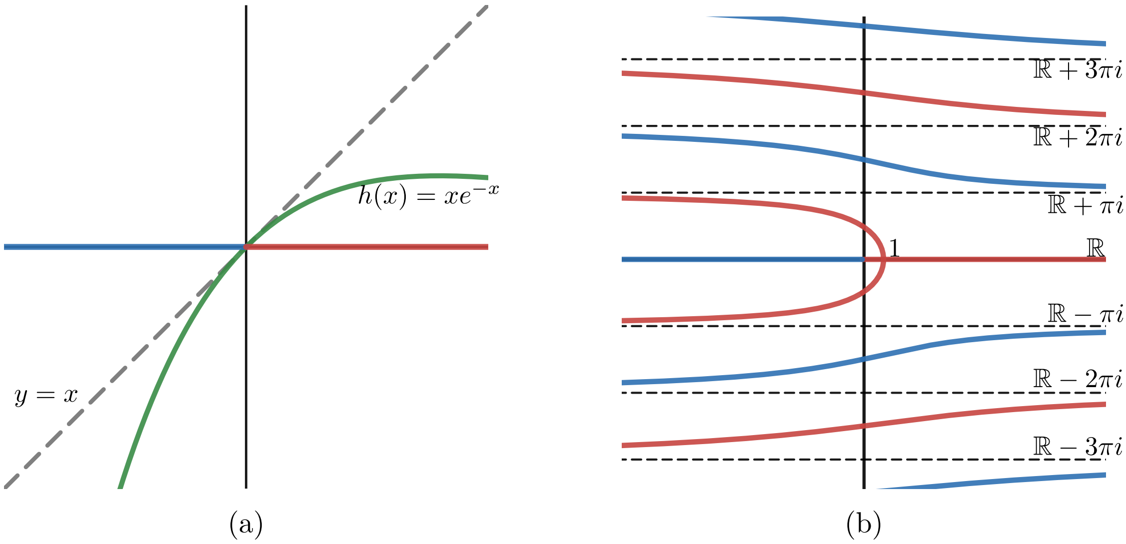



Figure 1.2: In the left, Figure 2.7a shows the plot of the real function $h(x) = xe^{-x}$ (green), together with the diagonal $y = x$ (grey dotted line). The point $x = 0$ is a parabolic fixed point. Points in $\mathbb{R}_+$ (red) are attracted to 0, so $\mathbb{R}_+ \subset \mathcal{A}_0 \subset \mathcal{F}(h)$, while points in $\mathbb{R}_-$ (blue) converge to $-\infty$ exponentially fast and $\mathbb{R}_- \subset \mathcal{J}(h)$. In the right, Figure 2.7b shows in red the preimages of the positive real line $\mathbb{R}_+$; and, in blue, the preimages of the negative real line $\mathbb{R}_-$. By the invariance of the Fatou and the Julia sets, all red lines are contained in the Fatou set, while the blue ones are in the Julia set. One deduces that the immediate parabolic basin $\mathcal{A}_0$ is contained in the region bounded by the two blue lines lying in the strips $\{\pi < y < 2\pi\}$ and $\{-2\pi < y < -\pi\}$ respectively.

### General dynamics of $f$

To give a first approach to the dynamics, one may consider the semiconjugacy $w = e^{-z}$ between $f(z) = z + e^{-z}$ and $h(w) = we^{-w}$. Observe that $w = 0$ is a fixed point of multiplier 1, and $h(w) = w - w^2 + \mathcal{O}(w^3)$ near 0, implying that 0 is a parabolic fixed point having one attracting and one repelling direction. From the fact that $\mathbb{R}$ is invariant by $h$ and from the action of $h$ in $\mathbb{R}$, it is deduced that the repelling direction is $\mathbb{R}_-$, which belongs to the Julia set $\mathcal{J}(h)$, and the attracting direction is $\mathbb{R}_+$, which belongs to the immediate parabolic basin of 0, and hence to the Fatou set $\mathcal{F}(h)$. See Figure 1.2. We denote by $\mathcal{A}_0$ the immediate basin of 0.

We note that all preimages of $\mathbb{R}_-$ are in the Julia set and, since 0 is an asymptotic value, they separate the plane into infinitely many components. It follows that the Fatou set $\mathcal{F}(h)$ has infinitely many connected components.

It is not hard to see that the only two singular values of $h$ are 0 and $e^{-1}$, the latter being contained in the immediate parabolic basin $\mathcal{A}_0$. Therefore, the Fatou set $\mathcal{F}(h)$ is precisely $\mathcal{A}_0$ and its preimages under $h$. Indeed, since $h$ has only a finite number of singular values, it cannot have Baker nor wandering domains [EL92, Sect. 5], and the presence of any other invariant Fatou component (either a basin or a Siegel disk) would require an additional singular value (see e.g. [Ber93, Thm. 7]). Since there are infinitely many Fatou components for $h$, $\mathcal{A}_0$ has infinitely many preimages, separated by the preimages of $\mathbb{R}_-$.

We lift these results to the dynamical plane of $f$, using Bergweiler's result [Ber95b], which ensures that the Fatou and Julia sets of $f$ and $h$ are in correspondence under the projection $w = e^{-z}$. Preimages of $\mathbb{R}_+$ under $e^{-z}$, which are precisely the forward invariant horizontal lines $\{\text{Im } z = 2k\pi i\}_{k\in\mathbb{Z}}$, are in the Fatou set and their points escape to $\infty$ to the right. Preimages of $\mathbb{R}_-$ under the exponential projection are the forward invariant horizontal lines $\{\text{Im } z = (2k+1)\pi i\}_{k\in\mathbb{Z}}$, which are in the Julia set and whose points escape to $-\infty$ exponentially fast. The horizontal strips $S_k := \{(2k-1)\pi \leq \text{Im } z \leq (2k+1)\pi\}$ contain a preimage $U_k$ of $\mathcal{A}_0$ which, in turn, contains a preimage of $\mathbb{R}_+$ under $e^{-z}$, that is $\{\text{Im } z = 2k\pi i\}$. Such horizontal line is forward invariant, so this implies that $U_k$ is forward invariant and iterates tend to $\infty$, so $U_k$ is a Baker domain.

Moreover, we note that $\mathcal{F}(f)$ is precisely the union of these Baker domains $U_k$ and their preimages under $f$. Indeed, any Fatou component $V$ of $f$ must project by $w = e^{-z}$ to a preimage of $\mathcal{A}_0$, implying that $V$ is mapped to some $U_k$ in a finite number of steps. Hence, the presence of wandering domains is ruled out.

Finally, we note that the function $f$ satisfies the relation $f(z + 2k\pi i) = f(z) + 2k\pi i$, for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, so it is enough to study it in the central strip $S := S_0$ and the corresponding Baker domain $U := U_0$. To do so, we consider the conformal branch of the semiconjugacy $w = e^{-z}$, defined on Int$S$, i.e.

$$E(z) := e^{-z} \colon \text{Int}S \longrightarrow \mathbb{C} \setminus \mathbb{R}_-,$$

$$E^{-1}(w) := -\text{Log}(w) \colon \mathbb{C} \setminus \mathbb{R}_- \longrightarrow \text{Int}S,$$

where $\text{Log} \colon \mathbb{C}\setminus\mathbb{R}_- \to \text{Int}S$ denotes the principal branch of the logarithm. Since $U \subset \text{Int } S$ and $\mathcal{A}_0 \subset \mathbb{C} \setminus \mathbb{R}_-$, this gives a conformal conjugacy between $f_{|U}$ and $h_{|\mathcal{A}_0}$. Hence, we deduce that the Baker domain $U$ is of doubly parabolic type and of degree two.

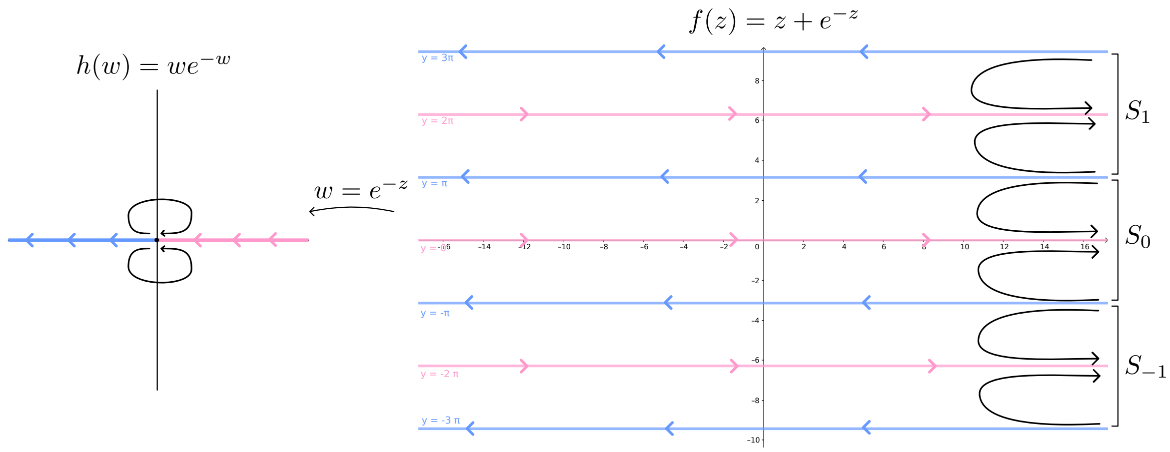



Figure 1.3: Schematic representation of the dynamics of $h$ and $f$ and how the exponential projection $w = e^{-z}$ relates both of them. In the left, $\mathbb{R}_+$ (in pink) is contained in the immediate parabolic basin $\mathcal{A}_0$. Its preimages by $w = e^{-z}$, the lines $\{\text{Im } z = 2k\pi i\}_{k\in\mathbb{Z}}$ (also in pink), lie each of them in a Baker domain $U_k$. In blue, in the left there is $\mathbb{R}_- \subset \mathcal{J}(h)$. Its preimages $\{\text{Im } z = (2k+1)\pi i\}_{k\in\mathbb{Z}}$ lie in $\mathcal{J}(f)$ and separate the plane into the strips $S_k$.

**Remark 1.1.1.** Although working with the function $h$ may seem easier, for having a finite number of singular values and being postsingularly bounded, the fact that one asymptotic value lies in the Julia set reduces this advantage. In general, we shall work with $f$, its logarithmic lift.

### Action of $f$ in the strip $S$

As seen before, it is enough to consider $f$ in the strip $S = \{z \in \mathbb{C}\colon\ |\text{Im } z| \leq \pi\}$, delimited by the horizontal lines $L^{\pm} := \{z\colon \text{Im } z = \pm\pi\}$. See Figure 1.4.

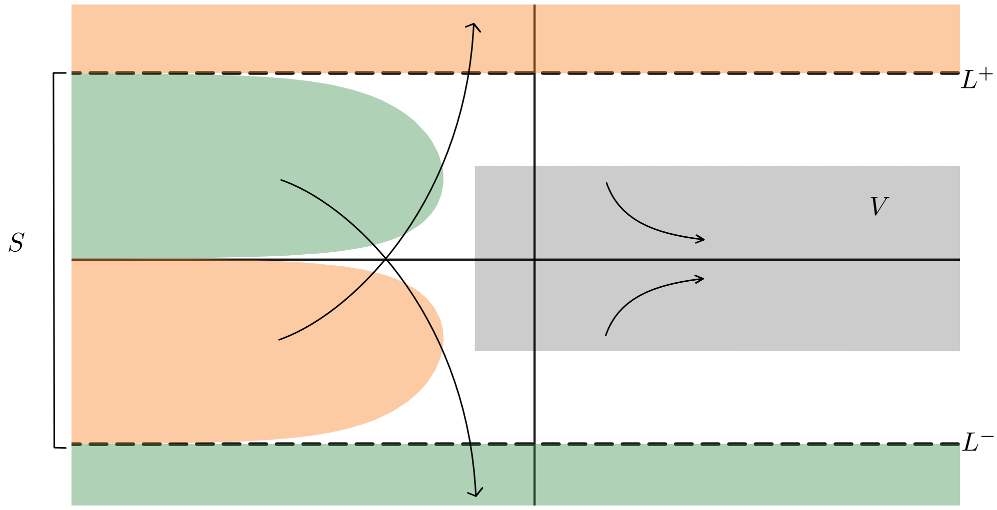



Figure 1.4: Schematic representation of how $f$ acts on the strip $S$ and of the absorbing domain $V$.

Observe that, to the left, $f$ behaves like the exponential and, to the right, like the identity. Moreover, if one writes $f$ as

$$f(x, y) = (x + e^{-x} \cos y, y - e^{-x} \sin y),$$

preimages of $L^{\pm}$ can be computed explicitly as the curves of the form $\{y - e^{-x} \sin y = \pm\pi\}$. In $S$ they consist precisely of two bent curves converging to $-\infty$ in both ends, being asymptotic to $\mathbb{R}$ and to $L^{\mp}$ (see Fig. 1.4). The region delimited by these curves is mapped outside $S$ in a one-to-one fashion. On the other hand, the map $f\colon f^{-1}(S) \cap S \to S$ is a proper map of degree two, which can be deduced for instance by computing the preimages of $\mathbb{R}$ in $S$.

Next, we define the set

$$\widehat{S} := \{z \in S\colon f^n(z) \in S, \text{ for all } n\}.$$

Clearly, $U \subset \widehat{S}$, since $U$ is forward invariant under $f$. Moreover, since both $f\colon f^{-1}(S) \cap S \to S$ and $f_{|U}$ have degree 2, there cannot be preimages of $U$ in $S$ other than itself. Therefore, $\mathcal{F}(f) \cap \widehat{S} = U$. On the other hand, $\partial U \subset \mathcal{J}(f) \cap \widehat{S}$. The other inclusion, which is going to be proved in Proposition 1.2.4, cannot be claimed directly to be true, for the possible existence of buried points in $\widehat{S}$, i.e. points in $\mathcal{J}(f)$ which are not eventually mapped to the boundary of any Baker domain $U_k$.

### Absorbing domains and expansion of $f$

Let us define the following set

$$V := \left\{ z \in S \colon \text{Re } z > -1, |\text{Im } z| < \frac{\pi}{2} \right\}.$$

**Lemma 1.1.2.** *The set $V$ is an absorbing domain for $f$ in $U$.*

*Proof.* Clearly, $V$ is open and connected. For the forward invariance, consider $z = x+iy \in V$, so $x > -1$ and $|y| < \frac{\pi}{2}$, then

$$\text{Re } f(x+iy) = x + e^{-x} \cos y > x > -1,$$

$$|\text{Im } f(x+iy)| < \left| \frac{\pi}{2} - e^{-x} \right| < \frac{\pi}{2}.$$

Finally, the fact that $V$ is absorbing, i.e. that all compact sets in $U$ must eventually enter in $V$, can be deduced from the dynamics on the conjugate parabolic basin $\mathcal{A}_0$. Indeed, $E(V)$ is the following forward invariant set,

$$E(V) = \left\{ w \in \mathbb{C} \colon \; |w| < \frac{1}{e}, |\arg w| < \frac{\pi}{2} \right\}.$$

Observe that $E(V)$ is an circular sector of angle $\frac{\pi}{2}$ containing the real interval $(0, e)$, which is in the attracting direction of the parabolic point $w = 0$. Hence, $E(V)$ is a parabolic petal, so all compact sets in $\mathcal{A}_0$ must eventually enter in $E(V)$. Hence, applying back the conjugacy, we get that $V$ is an absorbing domain for $f$ in $U$. □

**Remark 1.1.3.** Since it contains the critical point 0, $V$ is not a fundamental set. It can be turned into one making it smaller, for instance taking $\left\{ z \in S \colon \text{Re } z > 0, |\text{Im } z| < \frac{\pi}{2} \right\}$. On the other hand, fundamental sets, and absorbing domains, can be chosen bigger, although we have no need to do that. In fact, using local theory around parabolic fixed points, there exist fundamental sets which approach tangentially $L^{\pm}$.

One of the advantages of choosing $V$ as we have done is that the map is expanding outside it (although not uniformly expanding). Indeed, a simple computation yields:

$$f'(x+iy) = 1 - e^{-x} \cos y + i e^{-x} \sin y,$$

$$|f'(x+iy)| = \sqrt{1 + e^{-2x} - 2e^{-x} \cos y}.$$

Therefore, $|f'(x+iy)| > 1$ if and only if $e^{-x} - 2\cos y > 0$. This last inequality is satisfied if $\frac{\pi}{2} < |y| < \pi$ or if $x < -1$. Therefore, $|f'(z)| > 1$ for all $z \in S \setminus \overline{V}$.

Since $S \setminus \overline{V}$ is not convex, in order to apply the expansion of $f$ as an augmenter of the distance between points, we need to consider a more appropriate distance than the Euclidean one. To this aim, we define the following metric.

**Definition 1.1.4.** **($\rho$-distance in $S \setminus \overline{V}$)** Given $z, w \in S \setminus \overline{V}$, let us define its $\rho$-distance as:

$$\rho(z, w) := \inf l(\gamma),$$

where the infimum is taken over all paths $\gamma \subset S \setminus \overline{V}$ with endpoints $z$ and $w$, and $l$ denotes the length of the path with respect to the Euclidean metric.

Given a set $K \subset S \setminus \overline{V}$, we denote by $\operatorname{diam}_\rho(K)$ the diameter of $K$ with respect to the $\rho$-distance, i.e.

$$\operatorname{diam}_\rho(K) = \sup_{x,y \in K} \rho(x, y).$$

Observe that the Euclidean distance is always smaller than the $\rho$-distance, i.e.

$$|z - w| \leq \rho(z, w), \quad \text{for all } z, w \in S \setminus \overline{V},$$

with equality if both $z$ and $w$ are contained in a convex subset of $S \setminus \overline{V}$.

Notice also that the $\rho$-distance between two points can be arbitrarily large, although the Euclidean distance between them remains bounded. However, we are going to restrict the use of the $\rho$-distance to particular subsets of $S \setminus \overline{V}$, where we do have an upper bound for the $\rho$-distance in terms of the Euclidean one (see Lemma 1.1.10).

**Remark 1.1.5.** Let us observe that, instead of considering the dynamical system defined by $f$ in $\mathbb{C}$, we can restrict to the one given by $f$ in $\widehat{S}$. For it we have a similar situation that the one for $\lambda e^z$, $0 < \lambda < \frac{1}{e}$, in [DG87], and the corresponding generalization in [Bar07, BK07]: a unique Fatou component which contains the postsingular set. Mainly, two things distinguish our situation from theirs. First, $f$ in $\widehat{S}$ has degree two, and the functions they are dealing with have infinite degree. Second, they have uniform expansion (at least in the logarithmic tracts), while our expansion is not uniform (compare with Prop. 1.1.7). Hence, the results on next sections are meant to overcome this difficulty.

### Itineraries in $\widehat{S}$ and symbolic dynamics

Recall that $f\colon f^{-1}(S) \cap S \to S$ has degree two and the critical value is 1. Therefore, the two branches of the inverse of $f$ in $S$, say $\phi_0$ and $\phi_1$, are well-defined in $S \setminus [1, +\infty)$. More precisely

$$\phi_0\colon S \setminus [1, +\infty) \to \Omega_0 := S \cap \mathbb{H}_+,$$
$$\phi_1\colon S \setminus [1, +\infty) \to \Omega_1 := S \cap \mathbb{H}_-,$$

where $\mathbb{H}_+$ and $\mathbb{H}_-$ denote the upper and the lower half plane, respectively (see Fig. 1.5).

We claim that $\phi_0$ and $\phi_1$ do not increase the $\rho$-distance between points, as shown in the following proposition.

**Proposition 1.1.6.** **(Contraction and uniform contraction in $S \setminus \overline{V}$)** *The following properties hold true.*

(a) *Let $z, w \in S \setminus \overline{V}$. Then, for $i \in \{0, 1\}$,*

$$\rho(\phi_i(z), \phi_i(w)) \leq \rho(z, w).$$

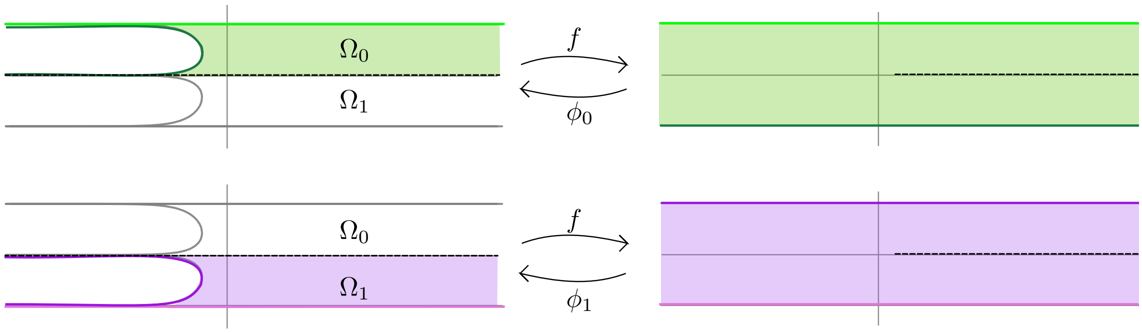



Figure 1.5: Action of the inverses $\phi_0$ and $\phi_1$ on the strip S.

(b) *Let $k \in \mathbb{R}$ and let $S_k := \left\{z = x + iy \in S \smallsetminus \overline{V} \colon x \leq k\right\}$. Then, there exists $\lambda := \lambda(k) < 1$ such that, if $z, w \in S \smallsetminus \overline{V}$, then, for $i \in \{0, 1\}$,*

$$\rho(\phi_i(z), \phi_i(w)) \leq \lambda \rho(z, w).$$

*Moreover, if $K \subset S_k$ is a compact set, then*

$$\operatorname{diam}_\rho(\phi_i(K)) \leq \lambda \operatorname{diam}_\rho(K).$$

*Proof.* (a) As observed above, it holds $|f'(z)| > 1$ for all $z \in S \smallsetminus \overline{V}$. Therefore, if $\gamma \subset S \smallsetminus \overline{V}$ is a geodesic (in $S \smallsetminus \overline{V}$) joining $z$ and $w$, then $\phi_i(\gamma)$ is a curve joining $\phi_i(z)$ and $\phi_i(w)$, and

$$\rho(\phi_i(z), \phi_i(w)) \leq \int_{\phi_i(\gamma)} ds = \int_\gamma |\phi_i'(s)|\, ds < \int_\gamma ds = \rho(z, w),$$

as desired.

(b) We start by noticing that $|f'|$ is uniformly bounded in $S \smallsetminus \overline{V}$. Indeed, on the one hand, for all $z = x + iy$ with $x \leq -1$, it holds

$$|f'(x + iy)| = \sqrt{1 + e^{-2x} - 2e^{-x}\cos y} \geq \sqrt{1 + e^2 - 2e} > 1.$$

On the other hand, assuming $k > -1$ and $-1 < x < k$, necessarily $\frac{\pi}{2} \leq |y| \leq \pi$, so

$$|f'(x + iy)| = \sqrt{1 + e^{-2x} - 2e^{-x}\cos y} \geq \sqrt{1 + e^{-2x}} \geq \sqrt{1 + e^{-2k}} > 1.$$

Hence, there exists a constant $\lambda$, depending only on $k$, such that $|f'(z)| \geq \lambda$, for all $z \in \left\{z = x + iy \in S \smallsetminus \overline{V} \colon x \leq k\right\}$. Hence, the first statement follows applying the same reasoning as in (a).

Finally, let $K \subset S_k$ and denote by $\lambda$ the constant of contraction in $S_k$. Then, for all $z, w \in \phi_i(K)$, we have $f(z), f(w) \in K$, and

$$\rho(z, w) \leq \lambda \rho(f(z), f(w)) \leq \lambda \operatorname{diam}_\rho(K).$$

Hence, $\operatorname{diam}_\rho(\phi_i(K)) \leq \lambda \operatorname{diam}_\rho(K)$, as desired.

□

**Remark 1.1.7. (Expansion and uniform expansion in $S \setminus \overline{V}$)** We note that, as a direct consequence of Proposition 1.1.6 (a), if $z, w \in \Omega_i$ and $f(z), f(w) \in S \setminus \overline{V}$, then

$$\rho(z, w) \leq \rho(f(z), f(w)).$$

Likewise, the expansion is uniform in any half-strip $S_k$. In particular, if $K$ is a compact set such that $\text{diam}_\rho(K) > 0$ and $f^n(K) \subset S_k \cap \Omega_{i_n}$, $i_n \in \{0, 1\}$, then $\text{diam}_\rho(f^n(K)) \to \infty$, as $n \to \infty$.

Next, we use this subdivision of the strip in $\Omega_0$ and $\Omega_1$ to define the itinerary for points in $\widehat{S}$, where $\Sigma_2$ denotes the space of infinite sequences of two symbols, taken to be 0's and 1's.

**Definition 1.1.8. (Itineraries in $\widehat{S}$)** Let $z \in \widehat{S}$ be such that $f^n(z) \notin \mathbb{R}$, for all $n \geq 0$. The sequence $I(z) = \underline{s} = \{s_n\}_n \in \Sigma_2$ satisfying $f^n(z) \in \Omega_{s_n}$ is called the *itinerary* of $z$.

**Remark 1.1.9.** For points in $\widehat{S}$ which are eventually mapped to $\mathbb{R}$, the itinerary is not defined. However, this can be neglected because they are in the Baker domain and their dynamics are already understood.

We will need a further subdivision of the strip. Let us define the regions

$$\Omega_{ij} := \phi_i(\phi_j(S)) \setminus \overline{V}.$$

For instance, the region $\Omega_{00}$ has to be seen as the set of points in $\Omega_0$ which remain in $\Omega_0$ after one iteration of the function, while points in $\Omega_{01}$ are the points which change to $\Omega_1$. Clearly, if $z \in \widehat{S}$ belongs to $\Omega_{00}$, its itinerary starts with 00; while if $z \in \Omega_{01}$, then $I(z)$ begins with 01. The absorbing domain $V$ is removed from the regions for practical use: this has no effect on the study of $\partial U$, since its points are never in $V$, but it allows us to give better estimates on the function. See Figure 1.6.

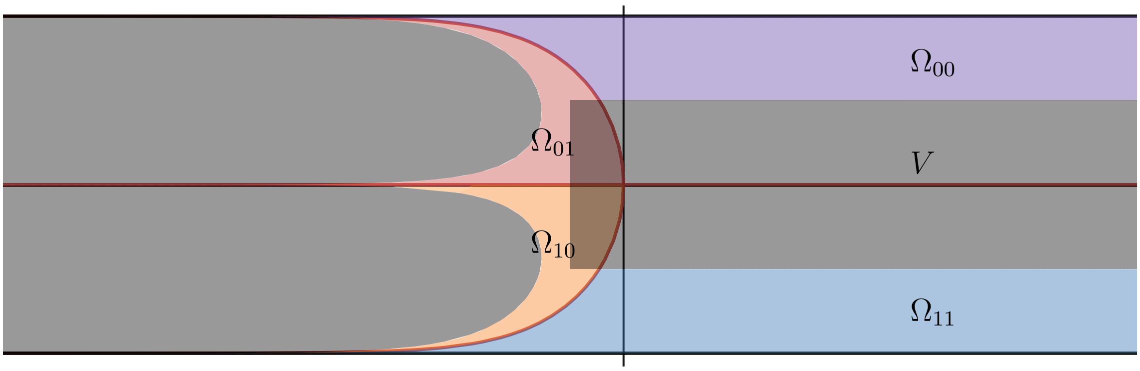



Figure 1.6: Graphic representation of the regions $\Omega_{ij}$, $i, j \in \{0, 1\}$.

**Lemma 1.1.10. (Properties of the regions $\Omega_{ij}$)** *The following properties hold true.*

(a) $\Omega_{01}, \Omega_{10} \subset \{z \in S \colon \text{Re z} < 0\}$. *Therefore, if* $z \in S \setminus \overline{V}$ *with* $\text{Re } z > 0$, *either* $z \in \Omega_{00}$ *or* $z \in \Omega_{11}$.

(b) *If* $z \in S \setminus \overline{V}$ *with* $-\frac{\pi}{2} < \text{Im } z < \frac{\pi}{2}$ *and* $f(z) \in S$, *then either* $z \in \Omega_{01}$ *or* $z \in \Omega_{10}$.

(c) *For* $z \in \Omega_{ii}$, $i \in \{0,1\}$, *we have* $|\text{Im } z| > \frac{\pi}{2}$. *In particular,* $\text{Re } f(z) < \text{Re } z$ *and, if* $z \notin L^{\pm}$, $|\text{Im } f(z)| < |\text{Im } z|$.

(d) *For* $z, w \in \Omega_{ij}$, $i, j \in \{0,1\}$, *it holds* $|z-w| \leq \rho(z,w) \leq |z-w| + \pi$.

*Proof.* The proof is direct from the definition of the regions. See also Figure 1.6. □

## 1.2 The escaping set in $\partial U$. Proof of Theorem 1.A

This section is devoted to the proof of Theorem 1.A, which asserts that escaping points in $\partial U$ are organized in curves, and $\partial U$ is precisely the closure of these curves. To do so, a detailed study of the escaping set is required, which is carried out in a several number of steps. First, it is proven that all escaping points in $\partial U$ are left-escaping (Lemma 1.2.1), and sufficiently to the left, curves of escaping points with the same itinerary are constructed (Prop. 1.2.2). Afterwards, these curves are enlarged by the dynamics to collect all points in $S$ with the same itinerary (Thm. 1.2.3); and, finally, all this construction is used to prove a characterization of $\partial U$ (Prop. 1.2.4), which is of independent interest. As indicated in the end of the section, Theorem 1.A will follow from Theorem 1.2.3 (a) and Proposition 1.2.4 (b).

First, recall that $\partial U \subset \widehat{S}$, where $\widehat{S}$ consists of all the points in $S$ which never leave $S$ under iteration; and observe that in $\widehat{S}$ there are three distinguished ways to escape to infinity. Indeed, points can escape to infinity to the left, to the right, or oscillating from left to right. This leads us to define the following sets:

$$\mathcal{I}_S^+ := \left\{ z \in \mathcal{I}(f) \cap \widehat{S} \colon \text{Re } f^n(z) \to +\infty \right\},$$

$$\mathcal{I}_S^- := \left\{ z \in \mathcal{I}(f) \cap \widehat{S} \colon \text{Re } f^n(z) \to -\infty \right\}.$$

By construction, these two sets are disjoint, but they may not contain all the escaping points: points which escape to $\infty$ oscillating from left to right belong neither to $\mathcal{I}_S^-$ nor to $\mathcal{I}_S^-$. However, this possibility is excluded, as it is shown in the following lemma. Intuitively, oscillations are not possible because, on the right, the map is close to the identity.

**Lemma 1.2.1. (No oscillating escaping points)** *There are no oscillating escaping points, i.e.*

$$\mathcal{I}(f) \cap \widehat{S} = \mathcal{I}_S^+ \cup \mathcal{I}_S^-.$$

*Moreover,* $\mathcal{I}_S^+ = U$.

*Proof.* Assume $z \in \mathcal{I}(f) \cap \widehat{S}$. For any $r > 0$, there exists $n_0$ such that, for all $n \geq n_0$, $f^n(z) \in S$ and $|f^n(z)| > r$. In particular, taking $r > \sqrt{\pi^2+1}$, there exists $R > 1$ such that $\text{Re } f^n(z) > R$ or $\text{Re } f^n(z) < -R$, for all $n \geq n_0$. Assuming that $\text{Re } z > R$, we are going to see that it is not possible to have Re $f(z) < -R$, so oscillating escaping orbits are not possible. Indeed,

$$\text{Re } f(x+iy) = x + e^{-x}\cos y \geq x - e^{-x} \geq R - e^{-R}.$$

Since $R > 1$, the right-hand side of the inequality is greater than 0, so it does not hold Re $f(z) < -R$, proving the first statement.

To prove the second statement, first observe that $U \subset \mathcal{I}_S^+$. It is left to show that, for $z \in \widehat{S} \smallsetminus U$, it cannot hold Re $f^n(z) \to +\infty$. Indeed, such a point never enters the absorbing domain, so, when Re $f^n(z) > 0$, either Im $f^n(z) > \frac{\pi}{2}$ or Im $f^n(z) < -\frac{\pi}{2}$. In both cases, Re $f^{n+1}(z) <$ Re $f^n(z)$, so it is impossible for a point which is not in $U$ to belong to $\mathcal{I}_S^+$. □

Next we show that these left-escaping points are organized in curves, which eventually contain all left-escaping points with the same itinerary. To do so, we adapt the proof of [DG87, Prop. 3.2] for the exponential maps $\lambda e^z$, $0 < \lambda < \frac{1}{e}$, to our setting. Moreover, the construction is made in such a way that a parametrization of the curves appears implicitly, as the one introduced in [BDH+99] for the exponential family (see also [SZ03, Rem03, Rem07]). The main attribute of this parametrization is to conjugate the dynamics on the curve with the model of growth given by $F(t) = t - e^{-t}$, $t \in \mathbb{R}$. Observe that $F\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism of $\mathbb{R}$ without fixed points, where all iterates converge to $-\infty$ under iteration.

**Proposition 1.2.2. (Escaping tails)** *For every sequence $\underline{s} = \{s_n\}_n \in \Sigma_2$ there exists a curve of left-escaping points $\gamma_{\underline{s}}\colon (-\infty, -2] \to \mathcal{I}_S^-$, whose points have itinerary $\underline{s}$ and $\gamma_{\underline{s}} \subset \partial U$. Such curve is called* escaping tail*. The following properties hold.*

(a) (Asymptotics and dynamics) *It holds that Re $\gamma_{\underline{s}}(t) \to -\infty$ , as $t \to -\infty$, and Re $f^n(\gamma_{\underline{s}}(t)) \to -\infty$, as $n \to \infty$. Moreover, Re $f^n(\gamma_{\underline{s}}(t)) \leq -2$ for all $n \geq 0$.*

(b) (Uniqueness) *Escaping tails are unique, in the sense that if $z \in \mathcal{I}_S^-$, with $I(z) = \underline{s}$, and Re $f^n(z) \leq -2-\pi$ for all $n \geq 0$, then $z \in \gamma_{\underline{s}}$.*

(c) (Internal dynamics) *For $t \leq -2$, it is satisfied*

$$f(\gamma_{\underline{s}}(t)) = \gamma_{\sigma(\underline{s})}(F(t)),$$

*where $\sigma$ denotes the shift map and $F(t) = t - e^{-t}$.*

It is worth mentioning that the existence of such curves of escaping points can be deduced directly from [RRRS11, Thm. 1.2] for functions in class $\mathcal{B}$ of finite order, applied to $h(w) = we^{-w}$. Indeed, both functions $f$ and $h$ are semiconjugate by $w = e^{-z}$, so left-escaping points for $f$ correspond to the escaping set of $h$. Then, if $z \in \mathcal{I}_S^-$, then $w = e^{-z} \in \mathcal{I}(h)$ and, by [RRRS11, Thm. 1.2], it is connected to $\infty$ by a curve $\Gamma$ of escaping points. An appropriate lift $\gamma$ of $\Gamma$ is a curve of left-escaping points connecting $z$ to infinity. It is easy to see that points in $\gamma$ must have the same itinerary. Indeed, $\gamma$

must be contained in either $\Omega_0$ or in $\Omega_1$, since it cannot intersect $\mathbb{R}$ (because it is in the Fatou set) nor $L^{\pm}$ (since $L^{\pm}$ separate distinct preimages of the $w$-plane under $w = e^{-z}$). Moreover, this is also true for any iterated image of $\gamma$, implying that all points in $\gamma$ must have the same itinerary.

However, from this general result, it cannot be deduced which of these curves are in $\partial U$ and it does not give a parametrization for the curves, which will be important in the following sections. This is why we choose an alternative proof for Proposition 1.2.2, based on the more constructive approach of [DG87]. On the other hand, we do apply [RRRS11, Thm. 1.2] to deduce the uniqueness of the escaping tails.

*Proof of Proposition 1.2.2.* First, let us show that, to every $t \leq -2$ and $\underline{s} \in \Sigma_2$, we can find a left-escaping point $z^{t,\underline{s}}$, with itinerary $\underline{s}$, associated to $t$. To do so, fix $t \leq -2$ and $\underline{s} \in \Sigma_2$, and let $D_0^{t,\underline{s}}$ be the square of side length $\pi$ located in $\Omega_{s_0}$ and right-hand side at $t_0 := t$. We construct a sequence of squares $\{D_n^{t,\underline{s}}\}_n$, where $D_n^{t,\underline{s}}$ is a square of side length $\pi$, located in $\Omega_{s_n}$ and right-hand side $t_n := F^n(t)$, where $F(t) = t - e^{-t}$. Observe that $t_n \to -\infty$, as $n \to \infty$. Compare with Figure 1.7.

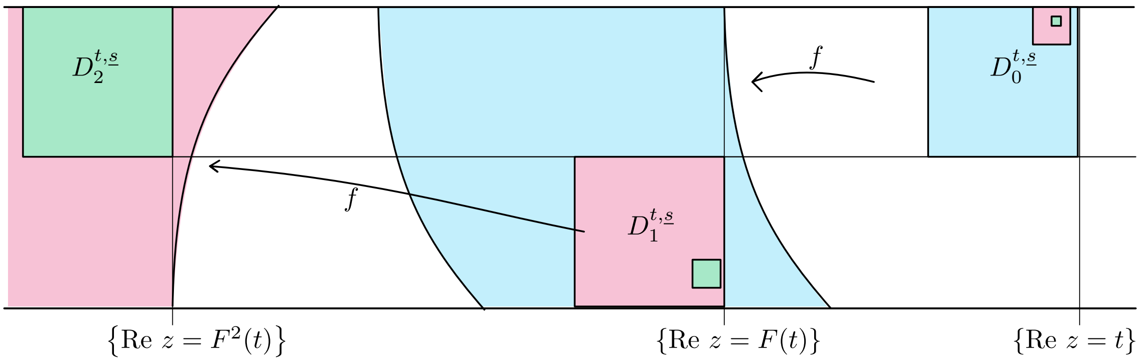



Figure 1.7: Schematic representation of the first three squares $\left\{D_n^{t,\underline{s}}\right\}_n$, for a given $t \leq -2$, showing how they satisfy $D_n^{t,\underline{s}} \subset f(D_{n-1}^{t,\underline{s}})$.

**Claim.** *The squares* $\{D_n^{t,\underline{s}}\}_n$ *satisfy* $D_n^{t,\underline{s}} \subset f(D_{n-1}^{t,\underline{s}})$, *for all* $n \geq 1$.

*Proof of the claim.* It is enough to show that $D_1^{t,\underline{s}} \subset f(D_0^{t,\underline{s}})$. Let us denote by $\partial^- D$ and $\partial^+ D$, the left and the right-hand sides of a square $D$, respectively.

First let us observe that, on the left, the map $f$ acts on a similar way than the exponential, sending vertical segments to circular curves, which start at $L^+$, ends at $L^-$ and have an auto-intersection in the negative real line. Compare with Figure 1.8.

Moreover, if Re $z = t \leq -2$, we have the following inequality controlling the modulus of the image:

$$|f(z)| = \left|z + e^{-z}\right| \geq \left|e^{-z}\right| - |z| = e^{-\text{Re } z} - |z| > \frac{1}{2}e^{-\text{Re } z} = \frac{1}{2}e^{-t} > -t.$$

To prove that $D_1^{t,\underline{s}} \subset f(D_0^{t,\underline{s}})$, it is enough to show that $\partial^- D_1^{t,\underline{s}}$ and $\partial^+ D_1^{t,\underline{s}}$ are contained in $f(D_0^{t,\underline{s}})$. In fact, we shall see that $\partial^- D_1^{t,\underline{s}}$ and $\partial^+ D_1^{t,\underline{s}}$ are contained in $f(D_0^{t,\underline{s}}) \cap S \cap \{\text{Re } z < 0\}$. Compare with Figure 1.8.

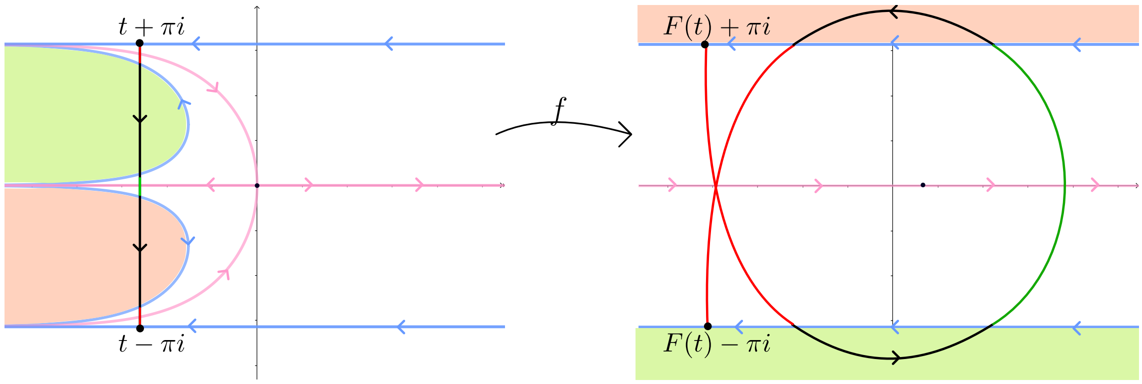



Figure 1.8: Schematic representation of how $f$ acts on the left-side on the strip.

First we see that $\partial^+ D_1^{t,\underline{s}}$ is located more to the left than $f(\partial^+ D_0^{t,\underline{s}})$. Indeed, points in $\partial^+ D_1^{t,\underline{s}}$ have real part $t - e^{-t}$, while for $z \in \partial^+ D_0^{t,\underline{s}}$ it is satisfied that $\text{Re } f(z) \geq t - e^{-t}$.

Finally, to see that $\partial^- D_1^{t,\underline{s}}$ is contained in $f(D_0^{t,\underline{s}}) \cap S \cap \{\text{Re } z < 0\}$, we shall see that $\partial^- D_1^{t,\underline{s}}$ is located more to the right than $f(\partial^- D_0^{t,\underline{s}}) \cap S \cap \{\text{Re } z < 0\}$. For $z \in \partial^- D_0^{t,\underline{s}}$ and such that $f(z) \in S \cap \{\text{Re } z < 0\}$, we have:

$$\text{Re } f(z) \leq -|f(z)| + \pi < -\frac{1}{2}e^{-\text{Re } z} + \pi = -\frac{1}{2}e^{-(t-\pi)} + \pi.$$

A point $z \in \partial^- D_1^{t,\underline{s}}$ has real part $t - e^{-t} - \pi$, which is easy to see that it is bigger than our previous bound. Indeed, the real function $h(x) = x - e^{-x} + \frac{1}{2}e^{-(x-\pi)} - 2\pi$ is positive, when $x < 0$. Therefore, the claim is proved. $\square$

Now, let us define

$$Q_n^{t,\underline{s}} := \phi_{s_0} \circ \cdots \circ \phi_{s_n}(\overline{D_{n+1}^{t,\underline{s}}}),$$

$$z^{t,\underline{s}} := \bigcap_{n \geq 0} Q_n^{t,\underline{s}}.$$

Notice that $z^{t,\underline{s}}$ is a unique point. Indeed, $\{Q_n^{t,\underline{s}}\}_n$ is a sequence of nested compact sets contained in $D_0^{t,\underline{s}}$. Its intersection is a connected compact set, and to prove that it consists precisely of a unique point, we shall see that the diameter of $Q_n^{t,\underline{s}}$ tends to 0, as $n \to \infty$. Indeed, since $\phi_{s_k} \circ \cdots \circ \phi_{s_n}(\overline{D_{n+1}^{t,\underline{s}}}) \subset \{\text{Re } z < 0\}$ for all $n \geq 0$ and $k \leq n$, each time we apply either $\phi_0$ or $\phi_1$ we are applying a contraction of constant $\frac{1}{\lambda} < 1$ with respect to the $\rho$-distance (see Prop. 1.1.6). Recall that, in the half-plane $\{\text{Re } z < -2\}$, the $\rho$-distance and the Euclidean distance coincide. Hence,

$$\text{diam } Q_n^{t,\underline{s}} = \text{diam}_\rho Q_n^{t,\underline{s}} \leq \frac{1}{\lambda^{n+1}}\sqrt{2}\pi \to 0, \text{ as } n \to \infty.$$

The point $z^{t,\underline{s}}$ satisfies the required conditions. Indeed, $z^{t,\underline{s}}$ follows the itinerary prescribed by $\underline{s}$ and converges to $-\infty$ under iteration. Moreover, we claim that $z^{t,\underline{s}} \in \partial U$. Indeed, since $f(\overline{D_{n+1}^{t,\underline{s}}})$ intersects $U$, then $\overline{D_{n+1}^{t,\underline{s}}}$ contains points of $U$, and so does $Q_n^{t,\underline{s}}$. Since the sets $\{Q_n^{t,\underline{s}}\}_n$ shrink to $z^{t,\underline{s}}$, this gives a sequence of points in $U$ approximating $z^{t,\underline{s}}$.

Therefore, we associate to any $t \leq -2$ and $\underline{s} \in \Sigma_2$ the point $z^{t,\underline{s}}$. Observe that the resulting point $z^{t,\underline{s}}$ depends continuously on $t$, since the entire construction depends continuously on $t$. Hence, letting $t \to -\infty$, the points $z_{\underline{s},t}$ describing the required curve $\gamma_{\underline{s}}$ of left-escaping points with itinerary $\underline{s}$. This induces naturally a parametrization on $\gamma_{\underline{s}}$: we define $\gamma_{\underline{s}} \colon (-\infty, -2] \to \mathbb{C}$ such that $\gamma_{\underline{s}}(t) \coloneqq z^{t,\underline{s}}$.

Finally, let us prove that, with this parametrization, the announced properties actually hold.

(a) (Asymptotics and dynamics)

It is clear by the construction of the squares that $\gamma_{\underline{s}}(t) \to -\infty$ , as $t \to -\infty$, and, for every $t \leq -2$, $f^n(\gamma_{\underline{s}}(t)) \to -\infty$, as $n \to \infty$. Moreover, since the orbit of a point is contained in the corresponding squares, we have Re $f^n(\gamma_{\underline{s}}(t)) \leq -2$ for all $n \geq 0$.

(b) (Uniqueness)

Uniqueness follows from the results in [RRRS11], which imply that every point in $\mathcal{I}_S^-$ can be connected to infinity by a curve of left-escaping points with the same itinerary.

Assume, on the contrary, that there exists $z_0 \in \mathcal{I}_S^-$, with $I(z_0) = \underline{s}$, and Re $f^n(z_0) \leq -2-\pi$ for all $n \geq 0$, but $z_0 \notin \gamma_{\underline{s}}$. Then, there would exist another curve $\widetilde{\gamma}_{\underline{s}}$ of left-escaping points with itinerary $\underline{s}$ connecting $z_0$ to $\infty$. Consider an open set $W$ placed in the left-unbounded region delimited by $\gamma_{\underline{s}}$, $\widetilde{\gamma}_{\underline{s}}$ and $\{z \in S \colon \text{Re } z = \text{Re } z_0\}$.

We claim that $f^n(W) \subset S \cap \{\text{Re } z < -2\}$, for all $n \geq 0$. Indeed, note that $\gamma_{\underline{s}}, \widetilde{\gamma}_{\underline{s}} \subset \left\{|\text{Im } z| > \frac{\pi}{2}\right\}$. Then, $W \subset S \cap \left\{|\text{Im } z| > \frac{\pi}{2}\right\}$. Recall that, for $z \in S \cap \left\{|\text{Im } z| > \frac{\pi}{2}\right\}$, Re $f(z) <$ Re $z$. Hence, $f(W) \subset S \cap \{\text{Re } z < -2\}$, and, by continuity, $f(W)$ is the left-unbounded region delimited by $f(\gamma_{\underline{s}})$, $f(\widetilde{\gamma}_{\underline{s}})$ and $f(\{z \in S \colon \text{Re } z = \text{Re } z_0\}) \subset S \cap \{\text{Re } z < -2\}$. We can apply the same argument inductively to see that $f^n(W) \subset S \cap \{\text{Re } z < -2\}$, for all $n \geq 0$, as claimed.

Therefore, $W$ is an open set which never enters the Baker domain, so $W \subset \mathcal{J}(f)$, leading to a contradiction.

(c) (Internal dynamics)

We have to prove that, for $t \leq -2$,

$$f(\gamma_{\underline{s}}(t)) = \gamma_{\sigma(\underline{s})}(F(t)).$$

First observe that, since $F$ is an increasing map, $F(t) < -2$ for $t \leq -2$, so $\gamma_{\sigma(\underline{s})}(F(t))$ is defined.

To construct the point $\gamma_{\sigma(\underline{s})}(F(t))$ we use the sequence of squares $\left\{D_n^{F(t),\sigma(\underline{s})}\right\}_n$. Therefore, the $n$-th square has right-hand side located at $\{x = F^n(F(t)) = F^{n+1}(t)\}$ and it is in the half-strip $\Omega_{s_{n+1}}$. Hence, $D_n^{F(t),\sigma(\underline{s})} = D_{n+1}^{t,\underline{s}}$. Moreover,

$$Q_n^{F(t),\sigma(\underline{s})} = \phi_{s_1} \circ \cdots \circ \phi_{s_{n+1}} \left(\overline{D_{n+1}^{F(t),\sigma(\underline{s})}}\right) = \phi_{s_1} \circ \cdots \circ \phi_{s_{n+1}} \left(\overline{D_{n+2}^{t,\underline{s}}}\right) = f(Q_{n+1}^{t,\underline{s}}).$$

Then,

$$\gamma_{\sigma(\underline{s})}(F(t)) = \bigcap_{n\geq 0} Q_n^{F(t),\sigma(\underline{s})} = \bigcap_{n\geq 0} f(Q_{n+1}^{t,\underline{s}}) = f(\bigcap_{n\geq 0} Q_{n+1}^{t,\underline{s}}) = f(\gamma_{\underline{s}}(t)),$$

as desired.

□

Escaping tails are mapped among them following the symbolic dynamics given by its itinerary: if $\sigma$ denotes the shift map in $\Sigma_2$ and $\underline{s} \in \Sigma_2$, we have $f(\gamma_{\underline{s}}) \subset \gamma_{\sigma(\underline{s})}$. Moreover, we claim that, as a consequence of Proposition 1.2.2 (c), this last inclusion is strict. Indeed, recall that, for all $t_0 \leq -2$, it holds $F(t_0) < t_0$. Hence, Proposition 1.2.2 (c) implies

$$f(\gamma_{\underline{s}}(\{t\colon t \leq t_0\})) = \gamma_{\sigma(\underline{s})}(\{t\colon t \leq F(t_0)\}) \subset \gamma_{\sigma(\underline{s})}(\{t\colon t \leq t_0\}),$$

where the last inclusion is strict.

Next, we define the dynamic rays as the natural extension of the escaping tails: we enlarge a given escaping tail $\gamma_{\underline{s}}$ by adding to it all points in $\widehat{S}$ which are eventually mapped to $\gamma_{\sigma^n(\underline{s})}$, for some $n \geq 0$ (see Fig. 1.9). Next theorem includes the formal definition as well as the corresponding extension of the dynamical properties of the escaping tails. Moreover, a new property is proven, showing the continuity of the parametrization the hairs with respect to the itinerary, analogously to [Rem07, Lemma 3.2].

**Theorem 1.2.3. (Dynamic rays)** *Let $\underline{s} \in \Sigma_2$. Let us define the* dynamic ray *(or* hair*) of sequence $\underline{s}$ as $\gamma_{\underline{s}}^{\infty}\colon (-\infty, +\infty) \to \mathcal{I}_S^-$ such that, if $n \geq 0$ with $F^n(t) < -2$, then*

$$\gamma_{\underline{s}}^{\infty}(t) := \phi_{s_0} \circ \cdots \circ \phi_{s_{n-1}}(\gamma_{\sigma^n(\underline{s})}(F^n(t))).$$

*The following properties hold.*

(a) (Well-defined) *Dynamic rays are well-defined, in the sense that the definition does not depend on $n$. Moreover, $\gamma_{\underline{s}}^{\infty}$ is actually a curve and contains all left-escaping points with itinerary $\underline{s}$.*

(b) (Internal dynamics) *For $t \in \mathbb{R}$, it holds*

$$f(\gamma_{\underline{s}}^{\infty}(t)) = \gamma_{\sigma(\underline{s})}^{\infty}(F(t)),$$

*where $\sigma$ denotes the shift map and $F(t) = t - e^{-t}$.*

(c) (Continuity between rays) *Let $n_0 \in \mathbb{N}$ and $\underline{s} \in \Sigma_2$. Let us denote by $\Sigma_2(\underline{s}, n_0)$ the set of all sequences $\underline{\widetilde{s}} \in \Sigma_2$ which agree with $\underline{s}$ in the first $n_0 + 1$ entries. Then, for all $t_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $n_0$ such that*

$$\left|\gamma_{\underline{s}}^{\infty}(t) - \gamma_{\underline{\widetilde{s}}}^{\infty}(t)\right| < \varepsilon,$$

*for all $t \leq t_0$ and $\underline{\widetilde{s}} \in \Sigma_2(\underline{s}, n_0)$.*

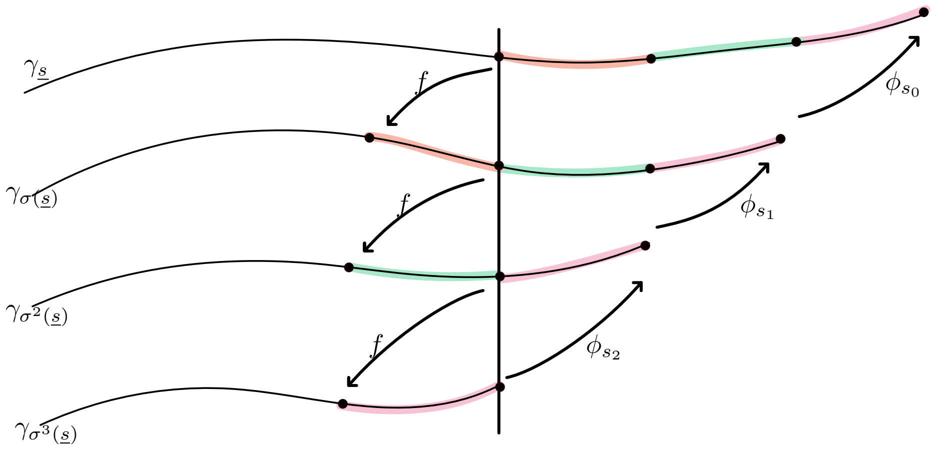



Figure 1.9: Construction of the hair $\gamma^\infty_{\underline{s}}$ from the escaping tail $\gamma_{\underline{s}}$. Intuitively, the process is clear: since the endpoint of the escaping tail is not mapped to the endpoint of the next escaping tail but to a point further to the left, the remaining piece of escaping tail can be added to the previous one by pulling back by the inverse. Repeating the process we get all the points in the ray.

*Proof.* (a) (Well-defined)

Fix $\underline{s} \in \Sigma_2$ and $t > -2$, and let $m > n$ be such that $F^m(t) < -2$ and $F^n(t) < -2$. Put $m = n + l$, with $l > 0$. We have to see that

$$\phi_{s_0} \circ \cdots \circ \phi_{s_{n-1}} \circ \phi_{s_n} \circ \cdots \circ \phi_{s_{n+l-1}} \left( \gamma_{\sigma^{n+l}(\underline{s})}(F^l(F^n(t))) \right) = \phi_{s_0} \circ \cdots \circ \phi_{s_{n-1}} \left( \gamma_{\sigma^n(\underline{s})}(F^n(t)) \right).$$

Since $\phi_i$, $i \in \{0, 1\}$, are univalent, this is equivalent to

$$\phi_n \circ \cdots \circ \phi_{s_{n+l-1}} \left( \gamma_{\sigma^{n+l}(\underline{s})}(F^l(F^n(t))) \right) = \gamma_{\sigma^n(\underline{s})}(F^n(t)),$$

and this last equality holds true by the internal dynamics of the escaping tail (Prop. 1.2.2(c)).

Finally, in view of Proposition 1.2.2, it is clear that dynamic rays are actually curves and contain all left-escaping points with the same itinerary, proving statement (a).

(b) (Internal dynamics)

We shall assume that $t > -2$, otherwise the point $\gamma^\infty_{\underline{s}}(t)$ is in the escaping tail, where we have already proven the statement. Let $n$ be such that $F^n(t) \leq -2$. Then, applying the known equality for the escaping tails, we have

$$f(\gamma^\infty_{\underline{s}}(t)) = f(\phi_{s_0} \circ \cdots \circ \phi_{s_{n-1}}(\gamma_{\sigma^n(\underline{s})}(F^n(t)))) =$$
$$= \phi_{s_1} \circ \cdots \circ \phi_{s_{n-1}}(\gamma_{\sigma^{n-1}(\sigma(\underline{s}))}(F^{n-1}(F(t)))) = \gamma^\infty_{\sigma(\underline{s})}(F(t)),$$

proving statement (b).

(c) (Continuity between rays)

Fix $\underline{s} \in \Sigma_2$ and $t_0 \in \mathbb{R}$. The goal is to determine $n_0 \in \mathbb{N}$ such that if $\widetilde{\underline{s}} \in \Sigma_2$ which agree with $\underline{s}$ in the first $n_0 + 1$ entries and $t \leq t_0$, then

$$\left| \gamma^\infty_{\underline{s}}(t) - \gamma^\infty_{\widetilde{\underline{s}}}(t) \right| < \varepsilon.$$

To do so, first assume $t_0 \leq -2$ and fix $\varepsilon > 0$. Let $\lambda > 1$ be the factor of expansion of $f$ in $S \cap \{\text{Re } z < 0\}$ (see Rmk. 1.1.7). Let $n_0$ be such that $\frac{1}{\lambda^{n_0}}\sqrt{2}\pi < \varepsilon$. We claim that for $\tilde{\underline{s}} \in \Sigma_2(\underline{s}, n_0)$ and $t \leq t_0$ it holds

$$\left|\gamma^{\infty}_{\underline{s}}(t) - \gamma^{\infty}_{\tilde{\underline{s}}}(t)\right| < \varepsilon.$$

Indeed, by construction we have

$$\gamma^{\infty}_{\underline{s}}(t), \gamma^{\infty}_{\tilde{\underline{s}}}(t) \in \bigcap_{n=0}^{n_0-1} Q_n^{t,\underline{s}} = Q_{n_0-1}^{t,\underline{s}} = \phi_{s_0} \circ \cdots \circ \phi_{s_{n_0-1}}(\overline{D_{n_0}^{t,\underline{s}}}).$$

Therefore,

$$\text{diam } Q_{n_0}^{t,\underline{s}} \leq \frac{1}{\lambda^{n_0}} \text{diam } \overline{D_{n_0}^{t,\underline{s}}} = \frac{1}{\lambda^{n_0}}\sqrt{2}\pi < \varepsilon,$$

implying that $\left|\gamma^{\infty}_{\underline{s}}(t) - \gamma^{\infty}_{\tilde{\underline{s}}}(t)\right| < \varepsilon$, as desired.

Now assume $t_0 > -2$. Choose $n_1$ such that $F^{n_1}(t_0) < -2$ (and, hence, $F^{n_1}(t) < -2$, for all $t \leq t_0$). By the previous reasoning, we can find $n_0$ such that $\left|\gamma^{\infty}_{\sigma^{n_1}(\underline{s})}(t) - \gamma^{\infty}_{\tilde{\underline{s}}}(t)\right| < \varepsilon$, for $\tilde{\underline{s}} \in \Sigma_2(\sigma^{n_1}(\underline{s}), n_0)$ and $t \leq -2$. Take $n := n_0 + n_1$ and let us check that the property of the lemma is satisfied.

Indeed, take $\tilde{\underline{s}} \in \Sigma_2(\underline{s}, n)$. Then, $\sigma^{n_1}(\tilde{\underline{s}}) \in \Sigma_2(\sigma^{n_1}(\underline{s}), n_0)$ and $F^{n_1}(t) < -2$, so

$$\left|\gamma^{\infty}_{\sigma^{n_1}(\underline{s})}(F^{n_1}(t)) - \gamma^{\infty}_{\sigma^{n_1}(\tilde{\underline{s}})}(F^{n_1}(t))\right| < \varepsilon.$$

Since applying the inverses $\phi_i$, $i \in \{0, 1\}$ does not increase the distance between points, we get

$$\left|\gamma^{\infty}_{\underline{s}}(t) - \gamma^{\infty}_{\tilde{\underline{s}}}(t)\right| = \left|\phi_{s_0} \circ \cdots \circ \phi_{n_1-1}(\gamma^{\infty}_{\sigma^{n_1}(\underline{s})}(F^{n_1}(t))) - \phi_{s_0} \circ \cdots \circ \phi_{n_1-1}(\gamma^{\infty}_{\sigma^{n_1}(\tilde{\underline{s}})}(F^{n_1}(t)))\right| \leq$$

$$\leq \left|\gamma^{\infty}_{\sigma^{n_1}(\underline{s})}(F^{n_1}(t)) - \gamma^{\infty}_{\sigma^{n_1}(\tilde{\underline{s}})}(F^{n_1}(t))\right| < \varepsilon,$$

ending the proof of statement (c).

□

Observe that, by uniqueness, we have $L^+ = \gamma^{\infty}_{\overline{0}}$ and $L^+ = \gamma^{\infty}_{\overline{1}}$, implying, in particular, that $L^{\pm} \subset \partial U$. Next, we use it to prove new characterization of $\partial U$, which will be useful in the sequel.

**Proposition 1.2.4. (Characterizations of $\partial U$)**

(a) *The boundary of $U$ consists precisely of the points in $\mathcal{J}(f)$ which never escape from $S$, i.e.*

$$\partial U = \widehat{S} \cap \mathcal{J}(f).$$

(b) *Every point in $\partial U$ is in the closure of a dynamic ray, i.e.*

$$\partial U = \bigcup_{\underline{s} \in \Sigma_2} \overline{\gamma^{\infty}_{\underline{s}}}.$$

*Proof.* (a) Let us start by proving statement (a). To do so, we show the following chain of inclusions:

$$\partial U \subset \widehat{S} \cap \mathcal{J}(f) \subset \overline{\bigcup_{n\geq 0} \bigcup_{\underline{s}\in\Sigma_2^n} \Phi_{\underline{s}}(L^{\pm})} \subset \partial U,$$

where $\Sigma_2^n$ denotes the space of finite sequences of two symbols, $\{0,1\}$, of length $n+1$; and if $\underline{s} \in \Sigma_2^n$, $\underline{s} = s_0 \dots s_n$, then

$$\Phi_{\underline{s}} \coloneqq \phi_{s_0} \circ \cdots \circ \phi_{s_n}.$$

The first inclusion comes straightforward from the definitions. To prove the second inclusion, consider $z \in \widehat{S} \cap \mathcal{J}(f)$ and let $W$ be a neighborhood of $z$. Without loss of generality, we can assume $z \notin L^{\pm}$ and $W \subset S$. Since $z \in \mathcal{J}(f)$, by the blow-up property, there exists $n > 0$ such that $f^n(W) \not\subset S$. But $z \in \widehat{S}$, so $f^n(z) \in S$. Therefore, $f^n(W)$ intersects $L^{\pm}$, and the result follows.

Finally, regarding the third inclusion, it is enough to prove that $\Phi_{\underline{s}}(L^{\pm}) \subset \partial U$, for all $\underline{s} \in \Sigma_2^n$ and $n \geq 0$. Hence, fix $n \geq 0$ and $\underline{s} \in \Sigma_2^n$, and consider $z \in \Phi_{\underline{s}}(L^{\pm})$. Since $f^n(z) \in L^{\pm} \subset \partial U$, there exists a sequence of points $\{w_n\}_n \subset U$ such that $w_n \to f^n(z)$. Applying $\Phi_{\underline{s}}$ to the sequence $\{w_n\}_n$, we have $\Phi_{\underline{s}}(w_n) \to z$ with $\Phi_{\underline{s}}(w_n) \in U$, since $f^{-1}(U) \cap S = U$. Therefore, $\Phi_{\underline{s}}(L^{\pm}) \subset \partial U$, as desired. See Figure 1.10.

(b) To prove statement (b), it is enough to show that, given an itinerary $\underline{s} \in \Sigma_2$, all points in $\widehat{S} \smallsetminus U$ having this itinerary are precisely the ones in $\overline{\gamma_{\underline{s}}^{\infty}}$.

Let us assume first, that $\underline{s} = \overline{0}$ and there is $z \in \partial U$ with this itinerary and $z \notin L^{+}$. Then, Im $z < \pi$ and, since

$$\text{Im } f(x+iy) = y - e^{-x} \sin y,$$

it follows that there exists $n \geq 0$ such that $0 < \text{Im } f^n(z) < \frac{\pi}{2}$. Therefore, $f^n(z) \in \Omega_{01}$, so $I(z)$ cannot be constant. The analogous argument works for $\underline{s} = \overline{1}$ and, taking preimages, it also proves the statement for eventually constant sequences.

Now assume $\underline{s}$ is a non-eventually constant sequence and there is $z \in \widehat{S}$, with $I(z) = \underline{s}$ and $z \notin \overline{\gamma_{\underline{s}}^{\infty}}$. Since $\overline{\gamma_{\underline{s}}^{\infty}}$ is closed in $\mathbb{C}$, we have

$$\rho(z, \overline{\gamma_{\underline{s}}^{\infty}}) \coloneqq \inf_{w \in \overline{\gamma_{\underline{s}}^{\infty}}} \rho(z, w) > 0,$$

where $\rho$ is the distance in $S \smallsetminus \overline{V}$ defined in 1.1.4.

We note that, since $f$ is expanding in $S \smallsetminus \overline{V}$ with respect to $\rho$, and $f^n(z) \in S \smallsetminus \overline{V}$, $f^n(\gamma_{\underline{s}}^{\infty}) \subset S \smallsetminus \overline{V}$, for all $n \geq 0$, it holds

$$\rho(f^{n+1}(z), f^{n+1}(\overline{\gamma_{\underline{s}}^{\infty}})) > \rho(f^n(z), f^n(\overline{\gamma_{\underline{s}}^{\infty}})).$$

Moreover, if both $f^n(z)$ and $f^n(\overline{\gamma_{\underline{s}}^\infty})$ lie in $\{\text{Re } z < 0\}$, we have uniform expansion by constant $\lambda > 1$ (see Rmk. 1.1.7), i.e.

$$\rho(f^{n+1}(z), f^{n+1}(\overline{\gamma_{\underline{s}}^\infty})) \geq \lambda \rho(f^n(z), f^n(\overline{\gamma_{\underline{s}}^\infty})).$$

Since $\underline{s}$ is non-eventually constant, there exists an infite increasing sequence $\{n_k\}_k$ such that $f^{n_k}(z), f^{n_k}(\overline{\gamma_{\underline{s}}^\infty})$ lie in $\Omega_{01}$, so in particular they lie in the left half-plane $\{\text{Re } z < 0\}$, where $f$ expands uniformly by factor $\lambda > 1$. Hence, since $f$ is always expanding and expands infinitely many times uniformly by factor $\lambda > 1$, we get that

$$\rho(f^n(z), f^n(\overline{\gamma_{\underline{s}}^\infty})) \to \infty, \text{ as } n \to \infty.$$

Hence, we can choose $N > 0$ such that $\rho(f^n(z), f^n(\overline{\gamma_{\underline{s}}^\infty})) > 2 + \pi$ and $f^n(z) \in \Omega_{01}$, $f^n(\overline{\gamma_{\underline{s}}^\infty}) \subset \Omega_{01}$.

By construction, $f^N(\gamma_{\underline{s}}^\infty)$ contains the escaping tail $\gamma_{\sigma^N(s)}$, which intersects the vertical segment $\{z \in S \colon \text{Re } z = M\}$. Observe that there are no points in $\Omega_{01}$ at a distance greater than $2 + \pi$ of $\gamma_{\sigma^N(\underline{s})}$, so this leads to a contradiction.

□

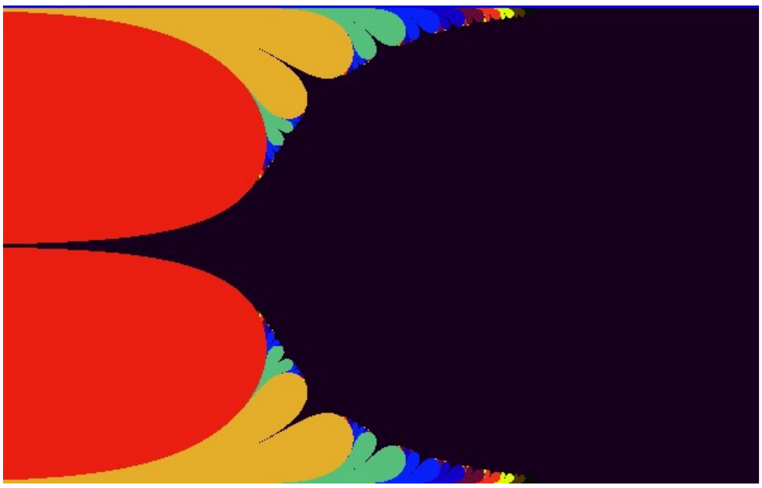


Figure 1.10: This picture shows the Baker domain (in black) and the regions (in different colors) which are eventually mapped outside $S$. The boundaries of these regions are precisely $\Phi_{\underline{s}}(L^{\pm})$, $\underline{s} \in \Sigma_2^n$, for some $n \geq 0$. Proposition 1.2.4 tells that $\partial U$ is precisely the accumulation of those curves.

We note that the previous proposition allow us to characterize the points in $\widehat{S}$. Indeed, as noted in Section 1.1, $U \cup \partial U \subset \widehat{S}$; and, from the fact that $U$ has no more preimages in $S$ apart from itself, $U \cap \mathcal{F}(f) = \widehat{S}$. The previous proposition characterizes $\mathcal{J}(f) \cap \widehat{S}$, implying the following corollary.

**Corollary 1.2.5. (Characterization of $\widehat{S}$)** *It holds:*

$$\widehat{S} = \overline{U} = U \cup \partial U.$$

From the results of this section, we shall deduce Theorem 1.A.

*Proof of Theorem 1.A.* The first statement of Theorem 1.A is deduced from statement (a) of Theorem 1.2.3, whereas the second statement of Theorem 1.A corresponds to statement (b) in Proposition 1.2.4. □

## 1.3 Landing and non-landing rays. Proof of Theorem 1.B

We shall discuss now the landing properties of the dynamic rays defined in the previous section. More precisely, we devote the section to prove Theorem 1.B, which asserts that for uncountably many sequences the dynamic ray land at some point; while for uncountable many others the dynamic ray does not land and its accumulation set (in the Riemann sphere) is an indecomposable continuum.

Let us recall first the the definition of indecomposable continuum and the following result, which gives a sufficient condition for the accumulation set of a curve to be an indecomposable continuum. Here, we shall understand simple curve as the continuous, one-to-one image of the non-negative real numbers.

**Definition 1.3.1. (Indecomposable continuum)** We say that $X \subset \widehat{\mathbb{C}}$ is a *continuum* if it is compact and connected. A continuum is *indecomposable* if it cannot be expressed as the union of two proper subcontinua.

**Theorem 1.3.2. (Curry,** [Cur91, Thm. 8]**)** *Let $X$ be a one-dimensional non-separating plane continuum which is the closure of a simple curve that limits upon itself. Then $X$ is indecomposable.*

We proceed as follows. First of all, we define precisely what we mean for a ray to land, introducing the notion of *landing set*. We also require the notion of *non-escaping set* to relate the accumulation set of a dynamic ray with the non-escaping points having the same itinerary. Afterwards, we classify the sequences $\underline{s} \in \Sigma_2$ according to the nature of its landing set, resulting in the different landing behaviours claimed in Theorem 1.B.

**Definition 1.3.3. (Landing set of a ray)** Let $\underline{s} \in \Sigma_2$ and let $\gamma_{\underline{s}}^{\infty}$ be the dynamic ray of sequence $\underline{s}$. We define the *landing set* $L_{\underline{s}}$ of the ray $\gamma_{\underline{s}}^{\infty}$ as the set of values $w \in \widehat{\mathbb{C}}$ for which there is a sequence $\{t_n\}_n \subset \mathbb{R}$ such that $t_n \to +\infty$ and $\gamma_{\underline{s}}^{\infty}(t_n) \to w$, as $n \to \infty$. If $L_{\underline{s}} = \{w\}$, we say that the dynamic ray $\gamma_{\underline{s}}^{\infty}$ *lands* at $w$.

Observe that, by Proposition 1.2.4(b), $\gamma_{\underline{s}}^{\infty} \cup L_{\underline{s}}$ contains all the points in $\partial U$ with itinerary $\underline{s}$, so

$$\gamma_{\underline{s}}^{\infty} \cup L_{\underline{s}} = \{z \in \partial U \colon I(z) = \underline{s}\}\,.$$

Therefore, all non-escaping points with itinerary $\underline{s}$ are in $L_{\underline{s}}$, but a priori $L_{\underline{s}}$ may contain escaping points. This leads us to define the following set.

**Definition 1.3.4. (Non-escaping set)** Let $\underline{s} \in \Sigma_2$. We define the *non-escaping set* $W_{\underline{s}}$ as the set of points in $\widehat{S}$ with itinerary $\underline{s}$ which do not escape to infinity.

Clearly, $W_{\underline{s}} \subset L_{\underline{s}} \cap \mathbb{C}$. Since all escaping points are in a ray, we have $W_{\underline{s}} = \overline{\gamma_{\underline{s}}^{\infty}} \setminus \gamma_{\underline{s}}^{\infty}$. Moreover, $L_{\underline{s}}$ is always non-empty, compact and connected, whereas $W_{\underline{s}}$ may be empty.

We start by describing $L_{\underline{s}}$ and $W_{\underline{s}}$ for eventually constant sequences.

**Lemma 1.3.5. (Eventually constant sequences)** *Let $\underline{s} \in \Sigma_2$. Then, $L_{\underline{s}} = \{\infty\}$ if, and only if, $\underline{s}$ is eventually constant. In this case, $W_{\underline{s}} = \emptyset$.*

*Proof.* Recall that $\gamma_{\overline{0}}^{\infty} = L^{+}$ and $\gamma_{\overline{1}}^{\infty} = L^{-}$, so $L_{\overline{0}} = L_{\overline{1}} = \{\infty\}$. Since preimages of curves landing at $\infty$ are again curves landing at $\infty$ and hairs with eventually constant sequence are the preimages of $L^{\pm}$, one implication is proven.

Now, assume $\underline{s}$ is a non-eventually constant sequence, and $\gamma_{\underline{s}}^{\infty}$ lands at $\infty$. Then, $\gamma_{\underline{s}}^{\infty}$ divides $S$ into two regions: $R_1, R_2$. The absorbing domain $V$ is contained in one of them, say $R_1$, so $R_1 \cap U \neq \emptyset$. We claim that $R_2 \cap U \neq \emptyset$. Indeed, $R_2 \cap \widehat{S} \neq \emptyset$, because the points that leave $S$ after applying $f$ are the ones enclosed by $f^{-1}(L^{\pm}) \cap S$, and $\gamma_{\underline{s}}^{\infty}$ is not a preimage of $L^{\pm}$ (see Fig. 1.4, 1.10). The fact that $U = \text{Int}(\widehat{S})$ (Corol. 1.2.5) gives that $R_2 \cap U \neq \emptyset$. This is a contradiction because $U$ is connected. □

The goal for the remaining part of the section is to describe the landing and the non-escaping sets for non-eventually constant sequences. First, we deal with the dynamics of the non-escaping points, whose orbit may be bounded or oscillating. It turns out that this only depends on its itinerary. Moreover, for certain types of sequence, we have a great control on the orbit of the ray and the non-escaping set, as the following results show.

**Definition 1.3.6. (Types of sequences)** Let $\underline{s} \in \Sigma_2$ be a non-eventually constant sequence. We say that $\underline{s}$ is *oscillating* if it contains arbitrarily large sequences of 0's or 1's. Otherwise, we say that $\underline{s}$ is *bounded*.

**Proposition 1.3.7. (Dynamics on the non-escaping sets)** *Let $\underline{s} \in \Sigma_2$ and let $W_{\underline{s}}$ be its corresponding non-escaping set. Then, $\left\{f^n(W_{\underline{s}})\right\}_n$ is contained in a compact set if and only if $\underline{s}$ is a bounded sequence. In this case, there exists $R > 0$ such that $f^n(\gamma_{\underline{s}}^{\infty}) \subset \{\text{Re } z < R\}$ and $f^n(W_{\underline{s}}) \subset \{|\text{Re } z| < R\}$.*

*Proof.* Assume first that $\underline{s} \in \Sigma_2$ is a bounded sequence and $z \in W_{\underline{s}}$. Then, there exists $N > 0$ such that $\underline{s}$ does not contain more than $N$ consecutive 0's and $N$ consecutive 1's. Take $R \coloneqq F^{-N}(0)$, where $F(t) = t - e^{-t}$. We claim that Re $f^n(z) \leq R$ for all $n$. Indeed, if it is not the case, there must exist $n_0$ such that Re $f^{n_0}(z) > R$. Then, since $F$ is increasing, we have

$$\text{Re } f^{n_0+1}(z) > \text{Re } f^{n_0}(z) - e^{\text{Re } f^{n_0}(z)} > R - e^{-R} = F(R).$$

Repeating the argument inductively, we get

$$\text{Re } f^{n_0+N}(z) > \text{Re } f^{n_0+N-1}(z) - e^{\text{Re } f^{n_0+N-1}(z)} > F^{N-1}(R) - e^{-F^{N-1}(R)} = F^{N}(R) = 0.$$

Therefore, by Lemma 1.1.10 (a), either $\left\{f^{n_0+k}(z)\right\}_{k=0}^{N} \subset \Omega_0$ or $\left\{f^{n_0+k}(z)\right\}_{k=0}^{N} \subset \Omega_1$, so $\underline{s}$ has $N+1$ consecutive 0's. Therefore, Re $f^n(z) < R$, for all $n \geq 0$. We note that the constant $R$ has been chosen to depend only on $N$ (but not on the particular point $z \in W_{\underline{s}}$), hence it holds Re $f^n(z) < R$, for all $n \geq 0$ and $z \in W_{\underline{s}}$.

We claim that, under the conditions described above, if $R$ has been chosen large enough, we also have Re $f^n(z) > -R$ for all $n \geq 0$ and $z \in W_{\underline{s}}$, and hence $\left\{f^n(W_{\underline{s}})\right\}_n$ is contained in a compact set.

Without loss of generality, we can assume $R > 3$ and large enough so that if $z \in \partial U$ and Re $z < -R$, then $|\text{Im } z| < \left(0, \frac{\pi}{3}\right)$ or $\frac{2\pi}{3} < |\text{Im } z| < 1$. We note that this is possible since $\phi_1(L^+)$ is a curve landing at $-\infty$ from both sides, approaching tangentially $L^-$ and $\mathbb{R}$; and $\phi_0(L^-)$ also at lands at $-\infty$, but approaching $L^+$ and $\mathbb{R}$ (see e.g. Fig. 1.4). Then, in order to show that Re $f^n(z) > -R$ for all $n \geq 0$ and $z \in W_{\underline{s}}$, we proceed by contradiction: let us assume that there exists $z \in W_{\underline{s}}$ and $n_0 \geq 0$ such that Re $f^{n_0}(z) < -R$. Then, since $z \in W_{\underline{s}}$, we can assume that Re $f^{n_0+1}(z) > -R$. Hence, $0 < |\text{Im } f^{n_0}(z)| < \frac{\pi}{3}$. Then,

$$\text{Re } f^{n_0+1}(z) = \text{Re } f^{n_0}(z) + e^{-\text{Re } f^{n_0}(z)} \cos(\text{Im } f^{n_0}(z)) \geq$$

$$\geq \text{Re } f^{n_0}(z) + \frac{1}{2} e^{-\text{Re } f^{n_0}(z)} > -\text{Re } f^{n_0}(z) > R.$$

This contradicts the assumption that Re $f^n(z) < R$, for all $n \geq 0$ and $z \in W_{\underline{s}}$, proving one implication.

For the other implication, let us assume that $z \in W_{\underline{s}}$ has a bounded orbit, and let us prove that then $\underline{s}$ is bounded. Let $R > 0$ be such that $-R \leq \text{Re } f^n(z) \leq R$, for all $n \geq 0$, and let $\varepsilon > 0$ be such that dist $(f^n(z), V) > \varepsilon$, for all $n \geq 0$. We note that, in this case, if $f^n(z) \in \Omega_{00} \cup \Omega_{01}$, $|\text{Im } f^n(z)| > \frac{\pi}{2} + \varepsilon$. Let $M = \left|e^{-R} \cos(\frac{\pi}{2} + \varepsilon)\right|$, and let $N$ be such that $R - NM < -R$. We claim that $\underline{s}$ cannot have more than $N$ consecutive 0's. On the contrary, assume $\{f^n(z)\}_{n=0}^N \subset \Omega_{00}$. Then,

$$\text{Re } f^n(z) = \text{Re } f^n(z) + e^{-\text{Re } f^n(z)} \cos(\text{Im } f^n(z)) < \text{Re } z - M,$$

for $0 \geq n \geq N - 1$, so

$$\text{Re } f^N(z) < \text{Re } z - NM < -R.$$

Therefore, $\underline{s}$ cannot have more than $N$ consecutive 0's. A similar argument can be used to prove that $\underline{s}$ cannot have more than $N$ consecutive 1's; and this proves the other implication.

The existence of $R > 0$ such that $f^n(\gamma_{\underline{s}}^\infty) \subset \{\text{Re } z < R\}$ and $f^n(W_{\underline{s}}) \subset \{|\text{Re } z| < R\}$ is deduced from the previous reasoning, taking into account that escaping points with bounded itinerary cannot go arbitrarily far to the right, since they have to be in $\Omega_{01}$ (or $\Omega_{10}$) in a bounded number of steps. □

Next, we use this control on the dynamic rays and the non-escaping sets for bounded sequences to prove that the non-escaping set is actually a point where the dynamic ray lands.

**Proposition 1.3.8. (Rays with bounded sequence land)** *Let $\underline{s} \in \Sigma_2$ be a bounded sequence. Then, there exists a point $w_{\underline{s}} \in \mathbb{C}$ such that*

$$L_{\underline{s}} = W_{\underline{s}} = \left\{w_{\underline{s}}\right\},$$

*i.e. the dynamic ray $\gamma_{\underline{s}}^{\infty}(t)$ lands at the point $w_{\underline{s}}$.*

*Proof.* First, let us prove that $W_{\underline{s}}$ consists of a single point. By Proposition 1.3.7, $W_{\underline{s}}$ is compact. Assume, on the contrary that $W_{\underline{s}}$ consists of more than one point, so $\operatorname{diam}_\rho(W_{\underline{s}}) > 0$. Recall that $f$ is uniformly expanding in any compact set $K \subset S \setminus \overline{V}$ with respect to $\rho$ (see Rmk. 1.1.7). Taking $K$ to be $W_{\underline{s}}$, we have $\operatorname{diam}_\rho(f^n(W_{\underline{s}})) \to \infty$, which contradicts the fact that $\left\{f^n(W_{\underline{s}})\right\}_n$ is contained in a compact set (Prop. 1.3.7). Therefore, $W_{\underline{s}}$ must consist only of one point, so $W_{\underline{s}} = \{w_s\}$.

To end the proof, it is enough to show that $L_{\underline{s}}$ cannot contain any escaping point. Indeed, this would imply, together with the previous lemma, that $L_{\underline{s}} \subset \left\{w_{\underline{s}}, \infty\right\}$ and, since $L_{\underline{s}}$ is connected and it cannot be equal to $\infty$, necessarily $L_{\underline{s}} = \left\{w_{\underline{s}}\right\}$.

By Proposition 1.3.7, $f^n(\gamma_{\underline{s}}^{\infty})$ and $f^n(W_{\underline{s}})$ are contained in the half-plane $\{\operatorname{Re} z < R\}$. Assume the dynamic ray $\gamma_{\underline{s}}^{\infty}$ accumulates at an escaping point $z$. Since $z$ is escaping and has itinerary $\underline{s}$, by Theorem 1.2.3(a), there exists $n_0 \geq 0$ such that $f^{n_0}(z) \in \gamma_{\sigma^{n_0}(\underline{s})}$ and $\operatorname{Re} f^{n_0}(z) < -R$.

We note that $\sigma^{n_0}(\underline{s})$ is also a bounded sequence satisfying that $f^n(\gamma_{\sigma^{n_0}(\underline{s})}^{\infty}) \subset \{\operatorname{Re} z < R\}$; and $f^{n_0}(z)$ is escaping and $f^{n_0}(z) \in L_{\sigma^{n_0}(\underline{s})}$. Therefore, there exists an increasing sequence $\{t_n\}_n \subset \mathbb{R}$ and $w_n \coloneqq \gamma_{\underline{s}}^{\infty}(t_n) \to f^{n_0}(z)$, as $n \to \infty$. Let us choose some $m$ such that $t_m \geq -2$, and hence $w_m \in \gamma_{\underline{s}}^{\infty} \setminus \gamma_{\underline{s}}$, and $\operatorname{Re} w_m < -R$. Since $w_m$ is not in the escaping tail, there exists $M > 0$ such that $\operatorname{Re} f^M(w_m) > -2 + \pi$, $M$ being the minimal integer satisfying this property. Hence, $\operatorname{Re} f^{M-1}(w_m) < -R$, so $\operatorname{Re} f^M(w_m) > R$ (since $|f(z)| > \operatorname{Re} z$, as shown in the proof of Prop. 1.2.2).

Therefore the property $f^n(\gamma_{\sigma^{n_0}(\underline{s})}^{\infty}) \subset \{\operatorname{Re} z < R\}$ does not hold, leading to a contradiction. □

To end the section, we prove that rays with oscillating sequences do not always land. In fact, we are going to prove that, for uncountably many sequences, $L_{\underline{s}}$ is an indecomposable continuum which contains the ray $\gamma_{\underline{s}}^{\infty}$. We follow the ideas of Rempe ([Rem03, Thm. 3.8.4], [Rem07, Thm. 1.2]).

**Proposition 1.3.9. (Some rays do not land)** *There exist uncountably many dynamic rays $\gamma_{\underline{s}}^{\infty}$ which do not land.*

*Proof.* First, by Lemma 1.3.5, if we show that, for a non-eventually constant sequence $\underline{s}$, the landing set $L_{\underline{s}}$ contains $\infty$, then the ray $\gamma_{\underline{s}}^{\infty}$ do not land. Hence, our goal is to construct a non-eventually constant sequence $\underline{s}$ with $\infty \in L_{\underline{s}}$.

Let us denote by $\overline{0}^n$ a block of $n$ zeroes and by $\overline{0}$ an infinite block of zeroes. Then, the itinerary $\underline{s}$ that we construct will be of the form $\underline{s} = 1\overline{0}^{n_1}1\overline{0}^{n_2}1\overline{0}^{n_3}\dots$ for an infinite sequence $\{n_j\}_j$. We choose the $n_j$'s inductively among countably many choices in each step, leading to uncountably many non-landing rays at the end.

Assume $n_1, \dots, n_{j-1}$ have been chosen, and consider the sequence $\underline{s}^j = 1\overline{0}^{n_1}\dots 1\overline{0}^{n_{j-1}}1\overline{0}$. Then, $\gamma_{\underline{s}^j}^{\infty}$ is a preimage of $L^+$, so it lands at $\infty$ in both ends. Let us choose $t_j > -2$ such that $\left|\gamma_{\underline{s}^j}(t_j)\right| > j$. By Theorem 1.2.3 (c), there exists $N_j \in \mathbb{N}$ such that $\left|\gamma_{\underline{s}}(t_j)\right| \geq j$ for all $s \in \Sigma_2(\underline{s}^j, N_j)$. We choose $n_j \geq N_j$.

Let $\underline{s}$ be the sequence constructed in this way. Then, $\underline{s}$ is clearly non-eventually constant, and $\infty \in L_{\underline{s}}$, since $\gamma_{\underline{s}}^{\infty}(t_j) \to \infty$, as $j \to \infty$, proving that the ray $\gamma_{\underline{s}}^{\infty}$ does not land. Evidently, by symmetry, the same construction interchanging 0's by 1's also gives non-landing rays. □

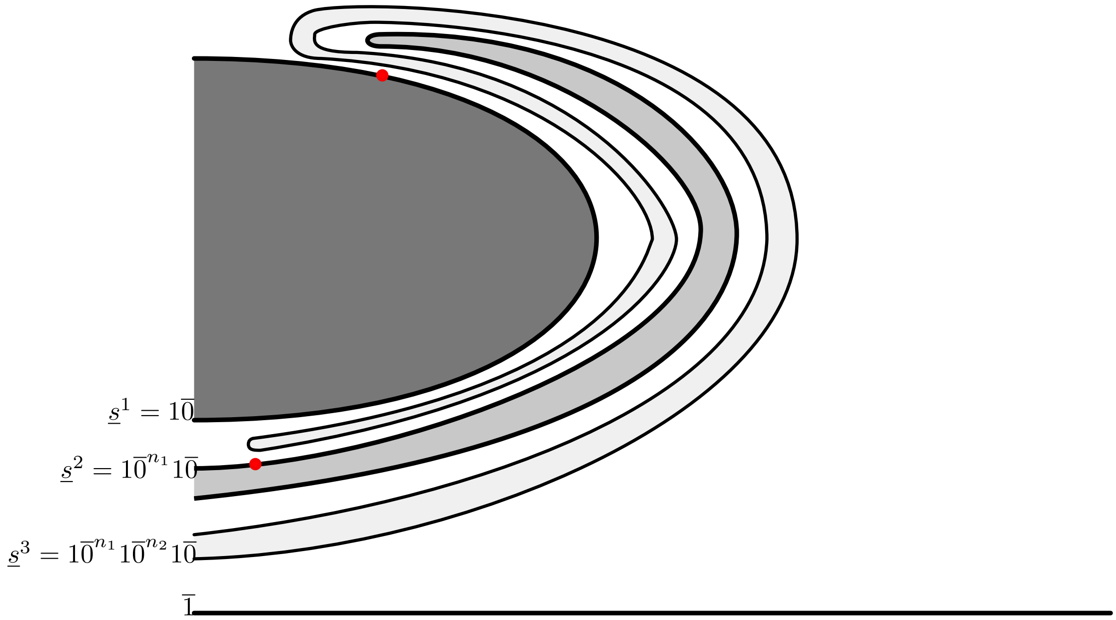



Figure 1.11: Schematic representation of the construction of the non-landing ray $\gamma_{\underline{s}}^{\infty}$, to give a geometric intuition of the proof, showing the first three steps of the induction. The sequence on the right indicates the itinerary of the ray. The first ray that is constructed is the one of sequence $\underline{s}^1$, which is a preimage of $L^+$. In red, it is marked the point $\gamma_{\underline{s}^1}^{\infty}(t_1)$. In the next step of the induction, it is chosen $\underline{s}_2$ in such a way that $\gamma_{\underline{s}^2}^{\infty}$ gets *close* to $\gamma_{\underline{s}^1}^{\infty}(t_1)$, so $\gamma_{\underline{s}^2}^{\infty}$ wraps along $\gamma_{\underline{s}^1}^{\infty}$. This wrapping is precisely what makes that, in the limit, we get a non-landing ray.

**Corollary 1.3.10. (Some landing sets are indecomposable continua)** *The landing set $L_{\underline{s}}$ of the non-landing rays of Proposition 1.3.9 is an indecomposable continuum.*

*Proof.* To prove that $L_{\underline{s}}$ is an indecomposable continuum, we shall invoke Curry's Theorem 1.3.2, after checking that $L_{\underline{s}}$ does not separate the plane and that $\gamma_{\underline{s}}^{\infty} \subset L_{\underline{s}}$.

On the one hand, let us observe that effectively $L_{\underline{s}}$ cannot separate the plane. We follow the same argument as in the proof of Lemma 1.3.5. Indeed, if $L_{\underline{s}}$ separates $\mathbb{C}$, it should also separate the strip $S$. Let $R_1$ be the connected component of $S \smallsetminus L_{\underline{s}}$ that contains the absorbing domain $V$, so $R_1 \cap U \neq \emptyset$. Let $R_2$ be any other component of $S \smallsetminus L_{\underline{s}}$. We claim that $R_2 \cap U \neq \emptyset$. Indeed, $R_2 \cap \widehat{S} \neq \emptyset$, because the points that leave $S$ after applying $f$ are the ones enclosed by $f^{-1}(L^{\pm}) \cap S$, and $L_{\underline{s}}$ is not a preimage of $L^{\pm}$. The fact that $U = \operatorname{Int}(\widehat{S})$ gives that $R_2 \cap U \neq \emptyset$. This is a contradiction because $U$ is connected. We note that this argument not only proves that $L_{\underline{s}}$ cannot separate the plane, but also that neither $\gamma_{\underline{s}}^{\infty}$ nor $\overline{\gamma_{\underline{s}}^{\infty}}$ can separate the plane.

On the other hand, the proof that $\gamma_{\underline{s}}^{\infty} \subset L_{\underline{s}}$ follows the idea of Rempe ([Rem07, Lemma 3.3]) based on the fact that dynamic rays accumulate among them. Indeed, $L_{\underline{s}}$ cannot intersect any dynamic ray different from $\gamma_{\underline{s}}^{\infty}$. In particular, $L_{\underline{s}}$ does not intersect the dynamic rays $\gamma_{\underline{r}^n}^{\infty}$, defined by

$$\underline{r}^n := s_0 s_1 \dots s_{n-1} r_n s_{n+1} s_{n+2} \dots,$$

where $r_n = 0$, if $s_n = 1$, and $r_n = 1$, if $s_n = 0$. By Theorem 1.2.3 (c), it is clear that $\gamma_{\underline{r}^n}^{\infty} \to \gamma_{\underline{s}}^{\infty}$, as $n \to \infty$, uniformly on every interval $(-\infty, t_0]$, $t_0 \in \mathbb{R}$. Moreover, from the fact that $\underline{s}$ is not eventually constant and escaping tails are ordered vertically following the (inverse) lexicographic order, it follows that $\left\{\gamma_{\underline{r}^n}^{\infty}\right\}_n$ approximates $\gamma_{\underline{s}}^{\infty}$ from above and from below. Therefore, we redefine the previous sequences as $\underline{r}^{n,+} := \underline{r}^m$, if $m \leq n$ is the maximal such that $\underline{r}^m > \underline{s}$ in the inverse lexicographic order; and $\underline{r}^{n,-} := \underline{r}^m$, if $m \leq n$ is the maximal such that $\underline{r}^m < \underline{s}$ in the inverse lexicographic order. Hence, the sequence of rays $\left\{\gamma_{\underline{r}^{n,+}}^{\infty}\right\}_n$ approximates $\gamma_{\underline{s}}^{\infty}$ from above; and $\left\{\gamma_{\underline{r}^{n,-}}^{\infty}\right\}_n$ from below.

Now, assume that $\gamma_{\underline{s}}^{\infty} \not\subset L_{\underline{s}}$, so we can find $t_0$ such that $\varepsilon := \operatorname{dist}\left(\gamma_{\underline{s}}^{\infty}(t_0), L_{\underline{s}}\right) > 0$. Since $\infty \in L_{\underline{s}}$ and points in $L_{\underline{s}}$ must have itinerary $\underline{s}$, it follows that $L_{\underline{s}}$ is contained in the connected component $U_n$ of

$$\mathbb{C} \smallsetminus \left(D(\gamma_{\underline{s}}^{\infty}(t_0), \varepsilon) \cup \gamma_{\underline{r}^{n,+}}^{\infty} \cup \gamma_{\underline{r}^{n,-}}^{\infty}\right),$$

which contains $\gamma_{\underline{s}}^{\infty}(t)$, for all $t \leq t_1$, for some $t_1 < t_0$. Therefore, $L_{\underline{s}} \subset \bigcap_n U_n \subset \gamma_{\underline{s}}^{\infty}((-\infty, t_0])$. In such a case, $\overline{\gamma_{\underline{s}}^{\infty}}$ would separate the plane into (at least) two different connected components, what we have proved before that it is not possible. Therefore, $\gamma_{\underline{s}}^{\infty} \subset L_{\underline{s}}$, as desired.

Then, it follows from Curry's Theorem 1.3.2 that $L_{\underline{s}}$ is an indecomposable continuum, as desired. □

Finally, we prove Theorem 1.B.

*Proof of Theorem 1.B.* The existence of uncountably many rays that land follows from Proposition 1.3.8 (observe that there are uncountably many bounded sequences), whereas the existence of uncountably many non-landing rays follows from Proposition 1.3.9. On Corollary 1.3.10, we prove that the accumulation set of such non-landing rays is an indecomposable continuum. □

## 1.4 Accessibility from $U$ of points on $\partial U$. Proof of Theorem 1.C

This section is devoted to the proof of Theorem 1.C, which relates the accessibility from $U$ with the previously studied sets: the escaping set, the non-escaping sets and the landing sets. In particular, Theorem 1.C asserts that all boundary points in the escaping set are non-accessible, while points in $\partial U$ having a bounded orbit are accessible.

First of all, let us choose as a Riemann map the function $\varphi \colon \mathbb{D} \to U$ such that $\varphi(0) = 0$ and $\varphi(\mathbb{R} \cap \mathbb{D}) = \mathbb{R}$, as in [BD99]. With this choice, the associated inner function is

$$g(z) = \frac{3z^2 + 1}{3 + z^2}.$$

It is easy to check that the Denjoy-Wolff point of $g$ is 1. Moreover, since $g$ is a Blaschke product of degree 2 (and hence there are no critical points in the unit circle), $g_{|\partial\mathbb{D}}$ is a 2-to-1 covering of $\partial\mathbb{D}$, being 1 the only fixed point. In particular, the preimages of 1 under $g$ are itself and $-1$, since $\varphi(\mathbb{R}\cap\mathbb{D}) = \mathbb{R}$ and $f(-\infty) = +\infty$.

Let us consider the following subsets of the (closed) unit disk

$$D_0 := \overline{\mathbb{D}} \cap \{\mathrm{Im}\ z > 0\} \qquad D_1 := \overline{\mathbb{D}} \cap \{\mathrm{Im}\ z < 0\},$$

as shown in Figure 1.12. We define the itinerary for a point on $\partial\mathbb{D}$ in the following way.

**Definition 1.4.1. (Itineraries on $\partial\mathbb{D}$)** Let $\xi \in \partial\mathbb{D}$. If $g^n(\xi) \neq 1$, for all $n \geq 0$, then the *itinerary* of $\xi$ is defined as the sequence $\mathscr{S}(\xi) = \underline{s} = \{s_n\}_n \in \Sigma_2$ satisfying $g^n(\xi) \in D_{s_n}$.

If there exists $n_0 \geq 0$ such that $g^{n_0}(\xi) = 1$, then the *itineraries* of $\xi$, $\mathscr{S}(\xi)$, are defined as the two sequences $\underline{s}^j = \{s_n^j\}_n \in \Sigma_2$, $j = 0, 1$, satisfying $g^n(\xi) \in D_{s_n}$ for $n \leq n_0 - 2$, $s^0_{n_0-1} = 1$, $s^1_{n_0-1} = 0$ and $s^j_n = j$, for $n \geq n_0$.

Hence, we have just defined a multivalued function

$$\mathscr{S} : \partial\mathbb{D} \longrightarrow \Sigma_2.$$

We note that, since every point in $\partial\mathbb{D}$ has an itinerary, the domain of $\mathscr{S}$ is $\partial\mathbb{D}$. Moreover, we claim that $\mathscr{S}$ is injective, i.e. that two different points in the unit circle cannot have the same itinerary. This is due to the expansiveness of the map $g_{|\partial\mathbb{D}}$. Indeed,

$$g'(z) = \frac{-16z}{(3z^2+1)^2},$$

and hence, for $\xi \in \partial\mathbb{D}$, it holds

$$|g'(\xi)| = \frac{16\,|\xi|}{|3\xi^2+1|^2} \geq \frac{16}{(3\,|\xi^2|+1)^2} \geq 1.$$

We also shall consider its inverse

$$\mathscr{S}^{-1} : \Sigma_2 \longrightarrow \partial\mathbb{D},$$

which is a single-valued function. Moreover, $\mathscr{S}^{-1}$ is surjective, but not injective, and commutes with the shift map $\sigma$ in $\Sigma_2$.

Since $\mathscr{S}$ is only multivalued when considering eventual preimages of 1, it follows that $\mathscr{S}$ is a bijection if we restrict ourselves to non-eventually constant sequences in $\Sigma_2$ and points in $\partial\mathbb{D}$ which are not eventual preimages of 1.

The following proposition is the key result which relates itineraries in $\partial\mathbb{D}$ and in $\widehat{S}$, and will clarify the choice of the itineraries in $\partial\mathbb{D}$.

**Proposition 1.4.2. (Correspondence between itineraries)** *Let $\xi \in \partial\mathbb{D}$. If $g^n(\xi) \neq 1$, for all $n \geq 0$, and $\underline{s} = \mathscr{S}(\xi)$ then $Cl(\varphi, \xi) = \overline{\gamma^\infty_{\underline{s}}}$. If there exists $n_0 \geq 0$ such that $g^{n_0}(\xi) = 1$ and $\{\underline{s}^0, \underline{s}^1\} = \mathscr{S}(\xi)$, then $Cl(\varphi, \xi) = \overline{\gamma^\infty_{\underline{s}^0}} \cup \overline{\gamma^\infty_{\underline{s}^1}}$.*

*Proof.* Observe that, according to the chosen Riemann map $\varphi\colon \mathbb{D} \to U$, it holds that $\varphi(\text{Int } D_0) \subset \Omega_0$ and $\varphi(\text{Int } D_1) \subset \Omega_1$ (see Fig. 1.12). Moreover, $\varphi((-1,1)) = \mathbb{R} \subset U$.

Hence, if $\xi \in \partial\mathbb{D}$ and $\xi \notin \{-1,1\}$, then $\xi \in D_i$, and so does a neighbourhood of $\xi$ in $\overline{\mathbb{D}}$. Hence, $Cl(\varphi,\xi) \subset \Omega_i$, for some $i \in \{0,1\}$. By continuity of $g$, every sequence in $\mathbb{D}$ converging to $\xi$ maps under $g$ to a sequence converging to $g(\xi)$. If $\xi \in \partial\mathbb{D}$ is not a preimage of 1, then $g(\xi) \notin \{-1,1\}$, so $f(Cl(\varphi,\xi) \cap \mathbb{C}) \subset \Omega_j$, for some $j \in \{0,1\}$. Repeating inductively the same argument, we get that the itinerary of $\xi$ determines completely the itinerary of points in $Cl(\varphi,\xi)$, so

$$Cl(\varphi,\xi) \subset \{z \in \partial U\colon I(z) = \mathscr{S}(\xi)\} \cup \{\infty\},$$

if $\xi \in \partial\mathbb{D}$ is not an eventual preimage of 1.

On the other hand, consider $1 \in \partial\mathbb{D}$, $\mathscr{S}(1) = \{\overline{0},\overline{1}\}$. We note that, for any sequence of points $\{w_k\}_k \subset D_0$ converging to 1, and for all $n \geq 0$, there exists $k_0 = k_0(n)$ such that $\{g^n(w_k)\}_{k \geq k_0} \subset D_0$; and we observe that 1 is the only point in $\partial\mathbb{D}$ with this property. Similarly, if $z \in \partial U$ and for any sequence $\{z_k\}_k \subset \Omega_0$ converging to $z$, for all $n \geq 0$ there exists $k_0$ such that $\{f^n(z_k)\}_{k \geq k_0} \subset \Omega_0$, then $z \in L^+$. Therefore, for any sequence $\{w_n\}_n \subset D_0$, $w_n \to 1$, any accumulation point of $\{\varphi(w_n)\}_n$ must be in $L^+ \cup \{\infty\}$. The analogous argument works similarly with $D_1$ and $L^-$. Hence,

$$Cl(\varphi,\xi) \subset L^+ \cup L^- \cup \{\infty\} = \{z \in \partial U\colon I(z) \in \{\overline{0},\overline{1}\}\} \cup \{\infty\}.$$

Therefore, if $\xi$ is an eventual preimage of 1, and hence $\mathscr{S}(\xi) = \{\underline{s}^0, \underline{s}^1\}$, it holds

$$Cl(\varphi,\xi) \subset \left\{z \in \partial U\colon I(z) \in \left\{\underline{s}^0, \underline{s}^1\right\}\right\} \cup \{\infty\}.$$

We note that, given two different sequences $\underline{r}, \underline{s} \in \Sigma_2$, the sets of points in $\partial U$ having these itineraries are disjoint, i.e.

$$\{z \in \partial U\colon I(z) \in \underline{r}\} \cap \{z \in \partial U\colon I(z) \in \underline{s}\} = \emptyset,$$

since a point in $\partial U$ has a unique itinerary. Moreover, any point $z \in \partial U$ must belong to at least one cluster set, hence the previous three inclusions are in fact equalities. The fact that all points in $\partial U$ are in the closure of a hair ends the proof of the proposition. □

Let us observe that the previous proposition gives, in particular, a way to compute the impression of the prime end at 1, alternative to the one in [BD99, Thm. 6.1].

**Corollary 1.4.3. (Prime end at 1)** *The prime end of $U$ which corresponds by the Riemann map $\varphi$ to 1 has the impression $L^+ \cup L^- \cup \{\infty\}$. Equivalently, $Cl(\varphi,1) = L^+ \cup L^- \cup \{\infty\}$.*

The previous correspondence between itineraries and the fact that in each cluster set $Cl(\varphi,\xi)$ there is at most one accessible point, imply that there is at most one accessible point per itinerary. In particular, in each hair and its landing set there is at most one accessible point.

A first study on accessibility and radial limits was carried out by Baker and Domínguez, characterizing the accesses to infinity.

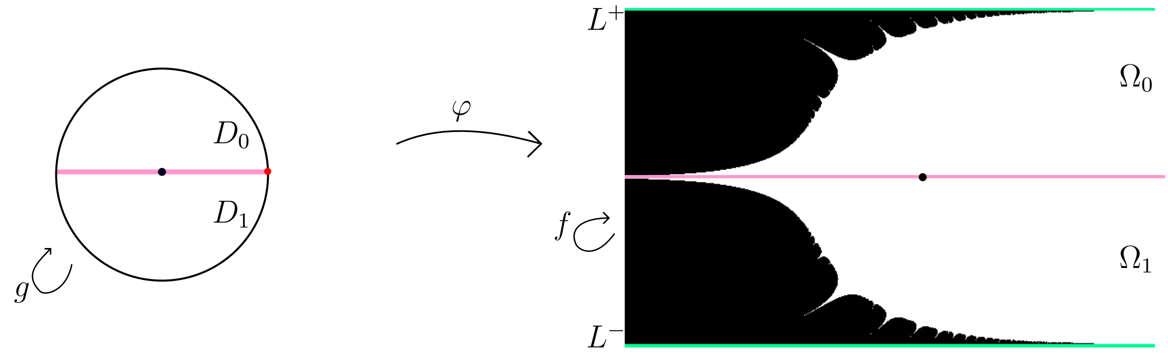



Figure 1.12: Representation of the Riemann map $\varphi\colon \mathbb{D} \to U$, which fixes the real axis. The regions $D_0, D_1, \Omega_0$ and $\Omega_1$ are also represented, and it is clear that $\varphi(D_0) \subset \Omega_0$ and $\varphi(D_1) \subset \Omega_1$ implying the correspondence between itineraries.

**Theorem 1.4.4. (Accesses to infinity,** [BD99]**)** *Accesses from $U$ to infinity are characterized by the eventual preimages of 1, i.e.*

$$\{\xi\colon \varphi^*(\xi) = \infty\} = \{\xi\colon g^n(\xi) = 1, \text{ for some } n \geq 0\}.$$

Next, we prove Theorem 1.C, which asserts that escaping points are non-accessible from $U$, while points in $\partial U$ having a bounded orbit are all accessible from $U$. Using the Correspondence Theorem II.4.6 between accesses and radial limits, we rewrite the statement of Theorem 1.C as follows.

**Theorem 1.C.** (a) *Let $\xi \in \partial\mathbb{D}$ such that the radial limit $z := \varphi^*(\xi)$ exists. Then, $z$ is non-escaping.*

(b) *Let $z \in \partial U$ be a point whose orbit is bounded. Then, there exists $\xi \in \partial\mathbb{D}$ such that $\varphi^*(\xi) = z$, i.e. $z$ is accessible from $U$.*

*Proof.* (a) The proof is based on the one developed by Baker and Domínguez in [BD99, Thm. 6.3].

Assume $z := \varphi^*(\xi)$ is an escaping point and let us define the open set

$$W := \left\{ z \in S\colon \text{Re } z < -2 \text{ and } |\text{Im } z| > \frac{\pi}{2} \right\}.$$

Iterating the function if needed, we can assume $f^n(z) \in W$, for all $n \geq 0$. Since the radial segment

$$\varphi_\theta := \{\varphi(r\xi)\colon r \in (0,1)\}$$

lands at $z$, one can choose $r_0 \in (0,1)$ such that $\gamma := \{\varphi(r\xi)\colon r \in (r_0, 1)\} \subset W$. For points in $W$ we have $\text{Re } f(z) < \text{Re } z$. Hence, since $\gamma$ is connected and $f^n(z) \in W$ for $n \geq 0$, we have $f^n(\gamma) \subset W$, for all $n \geq 0$. This is a contradiction because $\gamma \subset U$, so points in $\gamma$ must converge to $+\infty$.

(b) First, we note that, by the the results in Section 1.3, the only points in $\widehat{S}$ with bounded orbit are endpoints $w_{\underline{s}}$ for bounded sequences $\underline{s} \in \Sigma_2$. Therefore, the goal is to prove that, if $\xi_{\underline{s}} \in \partial\mathbb{D}$ has itinerary $\underline{s} \in \Sigma_2$, and $\underline{s}$ is a bounded sequence, then $\varphi^*(\xi_{\underline{s}}) = w_{\underline{s}}$. We note that the radial cluster set $Cl_R(\varphi, \xi_{\underline{s}})$, which is connected, is contained in the cluster set $Cl(\varphi, \xi_{\underline{s}})$, and for a bounded sequence, it holds

$$Cl(\varphi, \xi_{\underline{s}}) = \overline{\gamma_{\underline{s}}^{\infty}} = \gamma_{\underline{s}}^{\infty} \cup \left\{w_{\underline{s}}\right\} \cup \{\infty\},$$

by Propositions 1.3.8 and 1.4.2.

Hence, it is enough to show that, if $\underline{s}$ is a bounded sequence, then the radial cluster set $Cl_R(\varphi, \xi_{\underline{s}})$ cannot contain any escaping point.

Recall that $g|_{\partial\mathbb{D}}$ is conjugate to the doubling map. Moreover, since $\underline{s}$ contains at most $N$ consecutive 0's and 1's, there exist $0 < \theta_1 < \theta_2 < \pi$ such that $\theta_1$ and $\theta_2$ are eventual preimages of 1 and $g^n(e^{i\theta_{\underline{s}}}) \in \left(e^{i\theta_1}, e^{i\theta_2}\right) \cup \left(e^{-i\theta_2}, e^{-i\theta_1}\right)$. Then, $R_{\theta_1}$ and $R_{\theta_2}$ are curves starting at 0 and landing at $-\infty$ approaching $L^+$. Since $\varphi$ is a bijection, $f^n(R_{\underline{s}})$ is contained in the region bounded by $R_{\theta_1}$ and $R_{\theta_2}$ and its reflection along the real axis. Therefore, there exists $R > 0$ such that, if we consider the open set $W$ defined as before and

$$W' := \left\{z \in S \colon \text{Re } z < -R \text{ and } |\text{Im } z| < \frac{\pi}{2}\right\},$$

then $f^n(R_{\underline{s}}) \cap W' = \emptyset$, for all $n \geq 0$. Compare with Figure 1.13.

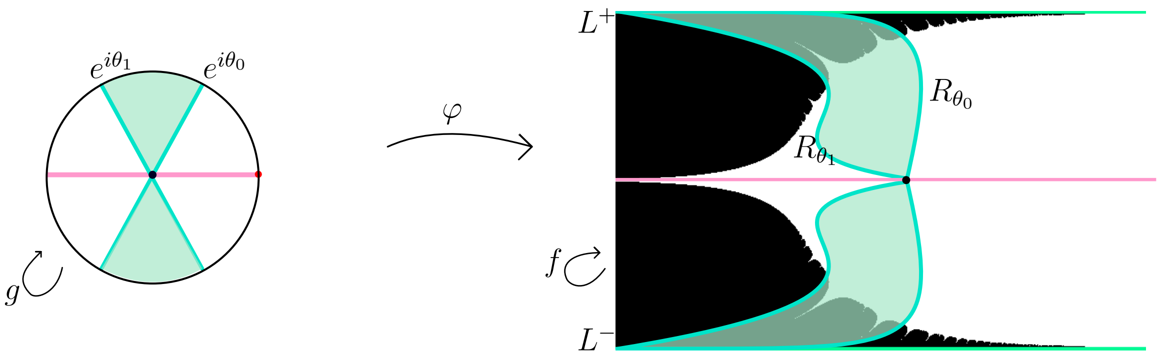



Figure 1.13: Schematic representation of the region bounded by $R_{\theta_1}$ and $R_{\theta_2}$ and its reflection along the real axis, where $f^n(R_{\underline{s}})$ is contained, for all $n \geq 0$.

Assume the radial cluster set contains an escaping point $z$. Iterating the function if needed, we can assume Re $f^n(z) < -R$, for all $n \geq 0$, so $z \in W$. Then, there exists a sequence of real numbers $\{t_n\}_n$ such that $t_n \to +\infty$ and $z_n := \gamma_{\underline{s}}^{\infty}(t_n) \to z$, as $n \to \infty$. Without loss of generality, since $z \in W$, we shall assume $\{z_n\}_n \subset W$. For points in $W$ we have Re $f(z) <$ Re $z$, so they either belong to $W$ or to $W'$. But $W'$ has been defined so that $f^n(R_{\underline{s}}) \cap W' = \emptyset$, so $\left\{f^k(z_n)\right\} \in W$ for all $k \geq 0$: a contradiction, since $\{z_n\}_n \subset U$, and points in $U$ converge to $+\infty$.

□

**Remark 1.4.5.** Alternatively, Theorem 1.C can be seen as a consequence from the results of [BR20]. Indeed, in [BR20, Sect. 6], it is proved that, for functions in class $\mathcal{B}$ and bounded postsingular set, accessible points in the boundary of an invariant Fatou component coincide with the endpoints of the hairs lying in its boundary (Remark 6.11). Such result can be applied to $h(w) = we^{-w}$, semiconjugate to $f(z) = z + e^{-z}$ (Sect. 1.1), to deduce that points with bounded orbit are accessible from $U$, since they are the endpoint of a hair in $\partial U$.

Nevertheless, although Theorem 1.C can be seen as a consequence of this more general result, it relies strongly on the study of the landing sets of the dynamic rays, carried out in the previous section, which has to be done specifically for our function. Moreover, our construction shows explicitly the relation between the dynamics of the inner function in $\partial\mathbb{D}$ and the dynamics of $f$ in $\partial U$, which was the main goal of the paper.

## 1.5 Periodic points in $\partial U$. Proof of Theorem 1.D

This last section of the paper is dedicated to prove Theorem 1.D, which asserts that periodic points are dense in $\partial U$. Although it is known that periodic points are dense in the Julia set, if we restrict ourselves to the boundary of a Baker domain, it is not known, in general, the existence of a single periodic point.

The general argument used to prove that periodic points are dense in the Julia set (e.g. [CG93, Thm. III.3.1]) cannot be used, since it gives no control about the resulting periodic point. The proof we present allows us to find a periodic point in any neighborhood of any point in $\partial U$, whose orbit is entirely contained in $S$, and hence implying that the periodic point is in $\partial U$.

**Theorem 1.D.** *Periodic points are dense in $\partial U$.*

*Proof.* In view of Theorem II.5.4, it is enough to approximate $z \in \partial U$ having a dense orbit by periodic points in $\partial U$. Let us fix $\varepsilon > 0$ and consider the disk $D(z, \varepsilon)$. Without loss of generality, we can assume $D(z, \varepsilon) \subset S$ and $D(z, \varepsilon) \cap \overline{V} = \emptyset$, where $V$ is the absorbing domain defined in Section 1.1. We also assume $\varepsilon < 1$.

Recall that $f$ is expanding in $S \smallsetminus \overline{V}$ and uniformly expanding in any left-half plane intersected with it (see Rmk. 1.1.7). In particular, the map is uniformly expanding in $S \cap \{\text{Re } z < -2 + \varepsilon\}$ with constant of expansion $\lambda > 1$.

Take $n_0 > 0$ such that $\lambda^{n_0} > 2$. Since the orbit of $z$ is assumed to be dense in $\partial U$, it visits infinitely many times $S \cap \{\text{Re} z < -2\}$. Let $n_1$ be such that

$$\# \{n < n_1 \colon \text{Re } f^n(z) < -2\} \geq n_0.$$

Since the orbit of $z$ is dense, there exists $n_2 > n_1$ with $z_{n_2} := f^{n_2}(z) \in D(z, \varepsilon)$. Then, $\phi_{s_0} \circ \cdots \circ \phi_{s_{n_2-1}}(z_{n_2}) = z$, for a suitable choice of $s_0, \ldots, s_{n_2-1} \in \{0, 1\}$.

We claim that $\phi_{s_0} \circ \cdots \circ \phi_{s_{n_2-1}}(D(z, \varepsilon)) \subset D(z, \varepsilon)$. Indeed, since $D(z, \varepsilon) \cap \overline{V} = \emptyset$, we have $D(z, \varepsilon) = D_\rho(z, \varepsilon)$, for the $\rho$-distance defined in 1.1.4. The forward invariance of $V$ gives $\phi_{s_0} \circ \cdots \circ \phi_{s_n}(D(z, \varepsilon)) \subset S \smallsetminus \overline{V}$, for all $n \geq 0$. Moreover, since inverses are contracting, if

$\phi_{s_0}\circ\cdots\circ\phi_{s_n}(z)\in S\cap\{\text{Re } z<-2\}$, we have $\phi_{s_0}\circ\cdots\circ\phi_{s_n}(D(z,\varepsilon))\subset S\cap\{\text{Re } z<-2+\varepsilon\}$. Hence, after applying $n_2$ inverses, since the iterated preimages of $D(z,\varepsilon)$ are contained in $\{\text{Re } z<-2+\varepsilon\}$ at least $n_0$ times, $\rho$-distances in $D(z,\varepsilon)$ are contracted by a factor less than $\frac{1}{\lambda^{n_0}}$. Therefore we have:

$$\rho(\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}(z),z)=\rho(\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}(z),\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}(z_{n_2}))\leq\frac{1}{\lambda^{n_0}}\rho(z,z_{n_2})\leq\frac{1}{2}\varepsilon.$$

Now let $w\in D(z,\varepsilon)$, then

$$\rho(\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}(w),\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}(z))\leq\frac{1}{\lambda^{n_0}}\rho(w,z)\leq\frac{1}{2}\varepsilon.$$

Therefore, applying the triangle inequality, one deduces that $\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}(w)\in D(z,\varepsilon)$, for any $w\in D(z,\varepsilon)$, as desired.

Finally, observe that $\rho(\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}$ is well-defined in $\overline{D(z,\varepsilon)}$, and

$$\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}(\overline{D(z,\varepsilon)})\subset\overline{D(z,\varepsilon)}.$$

Hence, Brouwer fixed-point theorem guarantees the existence of a fixed point $z_0$ for $\phi_{s_0}\circ\cdots\circ\phi_{s_{n_2-1}}$ in $D(z,\varepsilon)$. This point is periodic for $f$. Moreover, since its orbit is all contained in $S$, we have $z_0\in\partial U$, by Proposition 1.2.4. This ends the proof of Theorem 1.D. $\square$

# Chapter 2

# Boundary dynamics in unbounded Fatou components of entire maps

In this chapter, we aim to generalize some of the results obtained in Chapter 1 for the doubly parabolic Baker domain of $f(z) = z+e^{-z}$ to a wider class of Fatou components. As mentioned in the introduction, the topological structure given by the accesses to infinity, together with some control on the singular values, will be enough to deduce topological and dynamical properties of the boundary of certain Fatou components.

The objectives of this chapter are threefold. First, we aim to give a topological description of the boundary of $U$ from the point of view of its Riemann map; second, we wish to give analytical properties for the associated inner function $g$; and last, we wish to explore the existence, density and accessibility of periodic points and *escaping points* in $\partial U$, i.e. points that converge to $\infty$ under iteration.

**Remark. (Ergodic and recurrent Fatou components)** In this section, we shall use the following terminology in order to simplify the notation: we say that $U$ is an *ergodic* (resp. *recurrent*) if $g^*\colon \partial\mathbb{D} \to \partial\mathbb{D}$ is ergodic (resp. recurrent). Siegel disks, attracting and parabolic basins are both ergodic and recurrent. Hyperbolic and simply parabolic Baker domains are never ergodic nor recurrent. Doubly parabolic Baker domains are always ergodic, but they may be recurrent or not. Note that the properties of the boundary map of a Fatou component (both from the topological and the dynamical point of view) depend essentially on this ergodic classification, rather than on the precise type of Fatou component.

### Topology of the boundary of unbounded invariant Fatou components

In the setting described above, understanding the boundary behaviour of the Riemann map $\varphi\colon \mathbb{D} \to U$ is used not only to study the dynamics of $f$ on the boundary, but also to describe the topology of $\partial U$. We note that, *a priori*, continuity of the Riemann map in $\overline{\mathbb{D}}$ cannot be assumed, since the continuous extension only exists if $\partial U$ is locally connected, something impossible if, for example, $U$ is unbounded and it is not a univalent Baker domain [BW91]. Hence, given a point $\xi \in \partial\mathbb{D}$, we shall work with its radial limit $\varphi^*(\xi)$ (if it exists), and its cluster set $Cl(\varphi, \xi)$. We prove the following.

**Theorem 2.A. (Topological structure of $\partial U$)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Assume $U$ is ergodic. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map. Then, $\partial U$ is the disjoint union of cluster sets $Cl(\varphi, \cdot)$ of $\varphi$ in $\mathbb{C}$, i.e.*

$$\partial U = \bigsqcup_{\xi \in \partial\mathbb{D}} Cl(\varphi, \xi) \cap \mathbb{C}.$$

*Moreover, either $Cl(\varphi, \xi) \cap \mathbb{C}$ is empty, or has at most two connected components. If $Cl(\varphi, \xi) \cap \mathbb{C}$ is disconnected, then $\varphi^*(\xi) = \infty$.*

Observe that this is quite a strong property. For example, there cannot be points in $\partial U$ with more than one access from $U$, since they would belong to the cluster set of at least two points in $\partial\mathbb{D}$.

We shall see that Theorem 2.A plays an important role in the proofs of the main dynamical results in this paper, but it also has some interesting more direct consequences, like for example the following generalization of the result of Bargmann [Bar08, Corol. 3.15], which states that the boundary of a Siegel disk cannot have accessible periodic points.

**Corollary 2.B. (Periodic points in Siegel disks)** *Let $f$ be a transcendental entire function, and let $U$ be a Siegel disk, such that $\infty$ is accessible from $U$. Then, there are no periodic points on $\partial U$.*

Theorem 2.A improves the understanding of the topology of the boundary of unbounded Fatou components for transcendental entire functions, initiated by the work of Devaney and Golberg [DG87], on the completely invariant attracting basin $U_\lambda$ of $E_\lambda(z) := \lambda e^z$, with $0 < \lambda < \frac{1}{e}$. It was shown, on the one hand, that points $e^{i\theta} \in \partial\mathbb{D}$ such that $\varphi_\lambda^*(e^{i\theta}) = \infty$ are dense in $\partial\mathbb{D}$; and, on the other hand, that each cluster set $Cl(\varphi_\lambda, e^{i\theta})$ is either equal to $\{\infty\}$ or it consists of an unbounded curve landing at a finite accessible endpoint. This result was generalized to totally invariant attracting basins of transcendental entire function $f$, with connected Fatou set [BK07]. We note that both in [DG87] and in [BK07], symbolic dynamics (and tracts) play an important role in their proofs, which depend essentially on the class of functions they consider, and it does not lead to an obvious generalization to arbitrary Fatou components.

In this context, Theorem 2.A should be viewed as a susbtantial generalization of the results in [BK07], since it applies to arbitrary ergodic invariant Fatou components with infinity accessible. We remark that our proof does not rely on symbolic dynamics, but on the fact that radii landing at infinity under the Riemann map are dense in $\partial\mathbb{D}$, and they separate the plane into infinitely many regions, each of them containing a different cluster set, as well as on a deep analysis of clusters sets using null-chains of crosscut neighbourhoods.

### Inner functions associated to unbounded invariant Fatou components

As in Chapter II, to each invariant Fatou component $U$ of a transcendental entire function $f$ we associate an inner function $g\colon \mathbb{D} \to \mathbb{D}$ via a Riemann map $\varphi\colon \mathbb{D} \to U$. Such inner function is unique up to conformal conjugacy in the unit disk, and it is well-known that it is either a finite Blaschke product (when $f|_U$ has finite degree); or conjugate to an infinite Blaschke product (when $f|_U$ has infinite degree) (Frostman, [Gar07, Thm. II.6.4]). In the former case, $g$ extends to a rational function $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, whereas in the latter there exists at least a point $\xi \in \partial\mathbb{D}$ where $g$ does not extend holomorphically to any neighbourhood of it, what we call a *singularity* (Def. II.1.4).

A natural problem in this setting is to relate the inner function $g$ with the function $f|_U$, in the sense of understanding if every inner function can be realised for some $f|_U$, or if considering a particular class of functions $f$ limits the possible associate inner functions $g$; for instance if a bound on the number of singularities of $g$ in $\partial\mathbb{D}$ exists. Note that, in general, there exist inner functions for which every point in $\partial\mathbb{D}$ is a singularity.

A first (naive) remark is that singularities of $g$ are related to accesses from $U$ to infinity, since the singularities of $g$ share many properties with the essential singularity of $f$ (e.g. both are the only accumulation points of preimages of almost every point). In particular, bounded invariant Fatou components are always associated with finite Blaschke products. In fact, it is shown in [BFJK17, Prop. 2.7] that, if $\infty$ is accessible from $U$, then

$$E(g) \subset \overline{\Theta_\infty} = \overline{\{\xi \in \partial\mathbb{D}\colon \text{ the radial limit } \varphi^*(\xi) \text{ is equal to } \infty\}}.$$

We note that, by the results of [BD99, Bar08], when $U$ is ergodic, the latter set is the whole unit circle, and hence the result does not give actual information on the singularities of $g$.

A different approach is found in [EFJS19, ERS20], which relies on having a great control on the singular values of $f$, i.e. points for which not every branch of the inverse is locally well-defined around it. Indeed, assuming that the orbits of singular values belong to a compact set in $\mathcal{F}(f)$ (i.e. assuming $f$ to be hyperbolic), they give explicit bounds for the number of singularities. One can see from the proof that it is enough to assume that $f$ behaves as if it was a hyperbolic function when restricted to $U$ i.e. that the orbits of singular values in $U$ are compactly contained in $U$. We shall not enter into the details at this point, but keep in mind the main idea: controlling the singular values of $f$ allows to bound the singularities of $g$, and it is enough to have this control on the singular values which actually lie inside $U$.

**Definition. (Postsingularly separated Fatou components)** Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component. We say that $U$ is a *postsingularly separated Fatou component* (PS Fatou component) if there exists a domain $V$, such that $\overline{V} \subset U$ and

$$P(f) \cap U \subset V.$$

Hence, a PS Fatou component is a Fatou component whose postsingular values are allowed to accumulate at $\infty$, as long as they accumulate through accesses to $\infty$. Indeed, the role of the domain $V$ is precisely to control in which accesses of $U$ do the postsingular values accumulate.

Observe that PS Fatou components can be seen as a generalization of Fatou components of hyperbolic functions. In fact, if $U$ is a Fatou component of a hyperbolic function $f$, then $P(f) \cap U$ is contained in a compact set $V$ in $U$. PS Fatou components allow $V$ not to be compact, we only ask $\overline{V} \subset U$.

Note that there is no requirement for $P(f)$ outside $U$; in particular, $P(f)$ is allowed to accumulate in $\partial U$.

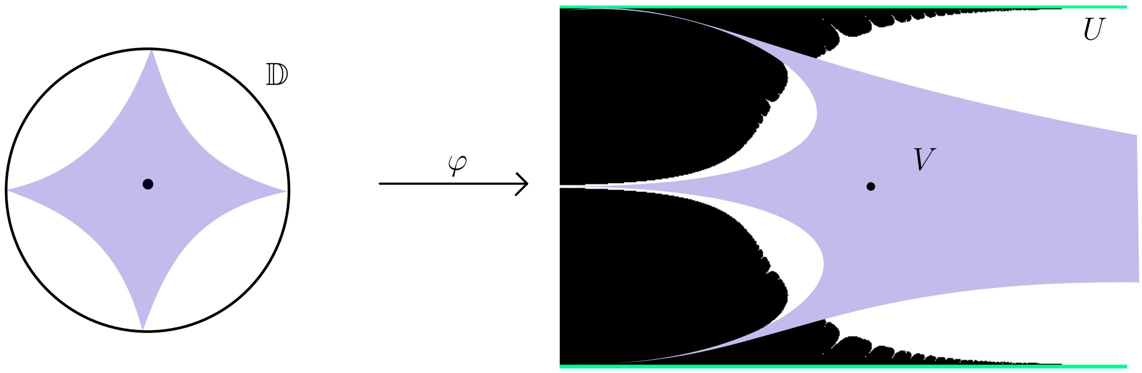



Figure 2.1: Schematic representation of how a postsingularly separated Fatou component would look like. The Fatou component of the left is a Baker domain of $z + e^{-z}$. For this particular example, the domain $V$ could have been taken simpler. However, we wanted to illustrate how $V$ looks like in general. As we will prove in the Technical Lemma 1, given a Riemann map $\varphi\colon \mathbb{D} \to U$, $\varphi^{-1}(V)$ is a domain enclosed by curves landing in $\partial\mathbb{D}$.

The postsingularly separated condition is sufficient to describe the singularities of $g$, as we show in the following theorem.

**Theorem 2.C. (Singularities for the associated inner function)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g := \varphi^{-1} \circ f \circ \varphi$ be the corresponding associated inner function. Assume $U$ is postsingularly separated.*
*Then, the set of singularities of $g$ has zero Lebesgue measure in $\partial\mathbb{D}$. Moreover, if $\xi \in \partial\mathbb{D}$ is a singularity for $g$, then $\varphi^*(\xi) = \infty$.*

Recall that we do allow postsingular values to accumulate at $\infty$. Hence, in contrast to the setting considered in [EFJS19, ERS20], we allow $\varphi^{-1}(SV(f))$ to accumulate in $\partial\mathbb{D}$. Moreover, Theorem 2.C strengthens the result in [BFJK17, Prop. 2.7], showing that a singularity not only must be approximated by points with radial limit infinity, but itself must have radial limit infinity, in accordance with the *a priori* naive idea of relating singularities with accesses to infinity.

**Dynamics on the boundary of unbounded invariant Fatou components**

Finally, we shall apply the techniques developed throughout the paper to study periodic points in $\partial U$, and also escaping points, in this general setting. To this end, and being very far from the rational case where the only singularities of $f^{-1}$ are a finite number of critical values, we need a slightly stronger condition on the orbits of the singular values of $f$. More precisely, we restrict to the following class of Fatou components, which is a subset of the previous ones.

**Definition. (Strongly postsingularly separated Fatou components)** Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component. We say that $U$ is a *strongly postsingularly separated Fatou component* (SPS Fatou component) if there exists a simply connected domain $\Omega$ and a domain $V$ such that $\overline{V} \subset U$, $\overline{U} \subset \Omega$, and

$$P(f) \cap \Omega \subset V.$$

Hence, the control on the postsingular set that we require is two-fold. On the one hand, to control the postsingular values inside $U$, we ask $U$ to be postsingularly separated. On the other hand, the existence of the simply connected domain $\Omega$ is needed to control the postsingular values in a neighbourhood of $\partial U$.

We note that this condition is analogous to the one given by Pérez-Marco [PM97] for the study of the boundary of Siegel disks.

**Example. (SPS Fatou components)** Many of the examples of unbounded invariant Fatou components that have been explicitly studied are SPS. Examples of basins of attraction include the ones of hyperbolic functions studied in [BK07], as well as the hyperbolic exponentials [DG87]. However, the results we prove are already known for this class of functions, since $\mathcal{J}(f) = \partial U$. A more significant example are the basins of attraction of $f(z) = z - 1 + e^{-z}$. Regarding Baker domains, consider Fatou's function $f(z) = z + 1 + e^{-z}$, studied in [Evd16]; the Baker domains of $f(z) = z + e^{-z}$, investigated in [BD99, FH06] and in Chapter 1; and the ones in [FH06, Ex. 4].

In all cases, singular values are all critical values and lie in the Baker domains, at a positive distance from the boundary, even though their orbits accumulate at $\infty$. Observe however that they do so through only one access, and that there is an absorbing domain $V \subset U$ with $P(f) \subset V$. This implies, in particular, that iterated inverse branches are globally well-defined around $\partial U$. Note that this situation is much simpler than the general case that we address in our theorem, and in fact, if this were always the case, our proofs could be simplified to a great extent (due to this global definition if inverse branches).

We prove the following.

**Theorem 2.D. (Boundary dynamics)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Assume $U$ is strongly postsingularly separated. Then, periodic points in $\partial U$ are accessible from $U$. Moreover, if $U$ is recurrent, then both periodic and escaping points are dense in $\partial U$.*

**Remark. (Parabolic basins)** Note that parabolic basins cannot be PS (and hence neither SPS), due to the fact that the parabolic fixed point is always in $P(f)$. However, as we see in Section 2.5, our results apply if the parabolic fixed point is the only point in $\partial U \cap P(f)$.

## 2.1 Ergodic Fatou components and boundary structure. Theorem 2.A and Corollary 2.B

In this section, we prove Theorem 2.A, which gives a detailed description of the topology of the boundary of *ergodic* Fatou components, i.e. Fatou components for which the boundary map $f\colon \partial U \to \partial U$ is ergodic. The main tool, and the reason to restrict to ergodic Fatou components, is Theorem II.5.6, which asserts that, for an ergodic Fatou component $U$ and $\varphi\colon \mathbb{D} \to U$ a Riemann map,

$$\overline{\Theta_\infty} = \overline{\{\xi \in \partial\mathbb{D}\colon \varphi^*(\xi) = \infty\}} = \partial\mathbb{D}.$$

**Remark 2.1.1. (Non-ergodic Fatou components)** We note that ergodicity is a sufficient condition, but not necessary. Indeed, there are examples of non-ergodic Fatou components that satisfy $\Theta_\infty$ is dense in $\partial\mathbb{D}$ [Bar08, Example 3.6]. Likewise, it is well-known that the Theorem II.5.6 does not hold for an arbitrary invariant Fatou components for which infinity is accessible, as shown for example by univalent Baker domains whose boundaries are Jordan curves [BF01].

*Proof of Theorem 2.A.* We shall prove first that all cluster sets are disjoint in $\mathbb{C}$ and its union is $\partial U$, i.e. if $p \in \partial U \cap \mathbb{C}$, there exists a unique $\xi \in \partial\mathbb{D}$ such that $p \in Cl_{\mathbb{C}}(\varphi, \xi)$.

To prove the existence of such $\xi$ it is enough to consider a sequence $\{z_n\}_n \subset U$ such that $z_n \to p$, and $\{w_n \coloneqq \varphi^{-1}(z_n)\}_n \subset \mathbb{D}$. Then, $\{w_n\}_n$ must have at least one accumulation point, which must be in $\partial\mathbb{D}$. For any such accumulation point $\xi$, we have $p \in Cl(\varphi, \xi)$.

To prove uniqueness, assume, on the contrary, that there exist $\xi_1, \xi_2 \in \partial\mathbb{D}$ such that $p \in Cl(\varphi, \xi_1) \cap Cl(\varphi, \xi_2)$, and $\xi_1 \neq \xi_2$. Since $\Theta_\infty$ is dense in $\partial\mathbb{D}$ (Thm. II.5.6), we can choose $\zeta_1, \zeta_2 \in \Theta_\infty$ such that $\zeta_1 < \xi_1 < \zeta_2 < \xi_2$ (in the circular order). The radial segments

$$R_{\zeta_i} = \{r\zeta_i\colon r \in [0, 1)\},$$

$i = 1, 2$, give a partition of $\mathbb{D}$. Since $\varphi^*(\zeta_i) = \infty$, $\varphi(R_{\zeta_1}) \cup \varphi(R_{\zeta_2})$ give a partition of $\mathbb{C}$ (see Fig. 2.2).

Therefore, given any two sequences $\{w_n^1\}_n, \{w_n^2\}_n \subset \mathbb{D}$, with $w_n^1 \to \xi_1$ and $w_n^2 \to \xi_2$, the corresponding sequences in $U$ lie in different connected components of $\mathbb{C} \setminus (\varphi(R_{\zeta_1}) \cup \varphi(R_{\zeta_2}))$, for $n$ large enough. Hence, they cannot accumulate at the same (finite) point, leading to a contradiction.

To prove the second statement notice that $Cl_{\mathbb{C}}(\varphi, \xi)$ is disjoint from any other cluster set. Therefore, any connected component of $Cl_{\mathbb{C}}(\varphi, \xi)$ is, in fact, a connected component of $\partial U$. Hence, by Proposition II.4.8, each cluster set has either one or two connected components, that must be unbounded, since the cluster set is connected in $\widehat{\mathbb{C}}$. □

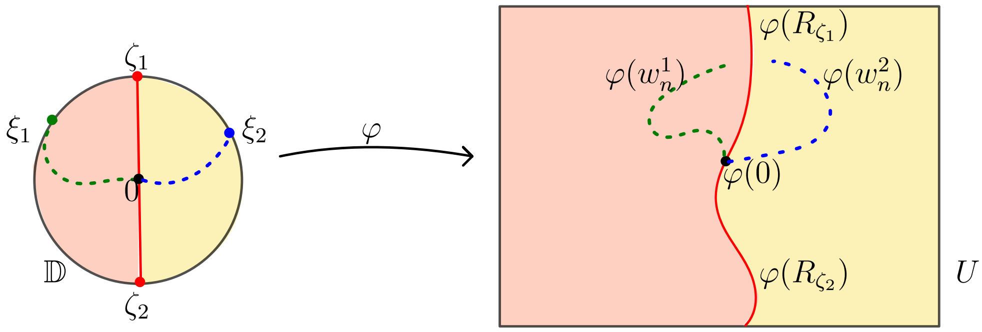



Figure 2.2: Diagram of the setup of the proof of Theorem 2.A.

**Remark 2.1.2. (Connected components per cluster set)** We note that, for the the exponential attracting basin in [DG87], all cluster sets have exactly one connected component in $\mathbb{C}$. However, the previous theorem is sharp, as shown by the function $f(z) = z + e^{-z}$. For the invariant Baker domains of this function, the cluster set of every point in $\Theta_\infty$ has two connected components (see Chapter 1).

### 2.1.1 Siegel disks. Proof of Corollary 2.B

For entire functions, it is known that Siegel disks have no accessible boundary periodic points [Bar08, Corol. 3.15]. An easy consequence of Theorem 2.A is that, if $\infty$ is accessible from the Siegel disk, in fact there are no periodic points at all.

*Proof of Corollary 2.B.* Assume there exists $p \in \partial U$ periodic, i.e. $f^n(p) = p$, for some $n \geq 1$. Then, $p \in Cl(\varphi, \xi)$, for a unique $\xi \in \partial\mathbb{D}$, since cluster sets are disjoint (Theorem 2.A). For a Siegel disk, the associated inner function $g$ is an irrational rotation, so it extends continuously to $\partial\mathbb{D}$. Hence, by Lemma II.5.2,

$$f^n(Cl_{\mathbb{C}}(\varphi, \xi)) \subset Cl_{\mathbb{C}}(\varphi, g^n(\xi)).$$

Now $f^n(p) = p \in Cl_{\mathbb{C}}(\varphi, \xi) \cap Cl_{\mathbb{C}}(\varphi, g^n(\xi))$. But this intersection is empty unless $g^n(\xi) = \xi$, and this is a contradiction because $g$ is an irrational rotation. $\square$

## 2.2 Technical Lemmas

In this section we prove some technical results which are the basis for the proofs of Theorems II.3 and II.4. Basically, we aim to relate the hyptothesis of being postsingularly separated (PS), or strongly postsingularly separated (SPS), with the possibility of defining inverse branches around points in $\partial U$. To do so, we construct in both cases appropriate neighbourhoods of each component of $\partial U$, in which we can define all inverse branches globally.

First, recall that if a Fatou component is PS, then there exists a domain $V$, such that $\overline{V} \subset U$ and

$$P(f) \cap U \subset V.$$

Note that, in this case, $P(f) \cap \partial U$ may be non-empty, so inverse branches may not be defined around points in $\partial U$. However, we can still define the inverse branches in a one-sided neighbourhood of each connected component $C$ of $\partial U$, or, equivalently, in sufficiently small crosscut neighbourhoods. This is the content of Technical Lemma 1.

**Technical Lemma 1.** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Assume $U$ is PS, and let $\varphi\colon \mathbb{D} \to U$ be a Riemann map.*
*Then, for any component $C$ of $\partial U$, there exists a domain $\Omega_C$ such that $C \subset \Omega_C$, $\Omega_C$ is simply connected, $\Omega_C \cap U$ is connected, and $\Omega_C$ is disjoint from $P(f) \cap U$.*
*In addition, for all $\xi$ such that $Cl_R(\varphi, \xi) \cap \mathbb{C} \subset C$, the set $\varphi^{-1}(\Omega_C \cap U)$ contains a crosscut neighbourhood of $\xi$.*

If, additionally, $U$ is SPS, i.e. if there exists a simply connected domain $\Omega$ such that $\overline{U} \subset \Omega$, and

$$P(f) \cap \Omega \subset V,$$

then inverse branches can be defined around each component of $\partial U$ globally, i.e. for each component $C$ of $\partial U$ there exists a simply connected domain $\Omega_C$ such that all inverse branches are well-defined in $\Omega_C$. Moreover, all inverse branches are locally contracting with respect to the hyperbolic metric in a certain neighbourhood of $\partial U$ (see Technical Lemma 2) and satisfy the following property, which is crucial in the proof of Theorem 2.D.

**Definition 2.2.1. (Proper invertibility)** Let $f$ be a holomorphic function, and let $U$ be an invariant Fatou component. Let $z \in \partial U$. We say $f$ is *properly invertible* (at $z$ with respect to $U$) if, there exists $r > 0$ such that for every $w \in \partial U$ such that $f^n(w) = z$ there exists a branch $F_n$ of $f^{-n}$ which is well-defined in $D(z, r)$, and satisfies

$$F_n(D(z, r) \cap U) \subset U.$$

The definition of the inverse branches and their properties are collected in Technical Lemma 2. In the sequel, let $W := \mathbb{C} \setminus P(f)$, and denote by $\rho_W$ the hyperbolic metric in $W$. We use standard properties of the hyperbolic metric, which can be found e.g. in [CG93, Sect. I.4], [BM07].

**Technical Lemma 2.** *Let $U$ be a Fatou component satisfying the assumptions of Technical Lemma 1. If, additionally, $U$ is SPS, then the domain $\Omega_C$ can be chosen to satisfy the following properties.*

1. *$\Omega_C \subset W$, so $\Omega_C \cap P(f) = \emptyset$.*

2. *For all $z \in \partial U$, there exists a neighbourhood $D_z \subset W$ of $z$ such that all branches $F_n$ of $f^{-n}$ are well-defined in $D_z$, $F_n(D_z) \subset W$ and*

$$\rho_W(F_n(x), F_n(y)) \leq \rho_W(x, y), \quad \textit{for all } x, y \in D_z.$$

*3. For all $z \in \partial U$, $f$ is properly invertible at $z$ with respect to $U$.*

The end of the Section, which is not needed for the proofs of Theorems C and D, is dedicated to further comments on the relationship between proper invertibility and SPS, and the connection with the concept of local surjectivity.

### Proof of Technical Lemma 1

We assume $U$ to be PS. Then, by definition, there exists a domain $V$ such that $\overline{V} \subset U$ and $P(f) \cap U \subset V$. Since $\infty$ is accessible from $U$, we can assume, without loss of generality, that $\infty$ is accessible from $V$. Indeed, if $\infty$ is not accessible from $V$, take a curve $\gamma\colon [0,1) \to U$, such that $\gamma(0) \in V$ and $\gamma$ lands at $\infty$. Then, redefine $V$ to contain $\gamma$.

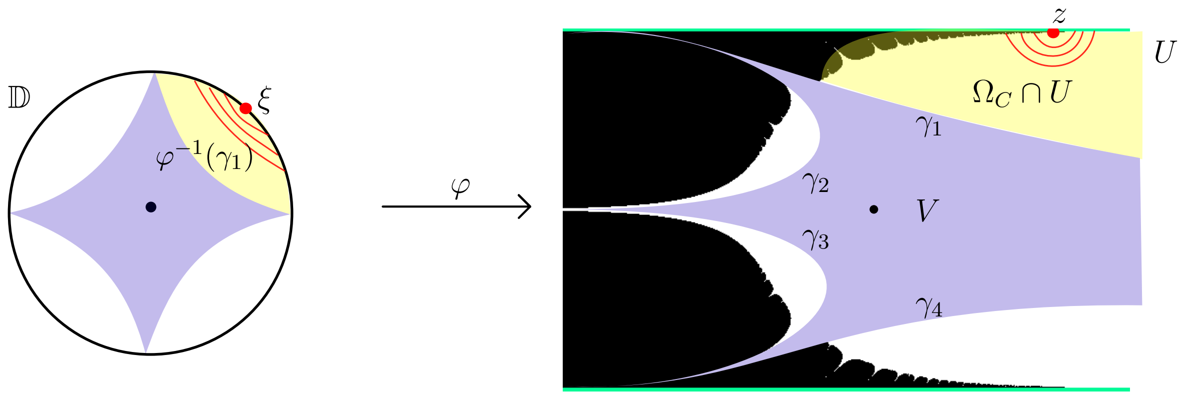



Figure 2.3: Set-up of the proof of Technical Lemma 1 for PS Fatou components.

Now, consider $\mathbb{C} \setminus \overline{V}$. Then, the boundary component $C$ is contained in a component of $\mathbb{C} \setminus \overline{V}$, say $\Omega_C$. By Theorem I.2.2, $\Omega_C$ is simply connected, because $\overline{V} \cup \{\infty\}$ is connected, and, since $\overline{V} \subset U$, $\Omega_C \cap U$ is connected. Since $P(f) \cap U \subset V$, it holds that $\Omega_C$ is disjoint from $P(f) \cap U$.

Next, let $\xi \in \partial\mathbb{D}$ be such that $Cl_\rho(\varphi, \xi) \cap \mathbb{C} \subset C$. We have to see that $\varphi^{-1}(\Omega_C \cap U)$ contains a crosscut neighbourhood of $\xi$. Without loss of generality, we assume that $V$ is bounded by a collection of disjoint curves $\{\gamma_i\}_{i \in I}$, each of them landing at $\infty$ from both ends. By the Correspondence Theorem II.4.6, each $\varphi^{-1}(\gamma_i)$ is a crosscut in $\mathbb{D}$ (possibly degenerate, i.e. such that both its endpoints in $\partial\mathbb{D}$ are the same).

To end the proof, we have to see that the crosscut corresponding to

$$\partial(\varphi^{-1}(\Omega_C \cap U)) \cap \mathbb{D} = \partial(\varphi^{-1}(\gamma_i)),$$

for some $i \in I$, is non-degenerate. Assume, on the contrary, that it is degenerate with common endpoint $\xi \in \partial\mathbb{D}$. By the Correspondence Theorem II.4.6, the two endpoints of $\gamma_i$ define the same access to infinity (the one corresponding to $\xi$), meaning that $\widehat{\gamma_i}$ is a Jordan curve in $\widehat{\mathbb{C}}$ and $\partial U$ is contained in the connected component of $\widehat{\mathbb{C}} \setminus \widehat{\gamma_i}$ which is not $\Omega_C$. Therefore, for any sequence $\{z_n\}_n \subset \Omega_C \cap U$ with $z_n \to \widehat{\partial} U$, we have $z_n \to \infty$. This is a contradiction with the fact that $\Omega_C$ is a neighbourhood of $C \subset \partial U$. This ends the proof of the Technical Lemma 1.

**Proof of Technical Lemma 2**

For SPS Fatou components, define $\Omega_C$ to be the connected component of

$$\Omega \smallsetminus \overline{V}$$

in which $C$ is contained (adjusting $V$ so that $\infty \in \overline{V}$). It is straightforward to see that $C \subset \Omega_C$, $\Omega_C$ is simply connected, $\Omega_C \cap U$ is connected, and $\Omega_C$ is disjoint from $P(f) \cap U$.

Now we prove that $f$ is locally expanding in $W = \mathbb{C} \smallsetminus P(f)$ with respect to the hyperbolic metric $\rho_W$. Note that $W$ is an open neighbourhood of $\partial U$ and $W$ is *backwards invariant* under $f$, i.e. $f^{-1}(W) \subset W$.

**Proposition 2.2.2. (Set of expansion)** *Under the assumptions of Technical Lemma 2, the following holds.*

1. $f\colon f^{-1}(W) \to W$ *is* locally expanding *with respect to the hyperbolic metric* $\rho_W$*, i.e.*

$$\rho_W(z) \leq \rho_W(f(z)) \cdot |f'(z)|, \quad \textit{for all } z \in f^{-1}(W).$$

2. *For all* $z \in \partial U$*, there exists a neighbourhood* $D_z \subset W$ *of* $z$ *such that all branches* $F_n$ *of* $f^{-n}$ *are well-defined in* $D_z$*,* $F_n(D_z) \subset W$ *and*

$$\rho_W(F_n(x), F_n(y)) \leq \rho_W(x, y), \quad \textit{for all } x, y \in D_z.$$

3. *Moreover, if* $z$ *and* $F_n(z)$ *belong to the same connected component of* $W$*, then there exists* $\lambda \in (0, 1)$ *such that*

$$\rho_W(F_n(x), F_n(y)) \leq \lambda \rho_W(x, y), \quad \textit{for all } x, y \in D_z.$$

*Proof.* Let us check that $W$ satisfies the required properties. Let $\rho_W$ denote the hyperbolic metric in $W$. Note that *a priori* $W$ cannot be assumed to be connected, so we define $\rho_W$ component by component. Indeed, each connected component $\widetilde{W}$ of $W$ is a hyperbolic domain, and hence admits a hyperbolic metric $\rho_{\widetilde{W}}$. Given $z \in W$, we define

$$\rho_W(z) \coloneqq \rho_{\widetilde{W}}(z),$$

where $\widetilde{W}$ stands for the connected component of $W$ with $z \in \widetilde{W}$. Given $z, w \in W$, the hyperbolic distance is defined as

$$\rho_W(z, w) \coloneqq \rho_{\widetilde{W}}(z, w),$$

if $z$ and $w$ lie in the same connected component $\widetilde{W}$ of $W$; and $\rho_W(z, w) = \infty$, otherwise.

1. Since $W$ does not contain singular values, given a connected component $W_1$ of $f^{-1}(W)$, $f\colon W_1 \to f(W_1)$ is a holomorphic covering. Note that $f(W_1)$ is a connected component of $W$. Indeed, $f(W_1)$ is connected, and hence contained in a connected

component $W_2$ of $W$. Since $W$ does not contain singular values, it holds $f(W_1) = W_2$.

By Schwarz-Pick lemma [CG93, Thm. I.4.1], $f$ is a local isometry, i.e. if $z \in W_1$,

$$\rho_{f^{-1}(W)}(z) = \rho_{W_1}(z) = \rho_{W_2}(f(z)) \left|f'(z)\right| = \rho_W(f(z)) \left|f'(z)\right|.$$

Since $f^{-1}(W) \subset W$, it holds

$$\rho_W(z) \leq \rho_{f^{-1}(W)}(z) = \rho_W(f(z)) \left|f'(z)\right|, \text{ for all } z \in f^{-1}(W).$$

2. Given $z \in \partial U$, we take $D_z$ to be a hyperbolic disk in $W$ of radius small enough so that $D_z$ is simply connected. Since $W$ is backwards invariant and $P(f) \cap W = \emptyset$, it follows that all branches $F_n$ of $f^{-n}$ are well-defined in $D_z$ and $F_n(D_z) \subset W$.

   Now, let $x, y \in D_z$. Since $D_z$ is a hyperbolic disk, it is hyperbolically convex, so there exists a geodesic $\gamma \subset D_z$ between $x$ and $y$, and $F_n(\gamma)$ is a curve joining $F_n(x)$ and $F_n(y)$. Hence, by statement 1,

$$\rho_W(F_n(x), F_n(y)) \leq \int_{F_n(\gamma)} \rho_W(s) ds = \int_\gamma \rho_W(F_n(t)) \left|F_n'(t)\right| dt =$$

$$= \int_\gamma \rho_W(F_n(t)) \frac{1}{\left|(f^n)'(F_n(t))\right|} dt \leq \int_\gamma \rho_W(t) dt = \rho_W(x, y),$$

   since $\gamma$ is taken to be a geodesic between $x$ and $y$.

3. Let $W_1$ be the connected component of $W$ in which $z$ and $F_n(z)$ lie. Hence, $f^{-n}(W_1) \cap W_1 \neq \emptyset$. We claim that any connected component of $f^{-n}(W_1)$ intersecting $W_1$ is strictly contained in $W_1$. Indeed, assume there exists $n \geq 1$ such that $f^n(W_1) = W_1$. Then, for the map $f^n$, $W_1$ is a neighbourhood of $z \in \mathcal{J}(f^n)$ for which

$$\bigcup_{m \geq 0} f^{n \cdot m}(W_1) = W_1 \subset W = \mathbb{C} \setminus P(f).$$

   Since $P(f)$ has more than one point, this would contradict the blow-up property of $\mathcal{J}(f^n)$.

   Therefore, $\rho_{W_1} < \rho_{f^{-n}(W_1)}$, and hence

$$\rho_W(w) < \rho_W(f^n(w)) \left|(f^n)'(w)\right|, \text{ for all } w \in f^{-n}(W_1) \cap W_1.$$

   Without loss of generality, let us assume that the neighbourhood $D_z$ of $z$ is compactly contained in $W_1$. Thus, the continuous function

$$0 < \frac{\rho_W(w)}{\rho_W(f^n(w)) \left|(f^n)'(w)\right|} < 1$$

   reaches a maximum in $D_z$. Therefore, there exists $\lambda \in (0, 1)$ such that

$$\rho_W(F_n(x), F_n(y)) \leq \lambda \rho_W(x, y), \quad \text{for all } x, y \in D_z,$$

   as desired.

Thus, the proof of Proposition 2.2.2 is complete. □

**Remark 2.2.3. (Strict expansion)** One may ask if this open set $W$ can be improved so that the function is strictly expanding on it. This is always the case for hyperbolic and subhyperbolic functions (see e.g [MB12, BFRG15, RGS17]). The answer is negative for arbitrary SPS Fatou components, as it can be seen for the doubly parabolic Baker domains of $f(z) = z + e^{-z}$ (see Chapter 1).

Finally, to end the proof we have to see that for all $z \in \partial U$, there exists $r > 0$ such that all branches $F_n$ of $f^{-n}$ with $F_n(z) \in \partial U$ are well-defined in $D(z, r)$, and satisfy

$$F_n(D(z, r) \cap U) \subset U.$$

We denote by $C$ the connected component of $\partial U$ with $z \in C$, and consider the neighbourhood $\Omega_C$ defined previously. Then, the proper invertibility follows from the properties of the set $\Omega_C$. Indeed, since $\Omega_C$ is simply connected and disjoint from $P(f)$, any inverse branch defined locally at $z \in \partial U$ extends conformally to $\Omega_C$. By construction, $\Omega_C \cap U$ is connected, and so is $F_n(\Omega_C \cap U)$. By the total invariance of the Julia and the Fatou set, it follows that $F_n(\Omega_C \cap U) \subset U$, as desired. This ends the proof of Technical Lemma 2.

#### Proper invertibility, strongly postsingular separation and local surjectivity

Finally, in this section we discuss the necessity of the condition of being SPS. To prove Theorem 2.D, not only we need to have all inverse branches well-defined locally around every point in $\partial U$, which could be achieved simply assuming

$$P(f) \cap \partial U = \emptyset,$$

but also to have proper invertibility.

Then, the following question arises: is it sufficient to add the assumption of $P(f) \cap \partial U = \emptyset$ to the PS condition to have proper invertibility? The answer is negative in general, so the hypothesis of being SPS is necessary. We prove this in Proposition 2.2.4.

We note that, if one could prove that any PS Fatou component with $P(f) \cap \partial U = \emptyset$ is, automatically, SPS, then results of Theorem 2.D (i.e. accessibility and density of periodic boundary points, and density of escaping boundary points) would hold only assuming the PS condition and $P(f) \cap \partial U = \emptyset$.

**Proposition 2.2.4. (Characterizations of proper invertibility)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Assume $U$ is PS and $P(f) \cap \partial U = \emptyset$. Then, the following are equivalent.*

(a) *$U$ is SPS.*

(b) *For each connected component $C$ of $\partial U$, there exists an open neighbourhood $\Omega_C$ of $C$ in which every branch $F_n$ of $f^{-n}$ is well-defined, and, if there exists $z \in \Omega_C \cap U$ such that $F_n(z) \in U$, then $F_n(\Omega_C \cap U) \subset U$.*

We shall rewrite the previous proposition in terms of boundary components and filled closures, defined as follows.

**Definition 2.2.5. (Filled closure)** Let $X \subset \mathbb{C}$ be any connected set in the complex plane. We define the *filled closure* of $X$ as

$$\text{fill}(X) := \overline{X} \cup \left(\text{components } U \text{of } \mathbb{C} \smallsetminus \overline{X} \text{ such that } \infty \text{ is not accessible from } U\right).$$

We note that $\text{fill}(X)$ is always closed, independently of whether $X$ is closed or not.

We observe that $U$ being SPS is equivalent to

$$P(f) \cap \text{fill}(U) \subset U, \quad \text{and to} \quad P(f) \cap \text{fill}(\partial U) = \emptyset.$$

Recall that $\text{fill}(A)$ is closed and does not include the unbounded components of $\mathbb{C} \smallsetminus \overline{A}$ from which $\infty$ is accessible. In particular, $U \cap \text{fill}(\partial U) = \emptyset$, if $\infty$ is accessible from $U$. Hence, in this case, being SPS is equivalent to

$$P(f) \cap \text{fill}(C) = \emptyset,$$

for all connected components $C$ of $\partial U$. See Figure 2.5 below to have a geometric intuition.

**Proposition 2.2.6. (Characterization of proper invertibility for boundary components)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Assume $U$ is PS and $P(f) \cap \partial U = \emptyset$. Let $C$ be a connected component of $\partial U$. Then, the following are equivalent.*

(a) *For all $z \in C$, $f$ is properly invertible at $z$ with respect to $U$.*

(b) $P(f) \cap \text{fill}(C) = \emptyset$.

(c) *There exists an open simply connected neighbourhood $\Omega_C$ of $C$ in which all branches $F_n$ of $f^{-n}$ are well-defined, and, either $F_n(\Omega_C \cap U) \cap U = \emptyset$, or $F_n(\Omega_C \cap U) \subset U$.*

*Proof.* We address first the equivalence between (b) and (c). To see that (b) implies (c), observe that, by Technical Lemma 2, there exists a simply connected domain $\Omega_C$, disjoint from $P(f)$, such that $C \subset \Omega_C$ and $\Omega_C \cap U$ is connected. Hence, $\Omega_C \cap U$ is simply connected, for being the connected intersection of two simply connected sets. In such a domain, all branches $F_n$ of $f^{-n}$ are well-defined. Moreover, since $\Omega_C \cap U$ is connected, so $F_n(\Omega_C \cap U)$ is connected. By the total invariance of the Fatou and Julia sets, it follows that either $F_n(\Omega_C \cap U) \cap U = \emptyset$ or $F_n(\Omega_C \cap U) \subset U$. Hence, (b) implies (c).

Conversely, we note that if all inverse branches are well-defined in $\Omega_C$, then $P(f) \cap \Omega_C = \emptyset$. In particular, since $\Omega_C$ is a simply connected neighbourhood of $C$, it must contain fill($C$) (recall that fill($C$) consists of $C$ and the components of its complement from which infinity is not accessible; hence, a simply connected neighbourhood of $C$ includes fill($C$)). Hence, we have

$$P(f) \cap \mathrm{fill}(C) = \emptyset.$$

Therefore, (c) implies (b).

It is left to prove the equivalence between (a) and (c). We note that (c) implies (a) trivially. Next, we prove that, if (c) does not hold, neither does (a).

By assumption, $P(f) \cap \partial U = \emptyset$. Hence, for every $z \in C$, there exists a sufficiently small disk $D(z,r)$, $r = r(z) > 0$, such that every branch $F_n$ of $f^{-n}$ is well-defined in $D(z,r)$. On the other hand, since $U$ is PS, by Technical Lemma 1, there exists a simply connected domain $\Omega_C$, such that $\Omega_U := \Omega_C \cap U$ is connected, simply connected and disjoint from $P(f) \cap U$.

Since we are assuming that (c) does not hold, we claim that there exists a point $z_0 \in C$ and $n \geq 1$, such that a branch $F_n$ of $f^n$, well-defined in $D(z_0, r)$ does not extend conformally to $\Omega_U$. Indeed, if (c) does not hold, then any neighbourhood $\Omega_C$ of $C$ given by the Technical Lemma 1 would meet $P(f)$ (outside $\overline{U}$). Equivalently, for such $\Omega_C$ and $\Omega_U := \Omega_C \cap U$, any neighbourhood of $\overline{\Omega_U}$ is multiply connected (otherwise it would be a simply connected neighbourhood of $C \subset \Omega_C$ disjoint from $P(f)$, so (c) would hold).

In particular, there exists a point $z_0 \in C$ and $r > 0$ so that $D(z_0, r) \cup \Omega_U$ is multiply connected and $D(z_0, r) \cup \Omega_U$ surrounds points in $P(f)$. Thus, there exists at least a branch $F_n$ of $f^n$, well-defined in $D(z_0, r)$ does not extend conformally to $\Omega_U$, as claimed.

Finally, we have to prove that, for such $F_n$, it does not hold

$$F_n(D(z,r) \cap U) \subset U.$$

Indeed, take $u_0 \in D(z_0, r) \cap U$, and consider the conformal extension of $F_n$ to $\Omega_U$ with basepoint $u_0$. Then, $F_n|_{D(z_0,r)}$ is univalent, as well as $F_n|_{\Omega_U}$. However, since $F_n$ does not extend conformally to $\Omega_U$, $F_n|_{D(z_0,r)\cup\Omega_U}$ is a multivalued function. Hence, there exists $w \in D(z_0, r) \cap \Omega_U$ such that $w = f^n(w_1) = f^n(w_2)$, with $w_1 \in F_n(D(z_0, r))$, $w_2 \in F_n(\Omega_U)$. Hence $w_1 \notin U$, because otherwise $w_1 \in \Omega_U$ and $F_n$ would be multivalued at $w \in \Omega_U$. Therefore $f^n$ is not properly invertible with respect to $U$, as desired. □

From Proposition 2.2.6 we deduce Proposition 2.2.4.

*Proof of Proposition 2.2.4.* Observe that $U$ being SPS is equivalent to say that, for every boundary component $C \subset \partial U$,

$$P(f) \cap \mathrm{fill}(C) = \emptyset.$$

Then, the equivalence (b)-(c) in Proposition 2.2.6 ends the proof. □

Finally, we shall give an intuition of how a non-SPS Fatou component would look like. First, let us look at the following example of a rational map $f$ which is not properly invertible.

**Example 2.2.7. (Non-properly invertible function,** [Sch97]**)** The function

$$f(z) = \frac{64}{(z+3)(z-3)^2} - 3$$

considered in [Sch97] is not locally surjective with respect to the invariant attracting basin $U$ (in black in Fig. 2.4). Indeed, the neighbourhood marked in Figure 2.4 is mapped conformally under $f$ onto the other marked neighbourhood. It is clear that the corresponding inverse branch send points in $U$ to points in $U$ and to points outside $U$ (in its preimage $V$) simultaneously. For a more precise description of the dynamics, we refer to [Sch97].

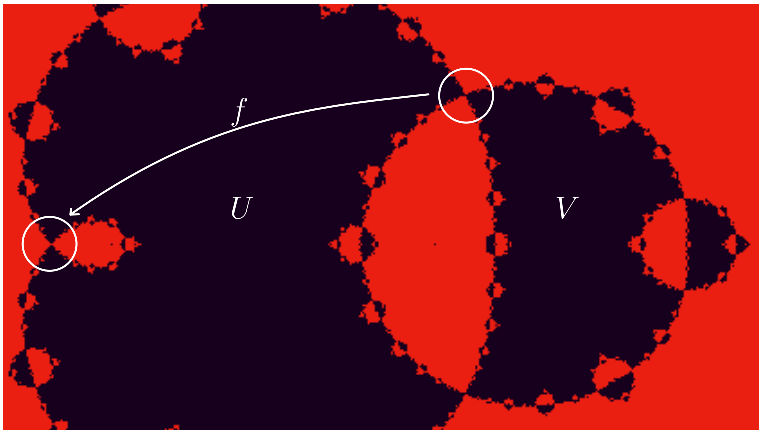



Figure 2.4: Dynamical plane of $f(z) = \frac{64}{(z+3)(z-3)^2} - 3$, which is not locally surjective.

Note that the previous Fatou component satisfies having a bi-accessible boundary point. This is never the case for Fatou components of transcendental entire functions: every finite point in the boundary has a unique access from the Fatou component [Bar08, Thm. 3.14]. Moreover, if $U$ is ergodic, every finite point $z \in \partial U$ is contained in a unique cluster set (Theorem 2.A). Hence, it seems plausible that any PS Fatou component is, automatically, SPS.

Indeed, a Fatou component not satisfying this condition would have a complicated boundary: there would exist a connected component $C$ of $\partial U$ such that

$$P(f) \cap \text{fill}(C) \neq \emptyset.$$

Since we are assuming that $P(f) \cap C = \emptyset$, it follows that there would exist a connected component $V$ of $\mathbb{C} \smallsetminus C$ for which infinity is not accessible. By invariance of $U$ and normality, $V$ must be a Fatou component; either an attracting basin, a preimage of it, or an escaping wandering domain.

Moreover, if $U$ is ergodic, by Theorem 2.A, any connected component $C$ of $\partial U$ is either a cluster set or it is contained in a cluster set. Hence, if $U$ is a non-SPS Fatou component, there would exist $\xi \in \partial\mathbb{D}$ such that

$$P(f) \cap \text{fill}(Cl_{\mathbb{C}}(\varphi, \xi)) \neq \emptyset.$$

See Figure 2.5.

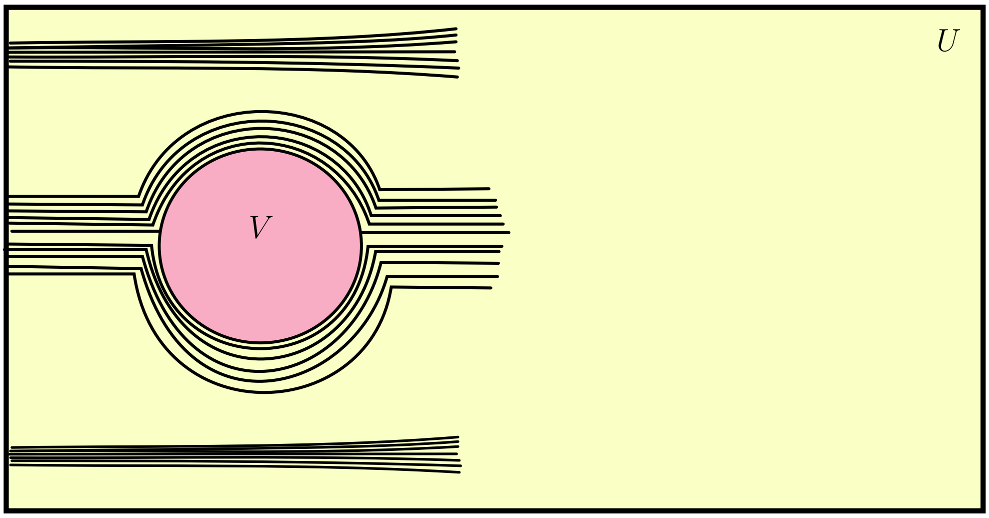



Figure 2.5: Schematic representation of how a non-SPS Fatou component would look like. One may imagine the boundary of $U$ (in $\mathbb{C}$) as a collection of curves (as in a Cantor bouquet). One of them, corresponding to, say, $Cl_{\mathbb{C}}(\varphi, \xi)$ is not a Jordan curve, but it encloses a bounded region, $V$, which is a Fatou component, by normality. For $U$ to be non-SPS, this enclosed Fatou component $V$ must contain a postsingular value. It is precisely the presence of this postsingular value in fill$(Cl_{\mathbb{C}}(\varphi, \xi))$ what prevents the definition of inverse branches around $Cl_{\mathbb{C}}(\varphi, \xi)$.

**Question 2.2.8.** *Let $f$ be a transcendental entire function, and let $U$ be an (ergodic) invariant Fatou component, such that $\infty$ is accessible from $U$. If $U$ is PS and $P(f) \cap \partial U = \emptyset$, then is $U$ SPS?*

Finally, we discuss the relation of proper invertibility with local surjectivity, a closely related notion used in [Sch97, Ima14], to study the accessibility of periodic points in the boundary of invariant Fatou components. The definition of local surjectivity reads as follows.

**Definition 2.2.9. (Local surjectivity)** Let $f$ be a holomorphic function, and let $U$ be an invariant Fatou component. Let $z \in \partial U$. We say $f$ is *locally surjective* (at $z$ with respect to $U$) if there exists $r > 0$ such that

$$f(D(z, r) \cap U) = f(D(z, r)) \cap U.$$

It is easy to see that $f$ being locally surjective on $\partial U$ (understood as it is locally surjective with respect to $U$ at every point of $\partial U$) is equivalent to $f$ being properly invertible on $\partial U$ (again, understood pointwise with respect to $U$). However, since we are interested in inverse branches, the concept of proper invertibility is more convenient.

## 2.3 Postsingularly separated Fatou components and the associated inner function. Theorem 2.C

We prove the following version of Theorem 2.C, which gives more details on the behaviour of boundary orbits for the associated inner function. One should view this general version of Theorem 2.C as a significantly stronger version of Lemma II.5.2 for PS Fatou components.

We use the notation $\Theta_\infty$ introduced in Section II.5.3. Let us recall the definition.

$$\Theta_\infty := \{\xi \in \partial\mathbb{D} \colon \varphi^*(\xi) = \infty\}$$

$$\partial\mathbb{D} \smallsetminus \Theta_\infty = \{\xi \in \partial\mathbb{D} \colon Cl_R(\varphi, \xi) \neq \{\infty\}\}$$

**Theorem II.3. (General version)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g := \varphi^{-1} \circ f \circ \varphi$ be the corresponding associated inner function. Assume $U$ is PS. Then, the following holds.*

(a) (Finite principal points avoid singularities) *Let $\xi \in \partial\mathbb{D}$. If $\xi \in \Theta_\mathbb{C}$, then $\xi$ is not a singularity of $g$ and $g(\xi) \in \Theta_\mathbb{C}$.*

*In particular, if, for some $n \geq 1$, $\xi \in \partial\mathbb{D}$ is a singularity for $g^n$, then $\xi \in \Theta_\infty$.*

(b) (Few singularities) *For almost every $\xi \in \partial\mathbb{D}$ (with respect to the Lebesgue measure), there exists $r := r(\xi) > 0$ such that $g$ is holomorphic in $D(\xi, r)$. In particular, the set $E(g)$ has zero Lebesgue measure.*

(c) (Backward and forward orbit at typical boundary points) *For almost every $\xi \in \partial\mathbb{D}$ (with respect to the Lebesgue measure), there exists $r := r(\xi) > 0$ such that all branches $G_n$ of $g^{-n}$ are well-defined in $D(\xi, r)$, for all $n \geq 0$. Moreover, for every $n \geq 1$, there exists $\rho := \rho(\xi, n) > 0$ such that $g^n$ is holomorphic in $D(\xi, \rho)$.*

(d) (Radial limit $g^*$ at a singularity) *Let $\xi \in \partial\mathbb{D}$ be a a singularity for $g$, and assume $g^*(\xi)$ exists. Then, either $g^*(\xi) \in \mathbb{D}$, and $\varphi(g^*(\xi)) \in U$ is an asymptotic value for $f$; or $g^*(\xi) \in \Theta_\infty$.*

*Proof.* We prove the different statements separately.

(a) Let $\xi \in \partial\mathbb{D} \smallsetminus \Theta_\infty$. By Lemma II.4.8, $Cl_\mathbb{C}(\varphi, \xi)$ is connected, so is $f(Cl_\mathbb{C}(\varphi, \xi))$, and hence it is contained in a component $C$ of $\partial U$. Since $U$ is assumed to be PS, by the Technical Lemma 1, there exists a domain $\Omega_C$ such that $C \subset \Omega_C$, $\Omega_C$ is simply connected, $\Omega_C \cap U$ is connected, and $\Omega_C$ is disjoint from $P(f) \cap U$. Moreover, $\Omega_C$ can be chosen so that $\varphi^{-1}(\Omega_C \cap U)$ is a crosscut neighbourhood of some $\zeta \in \partial\mathbb{D}$. Note that there are no postsingular values of $g$ in $\varphi^{-1}(\Omega_C \cap U)$.

Note that, since $\xi \in \partial\mathbb{D} \smallsetminus \Theta_\infty$, we can choose $z \in Cl_R(\varphi, \xi) \cap \mathbb{C}$. Since $z$ is a finite principal point, for any $r > 0$, there exists a null-chain $\{D_n\}_n \subset D(z, r)$. We choose $r$ small enough so that $f(D(z, r)) \subset \Omega_C$. Hence, for all $n \geq 0$, $f(D_n)$ is a crosscut of $U$ contained in $\Omega_C$.

Since $\Omega_C \cap U$ is simply connected and disjoint from $P(f) \cap U$, all inverse branches are well-defined in $\Omega_C \cap U$. In particular, there exists a branch $F_1$ of $f^{-1}$ and an inverse branch $G_1$ of $g^{-1}$ such that

$$\varphi^{-1}(F_1(\Omega_C \cap U)) = G_1(\varphi^{-1}(\Omega_C \cap U))$$

contains a crosscut neighbourhood of $\xi$. Hence, any sufficiently small crosscut neighbourhood $\mathbb{D}_D$ around $\xi$ is mapped conformally under $g$ to a crosscut neighbourhood $g(\mathbb{D}_D)$ around $\zeta$. Thus, $\overline{g(\mathbb{D}_D)} \neq \mathbb{D}$. This already implies that $\xi$ is not a singularity (Lemma II.3.11). By Lemma II.5.2, it is clear that $g(\xi) \in \partial\mathbb{D} \smallsetminus \Theta_\infty$.

Finally, note that, if $\zeta$ is a singularity for $g$, then $\zeta \in \Theta_\infty$. The last statement follows directly from the backwards invariance of $\Theta_\infty$. Indeed, if $\zeta$ is a singularity for the inner function $g^n$ ($n \geq 1$ taken minimal), then $g^{n-1}(\zeta)$ is a singularity for $g$. Then, $g^{n-1}(\zeta) \in \Theta_\infty$, so $\zeta \in \Theta_\infty$.

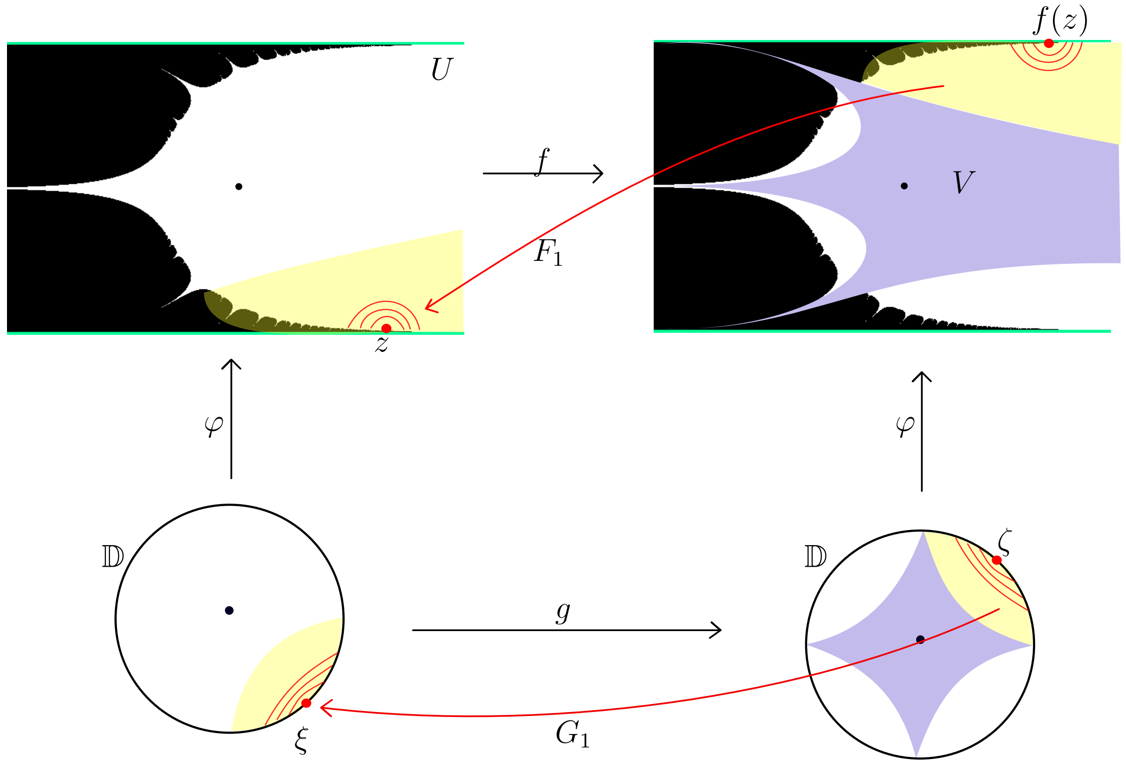



Figure 2.6: Set-up of the proof of (a) for PS Fatou components. Given $\xi \in \partial\mathbb{D} \smallsetminus \Theta_\infty$, we find a crosscut neighbourhood around it which is mapped conformally onto another crosscut neighbourhood, implying that $\xi$ is not a singularity for $g$.

(b) Since $E(g) \subset \Theta_\infty$, and $\lambda(\Theta_\infty) = 0$, it follows that $\lambda(E(g)) = 0$. Hence, for $\lambda$-almost every $\xi$, there exists $r = r(\xi) > 0$ such that $g$ is holomorphic in $D(\xi, r)$.

(c) Again, the first statement follows from the same argument than in (a), applied to the set of full measure $\partial\mathbb{D} \smallsetminus \Theta_\infty$. Indeed, for every point $\xi \in \partial\mathbb{D} \smallsetminus \Theta_\infty$, there exists a boundary component $C$ such that $Cl_{\mathbb{C}}(\varphi, \xi) \subset C$. by the Technical Lemma 1, there exists a domain $\Omega_C$ such that $C \subset \Omega_C$, $\Omega_C$ is simply connected, $\Omega_C \cap U$ is connected, and $\Omega_C$ is disjoint from $P(f) \cap U$. Moreover, $\varphi^{-1}(\Omega_C \cap U)$ contains a crosscut neighbourhood $N_\xi$ of $\xi$. Since $\Omega_C \cap U$ is simply connected and disjoint from $P(f) \cap U$, all inverse branches $F_n$ of $f^n$ are well-defined in $\Omega_C \cap U$. In particular, for each branch $F_n$ of $f^{-n}$, there exists an inverse branch $G_n$ of $g^{-n}$ such that

$$\varphi^{-1}(F_n(\Omega_C \cap U)) = G_n(\varphi^{-1}(\Omega_C \cap U)).$$

Thus, all inverse branches $G_n$ of $g^n$ are well-defined in $N_\xi$. By considering $g$ as its maximal meromorphic extension, and using Schwarz reflection, we get that there exists $r := r(\xi) > 0$ such that all branches $G_n$ of $g^{-n}$ are well-defined in $D(\xi, r)$, for all $n \geq 0$.

The second statement follows from the forward invariance of $\partial\mathbb{D} \setminus \Theta_\infty$, by induction. Indeed, for $n = 1$, if $\xi \in \partial\mathbb{D} \setminus \Theta_\infty$, then $\xi$ is not a singularity for $g$, so there exists a disk around $\xi$ in which $g$ is holomorphic. Now, for all $n \geq 2$, assume $g^{n-2}$ is holomorphic in a neighbourhood of $\xi$. By (a), $g^{n-1}(\xi) \in \partial\mathbb{D} \setminus \Theta_\infty$, so $g^{n-1}(\xi)$ is not a singularity of $g$, meaning that there exists a neighbourhood of $g^{n-1}(\xi)$ in which $g$ is holomorphic. This already implies the existence of a neighbourhood of $\xi$ in which $g^n$ is holomorphic.

(d) We let $\xi \in \partial\mathbb{D}$ be a singularity, and assume $g^*(\xi)$ exists. There are two possibilities: either $g^*(\xi) \in \mathbb{D}$, or $g^*(\xi) \in \partial\mathbb{D}$. In the first case, it is clear that $\varphi(g^*(\xi))$ must be an asymptotic value for $f$. We shall prove that, in the second case, $g^*(\xi) \in \Theta_\infty$. Assume, on the contrary, that $g^*(\xi) \in \partial\mathbb{D} \setminus \Theta_\infty$. Then, by (c), for some $r > 0$, all branches of $g^{-1}$ are well-defined in $D(g^*(\xi), r)$. This is a contradiction because $\xi$ was assumed to be a singularity, and hence it cannot be mapped locally homeomorphically to $g^*(\xi)$.

□

**Remark 2.3.1.** We shall make the following remarks on Theorem II.3.

1. On [BFJK17, Prop. 2.7] it is proved that $E(g) \subset \overline{\Theta_\infty}$, for a general invariant Fatou component (i.e. without the PS assumption). Hence, (a) show how the result can be strengthened to $E(g) \subset \Theta_\infty$, for PS Fatou components. Taking into account that $\lambda(\Theta_\infty) = 0$ and $\lambda(\overline{\Theta_\infty}) = 1$ for a wide class of Fatou components, we believe that our result is a noteworthy improvement.

2. Regarding (c), we note that, for almost every $\xi$, inverse branches $G_n$ of $g^n$ are well in a disk of fixed radius (depending only on $\xi$, but not on $n$). However, when iterating forward, we can only ensure that $g^n$ is holomorphic on a disk whose radius depends on $n$. In the general case, the result cannot be improved. Indeed, if $g^*$ is ergodic and $g|_\mathbb{D}$ has infinite degree, it is shown in [BD99, Lemma 8], [Bar08, Thm. 1.4] that

$$\overline{\bigcup_{n\geq 0} E(g^n)} = \partial\mathbb{D},$$

where $E(g^n)$ stands for the set of singularities of the inner function $g^n$, as defined in Definition II.1.4.

Hence, in general, there is no open disk around a boundary point which is never mapped to a singularity of $g$.

3. Finally, as noted in Section II.3.2, there are two distinct ways of considering iteration in $\partial\mathbb{D}$ for a given inner function $g$. On the one hand, the approach followed in [BD99, Bar08] consists of truncating the orbit of a point when it falls into a singularity, as in the iteration of meromorphic functions in $\mathbb{C}$.

   On the other hand, there is the approach of [DM91] of considering iteration on the set
   $$\{\xi \in \partial\mathbb{D} \colon (g^*)^n(\xi) \text{ exists for all } n \geq 0\},$$
   which has full Lebesgue measure in $\partial\mathbb{D}$. This procedure allows us to iterate at singularities, as long as their radial limit under $g$ is well-defined.

   Using this approach, (d) tells us that, whenever we can iterate at a singularity, its orbit either eventually enters $\mathbb{D}$ and hence converges to the Denjoy-Wolff point, or its orbit is completely contained in $\Theta_\infty$.

## 2.4 Dynamics on the boundary of unbounded invariant Fatou components. Theorem 2.D

Finally, we use the machinery developed in the previous sections to prove this more general version of Theorem 2.D.

**Theorem II.4. (General version)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant Fatou component, such that $\infty$ is accessible from $U$. Assume $U$ is SPS. Then, the following holds.*

(a) *Periodic points in $\partial U$ are accessible from $U$.*

(b) *If a component $C$ of $\partial U$ contains a periodic point, then every other point in $C$ is escaping.*

(c) *If, additionally, $U$ is recurrent, then periodic points and escaping points are dense in $\partial U$.*

**Remark 2.4.1.** The previous statement deserves some comments.

- *(Radial limit at periodic points)* Theorem 2.D(a) states that, given a periodic point $p \in \partial U$, there exists $\xi \in \partial\mathbb{D}$ such that $\varphi^*(\xi) = p$. If $U$ is assumed additionally to be ergodic, such $\xi$ is unique, by Theorem 2.A, and it is radially periodic under $g$. The fact that $\xi$ is unique implies that it has exactly the same period as $p$.

  Furthermore, it is well-known that the boundary map $g \colon \partial\mathbb{D} \to \partial\mathbb{D}$ of any inner function of finite degree (i.e. of a finite Blaschke product) is semi-conjugate to the shift map $\sigma_d$ in the space $\Sigma_d$ of sequences of $d$ symbols (see e.g. [IU23, Sect. 8.1]). Therefore, each periodic point for $\sigma_d$ in $\Sigma_d$ (that is, periodic sequences) corresponds to a periodic point for $g$ in $\partial\mathbb{D}$, except for the ones corresponding to $\overline{0}$ and $\overline{d-1}$, which are identified. When $f|_U$ has finite degree, this gives an upper bound on the number of periodic points of a given period in $\partial U$. Indeed, since $g|_{\partial\mathbb{D}}$ has exactly $d^n - 1$ periodic points of period $n$, there are at most $d^n - 1$ periodic points of period $n$ in $\partial U$.

- It follows trivially from Theorem 2.D(b) that any boundary connected component of a SPS Fatou component can have at most one periodic point.
- *(Accessibility of escaping points)* In the case of escaping points, we may ask what can be said about their accessibility from $U$. In Chapter 1 it is proven that the Baker domains of $f(z) = z + e^{-z}$ have no accessible escaping point. However, the Baker domain of $f(z) = z + 1 + e^{-z}$ has accessible escaping points, as it follows easily from [Evd16]. Hence, it remains as an open problem to find conditions under which no escaping point is accessible from the Baker domain.

*Proof of Theorem 2.D.* We prove the different statements separately.

(a) *Periodic points in $\partial U$ are accessible from $U$.*

First note that, if $U$ is SPS, any periodic point in $\partial U$ must be repelling. Indeed, attracting and Siegel periodic points lie in the Fatou set, while parabolic and Cremer periodic points lie in $P(f)$.

Let $p$ be a periodic point in $\partial U$, which is repelling, and assume $f^n(p) = p$. Let $C$ be the connected component of $\partial U$ containing $p$. By the Technical Lemma 2, there exists a simply connected domain $\Omega_C$ containing the connected component $C \subset \partial U$, such that $\Omega_C \cap P(f) = \emptyset$, and $\Omega_C \cap U$ is connected.

Let $F_n$ be the branch of $f^{-n}$ fixing $p$. It extends conformally to $\Omega_C$ and, by the Technical Lemma 2, $F_n(\Omega_C \cap U) \subset U$. Note that, not only $F_n$ is well-defined in $\Omega_C$, but its iterates $F_n^m$, for all $m \geq 0$.

Let $r > 0$ be such that $D(p, r) \subset \Omega_C$. Since $p$ is repelling, choosing $r$ smaller if needed, we can assume $F_n(D(p, r)) \subset D(p, r)$.

Let us choose $z_0 \in D(p, r) \cap U$ and define $z_m := F_n^m(z_0) \in D(p, r) \cap U$. Since $\Omega_C \cap U$ is connected, there exists a curve $\gamma \subset \Omega_C \cap U$ connecting $z_0$ and $z_1$. Observe that $\{F_n^m\}_m$ is well-defined and normal in $\Omega_C$, because $F_n^m(\Omega_C) \subset W$, for all $m \geq 0$ (where $W = \mathbb{C} \smallsetminus P(f)$, as introduced in Sect. 2.2, which is backwards invariant).

Thus, any limit function $g$ must be constantly equal to $p$ in $D(p, r) \cap \Omega_C$, so $F_n^m \to g \equiv p$ uniformly on compact subsets of $\Omega_C$. In particular, $F_n^m|_\gamma \to p$ uniformly. Hence,

$$\bigcup_{m \geq 0} F_n^m(\gamma)$$

is a curve in $U$ landing at $p$, showing that $p$ is accessible from $U$, as desired.

(b) *If a component $C$ of $\partial U$ contains a periodic point, then every other point in $C$ is escaping.*

Let $p \in \partial U$ be a periodic point of $f$ (which must be repelling), and denote by $C$ the connected component of $\partial U$ for which $p \in C$. We have to prove that

$$C \smallsetminus \{p\} \subset \mathcal{I}(f).$$

Without loss of generality, assume $p$ is fixed by $f$. By the Technical Lemma 2, there exists an open neighbourhood of $C$, say $\Omega_C$, in which the branch $F_1$ of $f^{-1}$ fixing $p$ is well-defined. In fact, $\{F_1^n\}_n$ is well-defined in $\Omega_C$ and, as in the proof of (a), for every compact set $K \subset \Omega_C$, we have $F_1^n|_K \to p$ uniformly. Moreover, there exists $r > 0$ small enough so that $D(p, r) \subset \Omega_C$ and $F_1(D(p, r)) \subset D(p, r)$ (see Fig. 2.7a).

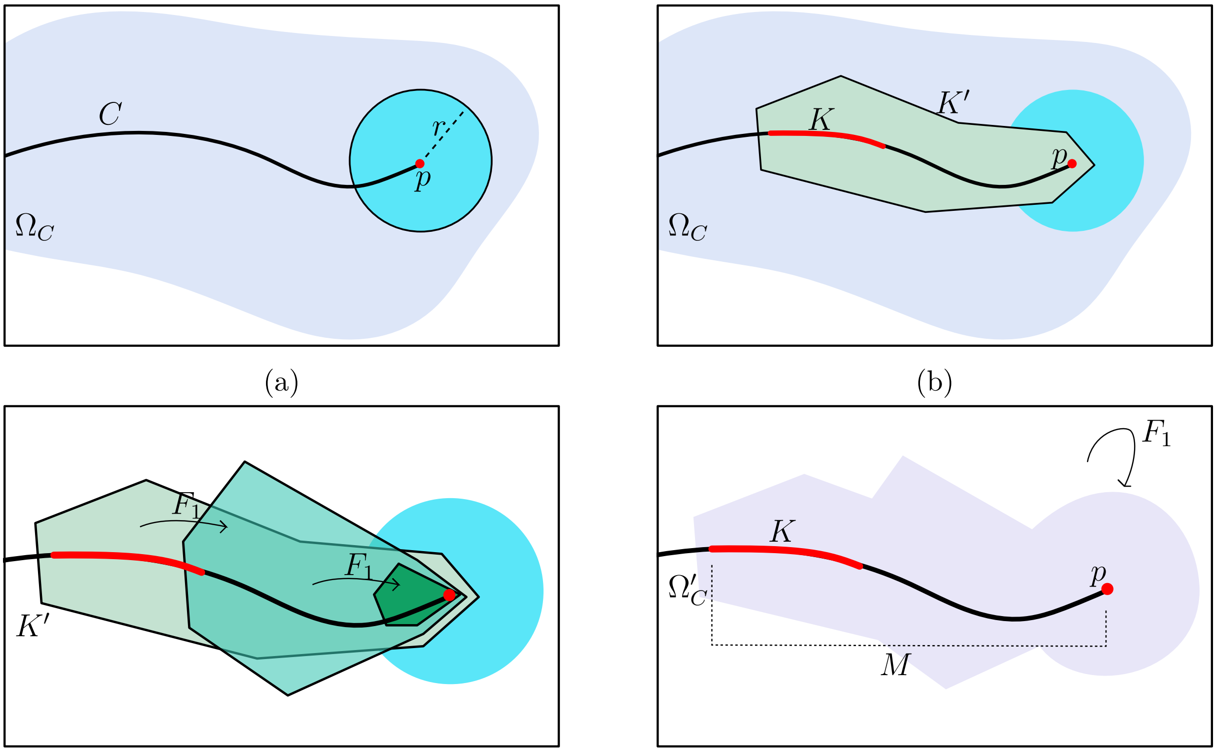



Figure 2.7: Steps on the proof of (b).

Now, let $z \in C \setminus \{p\}$. To prove that $z$ is escaping, we have to see that, for any compact set $K \subset C$, there exists $n_0$ such that $f^n(z) \notin K$, for all $n \geq n_0$. Since $p$ is assumed to be fixed, by continuity it follows that $f(C) \subset C$. Hence $f^n(z) \in C$ and, since $C$ is unbounded, it is enough to show that $z$ escapes from any compact set in $C$.

Hence, let us fix $K$ compact subset of $C$ and let us show that $z$ escapes from $K$. To do so, we will construct a domain $\Omega'_C \subset \Omega_C$, forward invariant under $F_1$, containing both $p$ and $K$. We will show that, if $z$ does not escape from $K$, then $\{f^n(z)\}_n$ should be entirely contained in $\Omega'_C$. Once we are in this situation, we will reach a contradiction using standard arguments based on Schwarz-Pick lemma.

We start by constructing the set $\Omega'_C$. To do so, we choose $K'$ compact connected subset of $\Omega_C$ such that $K \subset \text{Int } K'$ and $p \in K'$ (see Fig. 2.7b). Moreover, without loss of generality, we can choose $K'$ so that there is a connected component of $C \cap K'$ containing both $K$ and $p$. On the other hand, since $K'$ is a compact subset of $\Omega_C$, we have $F_1^n|_{K'} \to p$ uniformly, so there exists $N$ such that $F_1^N(K') \subset D(p, r)$.

Let us define the following sets:

$$V := \{z \in \text{Int } K' \colon F_1^n(z) \in \Omega_C, \text{ for all } n \leq N\} \cup D(p,r), \quad \Omega'_C := \bigcup_{n=0}^{N} F_1^n(V).$$

We note, on the one hand, that

$$C \cap \text{Int } K' \subset V,$$

because, as $f(C) \subset C$, points in $C \cap \text{Int } K'$ do not leave $\Omega_C$ under iteration. In particular, the connected component of $C \cap K'$ containing both $K$ and $p$ is in $V$. Hence, either $V$ is a connected open set, or $V$ can be redefined to be the connected component of $V$ containing $K$. In both cases, $K \subset V$ (see Fig. 2.7c).

Moreover, $\Omega'_C$ is also an open connected set, since it is the union of open connected sets, all containing $p$. By definition, it is forward invariant under $F_1$, and $K$ is compactly contained in $\Omega'_C$. Observe that both $F_1$ is well-defined and univalent in $\Omega'_C$.

Since $F_1(\Omega'_C) \subset \Omega'_C$, we have, by Schwarz-Pick lemma,

$$\rho_{\Omega'_C}(p, F_1(w)) = \rho_{\Omega'_C}(F_1(p), F_1(w)) < \rho_{\Omega'_C}(p, w),$$

for all $w \in \Omega'_C$. Equivalently, if $w, f(w) \in \Omega'_C$, it holds

$$\rho_{\Omega'_C}(p, w) < \rho_{\Omega'_C}(p, f(w)).$$

Let

$$M := \max_{w \in K} \rho_{\Omega'_C}(p, w),$$

(see Fig. 2.7d). We have $K \subset \overline{D_{\Omega'_C}(p, M)}$, and $F_1(D_{\Omega'_C}(p, M)) \subset D_{\Omega'_C}(p, M)$. Moreover, since $D_{\Omega'_C}(p, M)$ is compactly contained in $\Omega'_C$, there exists $\lambda_M > 1$ such that

$$\lambda_M \rho_{\Omega'_C}(w, p) \leq \rho_{\Omega'_C}(f(w), p),$$

for all $w$ such that $w, f(w) \in D_{\Omega'_C}(p, M)$.

Finally, let us show that the point $z$ should escape from the compact $K$. Assume, on the contrary, that $f^n(z)$ belongs to $K$, for infinitely many $n$'s. For these $n$'s it holds $f^n(z) \in D_{\Omega'_C}(p, M)$. Since $D_{\Omega'_C}(p, M)$ is backwards invariant under $f$, if $z$ does not escape from $K$, then

$$\{f^n(z)\}_n \subset D_{\Omega'_C}(p, M).$$

Hence,

$$\lambda_M^n \rho_{\Omega'_C}(z, p) \leq \rho_{\Omega'_C}(f^n(z), p),$$

for all $n \geq 0$, so $\rho_{\Omega'_C}(f^n(z), p) \to \infty$, as $n \to \infty$. In particular, there exists $n \geq 0$ such that $\rho_{\Omega'_C}(f^n(z), p) > M$, implying that $f^n(z) \notin D_{\Omega'_C}(p, M)$. This is a contradiction with the fact that $\{f^n(z)\}_n \subset D_{\Omega'_C}(p, M)$.

Hence, $z$ escapes from the compact set $K$, and applying the same argument to any compact set $K \subset C$, we get that $z \in \mathcal{I}(f)$, as desired.

(c) *If, additionally, $U$ is recurrent, then periodic points and escaping points are dense in $\partial U$.*

Let us start by proving the density of periodic points. Since $U$ is recurrent, then $\omega_U$-almost every boundary point has dense orbit in $\partial U$ (Thm. II.3.7), so it is enough to approximate points in $\partial U$ having dense orbit by periodic points in $\partial U$. Hence, we choose $z_0 \in \partial U$ with dense orbit and $\varepsilon > 0$, and we want to show that there is a periodic point in $D(z_0, \varepsilon) \cap \partial U$.

The idea of the proof is to see that there exists an appropriate branch $F_n$ of $f^{-n}$ defined in the hyperbolic disk $D_W(z_0, r_0)$ satisfying that

$$F_n(\overline{D_W(z_0, r_0)}) \subset D_W(z_0, r_0),$$

where $W = \mathbb{C} \smallsetminus P(f)$. Then, it follows straightforward from Brouwer fixed-point theorem that $F_n$ has a fixed point $p$ in $\overline{D_W(z_0, r_0)}$. The fact that $p \in \partial U$ is then due to proper invertibility.

Assume $z_0 \in \partial U$ has a dense orbit, and choose $r_0 > 0$ small enough so that

$$D_W(z_0, r_0) \subset D(z_0, \varepsilon)$$

and $D_W(z_0, r_0)$ is simply connected. By the Technical Lemma 2, $r_0$ can be chosen small enough so that, for all $n > 0$, any branch $F_n$ of $f^{-n}$ is defined in $D_W(z_0, r_0)$ and, if $F_n(z_0) \in \partial U$, then $F_n(D_W(z_0, r_0) \cap U) \subset U$.

Choose $F_n^*$ branch of $f^{-n}$ and $\lambda \in (0, 1)$ such that $F_n^*(z_0) \in \partial U$ and

$$\rho_W(F_n^*(z), F_n^*(w)) \leq \lambda \rho_W(z, w), \quad \text{for all } z, w \in D_W(z_0, r_0).$$

Note that this is possible by Proposition 2.2.2. Without loss of generality, we assume $n = 1$, so the inverse branch we consider is $F_1^*$.

Now, consider $W_1 := F_1^*(D_W(z_0, r_0))$. Hence,

$$F_1^* : D_W(z_0, r_0) \to W_1, \quad f : W_1 \to D_W(z_0, r_0),$$

are conformal. Moreover, for $r_0$ small enough, $W_1$ is disjoint from any other preimage of $D_W(z_0, r_0)$. Consider $r_1 > 0$ such that $D_W(F_1^*(z_0), r_1) \subset W_1$, and $r < \frac{r_1}{2} < r_0$. Define $W_2 := D_W(F_1^*(z_0), r)$. Observe that $D_W(z, r) \subset W_1$, for any $z \in W_2$. See Figure 2.8.

Since the orbit of $z_0$ is dense in $\partial U$, $\{f^n(z_0)\}_n$ visits infinitely many times $W_2$. Hence, we can choose $n_0$ such that $\lambda^{n_0} < \frac{1}{3}$, and $n_1$ such that $f^{n_1+1}(z_0) \in D_W(z_0, r)$ and

$$\# \{n \leq n_1 \colon f^n(z_0) \in W_2\} \geq n_0.$$

Consider $\{z_n := f^n(z_0)\}_{n=0}^{n_1+1} \subset W$. Let $F_{1,n}$ be the unique branch of $f^{-1}$ with $F_{1,n}(z_{n+1}) = z_n$, for $n = 0, \ldots, n_1$ (see Fig. 2.9). Each of these inverse branches $F_{1,n}$ is well-defined in $\Omega_{C_n}$, where $C_n$ is the boundary component with $z_n \in C_n$.

Define

$$F_{n_1} := F_{1,0} \circ \cdots \circ F_{1,n_1} \colon D_W(z_0, r) \longrightarrow \mathbb{C}.$$

Observe that $F_{n_1}$ is a branch of $f^{-n_1}$ defined in $D_W(z_0, r)$, such that $F_{n_1}(z_0) \in \partial U$. The Technical Lemma 2 yields that $F_{n_1}(D_W(z_0, r) \cap U) \subset U$.

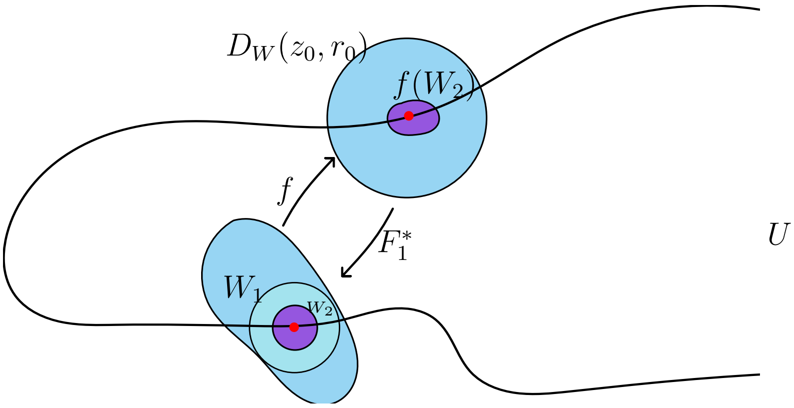



Figure 2.8: Setting of the proof of (c).

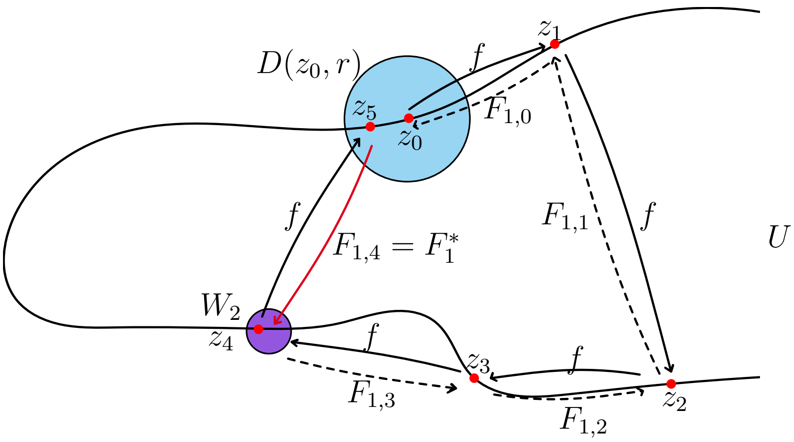



Figure 2.9: Schematic representation of $\{z_n\}_{n=0}^{n_1+1}$ in $\partial U$, with $n_0 = 1$ and $n_1 = 4$, and how $f$ maps these points. We note that, since $z_4 \in W_2$, then $F_1^4 = F_1^*$.

**Claim.** *It holds*

$$F_{n_1}(\overline{D_W(z_0, r)}) \subset D_W(z_0, r).$$

*Proof.* Indeed, by *Technical Lemma 2*, each time we apply an inverse branch $F_{1,n}$, the hyperbolic distance $\rho_W$ does not increase. That is, for all $z, w \in D_W(z_0, r)$ and $n \in \{0, \dots, n_1\}$,

$$\rho_W(F_{1,n} \circ \cdots \circ F_{1,n_1}(z), F_{1,n} \circ \cdots \circ F_{1,n_1}(w)) \leq \rho_W(F_{1,n+1} \circ \cdots \circ F_{1,n_1}(z), F_{1,n+1} \circ \cdots \circ F_{1,n_1}(w)).$$

Moreover, when the inverse branch we apply is $F_1^*$, the hyperbolic distance $\rho_W$ not only decreases, but it is contracted by the factor $\lambda$. We claim that this happens each time that $z_n$ lies in $W_2$, so at least $n_0$ times. Indeed, note that, when $z_n \in W_2$,

$$F_{1,n} \circ \cdots \circ F_{1,n_1}(D_W(z_0, r)) \subset D_W(z_n, r) \subset W_1,$$

and $z_{n+1} \in D_W(z_0, r)$. Hence, $F_{1,n}$ coincides with $F_1^*$, and it acts as a contraction by $\lambda$ in $F_{1,n} \circ \cdots \circ F_{1,n_1}(D_W(z_0, r))$. Then,

$$\rho_W(F_{n_1}(z), F_{n_1}(w)) \leq \lambda^{n_0} \rho_W(z, w) < \frac{1}{3} r.$$

In particular,

$$\rho_W(F_{n_1}(z_0), z_0) = \rho_W(F_{n_1}(z_0), F_{n_1}(z_{n_1+1})) \leq \lambda^{n_0}\rho_W(z_0, z_{n_1+1}) < \frac{1}{3}r.$$

Therefore, applying the triangle inequality, one deduces that $F_{n_1}(w) \in \overline{D_W(z_0, r)}$, for any $w \in D_W(z_0, r)$, as desired. ■

Finally, since

$$F_{n_1}(\overline{D_W(z_0, r)}) \subset D_W(z_0, r),$$

Brouwer fixed-point theorem ensures the existence of a fixed point $p$ for $F_{n_1}$ in $\overline{D_W(z_0, r)}$, which corresponds to a periodic point of $f$, which must be repelling for $f$ and hence belongs to the Julia set. Moreover, all $w \in D_W(z_0, r)$ converge to $p$ under iteration of $F_{n_1}$. In particular, if we choose $w \in D_W(z_0, r) \cap U$, we have $w_m := F_{n_1}^m(w) \in D_W(z_0, r) \cap U$ with $w_m \to p$ as $n \to \infty$, leading to a sequence of points in $U$ approximating $p$, so $p \in \partial U$, as desired.

Finally, to see that escaping points are dense in $\partial U$, note that, (b) implies that every periodic point in $\partial U$ is approximated by escaping points in $\partial U$. Hence, since periodic points are dense in $\partial U$ (under the assumption of $U$ recurrent), escaping points are also dense. □

## 2.5 Extension of the results to parabolic basins

Parabolic basins are always excluded when considering postsingulary separated Fatou components, since the parabolic fixed point $p$ is always in the postsingular set. However, if this is the only point of $P(f)$ in $\partial U$, i.e. if

$$P(f) \cap \partial U = \{p\},$$

then we shall see that we are in a similar situation than the one considered in the sections above and, with minor modifications, the proofs go through.

Next, we define PS and SPS parabolic basins, and we state Theorems II.3' and II.4'. Finally, we give an idea of the proof.

**Definition 2.5.1. (Postsingularly separated parabolic basins)** Let $f$ be a transcendental entire function, and let $U$ be an invariant attracting basin of a parabolic point $p \in \partial U$. We say that $U$ is *postsingularly separated* (PS) if there exists a domain $V$, such that $\overline{V} \subset U \cup \{p\}$ and

$$P(f) \cap U \subset V.$$

We say that $U$ is *strongly postsingularly separated* (SPS) if there exists a simply connected domain $\Omega$ and a domain $V$ such that $\overline{V} \subset U \cup \{p\}$, $\overline{U} \subset \Omega$, and

$$P(f) \cap \Omega \subset V.$$

The parabolic basin of $f(z) = \exp(\frac{z^2}{2} - 2z)$ considered in [FH06, Ex. 4] is SPS, as well as the one of $f(z) = ze^{-z}$, considered in [BD99] and in Chapter 1.

**Theorem 2.C'. (Singularities for the associated inner function)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant parabolic basin, such that $\infty$ is accessible from $U$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g := \varphi^{-1} \circ f \circ \varphi$ be the corresponding associated inner function. Assume $U$ is PS.*
*Then, the set of singularities of $g$ has zero Lebesgue measure in $\partial\mathbb{D}$. Moreover, if $\xi \in \partial\mathbb{D}$ is a singularity for $g$, then $\varphi^*(\xi) = \infty$.*

**Theorem 2.D'. (Boundary dynamics)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant parabolic basin, such that $\infty$ is accessible from $U$. Assume $U$ is SPS. Then, periodic points in $\partial U$ are accessible from $U$. Moreover, both periodic and escaping points in $\partial U$ are dense in $\partial U$.*

To prove Theorems 2.C' and 2.D' it is left to deal with the component of $\partial U$ containig the parabolic fixed point, and to explain how to adapt the proof periodic points are dense in $\partial U$, Thm. 2.D (c). In the sequel, we denote by $p$ the parabolic fixed point of $U$, and we fix the Riemann map that satisfies $\varphi^*(1) = p$. Note that $Cl_{\mathbb{C}}(\varphi, 1)$ is a connected component of $\partial U$.

First note that, if we define the set of expansion $W$ as in Technical Lemma 2, i.e.

$$W := \mathbb{C} \smallsetminus P(f),$$

it follows that $p \notin W$, since $p \in P(f)$. Hence, $W$ is no longer a neighbourhood of $\partial U$, but of $\partial U \smallsetminus \{p\}$, and we only have the expanding metric on $\partial U \smallsetminus \{p\}$. This is not a problem because the expanding metric $\rho_W$ is only needed in the proof of density of periodic points (Thm. D (c)), but in fact we do not need $\rho_W$ defined at $p$. Indeed, it is enough to have $\rho_W$ defined in a neighbourhood of points whose orbit is dense in $\partial U$, and the point $p$ does not have a dense orbit.

We remark that it may not be possible to find an open neighbourhood $\Omega_p$ of $Cl_{\mathbb{C}}(\varphi, 1)$ disjoint from $P(f)$ such that $\Omega_p \cap U$ is connected. As a counterexample, see Figure 2.10. However, we prove that we can find a neighbourhood of $Cl_{\mathbb{C}}(\varphi, 1)$ where one inverse branch is well-defined. This is enough to prove, in the PS case, that 1 is not a singularity for the associated inner function ending the proof of Theorem 2.C'; and, in the SPS case, that every point in $Cl_{\mathbb{C}}(\varphi, 1) \smallsetminus \{p\}$ is escaping. Hence, there are no periodic boundary points nor points with dense orbit in $Cl_{\mathbb{C}}(\varphi, 1)$. This would finish the proof of Theorem 2.D'.

**Lemma 2.5.2. (Cluster set of 1)** *Let $f$ be a transcendental entire function, and let $U$ be an invariant parabolic basin, such that $\infty$ is accessible from $U$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $\varphi^*(1) = p$ be the parabolic fixed point. Assume $U$ is PS. Then,* 1 *is not a singularity for the associated inner function.*

*In addition, if $U$ is SPS,*

$$Cl_{\mathbb{C}}(\varphi, 1) \smallsetminus \{p\} \subset \mathcal{I}(f).$$

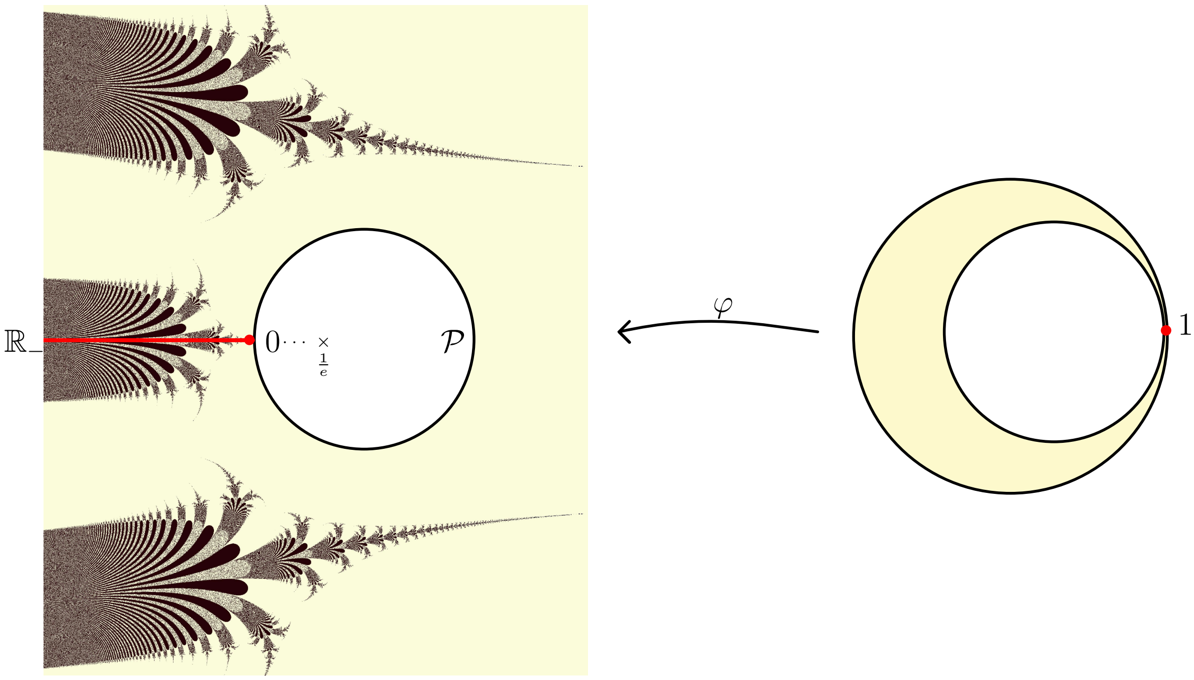



Figure 2.10: Dynamical plane of $f(z) = ze^{-z}$, which has a SPS parabolic basin $U$ with parabolic fixed point 0. The function has only one asymptotic value (0) which is fixed, and one critical value ($1/e$), which converges to 0. Hence $P(f) \subset \mathcal{P} \cup \{0\}$, where $\mathcal{P}$ is a parabolic petal containing $1/e$, and we can take $\Omega = \mathbb{C}$ in Definition 2.5.1. This shows that $U$ is SPS.

*Proof.* In the sequel, we let $F_1$ be the branch of $f^{-1}$ defined in $D(p, r)$, $r > 0$, such that $F_1(p) = p$. We shall prove the existence of a domain $\Omega_p$ analogous to the one of Technical Lemma 1.

Indeed, the construction of the domain $\Omega_p$ follows the procedure of Technical Lemma 1, applied not to $P(f) \cap U$, but to the following set of singular values

$$SV(f, p) = \{v \in \mathbb{C} \colon v \text{ is a singularity for } F_1\}.$$

We note that $SV(f, p) \subset P(f)$, and $SV(f, p) \cap D(p, r) = \emptyset$. Therefore, $SV(f, p) \cap U$ does not accumulate at any point of $Cl_{\mathbb{C}}(\varphi, 1)$, and the arguments of Technical Lemma 1 apply.

Note that $\Omega_p$ is a simply connected domain with $Cl_{\mathbb{C}}(\varphi, 1) \subset \Omega_p$, and $\Omega_p \cap U$ connected and disjoint from $SV(f, p) \cap U$. Hence, $F_1$ is well-defined in $\Omega_p \cap U$, and we apply the same arguments as in Theorem 2.C to the function $F_1|_{\Omega_p \cap U}$ to prove that 1 is not a singularity for the associated inner function.

In the SPS case, $\Omega_p$ is disjoint from $SV(f, p)$, and hence $F_1$ is well-defined in $\Omega_p$. To prove that every point in the cluster set is escaping (except for the parabolic point), we follow the proof of Theorem 2.D(b), considering $\Omega_p$ as defined above. □

# Chapter 3

## Periodic boundary points through recurrence

The goal of this chapter is to prove Theorem 3, which solves in a quite successful way the problem of showing density of periodic boundary points for Fatou components with recurrent boundary map. With the tools developed in Chapter II, we are now able to state the following more general version of Theorem 3.

**Theorem 3. (General version)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Fatou component for $f$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g\colon \mathbb{D} \to \mathbb{D}$ be the inner function associated with $(f, U)$ by $\varphi$.*
*Assume the following conditions are satisfied.*

(a) *$g^*|_{\partial\mathbb{D}}$ is recurrent with respect to $\lambda$.*

(b) *There exists $x_0 \in \partial U$ and $r_0 := r_0(x_0) > 0$ such that, for all $n \geq 0$, if $D_n$ is a connected component of $f^{-n}(D(x_0, r_0))$ such that $D_n \cap U \neq \emptyset$, then $f^n|_{D_n}$ is conformal.*

(c) *There exists a crosscut $C \subset \mathbb{D}$ and a crosscut neighbourhood $N_C$ with $N_C \cap P(g) = \emptyset$.*

*Then, accessible periodic points are dense on $\partial U$.*

This proof is inspired in the one given by F. Przytycki and A. Zdunik [PZ94] to prove density of periodic boundary points for basins of rational maps. However, as we show next, the extension of this result to transcendental maps is not straightforward, since many of the arguments are based on specific features of rational maps.

First of all, observe that, in the seminal paper [PZ94], two different proofs are provided: one for simply connected attracting basins, and a general one, which works in the non-simply connected or parabolic situations. The latter relies on a technique, known as geometric coding trees, which has been shown not to work well in the infinite degree case, even for hyperbolic maps (see [BK07, p. 405]).

From now on, we shall focus on the proof in [PZ94] for simply connected attracting basins. It relies on three specific features of rational maps: $f$ having finitely many singular values, $f$ extending analytically to the boundary of $U$ (taken in $\widehat{\mathbb{C}}$), and $f$ having finite degree. Note that these three assumptions are no longer satisfied for transcendental meromorphic maps. Indeed, a transcendental map $f$ can have infinitely many singular values, and it may have essential singularities on the boundary of $U$, implying that $f|_{\partial U}$ is no longer analytic. In addition, $f|_U$ may have infinite degree. Moreover, when dealing with transcendental meromorphic functions, one encounters other new challenges, namely a new type of Fatou components (Baker domains, on which iterates accumulate on the essential singularity), and the presence of asymptotic values.

Next, we shall outline the main steps on the proof of [PZ94], and explain how the new difficulties that appear for transcendental maps are overcome, showing at the same time the need for the hypotheses of the theorem. For simplicity, we shall assume that the Fatou component $U$ is invariant.

In the case of an attracting basin of a rational function, Pesin theory can be applied to prove that for $\omega_U$-almost every $x \in \partial U$, there exist inverse branches which are locally well-defined and contracting with respect to the Euclidean metric [PZ94, Lemma 1] (see also [PUZ91, Lemma 1], and [PU10, Thm. 11.2.3]). Crucial ingredients in this proof are the ergodic properties of $f|_{\partial U}$, studied in [Prz85, Prz86], together with $f|_{\partial U}$ being analytic and the finitude of critical values. None of the previous conditions is satisfied for a general transcendental meromorphic map, so *a priori* Pesin theory cannot be applied in our situation. We solve this by assuming that inverse branches are well-defined $\omega_U$-almost everywhere (this is a straightforward consequence of (b)), and we prove contraction of inverse branches with respect to the hyperbolic metric in a suitable domain.

Next, we extend the proof of [PZ94] to other Fatou components, apart from attracting basins. Indeed, our proof relies only on the ergodic properties of the map $f|_{\partial U}$, not on the precise type of Fatou component we are considering. More precisely, we only ask $f|_{\partial U}$ to be ergodic and recurrent with respect to the harmonic measure $\omega_U$, which implies that $\omega_U$-almost every orbit in $\partial U$ is dense in $\partial U$. Hence, all Fatou components for which the boundary map is ergodic and recurrent may be considered, and these include attracting and parabolic basins, rotation domains and certain types of Baker domains (for instance, doubly parabolic Baker domains with singular values compactly contained in $U$, Thm. II.5.4). However, note that rotation domains never satisfy the hypothesis of our theorem, since $P(f)$ is always dense in their boundary, and (b) is never fulfilled.

Finally, as it is common in constructions of this kind, and as we have seen throughout the thesis, the proof relies strongly on the inferred dynamics in the unit disk $\mathbb{D}$ via the Riemann map $\varphi\colon \mathbb{D} \to U$: the dynamics of the associated inner function.

Indeed, a careful study of the associated inner function is required. In the case of a rational attracting basin considered in [PZ94], $g$ is a finite Blaschke product, which can be chosen to satisfy $g(0) = 0$. We shall view $g$ as a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, extended by Schwarz reflection. Then, its critical values (which are finitely many) are compactly contained in $\mathbb{D}$ (and, by reflection, in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$) and their orbits converge uniformly to 0 (or to $\infty$), which are attracting fixed points. Hence, inverse branches of $g$ are well-defined for all points in $\partial\mathbb{D}$. Moreover, precise estimates on the behaviour of such inverse branches are given in [PZ94, Lemma 2].

In the case of infinite degree and inner functions which may not have a fixed point in $\mathbb{D}$, we should rely on Theorem A (proven in Sect. II.3.3), which provides good estimates on the behaviour of such inverse branches on $\partial\mathbb{D}$, whenever defined, under some mild assumptions on the postsingular set of the inner function. Note that, in particular, assumptions on Theorem A allow infinite degree and infinitely many singular values.

The remaining of the chapter is devoted to the proof of Theorem 3.

## 3.1 Proof of Theorem 3

We shall split the proof into several steps.

1. *Inverse branches well-defined $\omega_U$-almost everywhere.*

First note that hypothesis (b) implies that inverse branches of $f^n$ interplaying with $\partial U$ are well-defined in $D(x_0, r_0)$, being $r_0$ uniform for all $n \geq 0$ and all inverse branches. Let us start by proving that this actually holds for $\omega_U$-almost every $x \in \partial U$, as an easy consequence of (b) together with the ergodic properties of $f|_{\partial U}$.

**Claim.** *Under the assumptions of Theorem 3, for $\omega_U$-almost every $x \in \partial U$, there exists $r \coloneqq r(x) > 0$ such that, for all $n \geq 0$, $f^{-n}(D(x,r)) \cap U$ is non-empty, and, if $D_n$ is a connected component of $f^{-n}(D(x,r))$ such that $D_n \cap U \neq \emptyset$, then $f^n|_{D_n}$ is conformal.*

We shall refer to the inverse branches considered above as *relevant inverse branches* of $f^n$ at $x \in \partial U$. When we refer to a particular inverse branch, we write $F_{n,y,x}$ meaning that $F_n$ is an inverse branch of $f^n$ sending $y$ to $x$. When the points $x$ and $y$ are clear from the context, we shall write only $F_n$ to lighten the notation. Since we are interested in the study of $f|_{\partial U}$, relevant inverse branches are the only ones that play a role in our construction.

*Proof of the claim.* The first statement of the claim is deduced from the conjugacy between $f|_U$ and the inner function $g$, and the fact that inner functions associated to Fatou components of functions in class omit at most two values (see e.g. [Bol99, Thm. 1]).

For the second assertion, let $x_0$ and $r_0$ be the ones given by hypothesis (b). Since $\omega_U(D(x_0, r_0)) > 0$, by hypothesis (a) and Theorem II.5.4, the orbit of $\omega_U$-almost every point visits infinitely many times $D(x_0, r(x_0))$. Hence, $\omega_U$-almost every $x \in \widehat{\partial} U$ there exists $n_0 \coloneqq n_0(x)$ such that $f^{n_0}(x) \in D(x_0, r(x_0))$.

Fix $x \in \widehat{\partial} U$ with this property. Then, there exists $r \coloneqq r(x) > 0$ such that $f^{n_0}(D(x,r)) \subset D(x_0, r_0)$. By hypothesis (b), $f^{n_0}|_{D(x,r)}$ is conformal.

Let $D_m$ be a connected component of $f^{-m}(D(x,r))$ such that $D_m \cap U \neq \emptyset$. Then, $D_m$ is contained in a connected component of $f^{-m-n_0}(D(x_0, r_0))$, so there exists $x_m \in D_m$ and a relevant inverse branch of $f^{m+n_0}$

$$F_{m+n_0,x_0,x_m} \colon f^{n_0}(D(x,r)) \subset D(x_0, r_0) \longrightarrow D_m$$

which is well-defined, and hence conformal. Then,

$$F_{m,x,x_m} = F_{m+n_0,x_m,x_0} \circ f^m \colon D(x,r) \longrightarrow D_m$$

is a well-defined inverse branch of $f^m$. In particular, $f^m|_{D_m}$ is conformal, proving the claim. $\square$

2. *Construction of an expansive metric around $\partial U$.*

**Lemma.** *Under the assumptions of Theorem 3, there exists a hyperbolic open set $W \subset \mathbb{C}$ and a measurable set $X \subset W$ for which the following are satisfied.*

(2.1) *The set $X$ is contained in $\partial U$, and it has full $\omega_U$-measure.*

(2.2) *For all $x \in X$ and $n \geq 0$, there exists $r_W := r_W(x) > 0$ such that the hyperbolic disk $D_W(x, r_W)$ is simply connected and compactly contained in $W$, with all relevant branches $F_n$ of $f^{-n}$ at $x$ well-defined in $D_W(x, r_W)$. Moreover, $F_n(D_W(x, r_W)) \subset W$.*

(2.3) *Relevant inverse branches do not increase the hyperbolic distance $\text{dist}_W$ between points, i.e. for any $x \in W$ and $F_1$ relevant branch of $f^{-1}$ at $x$, if $z, w \in D_W(x, r_W)$,*

$$\text{dist}_W(F_1(z), F_1(w)) \leq \text{dist}_W(z, w).$$

*Proof.* For all $x \in \partial U$, let $r_n(x) \in [0, +\infty)$ be the radius of the maximal Euclidean disk $D(x, r_n(x))$ for which all relevant branches of $f^{-n}$ at $x$ are well-defined. We assume at least one such inverse branch exists, otherwise set $r_n(x) = 0$ (by the previous claim, this situation only happens on a set of zero $\omega_U$-measure). Clearly, $r_n(x) \geq r_{n+1}(x)$, so $\{r_n(x)\}_n$ is a convergent sequence for all $x \in \partial U$. Consider

$$X := \{x \in \partial U : r_n(x) \to r(x) > 0\}.$$

By the claim in the first step, $\omega_U(X) = 1$. Let

$$W := \bigcup_{x \in X} \bigcup_{n \geq 0} \{F_n(D(x, r(x))) : F_n \text{ is relevant at } x\}.$$

Note that $W$ is open, and $X \subset W$. Hence, (2.1) holds.

Taking $r(x) > 0$ smaller if needed, we can assume $W$ omits at least two points, so it is hyperbolic and admits a hyperbolic metric on it.

Note that $W$ may be disconnected. In this case, the hyperbolic density is defined on each connected component separatedly. Indeed, each connected component $W_1$ of $W$ is a hyperbolic domain, and hence admits a hyperbolic density $\rho_{W_1}$. Given $z \in W$, we define

$$\rho_W(z) := \rho_{W_1}(z),$$

where $W_1$ stands for the connected component of $W$ with $z \in W_1$. Given $z, w \in W$, the hyperbolic distance is defined as $\text{dist}_W(z, w) = \text{dist}_{W_1}(z, w)$, if $z$ and $w$ lie in the same connected component $W_1$ of $W$; and $\text{dist}_W(z, w) = \infty$, otherwise.

By construction, it holds that, for every $x \in X$, all relevant branches of $f^{-n}$ are well-defined in $D(x, r(x)) \subset W$. Since the Euclidean and the hyperbolic metrics are locally equivalent, there exists $r_W := r_W(x)$ such that $\overline{D_W(x, r_W)} \subset D(x, r)$. Hence, $D_W(x, r_W)$

is simply connected and compactly contained in $W$, and all relevant inverse branches $F_n$ are well-defined in $D_W(x, r_W)$, and $F_n(D_W(x, r_W)) \subset F_n(D(x, r)) \subset W$. Thus, (2.2) holds.

Finally, we claim that $f$ does not decrease the hyperbolic distance between points. Consider

$$W' := \bigcup_{x\in X} \bigcup_{n\geq 1} \{F_n(D(x, r(x))) \colon F_n \text{ is relevant at } x\} \subset W.$$

Consider the hyperbolic density $\rho_{W'}$ in $W'$, defined component by component if needed. Note that each connected component $W_1$ of $W'$ is mapped onto a connected component $W_2$ of $W$ as a holomorphic covering. Hence, if $x \in W_1 \subset W'$ and $f(x) \in W_2 \subset W$, then

$$\rho_{W'}(x) = \rho_{W_1}(x) = \rho_{W_2}(f(x)) \cdot |f'(x)| = \rho_W(f(x)) \cdot |f'(x)| .$$

Since each connected component of $W'$ is contained in a connected component of $W$, we have $\rho_W \leq \rho_{W'}$ and, in particular, if $x \in W'$,

$$\rho_W(x) \leq \rho_W(f(x)) \cdot |f'(x)| .$$

Now, let $x \in W$ and let $F_1$ be a relevant branch of $f^{-1}$ at $x$. Since $F_1$ is well-defined in $D_W(x, r_W)$, it holds

$$\rho_W(F_1(z)) \, |F_1'(z)| \leq \rho_W(z),$$

for all $z \in D_W(x, r_W)$. Next, take $z, w \in D_W(x, r_W)$. Note that $z, w \in W_1$, and $F_1(z), F_1(w) \in W_2$, for two connected components $W_1, W_2$ of $W$. Moreover, since hyperbolic disks are hyperbolically convex (i.e. a geodesic joining two points in the disk is contained in the disk), we can take $\gamma \subset D_W(x, r_W)$ geodesic between $z$ and $w$. Then,

$$\text{dist}_W(F_1(z), F_1(w)) = \text{dist}_{W_2}(F_1(z), F_1(w)) \leq \int_{F_1(\gamma)} \rho_W(t)dt =$$

$$= \int_\gamma \rho_W(F_1(t)) \, |F_1'(t)| \, dt \leq \int_\gamma \rho_W(t)dt = \text{dist}_{W_1}(z, w) = \text{dist}_W(z, w).$$

proving the claim. □

3. *Control of radial limits in terms of Stolz angles.*

Let us fix $\alpha \in (0, \frac{\pi}{2})$, and let $p \in \overline{\mathbb{D}}$ be the Denjoy-Wolff point of the associated inner function $g$. It follows from Theorem II.3.18 that, $\lambda$-almost every $\xi \in \partial\mathbb{D}$, there exists $\rho := \rho(\xi) > 0$ such that:

(3.1) for all $n \geq 0$, every branch $G_n$ of $g^{-n}$ is well-defined in $D(\xi, \rho)$;

(3.2) there exists $\rho_1 := \rho_1(\xi)$ such that, for all $n \geq 0$,

$$G_n(R_{\rho_1}, p) \subset \Delta_{\alpha,\rho_1}(G_n(\xi), p),$$

where $R_\rho(\cdot, p)$ and $\Delta_{\alpha,\rho}(\cdot, p)$ stand for the radial segment and the Stolz angle with respect to $p$ (Def. II.2.2).

In the sequel, to lighten the notation we denote the radial segments and the Stolz angles just by $R_\rho$ and $\Delta_\rho$, respectively. However, one should keep in mind that they are radial segments and Stolz angles with respect to the Denjoy-Wolff point, and that the opening $\alpha$ of the Stolz angles is fixed throughtout the proof.

4. *Choice of a set $K_\varepsilon \subset \partial\mathbb{D}$, where bounds on $\varphi$ and $G_n$ are uniform.*

**Lemma.** *Fix $\varepsilon > 0$. There exists a measurable set $K_\varepsilon \subset \partial\mathbb{D}$ with $\lambda(K_\varepsilon) \geq 1-\varepsilon$ such that the following holds.*

(4.1) *For all $\xi \in K_\varepsilon$, $\varphi^*(\xi)$ exists and $\varphi^*(\xi) \in X$. Moreover,*

$$(g^n)^*(\xi) \in \Theta_\Omega := \{\xi \in \partial\mathbb{D} \colon \varphi^*(\xi) \in \Omega(f)\},$$

*for all $n \geq 0$.*

(4.2) *There exists $r_\varepsilon > 0$ such that for all $\xi \in K_\varepsilon$ and $n \geq 0$, all relevant inverse branches of $f^n$ are well-defined in $D_W(\varphi^*(\xi), r_\varepsilon)$.*

(4.3) *There exist $\rho_\varepsilon > 0$ such that*

i. *For every $\xi \in K_\varepsilon$ and $n \geq 0$, every branch $G_n$ of $g^{-n}$ is well-defined in $D(\xi, \rho_\epsilon)$.*

ii. *For every $\xi \in K_\varepsilon$,*

$$G_n(R_{\rho_\varepsilon}(\xi)) \subset \Delta_{\rho_\varepsilon}(G_n(\xi)).$$

iii. *For every $\xi \in K_\varepsilon$, if $z \in \Delta_{\rho_\varepsilon}(\xi)$, then $\varphi(z) \in W$ and*

$$\text{dist}_W(\varphi(z), \varphi^*(\xi)) < \frac{r_\varepsilon}{3}.$$

(4.4) *There are no isolated points in $K_\varepsilon$. In fact, for every $\xi \in K_\varepsilon$, there exists a subsequence $\{n_k := n_k(\xi)\}_k$, $n_k \to \infty$, such that $(g^{n_k})^*(\xi) \in K_\varepsilon$ and $(g^{n_k})^*(\xi) \to \xi$.*

(4.5) *For every $\xi \in K_\varepsilon$, the orbit of $\varphi^*(\xi)$ under $f$ is dense in $\partial U$.*

*Proof.* Consider $K := (\varphi^*)^{-1}(X) \subset \partial\mathbb{D}$. Observe that $\lambda(K) = 1$. There is no loss of generality on assuming that $(g^n)^*(\xi) \in \Theta_\Omega$, for all $n \geq 0$, since this holds $\lambda$-almost everywhere. Hence, all points in $K$ satisfy (4.1).

Next, by (2.2), for all $\xi \in K$, there exists $r := r(\xi) > 0$, such that all relevant inverse branches $F_n$ are well-defined in $D_W(\varphi^*(\xi), r(\xi))$. Hence, we can write $K$ as the countable union of the nested measurable sets

$$K_m := \{\xi \in K \colon \text{ all relevant } F_n \text{ are well-defined in } D_W(\varphi^*(\xi), 1/m)\} \subset K_{m-1}.$$

Choose $m_0$ such that $\lambda(K_{m_0}) \geq 1 - \varepsilon/3$, which satisfies (4.2) with $r_\varepsilon = 1/m_0$.

Now, by (3.1) and (3.2), we can assume that, for all $\xi \in K_{m_0}$, there exists $\rho := \rho(\xi) > 0$ such that $G_n$ is well-defined in $D(\xi, \rho)$ and $G_n(R_\rho) \subset \Delta_\rho(G_n(\xi))$, for all $n \geq 0$. Hence, $K_{m_0}$ can be written as the countable union of the nested measurable sets

$$K^k_{m_0} := \left\{\xi \in K_{m_0} \colon G_n \text{ well-defined in } D(\xi, 1/k) \text{ and } G_n(R_{1/k}) \subset \Delta_{\frac{1}{k}}(G_n(\xi))\right\}.$$

Choose $k_0$ such that $\lambda(K^{k_0}_{m_0}) \geq 1-\varepsilon/2$. Finally, note that the angular limit of $\varphi$ exists at every $\xi \in K^{k_0}_{m_0}$, that is, for all $\xi \in K^{k_0}_{m_0}$ there exists $\rho_1(\xi) < \rho(\xi)$ such that, for all $z \in \Delta_{\rho_1}$,

$$\operatorname{dist}_W(\varphi(z), \varphi^*(\xi)) < \frac{r_\varepsilon}{3}.$$

Hence, proceeding as before, we find $K_\varepsilon \subset K^{k_0}_{m_0}$, with $\lambda(K_\varepsilon) \geq 1-\varepsilon$ and satisfying (4.3) for some $\rho_\varepsilon > 0$, uniform for $\xi \in K_\varepsilon$.

Since $\lambda$-almost every point in $K_\varepsilon$ is a Lebesgue density point, there is no loss of generality on assuming that every $\xi \in K_\varepsilon$ is a Lebesgue density point. In particular, there are no isolated points in $K_\varepsilon$.

Moreover, since $g^*|_{\partial\mathbb{D}}$ is recurrent, every measurable set $E \subset X$ with $\lambda(E) > 0$, we have that for $\lambda$-almost every point $\xi \in K_\varepsilon$, there exists a subsequence $\{n_k\}_k$, $n_k \to \infty$, with $(g^{n_k})^*(\xi) \in E$.

Now, take a countable sequence $\{\xi_n\}_n \subset K_\varepsilon$, such that $\{\xi_n\}_n$ is dense in $K_\varepsilon$ and each $\xi_n$ is a Lebesgue density point for $K_\varepsilon$. Consider $E_{j,n} := D(\xi_n, 1/j) \cap K_\varepsilon$, for $j, n \geq 1$. Since each $\xi_n$ is a Lebesgue density point for $K_\varepsilon$, $\lambda(E_{j,n}) > 0$, for all $j, n \geq 1$.

Applying the previous property to the sequence $\{E_{j,n}\}_{j,n}$, we have that, for $\lambda$-almost every $\xi \in K_\varepsilon$, there exists a subsequence $\{n_k\}_k$, $n_k \to \infty$, with $(g^{n_k})^*(\xi) \in E_{k,k}$. Hence, there exists a subsequence $\{n_k\}_k$, $n_k \to \infty$, with $(g^{n_k})^*(\xi) \in K_\varepsilon$, $(g^{n_k})^*(\xi) \to \xi$, proving (4.4).

Finally, since points in $\partial U$ with dense orbit have full harmonic measure, we can assume that $K_\varepsilon$ is chosen so that (4.5) holds. □

5. *Construction of a periodic point in $D_W(\varphi^*(\xi), r)$, for all $\xi \in K_\varepsilon$ and $r \in (0, r_\varepsilon)$.*

Set $\xi \in K_\varepsilon$ and $r \in (0, r_\varepsilon)$. The goal in this section is to find a periodic point in $D_W(\varphi^*(\xi), r) \cap \partial U$.

Write $\xi_n := (g^n)^*(\xi)$, for all $n \geq 0$. By (4.1), $\varphi^*(\xi_n)$ exists and belongs to $\Omega$, for all $n \geq 0$. By (4.4) and (4.2), $\xi_n \in K_\varepsilon$ infinitely often, and all relevant inverse branches are well-defined in $D_W(\varphi^*(\xi_n), r_\varepsilon)$. In particular, for all $0 \leq m < n$, there is a relevant inverse branch $F_{n-m}$ of $f^{n-m}$ in $D(\varphi^*(\xi_n), r_\varepsilon)$. Hence, for all $n \geq 0$, $f$ maps conformally a neighbourhood of $\varphi^*(\xi_n)$ onto a neighbourhood of $\varphi^*(\xi_{n+1})$.

Now, consider $D_W(\varphi^*(\xi_0), r_\varepsilon)$, and let $W_1$ be the connected component of $W$ such that $\varphi^*(\xi_0) \in W_1$. Note that, by (4.2), all relevant branches of $f^{-n}$ at $\varphi^*(\xi_0)$ are well-defined in $D_W(\varphi^*(\xi_0), r_\varepsilon)$. In particular, there exists $n_0 \geq 1$ and a relevant inverse branch of $f^{n_0}$, say $F_{n_0}$, such that $F_{n_0}(\varphi^*(\xi_0)) \in W_1$ (recall that preimages of any point are dense in the Julia set, with at most two exceptions). Consider $D_{n_0} := F_{n_0}(D_W(\varphi^*(\xi_0), r_\varepsilon))$. Therefore,

$$F_{n_0}\colon D_W(\varphi^*(\xi_0), r_\varepsilon) \longrightarrow D_{n_0}$$

conformally.

**Claim.** *There exists $k \in (0,1)$ such that, for all $z, w \in D_W(\varphi^*(\xi_0), r_\varepsilon)$,*

$$\operatorname{dist}_W(F_{n_0}(z), F_{n_0}(w)) \leq k \cdot \operatorname{dist}_W(z, w).$$

*Proof.* The proof follows the same idea as in (2.3) (and we use the same notation). Indeed, let $W_1$ be the connected component of $W$ in which $\varphi^*(\xi_0)$ lies, and let $W_1'$ be the connected component of $W'$ that contains $F_{n_0}(\varphi^*(\xi_0))$. Then, $W_1' \subset W_1$, and $f^{n_0}\colon W_1' \to W_1$ is a holomorphic covering.

Note that the inclusion $W_1' \subset W_1$ is strict. Otherwise $f^{n_0}(W_1) = W_1$, and this is impossible since $W_1$ contains points of $\mathcal{J}(f)$ (and, by the blow-up property of the Julia set, the neighbourhood of any point in $\mathcal{J}(f)$ should cover the Riemann sphere minus at most two points under iteration – taking $W_1$ to be that neighbourhood, the fact that $f^{n_0}(W_1) = W_1$ causes a contradiction). Hence, $\rho_{W_1} < \rho_{W_1'}$, so

$$\rho_{W_1}(x)| < \rho_{W_1}(f^{n_0}(x)) \cdot |(f^{n_0})'(x)| \,.$$

Moreover, since $D_W(\varphi^*(\xi_0), r_W)$ is compactly contained in $W$, there exists $k \in (0,1)$ such that, for all $x \in D_W(\varphi^*(\xi_0), r_W)$,

$$\rho_{W_1}(F_{n_0}(x)) \cdot \left|F_{n_0}'(x)\right| \leq k \cdot \rho_{W_1}(x).$$

With this in mind, the end of the proof is straightforward, as in (2.3). □

Now, we claim that we can find $N \geq 1$ satisfying the following properties.

(5.1) If $N_0 \coloneqq \#\{n \leq N \colon \varphi^*(\xi_n) \in D_{n_0}\}$, then $k^{N_0} < \dfrac{r}{3r_\varepsilon}$.

(5.2) $\xi_N \coloneqq (g^N)^*(\xi_0) \in K_\varepsilon$

(5.3) There exists $t_N \in (0,1)$ such that $t_N\xi_N \in R_{\rho_\varepsilon}(\xi_N) \cap \Delta_{\rho_\varepsilon}(\xi_0)$.

Indeed, by (4.5), the orbit of $\varphi^*(\xi_0)$ is dense in $\partial U$. In particular, it visits $D_{n_0}$ infinitely many times. Hence, there exists $N'$ so that (5.1) is satisfied for $N'$. By (4.4), there exists a subsequence $\{n_k\}_k$, $n_k \to \infty$, such that $\xi_{n_k} \in K_\varepsilon$ and $\xi_{n_k} \to \xi_0$. Thus, we can find $N \geq N'$ for which conditions (5.2) and (5.3) are also satisfied. Note that the geometric condition in (5.3) is satisfied as long $\xi_N$ is close enough to $\xi_0$, since the radius $\rho_\varepsilon$ and the angle $\alpha$ are uniform (see Fig. 3.1 for a geometric intuition).

**Claim.** *There exists a relevant inverse branch $F_N$ of $f^N$ at $\varphi^*(\xi_N)$ defined in $D_W(\varphi^*(\xi_N), r_\varepsilon)$, which satisfies $F_N(\varphi^*(\xi_N)) = \varphi^*(\xi_0)$ and*

$$F_N(D_W(\varphi^*(\xi_N), r_\varepsilon)) \subset D_W\left(\varphi^*(\xi_0), \frac{r}{3}\right) \subset D_W(\varphi^*(\xi_N), r_\varepsilon).$$

We note that, in particular, $F_N(D_W(\varphi^*(\xi_N), r_\varepsilon)) \subset D_W(\varphi^*(\xi_0), r)$.

*Proof.* First note that $\xi_N \in K_\varepsilon$ (5.2), so all relevant inverse branches are well-defined in $D_W(\varphi^*(\xi_N), r_\varepsilon)$ (4.2). Since $\xi_N = (g^N)^*(\xi_0)$, by Lemma II.5.2, we have $f^N(\varphi^*(\xi_0)) = \varphi^*(\xi_N)$. Hence, there exists a relevant inverse branch $F_N$ of $f^N$ at $\varphi^*(\xi_N)$ defined in $D_W(\varphi^*(\xi_N), r_\varepsilon)$, which satisfies $F_N(\varphi^*(\xi_N)) = \varphi^*(\xi_0)$.

Note that $F_N$ is the composition of different inverse branches of $f$, and each of them does not increase the hyperbolic distance $\text{dist}_W$ between points (2.3). Moreover, $\{f^n(\xi_0)\}_{n=0}^N$

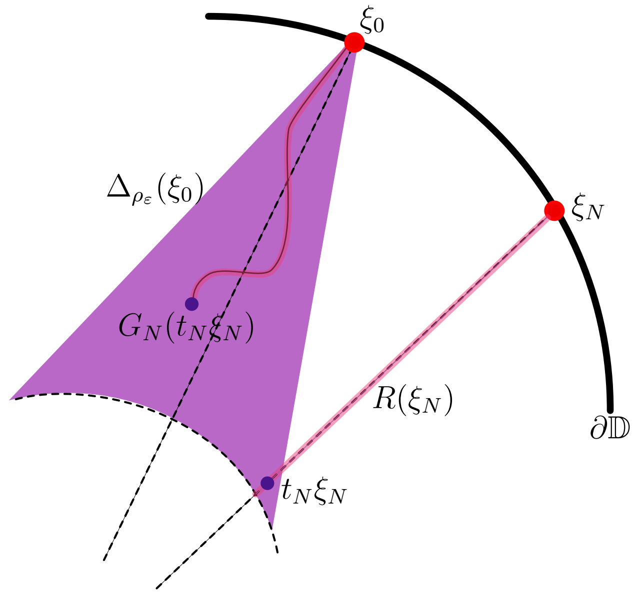



Figure 3.1: The choice of the point $t_N\xi_N \in R_{\rho_\varepsilon}(\xi_N) \cap \Delta_{\rho_\varepsilon}(\xi_0)$.

visits $D_{n_0}$ at least $N_0$ times (5.1). This means that applying $F_N$ corresponds to applying the inverse $F_{n_0}$, which acts as a contraction by $k$, at least $N_0$ times. Thus, we have

$$F_N(D_W(\varphi^*(\xi_N), r_\varepsilon)) \subset D_W\left(\varphi^*(\xi_0), k^{N_0} r_\varepsilon\right) \subset D_W\left(\varphi^*(\xi_0), \frac{r}{3}\right).$$

To see the remaining inclusion, note that, by (4.3) and (5.1), we have

$$\varphi(t_N\xi_N) \in D_W\left(\varphi^*(\xi_0), \frac{r_\varepsilon}{3}\right) \cap D_W\left(\varphi^*(\xi_N), \frac{r_\varepsilon}{3}\right).$$

Hence, applying the triangle inequality,

$$\mathrm{dist}_W(\varphi^*(\xi_0), \varphi^*(\xi_N)) \leq \mathrm{dist}_W(\varphi^*(\xi_0), \varphi(t_N\xi_N)) + \mathrm{dist}_W(\varphi(t_N\xi_N), \varphi^*(\xi_N)) < \frac{2r_\varepsilon}{3},$$

implying the desired inclusion. □

Finally, we end the proof finding a periodic point in $D_W(\varphi^*(\xi_0), r) \cap \partial U$. This final argument is essentially the same as in [PZ94], which we include for the sake of completeness.

**Claim.** *The map*

$$F_N\colon D_W(\varphi^*(\xi_N), r_\varepsilon) \longrightarrow D_W(\varphi^*(\xi_N), r_\varepsilon)$$

*has an attracting fixed point in $D_W(\varphi^*(\xi_0), r)$, which is accessible from $U$. Hence, $f$ has a repelling $N$-periodic point in $D_W(\varphi^*(\xi_0), r) \cap \partial U$.*

*Proof.* Since $F_N(D_W(\varphi^*(\xi_N), r_\varepsilon)) \subset D_W(\varphi^*(\xi_0), r)$, by the Denjoy-Wolff Theorem, $F_n$ as a fixed point $p \in D_W(\varphi^*(\xi_0), r)$, which attracts all points in $D_W(\varphi^*(\xi_N), r_\varepsilon)$ under the iteration of $F_n$. Hence it is repelling under $f^n$ and thus belongs to $\mathcal{J}(f)$.

It is left to see that $p$ is accessible from $U$. To do so, first note that, by (4.3), there exists a branch $G_N$ of $g^{-N}$ such that $G_N$ is well-defined in $D(\xi_N, \rho_\varepsilon)$ and $G_N(\xi_N) = \xi_0$. It holds that $\varphi \circ G_N = F_N \circ \varphi$ in $\Delta_{\rho_\varepsilon}(\xi_N)$. Moreover, $G_N(R_{\rho_\varepsilon}(\xi_N)) \subset \Delta_{\rho_\varepsilon}(\xi_0)$. In particular, $G_N(t_N\xi_N) \in \Delta_{\rho_\varepsilon}(\xi_0)$.

Since $t_N\xi_N \in \Delta_{\rho_\varepsilon}(\xi_0)$, we can find a curve $\gamma \subset \Delta_{\rho_\varepsilon}(\xi_0)$ joining $t_N\xi_N$ and $G_N(t_N\xi_N)$. Then, $\varphi(\gamma) \subset D_W(\varphi^*(\xi_N), r_\varepsilon)$ joins $\varphi(t_N\xi_N)$ with $F_N(\varphi(t_N\xi_N))$. Define

$$\Gamma \coloneqq \bigcup_{m \geq 0} F_N^m(\gamma).$$

Then, $\Gamma \subset \partial U$ lands at $p$, proving the claim. $\square$

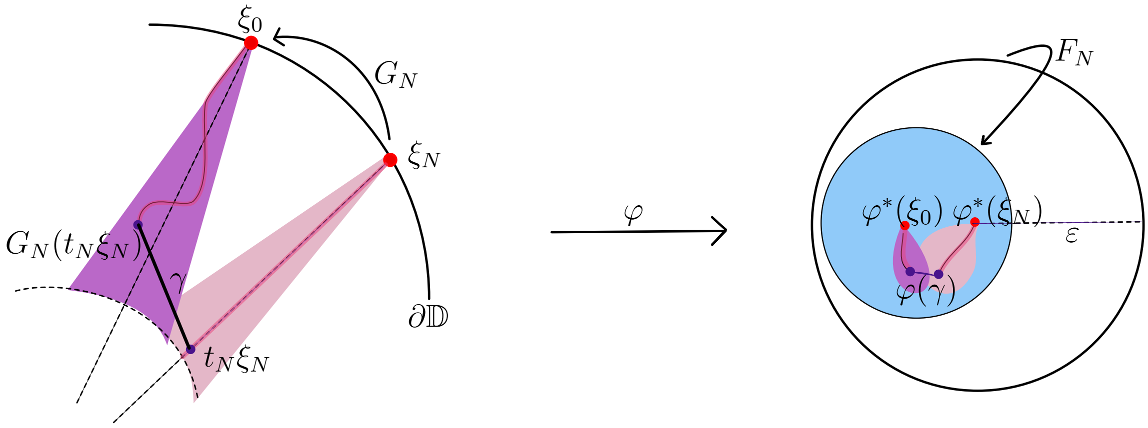



Figure 3.2: The construction of the curve $\gamma$ in $\mathbb{D}$, and its image $\varphi(\gamma)$ in the dynamical plane.

5. *Periodic points are dense in $\partial U$.*

Finally, to see that the previous construction leads to density of periodic points in $\partial U$, one should take into account that supp $\omega_U = \widehat{\partial} U$ (Lemma **??**). Hence, for all $x \in \partial U$ and $\delta > 0$, it holds $\omega_U(D(x, \delta)) > 0$. Take $\varepsilon \coloneqq \omega_U(D(x, \delta))/2$, and consider $K_\varepsilon$ as before. Note that, by the choice of $\varepsilon$, we have $\omega_U(D(x, \delta) \cap \varphi^*(K_\varepsilon)) > 0$.

Let $\xi \in K_\varepsilon$ be such that $\varphi^*(\xi) \in D(x, \delta)$, and let $r \leq r_\varepsilon$. In the previous step, we proved the existence of a periodic point in $D_W(\varphi^*(\xi), r)$. Taking $r$ small enough, since $\varphi^*(\xi) \in D(x, \delta)$, we can ensure that the periodic point is in $D(x, \delta)$.

This ends the proof of Theorem 3. $\square$

# Chapter 4

# Pesin theory for transcendental maps and applications

In the setting of smooth dynamical systems, *hyperbolic dynamical systems* play a distinguished role, since they are the easiest to study and exhibit the simplest possible behaviour. Indeed, hyperbolic dynamics are characterized by the presence of expanding and contracting directions for the derivative at every point, which provides strong local, semilocal or even global information about the dynamics. However, the assumption of hyperbolicity is quite restrictive. A weaker (and hence, more general) form of hyperbolicity, known as *non-uniform hyperbolicity*, was initially developed by Yakov Pesin in his seminal work [Pes76, Pes77]. Since then, Pesin's approach to hyperbolicity, also known as *Pesin theory*, has been extended, generalized and refined in numerous articles and research books (see e.g. [Pol93], [KH95, Supplement], [BP23]). Although results apply to both discrete and continuous dynamical systems, in this paper we focus on the discrete ones.

Roughly speaking, Pesin studied originally $\mathcal{C}^1$-diffeomorphisms on compact smooth Riemannian manifolds. Under the assumption that such a map is measure-preserving and ergodic, and no Lyapunov exponent vanishes except on a set of zero measure, the forward and backwards contraction or expansion around almost every point is controlled asymptotically by the Lyapunov exponents. Applications of this theory include periodic points, homoclinic points, and stable manifold theory [Pol93, Part II].

One of the natural generalizations of Pesin theory is to the setting of iteration of rational maps in the Riemann sphere $\widehat{\mathbb{C}}$. That is, let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic, and consider the discrete dynamical system generated by $f$. The phase space $\widehat{\mathbb{C}}$ is commonly split into two totally invariant sets: the Fatou set $\mathcal{F}(f)$, where the family of iterates is normal, and hence the dynamics are in some sense stable; and its complement, the Julia set $\mathcal{J}(f)$. Although the Fatou set is well-understood, the dynamics in the Julia set are more intricate and worthy of study. For general background in rational iteration we refer to [CG93, Mil06]. In contrast with the setting of $\mathcal{C}^1$-diffeomorphisms considered by Pesin, now the iterated function is no longer bijective, which is overcome by assuming a higher degree of regularity on the function.

A rational map is said to be *hyperbolic* if all orbits of critical values (i.e. images of zeros of $f'$) are compactly contained in the Fatou set, which already implies that *all* inverse branches around points in $\mathcal{J}(f)$ are well-defined and uniformly contracting (see e.g. [CG93, Sect. V.2], [Mil06, Sect. 19]). Hence, following Pesin's approach for diffeomorphisms, it is natural to ask whether, for a general map (not necessarily hyperbolic), generic inverse branches are well-defined and contracting. Note that one should make precise the notion of generic inverse branches, by defining the abstract space of backward orbits for points in $\mathcal{J}(f)$ and endow it with a measure (using Rokhlin's natural extension, see Sect. 4.1).

One can prove that, under the assumption of existence of an ergodic invariant probability with positive Lyapunov exponent, for almost every backbard orbit $\{x_n\}_n$ there exists a disk around the initial point $x_0$, such that the corresponding inverse branches of $f^n$ are well-defined and contracting in this disk (see [Led81, Dob12], and also [PU10, Sect. 11.2], [KU23, Chap. 9.3], [URM23, Sect. 28.3], among others). The proof relies strongly on the fact that $\widehat{\mathbb{C}}$ is compact (and hence, $\mathcal{J}(f)$ is also compact), and the finiteness of the set of critical values.

We note that the existence of ergodic invariant probabilities supported on the Julia set of rational maps has been historically a topic of wide interest, in connection with the measure of maximal entropy. For polynomials, the existence of such a measure was already proved by Brolin [Bro65], whereas for rational maps it was done by Freire, Lopes and Mañé [FLM83], and Lyubich [Lyu83], independently. Such a measure of maximal entropy is known to be an ergodic invariant probability, and hence it can be used as an initial cornerstone to develop Pesin theory. Moreover, Lyapunov exponents with respect to any ergodic invariant probability supported on $\mathcal{J}(f)$ have been studied in depth [Prz85, Prz93, Dob12].

The goal of this paper is to extend these well-known results for rational maps to the transcendental setting, that is, for maps $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ (transcendental) meromorphic, including the entire case. Although under the presence of poles some orbits get truncated, one can define the Fatou and Julia set for $f$ in a similar way than for rational maps (as shown in Section II.1). As in the rational case, the question we want to address is whether generic inverse branches are well-defined and contracting around points in $\mathcal{J}(f)$.

Note that the cornerstones from which the rational Pesin theory is built (namely, compact phase space, finitely many critical values, and existence of ergodic invariant probabilities) no longer hold in general. Indeed, first, the phase space is now $\mathbb{C}$, which is no longer compact, and nor is the Julia set. In fact, this lack of compactness causes difficulties even for the extension of the notion of hyperbolicity from the rational setting [RGS17].

Additionally, critical values are not the only values where inverse branches fail to be defined. Indeed, one shall consider the set of *singular values* (i.e. critical and asymptotic values, and accumulation thereof, denoted by $SV$), and it may be uncountable.

Finally, the existence of invariant measures on the Julia set is much more delicate and remains somewhat unexplored, as well as Lyapunov exponents (which depend on the existence of the previous measures). Indeed, although the existence of invariant ergodic probabilities supported on the Julia set has been proved for certain families (such as the hyperbolic exponential family [UZ03]), in other cases it is known that they do not exist [DS08]. Hence, in contrast with rational maps, the existence of an ergodic invariant measure supported in the Julia set is unknown in the general setting.

In order to overcome some of these difficulties, we restrict ourselves to some forward invariant subsets of the Julia set which are of special interest: the boundaries of invariant (or periodic) connected components of the Fatou set (known as *Fatou components*). If we let $U$ be an invariant Fatou component for $f$, then its boundary $\partial U$ is forward invariant under $f$. In the seminal work of Doering and Mañé [DM91], invariant ergodic measures for $f\colon \partial U \to \partial U$ supported on $\partial U$ are given, following the approach initiated by Przytycki to study rational maps restricted to the boundary of attracting basins [Prz85].

Taking advantage of these invariant measures, under some mild assumptions on the singular values, we are able to overcome the difficulties arising from the lack of compactness, the infinite degree and the presence of infinitely many singular values. Our techniques include refined estimates on harmonic measure and the construction of an appropriate conformal metric. In this manner, we can develop Pesin theory in the boundary of some transcendental Fatou components in a quite successful way, which is presented next.

**Statement of results**

Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function, i.e. so that $\infty$ is an essential singularity for $f$, and let $U$ be an invariant Fatou component for $f$. Such an invariant Fatou component is either an attracting basin, a parabolic basin, a rotation domain or a Baker domain (see Sect. II.1). As in the previous sections, we denote by $\partial U$ the boundary of $U$ in $\mathbb{C}$, and $\widehat{\partial} U$ the boundary in $\widehat{\mathbb{C}}$. All the derivatives and absolute values are understood to be with respect to the spherical metric in $\widehat{\mathbb{C}}$, and hence $|f'|$ is bounded on compact subsets of the plane.

Attracting basins are the natural candidates to perform Pesin theory on their boundary, since the harmonic measure $\omega_U$ (with basepoint the fixed point $p \in U$) is invariant under $f$ and ergodic. The transversal assumption throughout the paper is that singular values are 'not too dense' on $\partial U$, condition we make formal by requiring

$$\int_{\partial U} \log |x - SV|^{-1} \, d\omega_U(x) < \infty$$

(see Sect. 4.2). We note that this assumption is always satisfied if there are only finitely many singular values on $\widehat{\partial} U$ (Remark 4.2.2).

Our main result is the following.

**Theorem 4.A. (Pesin theory for attracting basins of transcendental maps)** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be a simply connected attracting basin,*

*with fixed point* $p \in U$*. Let* $\omega_U$ *be the harmonic measure on* $\partial U$ *with base point* $p$*. Assume* $f$ *has positive Lyapunov exponent, that is* $\log|f'| \in L^1(\omega_U)$ *with* $\int_{\partial U} \log|f'(x)|\, d\omega_U(x) > 0$*. Suppose also that* $\int_{\partial U} \log|x - SV|^{-1}\, d\omega_U(x) < \infty$*.*
*Then, for every countable collection of measurable sets* $\{A_k\}_k \subset \partial U$ *with* $\omega_U(A_k) > 0$*, and for* $\omega_U$*-almost every* $x_0 \in \partial U$*, there exists a backward orbit* $\{x_n\}_n \subset \partial U$ *and* $r > 0$ *such that*

(a) $x_{n_k} \in A_k$ *for some sequence* $n_k \to \infty$*;*

(b) *the inverse branch* $F_n$ *sending* $x_0$ *to* $x_n$ *is well-defined in* $D(x_0, r)$*;*

(c) $\operatorname{diam} F_n(D(x_0, r)) \to 0$*, as* $n \to \infty$*.*

Note that, in particular, for $\omega_U$-almost every $x_0 \in \partial U$ there exists a backward orbit $\{x_n\}_n$, and inverse branches $\{F_n\}_n$ of $f^n$, well-defined in $D(x_0, r)$, such that $\{x_n\}_n$ is dense on $\partial U$.

If we consider parabolic basins or Baker domains, the situation is even more unfavorable, since no harmonic measure on $\partial U$ is $f$-invariant. Nevertheless, there exists a $\sigma$-finite measure which is absolutely continuous with respect to it, invariant under $f$, recurrent and ergodic. By means of the first return map, we develope a similar result for parabolic basins and Baker domains. As far as we are aware of, this result is new even for parabolic basins of polynomials, in which case the assumptions are always trivially satisfied.

**Theorem 4.B. (Parabolic Pesin theory)** *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *be a meromorphic function, and let* $U$ *be a simply connected parabolic basin or Baker domain, such that* $SV \cap U$ *are compactly contained in* $U$*. Let* $\omega_U$ *be a harmonic measure on* $\partial U$*, such that* $\log|f'| \in L^1(\omega_U)$ *with* $\int_{\partial U} \log|f'|\, d\omega_U > 0$*. Assume there exists* $\varepsilon > 0$ *such that, if* $\partial U_{+\varepsilon} := \{z \in \mathbb{C}\colon \operatorname{dist}(z, \partial U) < \varepsilon\}$*, singular values of* $f$ *in* $\partial U_{+\varepsilon}$ *are finite.*
*Then, for every countable collection of measurable sets* $\{A_k\}_k \subset \partial U$ *with* $\omega_U(A_k) > 0$*, and for* $\omega_U$*-almost every* $x_0 \in \partial U$ *there exists a backward orbit* $\{x_n\}_n \subset \partial U$ *and* $r > 0$ *such that*

(a) $x_{n_k} \in A_k$ *for some sequence* $n_k \to \infty$*;*

(b) *the inverse branch* $F_n$ *sending* $x_0$ *to* $x_n$ *is well-defined in* $D(x_0, r_0)$*;*

(c) *for every subsequence* $\left\{x_{n_j}\right\}_j$ *with* $x_{n_j} \in D(x_0, r)$*,* $\operatorname{diam} F_{n_j}(D(x_0, r)) \to 0$*, as* $j \to \infty$*.*

**Remark.** In the particular case when $f$ is an entire function (polynomial or transcendental), instead of assuming that singular values of $f$ in $\partial U_{+\varepsilon}$ are finite, it is enough to assume that critical values in $\partial U_{+\varepsilon}$ are finite (see Section 4.4.4).

Next we present two applications of the theorems above: developing Pesin theory for centered inner functions, and finding periodic points for transcendental maps.

**Application. Pesin theory for centered inner functions**

Let $\mathbb{D}$ denote the unit disk, and $\partial\mathbb{D}$ the unit circle, and let $\lambda$ be the normalized Lebesgue measure in $\partial\mathbb{D}$. An *inner function* is, by definition, a holomorphic self-map of the unit disk, $g\colon \mathbb{D} \to \mathbb{D}$, which preserves the unit circle $\lambda$-almost everywhere in the sense of radial limits. If, in addition, we have that $g(0) = 0$, we say that the inner function is *centered*. A point $\xi \in \partial\mathbb{D}$ is called a *singularity* of $g$ if $g$ cannot be continued analitically to any neighbourhood of $\xi$. Denote the set of singularities of $g$ by $E(g)$.

It is well-known that the radial extension of a centered inner function preserves the Lebesgue measure $\lambda$ and is ergodic (see e.g. [DM91, Thm. A, B]). For these reasons, centered inner functions have been widely studied as measure-theoretical dynamical systems [Aar78, DM91, Cra91, Cra92, Aar97, IU23, IU24].

An important subset of centered inner functions are the ones with finite entropy, or equivalently, when $\log|g'| \in L^1(\partial\mathbb{D})$ [Cra91]. Such a property translates to a greater control on the dynamics, from different points of view (see e.g. [Cra91, Cra92, IU23, IU24]). In particular, centered inner functions with finite entropy are natural candidates to apply the theory developed above. Moreover, due to its rigidity and symmetries, we will deduce some additional properties. In general, inner functions present a highly discontinuous behaviour in $\partial\mathbb{D}$, so it is noteworthy the great control we achieve, only by assuming that $\int_{\partial\mathbb{D}} \log|x - SV|^{-1} d\lambda(x) < \infty$, where $\lambda$ denotes the Lebesgue measure on $\partial\mathbb{D}$.

Let us denote the *radial segment* at $\xi$ of length $\rho > 0$ by

$$R_\rho(\xi) := \{r\xi\colon r \in (1-\rho, 1)\}.$$

and the *Stolz angle* at $\xi$ of length $\rho > 0$ and opening $\alpha \in (0, \frac{\pi}{2})$ by

$$\Delta_{\alpha,\rho}(\xi) = \{z \in \mathbb{D}\colon\ |\mathrm{Arg}\ \xi - \mathrm{Arg}\ (\xi - z)| < \alpha, |z| > 1-\rho\}.$$

Using the same construction as in Theorem 4.A, we deduce the following.

**Corollary 4.C. (Pesin theory for centered inner functions)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be a centered inner function, such that $\log|g'| \in L^1(\partial\mathbb{D})$ and $\int_{\partial\mathbb{D}} \log|x - SV|^{-1} d\lambda(x) < \infty$. Fix $\alpha \in (0, \pi/2)$. Then, for every countable collection of measurable sets $\{A_k\}_k \subset \partial\mathbb{D}$ with $\lambda(A_k) > 0$, and for $\lambda$-almost every $\xi_0 \in \partial\mathbb{D}$ there exists a backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$ and $\rho_0 > 0$ such that*

(a) *$\xi_{n_k} \in A_k$ for some sequence $n_k \to \infty$;*

(b) *the inverse branch $G_n$ of $g^n$ sending $\xi_0$ to $\xi_n$ is well-defined in $D(\xi_0, \rho_0)$;*

(c) *for all $\rho \in (0, \rho_0)$, $G_n(R_\rho(\xi_0)) \subset \Delta_{\alpha,\rho}(\xi_n)$.*

*In particular, the set of singularities $E(g)$ has zero $\lambda$-measure.*

**Application. Periodic boundary points in transcendental dynamics**

One possible application of Pesin theory is to find periodic points. In complex dynamics, this idea was already exploited by F. Przytycki and A. Zdunik to find periodic points on the boundaries of basins for rational maps [PZ94]. Hence, we aim to apply Theorem 4.A and Theorem 4.B to find periodic boundary points in the transcendental setting.

To do so, we need a stronger assumption on the orbits of singular values inside $U$. Recall that, given a simply connected domain $U$, we say that $C \subset U$ is a *crosscut* if $C$ is a Jordan arc such that $\overline{C} = C \cup \{a, b\}$, with $a, b \in \partial U$, $a \neq b$. Any of the two connected components of $U \smallsetminus C$ is a *crosscut neighbourhood*. We define the *postsingular set* of $f$ as

$$P(f) \coloneqq \overline{\bigcup_{s \in SV} \bigcup_{n \geq 0} f^n(s)}.$$

**Corollary 4.D.** (Periodic boundary points are dense) *Under the hypotheses of Theorem 4.A or Theorem 4.B, assume, in addition, that there exists a crosscut neighbourhood $N_C$ with $N_C \cap P(f) = \emptyset$. Then, periodic points are dense on $\partial U$.*

**Lyapunov exponents of transcendental maps**

Finally, we note that one essential hypothesis in our results is that $\log |f'|$ is integrable with respect to the harmonic measure $\omega_U$, and hence the Lyapunov exponent

$$\chi_{\omega_U}(f) = \int_{\partial U} \log |f'| \, d\omega_U$$

is well-defined. We also require that $\chi_{\omega_U}$ is positive. These facts are well-known for simply connected basins of attraction of rational maps [Prz85, Prz93], but unexplored for transcendental maps. In this paper we give several conditions, concerning the order of growth of the function and the shape of the Fatou component, which implies that the Lyapunov exponent is well-defined and non-negative.

One of the main challenges that appear when considering transcendental maps is that $|f'|$ may not be bounded in $\partial U$, even when taking the derivative with respect to the spherical metric. Indeed, $|f'|$ is not bounded around the essential singularity, and the growth can be arbitrarily fast. Thus, we introduce the following concept, which relates the growth of the function with the shape of the Fatou component.

**Definition. (Order of growth in a sector for meromorphic functions)** Let $f \colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function, and let $U \subset \mathbb{C}$ be an invariant Fatou component for $f$. We say that $U$ is *asymptotically contained in a sector of angle $\alpha \in (0,1)$ with order of growth $\beta > 0$* if there exists $R_0 > 0$, $\xi \in \partial \mathbb{D}$ and $\alpha \in (0, 1)$, such that, if

$$S_R = S_{R,\alpha} \coloneqq \{z \in \mathbb{C} \colon \; |z| > R, \; |\text{Arg } \xi - \text{Arg } (1/\overline{z})| < \pi\alpha\}$$

then,

(a) $U \cap \{z \in \mathbb{C} \colon \; |z| > R_0\} \subset S_{R_0}$;

(b) $f$ has order of growth $\beta > 0$ in $S_{R_0}$, i.e. there exists $A, B > 0$ such that, for all $R > R_0$ and $z \in S_{R_0} \setminus S_R$,

$$A \cdot e^{B \cdot |z|^{-\beta}} \leq |f'(z)| \leq A \cdot e^{B \cdot |z|^{\beta}}.$$

Under this asusmption on the growth, we are able to prove the following.

**Proposition 4.E. ($\log |f'|$ is $\omega_U$-integrable)** *Let $f: \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be an invariant Fatou component for $f$. Let $\omega_U$ be a harmonic measure on $\partial U$. Assume $U$ is asymptotically contained in a sector of angle $\alpha \in (0, 1)$, with order of growth $\beta \in (0, \frac{1}{2\alpha})$. Then, $\log |f'| \in L^1(\omega_U)$.*

**Proposition 4.F. (Non-negative Lyapunov exponents)** *Let $f: \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $U$ be a simply connected attracting basin, with fixed point $p \in U$. Let $\omega_U$ be the harmonic measure in $\partial U$ with base point $p$. Assume*

(a) *$U$ is asymptotically contained in a sector of angle $\alpha \in (0, 1)$, with order of growth $\beta \in (0, \frac{1}{2\alpha})$;*

(b) $\int_{\partial U} \log |x - SV|^{-1} \, d\omega_U(x) < \infty$.

*Then,*

$$\chi_{\omega_U} = \int_{\partial U} \log |f'| \, d\omega_U \geq 0.$$

**Remark.** The statements of Theorem 4.A and Theorem 4.B, and Corollary 4.C are a simplified version of the ones proved inside the paper (respectively, Thms. 4.3.1, 4.4.4 and 4.5.1). These stronger statements are formulated in terms of the Rohklin's natural extension of the corresponding dynamical systems. Since this construction is not common in transcendental dynamics (although it is standard in ergodic theory), we chose to present our results in this simplified (and weaker) form. For convenience of the reader, all the needed results about Rohklin's natural extension can be found in Section 4.1.

Although the results are stated here for meromorphic functions, we shall work in the more general class $\mathbb{K}$ of functions with countably many singularities; in particular, this allows us to consider *periodic* attracting basins of meromorphic maps, not only invariant ones.

**Remark.** It seems plausible to extend the previous results to multiply connected Fatou components, as long the harmonic measure is well-defined. This is always the case of Fatou components in class $\mathbb{K}$ [FJ25].

## 4.1 Rokhlin's natural extension

A useful technique in the study of non-invertible measure-preserving tranformations is the so-called Rokhlin's natural extension [Roh64], which allows us to construct a measure-preserving automorphism in an abstract measure space, mantaining its ergodic properties.

However, this technique is often developed for Lebesgue spaces with invariant probabilities (see e.g [PU10, Sect. 1.7] , [URM22, Sect. 8.5]). Since we work also with $\sigma$-finite measures, we sketch how we can develop the theory in this more general case.

Let $(X, \mathcal{A}, \mu)$ be a *Lebesgue space*, i.e. a measure space isomorphic (in the measure-theoretical sense) to an interval (equipped with the Lebesgue measure) together with countably many atoms. Let $T \colon X \to X$ be measure-preserving. The measure $\mu$ is either finite (and we assume it is a probability measure), or $\sigma$-finite.

Consider the space of backward orbits for $T$

$$\widetilde{X} = \{\{x_n\}_n \subset X \colon \ x_0 \in X,\ T(x_{n+1}) = x_n,\ n \geq 0\},$$

and define, in a natural way, the following maps. On the one hand, for $k \geq 0$, let $\pi_k \colon \widetilde{X} \to X$ be the projection on the $k$-th coordinate of $\{x_n\}_n$, that is $\pi_k(\{x_n\}_n) = x_k$. On the other hand, we define *Rokhlin's natural extension* of $T$ as $\widetilde{T} \colon \widetilde{X} \to \widetilde{X}$, with

$$\widetilde{T}(\{x_n\}_n) = \widetilde{T}(x_0 x_1 x_2 \dots) = f(x_0) x_0 x_1 \dots$$

It is clear that $\widetilde{T}$ is invertible and $\widetilde{T}^{-1}$ is the shift-map, i.e.

$$\widetilde{T}^{-1}(\{x_n\}_n) = \widetilde{T}^{-1}(x_0 x_1 x_2 \dots) = x_1 x_2 x_3 \dots = \{x_{n+1}\}_n .$$

Moreover, for each $k \geq 0$, the following diagram commutes.

$$\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ \widetilde{T}\ } & \widetilde{X} \\
\{x_{n+1}\}_n & & \{x_n\}_n \\
\big\downarrow \pi_k & & \big\downarrow \pi_k \\
X & \xrightarrow{\ T\ } & X \\
x_{k+1} & & x_k
\end{array}$$

Note that, up to here, the construction is purely symbolic and measures have not come out yet. In fact, the next step in the construction is to endow the space $\widetilde{X}$ with an appropriate $\sigma$-algebra $\widetilde{\mathcal{A}}$ and a measure $\widetilde{\mu}$, which makes the previous projections $\pi_k$ and the map $\widetilde{T}$ measure-preserving. To do so, we will need the following more general result.

**Theorem 4.1.1. (Kolmogorov Consistency Theorem,** [Par67, Thm. V.3.2]**)** *Let $(X_n, \mathcal{A}_n, \mu_n)$ be Lebesgue probability spaces, and let $T_n \colon X_{n+1} \to X_n$ be measure-preserving. Let*

$$\widetilde{X} = \{\{x_n\}_n \colon \ x_n \in X_n,\ T_n(x_{n+1}) = x_n,\ n \geq 0\}.$$

*and let $\pi_k \colon \widetilde{X} \to X_k$ be the projection on the $k$-th coordinate. Then, there exists a $\sigma$-algebra $\widetilde{A}$ and a probability measure $\widetilde{\mu}$ in $\widetilde{X}$ such that $(\widetilde{X}, \widetilde{A}, \widetilde{\mu})$ is a Lebesgue probability space and, for each $k \geq 0$,*

$$\widetilde{\mu}(\pi_k^{-1}(A)) = \mu_k(A), \quad A \in \mathcal{A}_k.$$

Notice that the theorem above holds whenever $(X_n, \mathcal{A}_n, \mu_n)$ are Lebesgue measure spaces with finite measure. The $\sigma$-algebra $\widetilde{A}$ can be taken to be the smallest which makes each projection $\pi_k \colon \widetilde{X} \to X_k$ measurable [Par67, Thm. V.2.5]. Note that $T_k \circ \pi_{k+1} = \pi_k$. Observe that now $\widetilde{X}$ stands for the space of backward orbits under the sequence of maps $\{T_n\}_n$. Hence, one has to think of $\widetilde{X}$ as the infinite product of the spaces $\{X_n\}_n$, since the spaces in $\{X_n\}_n$ are *a priori* different, and hence there is no endomorphism $\widetilde{T} \colon \widetilde{X} \to \widetilde{X}$ in general. However, we will use these extensions $(\widetilde{X}, \widetilde{A}, \widetilde{\mu})$ of some appropriate spaces as building blocks for Rokhlin's natural extension for transformations with $\sigma$-finite invariant measures.

**Theorem 4.1.2. (Rokhlin's natural extension for $\sigma$-finite invariant measures)** *let $(X, \mathcal{A}, \mu)$ be a Lebesgue space, and let $T \colon X \to X$ be a measure-preserving transformation. Assume $\mu$ is a $\sigma$-finite measure, and consider Rokhlin's natural extension $\widetilde{T} \colon \widetilde{X} \to \widetilde{X}$. Then, there exists a $\sigma$-algebra $\widetilde{\mathcal{A}}$ and a $\sigma$-finite measure $\widetilde{\mu}$ such that the maps $\pi_k$ and $\widetilde{T}$ are measure-preserving.*

*Proof.* In the case of $(X, \mathcal{A}, \mu)$ being a Lebesgue probability space, the statement follows from applying Kolmogorov Consistency Theorem 4.1.1 with $X_n = X$, for all $n \geq 0$, as indicated in [URM22, Thm. 8.4.2].

Otherwise, let $\left\{X_0^j\right\}_j$ be a partition of $X$ such that $\mu(X_0^j)$ is finite, for each $j \geq 0$. Without loss of generality, we assume $\mu(X_0^j) = 1$, for each $j \geq 0$, to simplify the computations. Then, for all $n \geq 0$, $\left\{X_n^j := T^{-n}(X_0^j)\right\}_j$ is also a partition of $X$ such that $\mu(X_n^j) = 1$, for each $j \geq 0$, since $T$ is measure-preserving and preimages of disjoint sets are disjoint.

If we write $\mathcal{A}_n^j$ and $\mu_n^j$ for the restrictions of $\mathcal{A}$ and $\mu$ to $X_n^j$, we have that, for each $j \geq 0$, $(X_n^j, \mathcal{A}_n^j, \mu_n^j)$ is a Lebesgue probability space, and $T \colon X_{n+1}^j \to X_n^j$ is measure-preserving. Hence, by Theorem 4.1.1, there exists a Lebesgue probability space $(\widetilde{X^j}, \widetilde{\mathcal{A}^j}, \widetilde{\mu^j})$ such that

$$\widetilde{X^j} = \left\{\{x_n\}_n : \ x_n \in X_n^j, \ T(x_{n+1}) = x_n, \ n \geq 0\right\}.$$

and the projections $\pi_k^j \colon \widetilde{X^j} \to X_k^j$ are measure-preserving. The space of backward orbits

$$\widetilde{X} = \{\{x_n\}_n : \ x_n \in X, \ T_n(x_{n+1}) = x_n, \ n \geq 0\}$$

is the disjoint union of the $\widetilde{X^j}$, $j \geq 0$. Let $\mathcal{A}$ to be the $\sigma$-algebra generated by $\left\{\widetilde{\mathcal{A}^j}\right\}_j$, and the measure $\widetilde{\mu}$ on $(\widetilde{X}, \widetilde{A})$ unambiguously determined by the $\widetilde{\mu^j}$'s. It is clear that the maps $\pi_k$ are measure-preserving, for all $k \geq 0$.

It is left to see that $\widetilde{T}$ is measure-preserving. To do so, note that we have the following measure-preserving commutative diagram.

$$\begin{array}{ccccccc}
\ldots \xrightarrow{\widetilde{T}} & \underset{\{x_{n+2}\}_n}{\widetilde{X^{j_2}} \subset \widetilde{X}} & \xrightarrow{\quad\widetilde{T}\quad} & \underset{\{x_{n+1}\}_n}{\widetilde{X^{j_1}} \subset \widetilde{X}} & \xrightarrow{\quad\widetilde{T}\quad} & \underset{\{x_n\}_n}{\widetilde{X^{j_0}} \subset \widetilde{X}} \\
& \Big\downarrow {\scriptstyle \pi_0^{j_2}} & & \Big\downarrow {\scriptstyle \pi_0^{j_1}} & & \Big\downarrow {\scriptstyle \pi_0^{j_0}} \\
\ldots \xrightarrow{T} & \underset{x_2}{X_2^j \subset X} & \xrightarrow{\quad T \quad} & \underset{x_1}{X_1^j \subset X} & \xrightarrow{\quad T \quad} & \underset{x_0}{X_0^{j_0} \subset X}
\end{array}$$

Since the sets $\{(\pi_n^j)^{-1}(A \cap X_n^j) \colon A \in \mathcal{A}\}_{n,j}$ generate the $\sigma$-algebra $\widetilde{A}$, it is enough to prove invariance for such sets. Thus, without loss of generality, let $A \subset X_n^j$, and then

$$\begin{aligned}\widetilde{\mu} \circ \widetilde{T}^{-1}((\pi_n^j)^{-1}(A)) = \widetilde{\mu} \circ (\pi_n^j \circ \widetilde{T})^{-1}(A) = \widetilde{\mu} \circ (T \circ \pi_n^j)^{-1}(A) = \widetilde{\mu} \circ (\pi_n^j)^{-1} \circ T^{-1}(A) \\ = \mu(T^{-1}(A)) = \mu(A) = \widetilde{\mu}((\pi_n^j)^{-1}(A)),\end{aligned}$$

as desired.

□

It follows from the previous theorem that $\widetilde{\mu}$ is a probability measure if and only if so is $\mu$. Natural extensions share many ergodic properties with the original map, as shown in the following proposition for probability spaces.

**Proposition 4.1.3. (Ergodic properties of Rokhlin's natural extension)** *Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space, endowed with a measure-preserving transformation $T \colon X \to X$, and consider its Rokhlin's natural extension $\widetilde{T}$ acting in $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\mu})$, given by Theorem 4.1.2. Then, the following holds.*

(a) *$\widetilde{T}$ is recurrent with respect to $\widetilde{\mu}$.*

(b) *$\widetilde{T}$ is ergodic with respect to $\widetilde{\mu}$ if and only if $T$ is ergodic with respect to $\mu$.*

*Proof.* Since $\mu$ is assumed to be a probability measure, $\widetilde{\mu}$ is also a probability measure, and the recurrence of $\widetilde{T}$ follows from Poincaré Recurrence Theorem. For (b), see [URM22, Thm. 8.4.3]. □

Under the assumption of ergodicity and recurrence, we can prove that every subset of positive measure in the phase space is visited by almost every backward orbit.

**Corollary 4.1.4. (Almost every backward orbit is dense)** *Let $(X, \mathcal{A}, \mu)$ be a Lebesgue space, endowed with a measure-preserving transformation $T \colon X \to X$, and consider its Rokhlin's natural extension $\widetilde{T}$ acting in $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\mu})$, given by Theorem 4.1.2. Assume $\widetilde{T}$ is ergodic and recurrent with respect to $\widetilde{\mu}$, and $A \subset X$ is a measurable set with $\mu(A) > 0$. Then, for $\widetilde{\mu}$-almost every $\{x_n\}_n \in \widetilde{X}$, there exists a sequence $n_k \to \infty$ such that $x_{n_k} \in A$.*

*Proof.* Since $\widetilde{T}$ is ergodic and recurrent with respect to $\widetilde{\mu}$, by Theorem I.1.7, for every $\widetilde{A} \in \widetilde{\mathcal{A}}$ with $\widetilde{\mu}(\widetilde{A}) > 0$ and $\widetilde{\mu}$-almost every $\{x_n\}_n \in \widetilde{X}$, there exists a sequence $n_k \to \infty$ such that $\widetilde{T}^{-n_k}(\{x_n\}_n) \in \widetilde{A}$. Taking $\widetilde{A}$ to be $\pi_0^{-1}(A)$, we have that $\widetilde{\mu}(\widetilde{A}) > 0$, so for $\widetilde{\mu}$-almost every $\{x_n\}_n \in \widetilde{X}$, there exists a sequence $n_k \to \infty$ with

$$\widetilde{T}^{-n_k}(\{x_n\}_{n \geq 0}) = \{x_n\}_{n \geq n_k} \in \pi_0^{-1}(A).$$

Hence, $x_{n_k} \in A$, as desired. □

## 4.2 On the hypothesis

In this section we shall discuss the hypothesis we establish (specifically, $\int_{\partial U} \log|x - SV|^{-1} d\omega_U(x) < \infty$).

We need the following lemma, which gives a geometric intuition on the condition $\int_{\partial U} \log|x - SV|^{-1} d\omega_U(x) < \infty$, by showing that singular values are 'not too dense' on $\widehat{\partial} U$ with respect to harmonic measure.

**Lemma 4.2.1.** *Let $f \in \mathbb{K}$, and $U$ a Fatou component. Then, $\int_{\partial U} \log|x - SV|^{-1} d\omega_U(x) < \infty$ if and only if for any $C > 0$ and $t \in (0,1)$,*

$$\sum_{n\geq 0} \omega_U \left( \bigcup_{s\in SV} D(s, C\cdot t^n) \right) < \infty.$$

**Remark 4.2.2.** We note that $\int_{\partial U} \log|x - SV|^{-1} d\omega_U(x) < \infty$ always holds if $SV \cap \widehat{\partial} U$ is finite. Indeed, given any simply connected domain $U$, for every $a \in \mathbb{C}$, $\log|z-a| \in L^1(\omega_U)$ [Con95, Prop. 21.1.18].

*Proof of Lemma 4.2.1.* First note that, since we are working with the spherical metric, $\log|x - SV|^{-1}$ is uniformly bounded above. Hence, one has only to examine the previous integral close to singular values. Let $0 < t < 1$, and

$$A_n := \left\{ z \in \mathbb{C} \colon t^{n+1} \leq |z - SV| < t^n \right\}; \quad D_n := \bigcup_{s\in SV} D(s, C\cdot t^n) = \{ z \in \mathbb{C} \colon |z - SV| < t^n \}.$$

Then,

$$\int_{\partial U} \log|x - SV|^{-1} d\omega_U(x) \geq \sum_n \log(t^{n+1}) \cdot \omega_U(A_n) = -\log t \cdot \sum_n (n+1) \cdot \omega_U(A_n).$$

This already implies that $\sum_{n\geq 0} \omega_U(D_n) = \sum_n n \cdot \omega_U(A_n) < \infty$, for every $t \in (0,1)$. Since for every $C > 0$ and $0 < t < 1$ exists $0 < s < 1$ with $C \cdot t^n < s^n$ for $n$ sufficiently large, the claim of the lemma follows.

For the converse, note that $\sum_{n\geq 0} \omega_U(D_n) = \sum_n n \cdot \omega_U(A_n) < \infty$, implying that

$$\infty > -\log t \cdot \sum_n n \cdot \omega_U(A_n) = -\sum_n \log(t^n) \cdot \omega_U(A_n) \geq \int_{\partial U} \log|x - SV|^{-1} d\omega_U(x),$$

as desired. □

**Remark 4.2.3.** Note that Lemma 4.2.1 already implies that $SV \cap \widehat{\partial} U$ have zero harmonic measure.

## 4.3 Pesin theory for attracting basins. Theorem 4.A

In this section, we take on the main challenge of this paper: developing Pesin theory for a simply connected attracting basin $U$ of a function of class $\mathbb{K}$, or, in other words, proving that generic infinite inverse branches are well-defined on $\partial U$.

The easiest assumption one shall make to get that generic infinite inverse branches are well-defined in $\partial U$, is that there exists $x \in \partial U$ and $r > 0$ so that $D(x,r) \cap P(f) = \emptyset$. Indeed, in such case, all iterated inverse branches are well-defined in $D(x,r)$. Moreover, since $f|_{\partial U}$ is ergodic and recurrent, and $D(x,r)$ has positive harmonic measure, it follows that the forward orbit of $\omega_U$-almost every $y \in \partial U$ eventually falls in $D(x,r)$, so all iterated inverse branches are well-defined around $y$.

The previous method has a main limitation: it does not work when $\partial U \subset P(f)$. Even in the case where $f$ is a polynomial, one can find examples for which $\partial U \subset P(f)$, or even $\mathcal{J}(f) \subset P(f)$. Our goal is precisely to show that, even in the case where $\partial U \subset P(f)$, if $\int_{\partial U} \log |x - SV|^{-1} d\omega_U(x) < \infty$, generic inverse branches are well-defined. Hence, one should work with each infinite backward orbit separatedly, and try to find a disk where the inverse branches corresponding to this backward orbit are well-defined, but other inverse branches may fail to be defined. Here is where Rohklin's natural extension plays a crucial role.

Therefore, let $U$ be a simply connected attracting basin for a map $f \in \mathbb{K}$ with fixed point $p \in U$, and consider the measure-theoretical dynamical system given by $(\partial U, \omega_U, f)$, where $\omega_U$ is the harmonic measure with basepoint $p$. Note that, through this section, $\omega_U$ stands for the harmonic measure with basepoint $p$, although we do not write it explicitly. Recall that $\omega_U$ is $f$-invariant, ergodic and recurrent. Note also that we omit the dependence of the previous dynamical system on the $\sigma$-algebra $\mathcal{B}(\widehat{\mathbb{C}})$, in order to lighten the notation.

Now, consider the natural extension of $(\partial U, \omega_U, f)$, denoted by $(\widetilde{\partial U}, \widetilde{\omega_U}, \widetilde{f})$, and given by the projecting morphisms $\{\pi_{U,n}\}_n$. We note that $(\partial U, \omega_U, f)$ is indeed a Lebesgue probability space (in fact, it is isomorphic, in the measure-theoretical sense, to the unit interval), and hence Theorem 4.1.2 can be applied to guarantee the existence of Rokhlin's natural extension. Thus, $\widetilde{\partial U}$ is the space of backward orbits $\{x_n\}_n \subset \partial U$, with $f(x_{n+1}) = x_n$ for $n \geq 0$, and $\widetilde{f} \colon \widetilde{\partial U} \to \widetilde{\partial U}$ is the automorphism which makes the following diagram commute.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\widetilde{f}} & \underset{\{x_{n+2}\}_n}{\widetilde{\partial U}} & \xrightarrow{\widetilde{f}} & \underset{\{x_{n+1}\}_n}{\widetilde{\partial U}} & \xrightarrow{\widetilde{f}} & \underset{\{x_n\}_n}{\widetilde{\partial U}} & \xrightarrow{\widetilde{f}} & \cdots \\
 & & \downarrow{\scriptstyle \pi_{U,n}} & & \downarrow{\scriptstyle \pi_{U,n}} & & \downarrow{\scriptstyle \pi_{U,n}} & & \\
\cdots & \xrightarrow{f} & \underset{x_{n+2}}{\partial U} & \xrightarrow{f} & \underset{x_{n+1}}{\partial U} & \xrightarrow{f} & \underset{x_n}{\partial U} & \xrightarrow{f} & \cdots
\end{array}$$

Since the natural extension inherits the ergodic properties of the original dynamical system, we have that $\widetilde{\omega_U}$ is an $\widetilde{f}$-invariant, ergodic and recurrent probability (Prop. 4.1.3). Moreover, for every measurable set $A \subset \partial U$ with $\mu(A) > 0$ and $\widetilde{\omega_U}$-almost every $\{x_n\}_n \in \widetilde{\partial U}$, there exists a sequence $n_k \to \infty$ such that $x_{n_k} \in A$ (Corol. 4.1.4).

We shall rephrase Theorem 4.A in terms of Rokhlin's natural extension as follows.

**Theorem 4.3.1. (Inverse branches are well-defined almost everywhere)** *Let $f \in \mathbb{K}$, and let $U$ be a simply connected attracting basin for $f$, with fixed point $p \in U$. Let $\omega_U$ be the harmonic measure in $\partial U$ with base point $p$. Assume:*

(a) $\log|f'| \in L^1(\omega_U)$, *and* $\chi_{\omega_U}(f) > 0$;

(b) $\sum_n \omega_U(D(x, M^n)) < \infty$, *for every* $M \in (0,1)$.

*Then, for* $\widetilde{\omega_U}$*-almost every* $\{x_n\}_n \in \widetilde{\partial U}$, *there exists* $r := r(\{x_n\}_n) > 0$ *such that*

(i) *the inverse branch* $F_n$ *sending* $x_0$ *to* $x_n$ *is well-defined in* $D(x_0, r)$;

(ii) *for every* $\chi \in (-\chi_{\omega_U}, 0)$, *there exists* $C > 0$ *such that, for all* $n \in \mathbb{N}$, $|F_n'(x_0)| < C \cdot e^{\chi \cdot n}$;

(iii) *for every* $r_0 \in (0, r)$, *there exists* $m \in \mathbb{N}$ *such that* $F_m(D(x_0, r)) \subset D(x_0, r_0)$.

We show now how to deduce Theorem 4.A from Theorem 4.3.1, and later we give the proof of it.

*Proof of Theorem 4.A.* The assumptions of Theorem 4.A and 4.3.1 are equivalent, by Lemma 4.2.1. We have to see that the conclusions of Theorem 4.A can be derived from the ones of Theorem 4.3.1. But this follows straightforward from Corollary 4.1.4. Indeed, since $\widetilde{f}$ is ergodic and recurrent with respect to $\widetilde{\omega_U}$, for any $A \subset X$ measurable set with $\omega_U(A) > 0$, for $\widetilde{\omega_U}$-almost every $\{x_n\}_n \in \widetilde{X}$, there exists a sequence $n_k \to \infty$ such that $x_{n_k} \in A$. It follows that, for every countable collection of measurable sets $\{A_k\}_k \subset \partial U$ with $\omega_U(A_k) > 0$, then for $\widetilde{\omega_U}$-almost every $\{x_n\}_n \in \widetilde{X}$, there exists a sequence $n_k \to \infty$ such that $x_{n_k} \in A_k$. □

**Remark 4.3.2.** Before starting the proof let us note that we are assuming $f \in \mathbb{K}$ just because it is the largest class of functions in which Fatou components are defined. We do not use the fact that functions in class $\mathbb{K}$ have only countably many singularities, we only use that singular values are 'not too dense' on $\partial U$ (hypothesis (b)).

The remaining of the section is devoted to prove Theorem 4.3.1.

### 4.3.1 Proof of Theorem 4.3.1

Recall that $\omega_U$ is a $f$-invariant ergodic probability in $\partial U$. We fix $M \in (e^{\frac{1}{4}\cdot\chi}, 1)$.

**Lemma 4.3.3. (Almost every backward orbit does not come close to singular values)** *For* $\widetilde{\omega_U}$*-almost every* $\{x_n\}_n \in \widetilde{\partial U}$, *it holds*

(1.1) $x_0 \notin \bigcup\limits_{s \in SV} \bigcup\limits_{n \geq 0} f^n(s)$,

(1.2) $\lim\limits_n \dfrac{1}{n} \log|(f^n)'(x_n)| = \chi_{\omega_U}(f)$,

(1.3) *if* $D_n := \bigcup\limits_{s \in SV} D(s, M^n)$, *then* $x_n \in D_n$ *only for a finite number of* $n$*'s.*

*Proof.* Since the finite intersection of sets of full measure has full measure, it is enough to show that each of the conditions is satisfied in a set of full measure.

Condition (1.1) follows from $\omega_U(SV(f)) = 0$. Indeed, $f$ is holomorphic, and hence absolutely continuous, we have $\omega_U(\bigcup_{s\in SV(f)} \bigcup_{n\geq 0} f^n(s)) = 0$.

Requirement (1.2) follows from Birkhoff Ergodic Theorem I.1.9 applied to the map $\log|f'|$, which is integrable by assumption (a). Indeed, for $\widetilde{\omega_U}$-almost every $\{x_n\}_n \in \widetilde{\partial U}$, it holds

$$\chi_{\omega_U}(f) = \int_{\partial U} \log|f'|\, d\omega_U = \lim_m \frac{1}{m}\sum_{k=0}^{m-1} \log\left|f'(f^k(x_0))\right| =$$
$$= \lim_m \frac{1}{m}\sum_{k=0}^{n-1} \log\left|f'(f^k(\pi_{U,0}(\{x_n\}_n)))\right| = \lim_m \frac{1}{m}\sum_{k=0}^{m-1} \log\left|f'(\pi_{U,0}(\widetilde{f}^k(\{x_n\}_n)))\right|,$$

where in the last two equalities we used the properties of Rokhlin's natural extension.

Now, $\widetilde{f}$ is a measure-preserving automorphism, and, since $\log|f'| \in L^1(\omega_U)$, $\log|f'\circ\pi_{U,0}| \in L^1(\widetilde{\omega_U})$. Then, Birkhoff Ergodic Theorem yields

$$\lim_m \frac{1}{m}\sum_{k=0}^{m-1} \log\left|f'(\pi_{U,0}(\widetilde{f}^k(\{x_n\}_n)))\right| = \lim_m \frac{1}{m}\sum_{k=0}^{m-1} \log\left|f'(\pi_{U,0}(\widetilde{f}^{-k}(\{x_n\}_n)))\right| =$$
$$= \lim_m \frac{1}{m}\sum_{k=0}^{m-1} \log|f'(x_k)| = \lim_m \frac{1}{m}\log(|f'(x_0)|\ldots|f'(x_m)|) = \lim_m \frac{1}{m}\log|(f^m)'(x_m)|.$$

Putting everyting together, we get that for $\widetilde{\omega_U}$-almost every $\{x_n\}_n$, it holds

$$\lim_n \frac{1}{n}\log|(f^n)'(x_n)| = \chi_{\omega_U}(f).$$

For condition (1.3), note that by hypothesis (b),

$$\sum_{n\geq 1} \widetilde{\omega_U}(\pi_{U,n}^{-1}(D_n)) = \sum_{n\geq 1} \omega_U(D_n) < \infty,$$

Thus, by the Borel-Cantelli Lemma I.1.4, for $\widetilde{\omega_U}$-almost every $\{x_n\}_n \in \widetilde{\partial U}$, $x_n \in D_n$ for only finitely many $n$'s, as desired. This ends the proof of the Lemma. □

Let us fix a backward orbit $\{x_n\}_n$ satifying the conditions of the previous lemma. By (1.3), there exists $n_1 \in \mathbb{N}$ such that, for $n \geq n_1$, $x_n \notin \bigcup_{s\in SV} D(s, M^n)$. Moreover, by (1.2), there exists $n_2 \in \mathbb{N}$, $n_2 \geq n_1$ such that, for $n \geq n_2$,

$$|(f^n)'(x_n)|^{-\frac{1}{4}} < M^n < 1.$$

Note that both $n_1$ and $n_2$ do not only depend on the starting point $x_0$, but on all the backward orbit $\{x_n\}_n$. Two different backward orbits starting at $x_0$ may require different $n_1$ or $n_2$.

Let

$$b_n := \left|(f^{n+1})'(x_n)\right|^{-\frac{1}{4}}, \qquad P := \prod_{n\geq 1}(1-b_n).$$

Observe that, since $\sum_n b_n \leq \sum_n M^n < \infty$, the infinite product in $P$ is convergent. Choose $r := r(\{x_n\}_n) > 0$ such that

(2.1) $2rP < 1$,

(2.2) for $1 \leq n \leq n_2$, the branch $F_n$ of $f^{-n}$ sending $x_0$ to $x_n$ is well-defined in $D(x_0, r)$,

(2.3) $F_{n_2}(D(x_0, r \prod_{m=1}^{n_2}(1-b_m))) \subset D(x_{n_2}, M^{n_2})$.

The remaining inverse branches will be constructed by induction (Claim 4.3.4), but first let us note that such a $r > 0$ exists. Indeed, it follows from the fact that the inverse branch $F_{n_2}$ sending $x_0$ to $x_{n_2}$ is well-defined in an open neighbourhood of $x_0$, since the set of singular values of $f^{n_2}$ is closed, and $x_0 \notin SV(f^{n_2})$.

**Claim 4.3.4.** *For every $n \geq n_2$, there exists an inverse branch $F_n$ sending $x_0$ to $x_n$, defined in $D(x_0, r \prod_{m=1}^{n}(1-b_m))$, and such that*

$$F_n(D(x_0, r \prod_{m=1}^{n}(1-b_m))) \subset D(x_n, M^n).$$

Note that proving the claim ends the proof of the theorem. Indeed, letting $n \to \infty$ we get that all inverse branches are well-defined in $D(x_0, rP)$, i.e. in a disk centered at $x_0$ of positive radius. The estimate on the derivative follows from $|(f^n)'(x_n)|^{-\frac{1}{4}} < M^n < 1$, with $M \in (e^{\frac{1}{4}\chi}, 1)$, for $n \geq n_2$.

*Proof of the claim.* Suppose the claim is true for $n \geq n_2$, and let us see that it also holds for $n+1$. First, note that $D(x_n, M^n) \cap SV = \emptyset$ for all $n \geq n_2$ (by the choice of $n_2$). Hence, there exists a branch $F$ of $f^{-1}$ satisfying $F(x_n) = x_{n+1}$, well-defined in $D(x_n, M^n)$. By the inductive hypothesis, there exists an inverse branch $F_n$ sending $x_0$ to $x_n$, defined in $D_n \coloneqq D(x_0, r \prod_{m=1}^{n}(1-b_m))$, and such that $F_n(D_n) \subset D(x_n, M^n)$. Set $F_{n+1} = F \circ F_n$. Then, $F_{n+1}$ is well-defined in $D_n$, and sends $x_0$ to $x_{n+1}$.

Now we use Koebe's distortion estimates (Thm. I.3.3) to prove the bound on the size of $F_{n+1}(D_{n+1})$, where $D_{n+1} \coloneqq D(x_0, r \prod_{m=1}^{n+1}(1-b_m)) \subset D_n$. Note that $F_{n+1}$ is well-defined in $D_n$, which is strictly larger than $D_{n+1}$, and the ratio between the two radii of both disks is $(1-b_n)$. Since $F_{n+1}|_{D_n}$ is univalent, we have $F_{n+1}(D_{n+1}) \subset D(x_{n+1}, R)$, where

$$R = r \cdot \prod_{m=1}^{n+1}(1-b_m) \cdot |(F_{n+1})'(x_0)| \cdot \frac{2}{b_n^3} \leq 2r \cdot \frac{|(f^{n+1})'(x_{n+1})|^{-1}}{|(f^{n+1})'(x_{n+1})|^{-\frac{3}{4}}} \leq \left|(f^{n+1})'(x_{n+1})\right|^{-\frac{1}{4}} \leq M^{n+1},$$

as desired. □

As noted before, this last claim ends the proof of Theorem 4.3.1. □

## 4.4 Entire functions and the first return map. Theorem 4.B

In this section, we extend Theorem 4.A to parabolic and Baker domains of entire maps. The main challenge is that there does not exist an invariant probability which is absolutely continuous with respect to harmonic measure. However, the existence of an invariant $\sigma$-finite measure in $\partial U$ absolutely continuous with respect to $\omega_U$ will allow us to perform Pesin theory, by means of the first return map.

We shall start by constructing Rokhlin's natural extension (note that this is indeed possible due to the existence of the $\sigma$-invariant measure). We do this by showing that Rokhlin's natural extension is compatible with the use of first return maps if the transformation we consider is recurrent. This allows us to move from our problem of finding inverse branches in a space endowed with a $\sigma$-invariant measure to a probability space, where we can perform Pesin theory in a stardard way. We do this construction of the first return map and Rokhlin's natural extension in Section 4.4.1, and finally we develop Pesin theory in Section 4.4.2.

### 4.4.1 The first return map and Rokhlin's natural extension

Assume $U$ is a parabolic basin or a Baker domain, such that $f|_{\partial U}$ is recurrent. The measure

$$\lambda_{\mathbb{R}}(A) = \int_A \frac{1}{|w-1|^2} d\lambda(w), \qquad A \in \mathcal{B}(\partial\mathbb{D}),$$

is invariant under the radial extension of the associated inner function $g$ (taken such that 1 is the Denjoy-Wolff point) and its push-forward $\mu = (\varphi^*)_*\lambda_{\mathbb{R}}$ is an infinite $\sigma$-finite invariant measure supported in $\widehat{\partial} U$ (see Thm. II.5.5).

One can consider the Rokhlin's natural extension. Indeed, let $\widetilde{\partial U}$ be the space of backward orbits $\{x_n\}_n \subset \partial U$, with $f(x_{n+1}) = x_n$ for $n \geq 0$, and let $\widetilde{f}\colon \widetilde{\partial U} \to \widetilde{\partial U}$ be the automorphism which makes the following diagram commute.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\widetilde{f}} & \underset{\{x_{n+2}\}_n}{\widetilde{\partial U}} & \xrightarrow{\widetilde{f}} & \underset{\{x_{n+1}\}_n}{\widetilde{\partial U}} & \xrightarrow{\widetilde{f}} & \underset{\{x_n\}_n}{\widetilde{\partial U}} & \xrightarrow{\widetilde{f}} & \cdots \\
 & & \downarrow{\scriptstyle \pi_{U,n}} & & \downarrow{\scriptstyle \pi_{U,n}} & & \downarrow{\scriptstyle \pi_{U,n}} & & \\
\cdots & \xrightarrow{f} & \underset{x_{n+2}}{\partial U} & \xrightarrow{f} & \underset{x_{n+1}}{\partial U} & \xrightarrow{f} & \underset{x_n}{\partial U} & \xrightarrow{f} & \cdots
\end{array}$$

One can get an equivalent construction of backward orbits by means of the first return map. Indeed, let $E \subset \partial U$ be a measurable set with $\mu(E) \in (0, \infty)$ (we will fix $E$ later). Consider the *first return map* to $E$, i.e.

$$\begin{aligned} f_E\colon E &\longrightarrow E \\ x &\longmapsto f^{T(x)}(x), \end{aligned}$$

where $T(x)$ denotes the first return time of $x$ to $E$. We consider the measure-theoretical dynamical system $(E, \mu_k, f_{X_k})$, where

$$\mu_E(A) \coloneqq \frac{\mu(A \cap E)}{\mu(E)},$$

for every measurable set $A \subset \partial U$. Note that $(E, \mu_E)$ is a probability space. The following properties of the first return map $f_E$ will be needed.

**Lemma 4.4.1. (First return map)** *Let $f_E\colon E \to E$ be defined as above. Then, the following holds.*

(1.1) *$\mu_k$ is invariant under $f_E$. In particular, $f_E$ is recurrent with respect to $\mu_E$.*

(1.2) *$f_E$ is ergodic with respect to $\mu_E$.*

(1.3) *If $\log|f'| \in L^1(\mu)$ , then $\log|f'_E(x)| := \log\left|(f^{T(x)})'(x)\right| \in L^1(\mu_E)$ and*

$$\int_{X_k} \log|f'_E|\, d\mu_k = \frac{1}{\mu(E)} \int_{\partial U} \log|f'|\, d\mu.$$

*Proof.* The three claims are standard facts of measure-theoretical first return maps. More precisely, (1.1) and (1.2) follow from [URM22, Prop. 10.2.1] and [URM22, Prop. 10.2.7], respectively. Statement (1.3) comes from [URM22, Prop. 10.2.5], applied to $\varphi = \log|f'|$ and $\varphi_E = \log|f'_E|$ . □

Since $(X_E, \mu_E)$ is a Lebesgue probability space, and $\mu_E$ is $f_E$-invariant, we shall consider its Rohklin's natural extension, denoted by $(\widetilde{E}, \widetilde{f_E})$, and given by the projecting morphisms $\{\pi_{E_n}\}_n$. Thus, $\widetilde{E}$ is the space of backward orbits $\left\{x_n^E\right\}_n \subset \partial U$, with $f_E(x_{n+1}^E) = x_n^E$ for $n \geq 0$, and $\widetilde{f_E}\colon \widetilde{E} \to \widetilde{E}$ is the automorphism which makes the following diagram commute.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\widetilde{f_E}} & \widetilde{E} & \xrightarrow{\widetilde{f_E}} & \widetilde{E} & \xrightarrow{\widetilde{f_E}} & \widetilde{E} & \xrightarrow{\widetilde{f_E}} & \cdots \\
 & & \left\{x_{n+2}^E\right\}_n & & \left\{x_{n+1}^E\right\}_n & & \left\{x_n^E\right\}_n & & \\
 & & \downarrow \pi_{E_n} & & \downarrow \pi_{E_n} & & \downarrow \pi_{E_n} & & \\
\cdots & \xrightarrow{f_E} & E & \xrightarrow{f_E} & E & \xrightarrow{f_E} & E & \xrightarrow{f_E} & \cdots \\
 & & x_{n+2}^E & & x_{n+1}^E & & x_n^E & &
\end{array}$$

Since the natural extension of a probability space inherits the ergodic properties of the original system, we have that $\widetilde{\mu_E}$ is $\widetilde{f_E}$-invariant, ergodic and recurrent (Prop. 4.1.3).

We claim that both constructions of spaces of backward orbits are essentially the same, with the only difference that, when considering the first return map, orbits starting at the set $E$ are written 'packed' according to their visits to $E$.

Indeed, given a backward orbit $\left\{x_n^E\right\}_n \subset E$ for $f_E$, we can associate to it unambiguously a backward orbit $\{x_n\}_n \subset \partial U$ for $f$ as follows. Let $x_0 := x_0^E$, and let $x_{T(x_1^E)} := x_1^E$. Since $f_E(x_1^E) = f^{T(x_1^E)}(x_1^E) = x_0^E$, for $n = 1, \dots, T(x_1^E) - 1$, let $x_n := f^{T(x_1^E)-n}(x_1^E)$. The rest of the backward orbit is defined recursively. We say that the $f$-backward orbit $\{x_n\}_n \subset \partial U$ is *associated* to the $f_E$-backward orbit $\left\{x_n^E\right\}_n \subset E$. In the same way, if a $f$-backward orbit visits $E$ infinitely often, we can associate a $f_E$-backward orbit to it.

As noted above, for every $f_E$-backward orbit we can associate a $f$-backward orbit. Moreover, due to recurrence, the converse is true $\widetilde{\mu}$-almost everywhere. Hence, it is enough to consider $f_E$-backward orbits.

**Lemma 4.4.2. (Distribution of $f_E$-backward orbits in $\partial U$)** *Let $(\partial U, \mu, f)$ and $(E, \mu_E, f_E)$, and consider their natural extensions as before. Then, for $\widetilde{\mu}$-almost every $\{x_n\}_n \subset \partial U$ with $x_0 \in E$, $x_n \in E$ infinitely often, so we can associate a $f_E$-backward orbit to it.*

Let us now fix the set $E$, whose first return map will enjoy specific properties.

**Proposition 4.4.3. (The set $E$)** *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a meromorphic function, and let $U$ be a simply connected parabolic basin or Baker domain. Consider $\varphi\colon \mathbb{H} \to U$, and let $h\colon \mathbb{H} \to \mathbb{H}$ be the inner function associated with $(f, U)$, with the Denjoy-Wolff point placed at $\infty$, which we assume not to be a singularity. Let $p_1^{\pm}, p_2^{\pm} \dots$ be the (radial) preimages of $\infty$, ordered such that $p_1^- < p_2^- < \cdots < p_2^+ < p_1^+$.*

*Let $I \coloneqq \left[p_1^-, p_1^+\right]$, and $E \coloneqq \varphi^*(I)$. Then, as $n \to \infty$,*

$$\mu(\{x \in E\colon T(x) \geq n\}) \sim \frac{2}{\sqrt{n}}.$$

*Proof.* The existence of the set $I$ is proven in [IU23, Sect. 9.2], together with the fact that $\lambda(I^C \cap \{T(x) = n\}) \sim \frac{2}{\sqrt{n}}$. The standard fact for $\sigma$-finite measures $\lambda$ and sweep-out sets $I$

$$\lambda(I \cap \{T(x) = n\}) = \lambda(I^C \cap \{T(x) > n\})$$

(see e.g. [Tha01]) gives that $\lambda(I \cap \{T(x) > n\}) \sim \frac{2}{\sqrt{n}}$. The estimates for the system $(f|_{\partial U}, \mu)$ follow from the definition of the measure $\mu$ as the push-forward of $\lambda_{\mathbb{R}}$ under $\varphi^*$. $\square$

### 4.4.2 Pesin theory for the first return map. Proof of Theorem 4.B

We shall start by rewriting Theorem 4.B in terms of the space of backward orbits given by Rokhlin's natural extension.

**Theorem 4.4.4. (Inverse branches are well-defined almost everywhere)** *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a meromorphic function, and let $U$ be a simply connected parabolic basin or Baker domain. Assume*

(a) *the Denjoy-Wolff point of the associated inner function is not a singularity;*

(b) $\log|f'| \in L^1(\mu)$, *and* $\int_{\partial U} \log|f'|\, d\mu > 0$;

(c) *there is $\varepsilon > 0$ such that, if $\partial U_{+\varepsilon} \coloneqq \{z \in \mathbb{C}\colon \operatorname{dist}(z, \partial U) < \varepsilon\}$, $SV \cap \partial U_{+\varepsilon}$ is finite.*

*Then, for $\widetilde{\mu}$-almost every backward orbit $\{x_n\}_n \in \widetilde{\partial U}$, there is $r \coloneqq r(\{x_n\}_n) > 0$ such that*

(i) *the inverse branch $F_n$ sending $x_0$ to $x_n$ is well-defined in $D(x_0, r_0)$;*

(ii) *for every $r \in (0, r_0)$, there exists $m \in \mathbb{N}$ such that $F_m(D(x_0, r_0)) \subset D(x_0, r)$, and* $\operatorname{diam} F_m^j(D(x_0, r)) \to 0$, *as $j \to \infty$.*

It is clear that 4.4.4 implies Theorem 4.B (for hypothesis (a), see Thm. II.5.4; for (b), see Prop. 4.8.9). Going one step further, using the set $E$ defined above, we shall write Theorem 4.4.4 in terms of the first return maps $f_E\colon E \to E$ as follows.

**Proposition 4.4.5. (Generic inverse branches are well-defined for the first return map)** *Under the assumptions of Theorem 4.4.4, consider the set $E$ as in Section 4.4.1. Then, for $\widetilde{\mu_E}$-almost every backward orbit $\left\{x_n^E\right\}_n \in \widetilde{\partial U}$, there exists $r := r(\left\{x_n^k\right\}_n) > 0$ such that*

(i) *the inverse branch $F_n^E$ sending $x_0^E$ to $x_n^E$ is well-defined in $D(x_0^E, r_0)$;*

(ii) *for every $r \in (0, r_0)$, there exists $m \in \mathbb{N}$ such that*

$$F_m^E(D(x_0^E, r_0)) \subset D(x_0^E, r).$$

### 4.4.3 Proof of Proposition 4.4.5

We shall establish who are the 'singular values' for the first return map $f_E$. The only obstructions when defining inverse branches come from the singular values of $f$. Note that, if there are no singular values of $f$ in $D(f_E(x), \varepsilon)$ and the first return time of $x$ is 1, then the corresponding branch of $f_E$ is well-defined in $D(f_E(x), \varepsilon)$. Inductively, if there are no critical values of $f^n$ in $D(f_E(x), \varepsilon)$ and the first return time of $x$ is $n$, then the corresponding branch of $f_E$ is well-defined in $D(f_E(x), \varepsilon)$. Hence, we observe an interplay between the points in the orbit of singular values of $f$ and the first return times, as the limitation to define the inverse branches of $f_E$.

Next we aim to give estimates on the first return times and the size of disks centered at 'singular values of $f_E$'. This is the content of Lemma 4.4.6.

We use the following notation: let $\{v_1, \ldots, v_N\}$ be the singular values of $f$ in $\partial U_{+\varepsilon}$ (we assumed there are finitely many– other singular values do not play a role in the considered inverse branches), and denote them by $SV(f)$. $T(x)$ stands for the first return time to $E$ of $x \in E$.

$$A_n := \{x \in E\colon T(x) = n\}$$

$$B_n := \{x \in E\colon T(x) \geq n\}$$

**Lemma 4.4.6. (Estimates on critical values and first returns)** *In the previous setting, the following holds.*

(2.1) $\sum\limits_n \mu_E(B_{n^4}) < \infty$.

(2.2) $\sum_n \mu_E(D(CV(f^{n^4}), \varepsilon \cdot \lambda^n)) < \infty$*, for any $\lambda \in (0, 1)$.*

*Proof.* (2.1) follows directly from the estimate in Lemma 4.4.3. For (2.2) note that, since $E$ has finite measure, the measures $\mu_E$ and $\omega_U$ are comparable. Note also that $f^{n^4}$ has $n^4 \cdot N$ singular values (where $N$ stands for the number of singular values of $f$). Then, applying a standard estimate of the harmonic measure of disks (see Lemma 4.8.5), we have

$$\mu(E)\sum_n \mu_k(D(CV(f^{n^4}),\varepsilon\cdot\lambda^n)) \lesssim \sum_n \mu(D(CV(f^{n^4}),\varepsilon\cdot\lambda^n))$$
$$\lesssim \sum_n \omega_U(D(CV(f^{n^4}),\varepsilon\cdot\lambda^n)) \leq \sum_n \varepsilon^{1/2}\cdot N\cdot n^4\cdot\lambda^{n/2} < \infty.$$

□

From here, the proof ends as the one of Theorem 4.3.1: proving that orbits under $f_E$ do not come close to the 'singular values of $f_E$', and finally constructing inductively the required inverse branches of $f_E$, which turn out to be a composition of inverse branches for the original map $f$, as explained in Section 4.4.1. For convenience, we outline the steps of the proof, although not giving all the details as in Theorem 4.3.1.

Set

$$\chi := \int_E \log|f_E'|\, d\mu_E \in (0,+\infty),$$

and let $M \in (e^{\frac{1}{4}\cdot\chi}, 1)$.

**Lemma 4.4.7. (Almost every orbit does not come close to singular values)** *For $\widetilde{\mu_E}$-almost every $\left\{x_n^E\right\}_n \in \widetilde{E}$, it holds*

(3.1) $x_0^E \notin \bigcup_{s\in SV(f)} \bigcup_{n\geq 0} f^n(s)$,

(3.2) $\lim_n \dfrac{1}{n}\log\left|(f_E^n)'(x_n^E)\right| = \chi$,

(3.3) *inverse branches of $f_E$ are well-defined in $D(x_n^E,\varepsilon\cdot M^n)$, except for finitely many $n$'s.*

*Proof.* Since the finite intersection of sets of full measure has full measure, it is enough to show that each of the conditions is satisfied in a set of full measure.

For condition (3.1), note that $\bigcup_{s\in SV(f)}\bigcup_{n\geq 0} f^n(s)$ is countable, and hence has zero $\mu_E$-measure. Requirement (3.2) follows from Birkhoff Ergodic Theorem I.1.9 applied to the map $\log|f_E'|$ (note that $\mu_E$ is an ergodic probability).

Condition (3.3) follows from Lemma 4.4.6 together with the first Borel-Cantelli Lemma I.1.4. Indeed, if $D_n = (D(CV(f^{n^4}),\varepsilon\cdot M^n))$, then

$$\sum_{n\geq 1}\widetilde{\mu_E}(\pi_{U,n}^{-1}(D_n)) = \sum_{n\geq 1}\mu_E(D_n) < \infty,$$

implying that $x_n^E \notin D_n$, for all $n \geq n_0$, for some $n_0$ and $\widetilde{\mu_E}$-almost every backward orbit. But according to (2.1) in Lemma 4.4.6,

$$\sum_{n\geq 1}\widetilde{\mu_E}(\pi_{U,n}^{-1}(B_{n^4})) = \sum_n \mu_E(B_{n^4}) < \infty,$$

so $x_{n+1}^E \notin B_{n^4}$, for all $n \geq n_0$ (maybe taking $n_0$ larger), and $\widetilde{\mu_E}$-almost every backward orbit. Thus, for all $n \geq n_0$, the return time of $x_{n+1}^E$ is less than $n^4$, so we only have to take into account the singular values of $f^{n^4}$ in order to define the inverse branch from $x_n^E$ to $x_{n+1}^E$. Since $x_n^E \notin D_n$, the claim follows. □

Fix $\left\{x_n^E\right\}_n$ satifying the conditions of Lemma 4.4.9. By (3.2) and (3.3), there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, inverse branches of $f_E$ are well-defined in $D(x_n^E, \varepsilon \cdot M^n)$, and

$$\left|(f_E^n)'(x_n^E)\right|^{-\frac{1}{4}} < M^n < 1.$$

We set the following notation.

$$b_n \coloneqq \left|(f_E^n)'(x_n^E)\right|^{-\frac{1}{4}} \qquad P = \prod_{n\geq 1}(1-b_n)$$

Choose $r \coloneqq r(\left\{x_n^E\right\}_n) > 0$ such that

(4.1) $2rP < \varepsilon$,

(4.2) the branch $F_{n_0}^k$ of $f_E^{-n_0}$ sending $x_0^E$ to $x_{n_0}^E$ is well-defined in $D(x_0, r)$,

(4.3) $F_{n_0}^E(D(x_0^E, r\prod_{m=1}^{n_0}(1-b_m))) \subset D(x_{n_0}^E, M^{n_0})$.

Using the same procedure as in Theorem 4.3.1 (Claim 4.3.4), one can prove inductively the following claim.

**Claim 4.4.8. (Inductive construction of the inverse branches)** *For every $n \geq n_0$, there exists a branch $F_n^E$ of $f_E^{-n}$ sending $x_0^E$ to $x_n^E$, defined in $D(x_0, r\prod_{m=1}^{n}(1-b_m))$, and such that*

$$F_n^E(D(x_0^E, r\prod_{m=1}^{n}(1-b_m))) \subset D(x_n^E, \varepsilon \cdot M^n).$$

Letting $n \to \infty$, we get that all inverse branches of $f_E$ sending $x_0^E$ to $x_n^E$ are well-defined in $D(x_0, rP)$, with $r > 0$. Moreover, as $n \to \infty$, diam $\left(F_n^E(D(x_0, rP))\right) \leq \varepsilon \cdot M^n \to 0$.

This ends the proof of Proposition 4.4.5. □

### 4.4.4 Parabolic Pesin theory for entire functions

We end this section by showing that, when we are dealing with an entire function, it is enough to ask that there are finitely many critical values in $\partial U_{+\varepsilon}$. To see this, it it enough to show that inverse branches for $f^E$ are well-defined far from the orbit of critical values of $f$ and of exceptional points (points with finite backwards orbit; any entire function has at most two exceptional points [Ber93, p. 6]).

**Lemma 4.4.9.** *Let $x \in E$. Then, the inverse branch $F_1^E$ sending $f^E(x)$ to $x$ is well-defined in $D(f^E(x), r)$, $r < \varepsilon$, as long as $D(f^E(x), r) \cap CV(f^{T(x)}) = \emptyset$, and there are no exceptional points in $D(f^E(x), r)$.*

*Proof.* If there are no exceptional points in $D(f^E(x), r)$, it is easy to see that there exists $R > 0$ such that $f^{T(x)}|_{D(x,R)}$ is holomorphic, and $f^{T(x)}(D(x, R)) \supset D(f^{T(x)}(x), r)$ (see e.g. [Mil06, Corol. 14.2]). Hence, all obstructions to define the inverse branch $F_1^E$ come from the critical values of $f^{T(x)}$, but we assumed there are not. $\square$

Lemma 4.4.9 is telling us that critical values together with exceptional points are the 'singular values of $f^E$', and there are finitely many of them. Hence, one can prove a result analogous to Proposition 4.4.5.

**Theorem 4.4.10. (Inverse branches are well-defined almost everywhere)** *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function, and let $U$ be a parabolic basin or Baker domain. Assume*

(a) *the Denjoy-Wolff point of the associated inner function is not a singularity;*

(b) $\log|f'| \in L^1(\mu)$, *and* $\int_{\partial U} \log|f'|\, d\mu > 0$*;*

(c) *there exists $\varepsilon > 0$ such that critical values of $f$ in $\partial U_{+\varepsilon}$ are finite.*

*Then, for $\widetilde{\mu}$-almost every backward orbit $\{x_n\}_n \in \widetilde{\partial U}$, there is $r := r(\{x_n\}_n) > 0$ such that*

(i) *the inverse branch $F_n$ sending $x_0$ to $x_n$ is well-defined in $D(x_0, r_0)$;*

(ii) $\operatorname{diam} F^j_m(D(x_0, r)) \to 0$, *as $j \to \infty$.*

## 4.5 Dynamics of centered inner functions. Corollary 4.C

In this section we apply the techniques developed previously to a particular type of self-maps of the unit disk $\mathbb{D}$, the so-called *inner functions.* Recall that $g(0) = 0$, we say that $g$ is a *centered inner function.*

Every inner function induces a measure-theoretical dynamical system $g^*\colon \partial\mathbb{D} \to \partial\mathbb{D}$ defined $\lambda$-almost everywhere. For centered inner functions, $g^*|_{\partial\mathbb{D}}$ preserves the Lebesgue measure $\lambda$ in $\partial\mathbb{D}$, and $g^*|_{\partial\mathbb{D}}$ is ergodic. Hence, the radial extension of a centered inner functions is a good candidate to perform Pesin theory. Therefore, we shall see Corollary 4.C as an application of the work done in Theorem 4.A, for a particular class of inner functions (centered inner functions with finite entropy, i.e. $\log|g'| \in L^1(\lambda)$).

As in the previous sections, we rewrite Corollary 4.C in terms of Rokhlin's natural extension (Thm. 4.5.1). Indeed, $(\partial\mathbb{D}, \mathcal{B}(\partial\mathbb{D}), \lambda)$ is a Lebesgue space (it is isomorphic, in the measure-theoretical sense, to the unit interval), and hence Theorem 4.1.2 guarantees the existence of Rokhlin's natural extension. Thus, $\widetilde{\partial\mathbb{D}}$ is the space of backward orbits $\{\xi_n\}_n \subset \partial\mathbb{D}$, with $g^*(\xi_{n+1}) = \xi_n$ for $n \geq 0$, and $\widetilde{g^*}\colon \widetilde{\partial\mathbb{D}} \to \widetilde{\partial\mathbb{D}}$ is the automorphism which makes the following diagram commute.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\widetilde{g}} & \widetilde{\partial\mathbb{D}} & \xrightarrow{\widetilde{g}} & \widetilde{\partial\mathbb{D}} & \xrightarrow{\widetilde{g}} & \widetilde{\partial\mathbb{D}} & \xrightarrow{\widetilde{g}} & \cdots \\
 & & \{\xi_{n+2}\}_n & & \{\xi_{n+1}\}_n & & \{\xi_n\}_n & & \\
 & & \Big\downarrow{\scriptstyle \pi_{\mathbb{D},n}} & & \Big\downarrow{\scriptstyle \pi_{\mathbb{D},n}} & & \Big\downarrow{\scriptstyle \pi_{\mathbb{D},n}} & & \\
\cdots & \xrightarrow{g} & \partial\mathbb{D} & \xrightarrow{g} & \partial\mathbb{D} & \xrightarrow{g} & \partial\mathbb{D} & \xrightarrow{g} & \cdots \\
 & & \xi_{n+2} & & \xi_{n+1} & & \xi_n & &
\end{array}$$

This way, we rephrase Corollary 4.C as follows.

**Theorem 4.5.1. (Pesin theory for centered inner function)** *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function, such that $g(0) = 0$, and $\log|g'| \in L^1(\partial\mathbb{D})$. Fix $\alpha \in (0, \pi/2)$. Then, for $\widetilde{\lambda}$-almost every backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$, there exists $\rho > 0$ such that the inverse branch $G_n$ of $g^n$ sending $\xi_0$ to $\xi_n$ is well-defined in $D(\xi_0, \rho)$, and, for all $\rho_1 \in (0, \rho)$,*

$$G_n(R_{\rho_1}(\xi_0)) \subset \Delta_{\alpha,\rho_1}(\xi_n).$$

*Moreover, the set of singularities $E(g)$ has zero $\lambda$-measure.*

Using that $g^*|_{\partial\mathbb{D}}$ is ergodic and recurrent with respect to $\lambda$, it follows that for $\widetilde{\lambda}$-almost every backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$ and every set $A \subset \partial\mathbb{D}$ of positive measure, there exists a sequence $n_k \to \infty$ such that $\xi_{n_k} \in A$ (Prop. 4.1.3). Hence, it is clear that Theorem 4.5.1 implies Corollary 4.C.

*Proof of Theorem 4.5.1.* Proceeding exactly as in Theorem 4.A, we find that, for $\widetilde{\lambda}$-almost every backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$, there exists $\rho_0 > 0$ such that the inverse branch $G_n$ of $g^n$ sending $\xi_0$ to $\xi_n$ is well-defined in $D(\xi_0, \rho_0)$. Note that all inverse branches $\{G_n\}_n$ are well-defined in a disk of uniform radius, namely in $D(\xi_0, \rho_0)$. Hence, we can apply Proposition II.3.17, to see that, for all $\alpha \in (0, \pi/2)$ there exists $\rho < \rho_0$ such that for all $\rho_1 \in (0, \rho)$,

$$G_n(R_{\rho_1}(\xi_0)) \subset \Delta_{\rho_1}(\xi_n).$$

It is left to see that singularities have zero Lebesgue measure. Assume on the contrary that the set of singularities $E(g)$ has positive measure. Then, we can take $\{\xi_n\}_n$ visting $E(g)$ infinitely often, and satisfying that the inverse branches $\{G_n\}_n$ realizing such backward orbit are well-defined in $D(\xi_0, \rho)$. Consider

$$K \coloneqq \bigcup_{n\geq 1} G_n(D(\xi_0, \rho)).$$

We claim that no point in $K$ is a singularity for $g$. Indeed, for any $\xi \in K$, there exists $n \geq 1$ such that $\xi \in G_n(D(\xi_0, \rho))$. Hence,

$$g|_{G_n(D(\xi_0,\rho))}\colon G_n(D(\xi_0, \rho)) \longrightarrow G_{n-1}(D(\xi_0, \rho))$$

is univalent, so $\xi$ cannot be a singularity for $g$. This is a contradiction with the fact that $K \cap E(g) \neq \emptyset$, and ends the proof of Corollary 4.C. $\square$

## 4.6 Associated inner functions and Rokhlin's natural extension

Let $f \in \mathbb{K}$, and let $U$ be an invariant Fatou component for $f$, which we assume to be simply connected. Consider $\varphi\colon \mathbb{D} \to U$ to be a Riemann map. Then, $f\colon U \to U$ is conjugate by $\varphi$ to a holomorphic map $g\colon \mathbb{D} \to \mathbb{D}$, i.e. $f \circ \varphi = \varphi \circ g$.

The conjugacy $f \circ \varphi = \varphi \circ g$ extends almost everywhere to $\partial\mathbb{D}$ by means of the radial extensions $\varphi^*\colon \partial\mathbb{D} \to \partial U$ and $g^*\colon \partial\mathbb{D} \to \partial\mathbb{D}$. More precisely, consider the following subsets of $\partial\mathbb{D}$.

$$\Theta_E := \{\xi \in \partial\mathbb{D}\colon \varphi^*(\xi) \in E(f)\}$$

$$\Theta_\Omega := \{\xi \in \partial\mathbb{D}\colon \varphi^*(\xi) \in \Omega(f)\}$$

Since $E(f)$ is countable, $\lambda(\Theta_E) = 0$, so $\lambda(\Theta_\Omega) = 1$. Then, $f \circ \varphi = \varphi \circ g$ extends for the radial extensions in $\Omega_\Theta$, as shown in Lemma II.5.2. That is, for $\xi \in \Theta_\Omega$, then $g^*(\xi)$ and $\varphi^*(g^*(\xi))$ are well-defined, and

$$f(\varphi^*(\xi)) = \varphi^*(g^*(\xi)).$$

In this section we show that one can go further and relate backward orbits for the radial extension of the inner function $g^*$ with backward orbits for the boundary map $f|_{\partial U}$. Moreover, we will show how the natural extensions of $(\partial\mathbb{D}, \lambda, g^*)$ and $(\partial U, \omega_U, f)$ are related.

To do so, first we have to establish, in the spirit of Lemma II.5.2, a relation between backward orbits for $g^*$ and backward orbits for $f$. More precisely, we prove that backward orbits associated to a well-defined sequence of inverse branches indeed commute by the Riemann map, as long as the radial limit at the initial point exists.

**Proposition 4.6.1. (Backward orbits commute)** *Let $\{\xi_n\}_n \subset \partial\mathbb{D}$ be a backward orbit for $g^*$. Assume $\varphi^*(\xi_0)$ exists. Then, $\varphi^*(\xi_n)$ exists for all $n \geq 1$ and*

$$f(\varphi^*(\xi_{n+1})) = \varphi^*(g^*(\xi_{n+1})) = \varphi^*(\xi_n).$$

*Proof.* We note that, using an inductive argument, it is enough to prove that, if $\varphi^*(\xi_0)$ exists and $\xi_1 \in \partial\mathbb{D}$ is such that $g^*(\xi_1) = \xi_0$, then $\varphi^*(\xi_1)$ is well-defined, and

$$f(\varphi^*(\xi_1)) = \varphi^*(g^*(\xi_1)) = \varphi^*(\xi_0).$$

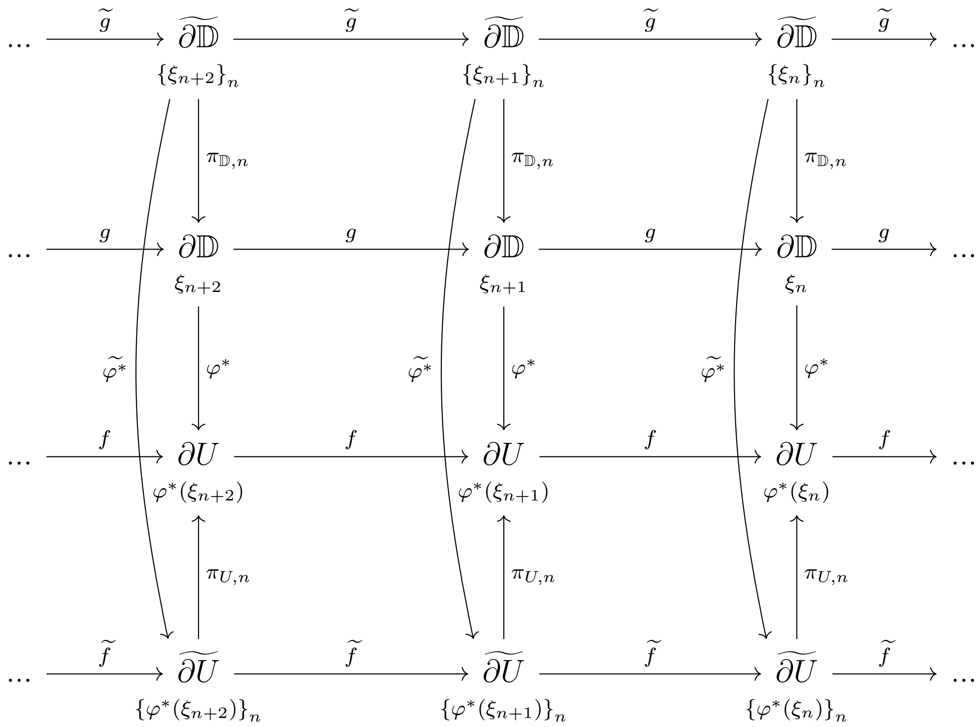
Let $R_{\xi_0}$ be the radius at $\xi_0$. Then, $\varphi(R_{\xi_0})$ is a curve landing at $\varphi^*(\xi_0)$, and there is a curve $\gamma$ landing at $\xi_1$, with $g(\gamma) = R_{\xi_0}$. Then, $f(\varphi(\gamma)) = \varphi(R_{\xi_0})$ is a curve landing at $\varphi^*(\xi_0)$. Since preimages of a point under a holomorphic map are discrete and the singularities of $f$ are countable, $\varphi(\gamma)$ lands at a point on $\widehat{\partial} U$, which, by Lindelöf Theorem II.4.5 coincides with $\varphi^*(\xi_1)$ (which in particular is well-defined and satisfies $f(\varphi^*(\xi_1)) = \varphi^*(\xi_0)$). □

We are interested now in the interplay between the backward orbits for the associated inner function $g$, and the backward orbits for $f$ in the dynamical plane. According to Section 4.1, we can consider the natural extension $(\widetilde{\partial\mathbb{D}}, \widetilde{\lambda}, \widetilde{g^*})$ of $(\partial\mathbb{D}, \lambda, g^*)$, given by the projecting morphisms $\{\pi_{\mathbb{D},n}\}_n$, and the natural extension $(\widetilde{\partial U}, \widetilde{\omega_U}, \widetilde{f})$ of $(\partial U, \omega_U, f)$, given by the projecting morphisms $\{\pi_{U,n}\}_n$. We are interested in relating both natural extensions.

In views of Proposition 4.6.1, it is clear that the transformation

$$\begin{aligned} \widetilde{\varphi^*}\colon \widetilde{\partial \mathbb{D}} &\longrightarrow \widetilde{\partial U} \\ \{\xi_n\}_n &\mapsto \{\varphi^*(\xi_n)\}_n \end{aligned}$$

is well-defined, and the following diagram commutes almost everywhere.




Now we claim that $\widetilde{\varphi^*}$ is measure-preserving. Indeed, one may take a basis for the $\sigma$-algebra in $\widetilde{\partial U}$ made of sets of the form $\pi_{U,n}^{-1}(A)$, where $A \subset \partial U$ measurable, and $n \geq 0$. It is enough to prove that $\widetilde{\varphi^*}$ preserves the measure of these sets. Indeed, using that $\varphi^* \circ \pi_{\mathbb{D},n} = \pi_{U,n} \circ \widetilde{\varphi^*}$ $\widetilde{\lambda}$-almost everywhere, we have

$$\widetilde{\omega_U}(\pi_{U,n}^{-1}(A)) = \omega_U(A) = \lambda(\varphi^*(A)) = \widetilde{\lambda}(\pi_{\mathbb{D},n}^{-1} \circ (\varphi^*)^{-1}(A)) = \widetilde{\lambda}((\widetilde{\varphi^*})^{-1} \circ \pi_{U,n}^{-1}(A)),$$

where $A \subset \partial U$ measurable, and $n \geq 0$, as desired. In other words, $\widetilde{\omega_U}$ is the push-forward of $\widetilde{\lambda}$ by $\widetilde{\varphi^*}$.

Hence, the following diagram

$$\begin{array}{ccc} (\partial\mathbb{D}, \lambda, g^*) & \xleftarrow{\{\pi_{\mathbb{D},n}\}_n} & (\widetilde{\partial\mathbb{D}}, \widetilde{\lambda}, \widetilde{g^*}) \\ \varphi^* \downarrow & & \downarrow \widetilde{\varphi^*} \\ (\partial U, \omega_U, f) & \xleftarrow{\{\pi_{U,n}\}_n} & (\widetilde{\partial U}, \widetilde{\omega_U}, \widetilde{f}). \end{array}$$

commutes almost everywhere.

**Proposition 4.6.2. (Generic inverse branches commute)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Fatou component for $f$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g\colon \mathbb{D} \to \mathbb{D}$ be the inner function associated with $(f, U)$ by $\varphi$. Assume the following conditions are satisfied.*

(a) *For $\widetilde{\omega_U}$-almost every backward orbit $\{x_n\}_n \subset \partial U$, there exists $r > 0$ such that the inverse branch $F_n$ sending $x_0$ to $x_n$ is well-defined in $D(x_0, r)$.*

(b) *For $\widetilde{\lambda}$-almost every backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$ there exists $\rho > 0$ such that the inverse branch $G_n$ sending $\xi_0$ to $\xi_n$ is well-defined in $D(\xi_0, \rho)$.*

*Then, $\widetilde{\lambda}$-almost every backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$ there exists $\rho, r > 0$ such that the inverse branch $G_n$ sending $\xi_0$ to $\xi_n$ is well-defined in $D(\xi_0, \rho)$, the inverse branch $F_n$ sending $\varphi^*(\xi_0)$ to $\varphi^*(\xi_n)$ is well-defined in $D(\varphi^*(\xi_0), r)$, and $\varphi^* \circ G_n(\xi_0) = F_n \circ \varphi^*(\xi_0)$, for all $n \geq 0$.*

We note that, if $\widetilde{\omega_U}$-almost every backward orbit $\{x_n\}_n \subset \partial U$ satisfies an additional property (such as the ones proved in Theorem 4.A), then it is straightforward to see that, for $\widetilde{\lambda}$-almost every backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$, the backward orbit $\{x_n \coloneqq \varphi^*(\xi_n)\}_n$ satisfies this additional property.

*Proof of Proposition 4.6.2.* The proof follows directly from the previous construction. Indeed, one shall write the first assumption as: for $\widetilde{\lambda}$-almost every backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$, there exists $r > 0$ such that the inverse branch $F_n$ sending $\varphi^*(\xi_0)$ to $\varphi^*(\xi_n)$ is well-defined in $D(\varphi^*(\xi_0), r)$. Since the intersection of sets of full measure has full measure, we have that inverse branches $G_n$ and $F_n$ are well-defined along the backward orbit of $\xi_0$ and $\varphi^*(\xi_0)$. By Proposition 4.6.1, such inverse branches commute. □

**Remark.** It follows from the previous construction that one can find first the backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$ and define the backward orbit in the dynamical plane as their image by $\varphi^*$. Moreover, one can choose a countable collection of sets $\{K_k\}_k \subset \partial\mathbb{D}$ and ask that there exists a sequence $n_k \to \infty$ with $\xi_{n_k} \in K_k$.

## 4.7 Application: periodic boundary points. Corollary 4.D

One application of Pesin theory in holomorphic dynamics is to prove that for some invariant Fatou components, periodic points are dense in their boundary. This was done in the seminal paper of Przytycki and Zdunik [PZ94] for simply connected attracting basins of rational maps (note that in this paper it is proved that periodic points are dense in the boundary of *every* attracting or parabolic basin of a rational map, but the proof relies on a different technique). In the spirit of Section 3, we aim to prove a similar result for transcendental maps.

The goal in this section is to prove Corollary 4.D, which states that, under the hyptotheses of either Theorem 4.A or Theorem 4.B, plus an extra hyptothesis on the singular values in $U$, accessible periodic boundary points are dense.

In view of the theory developed in the previous sections based on working in the space of backward orbits given by Rokhlin's natural extension, we shall formulate an alternative (and more natural) version of Corollary 4.D, in terms of backward orbits.

**Theorem 4.7.1. (Periodic points are dense)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Fatou component for $f$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g\colon \mathbb{D} \to \mathbb{D}$ be the inner function associated with $(f, U)$ by $\varphi$. Assume the following conditions are satisfied.*

(a) *For $\widetilde{\omega_U}$-almost every backward orbit $\{x_n\}_n \subset \partial U$, there exists $r > 0$ such that the inverse branch $F_n$ sending $x_0$ to $x_n$ is well-defined in $D(x_0, r)$, for every subsequence $\left\{x_{n_j}\right\}_j$ with $x_{n_j} \in D(x_0, r)$, $\text{diam } F_{n_j}(D(x_0, r)) \to 0$, as $j \to \infty$.*

(b) *For $\widetilde{\lambda}$-almost every backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$ there exists $\rho > 0$ such that the inverse branch $G_n$ sending $\xi_0$ to $\xi_n$ is well-defined in $D(\xi_0, \rho)$.*

*Then, accessible periodic points are dense in $\partial U$.*

However, we aim to give a proof of the density of periodic boundary points which does not use Rohklin's natural extension. To do so, we state Theorem 4.7.1 in a slightly different (and stronger) way.

**Theorem 4.7.2. (Periodic points are dense)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant simply connected Fatou component for $f$. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g\colon \mathbb{D} \to \mathbb{D}$ be the inner function associated with $(f, U)$ by $\varphi$. Assume that for every countable sequence of measurable sets $\{K_k\}_k \subset \partial\mathbb{D}$ with $\lambda(K_k) > 0$ and $\lambda$-almost every $\xi \in \partial\mathbb{D}$, there exists a backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$, such that*

(a) *$\xi = \xi_0$ and there exists $\rho > 0$ such that the inverse branch $G_n$ sending $\xi_0$ to $\xi_n$ is well-defined in $D(\xi_0, \rho)$, and there exists $n_k \to \infty$ with $\xi_{n_k} \in K_k$;*

(b) *for the backward orbit $\{x_n := \varphi^*(\xi_n)\}_n \subset \partial U$, there exists $r > 0$ such that the inverse branch $F_n$ sending $x_0$ to $x_n$ is well-defined in $D(x_0, r)$, for every subsequence $\left\{x_{n_j}\right\}_j$ with $x_{n_j} \in D(x_0, r)$, $\text{diam } F_{n_j}(D(x_0, r)) \to 0$, as $j \to \infty$.*

*Then, accessible periodic points are dense on $\partial U$.*

According to Proposition 4.6.2, it is clear that Theorem 4.7.2 implies 4.7.1. We show now how to deduce Corollary 4.D from Theorem 4.7.1, and later we give the proof of Theorem 4.7.2.

*Proof of Corollary 4.D.* On the one hand, it is clear that, by the conclusion of Theorem 4.A and Theorem 4.B, the second requirement of Theorem 4.7.1 holds.

On the other hand, we have to see the assumption of the existence of a crosscut neighbourhood $N_C$ in $U$ with $N_C \cap P(f) = \emptyset$ implies (b). Indeed, $\varphi^{-1}(N_C)$ is a crosscut neighbourhood in $\mathbb{D}$ which contains no postsingular value for the inner function. Since

$g^*|_{\partial\mathbb{D}}$ is ergodic and recurrent, for $\lambda$-almost $\xi \in \partial\mathbb{D}$, there exists $\rho > 0$ such that, for all $n \geq 0$, all inverse branches of $g^n$ are well-defined in $D(\xi, \rho)$. Denote this set of backward orbits by $\widetilde{A}$. We have to see that $\widetilde{A}$ has full $\widetilde{\lambda}$-measure in $\widetilde{\partial\mathbb{D}}$. Indeed, note that $\widetilde{A} = \pi_{\mathbb{D},0}^{-1}(\pi_{\mathbb{D},0}(\widetilde{A}))$, since the set $\widetilde{A}$ is made of *all* backward orbit with initial point in $\pi_{\mathbb{D},0}(\widetilde{A})$. Since $\lambda(\pi_{\mathbb{D},0}(\widetilde{A})) = 1$ and $\pi_{\mathbb{D},0}$ is measure-preserving, this already implies the requirement (b) in Theorem 4.7.1. □

### Proof of Theorem 4.7.2

Let $x \in \partial U$ and $R > 0$, we have to see that $f$ has a repelling periodic point in $D(x, R) \cap \partial U$, which is accessible from $U$.

We split the proof in several intermediate lemmas. We start by proving the existence of a backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$ such that for both $\{\xi_n\}_n$ and $\{\varphi^*(\xi_n)\}_n$ the corresponding inverse branches are well-defined (and conformal), and certain estimates on the contraction are achieved.

In the sequel, we fix $\alpha \in (0, \pi/2)$, and we take all Stolz angles of opening $\alpha$, although in the notation we omit the dependence.

**Lemma 4.7.3.** *There exists a backward orbit $\{\xi_n\}_n \subset \partial\mathbb{D}$, and constants $m \in \mathbb{N}$, $0 < \rho_m \leq \rho$, and $r \in (0, R/2)$ such that:*

(1.1) *$x_0 := \varphi^*(\xi_0)$ and $x_m := \varphi^*(\xi_m)$ are well-defined, and $x_0 \in D(x, R/2)$ and $x_m \in D(x_0, r/3)$;*

(1.2) *the inverse branch $F_m$ of $f^m$ sending $x_0$ to $x_m$ is well-defined in $D(x_0, r)$, and* $\operatorname{diam} F_m(D(x_0, r)) < r/3$*;*

(1.3) *the inverse branch $G_m$ of $g^m$ sending $\xi_0$ to $\xi_m$ is well-defined in $D(\xi_0, \rho_m)$, and satisfies*

$$G_m(R_{\rho_m}(\xi_0)) \subset \Delta_{\rho_m}(\xi_m);$$

(1.4) *$\Delta_\rho(\xi_0) \cap \Delta_\rho(\xi_m) \neq \emptyset$, and, if $z \in \Delta_\rho(\xi_0) \cup \Delta_\rho(\xi_m)$, then $\varphi(z) \in D(x_0, r)$.*

*Proof.* Let $A_n = D(x^n, r_n)$ be a countable basis for $D(x, R)$ with the Euclidean topology, where $x^n \in \partial U$ and $A_n \subset D(x, R)$.

In order to apply the hyptothesis of the theorem, we shall construct an appropriate countable sequence of measurable sets $\{K_k\}_k$ of $\partial\mathbb{D}$. We do it as follows.

For all $n \geq 0$, let

$$K^n = \{\xi \in \partial\mathbb{D} \colon \varphi^*(\xi) \in D(x^n, r_n/2)\}.$$

It is clear that $\lambda(K^n) > 0$. By the Lehto-Virtanen Theorem II.2.5, the angular limit exists whenever the radial limit exists. Therefore, there exists $\rho_n > 0$ small enough so that

$$K^n_{\rho_n} = \{\xi \in K^n \colon \Delta_{\rho_n}(\xi) \subset D(x^n, r_n/2)\}$$

has positive $\lambda$-measure. We can assume that every point in $K^n_{\rho_n}$ is a Lebesgue density point for $K^n_{\rho_n}$. Then, if we take $\xi^n \in K^n_{\rho_n}$, there exists a circular interval $I_{\xi^n}$ around $\xi^n$ such that for any $\zeta_1, \zeta_2 \in I_{\xi^n}$,

$$\Delta_{\rho_n}(\zeta_1) \cap \Delta_{\rho_n}(\zeta_2) \neq \emptyset.$$

Then, $K^n_{\rho_n} \cap I_{\xi^n}$ has positive $\lambda$-measure. Note that this property only depends on the length of the interval, as long as $\xi^n$ is a Lebesgue density point for $K^n_{\rho_n}$. Then, it is clear that there exist finitely many circular intervals $I^n_1, \dots, I^n_{i_n}$ with this property.

Let

$$K^{1,n}_{*,i} := K^n_{\rho_n} \cap I^n_i, \quad i = 1, \dots, i_n,$$
$$K^{1,n}_{*} := \left\{ K^{1,n}_{*,1}, \dots, K^{1,n}_{*,i_n} \right\}.$$

Then, we define the set $K^{j,n}_*$, as before, but replacing $\rho_n$ by $\rho_n/2^j$.

Having introduced all this notation of the sets $\{K^{j,n}_*\}_{n,j}$, we arrange the sequence $\{K_k\}_k$ as follows. We construct this sequence of sets inductively, adding at each step finitely many sets. Indeed, let us start by putting the block $K^{1,1}_* := \left\{ K^{1,1}_{*,1}, \dots, K^{1,1}_{*,i_1} \right\}$ as the first elements of the sequence. Then, for the $k$-th step of the induction, we consider $A_k$ and let $A_{k_1}, \dots, A_{k_n}$ be all the sets of $A_1, \dots, A_n$ such that $A_n \subset A_{k_i}$. Then, we add to the sequence the blocks

$$K^{1,k_1}_*, \dots K^{1,k_n}_*, \dots, K^{k,k_1}_*, \dots, K^{k,k_n}_*.$$

Basically, the idea is that, when one set is in the sequence $\{K_k\}_k$ for the first time, then it appears infinitely often. Moreover, the set of points in $\{K_k\}_k$ has measure $\lambda((\varphi^*)^{-1}(D(x,R)))$. Indeed, the set of points in $\partial\mathbb{D}$ for which the radial limit exists has full measure. Let $\zeta$ be one of such points. Then, $\varphi^*(\zeta) \in A_j$, for some $j$, and for $\rho > 0$ small enough, $\Delta_\rho(\zeta) \subset A_j$. Then, there exists $n \geq 0$ such that $A_n \subset A_j$ and $\rho < \rho_j/2^n$, so $\zeta \in K^n_{*,k_n}$, as desired.

By the assumption of the theorem, for $\lambda$-almost every $\xi_0 \in \partial\mathbb{D}$, there exists a backward orbit $\{\xi_n\}_n$ such that the hypothesis on the definition of the inverse branches for $\{\xi_n\}_n$ and $\{x_n := \varphi^*(\xi_n)\}_n$ are accomplished, and there exists $n_k \to \infty$ with $\xi_{n_k} \in K_k$.

Without loss of generality, we assume $\xi_0$ is chosen so that $x_0 \in D(x, R/2)$. Let $r > 0$ be such that the inverse branches realizing the backward orbit $\{x_n\}_n$ are well-defined in $D(x_0, r)$. There is no loss of generality on assuming $r \in (0, R/2)$.

On the one hand, since $\{A_n\}_n$ is a basis for $D(x, R)$, there exists $n_0$ such that

$$x_0 \in A_{n_0} \subset D(x_0, r/3),$$

and $\xi_0 \in K^{n_0}_*$, by the previous remark. In particular, for $\rho_{n_0}$,

$$\Delta_{\rho_{n_0}}(\xi_0) \subset A_{n_0} \subset D(x_0, r/3).$$

On the other hand, by the construction of the sets $\{K_n\}_n$, the backward orbit visits $D(x_0, r)$ infinitely many times. Let $n_1$ be large enough so that, for all $n \geq n_1$, if $x_n \in D(x_0, r)$, then diam $F_n(D(x_0, r)) < r/3$.

By the construction of the sets $\{K_n\}_n$, there exists $m \geq \max\{n_0, n_1\}$ such that $\xi_m \in K_*^{n_0}$. Hence, we take $r > 0$, $\rho = \rho_{n_0}$, and $\xi_0$ and $\xi_m$ as above, and define $\rho_m > 0$ as the radius such that the inverse branch $G_m$ sending $\xi_0$ to $x_m$ is defined around $\xi_0$ (such a radius exists by our assumptions on the orbit $\{\xi_n\}_n$). We have to check that, with these choices, the requirements are accomplished.

First, by the choice of $\{\zeta_n\}_n$, $\varphi^*(\xi_0) =: x_0$ and $\varphi^*(\xi_m) =: x_m$ are well-defined. Moreover, by the choice of $r$, we have $x_0 \in D(x, R/2)$. Since $\xi_0, \xi_m \in K_*^{n_0}$, we have

$$\Delta_\rho(\xi_0) \cap \Delta_\rho(\xi_n) \neq \emptyset,$$

and $\Delta_\rho(\xi_0), \Delta_\rho(\xi_m) \subset A_{n_0} \subset D(x_0, r/3)$. In particular, $x_m \in D(x_0, r/3)$, so (1.1) and (1.4) holds. To see (1.2), note that $r$ has been chosen so that the inverse branches corresponding to $\{x_n\}_n$ are well-defined in $D(x_0, r)$, and $m$ is large enough to that diam $F_m(D(x_0, r)) < r/3$, as desired. Requirement (1.3) is directly satisfied by the choice of $\rho_m$. Therefore, we have proved the lemma. □

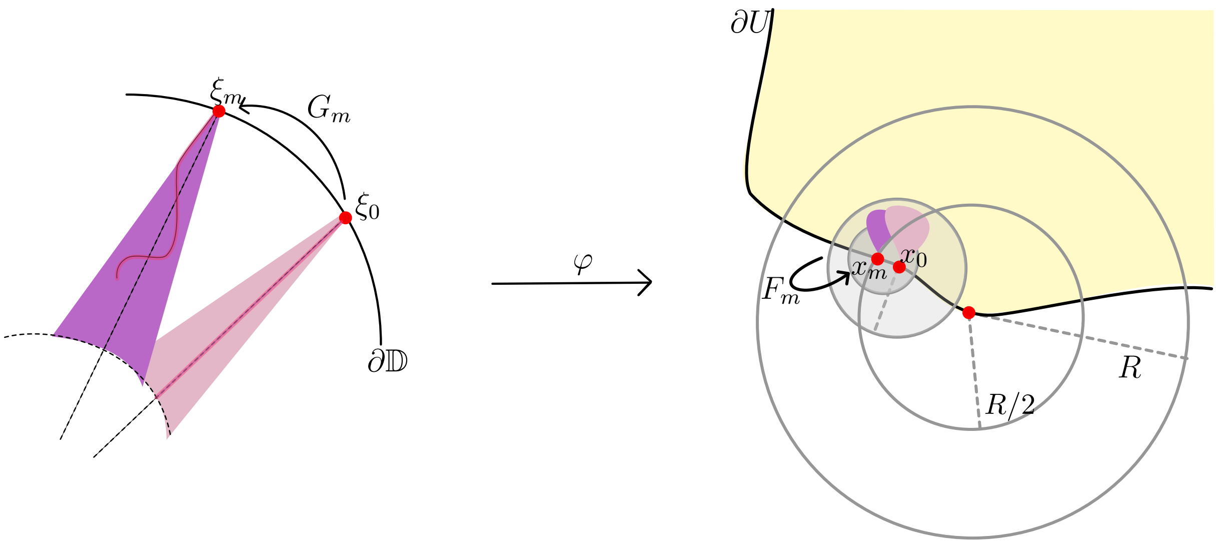



Figure 4.1: Situation after Lemma 4.7.3.

Next we prove the existence of a repelling periodic point in $D(x_0, r)$. Note that, since $D(x_0, r) \subset D(x, R)$, the proof of the next lemma ends the proof of the theorem.

**Lemma 4.7.4.** *The map $F_m$ has an attracting fixed point in $D(x_0, r)$ which is accessible from $U$. Hence, $f$ has a repelling $m$-periodic point in $D(x_0, r) \cap \partial U$.*

*Proof.* First note that $F_m(D(x_0, r)) \subset D(x_0, r)$. Indeed, by (1.1) and (1.2), we have that $x_m \in D(x_0, r/3)$ and diam $F_m(D(x_0, r)) < r/3$, so

$$F_m(D(x_0, r)) \subset D(x_m, 2r/3) \subset D(x_0, r).$$

Therefore, by the Denjoy-Wolff Theorem, there exists a fixed point $p \in D(x_0, r)$, which attracts all points in $D(x_0, r)$ under the iteration of $F_m$. Hence, it is repelling under $f^m$ and thus belongs to $\mathcal{J}(f)$.

It is left to see that $p$ is accessible from $U$. To do so, first note that, by (1.3), the inverse branch $G_m$ of $g^{-m}$ is well-defined in $D(\xi_0, \rho_m)$, and it holds that

$$\varphi \circ G_m = F_m \circ \varphi$$

in $\Delta_{\rho_m}(\xi_0)$. Moreover, we have that

$$G_m(R_{\rho_m}(\xi_0)) \subset \Delta_{\rho_m}(\xi_m) \subset \Delta_{\rho}(\xi_m).$$

By (1.4), $\Delta_\rho(\xi_0) \cup \Delta_\rho(\xi_m)$ is connected. Therefore, if we take $z \in R_{\rho_m}(\xi_0)$, then $G_m(z) \in \Delta_\rho(\xi_m)$, and we can find a curve $\gamma \subset \Delta_\rho(\xi_0) \cup \Delta_\rho(\xi_m)$ joining $z$ and $G_m(z)$. By (1.4), $\varphi(\gamma) \subset D(x_0, r)$, and joins $\varphi(z)$ with $F_m(\varphi(z))$. See Figure 4.2.

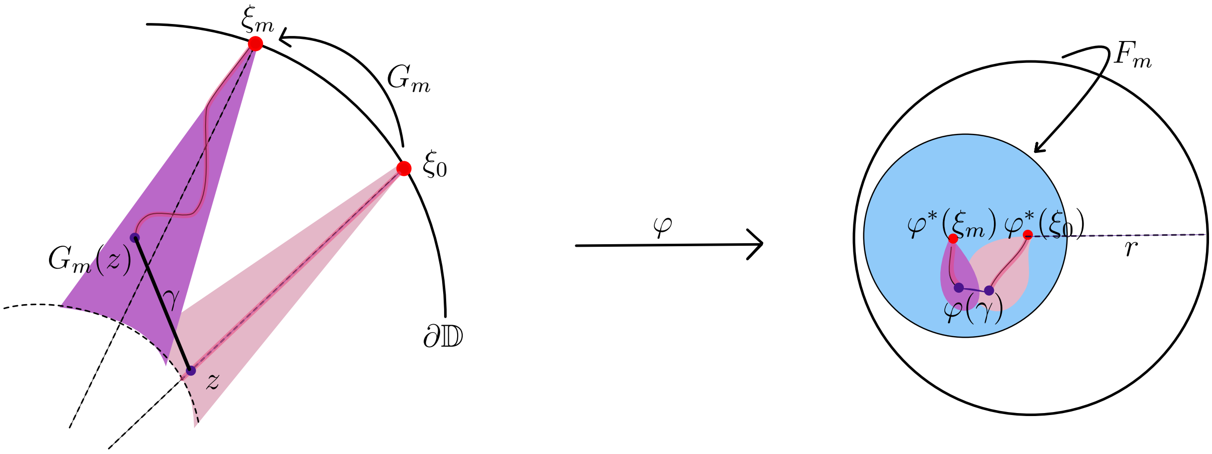



Figure 4.2: The construction of the curve $\gamma$ in $\mathbb{D}$, and its image $\varphi(\gamma)$ in the dynamical plane.

Define

$$\Gamma \coloneqq \bigcup_{k \geq 0} F_m^k(\gamma).$$

Then, $\Gamma \subset \partial U$ lands at $p$, ending the proof of Lemma 4.7.4, and hence of Theorem 4.7.1. □

## 4.8 Lyapunov exponents for transcendental maps

Let $f \in \mathbb{K}$, and let $X \subset \widehat{\mathbb{C}}$. Let $\mu$ be a measure supported on $X$, and assume $\log |f'| \in L^1(\mu)$. Then,

$$\chi_\mu \coloneqq \int_{\partial U} \log |f'| \, d\mu$$

is called the *Lyapunov exponent* of $f$ (with respect to the measure $\mu$). In the previous sections, we were interested in the particular case where $X$ is the boundary of an invariant Fatou component $U$, and $\mu = \omega_U$. We needed to assume $\log |f'| \in L^1(\omega_U)$. It is well-known that this holds for rational maps [Prz85], but it is not clear what happens in the transcendental case. In Section 4.8.1, we prove integrability of $\log |f'|$ with respect to $\omega_U$, under some assumptions on the shape of the Fatou component and the growth of the function. In Section 4.8.2 we give conditions under which Lyapunov exponents are non-negative. Again, this is well-known for rational maps [Prz93], but unexplored in the transcendental case. Finally, Section 4.8.3 is devoted to extend some of the results to parabolic basins and Baker domains.

### 4.8.1 Integrability of $\log|f'|$. Proposition 4.E

We examine the integrability of $\log|f'|$ with respect to harmonic measure. First observe that, by Harnack's inequality, if $\log|f'| \in L^1(\omega_U(p,\cdot))$, then $\log|f'| \in L^1(\omega_U(q,\cdot))$, for all $q \in U$. Hence, we simply write $\log|f'| \in L^1(\omega_U)$.

To begin with, we prove that the integrability of $\log|f'|$ and the Lyapunov exponent is invariant under conjugating $f$ by Möbius transformations. This is the content of the following lemma, which follows from [Prz85, p. 165].

**Lemma 4.8.1. (Lyapunov exponent invariant under Möbius transformations)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant Fatou component for $f$. Let $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Möbius transformation, and let $g \in \mathbb{K}$ be defined as $g := M \circ f \circ M^{-1}$. Then, $\log|f'| \in L^1(\omega_U)$ if and only if $\log|g'| \in L^1(\omega_{M(U)})$. Moreover, if $\omega_U$ is $f$-invariant, then $\omega_{M(U)}$ is $g$-invariant, and*

$$\chi_{\omega_U}(f) = \chi_{\omega_{M(U)}}(g).$$

Observe that, for rational maps, $|f'|$ is bounded, so $\chi_\mu(f)$ is well-defined (although *a priori* may be equal to $-\infty$). By a careful study of $f$ around critical points, it is established that it is never the case, and in fact Lyapunov exponents are always non-negative ([Prz93], see also [URM23, Sect. 28.1]). In the case of transcendental maps, $|f'|$ may not be bounded, and this is why we need the assumption on the growth.

To simplify the notation, in the sequel we shall assume $\infty \in U$, hence $\widehat{\partial} U$ is a compact subset of the plane, and that none of the singularities is placed at $\infty$.

**Definition 4.8.2. (Order of growth in sectors)** Let $f \in \mathbb{K}$, and let $U \subset \widehat{\mathbb{C}}$ be an invariant Fatou component for $f$. We say that $U$ is *asymptotically contained in a sector of angle $\alpha \in (0,1)$ with order of growth $\beta > 0$* if there exists $r_0 > 0$, $s_1, \dots, s_k \in \widehat{\partial} U \setminus \{\infty\}$ and $\xi_1, \dots, \xi_k \in \partial\mathbb{D}$, such that, if

$$S_{\alpha,r} = \bigcup_{i=1}^{k} \{z \in \mathbb{C}\colon\ |z - s_i| < r, |\text{Arg }\xi_i - \text{Arg }(z - s_i)| < \pi\alpha\}$$

satisfies

(a) $\overline{U} \cap \bigcup\limits_{i=1}^{k} D(s_i, r_0) \subset S_{\alpha,r_0}$;

(b) $f$ has order of growth $\beta > 0$ in $S_{\alpha,r_0}$, i.e. there exist $A, B > 0$ such that, for all $r < r_0$ and $z \in S_{\alpha,r_0} \setminus S_{\alpha,r}$,

$$A \cdot e^{B \cdot r^{\beta}} \leq |f'(z)| \leq A \cdot e^{B \cdot r^{-\beta}}.$$

Geometrically, each of the sets

$$S_i = \{z \in \mathbb{C}\colon \ |z - s_i| < r, |\text{Arg } \xi_i - \text{Arg } (z - s_i)| < \alpha\}$$

is a sector of angle $\alpha \in (0, \pi)$, and side-length $r > 0$, with vertex at $s_i \in \widehat{\partial} U$. Note that this notion of order of growth in sectors is invariant under conjugating $f$ by a Möbius transformation $M$, as long as $M(U) \subset \mathbb{C}$ and $M(s_i) \neq \infty$, $i = 1, \ldots, k$. Indeed, $M'$ is uniformly bounded around $\widehat{\partial} U$ and hence distances are distorded in a controlled way when applying $M$.

Next we check that, if $U$ is asymptotically contained in a sector of angle $\alpha \in (0, 1)$, with order of growth $\beta \in (0, 1/2\alpha)$, then $\log |f'| \in L^1(\omega_U)$.

**Proposition 4.8.3.** **($\log |f'|$ is $\omega_U$-integrable)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant Fatou component for $f$. Assume $U$ is asymptotically contained in a sector of angle $\alpha \in (0, 1)$, with order of growth $\beta \in (0, \frac{1}{2\alpha})$. Then, $\log |f'| \in L^1(\omega_U)$.*

Note that Proposition 4.8.3 implies Proposition Proposition 4.E. Before proving it, we need some estimates on the harmonic measure of sectors.

### Estimates on the harmonic measure of sectors

We start by recalling the following estimate on harmonic measure of disks for simply connected domains, which follows from Beurling's Projection Theorem [GM05, Thm. 9.2].

**Theorem 4.8.4. (Harmonic measure of disks,** [GM05, p. 281]**)** *Let $U \subset \widehat{\mathbb{C}}$ be a simply connected domain, such that $\infty \in U$ and $\text{diam}(\partial U) = 2$. Then, for all $x \in \partial U$ and $r > 0$, it holds*

$$\omega_U(\infty, D(x, r)) \leq \sqrt{r}.$$

Assuming that $U$ is contained in some sector

$$S_{\alpha,r}(x, \xi) = \{z \in \mathbb{C}\colon \ |z - x| < r, |\text{Arg } \xi - \text{Arg } (z - x)| < \pi\alpha\},$$

with vertex at $x \in \partial U$ (see Fig. 4.3), we obtain improved estimates of harmonic measure for disks centered at $x$.

**Lemma 4.8.5. (Harmonic measure of sectors)** *Let $U \subset \mathbb{C}$ be a simply connected domain, and let $z_0 \in U$, $x \in \partial U$. Assume there exists $r_0 > 0$, $\alpha \in (0, 1)$ and $\xi \in \partial \mathbb{D}$, such that*

$$D(x, r_0) \cap U \subset S_{\alpha, r_0}(x, \xi).$$

*Then, there exists $C > 0$ and $r_1 \in (0, r_0)$ such that, for all $r \in (0, r_1)$,*

$$\omega_U(z_0, D(x, r)) \leq C \cdot r^{\frac{1}{2\alpha}}.$$

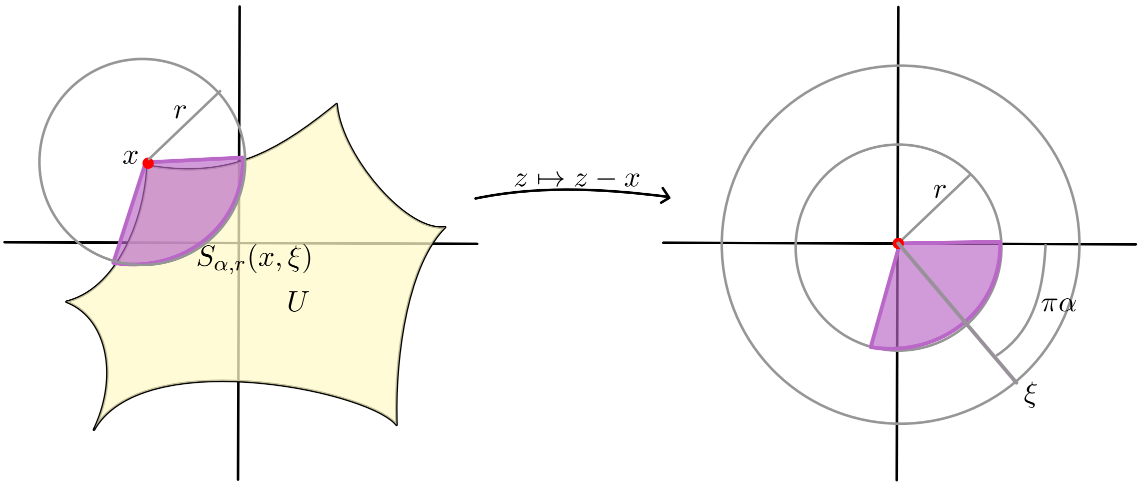



Figure 4.3: A visual representation of the definition of the sector $S_{\alpha,r}(x,\xi)$.

*Proof.* Without loss of generality we assume $z_0 \notin S_{\alpha,r_0}(x,\xi)$, and let $r \in (0, r_0)$. First observe that, if $V$ denotes the connected component of $U \setminus D(x,r)$ that contains $z_0$, we have that

$$\omega_U(z_0, D(x,r)) = \omega_U(z_0, D(x,r) \cap \partial U) \leq \omega_V(z_0, \partial D(x,r) \setminus \partial U) \leq \omega_V(z_0, \partial D(x,r)),$$

where in the first inequality we applied the Comparison Lemma [Con95, Prop. 21.1.13] (note that we apply it to the complements, and hence the inequality is reversed), and the second follows from the inclusion of the measured sets.

Next we observe that, without loss of generality, we can assume that

$$V \subset S_\alpha(x,\xi) = \{z \in \mathbb{C} \colon \ |\text{Arg } \xi - \text{Arg } (z-x)| < \pi\alpha\}.$$

Indeed, since we want to estimate the harmonic measure of disks $D(x,r)$ centered at $x \in \partial U$ (which is a local property of the boundary around the point $x$), and $D(x,r_0) \cap U \subset S_{\alpha,r_0}(x,\xi)$, for $r > 0$ small enough, we can disregard the part of $\partial U$ outside $D(x,r_0)$.

Therefore, up to composing by appropriate Möbius transformations, it is left to see that, if

$$S = \{z \in \mathbb{C} \colon \ |\text{Arg } z| < \pi\alpha\},$$

then, for some constant $C > 0$, we have

$$\omega_{S \setminus D(0,r)}(1, \partial D(0,r)) \leq C \cdot r^{\frac{1}{2\alpha}}.$$

However, since the length of a circumference of radius $r$ is $2\pi r$ (i.e. proporcional to the radius), it is enough to see that $\omega_S(1, D(0,r))$ decays to 0 like $r^{\frac{1}{2\alpha}}$, when $r \to 0$ (see Fig. 4.4). But, since $M(z) = z^{\frac{1}{2\alpha}}$ is a conformal map from $S$ to the right half-plane fixing 1, this follows straighforward (see again Fig. 4.4)

□

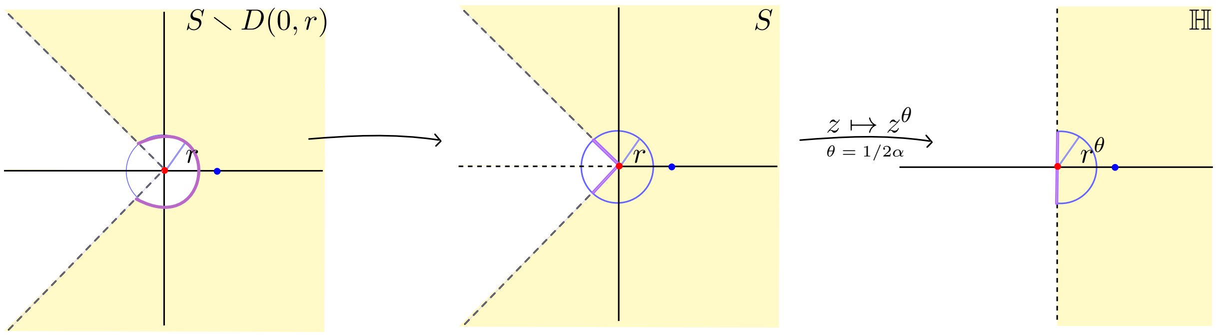



Figure 4.4: A visual scheme to approximate harmonic measure of sectors.

### Proof of Proposition 4.8.3

*Proof of Proposition 4.8.3.* By conjugating by a Möbius transformation if needed, we can assume $\infty \notin U$. Note that $\log |f'|$ is integrable with respect to harmonic measure when restricted to compact subsets of the domain $\widehat{\mathbb{C}} \setminus \{s_1, \ldots, s_k\}$. Indeed, the only difficulty is to see that $\log |f'|$ is integrable around critical points. It is easy to check this by considering the Taylor expansion of $f$ around the critical point, and using that $\log |z - a|$ is integrable with respect to $\omega_U$ for all $a \in \mathbb{C}$ (see e.g. [PU10, Sect. 11.2]).

It is left to check integrability near the singularities, and here is where we use the estimates on the growth. Let us use the notation

$$\log^+ |f'(z)| := \max(0, \log |f'(z)|), \qquad \log^- |f'(z)| := -\min(0, \log |f'(z)|),$$

so that $|\log |f'|| = \log^+ |f'| + \log^- |f'|$ . Since $\log^+ |f'|$ and $\log^- |f'|$ satisfy analogous estimates, we check only that $\log^+ |f'| \in L^1(\omega_U)$. In fact, we only need to check integrability near $s_i$, say in a disk $D(s_i, r)$ for some $r > 0$. Write $D_n = D(s_i, 1/n)$, for $n$ small enough. We have

$$\int_{\partial U \cap D(s_i,r)} \log^+ |f'| \, d\omega_U \lesssim \sum_n n^\beta \left(\omega_U(D_n) - \omega_U(D_{n+1})\right) \lesssim \sum_n ((n+1)^\beta - n^\beta) \omega_U(D_{n+1})$$
$$\lesssim \sum_n n^{\beta-1} \cdot \frac{1}{n^{2\alpha}} = \sum_n \frac{1}{n^{-\beta+1+2\alpha}}.$$

The hypothesis $\beta \in (0, \frac{1}{2\alpha})$ guarantees the convergence of the sum, and hence of the integral, as desired. □

### 4.8.2 Non-negative Lyapunov exponents. Proposition 4.F

Next, we give conditions under which $\chi_{\omega_U}$ is non-negative. Our result is inspired in [KU23, Lemma 9.1.2, Corol. 9.1.3], but we remark that we do not assume that $f$ extends holomorphically (in fact, not even continuously) to a neighbourhood of $\widehat{\partial} U$.

**Proposition 4.8.6. (Lyapunov exponents are non-negative)** *Let $f \in \mathbb{K}$, and let $U$ be an invariant Fatou component for $f$, such that $\omega_U$ is $f$-invariant. Assume*

(a) *$U$ is asymptotically contained in a sector of angle $\alpha \in (0,1)$, with order of growth $\beta \in (0, \frac{1}{2\alpha})$;*

(b) *$\int_{\partial U} \log |x - SV|^{-1} d\omega_U(x) < \infty$.*

*Then,*

$$\chi_{\omega_U} = \int_{\partial U} \log |f'| \, d\omega_U \geq 0.$$

Note that Proposition 4.8.6 implies Proposition 4.F.

*Proof.* If $\omega_U$ is $f$-invariant, then $U$ is either an attracting basin or a Siegel disk, and $\omega_U$ is precisely the harmonic measure with basepoint the fixed point $p \in U$. In particular, $f|_{\partial U}$ is ergodic with respect to $\omega_U$.

*1. Asymptotic contraction of $f^n|_{\partial U}$, $\omega_U$-almost everywhere.* By Proposition 4.8.3, the integral $\chi_{\omega_U} = \int_{\partial U} \log |f'| \, d\omega_U$ is well-defined. Since $f|_{\partial U}$ is ergodic, Birkhoff Ergodic Theorem I.1.9, for $\omega_U$-almost every $x \in \partial U$,

$$\lim_n \frac{1}{n} \log |(f^n)'(x)| = \chi_{\omega_U}(f).$$

We want to see that $\chi_{\omega_U}(f) \geq 0$. We shall assume, on the contrary, that $\chi_{\omega_U}(f) < 0$, and seek for a contradiction.

Since $\chi_{\omega_U}(f) < 0$, it follows that there exists $M \in (e^{\frac{\chi_{\omega_U}}{4}}, 1)$ and $n_0 \coloneqq n_0(x) \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$|(f^n)'(x)|^{\frac{1}{4}} \leq M^n < 1.$$

We fix $x \in \partial U$ satisfying the previous property, and we denote by $\{x_n\}_n$ its forward orbit.

*2. Shrinking domains where $f|_{\partial U}$ is univalent, $\omega_U$-almost everywhere.* Let $M \in (0,1)$ be the constant fixed in the previous step, and let $x_n = f^n(x)$, for $n \geq 0$.

**Lemma 4.8.7.** *For $\omega_U$-almost every $x \in \partial U$ and $\lambda \in (M, 1)$, there exists $n_1 \geq n_0$ such that $f|_{D(x_n, \lambda^n)}$ is univalent, for all $n \geq n_1$.*

In particular, since $\lambda > M$, $f|_{D(x_n, M^n)}$ is univalent, for all $n \geq n_1$.

*Proof.* Since $\beta < \frac{1}{2\alpha}$, we can choose $\gamma \in (\beta, \frac{1}{2\alpha})$. Then, applying the estimates of Lemma 4.8.5, we have

$$\omega_U\left(S_{\alpha, n^{-\frac{1}{\gamma}}}\right) \leq C \cdot n^{-\frac{1}{\gamma \cdot 2\alpha}},$$

and therefore

$$\sum_{n \geq 1} \omega_U\left(S_{\alpha, n^{-\frac{1}{\gamma}}}\right) \leq \sum_{n \geq 1} C \cdot n^{-\frac{1}{\gamma \cdot 2\alpha}} < +\infty.$$

By the assumption on the growth, for all $z \notin S_{\alpha, n^{-\frac{1}{\gamma}}}$ and $n \in \mathbb{N}$ large enough, we have

$$|f'(z)| \leq C \cdot e^{n^{\frac{\beta}{\gamma}}},$$

for some constant $C > 0$. Then, it is easy to see that there exists $C' > 0$ such that, for $z \notin S_{\alpha, n^{-\frac{1}{\gamma}}}$ and $n \in \mathbb{N}$ large enough,

$$|f'(z)| \leq C' \cdot \lambda^{-n/4},$$

where $\lambda \in (M, 1)$ is the constant given in the statement of the lemma.

Now, using the previous computations and assumption (b), the first Borel-Cantelli Lemma I.1.4 yields that, for $\omega_U$-almost every $x \in \partial U$ and $n$ large enough (depending on $x$), it holds

(1.1) $x_{n+1} \notin \bigcup_{s \in SV} D(s, \lambda^{(n+1)/2})$,

(1.2) $x_n \notin S_{\alpha, n^{-\frac{1}{\gamma}}}$.

By (1.1), all inverse branches of $f$ are well-defined in $D(x_{n+1}, \lambda^{(n+1)/2})$ and are univalent. Denote by $F$ the inverse branch of $f$ defined in $D(x_{n+1}, \lambda^{(n+1)/2})$ such that $F(x_{n+1}) = x_n$. By Koebe's distortion estimates I.3.3, we have

$$F(D(x_{n+1}, \lambda^{(n+1)/2})) \supset D(x_n, R),$$

where

$$R = \frac{1}{4} \cdot |F'(x_{n+1})| \cdot \lambda^{(n+1)/2} = \frac{\lambda^{(n+1)/2}}{4} \cdot \frac{1}{|f'(x_n)|} \geq \frac{\lambda^{(n+1)/2}}{4} \cdot \lambda^{n/4} = K \cdot \lambda^{\frac{3n}{4}},$$

for some constant $K > 0$. It follows that there exists $n_1 := n_1(x)$ large enough so that, for $n \geq n_1$,

$$F(D(x_{n+1}, \lambda^{(n+1)/2})) \supset D(x_n, \lambda^n).$$

Hence, $f|_{D(x_n, \lambda^n)}$ is univalent, for all $n \geq n_1$. □

Hence, we fix a point $x \in \partial U$ such that its forward orbit $\{x_n\}_n$ satisfies the following conditions, with $M \in (0, 1)$ and $n_1 := n_1(x)$ as above:

(2.1) $|(f^n)'(x)|^{\frac{1}{4}} \leq M^n < 1$, for all $n \geq n_1$;

(2.2) $f|_{D(x_n, M^n)}$ is univalent, for all $n \geq n_1$.

*3. Quantitative contraction of $f^n|_{D(x, \lambda^n)}$, for $n$ large enough.* Let

$$b_n := \left|(f^{n+1})'(x)\right|^{\frac{1}{4}}, \qquad P := \prod_{n \geq 1} (1 - b_n).$$

Observe that, since $\sum b_n \leq \sum M^n < \infty$, $P$ is convergent. For all $n$, let

$$D_n := D(x, r \cdot \prod_{m=1}^{n} (1 - b_m)).$$

We can choose $r := r(x) > 0$ small enough so that $2r < 1$, $f^{n_1}|_{D_{n_1}}$ is univalent, and $f^{n_1}(D_{n_1}) \subset D(x_{n_1}, M^{n_1})$.

**Claim 4.8.8.** *For $n \geq n_1$, $f^n|_{D_n}$ is univalent, and $f^n(D_n) \subset D(x_n, M^n)$.*

It follows from the claim that, for all $n \in \mathbb{N}$, $f^n$ is univalent in $D(x, rP)$ and $f^n|_{D(x,rP)} \subset D(x_n, M^n)$.

*Proof.* We prove the claim inductively: assume the claim is true for $n \geq n_1$, and let us see that it also holds for $n+1$.

First note that, since $f^n(D_n) \subset D(x_n, M^n)$ (by inductive assumption) and $f$ is univalent in $D(x_n, M^n)$ (by Lemma 4.8.7), it follows that $f^{n+1}|_{D_n}$ is univalent. In particular, since $D_{n+1} \subset D_n$, we have that $f^n|_{D_{n+1}}$ is univalent.

Now we use Koebe's distortion estimates (Thm. I.3.3) to prove the bound on the size of $f^{n+1}(D_{n+1})$. Indeed, since $f^{n+1}|_{D_n}$ is univalent, we have $f^{n+1}(D_{n+1}) \subset D(x_{n+1}, R)$, where

$$R = r \cdot \prod_{m\geq 1}^{n} (1 - b_m)) \cdot \left|(f^{n+1})'(x)\right| \cdot \frac{2}{b_n^3} \leq 2r \cdot \frac{|(f^{n+1})'(x)|}{|(f^{n+1})'(x)|^{\frac{3}{4}}} \leq \left|(f^{n+1})'(x)\right|^{\frac{1}{4}} \leq M^{n+1},$$

as desired. □

*4. Contradiction with the blow-up property of the Julia set.* Let $R > 0$ be small enough so that $D(p, R) \subset U$, where $p$ is the fixed point of $f$ in $U$. Let $n_2 \geq n_1$ be such that $M^{n_2} < \frac{R}{2}$ (recall that $M \in (0,1)$, so such $n_2$ exists).

Then, $f^{n_2}(D(x, rP))$ is a neighbourhood of $x_{n_2} = f^{n_2}(x) \in \mathcal{J}(f)$. By the previous step,

$$\bigcup_{n\geq n_2} f^n(D(x, rP)) \subset \bigcup_{n\geq n_2} D(x_n, M^n) \subset \bigcup_{n\geq n_2} D(x_n, M^{n_2}) \subset \widehat{\mathbb{C}} \setminus D(p, R/2).$$

This is a contradiction of the blow-up property of the Julia set. Notice that the contradiction comes from assuming $\chi_{\omega_U} < 0$. Therefore, $\chi_{\omega_U} \geq 0$, and this ends the proof. □

### 4.8.3 The Lyapunov exponent for parabolic basins and Baker domains

The boundary of parabolic basins and doubly parabolic Baker domains do not support invariant probabilities which are absolutely continuous with respect to the harmonic measure $\omega_U$. However, the measure

$$\lambda_{\mathbb{R}}(A) = \int_A \frac{1}{|w-1|^2} d\lambda(w), \qquad A \in \mathcal{B}(\partial\mathbb{D}),$$

is invariant under the radial extension of the associated inner function $g$ (taken such that 1 is the Denjoy-Wolff point) and its push-forward $\mu = (\varphi^*)_* \lambda_{\mathbb{R}}$ is an infinite invariant measure supported on $\widehat{\partial} U$.

Hence, in the case of parabolic basins and doubly parabolic Baker domains, we shall consider the Lyapunov exponent of $f$ with respect to $\mu$

$$\chi_\mu(f) := \int_{\partial U} \log |f'| \, d\mu.$$

We show that, for parabolic basins, if $\log|f'|$ is integrable with respect to harmonic measure, then it is also integrable with respect to the invariant measure $\mu$. Note also that Lemma 4.8.1 and Proposition 4.8.3 give conditions for $\log|f'| \in L^1(\omega_U)$ and do not assume that $\omega_U$ is invariant, so they still hold in the parabolic setting.

**Proposition 4.8.9. (Parabolic Lyapunov exponents)** *Let $f \in \mathbb{K}$, and let $U$ be a parabolic basin. If $\log|f'| \in L^1(\omega_U)$, then $\log|f'| \in L^1(\mu)$. If, in addition, $\chi_{\omega_U} > 0$, then $\chi_\mu > 0$.*

*Proof.* For the first statement note that, since $\omega_U$ and $\mu$ are comparable except in a neighbourhood of the parabolic fixed point $p \in \partial U$, it is enough to check that

$$\int_{D(p,r)} \log|f'|\, d\mu < \infty,$$

for some $r > 0$. We note that, in contrast with the situation considered in Proposition 4.8.3, $\log|f'|$ achieves a (finite) maximum and minimum around $p$ (since $f'(p) = 1$), but now the difficulty comes from the fact that $\mu$ is an infinite measure.

On the one hand, around the parabolic fixed point (which we assume to be the origin), we have the following normal form, $f(z) = z + az^q + \dots$, with $a \in \mathbb{C}$ and $q \geq 2$ (see e.g. [Mil06, Sect. 10]). Therefore, $\log|f'(z)| \sim \log(1 + qa\,|z|^{q-1})$.

On the other hand, by Lemma 4.8.4, we have that

$$\lambda((\varphi^*)^{-1}(D(p,r))) = \omega_U(D(p,r)) \leq C \cdot \sqrt{r}.$$

Therefore, setting $D_n = D(p, 1/n)$,

$$\begin{aligned}\int_{D(p,r)} \log|f'|\, d\mu &\lesssim \sum_{n=n_0}^{\infty} \log\left|1 + qa\frac{1}{n^{q-1}}\right| (\mu(D_n) - \mu(D_{n+1})) \\ &\lesssim \sum_{n=n_0}^{\infty} \left(\frac{1}{n^{q-1}} - \frac{1}{(n+1)^{q-1}}\right) \cdot \mu(D_n) \\ &\lesssim \sum_{n=n_0}^{\infty} \frac{n^{q-2}}{n^{2(q-1)}} \cdot \sqrt{n} < \infty,\end{aligned}$$

as desired.

For the second statement, applying the Leau-Fatou Flower Theorem (see e.g. [Mil06, Sect. 10]), we have that $\log|f'| > 0$ in $D(p,r) \cap \partial U$, for $r$ small enough. Then the statement follows directly. □

**Remark 4.8.10.** Proposition 4.8.9 is stated only for parabolic basins, and its proof used the normal form around a parabolic fixed point. For a Baker domain, there is no longer a normal form around the convergence point, since it is an essential singularity for $f$, and hence the argument cannot be applied in general. However, for some explicit Baker domains, similar estimates can be obtained and the argument may work *ad hoc*. Indeed, consider for instance the Baker domain of the map $f(z) = z + e^{-z}$ (see Chapter 1). Since it is contained in a strip and $f$ has finite order, by Proposition 4.8.3, $\log|f'| \in L^1(\omega_U)$. To see that $\log|f'| \in L^1(\mu)$, it is enough to check integrability in a neighbourhood of infinity. Note that $f'(z) = 1 - e^{-z}$, so the estimates on $|f'|$ are even better than in the parabolic case, and the same argument can be applied. Moreover, $|f'| > 1$ in a neighbourhood of $\partial U$, so $\chi_\mu(f) > 0$.

# Chapter 5

# Boundaries of hyperbolic and simply parabolic Baker domains

The goal of this chapter is to understand both topologically and dynamically the boundaries of hyperbolic and simply parabolic Baker domains.

We note that all the Fatou components studied in the previous chapters (and in the works of [DG87, BD99, BW91, BK07, Bar08, BFJK17]) are attracting and parabolic basins, and doubly parabolic Baker domains, which share the same ergodic properties of the boundary map ($f|_{\partial U}$ is ergodic and recurrent with respect to harmonic measure $\omega_U$, and $\omega_U$-almost every point has a dense orbit in $\partial U$ – aspects that are used in a fundamental way in the previous works).

On the other hand, hyperbolic and simply parabolic Baker domains may exhibit a completely different boundary behaviour. For instance, for the simply parabolic Baker domain of the function

$$f(z) = z + 2\pi i\alpha + e^z,$$

for appropriate $\alpha \in [0, 1] \setminus \mathbb{Q}$, studied in [BW91, Thm. 4] (see also Ex. II.5.12), all points in the boundary, which is a Jordan curve, converge to infinity under iteration. Thus, the previous techniques do not apply and such Baker domains remain somehow unexplored, except for the results in [RS18], [BFJK19, Thm. A] (Thm. II.5.4), which establish the measure of the escaping set in $\partial U$, under certain conditions on the associated inner function.

First, from an ergodic point of view, a hyperbolic or simply parabolic inner function is non-ergodic and non-recurrent with respect to the Lebesgue measure $\lambda$ (Thm. II.5.4). By relying on properties of the Riemann map, we show that both non-ergodicity and non-recurrence transfer to the boundary map, with respect to harmonic measure.

**Theorem 5.A. (Ergodic properties of the boundary map)** *Let $f \in \mathbb{K}$, and let $U$ be a simply connected Baker domain, of hyperbolic or simply parabolic type. Then, $f|_{\partial U}$ is non-ergodic and non-recurrent with respect to the harmonic measure $\omega_U$.*

Now, let us turn to analyze the Carathéodory set of such Baker domains. Let us recall that, in Chapter II, we introduce the notion of *Carathéodory set* of the Baker domain $U$ as the points $x \in \partial U$ such that, for any crosscut neighbourhood $N \subset \mathbb{D}$ at the Denjoy-Wolff point $p \in \partial\mathbb{D}$, there exists $k_0$ such that, for all $k \geq k_0$,

$$f^k(x) \in \overline{\varphi(N)}.$$

In other words, the Carathéodory set is the set of points in $\partial U$ whose orbit converges to the Denjoy-Wolff point in the Carathéodory's topology of $\partial U$.

By the results of Doering and Mañé for inner functions (Thm. II.3.7), the Carathéodory set has full harmonic measure (and, in particular, it is dense in $\partial U$) for simply connected Baker domains, of hyperbolic or simply parabolic type. Moreover the escaping points constructed in [RS18], [BFJK19, Thm. A] are also in the Carathéodory set (and since they are escaping, they are also in the Denjoy-Wolff set). However, points in the Carathéodory set may fail to converge to infinity in general, if the cluster set of the Denjoy-Wolff point is non-degenerate.

In view of these results, one shall ask if there exist points in $\partial U$ which are not in the Carathéodory set. The answer is negative in general, as shown by the univalent Baker domain of $f(z) = z + 2\pi i\alpha + e^z$ introduced above (Ex. II.5.12). In Section II.5.4, we analyse other examples of univalent Baker domains which have either none or a single non-Carathéodory point in the boundary. We note that, in this case, the associated inner function is a Möbius transformation, and every point in $\partial\mathbb{D}$ (with at most one exception) converges to the Denjoy-Wolff point.

On the other hand, if the Baker domain $U$ is non-univalent, there exists a perfect set in $\partial\mathbb{D}$ in which iterates do not converge to the Denjoy-Wolff point locally uniformly (the Julia set $\mathcal{J}(g)$), and thus one expects plenty of non-Carathéodory points in the boundary of $U$.

Our result reads as follows.

**Theorem 5.B. (Non-empty non-Carathéodory set)** *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function, and let $U$ be a non-univalent hyperbolic or simply parabolic Baker domain. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g = \varphi^{-1} \circ f \circ \varphi$ be the inner function associated with $(f, U)$ by $\varphi$. Assume there exists a crosscut neighbourhood $N_\xi$ of $\xi \in \mathcal{J}(g)$ such that $\varphi(N_\xi) \cap P(f) = \emptyset$. Then, there are uncountably many points which are not in the Carathéodory set, which are furthermore accessible from $U$.*

The hypothesis of the existence of a crosscut neighbourhood $N_\xi$ of $\xi \in \mathcal{J}(g)$ such that $\varphi(N_\xi) \cap P(f) = \emptyset$ is always satisfied when $SV(f) \cap U$ is compactly contained in $U$, in particular, if $f|_U$ has finite degree (see Prop. II.5.1). Note also that, if $SV(f) \cap U$ is compactly contained in $U$, then the Denjoy-Wolff point of the inner function is not a singularity (Prop. II.5.1), and by [BFJK19, Thm. A], the Denjoy-Wolff set (and the Carathéodory set) has full harmonic measure. However, it follows from Theorem 5.B that not all points in $\partial U$ are in the Carathéodory set.

The previous result relies strongly on the topology of the boundary of unbounded Fatou components of entire functions, studied in [BD99, Bar08], following the same argument as in Theorem 2. Recall that, for non-univalent Fatou components of entire functions, the set of accesses to infinity

$$\Theta_\infty := \{\xi \in \partial\mathbb{D} : \varphi^*(\xi) = \infty\}$$

is related with the Julia set $\mathcal{J}(g)$ in the sense that $\mathcal{J}(g) \subset \overline{\Theta_\infty}$ (see Chapter II). For hyperbolic and simply parabolic Baker domains, this translates into the following properties, concerning the topology of the boundary, the Julia set $\mathcal{J}(g)$ and the singularities of the associated inner function.

**Proposition 5.C. (Baker domains of entire functions)** *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and let $U$ be a non-univalent hyperbolic or simply parabolic Baker domain. Let $\varphi : \mathbb{D} \to U$ be a Riemann map, and let $g = \varphi^{-1} \circ f \circ \varphi$ be the inner function associated with $(f, U)$ by $\varphi$. Then, the following holds.*

(a) (Topology of $\partial U$) *$\partial U$ has infinitely many components. Moreover, for all $\xi \in \partial\mathbb{D}$, the cluster set $Cl(\varphi, \xi) \cap \mathbb{C}$ is contained in either one or two components of $\partial U$; in the later case, $\varphi^*(\xi) = \infty$.*

*Each component $C$ of $\partial U$ contains points of the cluster set $Cl(\varphi, \xi) \cap \mathbb{C}$ of a unique $\xi \in \mathcal{J}(g)$, with at most countably many exceptions.*

(b) (Associated inner function) *Assume $\overline{\Theta} \neq \partial\mathbb{D}$. Then, $\mathcal{J}(g)$ is a Cantor set, and the set of singularities of $g$, $E(g)$, is nowhere dense in $\partial\mathbb{D}$. If, in addition, the Denjoy-Wolff point $p \in \partial\mathbb{D}$ is not a singularity, then both $\mathcal{J}(g)$ and $E(g)$ have zero $\lambda$-measure.*

(c) (Periodic points) *Assume $\mathcal{J}(g) \neq \partial\mathbb{D}$. Then, periodic points are not dense on $\partial U$.*

Examples of non-univalent hyperbolic or simply parabolic Baker domains of entire functions are given in [RS18], [Rip06], [Bar08, Ex. 3.6] [BZ12], compare also with [Bar08, Sect. 2.5] for examples of meromorphic functions (thought as inner functions with a unique singularity). The examples in [BZ12] satisfy that $\overline{\Theta} \neq \partial\mathbb{D}$, and in fact $\varphi$ extends continuously to an arc.

The question on the size of the singularities of the inner function associated with a Fatou component has been widely studied [EFJS19, ERS20], see also [IU23, Part III] and Theorem 2 and Theorem A, although all these inner functions are associated with attracting or parabolic basins, or doubly parabolic Baker domains. This is the first time that inner functions of hyperbolic or simply parabolic type are addressed. Note also that, although for finite degree Blaschke products the Julia set is either a Cantor set or the unit circle, this no longer holds for inner functions of infinite degree [Bar08, Sect. 2.5], and hence (b) provides additional requirements on the inner functions that can be associated to Fatou components.

Finally, one can go one step further and ask whether there are periodic points in the boundary of such Baker domains. Even if periodic points are not dense in the boundaries in general, we prove the following.

**Theorem 5.D. (Boundary dynamics)** *Let $f \in \mathbb{K}$ and let $U$ be a non-univalent simply connected Baker domain, of hyperbolic or simply parabolic type. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g = \varphi^{-1} \circ f \circ \varphi$ be the inner function associated with $(f, U <)$ by $\varphi$. Assume there exists a crosscut neighbourhood $N_\xi$ of $\xi \in \mathcal{J}(g)$ such that $\overline{\varphi(N_\xi)} \cap P(f) = \emptyset$. Then, the following holds.*

(a) *There exist countably many accessible periodic points on $\partial U$. More precisely, for any $\zeta \in \mathcal{J}(g)$ and any crosscut neighbourhood $N_\zeta$ of $\zeta$, there is an accessible periodic point in $\overline{\varphi(N_\zeta)}$.*

(b) *For any countable collection of crosscut neighbourhoods $\{N_{\xi_k}\}_k$ of $\xi_k \in \mathcal{J}(g)$, there exists an accessible point $x \in \partial U$ and a sequence $n_k \to \infty$ such that $f^{n_k}(x) \in \overline{\varphi(N_{\xi_k})}$. Moreover, if $f|_U$ has infinite degree, $x$ can be taken* bungee, *i.e. such that $\{f^n(x)\}_n$ neither escapes nor is compactly contained in $\Omega(f)$.*

(c) *If $\mathcal{J}(g) = \partial\mathbb{D}$ and $\omega_U(P(f)) = 0$, then periodic points are dense on $\partial U$. Moreover, if $f|_U$ has infinite degree, there are accessible points on $\partial U$ whose orbit is dense on $\partial U$.*

The assumptions in (c) are satisfied for the hyperbolic Baker domain of the function

$$f(z) = 2z - 3 + e^z,$$

studied by Bargmann [Bar08, Ex. 3.6] (see Ex. II.5.14). Note the wide range of boundary dynamics for hyperbolic and simply parabolic Baker domains: from examples for which every point in $\partial U$ is escaping, to others for which both periodic and bungee points are dense on $\partial U$.

We remark that the construction of periodic boundary points for basins of rational maps in [PZ94] relies strongly on the ergodic properties of the boundary map (more precisely, ergodicity and recurrence), as well as the constructions in Theorems II and III. Instead, we use the topological properties of $\mathcal{J}(g)$, but the complexity of the proof increases substantially.

## 5.1 Ergodic properties of $f\colon \partial U \to \partial U$. Proof of Theorem 5.A

Now we prove Theorem 5.A, which states that, for a hyperbolic or simply parabolic Baker domain $U$, the boundary map $f\colon \partial U \to \partial\partial U$ is non-ergodic and non-recurrent with respect to harmonic measure $\omega_U$.

Note that, even though the diagram

$$\begin{array}{ccc} \partial U & \xrightarrow{f} & \partial U \\ {\scriptstyle\varphi^*}\uparrow & & \uparrow{\scriptstyle\varphi^*} \\ \partial\mathbb{D} & \xrightarrow{g^*} & \partial\mathbb{D} \end{array}$$

commutes $\lambda$-almost everywhere, and thus $f|_{\partial U}$ is a factor of $g^*|_{\partial \mathbb{D}}$, the map $\varphi^* \colon \partial\mathbb{D} \to \partial U$ need not be an isomorphism. Thus, the non-ergodicity and non-recurrence of $f|_{\partial U}$ can not be deduced straightforward from the non-ergodicity and non-recurrence of $g^*|_{\partial \mathbb{D}}$ (Thm. II.3.7), and our proof relies on properties of the Riemann map (more precisely, Thm. II.4.2), and of hyperbolic and simply parabolic inner functions (Thm. II.3.7, II.3.9). For more details on the ergodic properties of measure-theoretical isomorphisms and factor maps see e.g. [KH95, Sect. 4.1.g].

**Remark 5.1.1.** In the particular case of hyperbolic and simply parabolic Baker domains of entire maps, the map $\varphi^* \colon \partial\mathbb{D} \to \partial U$ is one-to-one (Thm. II.5.7), and thus the non-ergodicity and non-recurrence follow straightforward from the analogous properties of the associated inner function.

*Proof of Theorem 5.A.* Let us start by proving that $f|_{\partial U}$ is non-recurrent with respect to $\omega_U$. Note that if

$$\varphi^* \colon \partial\mathbb{D} \to \partial U$$

is a measure-theoretical isomorphism (i.e. a bijection up to sets of zero measure), non-recurrence follows from the same property of the associated inner function. Therefore, we shall assume that there exist $\xi_1, \xi_2 \in \partial\mathbb{D}$ and $x \in \partial U \subset \Omega(f)$ such that $\varphi^*(\xi_1) = \varphi^*(\xi_2) = x$. Then, the image under $\varphi$ of the radial segments at $\xi_1$ and $\xi_2$, together with $x$, i.e.

$$\varphi(R(\xi_1)) \cup \varphi(R(\xi_2)) \cup \{x\}$$

is a Jordan curve in $\widehat{\mathbb{C}}$, and divides $\partial U$ in two sets $X_1, X_2$ of positive measure. Moreover, $\xi_1$ and $\xi_2$ delimit two (non-degenerate) open circular arcs, say $I_1$ and $I_2$, and $(\varphi^*)^{-1}(X_1) \subset I_1$ and $(\varphi^*)^{-1}(X_2) \subset I_2$. We assume the Denjoy-Wolff point of the inner function lies in $I_1$. Since $\lambda$-almost $\xi \in \partial\mathbb{D}$ converges to the Denjoy-Wolff point under the iteration of $g^*$ (Thm. II.3.7), it follows that $\omega_U$-almost every point in $X_2$ does not come back to $X_2$ infinitely often (since its orbit is eventually contained in $X_1$), and proves non-recurrence. See Figure 5.1.

Let us turn now to prove non-ergodicity, i.e. the existence of an invariant set of neither full nor zero measure. To do so, let

$$X \coloneqq \{\xi \in \partial\mathbb{D} \colon \varphi^*(\xi) = a,\ \varphi^*(\xi_j) = a \text{ for three distinct points } \xi_j \in \partial\mathbb{D}\}\,.$$

By Theorem II.4.2, $\lambda(X) = 0$, since it is the countable union of sets of measure zero. Moreover, since inner functions are non-singular,

$$Z \coloneqq \partial\mathbb{D} \smallsetminus \bigcup_{n\in\mathbb{Z}} (g^*)^n(X)$$

is invariant under $g^*$ and has zero $\lambda$-measure.

Since $g^*|_{\partial\mathbb{D}}$ is non-ergodic, there exists $A \subset Z$ such that $\lambda(A) \in (0,1)$ and $(g^*)^{-1}(A) = A$. Since $f \circ \varphi^* = \varphi^* \circ g^*$ holds $\lambda$-almost everywhere, the set $\varphi^*(A)$ is $f$-invariant up to a set of zero measure. If $\omega_U(\varphi^*(A)) \in (0,1)$ we are done.

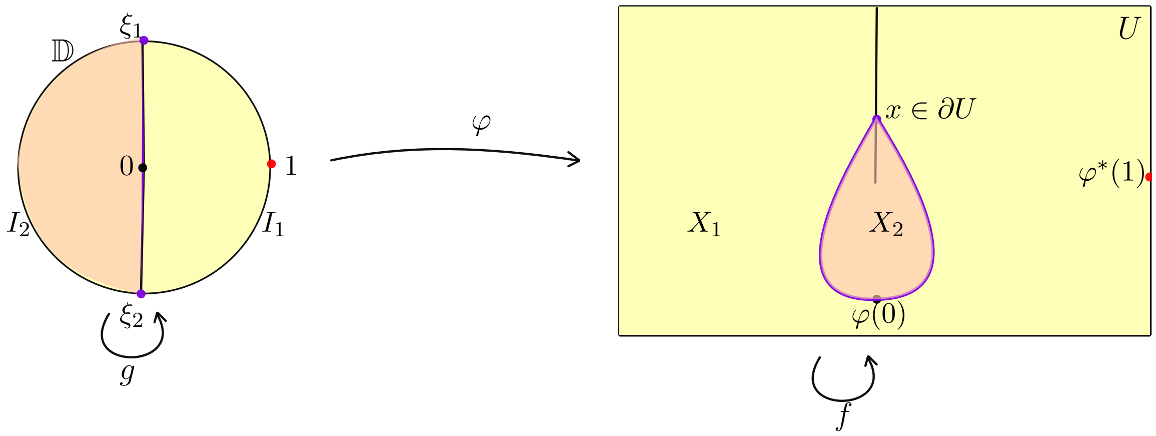



Figure 5.1: Set-up of the proof of non-recurrence: schematic representation the Riemann map $\varphi\colon \mathbb{D} \to U$, together with the choice of $\xi_1, \xi_2 \in \partial\mathbb{D}$.

Otherwise, $\omega_U(\varphi^*(A)) = 1$, and $\varphi^*|_A$ is one-to-one (up to a set of zero measure). We claim that there exists $A' \subset A$ such that $(g^*)^{-1}(A') = A'$ and $0 < \lambda(A') < \lambda(A)$. This would imply that $\omega_U(\varphi^*(A')) \in (0,1)$, since $\varphi^*|_A$ is one-to-one (up to a set of zero measure), and thus will end the proof of Theorem 5.A.

To prove the existence of $A'$ we rely on Theorem II.3.9, which claims that there exists $h$ inner function and $T$ Möbius transformation such that

$$\begin{array}{ccc} \partial\mathbb{D} & \xrightarrow{g^*} & \partial\mathbb{D} \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ \partial\mathbb{D} & \xrightarrow{T} & \partial\mathbb{D} \end{array}$$

$\lambda$-almost everywhere, and $h^*|_{\partial\mathbb{D}}$ is measure-preserving. With this diagram in mind it is easy to obtain the set $A'$ (since there exist $T$-invariant sets of arbitrarily small measure contained in $h^*(A)$). The proof is now complete. □

## 5.2 Hyperbolic and simply parabolic Baker domains of entire functions. Proof of Theorem 5.B and Proposition 5.C

In this section we prove Theorem 5.B, which asserts that the non-Carathéodory set is non-empty for non-univalent Baker domains of entire functions, of hyperbolic or simply parabolic type, under the assumption that there exists a crosscut neighbourhood $N_\xi$ of $\xi \in \mathcal{J}(g)$ with $\varphi(N_\xi) \cap P(f) = \emptyset$.

We split the proof into three steps: first, we prove that $\mathcal{J}(g)$ has positive Hausdorff dimension; second, we prove topological properties of the boundaries of such Baker domains (Proposition 5.C); and finally we prove Theorem 5.B, which will be a consequence of the two previous results.

### The Hausdorff dimension of $\mathcal{J}(g)$

We prove the following.

**Lemma 5.2.1. (Hausdorff dimension of $\mathcal{J}(g)$)** *Let $f \in \mathbb{K}$, and let $U$ be a non-univalent simply connected Baker domain, of hyperbolic or simply parabolic type. Let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, and let $g = \varphi^{-1} \circ f \circ \varphi$ be the inner function associated with $(f, U)$ by $\varphi$. Assume there exists a crosscut neighbourhood $N_\xi$ of $\xi \in \mathcal{J}(g)$ such that $\varphi(N_\xi) \cap P(f) = \emptyset$. Then, the Hausdorff dimension of $\overline{N_\xi} \cap \mathcal{J}(g)$ is positive.*

Note that, in the case where $g$ has finite degree (and thus it is a rational function), the fact that the Hausdorff dimension of $\mathcal{J}(g)$ is positive follows from a result of Garber [Gar78, Thm. 1].

Our proof follows the same ideas as in [Sta94, Thm. A], where Stallard prove that the Julia set of a meromorphic function has always positive Hausdorff dimension. In its turn, her result relies on the following lemma [Fal03, Prop. 9.7] (see also [Sta94, Lemma 2.1]).

**Lemma 5.2.2.** *Let $\phi_1$, $\phi_2$ be contractions on a closed set $D \subset \mathbb{C}$ such that for each $x, y \in D$, we have*

$$b_i\,|x-y| \leq |\phi_i(x) - \phi_i(y)|$$

*with $0 < b_i < 1$, $i = 1, 2$. Let $F \subset D$ be such that*

$$F = \phi_1(F) \sqcup \phi_2(F).$$

*Then, the Hausdorff dimension of $F$ is greater than $s$, where $b_1^s + b_2^s = 1$.*

*Proof of Lemma 5.2.1.* Since $\varphi(N_\xi) \cap P(f) = \emptyset$, according to Proposition II.3.14, there exists a disk $D$ such that $D \cap \mathcal{J}(g) \neq \emptyset$, and all branches $G_n$ of $g^n$ are well-defined (and univalent) in $\overline{D}$. Without loss of generality, let us assume that the Denjoy-Wolff point of $g$ is not in $D$.

Recall that preimages of any point in $\mathcal{J}(g)$ are dense in $\mathcal{J}(g)$ (Lemma II.3.12), and, as an straightforward consequence of Koebe distortion theorem together with the fact that Julia sets have empty interior, given a sequence of iterated inverse branches $\{G_n\}_n$, diam $G_n(D) \to 0$. Therefore, we can choose $\phi_1 = G_{n_1}|_D$ and $\phi_2 = G_{n_2}|_D$ such that $\phi_1(D) \sqcup \phi_2(D) = \emptyset$, $\phi_2(D) \subset D$ and $\phi_1(D) \subset D$ (for appropriate $n_1$, $n_2$ and suitable inverse branches).

Now the proof continues as in [Sta94, Thm. A]; we outline it for completeness. First, note that $\phi_1$, $\phi_2$ satisfy the hypothesis of Lemma 5.2.2 (indeed, the fact that they are contractions follows from Schwarz lemma; the existence of $b_i$ follows from the univalence of $\phi_i$ in $\overline{D}$). Next, define

$$D_{i_1 \dots i_n} = \phi_{i_1} \circ \cdots \circ \phi_{i_n}(\overline{D}),$$

$$H = \bigcup \bigcap_{n=1}^{\infty} D_{i_1 \dots i_n},$$

where the union is taken over all possible sequences $i_1 \dots i_n$, $i_j \in \{1, 2\}$. Then, $H$ is a non-empty compact set and

$$H = \phi_1(H) \sqcup \phi_2(H).$$

According to Lemma 5.2.2, $H$ has positive Hausdorff dimension. Moreover, the orbit under $g^*$ of points in $H$ comes back to $H \subset D$ infinitely often. Thus, points in $H$ do not converge to the Denjoy-Wolff point, meaning that $H \subset \mathcal{J}(g)$, as desired. □

Note that, more precisely, we proved that the set of points in $\overline{N_\xi} \cap \partial\mathbb{D}$ which come back to $\overline{N_\xi}$ infinitely often under iteration has positive Hausdorff dimension.

### Proof of Proposition 5.C

We prove the different statements separately.

(a) If $U$ is a non-univalent Baker domain, then $\mathcal{J}(g) \subset \overline{\Theta}$ and $\mathcal{J}(g)$ is uniformly perfect (Lemma II.3.10, Thm. II.5.6). Thus, $\Theta$ is infinite. Moreover, given $\xi_1, \xi_2 \in \Theta$, the image under $\varphi$ of the radial segments at $\xi_1$ and $\xi_2$, together with infinity, i.e.

$$\varphi(R(\xi_1)) \cup \varphi(R(\xi_2)) \cup \{\infty\}$$

is a Jordan curve in $\widehat{\mathbb{C}}$, say $\gamma$, and $\widehat{\mathbb{C}} \smallsetminus \gamma$ has two connected components, each containing points of $\partial U$. It follows that $\partial U$ has infinitely many connected components. The fact that, for all $\xi \in \partial\mathbb{D}$, $Cl(\varphi, \xi) \cap \mathbb{C}$ is contained in either one or two connected components, and $\varphi^*(\xi) = \infty$ in the latter case, follows from Lemma II.4.8.

It is left to prove that each component $C$ of $\partial U$ contains $Cl(\varphi, \xi) \cap \mathbb{C}$ for a unique $\xi \in \mathcal{J}(g)$, with at most countably many exceptions. Since prime ends are symmetric with at most countably many exceptions [Pom92, Prop. 2.21], we shall restrict to $\xi \in \partial\mathbb{D}$ such that $Cl(\varphi, \xi) \cap \mathbb{C}$ is contained in a unique component of $\partial U$.

Assume first that $\overline{\Theta_\infty} = \partial\mathbb{D}$. Let $\xi_1, \xi_2 \in \partial\mathbb{D}$. Then, there exists $\zeta_1, \zeta_2 \in \Theta_\infty$, the image under $\varphi$ of the radial segments at $\zeta_1$ and $\zeta_2$, together with infinity, i.e.

$$\varphi(R(\zeta_1)) \cup \varphi(R(\zeta_2)) \cup \{\infty\}$$

is a Jordan curve in $\widehat{\mathbb{C}}$, say $\gamma$, and $\widehat{\mathbb{C}} \smallsetminus \gamma$ has two connected components, one containing $Cl(\varphi, \xi_1) \cap \mathbb{C}$ and the other containing $Cl(\varphi, \xi_2) \cap \mathbb{C}$. This already implies that $Cl(\varphi, \xi_1) \cap \mathbb{C}$ and $Cl(\varphi, \xi_2) \cap \mathbb{C}$ lie in different components of $\partial U$. See Figure 5.2.

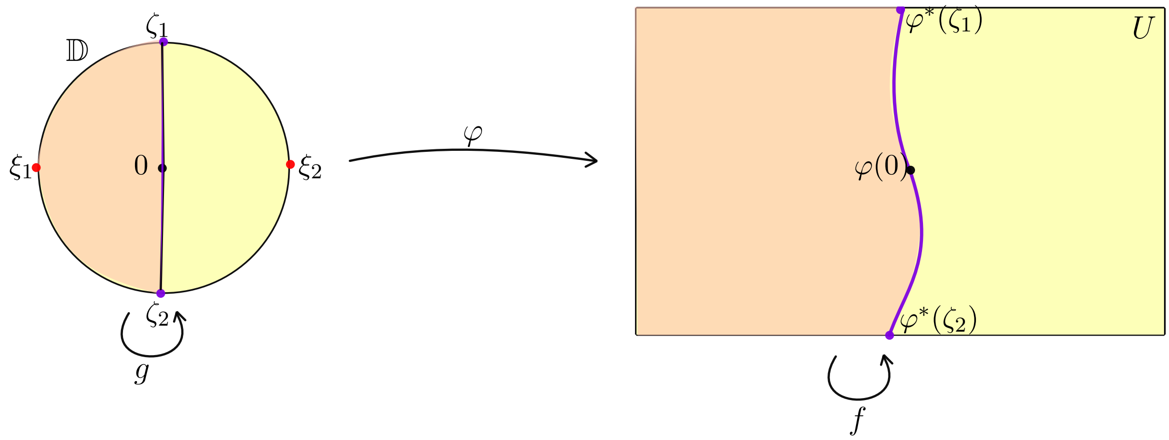



Figure 5.2: Set-up of the proof of Proposition 5.C(a): schematic representation the Riemann map $\varphi\colon \mathbb{D} \to U$, together with $\xi_1, \xi_2 \in \partial\mathbb{D}$, and the choice of $\zeta_1, \zeta_2 \in \Theta$.

Now, assume $\overline{\Theta_\infty} \neq \partial\mathbb{D}$. Then, $\partial\mathbb{D} \smallsetminus \overline{\Theta_\infty}$ is open, so consists of countably many (open) circular intervals on $\partial\mathbb{D}$. These open intervals (which are in $\mathcal{F}(g)$), together with their endpoints (in $\mathcal{J}(g)$), correspond to countably many components of $\partial U$, which we disregard. The remaining points are in $\overline{\Theta_\infty}$, and we can proceed as in the previous case. This ends the proof of (a).

(b) We rely on the following version of the Gross star theorem for iterates of a meromorphic function. The proof can be found in [MPRW25, Thm. 3.1], inspired in the general version of the Gross star theorem given by Kaplan [Kap54, Thm. 3].

**Theorem 5.2.3. (Gross star theorem for iterates)** *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and let $z_0 \in \mathbb{C}$ and $k \geq 1$ be such that $w_0 = f^k(z_0)$ is defined and $(f^k)'(z_0) \neq 0$. Let $W \ni w_0$ be a simply connected domain; for $\theta \in (0, 2\pi)$, denote by $\gamma_\theta$ the hyperbolic geodesic of $W$ starting at $w_0$ in the direction $\theta$. Then, for almost every $\theta \in (0, 2\pi)$, $\gamma_\theta$ is an arc connecting $w_0$ to an endpoint $w_\theta \in \partial W$, and the branch $F_k$ of $f^{-k}$ that maps $w_0$ to $z_0$ can be continued analytically along $\gamma_\theta$ into $w_\theta$.*

We have to prove that $\mathcal{J}(g)$ is has empty interior (then, since it is uniformly perfect, it follows that it is a Cantor set, and, since $E(g) \subset \mathcal{J}(g)$, $E(g)$ has empty interior). On the contrary, assume $I \subset \mathcal{J}(g)$ is a (non-degenerate) circular interval.

The set

$$X \coloneqq \bigcup \{\varphi(R(\xi))\colon \xi \in \Theta_\infty\} \cup \{\infty\}$$

divides $\widehat{\mathbb{C}}$ (and $\partial U$) into infinitely many components. Let $J \subset \partial\mathbb{D} \setminus \overline{\Theta_\infty}$ be a maximal circular interval. Then, $Cl(\varphi, J)$ is contained in one component of $\partial U$, say $C$, and any other interval in $\partial\mathbb{D} \setminus \overline{\Theta_\infty}$ is contained in a different one.

Since preimages of points in $\partial\mathbb{D}$ are dense in $\mathcal{J}(g)$ there exists $I_0 \subset I$ and $n \geq 1$ such that $g^n(I_0) \subset J$. Since critical points of $f^n$ are discrete, take $z_0 = \varphi^*(\xi_0) \in \mathbb{C}$, for some $\xi_0 \in I_0$, not a critical point of $f$. Then, $f^n(z_0) = w_0 \in C$, and let $W \ni w_0$ be a simply connected domain, such that $W \cap \partial U \subset C$ (note that this is possible because all other components of $\partial U$ lie in different components of $\widehat{\mathbb{C}} \setminus X$). Then, by Theorem 5.2.3, the branch $F_n$ of $f^n$ sending $w_0$ to $z_0$ can be continued along an open set containing a continua $K_0 \subset \mathbb{C}$ in $C$. This implies that, for an interval $I_1 \subset I_0$,

$$\{Cl(\varphi, \xi)\colon \xi \in I_1\} \subset K_0.$$

This contradicts the fact that $I_0 \subset \mathcal{J}(g) \subset \overline{\Theta_\infty}$ (and thus accesses to infinity are dense in $\{Cl(\varphi, \xi)\colon \xi \in I\}$), implying that $\mathcal{J}(g)$ is nowhere dense in $\partial\mathbb{D}$.

Finally, if the Denjoy-Wolff point $p \in \partial\mathbb{D}$ is not a singularity for $g$, it follows from the local dynamics around $p$ (which is either an attracting or a parabolic point, with one petal, for $g$), that the Fatou set $\mathcal{F}(g)$ is non-empty, and

$$\mathcal{J}(g) = \{\xi \in \partial\mathbb{D}\colon (g^*)^n(\xi) \not\to p\} \cup \bigcup_{n \geq 0} (g^*)^n(p).$$

In other words, the only way of converging to the Denjoy-Wolff point is either being a preimage of it, or being in the Fatou set. According to Theorem II.3.7, the right-hand side set has zero $\lambda$-measure. Therefore, $\lambda(\mathcal{J}(g)) = 0$ and, since $E(g) \subset \mathcal{J}(g)$, $\lambda(E(g)) = 0$, as desired.

(c) Assume $\mathcal{J}(g) \neq \partial\mathbb{D}$, we have to prove that periodic points are not dense in $\partial U$. As in (b), note that the set

$$X := \bigcup \{\varphi(R(\xi)) \colon \xi \in \Theta_\infty\} \cup \{\infty\}$$

divides $\widehat{\mathbb{C}}$ (and $\partial U$) into infinitely many connected components.

Let $I \subset \partial\mathbb{D} \setminus \mathcal{J}(g)$ be an (open) circular interval, which we can choose so that $I$ and the Denjoy-Wolff point $p \in \partial\mathbb{D}$ can be separated in $\partial\mathbb{D}$ by two different points $\xi_1, \xi_2 \in \Theta_\infty$, i.e. $p$ and $I$ lie in different circular arcs $I_1, I_2$ of $\partial\mathbb{D} \setminus \{\xi_1, \xi_2\}$, say $p \in I_1$, $I \subset I_2$. Note that

$$\varphi(R(\xi_1)) \cup \varphi(R(\xi_2)) \cup \{\infty\}$$

is a Jordan curve in $\widehat{\mathbb{C}}$, say $\gamma$, and $\widehat{\mathbb{C}} \setminus \gamma$ has two connected components, $X_1$ and $X_2$, with $Cl(\varphi, I_i) \cap \mathbb{C} \subset X_i$.

Since $g^n|_I \to p$ uniformly on compact sets, $g^n(I) \subset I_1$ for large $n$, and

$$Cl(\varphi, I) \subset Cl(\varphi, I_2) \subset X_2,$$

$$f^n(Cl(\varphi, I)) = Cl(\varphi, g^n(I)) \subset Cl(\varphi, I_1) \subset X_1.$$

This implies that there are no periodic points in $Cl(\varphi, I_2)$, and thus periodic points are not dense in $\partial U$.

The proof of Proposition 5.C is now complete. □

**Proof of Theorem 5.B**

Without loss of generality, assume that the Denjoy-Wolff point $p$ of $g$ does not belong to $\overline{N_\xi}$. Since $\mathcal{J}(g)$ is uniformly perfect, we can find a crosscut neighbourhood $M_\xi$ such that $M_\xi \subset N_\xi$, and there exists $\xi_1, \xi_2 \in \Theta$ such that $p$ and $M_\xi$ do not lie in the same circular arc of $\partial\mathbb{D} \setminus \{\xi_1, \xi_2\}$. Therefore, for any crosscut neighbourhood $N \subset \mathbb{D}$ of $p$ small enough,

$$\overline{\varphi(N)} \cap \overline{\varphi(M_\xi)} = \emptyset.$$

According to Lemma 5.2.1, the Hausdorff dimension of $\overline{M_\xi} \cap \mathcal{J}(g)$ is positive (more precisely, the set of points in $\overline{M_\xi} \cap \partial\mathbb{D}$ which come back to $\overline{M_\xi}$ infinitely often under iteration has positive Hausdorff dimension). Hence, there exists $\xi \in \overline{M_\xi} \cap \mathcal{J}(g)$ such that $\varphi^*(\xi) \in \Omega(f)$. We can assume that the orbit of $\xi$ comes back to $M_\xi$ infinitely often.

We claim that $\varphi^*(\xi)$ does not belong to the Carathéodory set. Indeed, if $\varphi^*(\xi)$ belongs to the Carathéodory set, for any crosscut neighbourhood $N \subset \mathbb{D}$ at the Denjoy-Wolff point $p \in \partial\mathbb{D}$, there exists $k_0 \geq 0$ such that, for all $k \geq k_0$,

$$f^k(x) \in \overline{\varphi(N)}.$$

Since $\overline{\varphi(N)} \cap \overline{\varphi(M_\xi)} = \emptyset$, this contradicts the fact that the orbit of $\xi$ visits $\overline{M_\xi}$ infinitely often. The proof of Theorem 5.B is now complete. □

## 5.3 Boundary dynamics. Proof of Theorem 5.D

*Proof of Theorem 5.D.* Before proceeding with the proof, note that, since there exists a crosscut neighbourhood $N_\xi$ of $\xi \in \mathcal{J}(g)$ such that $\overline{\varphi(N_\xi)} \cap P(f) = \emptyset$, Lemma 5.2.1 guarantees that the Hausdorff dimension of $\mathcal{J}(g) \cap \overline{(N_\xi)}$ is positive. In particular, there exists $\xi \in \mathcal{J}(g) \cap \overline{(N_\xi)}$ such that $\varphi^*(\xi)$ exists and belongs $\Omega(f)$.

We split the proof in several steps.

(1) *Construction of a region of expansion.* Let us consider an appropriate neighbourhood $W$ of $\overline{\varphi(N_\xi)}$, and prove that, with respect to the hyperbolic metric in $W$, inverse branches are contracting, and uniformly contracting when restricted to compact sets.

**Lemma 5.3.1.** *Let $W := \mathbb{C} \setminus P(f)$. Then, the following holds.*

(a) $f\colon f^{-1}(W) \to W$ *is* locally expanding *with respect to the hyperbolic metric $\rho_W$, i.e.*

$$\rho_W(z) \leq \rho_W(f(z)) \cdot |f'(z)|, \quad \textit{for all } z \in f^{-1}(W).$$

(b) *For all $z \in W$, there exists a neighbourhood $D_z \subset W$ of $z$ such that all branches $F_n$ of $f^{-n}$ are well-defined in $D_z$, $F_n(D_z) \subset W$ and*

$$\operatorname{dist}_W(F_n(x), F_n(y)) \leq \operatorname{dist}_W(x, y), \quad \textit{for all } x, y \in D_z.$$

(c) *Let $z \in W$, and let $D_W(z, R)$ be the hyperbolic disk centered at $z$ of radius $R$. Then, there exists $C \in (0, 1)$ such that, if $F_n(z) \in D_W(z, 2R)$ and $r \in (0, R)$, then*

$$F_n(D_W(z, r)) \subset D_W(F_n(z), C \cdot r) \subset D_W(z, 3R).$$

Note that $\overline{\varphi(N_\xi)} \subset W$, since $\overline{\varphi(N_\xi)} \cap P(f) = \emptyset$. Note also that the constant $C$ does not depend on $n$.

*Proof.* The first two items are standard in transcendental dynamics. For the third item, note that, by the Schwarz-Pick lemma [CG93, Thm. I.4.1] and the triangle inequality, we have that

$$F_n(D_W(z, r)) \subset D_W(F_n(z), r) \subset D_W(z, 3R).$$

This last disk is relatively compact in $W$, and hence there exists $C \in (0, 1)$ such that

$$\rho_W(z) \leq C \cdot \rho_W(f(z)) \cdot |f'(z)|, \quad \text{for all } z \in D_W(z, 3R).$$

Then, by the Schwarz-Pick lemma [CG93, Thm. I.4.1],

$$\begin{aligned}\rho_W(z) &\leq C \cdot \rho_W(f(z)) \cdot |f'(z)| \\ &\leq C \cdot \rho_W(f^{n-1}(f(z))) \cdot \left|(f^{(n-1)})'(f(z))\right| \cdot |f'(z)| \\ &\leq C \cdot \rho_W(f^n(z)) \cdot |(f^n)'(z)|,\end{aligned}$$

for all $z \in D_W(z, 3R)$, implying the first inclusion. The last inclusion follows from the triangle inequality. □

(2) *Construction of accessible periodic points.* Let us start with the following lemma.

**Lemma 5.3.2.** *Let $\zeta \in \mathcal{J}(g)$ be such that $x = \varphi^*(\zeta) \in \mathbb{C}$ and there exists a crosscut neighbourhood $N_\zeta$ such that $\overline{\varphi(N_\zeta)} \cap P(f) = \emptyset$. Then, for all $r > 0$, there exists a periodic point $p$ in $D_W(x, r)$, which is accessible from $U$.*
*Moreover, given a finite collection of crosscut neighbourhoods $\{N_{\xi_k}\}_{k=1}^n$, with $\xi_k \in \mathcal{J}(g)$, $p$ can be taken with $f^{n_k}(p) \in \overline{\varphi(N_{\xi_k})}$, for appropriate $n_k \in \mathbb{N}$.*

Note that, in particular, $p$ can be taken of arbitrarily large period.

*Proof.* Let us start by proving the following claim.

**Claim 5.3.3.** *There exists $\rho > 0$, a closed non-degenerate circular interval $I$ and a subset $K \subset I$ such that*

1. *$\overline{K} = I$;*
2. *for all $\eta \in K$, $\Delta_\rho(\zeta) \subset D_W(x, r/4)$;*
3. *for all $\eta_1 \in I$ and $\rho_1 > 0$, there exists $\eta_2 \in K$ so that $R_\rho(\eta_1) \cap \Delta_\rho(\eta_2) \neq \emptyset$;*
4. *for all $\eta \in I$, iterated inverse branches are well-defined in $D(\eta, \rho)$ and*
$$G_n(R_\rho(\eta)) \subset \Delta_\rho(G_n(\eta))$$
*where $R_\rho$ and $\Delta_\rho$ denote the radial segment and Stolz angle, respectively (for some opening $\alpha \in (0, \pi/2)$ which is fixed throughout the whole proof of Theorem 5.D).*

*Proof.* Note that $\omega_U(D_W(x, r/4)) > 0$, so $X := (\varphi^*)^{-1}(D(x, r))$ has positive Lebesgue measure. Without loss of generality, we can assume $X$ is contained in a circular interval for which condition (4) is satisfied, for some $\rho > 0$. Now, write

$$X^n := \left\{\xi \in X \colon \Delta_{\rho/n}(\xi) \subset D_W(x, r/4)\right\}.$$

Since radial and angular limits coincide, we have

$$X = \bigcup_{n \geq 0} X^n,$$

implying that $\lambda(X^n) > 0$ for some $n \geq 0$. Let $K$ be the set of Lebesgue density points for $X^n$. Note that $\lambda(K) = \lambda(X^n) > 0$. Without loss of generality, we can assume that $\overline{K}$ is connected (and hence it is a non-degenerate closed circular interval). Denote this closed circular interval by $I$.

Replacing $\rho$ by $\rho/n$, we constructed a circular interval $I$ and a subset of it, $K \subset I$, which satisfy conditions (1), (2) and (4). Then, condition (3) follows straightforward from the fact that $K$ is dense in $I$ (replacing $I$ by a smaller subinterval if needed). This ends the proof of the claim. □

Without loss of generality, we can assume $\zeta \in I$ (otherwise, replace $\zeta$ by a point in $I$, and prove the lemma for this new point; this already implies the theorem for the original $\zeta$).

Now, let $C$ be the constant of hyperbolic contraction given by Lemma 5.3.1(c), and let $n_0$ be such that $C^{n_0} < 1/2$. Since iterated preimages of any point in $\mathcal{J}(g)$ are dense in $\mathcal{J}(g)$ (Lemma II.3.12), we can find an inverse branch $G_n$ of $g^n$, well-defined in $\bigcup_{\eta\in I} D(\eta,\rho)$, such that, if we denote by $G_j$, $j = 1, \dots, n$, the inverse branch of $g^j$ sending $\zeta$ to $g^{n-j}(G_n(\zeta))$, it holds

1. $\#\{j = 1, \dots, n \colon G_j(\zeta) \in I\} \geq n_0$;
2. $G_n(\zeta) \in I$;
3. if we denote by $I_{\xi_k}$ the circular arc bounded by $N_{\xi_k}$, there exists $j \in \{1, \dots, n\}$ with $G_j(\zeta) \in I_{\xi_k}$.

Now, let $F_j$ be the inverse branch of $f^j$ corresponding to $G_j$ (i.e. the one such that $\varphi \circ G_j = F_j \circ \varphi$ in $\Delta_\rho(\zeta)$), which is well-defined in $D_W(x,r)$. We claim that

$$F_n(D_W(x,r)) \subset D_W(F_n(x), C^{n_0} \cdot r) \subset D_W(F_n(x), r/2).$$

According to Lemma 5.3.1, it is enough to see that

$$\#\{j = 1, \dots, n \colon F_j(x) \in D_W(x, 2r)\} \geq n_0.$$

Indeed, for all $j \in \{1, \dots, n\}$, we have that

$$F_j(D_W(x,r)) \subset D_W(F_j(x), r).$$

Moreover, for all $j \in \{1, \dots, n\}$ such that $G_j(\zeta) \in I$ (what happens at least $n_0$ times), there exists $\eta \in K$ such that

$$\emptyset \neq G_j(R_\rho(\zeta)) \cap \Delta_\rho(\eta) \ni z.$$

Thus, $w = g^j(z) \in \Delta_\rho(\zeta), G_j(w) = z \in \Delta_\rho(\eta)$, so

$$\varphi(w), \varphi(G_j(w)) = F_j(\varphi(w)) \in D_W(x, r/4).$$

Therefore, by the triangle inequality,

$$\operatorname{dist}_W(F_j(x), x) \leq \operatorname{dist}_W(F_j(x), F_j(\varphi(w))) + \operatorname{dist}_W(F_j(\varphi(w)), x) \leq 2r,$$

as desired.

Next we claim that

$$\overline{F_n(D_W(x,r))} \subset D_W(x,r).$$

Indeed, recall that $G_n$ has been chosen so that $G_n(\zeta) \in I$. Therefore, there exists $\eta \in K$ such that

$$\emptyset \neq G_n(R_\rho(\zeta)) \cap \Delta_\rho(\eta) \ni z,$$

and, as above, $w = g^n(z) \in \Delta_\rho(\zeta), G_n(w) = z \in \Delta_\rho(\eta)$, so

$$\varphi(w), \varphi(G_n(w)) = F_n(\varphi(w)) \in D_W(x, r/4).$$

Note that $\Delta_\rho(\eta) \cap \Delta_\rho(\zeta) \neq \emptyset$.

Since we just proved that $F_n(D_W(x,r)) \subset D_W(F_n(x), r/2)$, we have

$$\text{dist}_W(F_n(x), x) \leq \text{dist}_W(F_n(x), F_n(\varphi(w))) + \text{dist}_W(F_n(\varphi(w)), x) \leq \frac{r}{2} + \frac{r}{4} = \frac{3r}{4}.$$

Therefore, $F_n(D_W(x,r)) \subset D_W(x, 3r/4)$, implying the claim.

See Figure 5.3.

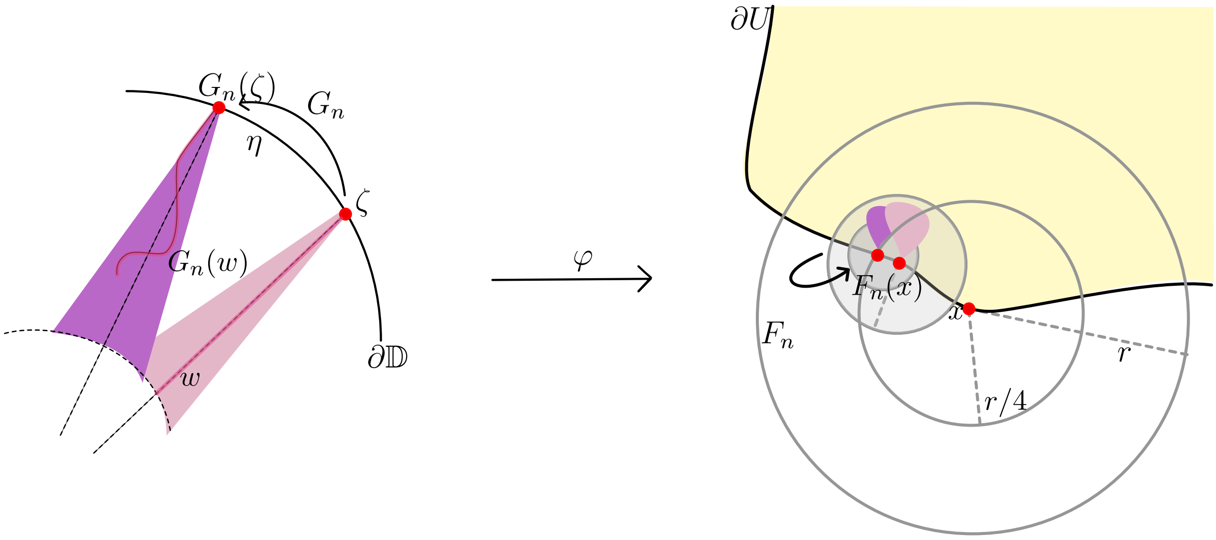



Figure 5.3: Set-up of the proof of the existence of a periodic point in $D(x,r)$, for $x = \varphi^*(\zeta) \in \mathbb{C}$, $\zeta \in \mathcal{J}(g)$.

Finally, by the Brouwer fixed point theorem, $F_n$ has a fixed point $p \in D_W(x,r)$. We claim that $p$ is accessible from $U$, and thus $p \in \partial U$. Indeed, let $\gamma$ be a curve connecting $w$ and $G_n(w)$ in $\Delta_\rho(\eta) \cup \Delta_\rho(\zeta)$, which exists because $\Delta_\rho(\eta) \cap \Delta_\rho(\zeta) \neq \emptyset$. Then, $\varphi(\gamma)$ joins $\varphi(w)$ and $F_n(\varphi(w))$ in $U \cap D_W(x,r)$. The curve

$$\Gamma = \bigcup_{m \geq 0} F_n^m(\varphi(\gamma))$$

lands at $p$, ending the proof of the lemma. □

(3) *Distribution of periodic points.* Statement (a) follows straightforward from Lemma 5.3.2, taking into account that the preimages of $\xi \in \partial\mathbb{D}$ are dense in $\mathcal{J}(g)$, and, if $\zeta = G_n(\xi)$, then $\varphi^*(\zeta)$ exists and it belongs to $W$. Indeed, this follows from the commutative relation $\varphi \circ G_n = F_n \circ \varphi$ in any sufficiently small Stolz angle at $\xi$. Then, for the radial segment $R(\xi)$, we have

$$F_n(\varphi(R(\xi))) = \varphi(G_n(R(\xi))).$$

By Lindelöf Theorem II.4.5, this already implies that $\varphi^*(\zeta)$ exists (and belongs to $W$ by the backward invariance of $W$). Therefore, one can apply Lemma 5.3.2 to $\zeta = G_n(\xi)$ and get the desired result.

If $\mathcal{J}(g) = \partial\mathbb{D}$ and $\omega_U(P(f)) = 0$, note that one can apply Lemma 5.3.2 to $\lambda$-almost every point on $\partial\mathbb{D}$.

(4) *Accessible points with prescribed orbit.* We find the desired point inductively. Indeed, let $x_0 = \varphi^*(\zeta_0) \in W$ as in Lemma 5.3.2, there exists an inverse branch $F_{n_1}$, well-defined in $D_W(x_0, r_0)$ such that

1. $G_{n_1}(\zeta_0) = \zeta_{n_1}$, $F_{n_1}(x_0) = \varphi^*(\zeta_{n_1}) = x_{n_1} \in W \cap \partial U$;
2. $F_{n_1}(D(x_0, r_0)) \subset D_W(x_{n_1}, r_1) \subset D_W(x_0, r_0)$, $r_1 = r_0/2$;
3. at least one of the sets $\{D_W(x_0, r_0), F_1(D_W(x_0, r_0)), \ldots, F_{n_1}(D_W(x_0, r_0))\}$ intersects $\overline{\varphi(N_{\xi_1})}$.

Since $x_{n_1} = \varphi^*(\zeta_{n_1}) \in W \cap \partial U$, we can apply again Lemma 5.3.2 to find an inverse branch $F_{n_2}$, well-defined in $D_W(x_{n_1}, r_1)$, satisfying the analogous properties. Therefore, proceeding inductively, we get a sequence of inverse branches $\{F_{n_k}\}_k$ and points $\{\zeta_{n_k}\}_k \subset \partial\mathbb{D}$, $\{x_{n_k}\}_k \subset \partial U$ satisfying that the inverse branch $F_{n_k}$, well-defined in $D_W(x_{n_{k-1}}, r_{k-1})$ and

1. $G_{n_k}(\zeta_{n_{k-1}}) = \zeta_{n_k}$, $F_{n_k}(x_{k-1}) = \varphi^*(\zeta_{n_k}) = x_{n_k} \in W \cap \partial U$;
2. $F_{n_k}(D(x_{n_{k-1}}, r_{k-1})) \subset D_W(x_{n_k}, r_k) \subset D_W(x_{n_{k-1}}, r_{k-1})$, $r_k = r_{k-1}/2$;
3. at least one of the sets
$$\left\{D_W(x_{n_{k-1}}, r_{k-1}), F_1(D_W(x_{n_{k-1}}, r_{k-1})), \ldots, F_{n_k}(D_W(x_{n_{k-1}}, r_{k-1}))\right\}$$
intersects $\overline{(N_{\xi_{n_k}})}$.

See Figure 5.4.

Let
$$x_* := \bigcap_k F_{n_k}(D(x_0, r_0)).$$
Note that $x_* \in \partial U$, since $x_{n_k} \to x_*$ (and $\partial U$ is closed).

It is left to prove that $x_*$ is accessible from $U$. To do so, note that, by the construction in Lemma 5.3.2, at step $k$, it is satisfied that there exists a measurable set $K_k$ and $\eta_k \in K_k$ (chosen as in the Claim 5.3.3) such that
$$\Delta_{\rho_k}(\zeta_{n_{k-1}}) \cap R_{\rho_k}(\zeta_{n_k}) \neq \emptyset;$$
$$G_{n_k}(R_{\rho_k}(\zeta_{n_{k-1}})) \subset \Delta_{\rho_k}(\zeta_{n_k});$$
$$G_{n_k}(R_{\rho_k}(\zeta_{n_{k-1}})) \cap \Delta_{\rho_k}(\eta_k) \neq \emptyset;$$

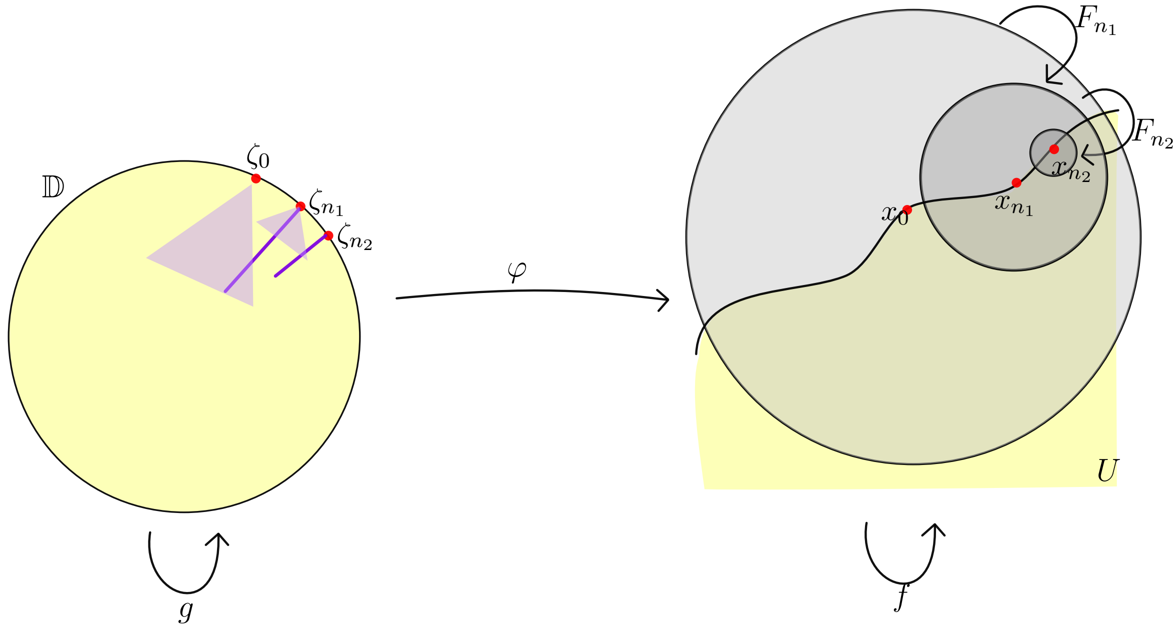



Figure 5.4: Set-up of the proof of the existence of an accessible point in $\partial U$ with a prescribed orbit.

$$\varphi(\Delta_{\rho_k}(\zeta_{n_{k-1}})), \varphi(\Delta_{\rho_k}(\eta_k)) \subset D_W(x_{n_{k-1}}, r_{k-1}/4);$$

for some appropriate radius $\rho_k$. Without loss of generality, we shall assume that the sequence $\{\rho_k\}_k$ is decreasing.

Thus, for all $n \geq 0$, take

$$z_{n_k} \in \Delta_{\rho_k}(\zeta_{n_k}) \cap R_{\rho_k}(\zeta_{n_{k-1}}).$$

We claim that there exists a curve $\gamma_k$ connecting $\varphi(z_{n_k})$ to $\varphi(z_{n_{k+1}})$ in $D_W(x_{n_{k-1}}, r_{k-1}) \cap U$. Indeed, since $z_{n_k} \in \Delta_{\rho_k}(\zeta_{n_{k-1}})$ and $z_{n_{k+1}} \in \Delta_{\rho_{k+1}}(\zeta_{n_k})$, if we prove that the set

$$\Delta_{\rho_k}(\zeta_{n_{k-1}}) \cup \Delta_{\rho_k}(\eta_k) \cup G_{n_k}(\Delta_{\rho_k}(\zeta_{n_{k-1}})) \cup \Delta_{\rho_{k+1}}(\zeta_{n_k})$$

is connected, since

$$\varphi(\Delta_{\rho_k}(\zeta_{n_{k-1}})), \varphi(\Delta_{\rho_k}(\eta_k)) \subset D_W(x_{n_{k-1}}, r_{k-1}),$$

$$\varphi(G_{n_k}(\Delta_{\rho_k}(\zeta_{n_{k-1}}))) = F_{n_k}(\varphi(\Delta_{\rho_k}(\zeta_{n_{k-1}}))) \subset F_{n_k}(D_W(x_{n_{k-1}}, r_{k-1})) \subset D_W(x_{n_{k-1}}, r_{k-1}),$$

$$\varphi(\Delta_{\rho_{k+1}}(\zeta_{n_k})) \subset D(x_{n_k}, r_k) \subset D(x_{n_{k-1}, r_{k-1}}),$$

then the existence of the curve $\gamma_k$ follows straightforward.

To see that $\Delta_{\rho_k}(\zeta_{n_{k-1}}) \cup \Delta_{\rho_k}(\eta_k) \cup G_n(\Delta_{\rho_k}(\zeta_{n_{k-1}})) \cup \Delta_{\rho_{k+1}}(\zeta_{n_k})$ is connected, note that, on the one hand, $\Delta_{\rho_k}(\zeta_{n_{k-1}}) \cap \Delta_{\rho_k}(\eta_k) \neq \emptyset$, by the Claim 5.3.3 in Lemma 5.3.2. Then, $G_{n_k}(R_{\rho_k}(\zeta_{n_{k-1}})) \cap \Delta_{\rho_k}(\eta_k) \neq \emptyset$, by the choice of $\eta_k$. On the other hand,

$$R_{\rho_k}(\zeta_{n_{k-1}}) \subset \Delta_{\rho_k}(\zeta_{n_{k-1}}),$$

and, according to Proposition II.3.17, $G_n(R_{\rho_k}(\zeta_{n_{k-1}}))$ is a curve landing at $\zeta_{n_k}$ inside $\Delta_{\rho_k}(\zeta_{n_k})$, and hence intersecting $\Delta_{\rho_{k+1}}(\zeta_{n_k})$. Hence, the intersection between $G_n(\Delta_{\rho_k}(\zeta_{n_{k-1}}))$ and $\Delta_{\rho_{k+1}}(\zeta_{n_k})$ is non-empty. This proves the claim.

Finally,

$$\Gamma = \bigcup_{k \geq 0} \gamma_k$$

is an access to $x_*$, as desired.

(5) *The case of infinite degree.* It is left to show that, in the case of infinite degree, the set of periodic points in $\partial U$ is unbounded and $x_*$ can be taken bungee.

This follows from the fact that the crosscut neighbourhood $N_\varsigma$ with $\overline{\varphi(N_\varsigma)} \cap P(f) = \emptyset$ has countably many preimages, and, for every compact subset $K$ of $\mathbb{C}$, only finitely many of them intersect $K$. Hence, in the choices of crosscuts in Lemma 5.3.2, we can construct a periodic point which is outside $D(0, n)$, leading to an unbounded sequence of periodic points. The same can be done to show that $x_*$ can be taken to be bungee. This ends the proof of the theorem.

□

# List of symbols

| | |
|---|---|
| $\mathbb{C}$ | complex plane. |
| $\widehat{\mathbb{C}}$ | Riemann sphere. |
| $C$ | (non-degenerate) crosscut. Def II.2.1. |
| $D(x,r)$ | disk of center $x$ and radius $r$. |
| $\mathbb{D}$ | unit disk. |
| $\partial\mathbb{D}$ | unit circle. |
| $\Delta_{\alpha,\rho}(\xi)$ | Stolz angle of openning $\alpha$ and length $\rho$ at $\xi \in \partial\mathbb{D}$. Def. II.2.1. |
| $\Delta_{\alpha,\rho}(\xi,p)$ | Stolz angle of openning $\alpha$ and length $\rho$ at $\xi \in \partial\mathbb{D}$, with respect to $p$. Def. II.2.2. |
| $E(g)$ | set of singularities of an inner function $g$. Def. II.1.4. |
| $\mathbb{H}_+$ | upper half-plane. |
| $\mathbb{H}_-$ | lower half-plane. |
| $\lambda$ | normalised Lebesgue measure on $\partial\mathbb{D}$. |
| $N$ | crosscut neighbourhood. Def II.2.1. |
| $R_\rho(\xi)$ | radial segment of length $\rho$ at $\xi \in \partial\mathbb{D}$. Def. II.2.1. |
| $R_\rho(\xi,p)$ | radial segment of length $\rho$ at $\xi \in \partial\mathbb{D}$. Def. II.2.2. |
| $\mathbb{R}$ | real axis. |
| $\mathbb{R}_+$ | positive real axis. |
| $\mathbb{R}_-$ | negative real axis. |
| $\omega_U$ | harmonic measure with respect to a simply connected domain $U$. Def. **??**. |

Moreover, given $f \in \mathbb{K}$, we use the following notation.

| | |
|---|---|
| $\Omega(f)$ | domain of holomorphicity of $f$. |
| $E(f)$ | set of singularities of $f$. |
| $\overline{A}$ | closure of a set $A \subset \Omega(f)$ taken in $\Omega(f)$. |
| $\widehat{A}$ | closure of a set $A \subset \Omega(f)$ taken in $\widehat{\mathbb{C}}$. |
| $\partial A$ | boundary of a set $A \subset \Omega(f)$ taken in $\Omega(f)$. |
| $\widehat{\partial} A$ | boundary of a set $A \subset \Omega(f)$ taken in $\widehat{\mathbb{C}}$. |

# Bibliography


[Aar78] J. Aaronson, *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. H. Poincaré Sect. B (N.S.) **14** (1978), no. 3, 233–253.

[Aar97] ______, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, vol. 50, American Mathematical Society, Providence, RI, 1997.

[Bak68] I. N. Baker, *Repulsive fixpoints of entire functions*, Math. Z. **104** (1968), 252–256.

[Bak84] ______, *Wandering domains in the iteration of entire functions*, Proc. London Math. Soc. (3) **49** (1984), no. 3, 563–576.

[Bar07] K. Barański, *Trees and hairs for some hyperbolic entire maps of finite order*, Mathematische Zeitschrift **257** (2007), no. 1, 33–59.

[Bar08] D. Bargmann, *Iteration of inner functions and boundaries of components of the Fatou set*, Transcendental dynamics and complex analysis, London Math. Soc. Lecture Note Ser., vol. 348, Cambridge Univ. Press, Cambridge, 2008, pp. 1–36.

[BD99] I. N. Baker and P. Domínguez, *Boundaries of unbounded Fatou components of entire functions*, Ann. Acad. Sci. Fenn. Math. **24** (1999), no. 2, 437–464.

[BDH+99] C. Bodelón, R. L. Devaney, M. Hayes, G. Roberts, L. R. Goldberg, and J. H. Hubbard, *Hairs for the complex exponential family*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **9** (1999), no. 8, 1517–1534.

[BDH01] I. N. Baker, P. Domínguez, and M. E. Herring, *Dynamics of functions meromorphic outside a small set*, Ergodic Theory Dynam. Systems **21** (2001), no. 3, 647–672.

[BDH04] ______, *Functions meromorphic outside a small set: completely invariant domains*, Complex Var. Theory Appl. **49** (2004), no. 2, 95–100.

[BE95] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana **11** (1995), no. 2, 355–373.

[Bea91] A. F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991.

[BEF+24] A. M. Benini, V. Evdoridou, N. Fagella, P. J. Rippon, and G. M. Stallard, *Boundary dynamics for holomorphic sequences, non-autonomous dynamical systems and wandering domains*, Adv. Math. **446** (2024), Paper No. 109673, 51.

[Ber93] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 2, 151–188.

[Ber95a] ______, *Invariant domains and singularities*, Math. Proc. Cambridge Philos. Soc. **117** (1995), no. 3, 525–532.

[Ber95b] ______, *On the Julia set of analytic self-maps of the punctured plane*, Analysis **15** (1995), no. 3, 251–256.

[BF01] K. Barański and N. Fagella, *Univalent Baker domains*, Nonlinearity **14** (2001), no. 3, 411–429.

[BFJK14] K. Barański, N. Fagella, X. Jarque, and B. Karpińska, *On the connectivity of the Julia sets of meromorphic functions*, Invent. Math. **198** (2014), no. 3, 591–636.

[BFJK17] ______, *Accesses to infinity from Fatou components*, Trans. Amer. Math. Soc. **369** (2017), no. 3, 1835–1867.

[BFJK18] ______, *Connectivity of Julia sets of Newton maps: a unified approach*, Rev. Mat. Iberoam. **34** (2018), no. 3, 1211–1228.

[BFJK19] ______, *Escaping points in the boundaries of Baker domains*, J. Anal. Math. **137** (2019), no. 2, 679–706.

[BFJK20] ______, *Fatou components and singularities of meromorphic functions*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 2, 633–654.

[BFJK25] ______, *Local connectivity of boundaries of tame Fatou components of meromorphic functions*, Math. Ann. **391** (2025), no. 2, 1779–1843.

[BFRG15] W. Bergweiler, N. Fagella, and L. Rempe-Gillen, *Hyperbolic entire functions with bounded Fatou components*, Comment. Math. Helv. **90** (2015), no. 4, 799–829.

[BK07] K. Barański and B. Karpińska, *Coding trees and boundaries of attracting basins for some entire maps*, Nonlinearity **20** (2007), no. 2, 391–415.

[BKL91] I. N. Baker, J. Kotus, and Y. Lü, *Iterates of meromorphic functions. I*, Ergodic Theory Dynam. Systems **11** (1991), no. 2, 241–248.

[BM07] A. F. Beardon and D. Minda, *The hyperbolic metric and geometric function theory*, Quasiconformal mappings and their applications, Narosa, New Delhi, 2007, pp. 9–56.

[Bog07] V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007.

[Bol96] A. Bolsch, *Repulsive periodic points of meromorphic functions*, Complex Variables Theory Appl. **31** (1996), no. 1, 75–79.

[Bol97] ______, *Iteration of meromorphic functions with countably many essential singularities*, Ph.D. thesis, Technischen Universität Berlin, 1997.

[Bol99] ———, *Periodic Fatou components of meromorphic functions*, Bull. London Math. Soc. **31** (1999), no. 5, 543–555.

[Bon97] P. Bonfert, *On iteration in planar domains*, Michigan Math. J. **44** (1997), no. 1, 47–68.

[BP23] L. Barreira and Y. Pesin, *Introduction to smooth ergodic theory*, second ed., Graduate Studies in Mathematics, vol. 231, American Mathematical Society, Providence, RI, 2023.

[BR20] A. M. Benini and L. Rempe, *A landing theorem for entire functions with bounded post-singular sets*, Geom. Funct. Anal. **30** (2020), no. 6, 1465–1530.

[Bro65] H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965), 103–144.

[BW91] I. N. Baker and J. Weinreich, *Boundaries which arise in the dynamics of entire functions*, Rev. Roumaine Math. Pures Appl. **36** (1991), no. 7-8, 413–420.

[BZ12] W. Bergweiler and J. Zheng, *Some examples of Baker domains*, Nonlinearity **25** (2012), no. 4, 1033–1044.

[CG93] L. Carleson and T. W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

[Con95] J. B. Conway, *Functions of one complex variable. II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995.

[Cow81] C. C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. **265** (1981), no. 1, 69–95.

[CP02] J. J. Carmona and C. Pommerenke, *On prime ends and plane continua*, J. London Math. Soc. (2) **66** (2002), no. 3, 641–650.

[Cra91] M. Craizer, *Entropy of inner functions*, Israel J. Math. **74** (1991), no. 2-3, 129–168.

[Cra92] ———, *The Bernoulli property of inner functions*, Ergodic Theory Dynam. Systems **12** (1992), no. 2, 209–215.

[Cur91] S.B. Curry, *One-dimensional nonseparating continua with disjoint $\varepsilon$-dense subcontinua*, Topology and its Applications **32** (1991), no. 2, 145–151.

[dB85] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), no. 1-2, 137–152.

[Dev89] R. L. Devaney, *An introduction to chaotic dynamical systems*, second ed., Addison-Wesley Studies in Nonlinearity, 1989.

[Dev93] ———, *Knaster-like continua and complex dynamics*, Ergodic Theory Dynam. Systems **13** (1993), no. 4, 627–634.

[Dev94] ———, *Complex dynamics and entire functions*, Complex dynamical systems (Cincinnati, OH, 1994), Proc. Sympos. Appl. Math., vol. 49, Amer. Math. Soc., Providence, RI, 1994, pp. 181–206.

[DG87] R. L. Devaney and L. R. Goldberg, *Uniformization of attracting basins for exponential maps*, Duke Math. J. **55** (1987), no. 2, 253–266.

[DJ02] R. L. Devaney and X. Jarque, *Indecomposable continua in exponential dynamics*, Conform. Geom. Dyn. **6** (2002), 1–12.

[DJR05] R. L. Devaney, X. Jarque, and M. M. Rocha, *Indecomposable continua and Misiurewicz points in exponential dynamics*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **15** (2005), no. 10, 3281–3293.

[DM91] C. Doering and R. Mañé, *The Dynamics of Inner Functions*, Ensaios Matemáticos (SBM) **3** (1991), 1–79.

[DMdOS22] P. Domínguez, M. A. Montes de Oca, and G. J. F. Sienra, *Extended escaping set for meromorphic functions outside a countable set of transcendental singularities*, Ann. Polon. Math. **129** (2022), no. 1, 25–41.

[Dob12] N. Dobbs, *Measures with positive Lyapunov exponent and conformal measures in rational dynamics*, Trans. Amer. Math. Soc. **364** (2012), no. 6.

[Dom10] P. Domínguez, *Residual Julia sets for meromorphic functions with countably many essential singularities*, J. Difference Equ. Appl. **16** (2010), no. 5-6, 519–522.

[DS08] N. Dobbs and B. Skorulski, *Non-existence of absolutely continuous invariant probabilities for exponential maps*, Fund. Math. **198** (2008), no. 3, 283–287.

[EFJS19] V. Evdoridou, N. Fagella, X. Jarque, and D. J. Sixsmith, *Singularities of inner functions associated with hyperbolic maps*, J. Math. Anal. Appl. **477** (2019), no. 1, 536–550.

[EL92] A. È. Erëmenko and M. Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020.

[ERS20] V. Evdoridou, L. Rempe, and D. J. Sixsmith, *Fatou's associates*, Arnold Math. J. **6** (2020), no. 3-4, 459–493.

[Evd16] V. Evdoridou, *Fatou's web*, Proc. Amer. Math. Soc. **144** (2016), no. 12, 5227–5240.

[Fal03] K. Falconer, *Fractal geometry*, second ed., John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[Fat20] P. Fatou, *Sur les équations fonctionnelles*, Bulletin de la Société Mathématique de France **48** (1920), 208–314.

[Fer23] G. R. Ferreira, *A note on forward iteration of inner functions*, Bull. Lond. Math. Soc. **55** (2023), no. 3, 1143–1153.

[FH06] N. Fagella and C. Henriksen, *Deformation of entire functions with Baker domains*, Discrete Contin. Dyn. Syst. **15** (2006), no. 2, 379–394.

[FJ25] G. R. Ferreira and A. Jové, *Boundaries of multiply connected Fatou components*, 2025, In preparation.

[FJT08] N. Fagella, X. Jarque, and J. Taixés, *On connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points. I*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 3, 599–622.

[FJT11] ______, *On connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points II*, Fund. Math. **215** (2011), no. 2, 177–202.

[FLM83] A. Freire, A. Lopes, and R. Mañé, *An invariant measure for rational maps*, Bol. Soc. Brasil. Mat. **14** (1983), no. 1, 45–62.

[Gar78] V. Garber, *On the iteration of rational functions*, Math. Proc. Cambridge Philos. Soc. **84** (1978), no. 3, 497–505.

[Gar07] J. B. Garnett, *Bounded analytic functions*, Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007.

[GM05] J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2005.

[Haw21] J. Hawkins, *Ergodic dynamics-from basic theory to applications*, Graduate Texts in Mathematics, vol. 289, Springer, 2021.

[Her85] M. Herman, *Are there critical points on the boundaries of singular domains?*, Comm. Math. Phys. **99** (1985), no. 4, 593–612.

[Ima14] M. Imada, *Periodic points on the boundaries of rotation domains of some rational functions*, Osaka J. Math. **51** (2014), no. 1, 215–224.

[IU23] O. Ivrii and M. Urbański, *Inner Functions, Composition Operators, Symbolic Dynamics and Thermodynamic Formalism*, 2023, available at `arXiv:2308.16063`.

[IU24] ______, *Inner functions and laminations*, 2024, available at `arXiv:2405.02878`.

[Ive14] F. Iversen, *Recherches sur les fonctions inverses des fonctions méromorphes*, Ph.D. thesis, Helsingfors, 1914.

[Kap54] W. Kaplan, *Extensions of the Gross star theorem*, Michigan Math. J. **2** (1954), 105–108.

[KH95] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.

[KU23] J. Kotus and M. Urbański, *Meromorphic dynamics. Vol. II. Elliptic functions with an introduction to the dynamics of meromorphic functions*, New Mathematical Monographs, vol. 47, Cambridge University Press, Cambridge, 2023.

[Kö99] H. König, *Conformal conjugacies in Baker domains*, J. London Math. Soc. (2) **59** (1999), no. 1, 153–170.

[Led81] F. Ledrappier, *Some properties of absolutely continuous invariant measures on an interval*, Ergodic Theory Dynam. Systems **1** (1981), no. 1, 77–93.

[Lyu83] M. Ju. Lyubich, *Entropy properties of rational endomorphisms of the riemann sphere*, Ergodic Theory and Dynamical Systems **3** (1983), no. 3, 351–385.

[MB12] H. Mihaljević-Brandt, *Semiconjugacies, pinched Cantor bouquets and hyperbolic orbifolds*, Trans. Amer. Math. Soc. **364** (2012), no. 8, 4053–4083.

[Mil06] J. Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[MPRW25] D. Martí-Pete, L. Rempe, and J. Waterman, *Bounded Fatou and Julia components of meromorphic functions*, Math. Ann. **391** (2025), no. 1, 95–111.

[Par67] K. R. Parthasarathy, *Probability measures on metric spaces*, Probability and Mathematical Statistics, vol. No. 3, Academic Press, Inc., New York-London, 1967.

[Pes76] Ja. B. Pesin, *Families of invariant manifolds that correspond to nonzero characteristic exponents*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 6, 1332–1379, 1440.

[Pes77] ———, *Characteristic Ljapunov exponents, and smooth ergodic theory*, Uspehi Mat. Nauk **32** (1977), no. 4(196), 55–112, 287.

[PM97] R. Pérez-Marco, *Fixed points and circle maps*, Acta Math. **179** (1997), no. 2, 243–294.

[Pol93] M. Pollicott, *Lectures on ergodic theory and Pesin theory on compact manifolds*, London Mathematical Society Lecture Note Series, vol. 180, Cambridge University Press, Cambridge, 1993.

[Pom92] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der mathematischen Wissenschaften, vol. 299, Springer-Verlag, Berlin, 1992.

[Prz85] F. Przytycki, *Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map*, Invent. Math. **80** (1985), no. 1, 161–179.

[Prz86] ———, *Riemann map and holomorphic dynamics*, Invent. Math. **85** (1986), no. 3, 439–455.

[Prz93] ———, *Lyapunov characteristic exponents are nonnegative*, Proc. Amer. Math. Soc. **119** (1993), no. 1, 309–317.

[PS22] L. Pardo-Simón, *Criniferous entire maps with absorbing Cantor bouquets*, Discrete Contin. Dyn. Syst. **42** (2022), no. 2, 989–1010.

[PU10] F. Przytycki and M. Urbański, *Conformal fractals: ergodic theory methods*, vol. 371, Cambridge University Press, 2010.

[PUZ91] F. Przytycki, M. Urbański, and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps. II*, Studia Math. **97** (1991), no. 3, 189–225.

[PZ94] F. Przytycki and A. Zdunik, *Density of periodic sources in the boundary of a basin of attraction for iteration of holomorphic maps: geometric coding trees technique*, Fund. Math. **145** (1994), no. 1, 65–77.

[Rem03] L. Rempe, *Dynamics of exponential maps*, Ph.D. thesis, Christian-Albrechts-Universität Kiel, 2003.

[Rem07] ———, *On nonlanding dynamic rays of exponential maps*, Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 2, 353–369.

[RGS17] L. Rempe-Gillen and D. Sixsmith, *Hyperbolic entire functions and the Eremenko-Lyubich class: class $\mathcal{B}$ or not class $\mathcal{B}$?*, Math. Z. **286** (2017), no. 3-4, 783–800.

[Rip06] P. J. Rippon, *Baker domains of meromorphic functions*, Ergodic Theory Dynam. Systems **26** (2006), no. 4, 1225–1233.

[Roh64] V. A. Rohlin, *Exact endomorphism of a Lebesgue space*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **14** (1964), 443–474. MR 228654

[RRRS11] G. Rottenfusser, J. Rückert, L. Rempe, and D. Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) **173** (2011), no. 1, 77–125.

[RS18] P. J. Rippon and G. M. Stallard, *Boundaries of univalent Baker domains*, J. Anal. Math. **134** (2018), no. 2, 801–810.

[Rud87] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

[RY08] P. Roesch and Y. Yin, *The boundary of bounded polynomial Fatou components*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 15-16, 877–880.

[RY22] ———, *Bounded critical Fatou components are Jordan domains for polynomials*, Sci. China Math. **65** (2022), no. 2, 331–358.

[Sch97] W. Schmidt, *Accessible fixed points on the boundary of stable domains*, Results Math. **32** (1997), no. 1-2, 115–120.

[Sta94] G. M. Stallard, *The Hausdorff dimension of Julia sets of meromorphic functions*, J. London Math. Soc. (2) **49** (1994), no. 2, 281–295.

[SZ03] D. Schleicher and J. Zimmer, *Escaping points of exponential maps*, J. London Math. Soc. (2) **67** (2003), no. 2, 380–400.

[Tha01] M. Thaler, *Infinite ergodic theory*, 2001, Marseille-Luminy CIRM, The dynamic odyssey.

[URM22] M. Urbański, M. Roy, and S. Munday, *Non-invertible dynamical systems. Vol. 1. Ergodic theory—finite and infinite, thermodynamic formalism, symbolic dynamics and distance expanding maps*, De Gruyter Expositions in Mathematics, vol. 69.1, De Gruyter, Berlin, 2022.

[URM23] M. Urbanski, M. Roy, and S. Munday, *Non-invertible dynamical systems. Vol. 3. Analytic Endomorphisms of the Riemann Sphere*, De Gruyter, Berlin, Boston, 2023.

[UZ03] M. Urbański and A. Zdunik, *The finer geometry and dynamics of the hyperbolic exponential family*, Michigan Math. J. **51** (2003), no. 2, 227–250.

[Wol26] J. Wolff, *Sur une généralisation d'un théorème de Schwarz*, Comptes Rendus Acad. Sci. **183** (1926), 918–920.