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Large and iterated finite group actions on aspherical manifolds

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Abstract

The theory of finite transformation groups investigates the finite symmetries of topological objects, such as manifolds or CW-complexes. In this thesis, we focus on actions on manifolds and we adopt the following approach: rather than directly studying the properties of a finite group G acting on a manifold M, we focus on the properties of the action restricted to suitable subgroups $H \subseteq G$ of bounded index. This index is controlled by a constant C that depends only on M. Several problems align with this philosophy, including determining whether the homeomorphism group of a manifold is Jordan, computing the discrete degree of symmetry of a manifold, establishing whether a manifold is almost-asymmetric, and examining the number and size of stabilizer subgroups for finite group actions on manifolds.

In the first part of the thesis, we provide solutions to these problems for two broad classes of manifolds, specifically:

(1) Closed connected aspherical manifolds whose fundamental group has Minkowski outer automorphism group (a group G is Minkowski if there exists a constant C such that every finite subgroup $H \leq G$ satisfies $|H| \leq C$).

(2) Closed connected orientable manifolds that admit a non-zero degree map to a nilmanifold.

We show that the outer automorphism group of a lattice in a connected Lie group is Minkowski, enabling our results to be applied to closed aspherical locally homogeneous spaces. Additionally, we provide the first known examples of manifolds M and M' with isomorphic cohomology rings $H^*(M,\mathbb{Z}) \cong H^*(M',\mathbb{Z})$ where $\mathrm{Homeo}(M)$ is Jordan but $\mathrm{Homeo}(M')$ is not. We establish two rigidity results for the discrete degree of symmetry: if M is a closed, connected aspherical manifold whose fundamental group has Minkowski outer automorphism group, or if M admits a non-zero degree map to a nilmanifold and its fundamental group is virtually solvable, then M is homeomorphic to a torus if its discrete degree of symmetry equals the dimension of M. The latter means that M supports effective actions of a sequence of groups $(\mathbb{Z}/a_i)^n$, where $\{a_i\}$ is a strictly increasing sequence of natural numbers and $n = \dim(M)$.

In the second part, we refine the concept of group actions to explore the topological and cohomological rigidity of manifolds in greater depth. We introduce the notion of iterated actions of a collection of finite groups $\{G_1, \ldots, G_n\}$ on a manifold M, which is a collection of effective group actions of G_i on M_{i-1} for $1 \leq i \leq n$, where $M_0 = M$ and $M_i = M_{i-1}/G_i$ for $i \geq 1$. This framework allows us to analyze the structure of closed aspherical manifolds and those admitting non-zero degree maps to nilmanifolds in more detail. We define new invariants, such as the iterated length of a manifold, which is closely related to its self-coverings, and introduce a refined version of the discrete degree of symmetry, termed the iterated discrete degree of symmetry. We demonstrate that if M is a closed oriented manifold admitting a non-zero degree map to a 2-step nilmanifold N/Γ, and both manifolds have the same iterated discrete degree of symmetry, then $H^*(M, \mathbb{Q}) \cong H^*(N/\Gamma, \mathbb{Q})$. Furthermore, if $\pi_1(M)$ is virtually solvable, then $M \cong N/\Gamma$. We also prove that if M is a closed aspherical locally homogeneous space with an iterated discrete degree of symmetry equal to its dimension, then M is homeomorphic to a 2-step nilmanifold.

Key words: finite group actions, geometric topology, aspherical manifolds, lattices of Lie groups

Resum en català

En aquesta tesi avancem en el camp de la teoria de grups finits de transformació obtenint resultats que segueixen el següent enfocament: en comptes d'estudiar directament les propietats d'un grup finit G que actua sobre una varietat M, ens centrem en les propietats de l'acció restringida a certs subgrups $H \subseteq G$ d'índex acotat per una constant C que depèn únicament de M. Hi ha diversos problemes que s'alineen amb aquesta filosofia, com ara determinar si el grup d'homeomorfismes d'una varietat és Jordan, calcular el grau discret de simetria d'una varietat, establir si una varietat és quasi asimètrica i examinar el nombre i la mida dels subgrups d'isotropia d'accions de grups finits sobre varietats. A la primera part de la tesi, oferim solucions a aquests problemes per a varietats tancades connexes asfèriques localment homogènies i per a varietats tancades connexes i orientables que admeten una aplicació de grau diferent de zero cap a una nilvarietat. Establim dos resultats de rigidesa per al grau discret de simetria: si M és una varietat tancada connexa asfèrica localment homogènia, o si M admet una aplicació de grau diferent de zero cap a una nilvarietat i el seu grup fonamental és virtualment resoluble, llavors M és homeomorfa a un tor si el seu grau discret de simetria és igual a la seva dimensió. A la segona part, refinem el concepte d'accions de grup per explorar la rigidesa topològica i cohomològica de les varietats amb més detall. Introduïm la noció d'accions iterades d'una col·lecció de grups finits, definim nous invariants, com la longitud iterada d'una varietat i introduïm una versió refinada del grau discret de simetria, anomenada grau discret iterat de simetria. Demostrem que si M és una varietat tancada orientada que admet una aplicació de grau diferent de zero cap a una nilvarietat N/Γ, on N té classe de nilpotència 2 i ambdues varietats tenen el mateix grau discret iterat de simetria, llavors $H^*(M, \mathbb{Q}) \cong H^*(N/\Gamma, \mathbb{Q})$. També provem que si M és una varietat tancada asfèrica i localment homogènia amb un grau discret iterat de simetria igual a la seva dimensió, llavors M és homeomorfa a una nilvarietat N/Γ, on N té classe de nilpotència 2.

Paraules clau: accions de grups finits, topologia geomètrica, varietats asfèrqiues, reticles de grups de Lie

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Contents

Introduction

The theory of finite transformation groups focuses on studying the symmetries of topological objects, such as manifolds, by analysing the actions of finite groups on them. A fundamental question in this field is: Given a closed topological manifold M, can we determine which finite groups act effectively on M? A complete answer to this question is completely out of reach in the vast majority of cases with the current tools of finite transformation groups theory. One way to modify this question to obtain a more tractable problem is to consider actions of a finite group G on M and to study properties not of the action of G on M, but on the restriction to some subgroup of G of index bounded by a constant C only depending on M. Let us recall some problems that follow this philosophy from [MiR24b], a recent survey on the topic.

We need the following definition to state the first problem:

Definition 1. *A group $\mathcal{G}$ is said to be Jordan if there exists a constant C such that every finite subgroup $G \leq \mathcal{G}$ has an abelian subgroup $A \leq G$ such that $[G : A] \leq C$.*

The name of this property is motivated by a classical theorem of Camille Jordan which states that $\mathrm{GL}(n, \mathbb{R})$ is Jordan. Around 30 years ago Étienne Ghys asked in a series of talks whether the diffeomorphism group of any closed connected smooth manifold M is Jordan. This question has been answered affirmatively for a lot of different manifolds like closed flat manifolds, integral homology spheres, closed connected manifolds with non-zero Euler characteristics or closed connected manifolds up to dimension 3 (see [MiR10, Zim14, Ye19, MiR19]). However, it has been shown that there are closed manifolds whose diffeomorphism group is not Jordan, like $S^2 \times T^2$ (see [CPS14, MiR17, Sza19, Sza23, Zar14] for various examples of non-Jordan groups). More recently, the same question has been studied for the homeomorphism group of closed topological manifolds, extending the results obtained in the smooth case (see [MiR24b] and references therein).

Note that this property does not tell us anything for finite groups G acting on M with $|G| \leq C$, but it becomes relevant for large enough finite groups acting effectively on M. There is no universal method to prove or disprove whether the homeomorphism group of a closed, connected manifold is Jordan, and many manifolds remain for which it is unknown whether their homeomorphism group has the Jordan property.

We remark that the Jordan property also has a rich history in the field of algebraic geometry, which started when J.-P. Serre proved that the Cremona group $Cr_2 = \mathrm{Bir}(\mathbb{P}^2)$ is Jordan in [Ser09] and asked if higher rank Cremona groups Cr_n also have this property (the question has been affirmatively answered in [PS16, Bir21]). For the most recent developments on the Jordan property in algebraic geometry we refer to the survey [BZ24], as well as [MiR24a, §2], the introductions of [CPS22] and [Gol23].

Another interesting problem, specially when $\mathrm{Homeo}(M)$ is Jordan, is to study the rank of finite abelian groups (defined as the smallest number of elements needed to generate the group) which can act effectively on M. Here again we follow the philosophy of looking for a statement that eventually requires to replace the group by a subgroup of bounded index.

Question 2. *Fix a natural number k. Does there exist a constant C such that every finite abelian group A acting effectively on M has an abelian subgroup B such that* $[A : B] \leq C$ *and* $\mathrm{rank}(B) \leq k$*?*

This question can be reformulated in terms of the following invariant introduced in [MiR24a].

Definition 3. *Given a manifold M let*

$$\mu(M) = \{r \in \mathbb{N} : \ M \text{ admits an effective action of } (\mathbb{Z}/a)^r \text{ for arbitrarily large } a\}.$$

More explicitly, $r \in \mu(M)$ *if there exists an increasing sequence of natural number* $\{a_i\}$ *and effective group actions of* $(\mathbb{Z}/a_i)^r$ *on M for each i.*

The discrete degree of symmetry of a manifold M is

$$\text{disc-sym}(M) = \max(\{0\} \cup \mu(M)).$$

Question 2 is equivalent to asking whether $\text{disc-sym}(M) \leq k$. By a theorem of L.N.Mann and J.C.Su (see [MS63] and theorem 1.1.32) we know that if M is a closed connected manifold then $\text{disc-sym}(M)$ is a well-defined natural number, but finding the exact value of $\text{disc-sym}(M)$ is probably difficult in most cases. Note that a manifold M can admit group actions of abelian group of higher rank than $\text{disc-sym}(M)$. For example, for any natural numbers a, b there exists a closed surface $\Sigma_{g(a,b)}$ of genus $g(a,b) \geq 2$ such that $(\mathbb{Z}/a)^b$ acts freely on $\Sigma_{g(a,b)}$. On the other hand $\text{disc-sym}(\Sigma_{g(a,b)}) = 0$. This fact is a consequence of the $84(g-1)$ theorem of Hurwitz, see [FM11, Chapter 7]).

The definition of $\text{disc-sym}(M)$ is analogous to the definition of the toral degree of symmetry

$$\text{tor-sym}(M) = \max(\{0\} \cup \{r \in \mathbb{N} : T^r \text{ acts effectively on } M\})$$

studied in [Hsi12, Chapter VII. §2]. A classical result states that if M is a closed manifold of dimension n then $\text{tor-sym}(M) \leq n$ and that $\text{tor-sym}(M) = n$ if and only if $M \cong T^n$

(see theorem 1.1.48). It is not known, however, whether disc-sym$(M) \leq \dim(M)$ for every M and whether disc-sym$(M) = \dim(M)$ implies $M \cong T^n$ (see [MiR24b, Question 3.4, Question 3.5]). For more results on the toral degree of symmetry we refer to [LR10, §11.7, §11.8] and the survey [Gro02].

The third problem relates to counting the number of stabilizers of a group action on a manifold.

Definition 4. *Let G be a finite group acting effectively on a manifold M. The set of stabilizer subgroups of the action of G on M is denoted by*

$$\mathrm{Stab}(G, M) = \{G_x : x \in M\}.$$

It is not possible to bound $|\mathrm{Stab}(G, M)|$ only depending on M. For example, for all n there is an effective action of the dihedral group D_n on S^1 such that $|\mathrm{Stab}(D_n, S^1)| \geq n/2$. On the other hand, it is proven in [CMiRPS21, Theorem 1.3] that for any closed connected manifold M there exists a constant C only depending on M such that any finite p-group G acting effectively on M has a subgroup H such that $[G : H] \leq C$ and $|\mathrm{Stab}(H, M)| \leq C$. This result was crucial in proving a generalized version of the Jordan property of the homeomorphism group of closed manifolds in [CPS22], which states that there exists a constant C such that every finite group F acting on M has a nilpotent subgroup N such that $[F : N] \leq C$. It is unknown whether the hypothesis of F being a p-group can be removed in [CMiRPS21, Theorem 1.3] (see [MiR24b, Question 12.2]).

To address the final question, we introduce the following definition:

Definition 5. *A group $\mathcal{G}$ is said to be Minkowski if there exists a constant C such that every finite subgroup $G \leq \mathcal{G}$ satisfies $|G| \leq C$.*

Remark 6. *This name is motivated by a classical result of Hermann Minkowski which states that* $\mathrm{GL}(n, \mathbb{Z})$ *is Minkowski. The Minkowski property was studied in [Pop18, Gol23, BZ24] under the name of bounded finite subgroups property.*

If M is a closed manifold and Homeo(M) is Minkowski then M is said to be almost-asymmetric. In the particular case where M does not admit any effective finite group action we say that M is asymmetric. This case has been extensively studied (see [Pup07] and references therein).

Compact group actions on closed aspherical manifolds have been widely studied since the seventies, beginning with an unpublished work by A.Borel where the first example of asymmetric manifold is constructed (see [Bor83]) and the seminal work of P.E.Conner and F.Raymond (see [Con70]). This area of research is still active at present (see, for example, [JL10, CLW18, Ye19, BT21, BK23b, BK23a]). However, the above questions have not been studied in full generality in the literature. These questions are particularly relevant for closed connected aspherical manifolds, since tori are the only compact connected Lie groups that act effectively on them [LR10, Theorem 3.1.16]. Therefore, tor-sym(M) and disc-sym(M) become important invariants.

We recall some notation before stating the main results of this paper. Given a group G we denote the automorphism group of G by $\operatorname{Aut}(G)$. Given $g \in G$, we denote by $c_g : G \longrightarrow G$ the conjugation by g, $c_g(h) = ghg^{-1}$. The normal subgroup $\{c_g : g \in G\} \trianglelefteq \operatorname{Aut}(G)$ is denoted by $\operatorname{Inn}(G)$. Recall that $\operatorname{Inn}(G) \cong G/\mathcal{Z}G$, where $\mathcal{Z}G$ denotes the center of G. The outer automorphisms group of G is $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$.

Our first theorem is the following:

Theorem 7. *(Theorem 2.0.1) Let M be a closed connected n-dimensional aspherical manifold such that $\mathcal{Z}\pi_1(M)$ is finitely generated and $\operatorname{Out}(\pi_1(M))$ is Minkowski. Then:*

1. $\operatorname{Homeo}(M)$ *is Jordan.*
2. disc-sym$(M) \leq \operatorname{rank} \mathcal{Z}\pi_1(M) \leq n$, *and* disc-sym$(M) = n$ *if and only if M is homeomorphic to T^n.*
3. *If $\chi(M) \neq 0$ then M is almost-asymmetric.*
4. *If* $\operatorname{Aut}(\pi_1(M))$ *is Minkowski, then there exists a constant C such that every finite group F acting effectively on M has a subgroup H such that $[F : H] \leq C$ and $|\operatorname{Stab}(H, M)| \leq C$.*

This result is mainly a combination of several known results. Note that items 2 and 4 partially answer affirmatively the questions [MiR24b, Question 3.4, Question 3.5] and [MiR24b, Question 12.2] respectively.

As an application of theorem 7, we prove:

Proposition 8. *(Proposition 2.0.5) There exists a closed connected aspherical manifold M such that* $\operatorname{Homeo}(M)$ *is Jordan and $H^*(M) \cong H^*(T^2 \times S^3)$.*

Note that $\operatorname{Homeo}(T^2 \times S^3)$ is not Jordan by [MiR17]. Proposition 8 provides the first example of two manifolds with the same cohomology with integer coefficients such that one has Jordan homeomorphism group and the other does not.

We also obtain a rigidity result for closed connected aspherical n-dimensional manifolds when disc-sym$(M) = n - 1$. Let K denote the Klein bottle and SK denote the only non-trivial principal S^1-bundle over K, then:

Proposition 9. *(Proposition 2.0.6) Let M be a closed connected n-dimensional aspherical manifold such that $\mathcal{Z}\pi_1(M)$ is finitely generated,* $\operatorname{Out}(\pi_1(M))$ *is Minkowski and* $\operatorname{Inn}\pi_1(M)$ *has an element of infinite order. If* disc-sym$(M) = n - 1$ *then $M \cong T^{n-2} \times K$ or $M \cong T^{n-3} \times SK$*

It is important to know when the hypothesis on the fundamental group of theorem 7 are satisfied. At the moment there are no known closed aspherical manifolds where $\mathcal{Z}\pi_1(M)$ is not finitely generated (see [LR10, Remark 3.1.19]). Regarding the second hypothesis, we prove:

Theorem 10. *(Theorem 2.0.2) Let* Γ *be a lattice in a connected Lie group* G*. Then* $\mathrm{Out}(\Gamma)$ *and* $\mathrm{Aut}(\Gamma)$ *are Minkowski.*

The proof of this results uses several theorems of the theory of lattices of Lie groups, like the Mostow-Prasad-Margulis rigidity theorem, the Borel density theorem, Margulis superrigidity theorem, and Margulis normal subgroup theorem, as well as the results in [Mal02], which are used to compute the outer automorphism group of a group extension.

If G is a connected Lie group, H is a maximal compact subgroup and Γ is a torsion-free cocompact lattice of G then the closed aspherical locally homogeneous space $H \setminus G/\Gamma$ satisfies the hypothesis of theorem 7. In particular, flat manifolds, nilmanifolds, almost-flat manifolds, solvmanifolds, infra-solvmanifolds and closed connected aspherical locally symmetric spaces satisfy the hypothesis of theorem 7. Note that if we remove the asphericity hypothesis then closed locally homogeneous spaces do not necessarily have Jordan homeomorphism group. Indeed, $T^2 \times S^2$ is homogeneous (and hence locally homogeneous) but $\mathrm{Homeo}(T^2 \times S^2)$ is not Jordan.

Theorem 10 generalizes [Gol23, Theorem 1.7], where it is proven that the outer automorphism group of a cocompact lattice on connected complex Lie groups is Minkowski. In the real case we need to be careful with the compact factors and factors isomorphic to $\mathrm{PSL}(2, \mathbb{R})$ of the semisimple part of G.

Note that theorem 10 is also valid for non-cocompact lattices, although it cannot be used to deduce properties of large finite groups actions on non-compact aspherical locally homogeneous spaces, since the compactness hypothesis in theorem 7 is essential.

To complement theorem 10, we also prove that the bound on the discrete degree of symmetry is reached for closed connected aspherical locally homogeneous spaces.

Theorem 11. *(Theorem 2.0.4) Let* $H \setminus G/\Gamma$ *be a closed connected aspherical locally homogeneous space, where* G *is a connected Lie group. Then* $\text{disc-sym}(H \setminus G/\Gamma) = \operatorname{rank} \mathcal{Z}\Gamma$.

This result is a combination of the results in [LR10, Section 11.7] on the toral degree of symmetry and the validity of the Borel conjecture for lattices of connected Lie groups [KLR16].

Finally, using similar arguments used to prove theorem 10 we also prove:

Proposition 12. *(Proposition 2.0.3) Let* $M = M_1 \times \cdots \times M_m$*, where* M_i *are a closed aspherical manifolds such that* $\pi_1(M_i)$ *is hyperbolic and* $\dim(M_i) \geq 3$*. Then* $\mathrm{Out}(\pi_1(M))$ *is finite and* $\mathrm{Aut}(\pi_1(M))$ *is Minkowski.*

Proposition 12, together with theorem 7 and the fact that $\pi_1(M)$ is centreless, implies that M is almost asymmetric.

There have been many efforts to generalize the results of group actions on closed connected aspherical manifolds to a wider class of manifolds, for example closed connected manifolds which admit a non-zero degree map to a closed aspherical manifold (see [Sch81, DS82, GLO85, KK83, WW83, MiR10, MiR24a]). In this thesis we also prove a generalization of theorem 7 for manifolds which admit a non-zero degree map to a nilmanifold. Recall that a nilmanifold is a manifold which admits a transitive group action of a simply connected nilpotent Lie group N. If a nilmanifold is compact then it is homeomorphic to the coset space N/Γ, where Γ is a lattice of N. If N is c-step nilpotent then we say that N/Γ is a c-step nilmanifold.

Theorem 13. *(Theorem 3.0.2) Let M be a closed connected orientable manifold of dimension n and let $f : M \longrightarrow N/\Gamma$ be a non-zero degree map to a nilmanifold. Then:*

1. $\mathrm{Homeo}(M)$ *is Jordan.*
2. $\text{disc-sym}(M) \leq \operatorname{rank} \mathcal{Z}\Gamma$ *and if* $\text{disc-sym}(M) = n$ *then* $H^*(M, \mathbb{Z}) \cong H^*(T^n, \mathbb{Z})$.
3. *If $\chi(M) \neq 0$ then M is almost-asymmetric.*
4. *There exists a constant C such that every finite group G acting effectively on M has a subgroup H such that $[G : H] \leq C$ and $|\operatorname{Stab}(H, M)| \leq C$.*

Theorem 13 is a generalization of [MiR24a, Theorem 1.3, Theorem 1.14, theorem 1.15]. To prove theorem 13, we introduce a new concept called exporting map, inspired by [MiR24a, Theorem 4.1]. A map between manifolds $f : M \longrightarrow M'$ is an exporting map if there exists a constant C such that every finite group G acting on M has a subgroup $H \leq G$ such that H acts on M', there exists a H-equivariant map $f_H : M \longrightarrow M'$ homotopic to f and $[G : H] \leq C$ (see definition 3.2.3). Using local systems, we will prove that a non-zero degree map to a nilmanifold is an exporting map. An accurate study of the relation between exporting maps and the Jordan property, the discrete degree of symmetry, asymmetry and the stabilizer set will complete the proof of theorem 13.

It is a natural question to ask whether $\text{disc-sym}(M) = \operatorname{rank} \mathcal{Z}\Gamma$ implies that $H^*(M, \mathbb{Z}) \cong H^*(N/\Gamma, \mathbb{Z})$. However, this is not true in general (see proposition 3.3.13). Thus, we want to find a new invariant refining $\text{disc-sym}(M)$ to study cohomological rigidity for manifolds admitting a non-zero degree map to a nilmanifold. In order to do so, we recall that nilmanifolds are precisely the iterated principal S^1-bundles, [Bel20]. This fact leads to the following definition:

Definition 14. *Let $\mathcal{G} = \{G_i\}_{i=1,\dots,n}$ be a collection of groups and let X be a topological space. An iterated action of $\mathcal{G}$ on X (denoted by $\mathcal{G} \curvearrowright X$) is a collection of group actions $\{\Phi_i : G_i \longrightarrow$*

$\text{Homeo}(X_{i-1})\}_{i=1,\dots,n}$, *where* $X_0 = X$ *and* $X_i = X_{i-1}/G_i$ *for all* $1 \leq i \leq n$. *The orbit maps will be denoted by* $p_i : X_{i-1} \longrightarrow X_i$ *for all* $1 \leq i \leq n$. *We will denote* $n = l(\mathcal{G})$.

A closed connected manifold M is homeomorphic to a nilmanifold if and only if it admits an iterated action of $\{T^{b_1}, \cdots, T^{b_c}\}$ such that each action is free and $\sum_{i=1}^{c} b_i = \dim(M)$. Indeed, since all the actions are free, the maps $p_i : M_{i-1} \longrightarrow M_i$ are principal T^{b_i}-bundles and the quotient M_{c-1} is a closed manifold of dimension $\dim(M) - \sum_{i=1}^{c-1} b_i = b_c$. Since M_{c-1} is closed and connected and it admits a free action of a torus $T^{\dim(M_{c-1})}$ we can conclude that $M_{c-1} \cong T^{b_c}$. This implies that M can be obtained as iterated principal torus bundles and hence M is a nilmanifold.

The concept of iterated group action has appeared implicitly in the literature. For example, towers of regular self-covering are studied in [BBS01, VL18, vL21, QSW24]. A regular self-covering of a closed manifold M is a map $p : M \longrightarrow M$ which is a regular covering. We can compose this map with itself to obtain a tower of regular self-coverings

$$M \xrightarrow{p} M \xrightarrow{p} \cdots \xrightarrow{p} M \xrightarrow{p} M.$$

Since $p : M \longrightarrow M$ is a regular covering, there exists a finite group G acting freely on M such that p can be seen as the orbit map $p : M \longrightarrow M/G$. Consequently, the study of towers of regular self-coverings is the study of iterated group actions of $\mathcal{G} = \{G, \overset{n}{\cdots}, G\}$ such that $M_i \cong M$ for all i and all the actions $\Phi_i : G \longrightarrow \text{Homeo}(M)$ are the same. In [FOM12] iterated group actions are used to describe and classify spin orbifolds of the form S^7/Γ where Γ is a finite group of $\text{SO}(8)$. Iterated group actions have also been studied when each group G_i is a connected Lie group. For example, in [BK23a] O.Baues and Y.Kamishima study iterated group actions on Riemannian aspherical manifolds where each group G_i is the solvable radical of $\text{Isom}(M_{i-1})^0$.

Nevertheless, it seems that a theory of iterated group actions have not been developed yet in full generality. In the last part of the thesis we start the development of a theory of finite iterated group actions. Recall that if a compact Lie group G acts freely on a manifold M then the quotient map M/G is also a manifold. Thus, our first focus is on free iterated actions of finite groups on manifolds. Let us briefly define some of the notions that we need to state our results on iterated actions.

Definition 15. *Let* $\mathcal{G} = \{G_i\}_{i=1,\dots,n}$ *and* $\mathcal{G}' = \{G'_i\}_{i=1,\dots,n'}$ *be two collections of groups supporting free iterated actions on* M. *We say that the iterated actions* $\mathcal{G} \curvearrowright M$ *and* $\mathcal{G}' \curvearrowright M$ *are equivalent (and we denote it by* $\mathcal{G} \curvearrowright M \sim \mathcal{G}' \curvearrowright M$*) if* $M_n \cong M_{n'}$ *and* $p = p_n \circ \cdots \circ p_1$ *and* $p'_{n'} \circ \cdots \circ p'_1 = p'$ *are isomorphic coverings. The equivalence class will be denoted by* $[\mathcal{G} \curvearrowright M]$. *If* $\mathcal{G} \curvearrowright M$ *is equivalent to an iterated group action of length 1 (a usual group action), then we say that* $\mathcal{G} \curvearrowright M$ *is simplifiable.*

With definition 15 is straightforward to see that a free iterated action $\mathcal{G} = \{G_1, \dots, G_n\} \curvearrowright M$ is simplifiable if and only if $\pi_1(M) \trianglelefteq \pi_1(M_n)$. In particular, all free iterated actions on simply-connected manifolds are simplifiable. However, there exist manifolds with non-trivial fundamental group where all free iterated actions on them are simplifiable (see lemma 4.2.5). There also many examples of non-simplifiable free iterated actions (see, for example, remark 4.2.6).

With this equivalence relation we can study free iterated group actions on nilmanifolds. In particular, we prove:

Theorem 16. *(Theorem 4.0.8) There exists a constant C only depending on Γ such that any free iterated action $\mathcal{G} \curvearrowright N/\Gamma$ is equivalent to a free iterated action $\mathcal{G}' \curvearrowright N/\Gamma$ where $\mathcal{G}' = \{A_1, \dots, A_c, G'\}$, A_i are finite abelian groups and $|G'| \leq C$.*

The proof of theorem 16 can be deduced from the properties of infranilmanifolds and the theory of almost-crystallographic groups. The conclusion of theorem 16 can be though as a generalized Jordan property. Indeed, given a closed connected manifold M we say that M has the iterated Jordan property if there exists a constant C depending only on M such that any free iterated action $\mathcal{G} \curvearrowright M$ is equivalent to a free iterated action of the form $\{A_1, \dots, A_c, G\} \curvearrowright M$, where A_i is abelian for $1 \leq i \leq c$ and $|G| \leq C$. Theorem 16 shows that nilmanifolds have the iterated Jordan property. Another example of manifolds satisfying this property are manifolds satisfying that all free iterated actions on them are simplifiable (for example, simply connected manifolds). This is a consequence of the validity of the generalized Jordan conjecture on closed connected manifolds (see [CPS22]).

With this equivalence relation we can define two invariants that measure the size of free iterated actions on a closed connected manifold. The first invariant is called the iterated length of a manifold.

Definition 17. *Given a free iterated action $\mathcal{G} \curvearrowright M$, the length of the iterated action is*

$$l(\mathcal{G} \curvearrowright M) = \min\{l(\mathcal{G}') : \mathcal{G}' \curvearrowright M \in [\mathcal{G} \curvearrowright M]\}.$$

The iterated length of a manifold M is

$$l(M) = \max\{l(\mathcal{G} \curvearrowright M) : \text{ free iterated action } \mathcal{G} \curvearrowright M\}.$$

Given a closed connected manifold M, it is an interesting question to study when $l(M)$ is bounded. If no bound exists then we will write $l(M) = \infty$.

Theorem 18. *(Theorem 4.0.10, cases where the length of a space is bounded)*

1. *If N/Γ is a c-step nilmanifold, then $l(N/\Gamma) \leq c + 1$.*
2. *Given a locally symmetric space $H \setminus G/\Gamma$, there exists C depending on Γ such that $l(M) \leq C$.*

The bound of theorem 18.1. is sharp. For example $l(T^n) = 2$ for $n \geq 3$ and $l(T^n) = 1$ for $n = 1, 2$ (see remark 4.3.9).

Theorem 19. *(Theorem 4.0.11, cases where the length of a space is not bounded)*

1. *There exists a closed solvmanifold M such that $l(M) = \infty$.*

2. *There exist a closed connected aspherical locally homogeneous space $H \setminus G/\Gamma$ such that the solvable radical of G is abelian and $l(H \setminus G/\Gamma) = \infty$.*

The proof of theorem 18 is based on the properties of lattices in nilpotent and semisimple Lie groups, specially the existence of lattices of minimal volume in semisimple Lie groups for item 3. Regular self-coverings play a key role in the proof of theorem 19.

The second invariant is a generalization of the discrete degree of symmetry of a manifold for free iterated actions of length 2. Recall that the rank of a finite group is the minimal number of elements needed to generate it.

Definition 20. *Given a free iterated action $\mathcal{A} \curvearrowright M$ of abelian groups, the rank of the iterated action is*

$$\operatorname{rank}_{ab}(\mathcal{A} \curvearrowright M) = \min\{\sum_{i=1}^{n} \operatorname{rank} A'_i : \{A'_1, \dots, A'_n\} \curvearrowright M \in [\mathcal{A} \curvearrowright M]\ A'_i \text{ abelian for all } i\}.$$

We define $\mu_2(M)$ as the set of all pairs $(f, b) \in (\mathbb{N})^2$ which satisfy:

1. *There exist an increasing sequence of prime numbers $\{p_i\}$, a sequence of natural numbers $\{a_i\}$ and a collection of free iterated actions $\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M$ for each $i \in \mathbb{N}$.*

2. $\operatorname{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M) = f + b$ *for each $i \in \mathbb{N}$.*

Consider the lexicographic order in $\mathbb{N}^2$ ($(a,b) \geq (c,d)$ if $a > c$, or $a = c$ and $b \geq d$). Define the iterated discrete degree of symmetry of M as

$$\text{disc-sym}_2(M) = \max\{(0,0) \cup \mu_2(M)\}.$$

Note that there are two significant differences between the conditions used in the definitions of the discrete degree of symmetry and the iterated discrete degree of symmetry. On the iterated discrete degree of symmetry we only consider actions of abelian p-groups, and all these actions are assumed to be free. While this hypothesis are made for technical reasons, they do not suppose a big loss of generality in our case, since we will see that the manifolds that we study have effective actions of abelian p-groups for arbitrarily large prime p and these actions are free for p large enough.

The iterated discrete degree of symmetry has similar properties to the degree of symmetry, and the following results shows that it is a suitable invariant to study rigidity problems related with nilmanifolds.

Theorem 21. *(Theorem 4.0.13) Let M be a closed connected n-dimensional aspherical manifold such that $\mathcal{Z}\pi_1(M)$ and $\mathcal{Z}(\operatorname{Inn}\pi_1(M))$is finitely generated, and* $\operatorname{Aut}(\operatorname{Inn}\pi_1(M))$ *and* $\operatorname{Out}(\operatorname{Inn}\pi_1(M))$

are Minkowski. If $\text{disc-sym}_2(M) = (f, b)$ *with* $f + b = n$ *then* $M \cong N/\Gamma$*, where* N/Γ *is the total space of a principal* T^f*-bundle over* T^b*.*

The proof is based on generalizing some arguments from theorem 7 and a careful analysis of the different group short exact sequences induced by the various free group actions. The key step is that there exists a constant C such that if $(\mathbb{Z}/p^a)^b$ acts freely on M and $p > C$ then $\text{Aut}(\pi_1(M/(\mathbb{Z}/p^a)^b))$ does not depend on p. We note that it is not known whether $\text{Aut}(\text{Inn}\,\pi_1(M))$ and $\text{Out}(\text{Inn}\,\pi_1(M))$ are Minkowski for every closed connected aspherical manifold M. The same arguments used in theorem 10 show that if Γ is a lattice in a connected Lie group G then $\text{Aut}(\text{Inn}\,\Gamma)$ and $\text{Out}(\text{Inn}\,\Gamma)$ are Minkowski. In particular, theorem 21 can be used on closed aspherical locally homogeneous spaces $H \setminus G/\Gamma$ with G connected.

We also compute the free iterated discrete degree of symmetry for closed connected aspherical 3-manifolds. Let H/Γ denote a Heisenberg manifold (see example 1.3.26), let K denote the Klein bottle and SK denote the total space of the unique non trivial principal S^1-bundle over K. Then:

Theorem 22. *(Theorem 4.0.15) Let* M *be a* 3*-dimensional closed connected aspherical manifold. Then:*

1. $\text{disc-sym}_2(M) = (3, 0)$ *if* $M \cong T^3$.
2. $\text{disc-sym}_2(M) = (2, 0)$ *if* $M \cong K \times S^1$ *or* $M \cong SK$.
3. $\text{disc-sym}_2(M) = (1, 2)$ *if* $M \cong H/\Gamma$.
4. $\text{disc-sym}_2(M) = (1, 0)$ *if* $\mathcal{Z}\pi_1(M) \cong \mathbb{Z}$ *and* $\text{Inn}\,\pi_1(M)$ *is centreless.*
5. $\text{disc-sym}_2(M) = (0, 0)$ *if* M *does not belong to one of the previous 4 cases.*

Finally, we obtain the following generalization of item (2) in theorem 13 for iterated group actions.

Theorem 23. *(Theorem 4.0.14) Let* M *be a closed connected orientable manifold admitting a non-zero degree map* $f : M \longrightarrow N/\Gamma$ *to a* 2*-step nilmanifold, which is the total space of a principal* T^a*-bundle over* T^b*. Then* $\text{disc-sym}_2(M) \leq (a, b)$ *and if* $\text{disc-sym}_2(M) = (a, b)$ *then* $H^*(M, \mathbb{Q}) \cong H^*(N/\Gamma, \mathbb{Q})$.

The proof of theorem 23 is based on refining the ideas used in theorem 13 and [MiR24a, Theorem 1.3], as well as in the generalization of some commutative algebra results proved in [MiR24a, §6] to a non-commutative algebra setting.

It is natural to ask if we can remove the freeness hypothesis in the study of iterated group actions. Assuming that each step of the iterated action is only effective is a hypothesis which is too weak to obtain meaningful results. Thus, we introduce the following hypothesis:

Definition 24. *Assume that we have an iterated action of $\mathcal{G}$ on a topological space X. An open subset $U \subseteq X$ is said to be $\mathcal{G}$-invariant if there exists a connected open subset $V \subseteq X/\mathcal{G}$ such that $p^{-1}(V) = U$, where $p : X \longrightarrow X/\mathcal{G}$ is the orbit map.*

An iterated action of $\mathcal{G} \curvearrowright X$ is said to be locally simplifiable if for every $x \in X$ there exists an open $\mathcal{G}$-invariant neighbourhood U of x such that the iterated action of $\mathcal{G}$ on U is simplifiable.

Theorem 25. *(Theorem 4.0.17) Assume that we have a locally simplifiable iterated action of $\mathcal{G}$ on a manifold M. Then $M/\mathcal{G}$ is an orbifold and $p : M \longrightarrow M/\mathcal{G}$ is an orbifold covering.*

As a corollary of theorem 25, if M is simply connected then every locally simplifiable iterated action is simplifiable. Using orbifolds, we can define an equivalence relation between locally simplifiable iterated actions as in definition 15. We can define the iterated discrete degree of symmetry for locally simplifiable actions.

Definition 26. *Given a locally simplifiable iterated action $\mathcal{A} \curvearrowright M$ of abelian groups, the rank of the iterated action is*

$$\operatorname{rank}_{ab}(\mathcal{A} \curvearrowright M) = \min\{\sum_{i=1}^{m} \operatorname{rank} A_i' : \{A_1', \dots, A_m'\} \curvearrowright M \in [\mathcal{A} \curvearrowright M] \text{ and all } A_i' \text{ are abelian}\}.$$

We define $\mu_2^{ls}(M)$ as the set of all pairs $(f, b) \in (\mathbb{N})^2$ which satisfy:

1. *There exist an increasing sequence of prime numbers $\{p_i\}$, a sequence of natural numbers $\{a_i\}$ and a collection of locally simplifiable iterated actions $\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M$ for each $i \in \mathbb{N}$.*
2. $\operatorname{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M) = f + b$ *for each* $i \in \mathbb{N}$.

Consider the lexicographic order in $\mathbb{N}^2$ ($(a,b) \geq (c,d)$ if $a > c$, or $a = c$ and $b \geq d$). Define the locally simplifiable iterated discrete degree of symmetry of M as

$$\text{disc-sym}_2^{ls}(M) = \max\{(0,0) \cup \mu_2^{ls}(M)\}.$$

Theorem 21, theorem 22 and theorem 23 also hold vacuously for locally simplifiable actions, since all manifolds appearing in the theorems satisfy that if a finite p-group acts on them for p a prime large enough, then the action is free. An example where $\text{disc-sym}_2^{ls}(M) \neq \text{disc-sym}_2^{free}(M)$ is the following.

Proposition 27. *We have* $\text{disc-sym}_2^{ls}(S^n) = ([\frac{n+1}{2}], 0)$ *and* $\text{disc-sym}_2^{free}(S^n) = (\frac{(-1)^{n+1}+1}{2}, 0)$*, where $[x]$ denotes the integer part of x.*

We finish the introduction with a discussion of open problems and possible research directions stemming from the results of this thesis.

The first questions are related to the hypothesis of theorem 7 and proposition 9.

1. Does there exist a closed connected aspherical manifold M such $\mathrm{Out}(\pi_1(M))$ is not Minkowski? Note that $\mathrm{Out}(\pi_1(M))$ not having the Minkowski property does not imply that $\mathrm{Homeo}(M)$ is Jordan, since, in general, not all subgroups $G \leq \mathrm{Out}(\pi_1(M))$ can be realized as an effective group action of G on M (this is the generalized Nielsen realization problem for closed connected aspherical manifolds, see [BW08]). There exist aspherical finite CW-complexes whose outer automorphism group does not have the Minkowski property(see remark 2.1.5).

2. Does there exist a closed connected aspherical manifold M such that $\mathrm{Inn}\,\pi_1(M)$ is infinite periodic? An affirmative answer to this question would be highly surprising, since $\pi_1(M)$ is finitely presented and torsion-free. An affirmative answer would also provide a counterexample to the finitely presented version of the Burnside problem, see [Sap07, pg. 3].

3. Given a closed connected aspherical manifold M, is $\text{disc-sym}(M) = \mathrm{rank}\,\mathcal{Z}\pi_1(M)$? This question is a revised conjecture of P.E.Conner and F.Raymond, which asked whether $\text{tor-sym}(M) = \mathrm{rank}\,\mathcal{Z}\pi_1(M)$ for every closed connected aspherical manifold (see [Con70]). This conjecture was disproved by S.Cappell, S.Weinberger and M.Yan in [CWY13] by constructing closed connected aspherical manifolds M satisfying $\text{tor-sym}(M) = 0$ and $\mathrm{rank}\,\mathcal{Z}\pi_1(M) = 1$. Nevertheless, these manifolds satisfy $\text{disc-sym}(M) \geq 1$ and they are not counterexamples to the proposed question.

The second set of questions is related to group actions on closed connected manifolds which generalize the class of aspherical manifolds.

4. Recall that if M is a closed connected aspherical manifold, then any compact connected Lie group acting effectively on M is a torus (see [LR87] for other examples of classes of manifolds on which only tori can act). On the other hand, if M is a closed connected manifold which admits actions of $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$ then $\mathrm{Homeo}(M \times T^2)$ is not Jordan (see [MiR17, Sza19, Sza23]). These two facts lead to the following question: If M is a closed connected manifold such that the only compact connected Lie groups acting effectively on M are tori and such that $\mathrm{Homeo}(M)$ is Jordan, is $\mathrm{Homeo}(M \times T^2)$ Jordan?

   Let us consider the smooth version of this question. If M is a closed connected smooth manifold such that the only compact connected Lie groups acting effectively and smoothly on M are tori and such that $\mathrm{Diff}(M)$ is Jordan, is $\mathrm{Diff}(M \times T^2)$ Jordan? This question would have interesting consequences independently on the answer being positive or negative. By a result of R.Schultz in [Sch72], there exists an exotic 10-sphere Σ where the only connected compact Lie group that act effectively on Σ is S^1. If the answer of the question was positive, $\mathrm{Diff}(\Sigma \times T^2)$ would be Jordan. On the other hand, since Σ is homeomorphic to S^{10}, Homeo$(\Sigma \times T^2)$ is not Jordan. This would provide the first example of a smooth manifold whose diffeomorphism group is Jordan and whose homeomorphism group is not Jordan. If the answer to the question was negative, we would have found new examples of smooth manifolds with non-Jordan diffeomorphism group which cannot be constructed using the techniques in [MiR17, Sza19, Sza23].

Lastly, we present some questions and problems on finite iterated actions:

5. The iterated discrete degree of symmetry has been defined for free or locally simplifiable iterated group actions of lenght 2. An interesting problem would be to extend its definition to free or locally simplifiable iterated group actions of length c, where c is an arbitrary natural number. More precisely, we define $\mu_c(M)$ as the set of tuples $(f_1, \ldots, f_c) \in (\mathbb{N})^c$ which satisfy:

   1. There exist an increasing sequence of prime numbers $\{p_i\}$, $c-1$ sequences of natural numbers $\{a_{i,j}\}_{i\in\mathbb{N}}$ and $1 \leq j \leq c-1$, and a collection of free iterated actions $\{(\mathbb{Z}/p_i^{a_{i,1}})^{f_1}, (\mathbb{Z}/p_i^{a_{i,2}})^{f_2}, \cdots, (\mathbb{Z}/p_i)^{f_c}\} \curvearrowright M$ for each $i \in \mathbb{N}$.
   2. $\mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i{}^{a_{i,1}})^{f_1}, (\mathbb{Z}/p_i^{a_{i,2}})^{f_2}, \cdots, (\mathbb{Z}/p_i)^{f_c}\} \curvearrowright M) = \sum_{k=1}^{c} f_k$ for each $i \in \mathbb{N}$.

   We define the c-iterated degree of symmetry as

$$\text{disc-sym}_c(M) = \max\left\{(0, \overset{c}{\cdots}, 0) \cup \mu_c(M)\right\}$$

   where the maximum is taken with respect to the lexicographic order. A first interesting question is whether there exists a natural number c_0 such that for all $c > c_0$ we have $\text{disc-sym}_c(M) = (f_1, \ldots, f_{c_0}, 0, \ldots, 0)$. We would like to generalize theorem 21 and theorem 23. Thus, we propose the following two questions:

   5A. Let M be a closed connected n-dimensional aspherical manifold and let $\mathcal{Z}_i\pi_1(M)$ be the i-th term of the lower central series of $\pi_1(M)$. Suppose that the groups $\mathrm{Aut}(\pi_1(M)/\mathcal{Z}_{c-1}\pi_1(M))$ and $\mathrm{Out}(\pi_1(M)/\mathcal{Z}_{c-1}\pi_1(M))$ are Minkowski and suppose that $\text{disc-sym}_c(M) = (f_1, \ldots, f_c)$ satisfies that $\sum f_i = n$. Is M a c-step nilmanifold obtained as iterated T^{f_i}-bundles?

   5B. Let M be a closed connected orientable manifold and $f : M \longrightarrow N/\Gamma$ a non-zero degree map to a c-step nilmanifold. If $\text{disc-sym}_c(M) = \text{disc-sym}_c(N/\Gamma)$, is $H^*(M, \mathbb{Q}) \cong H^*(N/\Gamma, \mathbb{Q})$?

6. It is conjectured in [MiR24a] that any closed connected manifold M satisfies that $\text{disc-sym}(M) \leq \dim(M)$. Is it true that if M is a closed connected manifold with $\text{disc-sym}_2(M) = (d_1, d_2)$ then $d_1 + d_2 \leq \dim(M)$? The answer is affirmative for the closed connected manifolds appearing in theorem 21, theorem 22 and theorem 23.

7. A possible application of iterated group actions could be the study of geometric structures which involve fibrations and fiber bundles. Sasakian manifolds are an odd dimensional analogue of Kähler manifolds. Given a closed connected regular Sasakian manifold M, there exists an S^1 action on M such that M/S^1 is a compact Kähler orbifold ($M \longrightarrow M/S^1$ is known as the Boothby-Wang fibration), see [BG07] and [BK20]. Recall that a Kähler nilmanifold is necessarily a torus (see [BG88]). Similarly, a Sasakian nilmanifold is necessarily a Heisenberg manifold (a non-trivial principal S^1-bundle over T^{2n}), by [Kas16].

   If M is closed connected Kähler manifold such that $\text{disc-sym}(M) = \dim(M)$ then M is biholomorphic to $T^{\dim(M)}$ (see [MiR24a]). Let M be a closed connected Sasaki manifold such that $\text{disc-sym}_2(M) = (d_1, d_2)$ with $d_1 + d_2 = \dim(M)$, does there exist a Sasaki automorphism between M and a Heisenberg manifold?

The thesis is divided in 4 chapters. The first chapter provides the necessary tools to prove the results of this thesis and it contains five sections. The first section is a succinct introduction to theory of finite transformations groups. We present general definitions, notation and results that will be used freely in the rest of the thesis. Thereafter, we also provide more background on the theory of large finite group actions on manifolds. In the second section we present group theory results aimed to study outer automorphism groups. These results are crucial to prove theorem 10. The third section is devoted to aspherical manifolds, focusing on the different types of aspherical locally homogeneous spaces. We also give the results of the theory of finite group actions on closed aspherical manifolds that will be used in extensively in this thesis. The fourth section contains a summary of the theory of non-commutative ring theory, which play a major role in the prove of theorem 23. Finally, the fifth section is an introduction to the theory of orbifolds, required to prove theorem 25. We have two important remarks about this chapter. First, most of the results of this chapter are given without proof and we refer to the literature for the detailed proofs of these statements. We only provide the proof of a result if it contains ideas that will be relevant to prove the original results of the thesis. Secondly, although most of the chapter is expository, it also contains some new definitions and results in section 1.1.3 and section 1.5. We will indicate clearly which results are new.

The second chapter is devote to the study of large finite group actions on closed aspherical manifolds, proving theorem 7, proposition 8, proposition 9, theorem 10, theorem 11 and proposition 12. The results of this chapter appear on the preprint [DS24], which has been accepted for publication in the journal *International Mathematics Research Notices* (https://doi.org/10.1093/imrn/rnaf126).

In the third chapter we study large finite group actions on manifolds which admit a non-zero degree map to a nilmanifold. In particular, we prove theorem 13.

Finally, in chapter 4 we study develop a theory of finite iterated actions, staring with free iterated group actions. We prove theorem 18, theorem 19, theorem 16, theorem 21, theorem 22, theorem 23, theorem 25 and proposition 27 in this chapter.

Chapter 1

Preliminaries

The goal of this chapter is to provide the necessary material required to explain and to prove the results of this thesis. The chapter is divided into six sections. The two largest sections (section 1.1,section 1.3) are introductions to the theory of finite transformation groups and the theory of aspherical manifolds, which are the main research topics of this thesis. There are three shorter sections dedicated to group cohomology (section 1.2), non-commutative ring theory (section 1.4) and orbifolds (section 1.5), which will become necessary tools to prove some of the thesis contributions.

1.1 Group actions on manifolds

1.1.1 Definitions and generalities

The aim of this section is to introduce the basic notions and results of the theory of compact transformation groups and to fix some notation for the rest of this thesis. The main references used are [Bor16, Bre72, Kaw91, LR10].

Definition 1.1.1. *A topological group G is a Hausdorff topological space together with a continuous map $G \times G \longrightarrow G$ (denoted by $(g,h) \mapsto gh$) which makes G into a group and such that the map $G \longrightarrow G$, where $g \mapsto g^{-1}$, is continuous. If G is a smooth manifold and the product and inverse maps are smooth we say that G is a Lie group.*

We are ready to define the concept of continuous group action on a topological space. In this thesis we will assume that all topological spaces are Hausdorff.

Definition 1.1.2. *A continuous left group action of G on a topological space X is a continuous map $\Phi : G \times X \longrightarrow X$ such that:*

(1) $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ *for all $g, h \in G$ and $x \in X$.*

(2) $\Phi(e,x) = x$ *for all* $x \in X$*, where* e *is the identity element of* G*.*

Similarly, a continuous right group action of G *on a topological space* X *is a continuous map* $\Phi : X \times G \longrightarrow X$ *such that:*

(1) $\Phi(x, gh) = \Phi(\Phi(x,g),h)$ *for all* $g, h \in G$ *and* $x \in X$*.*

(2) $\Phi(x,e) = x$ *for all* $x \in X$*, where* e *is the identity element of* G*.*

A topological space X *endowed with a group action of* G *is called a* G*-space.*

Notation 1.1.3. *To ease the notation, we denote* $\Phi(g,x)$ *by* gx*. Given* $g \in G$*, the map* $\Phi(g,\cdot) : X \longrightarrow X$ *will be denoted by* ϕ_g*.*

Note that given a left continuous action of G on X we can define a right action of G on X given by $xg = g^{-1}x$ for all $x \in X$ and $g \in G$. To ease notation, we will omit the adjective left and right from the statements on this thesis, since it will be clear from the context if the action is left or right.

Additionally, if X is a smooth manifold and G is a Lie group acting on X, then we say that the action of G on X is smooth if $\Phi : G \times X \longrightarrow X$ is a smooth map. All actions in this thesis will be assumed to be continuous.

Remark 1.1.4. *Note that a continuous group action of* G *on* X *induces a group morphism* $\phi : G \longrightarrow \mathrm{Homeo}(X)$ *such that* $\phi(g) = \phi_g$*.*

If X *has an extra geometric structure, we can impose to the action of* G *to preserve it. For example, if* X *is a smooth manifold then* G *acts smoothly on* X *if* ϕ_g *is a diffeomorphism for all* $g \in G$*. In this case we have a group morphism* $\phi : G \longrightarrow \mathrm{Diff}(X)$*.*

Example 1.1.5. *(1) The group* $\mathrm{GL}(n,\mathbb{R})$ *acts on* $\mathbb{R}^n$ *by linear transformations. Given* $A \in \mathrm{GL}(n,\mathbb{R})$ *and* $x \in \mathbb{R}^n$ *then* $\Phi(A,x) = Ax$ *is the multiplication of a matrix and a vector.*

(2) Let G *be a Lie group. Then* G *acts on itself (seeing* G *as a manifold) by left multiplication. More precisely, the action* $\Phi : G \times G \longrightarrow G$ *satisfies* $\Phi(g,h) = gh$ *for all* $g, h \in G$*.*

Definition 1.1.6. *Let* G *be a group acting on spaces and* X *and* Y*. A continuous map* $f : X \longrightarrow Y$ *is said to be equivariant if it satisfies* $f(gx) = gf(x)$ *for all* $g \in G$ *and* $x \in X$*.*

Definition 1.1.7. *Let* G *be a group acting on* X *and let* $x \in X$*, then:*

- *The stabilizer of* x *(also called isotropy subgroup of* x*) is* $G_x = \{g \in G : gx = x\}$*. Note that* G_x *is a closed subgroup of* G*.*
- *The orbit of* x *is* $\mathcal{O}(x) = Gx = \{gx : g \in G\}$*. The space of all orbits with the quotient topology will be denoted by* X/G*.*
- *The set of fixed points is* $X^G = \{x \in X : gx = x \text{ for all } g \in G\}$*.*

Note that we can define an equivalence relation on X such that $x \sim y$ if and only if $Gx = Gy$ (that is, there exists $g \in G$ such that $y = gx$). Thus, X/G is a topological space with the topology induced by the map $\pi : X \longrightarrow X/G$ that sends each point to its orbit.

Some basic properties of group actions are the following:

Definition 1.1.8. *Let G be a group acting on X. Then:*

(1) The action is said to be effective if $\bigcap_{x \in X} G_x = \{e\}$ (or equivalently, $\phi : G \longrightarrow \mathrm{Homeo}(X)$ is injective).

(2) The action is almost free if G_x is finite for all $x \in X$, and it is free if $G_x = \{e\}$ for all $x \in X$.

(3) The action is transitive if for all $x, y \in X$ there exists $g \in G$ such that $y = gx$ (or equivalently, $Gx = X$ for a $x \in X$). If a space X admits a transitive action of a group we will say that X is a homogeneous space.

The next proposition summarises some of the properties of compact group actions (see [Bre72, I.3.1.]).

Proposition 1.1.9. *Let G be a compact group acting on X, then:*

(1) The map $\Phi : G \times X \longrightarrow X$ is a closed map.

(2) The orbit space X/G is Hausdorff. The space X is compact if and only if X/G is compact.

(3) The quotient map $\pi : X \longrightarrow X/G$ is closed and proper.

Remark 1.1.10. *(Non-compact vs. compact group actions) We define an action $\mathbb{R}$ of T^2 such that $t(e^{2\pi i x}, e^{2\pi i y}) = (e^{2\pi i(x+t)}, e^{2\pi i(y+at)})$, with $a \in \mathbb{R}$. The action behaves differently depending on whether a is rational or irrational. If $a = r/s$ is rational, then the action is not effective, since the action of the integral multiples of $s \in \mathbb{R}$ induce the identity map on T^2. In consequence, this action induces an effective action of $S^1 = \mathbb{R}/(s\mathbb{Z})$ on T^2. On the other hand, if a is irrational the action of $\mathbb{R}$ on T^2 is free, but the orbit for any $(x, y) \in T^2$ is dense in T^2. In particular $T^2/\mathbb{R}$ is homeomorphic to the quotient of S^1 by the group generated by a rotation of infinite order, which has the trivial topology. In particular, $T^2/\mathbb{R}$ is not Hausdorff.*

If G is not compact we can impose the following conditions on the group action to avoid the situation of the above example.

Definition 1.1.11. *Let G be a Lie group acting on a space X. We say that the action is*

1. *locally proper if for each $x \in X$ there exists a neighbourhood U of x such that $\{g \in G : gU \cap U \neq \emptyset\}$ has compact closure. In particular, G_x is compact and G_x is finite if G is discrete.*

2. *proper if for each $x \in X$, there exists a neighbourhood U of x such that, for each $y \in X$, there exists a neighbourhood V of y such that $\{g \in G : gV \cap U \neq \emptyset\}$ has compact closure.*

Equivalently, the action of G on X is proper if the map $G \times X \longrightarrow X \times X$ *such that* $(g, x) \mapsto (gx, x)$ *is proper.*

3. *properly discontinuous if every* $x \in X$ *has a neighbourhood U such that* $\{g \in G : gU \cap U \neq \emptyset\}$ *only contains the identity element. Notice that a properly discontinuous action is free and the projection* $X \longrightarrow X/G$ *is a covering map.*

For example, if G is a Lie group and Γ is a discrete subgroup of G, then the action of Γ on G by multiplication is properly discontinuous. All non-compact group actions appearing in this thesis will be proper.

Given a space X with a group action of G, we want to understand and classify the different orbits that this action can produce. Given $x \in X$, we define the evaluation map at x to be the continuous map $\mathrm{ev}_x : G \longrightarrow X$ such that $\mathrm{ev}_x(g) = gx$. Note that we have a bijective map $\alpha_x : G/G_x \longrightarrow Gx$ such that $\alpha_x(gG_x) = gx$. Moreover if G is compact then α_x is a continuous map of a compact to a Hausdorff space which implies that α_x is closed. Therefore:

Lemma 1.1.12. *[Bre72, I.4.1] If G is compact, then* α_x *is a homeomorphism.*

Thus every orbit is homeomorphic to a coset space G/H where H is a closed subgroup of G. In consequence, we need to study coset spaces G/H.

Lemma 1.1.13. *[Bre72, I.4.2] Let G be a compact group and H and K closed subgroups. Then:*

(1) *There exists an equivariant map* $G/H \longrightarrow G/K$ *if and only if there exists* $a \in G$ *such that* $aHa^{-1} \subseteq K$.

(2) *Let H and K be closed subgroups of G and assume that there exists* $a \in G$ *such that* $aHa^{-1} \subseteq K$, *then the map* $R_a^{K,H} : G/H \longrightarrow G/K$ *such that* $R_a^{K,H}(gH) = ga^{-1}K$ *is well-defined and equivariant. Any equivariant map* $G/H \longrightarrow G/K$ *is of this form.*

(3) $aHa^{-1} \subseteq H$ *implies* $aHa^{-1} = H$

Corollary 1.1.14. *If there exists equivariant maps* $G/H \longrightarrow G/K$ *and* $G/K \longrightarrow G/H$ *then H and K are conjugate and these maps are homeomorphisms.*

Let G be a compact Lie group and let $\mathbf{G}$ denote the family of all homogeneous spaces of G. Given $X, Y \in \mathbf{G}$ we define an equivalence relation $X \sim Y \in \mathbf{G}$ if and only if there exists a G-equivariant homeomorphism $f : X \longrightarrow Y$. The equivalence classes under this relation are called orbits types and denoted by $\mathrm{type}(X)$. By lemma 1.1.12, for any $X \in \mathbf{G}$ there always exists a closed subgroup $H \leq G$ and a G-equivariant homeomorphism $X \cong G/H$. Thus, we can always choose a coset space G/H as a representative of each orbit type. Moreover, we can define a partial order relation in $\mathbf{G}/\sim$ given by $\mathrm{type}(X) \geq \mathrm{type}(Y)$ if and only if there exists a G-equivariant map $X \longrightarrow Y$ (note that it is a well-defined partial order because of lemma 1.1.13). Note that we have a minimum and maximum elements corresponding to type(G/G) and type(G) respectively.

Similarly, given a homogeneous space X equivalent to G/H, we define the isotropy type of X to be the conjugacy class of H in G, which we will denote by (H). In the set of conjugacy classes of closed subgroups of G there is a partial order given by $(H) \leq (K)$ if and only if H is conjugate to a subgroup of K. Then, the lemma 1.1.13 implies that we have an anti-isomorphism of partially ordered sets between $\mathbf{G}/\sim$ and the set of conjugacy classes of closed subgroups of G given by type$(G/H) \mapsto (H)$.

Given a manifold M, the union of all orbits with isotropy type (H) will be denoted by $M_{(H)}$. The next theorem is known as the principal orbit theorem.

Theorem 1.1.15. *Let G be a compact Lie group acting on a connected manifold M. There exists a maximal orbit type (also known as principal orbit type) G/H on M. The set $M_{(H)}$ is open and dense in M and the image of $M_{(H)}$ by the orbit map is connected in M/G.*

The proof can be found in [Kaw91, Theorem 4.27] or [Bre72, IV.3.1].

Theorem 1.1.16. *Any action of a compact Lie group on a compact connected manifold has finitely many orbit types.*

See [Kaw91, Theorem 4.23] for a proof of the theorem.

Now, we briefly explain the construction of associated bundles to a principal G-bundle. Let X be a G-space and $p : E \longrightarrow B$ a principal G-bundle. The associated bundle construction is a way to construct fiber bundles with fiber X by replacing X to each fiber of a principal G-bundle using the group action of G on X. More precisely, consider the diagonal G-action on $E \times X$ such that $(a, x)g = (ag, g^{-1}x)$ for all $g \in G$ and $(a, x) \in E \times X$. Note that this action is free. The quotient space $(E \times X)/G$ is denoted by $E \times_G X$ and its elements by $[a, x]$. We have a continuous map $q : E \times_G F \longrightarrow B$ defined by $q[e, f] = p(e)$, which is a fiber bundle over B with fiber X.

Definition 1.1.17. *Let G be a compact group and X be a G-space. A slice at x is a subspace $S \subseteq X$ containing x such that $G_x(S) = S$, and the map $\tau : G \times_{G_x} S \longrightarrow X$ given by $[g, s] \mapsto gs$ is a tube about Gx (that is, τ is a G-equivariant homeomorphism onto an open neighbourhood of $G(x)$).*

The slice theorem asserts that, assuming certain conditions on the topological space and the group action, there exists a slice for every $x \in X$. The theorem was first proved by Gleason in the case of free group actions. The proof of the general statement has contributions from Montgomery, Zippin, Koszul, Yang, Mostow and Palais.

Theorem 1.1.18. *Let G be a compact Lie group acting on a completely regular topological space X. Then there exists a slice for every $x \in X$.*

Recall that a space X is completely regular if for any closed subset $C \subseteq X$ and any point

$x \in X \setminus C$ there exists a continuous function $f : X \longrightarrow [0,1]$ satisfying that $f^(x) = 0$ and $f(C) = 1$. We refer to [Bre72, §2.5] for a proof of the theorem.

Remark 1.1.19. *If G is finite then X being Hausdorff is enough for the existence of a slice for every $x \in X$.*

The slice theorem has far-reaching consequences. As a first corollary, we have the following result:

Theorem 1.1.20. *Suppose that X is a completely regular space with an action of a compact Lie group G and that all orbits have type G/H. Then the orbit map $\pi : X \longrightarrow X/G$ is a fiber bundle with fiber G/H and structural group $N(H)/H$ acting by right translations on G/H.*

The proof can be found in [Bre72, Theorem 5.8]. In particular, if the action of G on X is free then $\pi : X \longrightarrow X/G$ is a principal G-bundle. If G is a compact group acting freely on a manifold M then the orbit space M/G is also a manifold of dimension $\dim M - \dim G$. We also note that the slice theorem is a key ingredient to prove theorem 1.1.15 and theorem 1.1.16.

We end this section with some comments on the relation between group actions and coverings. Let X be a topological space admitting a covering space theory (for example, X being connected, locally arcwise connected and semilocally 1-connected) and let $\pi : \tilde{X} \longrightarrow X$ denote the universal covering of X. Recall that $\pi_1(X)$ acts freely on $\tilde{X}$. If we have a finite group G acting effectively on X then we can lift (non uniquely, in general) each homeomorphism $g : X \longrightarrow X$ to a homeomorphism $\tilde{g} : \tilde{X} \longrightarrow \tilde{X}$ such that $\pi \circ \tilde{g} = g \circ \pi$. Then;

Lemma 1.1.21. *[Bre72, 9.3 Theorem] Let X and G be as above. There exists a short exact sequence*

$$1 \longrightarrow \pi_1(X) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

where $\tilde{G}$ is the group of all lifts of elements of G. The group $\tilde{G}$ acts effectively on $\tilde{X}$. If the action of G on X is free then the action of $\tilde{G}$ on $\tilde{X}$ is free and $\pi_1(X/G) = \tilde{G}$.

If the action of G is not free then the description of $\pi_1(X/G)$ is more difficult, as the next result shows:

Theorem 1.1.22. *[Arm82, Arm68] Let X and G be as above. Then $\pi_1(X/G) = \tilde{G}/K$, where K is the normal subgroup of $\tilde{G}$ generated by all elements $\tilde{g} \in \tilde{G}$ such that $\tilde{g} \in \tilde{G}_{\tilde{x}}$ for some $\tilde{x} \in \tilde{X}$.*

Corollary 1.1.23. *With X and G as in theorem 1.1.22, assume also that X is simply connected and that X^G is non-empty. Then X/G is simply connected.*

1.1.2 Transformation groups and cohomology

The aim of this section is to recall some results of the theory of finite transformation groups whose proofs require the use of cohomology. We review the Borel construction and equivariant cohomology. As an application, we provide a proof sketch of a theorem of L.N.Mann and J.C.Su (see [MS63]) and a proof of a result of P.A.Smith (originally proved in [Smi41] using different techniques). Our main references are [Bre72, AP93, Hsi12, LR10].

Theorem 1.1.24. *[Mil56a, Mil56b] For any Lie group G there exists a contractible space EG where G acts freely.*

Remark 1.1.25.

1. *EG is unique up to G-homotopy.*
2. *The quotient $EG/G = BG$ is called the classifying space of G. This name is a consequence of the fact that there is a bijection between the set of principal G-bundles over a finite CW-complex X, $Prin_G(X)$, and the set of homotopy classes of maps $[X, BG]$ (see [Mil56a, Mil56b]).*
3. *The principal bundle $EG \longrightarrow BG$ is called the universal principal G-bundle.*

Before sketching the main ideas of the proof (for a detailed discussion, we also refer to [AP93, §1.1]), we will focus on an important example:

Example 1.1.26. *Let p be a positive integer. We will construct EG when $G = \mathbb{Z}/p$. Firstly, we consider the sphere*

$$S^{2n-1} = \{(z_1, \dots, z_n) \in C^n : |z_1|^2 + \cdots + |z_n|^2 = 1\} \subseteq \mathbb{C}^n.$$

There is a group action of $\mathbb{Z}/p$ on S^{2n-1} given by $a(z_1, \dots, z_n) = (e^{\frac{2\pi ia}{p}} z_1, \dots, e^{\frac{2\pi ia}{p}} z_n)$ for any $a \in \mathbb{Z}/p$ and $(z_1, \dots, z_n) \in \mathbb{C}^n$. It is straightforward to check that this action is free. Moreover, form $m \leq n$, the inclusion maps $i_{n,m} : S^{2m-1} \longrightarrow S^{2n-1}$, $i_{m,n}(z_1, \dots, z_n) = (z_1, \dots, z_n, 0, \dots, 0)$ are $\mathbb{Z}/p$-equivariant.

We have obtained a family of groups actions of $\mathbb{Z}/p$ on spaces that are not contractible. We will use them to construct a $\mathbb{Z}/p$-action on a contractible space. Note that we have a chain of $\mathbb{Z}/p$-equivariant inclusions $S^1 \subseteq S^3 \subseteq S^5 \subseteq \cdots$. Then we can define the space

$$S^\infty = \bigcup_{n \in \mathbb{N}} S^{2n-1}$$

with this topology: $U \subseteq S^\infty$ is open if and only if $S^{2n-1} \cap U$ is open in S^{2n-1} for all $n \in \mathbb{N}$. A point in S^∞ is a sequence $(z_1, z_2, \dots)$ with $z_i \in \mathbb{C}$, only finitely many of them are non-zero and $\sum |z_i|^2 = 1$. The free group actions of $\mathbb{Z}/p$ on S^{2n-1} for $n \geq 1$ induce a free group action of $\mathbb{Z}/p$ on S^∞. In addition, we have the well-known result:

Lemma 1.1.27. *S^∞ is contractible.*

Proof. Let $\mathbf{1} : S^\infty \longrightarrow S^\infty$ be the constant map $\mathbf{1}(z_1, z_2, \dots) = (1, 0, \dots)$ and let $\sigma : S^\infty \longrightarrow S^\infty$ be the shift map $\sigma(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$. Then σ is continuous and the map

$H: S^\infty \times I \longrightarrow S^\infty$ defined by

$$H(z,t) = \frac{(1-t)z + t\mathbf{1}(z) + (t-t^2)\sigma(z)}{|(1-t)z + t\mathbf{1}(z) + (t-t^2)\sigma(z)|}$$

is a homotopy from the identity to $\mathbf{1}$. Hence, S^∞ is contractible.

□

In consequence, $E\mathbb{Z}/p \simeq S^\infty$. The orbit spaces $S^{2n-1}/(\mathbb{Z}/p) \cong L_p^n$ are known as lens spaces. The space $B\mathbb{Z}/p$ is usually denoted by L_p^∞.

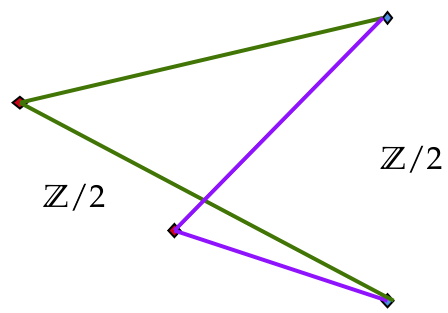
Note that the action of $\mathbb{Z}/p$ on S^{2n-1} is the restriction of the free action of $S^1 \subseteq \mathbb{C}^*$ on S^{2n-1}. We can use the same construction to show that $ES^1 \simeq S^\infty$ and $BS^1 \cong \mathbb{C}P^\infty$.

Recall that given two topological spaces X and Y, we define the join $X * Y$ to be $X \times Y \times I/\sim$, with the equivalence relation $(x,y,0) \sim (x,y',0)$ and $(x,y,1) \sim (x',y,1)$ for all $x,x' \in X$ and $y,y' \in Y$. Moreover, if X and Y are G-spaces then $X * Y$ is is a G-space where the action satisfies $g[x,y,t] = [gx,gy,t]$ for all $[x,y,t] \in X * Y$ and $g \in G$.

We are now ready to explain the Milnor construction of EG for an arbitrary Lie group G. Firstly, note that G (as a group) acts freely on itself (as a topological space) by right multiplication. Explicitly, given $g,h \in G$ we have $g(h) = gh$. We set $EG(0) = G$ and $EG(n) = EG(n-1) * G$ fro $n \geq 1$. The free action of G on itself induces a free action of G on each $EG(n)$. On the other hand, $EG(n)$ is $n-1$-connected for $n \geq 1$.

We have a sequence of G-equivariant inclusions $EG(0) \subseteq EG(1) \subseteq EG(2) \subseteq \cdots$. Therefore, we can define EG to be the colimit of the chain of inclusions. The space EG can be constructed as the infinite join $G * G * G * \cdots$. This space is contractible and has a free action of G.

Example 1.1.28. *As an example, we construct EG for $G = \mathbb{Z}/2$ using the Milnor construction. Since $\mathbb{Z}/2 \cong S^0$, we can use that $S^m * S^n \cong S^{m+n+1}$ to conclude that $E\mathbb{Z}/2(n) = S^n$ for all $n \geq 0$. It can also be shown that the action of $\mathbb{Z}/2$ induced on $E\mathbb{Z}/2(n)$ is the antipodal action, In consequence $E\mathbb{Z}/2(n)/(\mathbb{Z}/2) \cong \mathbb{R}P^n$. Therefore $E\mathbb{Z}/2 \cong S^\infty$ and $B\mathbb{Z}/2 \cong \mathbb{R}P^\infty$.*



$\mathbb{Z}/2 * \mathbb{Z}/2 \cong S^1$

Remark 1.1.29. *If G_1 and G_2 are two Lie groups, then $E(G_1 \times G_2) \simeq EG_1 \times EG_2$ and $B(G_1 \times G_2) \cong BG_1 \times BG_2$.*

Definition 1.1.30. *Let G be a Lie group and X a G-space, the Borel construction of X is the space $X_G = EG \times_G X$. The equivariant cohomology of X is*

$$H_G^*(X) = H^*(X_G).$$

Remark 1.1.31.
1. *By using the universal principal G-bundle and the associated bundle construction, we can find a fibration*

$$X \longrightarrow X_G \longrightarrow BG.$$

   This fibration is known as the Borel fibration.

2. *If the action of G on X is free, we can use the principal G-bundle $\pi : X \longrightarrow X/G$ to construct a fibration*

$$EG \longrightarrow X_G \longrightarrow X/G.$$

   Since EG is contractible, $H_G^*(X) \cong H^*(X/G)$.

An elementary p-group (or p-torus) is a group of the form $(\mathbb{Z}/p)^r$. The Borel construction can be used to prove the next theorem of L.N. Mann and J.C. Su.

Theorem 1.1.32. *[MS63] Let M be a closed manifold of dimension n. For a prime p, we define $b_p(M) = \sum_{i=0}^n \dim H^i(M, \mathbb{Z}/p)$. There exists a number C_p only depending on n and $b_p(M)$ such that if $(\mathbb{Z}/p)^r$ acts effectively on M then $r \leq C_p$.*

We will only give a sketch of the proof of this theorem with the extra assumption that the action is free.

Proof. Assume that we have a free action of $G = (\mathbb{Z}/p)^r$ on M. We consider the Serre spectral sequence of the Borel fibration $M \longrightarrow M_G \longrightarrow BG$. Thus, we have a convergent spectral sequence with second page

$$E_2^{s,t} = H^s(BG, \mathcal{H}^t(M, \mathbb{Z}/p)) \implies H_G^{s+t}(M),$$

where we use the calligraphic letter $\mathcal{H}$ to denote the cohomology with local coefficients. Since $H^i(M, \mathbb{Z}/p) = 0$ for all $i > n$, the spectral sequence collapses at the page $n+1$. Hence, $E_\infty = E_{n+1}$. Moreover, since the action of G on M is free, we have $H_G^i(M) = H^i(M/G) = 0$ for all $i > n$. This fact together with some computations on the spectral sequence implies that

$$\dim E_2^{s+1,0} \leq \sum_{j=0}^n \dim E_2^{s-j,j}.$$

On the other hand

$$\dim E_2^{s,t} \leq \binom{s-r-1}{r-1} \dim H^t(M, \mathbb{Z}/p),$$

where the first factor is precisely $\dim H^s(BG, \mathbb{Z}/p)$, which can be computed using the Künneth formula. Moreover

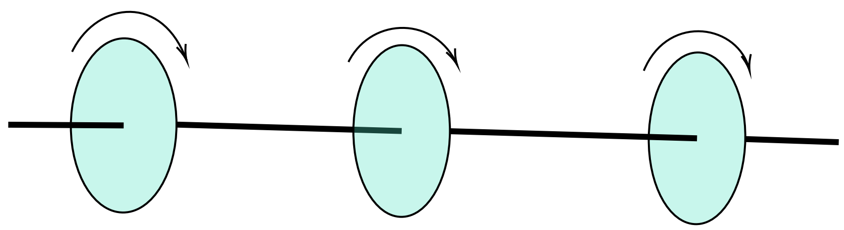
$$\dim E_2^{n+1,0} = \binom{n-r}{r-1}.$$

With these two inequalities and some straightforward computations we reach the conclusion that

$$r \leq \frac{\sqrt{n^2 + 4n(n+1)b_p(M)} - n}{2} = C_p.$$

□

Remark 1.1.33. 1. *The condition of M being a manifold is necessary. For any $r > 0$ we can construct a contractible 2-complex X_r which admits an action of $(\mathbb{Z}/p)^r$ as shown in the image below. Each component of $(\mathbb{Z}/p)^r$ rotates one of the disks and fixes the line through the origin.*



2. *We cannot bound r with a constant only depending on the dimension. For any $r > 0$, it is possible to construct a surface S_r which admit an action of $(\mathbb{Z}/p)^{2r}$.*

3. *The compactness of M is essential. For example, we have the following theorem due to V. Popov:*

   Theorem 1.1.34. *[Pop15, Theorem 1] For each $n \geq 4$, there exists a simply connected orientable non-compact smooth n-dimensional manifold M such that every finite group acts smoothly and freely on M.*

4. *We can ask how sharp the bound is. For example, if $M = S^1$ (so $n = 1$ and $b_p(S^1) = 2$) then C_p is approximately $1{,}47$. Therefore, if $(\mathbb{Z}/p)^r$ acts freely on S^1 then $r = 1$. However, it is possible to find much better bounds if we focus on some specific manifold:*

   Theorem 1.1.35. *[Smi60] Assume that $(\mathbb{Z}/p)^r$ acts effectively on S^n. If $p > 2$ then $r \leq \frac{n+1}{2}$. If $p = 2$ then $r \leq n + 1$.*

Note that this bound is for effective group actions.

Another result that can be proved using the Serre spectral sequence on the Borel fibration is the following:

Lemma 1.1.36. *Let M be a closed connected manifold and p a prime number. Assume that we have an effective action of a finite p-group G on M. Then*

$$\dim H_k(M/G, \mathbb{Z}/p) \leq \sum_{i+j=k} \dim H_i(BG, \mathbb{Z}/p) \dim H_j(M, \mathbb{Z}/p).$$

We refer to [SC19, Lemma 1.46] for a detailed proof of the lemma. The next result, which is also proved using the Borel construction, will be crucial to study group actions on aspherical manifolds. Recall that given a commutative ring A, a space X is A-acyclic if the reduced cohomology $\tilde{H}^i(X, A) = 0$ for all i.

Theorem 1.1.37. *[LR10, Lemma 3.1.6] Let X be a connected finite CW-complex. Then:*

1. *Assume that X is $\mathbb{Q}$-acyclic and that S^1 acts effectively on X. Then $F = X^{S^1}$ is not empty and is also $\mathbb{Q}$-acyclic.*
2. *Let p be a prime. Assume that X is $\mathbb{Z}/p$-acyclic and that a finite p-group G acts effectively on X. Then $F = X^G$ is not empty and is also $\mathbb{Z}/p$-acyclic.*

Proof. We start proving the first part. From the Borel construction we obtain a diagram

$$\begin{array}{ccccc} X & \xleftarrow[\pi_2]{} & S^\infty \times X & \xrightarrow{\pi_1} & S^\infty = ES^1 \\ \downarrow & & \downarrow & & \downarrow \\ X/S^1 & \xleftarrow[\overline{\pi}_2]{} & X_{S^1} & \xrightarrow{\overline{\pi}_1} & \mathbb{C}P^\infty = BS^1 \end{array}$$

Since $\overline{\pi}_1$ is a fibration with a $\mathbb{Q}$-acyclic fiber X, the morphism $\overline{\pi}_1^* : H^*(\mathbb{C}P^\infty, \mathbb{Q}) \longrightarrow H^*(X_{S^1}, \mathbb{Q})$ is an isomorphism. Suppose $F = \emptyset$. Given $x \in S^1$ and its orbit $\overline{x} \in X/S^1$, we have $\overline{\pi}_2^{-1}(\overline{x}) = S^\infty/S^1_x = BS^1_x$. Since S^1_x is either trivial or finite cyclic, we can conclude that BS^1_x is $\mathbb{Q}$-acyclic. Consequently, $\overline{\pi}_2^* : H^*(X/S^1, \mathbb{Q}) \longrightarrow H^*(X_{S^1}, \mathbb{Q})$ is an isomorphism by Vietoris mapping theorem. Thus, $H^*(\mathbb{C}P^\infty, \mathbb{Q}) \cong H^*(X/S^1, \mathbb{Q})$, which is not possible since $H^*(X/S^1, \mathbb{Q})$ is finite dimensional and $H^*(\mathbb{C}P^\infty, \mathbb{Q}) \cong \mathbb{Q}[a]$ with $\deg(a) = 2$. Therefore, $F \neq \emptyset$.

For $\overline{x} \in (X/S^1) \setminus F$, $\overline{\pi}_2^{-1}(\overline{x})$ is still $\mathbb{Q}$-acyclic. In consequence, $\overline{\pi}_2^* : H^*(X/S^1, F, \mathbb{Q}) \longrightarrow H^*(X_{S^1}, F_{S^1}, \mathbb{Q})$ is an isomorphism. There exists a long exact sequence

$$\cdots \longrightarrow H^q(X_{S^1}, F_{S^1}, \mathbb{Q}) \longrightarrow H^q(X_{S^1}, \mathbb{Q}) \xrightarrow{i^*} H^q(F_{S^1}, \mathbb{Q}) \longrightarrow \cdots$$

If $q > \dim(X/S^1)$ then $i^* : H^q(X_{S^1}, \mathbb{Q}) \longrightarrow H^q(F_{S^1}, \mathbb{Q})$ becomes an isomorphism. Using that $F_{S^1} = F \times \mathbb{C}P^\infty$ and that $\overline{\pi}_1^*$ is an isomorphism, we obtain that

$$H^q(\mathbb{C}P^\infty, \mathbb{Q}) = H^q(\mathbb{C}P^\infty \times F, \mathbb{Q}) \cong \bigoplus_{i+j=q} H^i(F, \mathbb{Q}) \otimes H^j(\mathbb{C}P^\infty, \mathbb{Q})$$

for $q > \dim(X/S^1)$. This implies that F must be $\mathbb{Q}$-acyclic.

For the second part, analogous arguments can be used replacing S^1 by $\mathbb{Z}/p$ and $\mathbb{Q}$ by $\mathbb{Z}/p$, obtaining the second result for $G = \mathbb{Z}/p$. For the general case of a finite p-group G, we use the fact that G is solvable and contains a nontrivial normal subgroup H. Consequently, using that $X^G = (X^H)^{G/H}$, we can reduce this case to the case $G = \mathbb{Z}/p$ by induction. □

Theorem 1.1.37 as well as the next two results can be proved using Smith theory. Their proofs can be found in [Kaw91, §5.5] or [Bre72, Chapter 3, §10].

Theorem 1.1.38. *Let p be a prime number, X a CW-complex and G a p-group acting effectively on X. Assume that $H_*(X, \mathbb{Z}/p) \cong H_*(S^n, \mathbb{Z}/p)$. Then $H_*(X^G, \mathbb{Z}/p) \cong H_*(S^m, \mathbb{Z}/p)$ with $-1 \leq m \leq n$ ($m = -1$ if $X^G = \emptyset$). If p is odd then $n - m$ is even.*

Theorem 1.1.39. *Let p be a prime and let X be a CW-complex with a free action of $\mathbb{Z}/p$. Then $\dim H_n(X/(\mathbb{Z}/p), \mathbb{Z}/p) \leq \sum_{i \geq n} H_i(X, \mathbb{Z}/p)$.*

1.1.3 Actions of large finite groups on manifolds

In this section we review some questions and results presented in the introduction.

Jordan property

Definition 1.1.40. *A group $\mathcal{G}$ is said to be Jordan if there exists a constant C such that every finite subgroup $G \leq \mathcal{G}$ has an abelian subgroup $A \leq G$ such that $[G : A] \leq C$.*

The name of this property is motivated by the following classical theorem of Camille Jordan (see [Rag12, Theorem 8.29] and [Bre23] for an overview of the original proof).

Theorem 1.1.41. *The group $\mathrm{GL}(n, \mathbb{R})$ is Jordan for all n.*

The next theorem summarises the known results about the Jordan property for homeomorphism groups of manifolds.

Theorem 1.1.42. *[MiR24b] Let M be a manifold. Assume that it satisfies one of the following conditions:*

1. *M is compact and $\dim M \leq 3$.*
2. *M has dimension n and $H_*(M, \mathbb{Z}) \cong H_*(S^n, \mathbb{Z})$.*

3. *M is connected,* $H_*(M, \mathbb{Z})$ *is finitely generated and* $\chi(M) \neq 0$.
4. *M is compact, connected and orientable and it admits a non-zero degree map* $M \longrightarrow T^n$.
5. *M is a closed flat manifold.*

Then $\mathrm{Homeo}(M)$ *is Jordan.*

On the other hand, if M admits an effective action of $\mathrm{SO}(3)$ *or* $\mathrm{SU}(2)$ *then* $\mathrm{Homeo}(M \times T^2)$ *is not Jordan.*

The method to show that $\mathrm{Homeo}(M \times T^2)$ is not Jordan when $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$ acts effectively on M can be generalized as follows:

Theorem 1.1.43. *[CPS22, Theorem 2.4] Let M be a closed connected manifold and G a connected compact Lie group with finite center. For any principal G-bundle* $E \longrightarrow T^2$*, the homeomorphism group of the total space of the associated bundle* $\mathrm{Homeo}(E \times_G M)$ *is not Jordan.*

However, Csikós, Pyber and Szabó proved that the homeomorphism group of closed connected manifolds have that following property:

Theorem 1.1.44. *[CPS22, Theorem 1.4] Let M be a closed connected manifold. There exists a constant C such that every finite group G acting effectively on M has a nilpotent subgroup* $H \leq G$ *such that* $[G : H] \leq C$.

We note that we can remove the hypothesis of being closed and connected by $H_*(M, \mathbb{Z})$ being finitely generated. It is interesting to study the class of nilpotency of the subgroup H. For example:

Theorem 1.1.45. *[MiRSC22, Theorem 1.1] Let M be a closed connected smooth 4-manifold. There exists a constant C such that every finite group G acting effectively and smoothly on M has a nilpotent subgroup* $H \leq G$ *of at most nilpotency class 2 such that* $[G : H] \leq C$.

Discrete degree of symmetry

Recall the definition of the discrete degree the introduction:

Definition 1.1.46. *Given a manifold M let*

$$\mu(M) = \{r \in \mathbb{N} : \text{ M admits an effective action of } (\mathbb{Z}/a)^r \text{ for arbitrarily large } a\}.$$

More explicitly, $r \in \mu(M)$ *if there exists an increasing sequence of natural number* $\{a_i\}$ *and effective group actions of* $(\mathbb{Z}/a_i)^r$ *on M for each i.*

The discrete degree of symmetry of a manifold M is

$$\text{disc-sym}(M) = \max(\{0\} \cup \mu(M)).$$

By theorem 1.1.32, we know that if M is a closed connected manifold then disc-sym(M) is a well-defined natural number, but finding the exact value of disc-sym(M) is probably difficult in most cases.

The definition of disc-sym(M) is related to the definition of the toral degree of symmetry tor-sym(M) (see [Hsi12, Chapter VII. §2]):

Definition 1.1.47. *Let M be a manifold. The toral degree of symmetry of M is*

$$\text{tor-sym}(M) = \max\{r : T^r \textit{ acts effectively on } M\}.$$

It is clear that tor-sym$(M) \leq$ disc-sym(M). We can bound the toral degree of symmetry of a manifold depending on its dimension.

Theorem 1.1.48. *Assume that M is a connected manifold of dimension n. Then* tor-sym$(M) \leq n$ *and* tor-sym$(M) = n$ *if and only if M is homeomorphic to T^n.*

Proof. Assume that M admits an effective action of T^r for some r and let H be the isotropy type of the principal orbits of the action. Thus, all points in $M_{(H)}$ have an isotropy subgroup conjugated to H. Since T^r is abelian, all points in $M_{(H)}$ have H as isotropy subgroup. Since $M_{(H)}$ is open and dense and H acts trivially on $M_{(H)}$, the action of H on M is also trivial. Because the action of T^r on M is effective we can conclude that H is the trivial subgroup. Therefore, we can choose $x \in M$ with trivial isotropy group. The evaluation map $f : T^r \longrightarrow M$ such that $f(t) = tx$ is continuous and injective, which implies that $r \leq \dim M = n$. If $r = \dim M$, then the map $f(T^r)$ is open. Since $f(T^r)$ is also closed and M is connected, we have $M = f(T^r)$. Since f is continuous, injective and open, we can conclude that f is a homeomorphism between M and T^r. □

Let $n = \dim(M)$. It is not known whether disc-sym$(M) \leq n$, and whether disc-sym$(M) = n$ if and only if $M \cong T^n$. However, we have the following bound:

Theorem 1.1.49. *[MiR24a, Theorem 1.2] For any closed connected n-dimensional manifold M we have* disc-sym$(M) \leq \frac{3}{2}n$*.*

If we put some restriction on the manifold we can obtain better bounds. Recall that $[x]$ denotes the integer part of a number $x \in \mathbb{R}$.

Theorem 1.1.50. *[MiR24b, Theorem 4.3] Let M be a closed connected n-dimensional manifold. If:*

1. *$H^*(M, \mathbb{Z}) \cong H^*(S^n, \mathbb{Z})$ then* disc-sym$(M) \leq [\frac{n+1}{2}]$*. In particular,* disc-sym$(S^n) = [\frac{n+1}{2}]$*.*
2. *$\chi(M) \neq 0$ then* disc-sym$(M) \leq [\frac{n}{2}]$*.*

For the next application of the discrete degree of symmetry we need to introduce the following notation:

Definition 1.1.51. *A group $\mathcal{G}$ is said to be Minkowski if there exists a constant C such that every finite subgroup $G \leq \mathcal{G}$ satisfies $|G| \leq C$.*

Remark 1.1.52. *This name is motivated by a classical result of Hermann Minkowski which states that* $\mathrm{GL}(n, \mathbb{Z})$ *is Minkowski (see [Ser10, §1]). The Minkowski property was studied in [Pop18, Gol23] under the name of bounded finite subgroups property.*

If M is a closed manifold and $\mathrm{Homeo}(M)$ is Minkowski then M is said to be almost-asymmetric. In the particular case where M does not admit any effective finite group action we say that M is asymmetric. This case has been extensively studied (see [Pup07] and references therein).

Lemma 1.1.53. *[MiR24b, Lemma 8.1] Given a closed connected manifold M,* $\text{disc-sym}(M) = 0$ *if and only if M is almost asymmetric.*

The relation between the Minkowski property of groups and short exact sequences is explained in the next elementary group-theoretic lemma, which will be used later.

Lemma 1.1.54. *Let* $1 \longrightarrow K \longrightarrow H \xrightarrow{p} Q \longrightarrow 1$ *be a short exact sequence of groups. If K and Q are Minkowski, then H is Minkowski. If K is finite and H is Minkowski then Q is Minkowski.*

Proof. Let us prove the first part of the statement. Let G be a finite subgroup of H. Then we have a short exact sequence $1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_3 \longrightarrow 1$, where G_3 is the image of G by the map $H \longrightarrow Q$ and $G_1 = G \cap K$. If C_1 and C_3 are the Minkowski constants of K and Q respectively then $|G| \leq |G_1||G_3| \leq C_1C_3$. Therefore H is Minkowski.

For the second part assume that G is a finite subgroup of Q, then $p^{-1}(G)$ is a finite subgroup of H of order $|K||G|$. Since H is Minkowski, $|K||G| \leq C_2$. Therefore $|G| \leq C_2/|K|$. Thus Q is Minkowski. □

Proposition 1.1.55. *[Gol23, Proposition 2.8] Let* $1 \longrightarrow K \longrightarrow H \xrightarrow{p} Q \longrightarrow 1$ *be a short exact sequence of groups. Suppose that K is Jordan and Q is Minkowski. Then H is Jordan.*

We finish this section by introducing other invariants which are similar to $\text{disc-sym}(M)$ and $\text{tor-sym}(M)$. Recall that an action of a Lie group G on a manifold M is said to be almost-free if G_x is finite for all $x \in M$.

Definition 1.1.56. *Let M be a manifold. We define the following invariants:*

1. *The rank of M is*

$$\mathrm{rank}(M) = \max\{\{0\} \cup \{r : T^r \text{ acts almost-freely on } M\}\}.$$

2. *Given a prime p, the p-rank of M is*

$$\mathrm{rank}_p(M) = \max\{\{0\} \cup \{r : (\mathbb{Z}/p)^r \text{ acts freely on } M\}\}.$$

3. *Let*

$$\mu_P(M) = \{r : M \text{ admits free actions of } (\mathbb{Z}/p)^r \text{ for arbitrarily high prime } p\}.$$

The stable rank of a space M is

$$\text{stable-rank}(M) = \max\{\{0\} \cup \mu_P(M)\}.$$

It is clear that $\text{rank}(M) \leq \text{tor-sym}(M)$ and that $\text{rank}(M) \leq \text{stable-rank}(M) \leq \text{disc-sym}(M)$. There are two important conjectures regarding $\text{rank}(M)$ and $\text{rank}_p(M)$.

Conjecture 1.1.57. *Let M be a closed connected manifold. Then:*

1. *(Toral rank conjecture)* $\dim H^*(M, \mathbb{Q}) \geq 2^{\text{rank}(M)}$.
2. *(Carlsson conjecture) For all prime p,* $\dim H^*(M, \mathbb{Z}/p) \geq 2^{\text{rank}_p(M)}$.

The toral rank conjecture (proposed by S. Halperin in [Hal85]) and the Carlsson conjecture (see [Car86]) have been studied extensively (see [FOT08, §7.3]). We can also ask if given a closed connected manifold M there exists a constant C such that $\dim H^*(M, \mathbb{Z}/p) \geq 2^{\text{rank}_p(M)}$ for all prime $p > C$. We will call this weaker version of the Carlsson conjecture the stable Carlsson conjecture. For example:

Theorem 1.1.58. *[Han09] Let* $M = S^{n_1} \times \cdots \times S^{n_o} \times S^{m_1} \times \cdots \times S^{m_e}$*, where all* n_i *are odd and all* m_i *are odd. Then* $\text{rank}_p(M) = o$ *for all* $p > 3\dim(M)$*. In particular,* $\text{stable-rank}(M) = o$ *and the stable Carlsson conjecture is true for M.*

Small and few stabilizers

We finish this section by introducing some problems about the stabilizers of finite group actions on manifolds.

Definition 1.1.59. *Let G be a finite group acting effectively on a manifold M. The set of all stabilizer subgroups is*

$$\text{Stab}(G, M) = \{G_x : x \in M\}.$$

We say that M has few stabilizers if there exists a constant C such that every a finite group G acting effectively on M has a subgroup $H \leq G$ *such that* $[G : H] \leq C$ *and* $|\text{Stab}(H, M)| \leq C$.

Note that it is not possible to bound $|\text{Stab}(G, M)|$ only depending on M. For example, the natural dihedral group D_n action on S^1 resulting from the inclusion $D_n \hookrightarrow \text{O}(2, \mathbb{R})$ is effective and satisfies $|\text{Stab}(D_n, S^1)| \geq n/2$. It is not known if all closed connected manifolds have few stabilizers. However, if we only consider actions of p-groups then the analogous property is known to be true:

Theorem 1.1.60. *[CMiRPS21, Theorem 1.3] There exists a constant C such that any finite p-group G acting effectively on a closed manifold M has a subgroup H such that $[G : H] \leq C$ and $|\operatorname{Stab}(H, M)| \leq C$.*

Note that C does not depend on the prime p. This result was crucial to prove theorem 1.1.44. An application of theorem 1.1.60 is:

Corollary 1.1.61. *[CMiRPS21, Corollary 1.5] Let M be a manifold such that $H_*(M, \mathbb{Z})$ is finitely generated. There exists a constant C depending only on $\dim(M)$ and $H_*(M, \mathbb{Z})$ such that every finite p-group acting effectively on M has a characteristic subgroup H containing the center such that $[G : H] \leq C$ and $\dim H_*(M'_H, \mathbb{Z}/p) \leq C$, where $M'_H = \{x \in M : H_x = \{e\}\}$.*

Given a finite group acting effectively on a closed connected manifold, we are also interested in how large the stabilizers of this action can be. Thus, we introduce the following two definitions:

Definition 1.1.62. *Let M be a closed connected manifold. Then:*

1. *We say that M has the small stabilizers property if there exist a constant C such that if G is a finite group acting effectively on M then $|G_x| \leq C$ for all $x \in M$.*
2. *We say that M has the almost fixed point property if there exists a constant C such that if G is a finite group acting effectively on M then there exist $x \in M$ such that $[G : G_x] \leq C$.*

The first property is new and it has not been studied yet. We will show that most manifolds studied in this thesis have this property. The second property was studied in [MiR24c].

Theorem 1.1.63. *[MiR24c, Theorem 1.5] Let M be a closed connected manifold with $\chi(M) \neq 0$. Then M has the almost fixed point property.*

The almost-fixed point property and the small stabilizers property are dual in the following sense:

Lemma 1.1.64. *A closed manifold M with the small stabilizers and the almost fixed point property is almost asymmetric.*

Proof. Let C be the constant of the almost fixed point property and D the constant of the small stabilizers property. Assume that we have a finite group G acting effectively on M. Then there exists $x \in M$ such that $[G : G_x] \leq C$. In addition, $|G_x| \leq D$. Consequently, $|G| \leq C \cdot D$. Thus, M is almost asymmetric. □

Another relation between these properties is the following:

Lemma 1.1.65. *Let M be a closed connected manifold with the small stabilizers property and such that $\operatorname{Homeo}(M)$ is Jordan. Then M has the few stabilizers property.*

Proof. Let C be the constant of the Jordan property and let C' be the constant of the small stabilizers property. Let G be a finite group acting effectively on M. Since $\mathrm{Homeo}(M)$ is Jordan there exists an abelian subgroup H such that $[G:H] \leq C$. We will see that we can bound $\mathrm{Stab}(H,M)$ by a constant C'' for any finite abelian group acting effectively on M and therefore the constant of the few stabilizer property will be $D = \max\{C, C''\}$. Recall that, by theorem 1.1.32, there exists a constant r such that any finite abelian group H which acts effectively on M satisfies $\mathrm{rank}(H) \leq r$.

We have an inclusion $\mathrm{Stab}(H,M) \subseteq \{L \leq H : |L| \leq C'\}$, thus we will bound the number of subgroups of H of order at most C' instead of the number of stabilizers of the action of H. Since H is abelian, we have a decomposition $H = H_1 \times \cdots \times H_l$, where each H_i is a p_i-Sylow subgroup, thus is of the form $H_i = \mathbb{Z}/p_i^{a_{i,1}} \times \cdots \times \mathbb{Z}/p_i^{a_{i,r_i}}$, where $a_{i,1} \leq \cdots \leq a_{i,r_i}$ and $r_i \leq r$ for all i. Then we have a bijective correspondence

$$\{L \leq H\} \longleftrightarrow \prod_{i=1}^{l} \{L_i \leq H_i\}$$

which induces an inclusion

$$\{L \leq H : |L| \leq C'\} \subseteq \prod_{i=1}^{l} \{L_i \leq H_i : |L_i| \leq C'\}.$$

For a prime p we define $e(p) = \max\{e \in \mathbb{N} : p^e \leq C'\}$ and we denote the set of all primes such that $e(p) \neq 0$ by $\mathcal{P}$. Note that $\mathcal{P}$ is precisely the set of primes which are equal or smaller than C' and therefore $|\mathcal{P}| \leq C'$. Moreover,

$$\{L_i \leq H_i : |L_i| \leq C'\} \subseteq S_i = \{L_i \leq H_i : \text{all elements of } L \text{ have order at most } e(p_i)\}$$

for each i. Note that if $p_i \notin \mathcal{P}$ then $|S_i| = 1$. Therefore

$$\{L \leq H : |L| \leq C'\} \subseteq \prod_{i=1}^{l} |S_i|.$$

For each H_i we choose an inclusion $H_i \longrightarrow (\mathbb{Z}/p_i^{a_{i,r_i}})^r$, which induce an inclusion

$$S_i \subseteq \{L_i \leq (\mathbb{Z}/p_i^{a_{i,r_i}})^r : \text{all elements of } L \text{ have order at most } e(p_i)\}.$$

Finally, we use that if L is a subgroup of $(\mathbb{Z}/p_i^{a_{i,r_i}})^r$ where all elements are of order at most $e(p_i)$ then $L_i \leq (\mathbb{Z}/p_i^{e(p_i)})^r \leq (\mathbb{Z}/p_i^{a_{i,r_i}})^r$. In consequence, we have an inclusion

$$\{L_i \leq (\mathbb{Z}/p_i^{a_{i,r_i}})^r : \text{all elements of } L_i \text{ have order at most } e(p_i)\} \subset \{L_i \leq (\mathbb{Z}/p_i^{e(p_i)})^r\}.$$

Since $p_i \leq C'$ and $e(p_i) \leq C'$, we have $|\{L_i \leq (\mathbb{Z}/p_i^{e(p_i)})^r\}| \leq 2^{|(\mathbb{Z}/p_i^{e(p_i)})^r|} \leq 2^{(C'^{C'})^r}$, which does not depend on the prime p_i. Therefore, $|S_i| \leq 2^{(C'^{C'})^r}$ if $p_i \in \mathcal{P}$ and $|S_i| = 1$ if $p_i \notin \mathcal{P}$. By using that $|\mathcal{P}| \leq C'$ we can conclude that $|\mathrm{Stab}(H,M)| \leq |\{L \leq H : |L| \leq C'\}| \leq \prod_{i=1, p_i \in \mathcal{P}}^{l} |S_i| \leq (2^{(C'^{C'})^r})^{C'}$, which completes the proof. $\square$

Finally, the following remark will be important:

Remark 1.1.66. *Let M be a closed connected manifold with the small stabilizers property with constant C. Every effective action of a finite p-group on M is free for all prime $p > C$.*

1.2 Group theory preliminaries

1.2.1 Group cohomology

In this section we summarize some basic results in group cohomology, focusing on the relationship between group extensions and low dimensional cohomology groups. We use [LR10, Bro12] as a reference.

Let G be a group and $\mathbb{Z}G$ be its group ring. Recall that the group ring $\mathbb{Z}G$ is a ring whose elements are formal sums $\sum_{g\in G} a_g g$ where only finitely many $a_g \in \mathbb{Z}$ are non-zero. Given another element $\sum_{g'\in G} b_{g'} g' \in \mathbb{Z}G$ we have

$$(\sum_{g\in G} a_g g)(\sum_{g'\in G} b_{g'} g') = \sum_{h\in G} c_h h$$

where

$$c_h = \sum_{gg'=h} a_g b_{g'}.$$

See section 1.4 for some examples. A module over $\mathbb{Z}G$ will be called a G-module. Recall also that a module P is projective if for any exact sequence $M \xrightarrow{i} M' \xrightarrow{j} M''$ and any map $\phi : P \longrightarrow M'$ such that $j \circ \phi = 0$ there exists a morphism $\psi : P \longrightarrow M$ such that $i \circ \psi = \phi$. Given a module M, a projective resolution of M is a long exact sequence

$$\cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_0 \xrightarrow{\epsilon} M \to 0$$

where F_i is projective for all $i \geq 0$. We will denote it by $\epsilon : F_* \longrightarrow M$.

Definition 1.2.1. *Let G be a group and $\epsilon : F_* \longrightarrow \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ by G-modules. The group homology $H_*(G, \mathbb{Z})$ is defined as $H_*(F_G)$, where $F_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$.*

If M is a G-module, then we define $H_*(G, M) = H_*(F \otimes_{\mathbb{Z}G} M)$. The group cohomology is $H^*(G, M) = H_*(Hom_G(F, M))$.

Remark 1.2.2. *There is a natural isomorphism between the group cohomology of G and the singular cohomology of the classifying space BG, $H^*(G, \mathbb{Z}) \cong H^*(BG, \mathbb{Z})$ (see [Bro12, §I.4]).*

Proposition 1.2.3. *[Bro12, III.(6.1)Proposition] Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of G-modules. Then there exists a long exact sequence:*

$$\cdots \to H^{i-1}(G, M'') \to H^i(G, M') \to H^i(G, M) \to H^i(G, M'') \to H^{i+1}(G, M') \to \cdots$$

Now we give an interpretation of the cohomology groups of dimension 0, 1 and 2.

Remark 1.2.4. *We have*

$$H_0(G, M) = M^G = \{m \in M : gm = m \textit{ for all } g \in G\}.$$

$$H^0(G, M) = M_G = M/IM,$$

where I is the augmentation ideal (that is, the kernel of the map $\mathbb{Z}G \longrightarrow \mathbb{Z}$, which send $\sum_{g \in G} a_g g$ to $\sum_{g \in G} a_g$.)

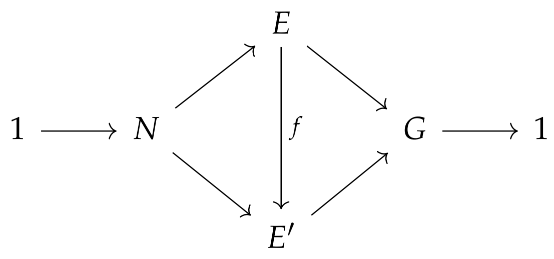
Consider a short exact sequence $1 \to N \to E \to G \to 1$. We have a group morphism $\phi : E \longrightarrow \mathrm{Aut}(N)$ given by $\phi(x) = c_x$ the conjugation by x. If N is abelian then $\phi(n) = Id_N$ for all $n \in N$ and therefore we obtain a new group morphism $\psi : E/N \cong G \longrightarrow \mathrm{Aut}(N)$. Hence, N becomes a G-module.

Notation 1.2.5. *When studying group extensions of $1 \to N \to E \to G \to 1$, we will write the cohomology group $H^i(G, N)$ as $H^i_\psi(G, N)$ to emphasize the structure of G-module of the group N induced by ψ. If ψ is trivial (thus N has the structure of a trivial G-module), then we will omit the subindex ψ.*

Definition 1.2.6. *We say that a short exact sequence $1 \to N \to E \xrightarrow{\pi} G \to 1$ splits if there exists a group morphism $s : G \longrightarrow E$ such that $\pi \circ s = Id_G$.*

Remark 1.2.7. *There always exist set-theoretic sections $s : G \longrightarrow E$, but they are not group morphisms in general.*

Two short exact sequences $1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1$ and $1 \longrightarrow N \longrightarrow E' \longrightarrow G \longrightarrow 1$ are said to be equivalent if there exists an isomorphism $f : E \longrightarrow E'$ such that



Proposition 1.2.8. *[Bro12, IV.(2.1)Proposition] A short exact sequence $1 \to N \to E \to G \to 1$ with N abelian is split if and only if it is equivalent to $1 \to N \to N \rtimes_\psi G \to G \to 1$, where $\psi : G \longrightarrow \mathrm{Aut}(N)$ is the map defined above.*

On the other hand, the extension $1 \to N \to E \to G \to 1$ is central if and only if $\psi = 0$. In consequence, if we have a central split extension, then $E \cong N \times G$.

We say that two splittings $s_1 : G \longrightarrow E$ and $s_2 : G \longrightarrow E$ are conjugate if there exist $n \in N$ such that $s_1(g) = ns_2(g)n^{-1}$ for all $g \in G$.

Theorem 1.2.9. *[Bro12, IV.(2.3)Proposition] Let N be an abelian group, there exists a one to one correspondence between conjugacy classes of splittings of the extension $1 \to N \to N \rtimes_\psi G \to G \to 1$ and the first cohomology group $H^1_\psi(G, N)$.*

We can also describe the first cohomology groups as follows.

Definition 1.2.10. *A crossed homomorphism (or 1-cocycle) is a map $f : G \longrightarrow N$ such that*

$$f(gh) = f(g) + \psi(g)(f(h))$$

for all $g, h \in G$. The set of 1-cocycles is denoted by $Z^1_\psi(G, N)$.

A principal crossed homomorphism (or 1-coboundary) is a crossed homomorphism $f : G \longrightarrow N$ such that there exists $n \in N$ satisfying

$$f(g) = \psi(g)(n) - n$$

for all $g \in G$.

Lemma 1.2.11. *[Bro12, IV.(2.1)Proposition] We have $Z^1_\psi(G, N)/B^1_\psi(G, N) \cong H^1_\psi(G, N)$.*

We say that a set-theoretic section $s : G \longrightarrow E$ is normalized if $s(1) = 1$. Given a normalized set-theoretic section $s : G \longrightarrow E$, we define the abstract kernel of the extension $\psi : G \longrightarrow \mathrm{Aut}(N)$ to be the group morphism $\psi(g) = c_{s(g)}$. We denote by $\mathcal{E}(G, N, \psi)$ the set of equivalence classes of short exact sequences with abstract kernel ψ. Then:

Theorem 1.2.12. *[Bro12, IV.(3.12)Theorem] There exists a one to one correspondence between $\mathcal{E}(G, N, \psi)$ and $H^2_\psi(G, N)$, where the subindex denotes that the G-module structure on N is induced by ψ.*

If N is not abelian, then the abstract kernel constructed as above is not a group morphism in general. But, if we only look at automorphisms up to conjugation, we obtain a group morphism $\psi : G \longrightarrow \mathrm{Out}(N)$ defined as $\psi(g) = [c_{s(g)}]$. In this case, we have the following theorem:

Theorem 1.2.13. *[Bro12, IV.(5.4)Theorem] Given an abstract kernel $\psi : G \longrightarrow \mathrm{Out}(N)$, choose a set-theoretic lift $\tilde{\psi} : G \longrightarrow \mathrm{Aut}(N)$. Then the set of all extensions with abstract kernel ψ is in one to one correspondence with the set of all maps $c : G \times G \longrightarrow N$ satisfying*

$$\tilde{\psi}(g) \circ \tilde{\psi}(h) = c_{c(g,h)} \circ \tilde{\psi}(gh),$$

$$c(g, 1) = 1 = c(1, h),$$

$$c(g, h)c(gh, k) = \tilde{\psi}(g)(c(h, k))c(g, hk).$$

Given a group extension $1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1$ with abstract kernel ψ, we can take a normalized section $s : G \longrightarrow E$ and define the map $c : G \times G \longrightarrow N$ as $c(g,h) = s(g)s(h)s^{-1}(gh)$. It can be easily verified that c is well defined and that it satisfies the conditions on the theorem. Conversely, given c and a lift $\tilde{\psi}$ we can construct the group $N \times_{(\tilde{\psi},c)} G$, with underlying set $N \times G$ and group operation

$$(a,g)(b,h) = (a\tilde{\psi}(g)c(g,h), gh).$$

One can check that we have a short exact sequence $1 \longrightarrow N \longrightarrow N \times_{(\tilde{\psi},c)} G \longrightarrow G \longrightarrow 1$ with abstract kernel ψ.

Theorem 1.2.14. *[Bro12, IV.(6.6)Theorem] If $\mathcal{E}(G,N,\psi) \neq \emptyset$ then we have a one to one correspondence between $\mathcal{E}(G,N,\psi)$ and $H^2_{\psi'}(G,\mathcal{Z}N)$, where $\psi' : G \longrightarrow \operatorname{Aut}(\mathcal{Z}N)$ is obtained by restricting the outer automorphism $\psi(g)$ to the center $\mathcal{Z}N$. Moreover $\mathcal{E}(G,N,\psi)$ is a $H^2_{\psi}(G,\mathcal{Z}N)$-torsor.*

Theorem 1.2.15. *[Bro12, IV.(6.7)Theorem] Any abstract kernel $\psi : G \longrightarrow \operatorname{Out}(N)$ has an associated element $u_\psi \in H^3(G,\mathcal{Z}N)$. Then $\mathcal{E}(G,N,\psi) \neq \emptyset$ if and only if $u_\psi = 0$.*

We end this section with the Lyndon-Hochschild spectral sequence. Given a short exact sequence $1 \to N \to E \xrightarrow{\pi} G \to 1$ and a E-module M then:

Theorem 1.2.16. *[Bro12, VII.(6.3)Theorem] There exists a convergent spectral sequence such that*

$$E_2^{p,q} = H^p(G, H^q(N,M)) \Longrightarrow H^{p+q}(E,M).$$

Corollary 1.2.17. *[Bro12, VII.(6.4)Corollary] We have a long exact sequence*

$$0 \to H^1(G,M^N) \to H^1(E,M) \to H^1(N,M) \to H^1(N,M)^G \to H^2(G,M^N) \to H^2(E,M).$$

1.2.2 Outer automorphism group

The aim of this section is to briefly explain the constructions in [Mal02] used to compute the outer automorphism group of a group extension.

Let

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\ p\ } Q \longrightarrow 1$$

be a short exact sequence of groups. The extension is determined by the morphism $\psi : Q \longrightarrow \operatorname{Out}(K)$ and a 2-cocycle $c \in H^2_\psi(Q,\mathcal{Z}K)$. Let $\operatorname{Aut}(G,K) = \{f \in \operatorname{Aut}(G) : f(K) = K\}$ and $\operatorname{Out}(G,K) = \operatorname{Aut}(G,K)/\operatorname{Inn} G$. Define the group morphism $\Theta : \operatorname{Aut}(G,K) \longrightarrow \operatorname{Aut}(K) \times \operatorname{Aut}(Q)$ such that $f \mapsto (f_{|K}, \overline{f})$, where $\overline{f} : Q \longrightarrow Q$ is the map induced by f on Q. Finally, recall that if H is a subgroup of G, the centralizer $C_G(H)$ is $\{g \in G : c_h(g) = g \text{ for all } h \in H\}$.

Theorem 1.2.18. *[Mal02, Theorem 3.3, Theorem 3.6] With the above notation, we have*

1. $\mathrm{Im}(\Theta)$ *are the elements of* $\mathrm{Aut}(K) \times \mathrm{Aut}(Q)$ *which fix the 2-cocycle* c.
2. *There exists an isomorphism* $\xi : \mathrm{Ker}(\Theta) \longrightarrow Z^1_\psi(Q, \mathcal{Z}K)$.

Definition 1.2.19. *[Mal02, Definition 3.7] We define*

$$\overline{B}^1_\psi(Q, \mathcal{Z}K) = \{\xi(c_g) : g \in C_G(K), p(g) \in \mathcal{Z}Q\}.$$

Proposition 1.2.20. *[Mal02, Proposition 3.8] We have*

$$B^1_\psi(Q, \mathcal{Z}K) \leq \overline{B}^1_\psi(Q, \mathcal{Z}K) \leq Z^1_\psi(Q, \mathcal{Z}K).$$

In consequence, there exists a surjective morphism

$$H^1_\psi(Q, \mathcal{Z}K) \longrightarrow \overline{H}^1_\psi(Q, \mathcal{Z}K) = Z^1_\psi(Q, \mathcal{Z}K)/\overline{B}^1_\psi(Q, \mathcal{Z}K).$$

We have an isomorphism $\overline{B}^1_\psi(Q, \mathcal{Z}K) \cong (p^{-1}(\mathcal{Z}Q) \cap C_G K)/\mathcal{Z}G$.

We can think $\overline{H}^1_\psi(Q, \mathcal{Z}K)$ as the subgroup of $\mathrm{Out}(G, K)$ whose automorphism classes induce inner automorphisms on K and Q. This interpretation is made precise in the next theorem.

Theorem 1.2.21. *[Mal02, Theorem 4.8] There exist short exact sequences*

$$1 \longrightarrow \mathcal{K} \longrightarrow \mathrm{Out}(G, K) \longrightarrow L_1 \longrightarrow 1,$$

and

$$1 \longrightarrow \overline{H}^1_\psi(Q, \mathcal{Z}K) \longrightarrow \mathcal{K} \longrightarrow L_2 \longrightarrow 1$$

where

$$L_1 = \{\overline{f} \in \mathrm{Aut}(Q) : f \in \mathrm{Aut}(G, K)\} / \mathrm{Inn}(Q) \leq \mathrm{Out}(Q)$$

and

$$L_2 \cong (\mathrm{Stab}_{\mathrm{Aut}(K)}\, c / \mathrm{Inn}(K)) / \mathcal{Z}\psi(Q) \leq C_{\mathrm{Out}(K)}\psi(Q)/\psi(\mathcal{Z}Q).$$

There is also a version for the automorphism group.

Theorem 1.2.22. *[Mal02, Theorem 4.8] There exist short exact sequences*

$$1 \longrightarrow \mathcal{K}' \longrightarrow \mathrm{Aut}(G, K) \longrightarrow L'_1 \longrightarrow 1,$$

and

$$1 \longrightarrow Z^1_\psi(Q, \mathcal{Z}K) \longrightarrow \mathcal{K}' \longrightarrow L'_2 \longrightarrow 1$$

where

$$L'_1 = \{\overline{f} \in \mathrm{Aut}(Q) : f \in \mathrm{Aut}(G, K)\} \leq \mathrm{Aut}(Q)$$

and

$$L'_2 = \mathrm{Stab}_{\mathrm{Aut}(K)}\, c \leq \mathrm{Aut}(K).$$

Remark 1.2.23. *If K is a characteristic subgroup of G then* $\mathrm{Aut}(G,K) = \mathrm{Aut}(G)$ *and* $\mathrm{Out}(G,K) = \mathrm{Out}(G)$.

Another result about outer automorphism groups we will extensively use is the following:

Lemma 1.2.24. *Let Γ and Γ' be finitely generated groups with finitely generated center such that $\Gamma' \trianglelefteq \Gamma$ and $\Gamma/\Gamma' = F$ is a finite group. If* $\mathrm{Out}(\Gamma')$ *is a Minkowski, then* $\mathrm{Out}(\Gamma)$ *is Minkowski.*

Proof. By [McC88, Lemma 1.(a)] we know that $[\mathrm{Out}(\Gamma) : \mathrm{Out}(\Gamma,\Gamma')] < \infty$, hence it is enough to prove that $\mathrm{Out}(\Gamma,\Gamma')$ is Minkowski.

By theorem 1.2.21 and lemma 1.1.54, if $\mathrm{Out}(F)$, $C_{\mathrm{Out}(\Gamma')}\psi(F)/\psi(\mathcal{Z}F)$ and $H^1_\psi(F,\mathcal{Z}\Gamma')$ are Minkowski then $\mathrm{Out}(\Gamma,\Gamma')$ is Minkowski. But $\mathrm{Out}(F)$ and $H^1_\psi(F,\mathcal{Z}\Gamma')$ are Minkowski since F is finite and $\mathcal{Z}\Gamma'$ is finitely generated. Finally, the group $C_{\mathrm{Out}(\Gamma')}\psi(F)/\psi(\mathcal{Z}F)$ is Minkowski because $C_{\mathrm{Out}(\Gamma')}\psi(F) \leq \mathrm{Out}(\Gamma')$ is Minkowski by hypothesis and $\psi(\mathcal{Z}F)$ is a finite group, hence we can use the second part of lemma 1.1.54. □

1.3 Aspherical manifolds

The aim of this section is to give an introduction to aspherical manifolds, which are one of the main object of study of this thesis.

Definition 1.3.1. *A connected manifold M is said to be aspherical if its universal cover $\tilde{M}$ is contractible.*

The definition implies that $\pi_i(M) = 0$ for all $i > 1$. There are also some restrictions on the fundamental group of M. For example:

Proposition 1.3.2. *If M is a closed aspherical manifold, then $\pi_1(M)$ is torsion-free.*

Proof. Suppose that $\pi_1(M)$ has torsion. Thus, there exists a prime p such that $\mathbb{Z}/p \leq \pi_1(M)$. Note that the universal cover $\tilde{M}$ is $\mathbb{Z}/p$-acyclic since it is contractible. Since the action of $\pi_1(M)$ on $\tilde{M}$ is free, the set $\tilde{M}^{\mathbb{Z}/p}$ is empty, contradicting theorem 1.1.37. Therefore $\pi_1(M)$ is torsion-free. □

Proposition 1.3.3. *Let M_1 and M_2 be closed manifolds of dimension $n \geq 3$ which are not homotopically equivalent to a sphere. Then $M_1 \# M_2$ is not aspherical.*

See [Lüc09, Lemma 3.2] for a proof. Despite these restrictions there is an abundance of aspherical manifolds. Let us show some ways to construct them.

Firstly, if G is a connected Lie group and K is a maximal compact subgroup of G then G/K homeomorphic to $\mathbb{R}^m$ for some m (see [Hel01, Chapter VI, Theorem 1]). If $\Gamma \leq G$ is a discrete torsion-free subgroup of G then the double coset space $\Gamma \setminus G/K$ is an aspherical manifold. These type of manifolds are known as aspherical locally homogeneous space (or classical aspherical manifolds, see [FJ90]). Closed connected surfaces of genus equal or greater than 2 and closed connected hyperbolic manifolds fall into this class.

There are two other important techniques to construct aspherical manifolds. Although they are not used in the thesis, we briefly review them for the sake of completeness. Firstly, there is the hyperbolization technique due to Gromov (see [Gro87]), which turns a cell complex into a non-positively curved (hence aspherical by the Cartan-Hadamard theorem) polyhedron. The version of this technique for manifolds is the following:

Theorem 1.3.4. *Let M be a closed oriented manifold. There exists a closed aspherical manifold $h(M)$ and a continuous map $f : h(M) \longrightarrow M$ which induces a surjection on the fundamental group and the integral homology.*

The hyperbolization process can be modified to obtain manifolds of negative curvature (the construction is known as strict hyperbolization [CD95]). In this setting, the fundamental group $\pi_1(h(M))$ is a hyperbolic group.

Finally, the Davis' reflection trick is a method to construct closed aspherical manifolds by using Coxeter groups. For a detailed exposition on this topic see [Dav12].

Theorem 1.3.5. *[Dav12, Chapter 9, Chapter 11] Let Γ be a group such that $B\Gamma$ has a finite model. Then there exists a closed aspherical manifold M with fundamental group $\pi_1(M) = \tilde{W} \rtimes \Gamma$, where $\tilde{W}$ is a torsion-free subgroup of a (maybe infinitely generated) Coxeter group.*

Davis' reflection trick was used to construct closed aspherical manifolds with exotic properties.

Theorem 1.3.6. *[Dav12, Chapter 10, Chapter 11] For every $n \geq 4$ there exist:*

(1) A closed connected n-dimensional aspherical manifold whose universal cover is not homeomorphic to $\mathbb{R}^n$.

(2) A closed connected n-dimensional aspherical manifold which is not homotopy equivalent to a PL-manifold.

(3) A closed connected n-dimensional aspherical manifold whose fundamental group is not residually finite.

(4) A closed connected n-dimensional aspherical manifold whose fundamental group contains an infinite divisible abelian group.

(5) A closed connected n-dimensional aspherical manifold whose fundamental group has an unsolvable word problem.

A recent application of Davis' trick is the following:

Theorem 1.3.7. *[DHHRS24] There exists a pair of closed connected aspherical* 4*-manifolds which are homeomorphic but not diffeomorphic.*

Note that a closed aspherical manifold M is an Eilenberg-Maclane space $K(\pi_1(M), 1)$. In consequence, it is natural that the fundamental group plays a crucial role in the understanding the topology of M. This idea is summarised in the next conjecture:

Conjecture 1.3.8. *(Borel conjecture) Homotopy equivalent closed aspherical manifolds are homeomorphic.*

This conjecture has been answered affirmatively for a large class of closed aspherical manifolds in [BL12]. In particular, the Borel conjecture holds for aspherical locally homogeneous spaces $K \setminus G/\Gamma$, where G is a connected Lie group, K is a maximal compact subgroup of G, Γ is a cocompact lattice of G and $\dim(\Gamma \setminus G/K) \neq 4$ (see [KLR16]). Note that the Borel conjecture is not true in the smooth category. There exist manifolds which are homeomorphic but not diffeomorphic to tori in all dimensions equal or bigger than 5, see [HS70]. While, it is not known if there exist manifolds homeomorphic but not diffeomorphic to T^4, theorem 1.3.7 shows that the smooth Borel conjecture is false also in dimension 4.

The rest of this section is devoted to the study of aspherical locally homogeneous spaces and group actions on aspherical manifolds.

1.3.1 Lattices of Lie groups

The aim of this section is to introduce the concept of lattice of a Lie group. In order to do so, we will need to introduce some preliminary notions.

Proposition 1.3.9. *[Mor01b, (A3.1) Proposition] Let G be a Lie group. There exists a unique (up to scalar multiple) measure μ on G such that:*

1. *$\mu(K)$ is finite, for every compact set K of G.*
2. *$\mu(gA) = \mu(A)$ for every Borel subset A of G and $g \in G$.*

The measure μ is called the left Haar measure on G.

Remark 1.3.10. *Analogously, there exists a unique (up to scalar multiple) measure μ' such which satisfies item 1 of proposition 1.3.9 and $\mu'(Ag) = \mu'(A)$ for every Borel subset A of G and $g \in G$. μ' is known as right Haar measure. We say that G is unimodular if the left and right Haar measure coincide, $\mu = \mu'$. For example, if G is simple, then G is unimodular.*

Lemma 1.3.11. *[Mor01b, (A3.8) Proposition] Let μ be the left Haar measure of a Lie group G. Then $\mu(G) < \infty$ if and only if G is compact.*

Using a Haar measure we can define fundamental domains and strict fundamental domains of discrete subgroups of Lie groups. Recall that Γ is a discrete subgroup of G if there exists a neighbourhood U of the identity element e such that $U \cap \Gamma = \{e\}$.

Lemma 1.3.12. *[Mor01b, (4.1.1) Lemma] Let G be a Lie group and Γ a discrete subgroup of G. There exists a Borel subset $\mathcal{F}$ such that the map $\mathcal{F} \longrightarrow G/\Gamma$, defined by $g \mapsto g\Gamma$, is bijective. The set $\mathcal{F}$ is known as a strict fundamental domain for G/Γ in G.*

There exists a Borel subset $\mathcal{F}'$, called fundamental domain for G/Γ in G, such that:

1. *$\mathcal{F}'\Gamma = G$.*
2. *$\mathcal{F}'$ is closed, the interior $\mathcal{F}'^o$ is dense in $\mathcal{F}'$ and $\mathcal{F}' \setminus \mathcal{F}'^o$ has measure 0.*
3. *$\mathcal{F}'\gamma \cap \mathcal{F}'^o = \emptyset$ for all non-trivial elements $\gamma \in \Gamma$.*

Remark 1.3.13. *Given a fundamental domain $\mathcal{F}'$, there exists a strict fundamental domain $\mathcal{F} \subseteq \mathcal{F}'$ such that $\mathcal{F}' \setminus \mathcal{F}$ has measure 0.*

Proposition 1.3.14. *[Mor01b, (4.1.3) Proposition] Let G be a Lie group and $\Gamma \leq G$ a discrete subgroup. There exists a unique (up to scalar multiple) G-invariant measure ν on G/Γ such that given any strict fundamental domain $\mathcal{F}$ for G/Γ in G and any Γ-invariant Borel set A of G we have*

$$\nu(A/\Gamma) = \mu(\mathcal{F} \cap A).$$

We are ready to define lattice of a Lie group.

Definition 1.3.15. *A discrete subgroup Γ of G is a lattice if $\nu(G/\Gamma) < \infty$. A lattice is said to be cocompact if G/Γ is compact.*

Lemma 1.3.16. *[Mor01b, (4.1.11) Proposition] Let G be a Lie group, Γ a discrete subgroup of G and μ the left Haar measure of G. The following are equivalent:*

1. *Γ is a lattice of G.*
2. *There exists a (strict) fundamental domain $\mathcal{F}$ for G/Γ on G such that $\mu(\mathcal{F}) \leq \infty$.*
3. *There exists a Borel subset A of G such that $A\Gamma = G$ and $\mu(A) < \infty$.*

We are interested in the following corollary, which will be extensively used in this thesis.

Corollary 1.3.17. *[Mor01b, (4.1.14) Corollary] Let G be a Lie group. Then:*

1. *Every cocompact discrete subgroup of G is a lattice.*
2. *Every finite index subgroup of a lattice is a lattice.*

Example 1.3.18. *The subgroup $\mathbb{Z}^2$ is a cocompact lattice of $\mathbb{R}^2$. Indeed, $\mathbb{Z}^2$ is clearly discrete, since the open set $(-1/3, 1/3)^2$ satisfies $\mathbb{Z}^2 \cap (-1/3, 1/3)^2 = \{(0,0)\}$. Moreover, $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to the torus T^2, which is compact. By corollary 1.3.17, $\mathbb{Z}^2$ is a lattice of $\mathbb{R}^2$. Note that $[0,1)^2$ is a strict fundamental domain and $[0,1]^2$ is a fundamental domain.*

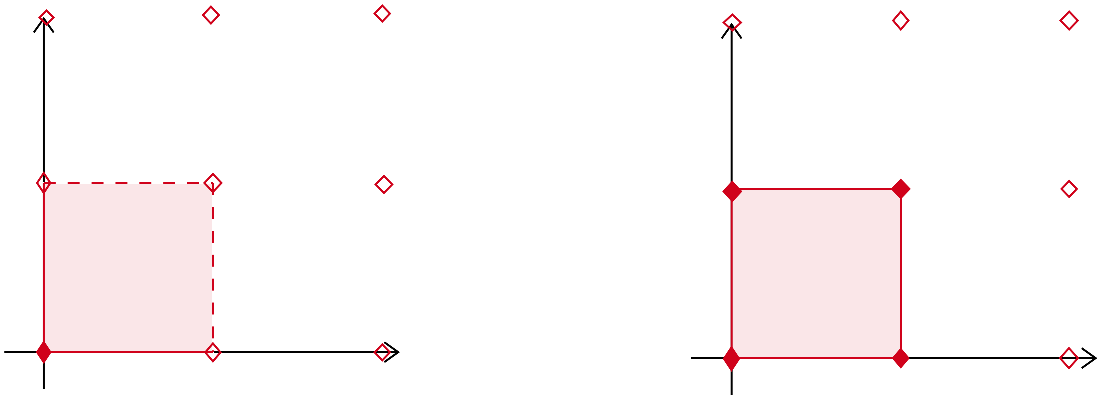


Figure 1.1: Strict fundamental domain and fundamental domain of the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$.

1.3.2 Nilmanifolds

Nilmanifolds were introduced in the seminal paper of Mal'cev [Mal51] and they have seen a wide range of applications in geometry and topology. There are several references that gives a detailed account of nilmanifold properties. For this section we mainly use [Rag12, Chapter 2] and [Dek16].

We start with a fact on homogeneous spaces. Given a homogeneous space X with a transitive action of a Lie group G, we have a transitive action of the universal cover $\tilde{G}$ of G on X. Indeed, we consider the universal covering map $p : \tilde{G} \longrightarrow G$, which is a group morphism. We can define a transitive group action of $\tilde{G}$ on X such that $\tilde{g}x = p(\tilde{g})x$. Consequently, given a homogeneous space X, we can always assume that the Lie group G which acts transitively on X is simply connected.

We now recall the definition and some of the properties of nilpotent groups. Given a group G and two elements $g, h \in G$, we define $[g, h] = ghg^{-1}h^{-1}$. If H and K are subgroups of G, then $[H, K]$ denotes the subgroup generated by elements of the form $[h, k]$ with $h \in H$ and $k \in K$. Recall that given a group G we define the lower central series $G \geq G^1 \trianglelefteq \cdots \trianglelefteq G^i \trianglelefteq \ldots$, where $G^0 = G$, $G^i = [G, G^{i-1}]$.

Similarly, for a Lie algebra $\mathfrak{g}$ and Lie subalgebras $\mathfrak{h}$ and $\mathfrak{k}$, we define $[\mathfrak{h}, \mathfrak{k}]$ to be the Lie subalgebra generated by elements of the form $[h, k]$, where $h \in \mathfrak{h}$ and $k \in \mathfrak{k}$ and $[\cdot, \cdot]$ is the Lie bracket. Given a Lie algebra $\mathfrak{g}$, we define the lower central series $\mathfrak{g} \supseteq \mathfrak{g}^1 \supseteq \cdots \supseteq \mathfrak{g}^i \supseteq \ldots$, where $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$.

Definition 1.3.19. *A group G is nilpotent if there exists a c such that $G^c = \{e\}$. A Lie algebra $\mathfrak{g}$ is nilpotent if there is a c such that $\mathfrak{g}^c = \{0\}$.*

The smallest c such that $G^c = \{e\}$ or $\mathfrak{g}^c = \{0\}$ is the nilpotency class of G or $\mathfrak{g}$. A c-step nilpotent Lie group (resp. Lie algebra) is a nilpotent Lie group (nilpotent Lie algebra) with class of nilpotency *c*.

Remark 1.3.20. *We can also define nilpotency using the upper central series. The upper central series is the subnormal series of subgroups of G* $\{e\} = Z_0 \trianglelefteq Z_1 \trianglelefteq Z_2 \trianglelefteq \cdots$, *where* $Z_{i+1}/Z_i \cong \mathcal{Z}(G/\mathbb{Z}_i)$ *for all* $i \geq 0$. *Then G is nilpotent if and only if* $G = Z_c$ *for some c.*

When we consider Lie groups, we have the following properties:

Lemma 1.3.21. *[Rag12, Chapter I.1.9] A Lie group N is nilpotent if and only if its Lie algebra* $\mathcal{L}(N)$ *is nilpotent. If N is a simply connected nilpotent Lie group then:*

1. *The exponential map* $\exp : \mathcal{L}(N) \longrightarrow N$ *is a diffeomorphism.*
2. *The exists a number n such that N is a Lie subgroup of the group of unipotent* $n \times n$ *matrices with real coefficients,* $\mathcal{U}(n, \mathbb{R})$.

Recall that a unipotent matrix is an upper triangular matrix with ones in the diagonal. We are ready to introduce nilmanifolds and to explain their properties.

Definition 1.3.22. *A nilmanifold M is a smooth manifold which admits a transitive action of a connected nilpotent Lie group N.*

Since M is a homogeneous space, there is a diffeomorphism between M and a coset space N/H. The work of A.Mal'cev (see [Mal51] and [Rag12, Chapter 2]) describes the structure of nilmanifolds:

Theorem 1.3.23. *Let N be a simply connected nilpotent Lie group, then:*

1. ***(Lattices of** N**)** Any lattice* Γ *of N is cocompact.*
2. ***(Structure of nilmanifolds)** If N acts transitively and effectively on a compact manifold M, then* $M \cong N/\Gamma$ *where* Γ *is a lattice of N. In general, a nilmanifold is diffeomorphic to the product of a compact nilmanifold and a simply connected nilpotent Lie group.*
3. ***(Rigidity)** Let* Γ_1 *and* Γ_2 *be lattices of simply connected nilpotent Lie groups* N_1 *and* N_2 *respectively. Assume that we have a group morphism* $f : \Gamma_1 \longrightarrow \Gamma_2$, *then there exists a group morphism* $F : N_1 \longrightarrow N_2$ *such that* $F_{|\Gamma_1} = f$. *In particular, F descends to a map* $N_1/\Gamma_1 \longrightarrow N_2/\Gamma_2$. *If* Γ_1 *and* Γ_2 *are isomorphic then* N_1/Γ_1 *and* N_2/Γ_2 *are diffeomorphic.*
4. ***(Existence of lattices)** N admits a lattice if and only if* $\mathcal{L}(N)$ *has a basis with rational constant structures. That is, there exists a basis* $\{e_1, \ldots, e_n\}$ *such that for every* i, j, k *the numbers* $c^k_{i,j}$ *such that* $[e_i, e_j] = \sum c^k_{i,j} e_k$ *are rational.*
5. ***(Abstract group properties of lattices)** A group* Γ *is a lattice of some simply connected nilpotent Lie group if and only if* Γ *is nilpotent, finitely generated and torsion-free.*

Remark 1.3.24. *By properties 2. and 3. of theorem 1.3.23, given a closed connected nilmanifold M, there exists a unique simply connected nilpotent Lie group N acting transitively and effectively on*

M. If N has nilpotency class c, we will say that $M \cong N/\Gamma$ is a c-step nilmanifold.

Remark 1.3.25. *Since N is contractible, a compact nilmanifold N/Γ is an aspherical manifold with fundamental group Γ.*

Example 1.3.26.

1. *(The abelian case) $\mathbb{R}^n$ is a 1-step simply connected nilpotent Lie group. Any lattice of $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^n$ and therefore the only closed 1-step nilmanifold is the torus T^n.*

2. *(Heisenberg manifolds) The matrix group*

$$H_{2n+1} = \{(x,y,z) = \left(\begin{array}{c|c|c} 1 & x & z \\ \hline 0 & Id_n & y^T \\ \hline 0 & 0 & 1 \end{array}\right) : x, y \in \mathbb{R}^n, z \in \mathbb{R}\}.$$

is called generalized Heisenberg group of dimension $2n+1$. It is straightforward to show that the product of two elements takes the form $(x,y,z)(x',y',z') = (x+x', y+y', z+z'+xy'^T)$ and $(x,y,z)^{-1} = (-x,-y,-z+xy^T)$, which implies that $[(x,y,z),(x',y',z')] = (0,0,-x'y^T+xy'^T)$. In consequence, $[H_n,H_n] = \{(0,0,z) : z \in \mathbb{R}\}$, which is also the center of H_n. This implies that $[H_n,[H_n,H_n]] = \{(0,0,0)\}$ and H_n is 2-step nilpotent. To construct a lattice we can consider the discrete subgroup $\Gamma_{2n+1} = \{(x,y,z) \in H_n : x,y \in \mathbb{Z}^n, z \in \mathbb{Z}\}$. Then the 2-step nilmanifold H_{2n+1}/Γ_{2n+1} of dimension $2n+1$ is called generalized Heisenberg manifold. The case $n=1$ is usually known as Heisenberg manifold.

Generalized Heisenberg manifolds admit non-isomorphic lattices. To construct them we take a vector of positive integers $r = (r_1, \dots, r_n)$ such that r_i divides r_{i+1} for all $1 \le i \le n-1$. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $rx = (r_1x_1, \dots, r_nx_n)$. We now consider a lattice of the form $\Gamma_n(r) = \{(rx,y,z) : x,y \in \mathbb{Z}^n, z \in \mathbb{Z}\}$. It can be seen that every lattice on a generalized Heisenberg manifold takes the form of $\Gamma_n(r)$ for some r and that the coefficients of r uniquely determine $H_n/\Gamma_n(r)$.

3. *(Filiform groups) We construct these groups from their Lie algebras. An n-dimensional nilpotent Lie algebra $\mathfrak{f}_n$ is said to be filiform if $\dim \mathfrak{f}_n^i = n-i-1$, $1 \le i \le n-1$. The simplest way to construct an example of filiform Lie algebra is by taking a basis $\{X_1, \dots, X_n\}$ of a n-dimensional vector space and defining the Lie bracket by the relations $[X_1, X_i] = X_{i+1}$ with $1 \le i \le n-1$, where the undefined brackets are 0 except the brackets obtained by antisymmetry. The Lie algebra $\mathfrak{f}_n$ is called the standard filiform algebra. We can describe $\mathfrak{f}_n$ with matrices*

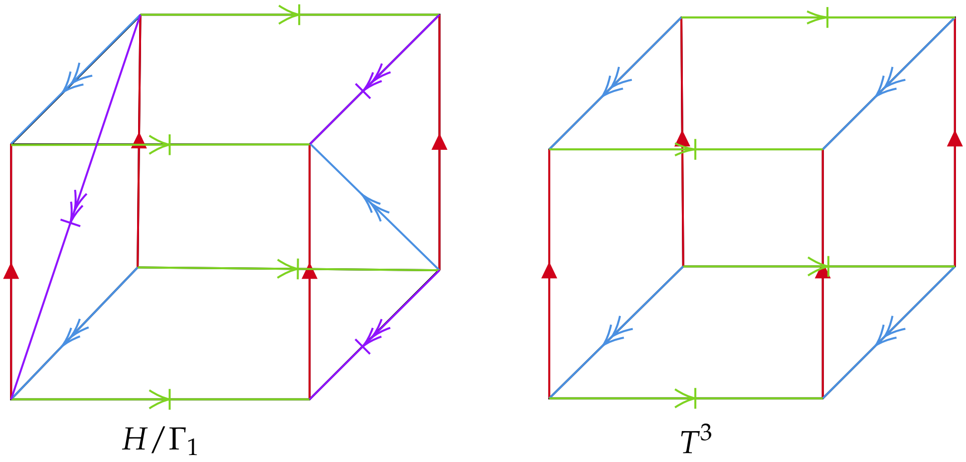


Figure 1.2: 3-dimensional Heisenberg manifold and torus

in the following way. We consider the $n-1 \times n-1$ *matrix*

$$\tau(x) = \begin{pmatrix} 0 & x & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $x \in \mathbb{R}$ *and we denote the canonical basis of* $\mathbb{R}^{n-1}$ *by* $e_1, \ldots, e_{n-1}$, *then*

$$\mathfrak{f}_n = \{ \left(\begin{array}{c|c} \tau(x) & v^T \\ \hline 0 & 0 \end{array} \right) : x \in \mathbb{R}, v \in \mathbb{R}^{n-1} \}$$

and

$$X_1 = \left(\begin{array}{c|c} \tau(1) & 0 \\ \hline 0 & 0 \end{array} \right) \text{ and } X_i = \left(\begin{array}{c|c} \tau(0) & e^T_{n+1-i} \\ \hline 0 & 0 \end{array} \right)$$

for $2 \leq i \leq n$. *By using the exponential map and the Baker-Campbell-Hausdorff formula we can describe explicitly the simply connected nilpotent Lie group* F_n *corresponding to* $\mathfrak{f}_n$. *Firstly, we define the* $n-1 \times n-1$ *matrix*

$$\sigma(x) = \begin{pmatrix} 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^{n-3}}{(n-3)!} & \frac{x^{n-2}}{(n-2)!} \\ 0 & 1 & x & \cdots & \frac{x^{n-4}}{(n-4)!} & \frac{x^{n-3}}{(n-3)!} \\ 0 & 0 & 1 & \cdots & \frac{x^{n-5}}{(n-5)!} & \frac{x^{n-4}}{(n-4)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then

$$F_n = \{ \left(\begin{array}{c|c} \sigma(x) & v^T \\ \hline 0 & 1 \end{array} \right) : x \in \mathbb{R}, v \in \mathbb{R}^{n-1} \}.$$

Note that $F_n = \mathbb{R}^{n-1} \rtimes_\sigma \mathbb{R}$, *where* $\sigma : \mathbb{R} \longrightarrow \mathrm{GL}(n-1, \mathbb{R})$ *sends* $x \in \mathbb{R}$ *to* $\sigma(x)$.

If we want to obtain a lattice, we can consider the $\mathbb{Z}$*-span of* $X_1, \ldots, X_n$ *in* $\mathfrak{f}_n$ *and then take* $\Gamma = \exp(\langle X_1, \ldots, X_n \rangle_{\mathbb{Z}})$. *The resulting nilmanifold* F_n/Γ *is a* $n-1$*-step nilmanifold of dimension* n.

The rest of this subsection is devoted to explain in more detail some of the tools used in the proof of the statements of theorem 1.3.23, which are also relevant to prove some of the results of this thesis.

Mal'cev completion

Given a finitely generated torsion-free nilpotent group Γ one can refine the upper central series to obtain a series

$$\Gamma = \Gamma_0 \trianglerighteq \Gamma_1 \trianglerighteq \cdots \trianglerighteq \Gamma_k = \{e\}$$

such that $\Gamma_i/\Gamma_{i+1} = \mathbb{Z}$ and $\Gamma_i/\Gamma_{i+1} \leq \mathcal{Z}(\Gamma/\Gamma_{i+1})$ for all $i \geq 0$. Now, we can choose an element $a_{i+1} \in \Gamma_i$ which projects to a generator of Γ_i/Γ_{i+1}. The set $\{a_1, \ldots, a_k\}$ generates Γ and satisfies that any $x \in \Gamma$ can be uniquely expressed as an element of the form

$$x = a_1^{z_1} a_2^{z_2} \cdots a_k^{z_k}.$$

The set $\{a_1, \ldots, a_k\}$ is known as the Mal'cev basis of Γ.

Proposition 1.3.27. *[Dek16, Proposition 2.4] Let* Γ *be a finitely generated torsion-free nilpotent group with Mal'cev basis* $\{a_1, \ldots, a_k\}$. *There exist polynomials* $p_i(x_1, x_2, \ldots, x_{i-1}, y_1, y_2, \ldots, y_{i-1})$ *for* $2 \leq i \leq k$ *with coefficients in* $\mathbb{Q}$ *such that if* $x = a_1^{x_1} \cdots a_k^{x_k}$ *and* $y = a_1^{y_1} \cdots a_k^{y_k}$ *then*

$$xy = a_1^{x_1+y_1} a_2^{x_2+y_2+p_2(x_1,y_1)} \cdots a_k^{x_k+y_k+p_k(x_1,\ldots,x_k,y_1,\ldots,y_k)}.$$

Using these polynomials we can define two new groups, $\Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{R}}$, known as the rational and real Mal'cev completion. These groups consist formally of elements of the form $a_1^{x_1} \ldots a_k^{x_k}$ with $x_i \in \mathbb{Q}$ or $x_i \in \mathbb{R}$ respectively. Both groups $\Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{R}}$ are torsion-free nilpotent groups, but they are not finitely generated. The real Mal'cev completion $\Gamma_{\mathbb{R}}$ is a simply connected nilpotent Lie group containing Γ as a lattice. We also have an extension property for the rational Mal'cev completion.

Proposition 1.3.28. *[Dek16, Proposition 2.5] Let* $f : \Gamma_1 \longrightarrow \Gamma_2$ *be a group morphism between torsion-free finitely generated nilpotent groups. Then there exists a unique group morphism* $f_{\mathbb{Q}} : \Gamma_{1_{\mathbb{Q}}} \longrightarrow \Gamma_{2_{\mathbb{Q}}}$ *such that* $f_{\mathbb{Q}|_{\Gamma_1}} = f$.

Note that if f is an isomorphism so it is $f_{\mathbb{Q}}$. To announce the next proposition we need the following group-theoretic definition.

Definition 1.3.29. *Two groups G and G' are said to be commensurable if there exist finite index subgroups $H \leq G$ and $H' \leq G'$ such that $H \cong H'$.*

Proposition 1.3.30. *[Dek16, Proposition 2.7] Let Γ_1 and Γ_2 be torsion-free finitely generated nilpotent groups. Then Γ_1 and Γ_2 are commensurable if and only if ${\Gamma_1}_{\mathbb{Q}} \cong {\Gamma_2}_{\mathbb{Q}}$.*

We want to use Lie algebras to study $\Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{R}}$. In order to do so, we first need the next proposition.

Proposition 1.3.31. *[Dek16, Theorem 2.10] Let Γ be a finitely generated torsion-free nilpotent group. Then there exists an embedding of Γ to the group of upper triangular matrices with 1s in the diagonal and integer coefficients, $\Gamma \longrightarrow \mathcal{U}(n, \mathbb{Z})$.*

Hence, we can assume that $\Gamma \leq \mathcal{U}(n, \mathbb{Z}) \leq \mathcal{U}(n, \mathbb{Q}) \leq \mathcal{U}(n, \mathbb{R})$. The group $\mathcal{U}(n, \mathbb{R})$ is a real simply connected nilpotent matrix group and hence we have bijections $\exp : \mathcal{L}(\mathcal{U}(n, \mathbb{R})) \longrightarrow \mathcal{U}(n, \mathbb{R})$ and $\log : \mathcal{U}(n, \mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{U}(n, \mathbb{R}))$ where $\mathcal{L}(\mathcal{U}(n, \mathbb{R}))$ is the Lie algebra of $\mathcal{U}(n, \mathbb{R})$, which can be described as the upper triangular matrices with 0s in the diagonal and Lie bracket $[A, B] = AB - BA$. The exponential and logarithm map can be explicitly described as

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$$

and

$$\log(A) = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i} (A_i - Id)^i.$$

Since $\mathcal{U}(n, \mathbb{R})$ is nilpotent both sums are finite. The Campbell-Backer-Hausdorff formula gives a relationship between the Lie bracket in $\mathcal{L}(\mathcal{U}(n, \mathbb{R}))$ and the matrix multiplication in $\mathcal{U}(n, \mathbb{R})$. Given $A, B \in \mathcal{L}(\mathcal{U}(n, \mathbb{R}))$, we have

$$\exp(A)\exp(B) = \exp(A + B + \tfrac{1}{2}[A, B] + \tfrac{1}{12}([A, [A, B]] + [B, [B, A]]) + \sum_{i=4}^{\infty} r_i)$$

where r_i is a rational combination of i-fold Lie brackets in A and B. Note that the sum also becomes finite. Since all coefficients are rational, we obtain:

Proposition 1.3.32. *[Dek16, Theorem 2.11, Theorem 2.12] Let $\Gamma \leq \mathcal{U}(n, \mathbb{Z})$ and let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$. Define $\mathcal{L}(\Gamma)_{\mathbb{F}} = \langle \log(\Gamma) \rangle_{\mathbb{F}}$, the $\mathbb{F}$-vector space generated by $\log(\Gamma)$ in $\mathcal{L}(\mathcal{U}(n, \mathbb{R}))$. Then $\mathcal{L}(\Gamma)_{\mathbb{F}}$ is a Lie algebra over $\mathbb{F}$ and $\Gamma_{\mathbb{F}} = \exp(\mathcal{L}(\Gamma)_{\mathbb{F}})$.*

Moreover, given a group morphism $f : \Gamma_1 \longrightarrow \Gamma_2$ between finitely torsion-free nilpotent group, *there is a unique Lie algebra morphism* $f_* : \mathcal{L}(\Gamma_1)_{\mathbb{F}} \longrightarrow \mathcal{L}(\Gamma_2)_{\mathbb{F}}$ *such that*

$$\begin{array}{ccc} {\Gamma_1}_{\mathbb{F}} & \xrightarrow{f_{\mathbb{F}}} & {\Gamma_2}_{\mathbb{F}} \\ \exp \Big\uparrow \Big\downarrow \log & & \exp \Big\uparrow \Big\downarrow \log \\ \mathcal{L}(\Gamma_1)_{\mathbb{F}} & \xrightarrow{f_*} & \mathcal{L}(\Gamma_2)_{\mathbb{F}} \end{array}$$

commutes. Conversely, given any $f_* : \mathcal{L}(\Gamma_1)_{\mathbb{F}} \longrightarrow \mathcal{L}(\Gamma_2)_{\mathbb{F}}$ *there exists a unique* $f_{\mathbb{F}} : {\Gamma_1}_{\mathbb{F}} \longrightarrow {\Gamma_2}_{\mathbb{F}}$ *making the above diagram commutative.*

Note that under this correspondence $f_{\mathbb{F}}$ is an isomorphisms if and only if f_* is a Lie algebra isomorphism. In consequence, we obtain

Corollary 1.3.33. *[Dek16, Corollary 2.13] Let* Γ *be a finitely generated torsion-free nilpotent group. Then* $\mathrm{Aut}(\Gamma_{\mathbb{F}}) = \mathrm{Aut}(\mathcal{L}(\Gamma)_{\mathbb{F}})$ *with* $\mathbb{F} = \mathbb{Q}$ *or* $\mathbb{R}$.

Nilmanifolds and coverings

Let G/H be a homogeneous space and $p : M' \longrightarrow G/H$ a covering. Then M' is also a homogeneous space since it admits a transitive action of the universal cover $\tilde{G}$ of G. If the homogeneous space is a nilmanifold N/Γ then M' is also a nilmanifold with a transitive of N, hence $M' \cong N/\Gamma'$.

If N/Γ is compact and the covering is finite then Γ' is a lattice of N. In this case the covering map $p : N/\Gamma' \longrightarrow N/\Gamma$ induces an injective map $p_* : \Gamma' \longrightarrow \Gamma$. The unique lift $p_{\mathbb{R}} : \Gamma'_{\mathbb{R}} \longrightarrow \Gamma_{\mathbb{R}}$ is the identity map and $\Gamma_R \cong N$. Thus, we obtain a new covering map of the form $q : N/\Gamma' \longrightarrow N/\Gamma$ such that $q(n\Gamma') = n\Gamma$. Note that p and q are homotopy equivalent. In general:

Proposition 1.3.34. *[Bel03, Proposition 5.4] Let* $f : N'/\Gamma' \longrightarrow N/\Gamma$ *be a non-zero degree map between nilmanifolds. Then* f *is homotopy equivalent to a covering.*

In particular, N and N' are isomorphic. An interesting question is to determine if given lattices $\Gamma, \Gamma' \leq N$, there exists a non-zero degree map $f : N/\Gamma \longrightarrow N'/\Gamma$. By proposition 1.3.30 and proposition 1.3.34, a necessary condition is $\Gamma_{\mathbb{Q}} \cong {\Gamma'}_{\mathbb{Q}}$.

Definition 1.3.35. *A group* G *is Hopfian if any surjective group morphism* $G \longrightarrow G$ *is an isomorphism. A group* G *is co-Hopfian if any injective morphism* $G \longrightarrow G$ *is an isomorphism. A group* G *is called compressible if any subgroup* $H \leq G$ *of finite index contains a subgroup* G_H *of finite index in* H *isomorphic to* G.

Any finitely generated torsion-free nilpotent group is Hopfian (see [Bel03, Proposition 5.1]).

Note that if a finitely generated torsion-free nilpotent group Γ is co-Hopfian then any non-zero degree map $f : N/\Gamma \longrightarrow N/\Gamma$ is homotopic to a diffeomorphism. We have the following criteria in order to check when Γ is co-Hopfian.

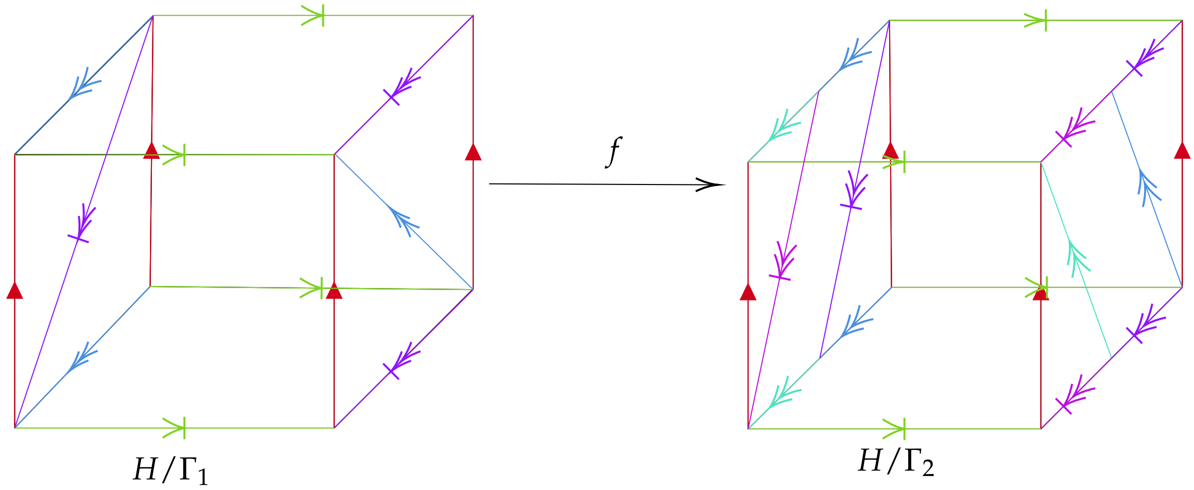


Figure 1.3: 2-covering of Heisenberg manifolds

Proposition 1.3.36. *[Bel03, Theorem 1.1] A finitely generated torsion-free nilpotent group* Γ *is co-Hopfian if and only if any Lie algebra automorphism of* $f : \mathcal{L}(\Gamma)_{\mathbb{Q}} \longrightarrow \mathcal{L}(\Gamma)_{\mathbb{Q}}$ *which maps* $\log(\Gamma)$ *into itself satisfies* $|\det(f)| = 1$.

Proposition 1.3.37. *[Smi85, Theorem A] Let* Γ *and* Γ' *be finitely generated torsion-free nilpotent groups such that* $\Gamma_{\mathbb{Q}} \cong \Gamma'_{\mathbb{Q}}$. *Then* Γ *is compressible if and only if* Γ' *is compressible.*

The next proposition implies that 2-step finitely generated torsion-free nilpotent groups are not co-Hopfian.

Proposition 1.3.38. *[Smi85, Proposition 2] Any 2-step nilpotent finitely generated torsion-free nilpotent group is compressible.*

Iterated principal S^1-bundles

Recall that we denote the upper central series of a group G by $\{e\} = Z_0 \trianglelefteq Z_1 \trianglelefteq Z_2 \trianglelefteq \cdots \trianglelefteq Z_i \trianglelefteq \cdots$, where $Z_1 = \mathcal{Z}G$ and $Z_{i+1}/Z_i = \mathcal{Z}(G_{i+1}/Z_i)$. We consider a c-step nilmanifold N/Γ. First we note that $\mathcal{Z}\Gamma = \mathcal{Z}N \cap \Gamma$. The inclusion $\mathcal{Z}\Gamma \subseteq \mathcal{Z}N \cap \Gamma$ is clear. To prove the other direction we use that any automorphism of Γ lifts uniquely to an automorphism of N. Then by lifting the conjugation automorphism $c_\gamma : \Gamma \longrightarrow \Gamma$ where $\gamma \in \mathcal{Z}\Gamma$ we obtain that $\gamma \in \mathcal{Z}N$. Thus the other inclusion also holds.

The action by left multiplication of $\mathcal{Z}N$ on N descends to a free $T^n = \mathcal{Z}N/\mathcal{Z}\Gamma$ action on N/Γ, where the quotient is the $c-1$ step nilmanifold $(N/\mathcal{Z}N)/(\Gamma/\mathcal{Z}\Gamma)$. Taking a subgroup $S^1 \leq T^n$ we obtain that N/Γ is the total space of a principal S^1-bundle over a nilmanifold. By repeating this process with the base nilmanifold, we have seen the 'if' part of the next theorem:

Theorem 1.3.39. *A compact manifold is the total space of an iterated principal circle bundle if and *only if it is a nilmanifold.*

For the 'only if' part, let $p : E \longrightarrow N'/\Gamma'$ be a principal S^1-bundle over a nilmanifold. By using the long exact sequence of homotopy, we can conclude the that E is aspherical and its fundamental group fits into the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(E) \longrightarrow \Gamma' \longrightarrow 1.$$

In particular $\Gamma = \pi_1(E)$ is a finitely generated torsion-free nilpotent group and therefore it is the fundamental group of a compact nilmanifold N/Γ, which is the total space of a principal S^1-bundle over $q : N/\Gamma \longrightarrow N'/\Gamma'$. By construction the first Chern classes $c_1(p : E \longrightarrow N'/\Gamma')$ and $c_1(q : N/\Gamma \longrightarrow N'/\Gamma')$ are equal, which implies that the principal S^1-bundles are isomorphic. This isomorphism provides the desired diffeomorphism between E and N/Γ.

2-step nilmanifolds

The aim of this part is to study principal torus bundle over a torus. These spaces are precisely compact nilmanifolds where the transitive nilpotent Lie group is 2-step nilpotent (see [PS61, Jak74, Bel20]). Let $T^f \hookrightarrow Y \xrightarrow{p} T^b$ be a principal bundle. We first recall how to construct a nilpotent Lie group acting transitively on Y. From the exact sequence of homotopy we obtain a central exact sequence

$$1 \longrightarrow \mathbb{Z}^f \longrightarrow \pi_1(Y) = \Gamma \longrightarrow \mathbb{Z}^b \longrightarrow 1.$$

The abstract kernel $\phi : \mathbb{Z}^b \longrightarrow \mathrm{Aut}(\mathbb{Z}^f)$ is trivial and Γ is determined by a cohomology class $c \in H^2(\mathbb{Z}^b, \mathbb{Z}^f) \cong H^2(T^b, \mathbb{Z}^f) \cong (H^2(T^b, \mathbb{Z}))^f$. Thus we can write $c = (c^1, \dots, c^f)$ with $c^i \in H^2(T^b, \mathbb{Z})$. Each c^i can be thought as the first Chern class of the circle bundle $S^1 \hookrightarrow Y/T_i^f \longrightarrow T^b$ where $T_i^f = \{(\theta_1, \dots, \theta_f) \in T^f : \theta_i = 0\}$.

Let $\{e_1, \dots, e_b\}$ be a basis of $H^1(T^b, \mathbb{Z})$. Since $H^2(T^b, \mathbb{Z}) = \bigwedge^2 H^1(T^b, \mathbb{Z})$ we have $c^i = \sum_{1 \le j < k \le b} c^i_{jk} e_j \wedge e_k$. Hence we can define coefficients d^i_{jk} for all $1 \le j, k \le b$ and $1 \le i \le f$ as $d^i_{jk} = c^i_{jk}$ if $j < k$, $d^i_{jk} = -c^i_{kj}$ if $j > k$ and $d^i_{jk} = 0$ if $j = k$.

We can construct a Lie algebra $\mathcal{L}(Y)$ generated by $b + f$ elements $\{x_1, \dots, x_b, y_1, \dots, y_f\}$ and a Lie bracket which satisfies that $[x_j, x_k] = \sum_{i=1}^f d^i_{jk} y_i$, $[x_j, y_k] = 0$ and $[y_j, y_k] = 0$ for all j, k. It is straightforward to check that $\mathcal{L}(Y)$ is a 2-step nilpotent Lie algebra. Then $\exp(\mathcal{L}(Y)) = N(Y)$ is a 2-step nilpotent simply connected Lie group which acts transitively on Y.

Cohomology of nilmanifolds

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ of characteristic 0. We define the Chevalley-Eilenberg cochain complex $(C^*(\mathfrak{g}), d)$ as follows. We set $C^k = \mathrm{Hom}(\bigwedge^k \mathfrak{g}, \mathbb{F})$. The differential d :

$C^1 \longrightarrow C^2$ is the dual of the Lie bracket $[\cdot,\cdot] : \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$. Using linearity and the Leibniz rule we can uniquely define the differential d, which satisfies that for an alternating form $f : \mathfrak{g}^n \longrightarrow \mathbb{F}$,

$$df(x_1, \ldots, x_n) = \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n).$$

where the caret signifies omitting that argument.

It is straightforward to prove that $d^2 = 0$. Therefore, we can define the Lie algebra cohomology $H^*(\mathfrak{g}, \mathbb{F}) = \operatorname{Ker} d / \operatorname{Im} d$. The cohomology of nilmanifolds can be computed using Lie algebra cohomology.

Theorem 1.3.40. *[Nom54] Let N/Γ be a closed nilmanifold. Then $H^*(N/\Gamma, \mathbb{Q}) \cong H^*(\mathcal{L}(\Gamma)_{\mathbb{Q}}, \mathbb{Q})$ and $H^*(N/\Gamma, \mathbb{R}) \cong H^*(\mathcal{L}(N), \mathbb{R})$.*

Note that the integer cohomology of a nilmanifold $H^*(N/\Gamma, \mathbb{Z})$ depends on the lattice. The nilmanifolds from example 1.3.26(1) have first homology group $H_1(H_3/\Gamma_3(r), \mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/r$. Therefore $H^*(H_3/\Gamma_3(r), \mathbb{Z}) \ncong H^*(H_3/\Gamma_3(r'), \mathbb{Z})$ if $r \neq r'$.

On the other hand $H^*(N/\Gamma, \mathbb{Q}) \cong H^*(N/\Gamma', \mathbb{Q})$ if Γ and Γ' are commensurable. Finally, $H^*(N/\Gamma, \mathbb{R})$ only depends on the Lie group N.

1.3.3 Solvmanifolds

Solvmanifolds are a generalization of nilmanifolds. In this section we give a brief introduction to the properties of solvmanifolds. The main references used in this section are [Rag12, Chapter III, Chapter IV] and [LR10, Chapter 6].

Definition 1.3.41. *A group S is solvable if there exists a number c such that $S^{(c)} = \{e\}$, where $S = S^{(0)}$ and $S^{(i)} = [S^{(i-1)}, S^{(i-1)}]$ for all $i > 0$. A Lie algebra $\mathfrak{s}$ is solvable if there exists a number c such that $\mathfrak{s}^{(c)} = \{0\}$, where $\mathfrak{s} = \mathfrak{s}^{(0)}$ and $\mathfrak{s}^{(i)} = [\mathfrak{s}^{(i-1)}, \mathfrak{s}^{(i-1)}]$ for all $i > 0$.*

Remark 1.3.42. *All nilpotent groups are solvable.*

If S is a Lie group, the conditions of S being solvable as an abstract group and of its Lie algebra $\mathfrak{s}$ being solvable are equivalent. Solvable Lie groups are more complicated than nilpotent Lie groups. For example, the exponential map $\exp : \mathfrak{s} \longrightarrow S$ of a simply connected solvable Lie group S is not bijective in general. However, there are some types of solvable Lie groups which resemble nilpotent Lie groups.

Definition 1.3.43. *Let S be a simply connected solvable Lie group, then:*

1. *S is of type (R) (also known as completely solvable or triangular) if all the eigenvalues of the linear map $ad(g) : \mathfrak{s} \longrightarrow \mathfrak{s}$ are real for all $g \in S$.*

2. *S is of type (E) if the exponential map $\exp : \mathfrak{s} \longrightarrow S$ is a diffeomorphism.*

Remark 1.3.44. *Let G be a simply connected Lie group, then we have the chain of implications*

$$G \text{ nilpotent} \Rightarrow G \text{ of type (R)} \Rightarrow G \text{ of type (E)}$$

Lemma 1.3.45. *[Rag12, Chapter I.1.9] If S is a simply connected solvable Lie group of dimension n, then S is diffeomorphic to $\mathbb{R}^n$.*

Definition 1.3.46. *A manifold M is a solvmanifold if there exists a connected solvable Lie group S acting transitively on M.*

Recall that we do not lose any generality if we assume that S is simply connected.

A group Γ is polycyclic if there exist a series $\Gamma = \Gamma_0 \trianglerighteq \Gamma_1 \trianglerighteq \cdots \trianglerighteq \Gamma_{n-1} \trianglerighteq \Gamma_n = \{e\}$ such that Γ_i/Γ_{i+1} is cyclic for all i, and it is strongly polycyclic if $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$ for all i. An equivalent definition is that a group is polycyclic if and only if it is solvable and all its subgroups are finitely generated. The next theorem summarises the most important results on solvmanifolds (see [Rag12, Chapter III, Chapter IV]):

Theorem 1.3.47.

1. ***(Lattices of solvable Lie groups)*** *Let S be a simply connected solvable Lie group and $\Gamma \leq S$ a lattice, then Γ is cocompact.*
2. ***(Lattices and compact solvmanifolds)*** *Let M be compact solvmanifold, then there exists a finite cover $p : S/\Gamma \longrightarrow M$, where S is a simply connected solvable Lie group and Γ is a lattice of S.*
3. ***(Structure of non-compact solvmanifolds)*** *A non-compact solvmanifold is the total space of a vector bundle over a compact solvmanifold.*
4. ***(Rigidity)*** *Let Γ_1 and Γ_2 be lattices of simply connected solvable Lie group S_1 and S_2 respectively. If $\phi : \Gamma_1 \longrightarrow \Gamma_2$ is an isomorphism, then there exists a diffeomorphism $\tilde{\phi} : S_1 \longrightarrow S_2$ such that $\tilde{\phi}_{|\Gamma_1} = \phi$ and $\tilde{\phi}(s\gamma) = \tilde{\phi}(s)\phi(\gamma)$ for any $s \in S_1$ and $\gamma \in \Gamma_1$. In particular, if two solvmanifolds have the same fundamental group then they are diffeomorphic.*
5. ***(Mostow fibration)*** *Let N be the nilradical of a simply connected solvable Lie group S (the maximal connected nilpotent Lie subgroup of S). If Γ is a lattice of S, then $N \cap \Gamma$ is a lattice of N and we have the homogeneous fibration:*

$$N/N \cap \Gamma \cong N\Gamma/\Gamma \longrightarrow S/\Gamma \longrightarrow S/N\Gamma \cong T^r.$$

6. ***(Lattices as abstract groups)*** *Let S be a simply connected solvable Lie group and $\Gamma \leq S$ a lattice, then Γ is strongly polycyclic. Conversely, any strongly polycyclic group has a finite index subgroup which is a lattice of a simply connected solvable Lie group.*

Remark 1.3.48. *Unlike nilmanifolds, not all solvmanifolds are coset spaces of a simply connected solvable Lie group by a lattice. The Klein bottle is one of such examples. Recall that the Klein bottle* *is $K = \mathbb{R}^2/\sim$, where $(x,y) \sim (x+n, (-1)^n y + m)$ for any $(n,m) \in \mathbb{Z}^2$. We can write it as a solvmanifold in the following way; let $S = \mathbb{C} \rtimes_\phi \mathbb{R}$, where $\phi : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathbb{C})$ with $\phi(t) = e^{\pi i t}$ so that $(w,t)(z,s) = (e^{i\pi t}z + w, t+s)$. Let $H = \{(p+iu, q) : p,q \in \mathbb{Z}, u \in \mathbb{R}\}$. We have $S/H \cong K$. If K was of the form S'/Γ where S' is a simply connected solvable Lie group and Γ a lattice of S', then K would be parallelizable and therefore orientable, but K is not orientable.*

Remark 1.3.49. *The diffeomorphism provided by theorem 1.3.47(4) is not in general an isomorphism. For example, let $S = \mathbb{C} \rtimes_\phi \mathbb{R}$ where $\phi : \mathbb{R} \longrightarrow Aut(\mathbb{C})$ is given by $\phi(t)z = e^{2\pi i t}z$. Consider $\Gamma = \{(n_1 + in_2, m) : n_1, n_2, m \in \mathbb{Z}\}$. Then $S/\Gamma \cong T^3$, and $\Gamma \cong \mathbb{Z}^3$. However, we cannot extend this isomorphism to an isomorphisms between S and $\mathbb{R}^3$, since S is not abelian.*

However, if S is a simply connected solvable Lie group of type (R), any automorphism between $\phi : \Gamma \longrightarrow \Gamma$ of a lattice Γ lifts to an isomorphism $\tilde{\phi} : S \longrightarrow S$.

The Mostow fibration motivates the next definition.

Definition 1.3.50. *[LR10, Definition 9.5.1.] A Mostow-Wang group Γ is a group Γ which fits in the short exact sequence $1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \mathbb{Z}^k \longrightarrow 1$ where Δ is torsion-free finitely generated and nilpotent. Moreover, we say that Γ is predivisible if Γ is Mostow-Wang and for every $\gamma \in \Gamma$ the automorphism $c_{\gamma *} : \mathcal{L}(\Delta_\mathbb{R}) \longrightarrow \mathcal{L}(\Delta_\mathbb{R})$ induced by the conjugation of $\gamma \in \Gamma$ on the Lie algebra of real Mal'cev completion satisfies that all its eigenvalues λ satisfy $\frac{\lambda}{|\lambda|} = \cos(2\pi\rho) + i\sin(2\pi\rho)$ with $\rho = 0$ or irrational.*

Predivisible groups will be useful because of the next theorem:

Theorem 1.3.51. *[LR10, §9.5] Any torsion-free polycyclic group contains a characteristic Mostow-Wang group of finite index. Any Mostow-Wang group contains a characteristic predivisible group of finite index.*

For a predivisible group Γ, there exists a connected solvable Lie group $G = S \rtimes K$ satisfying:

1. *S is a closed normal subgroup of G.*
2. *K is a torus and a maximal compact subgroup of G.*
3. *Γ is a lattice of G.*
4. *For any $f \in \mathrm{Aut}(\Gamma)$ there exists a unique $\tilde{f} \in \mathrm{Aut}(G)$ such that $\tilde{f}_{|\Gamma} = f$.*

The cohomology of solvmanifolds is harder to understand and to compute than the cohomology of nilmanifolds. For some results about the cohomology solvmanifolds we refer to [CF11, Wit95].

1.3.4 Flat manifolds and infranilmanifolds

In this subsection we review some properties of manifolds finitely covered by tori and nilmanifolds. They are aspherical and have remarkable geometrical properties, but we are mostly interested in the group theoretic properties of their fundamental group. For this section we refer to the books [Cha12, Dek06, Szc12], as well as the survey [Dek16] and [LR10, Chapter 8].

Let $E(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ denote the group of isometries of $\mathbb{R}^n$. An element $g \in E(n)$ is of the form $g = (a, A)$ and the action of $E(n)$ on $\mathbb{R}^n$ satisfies that $gx = Ax + a$ for all $g \in E(n)$ and $x \in \mathbb{R}^n$. There are group morphisms $r : E(n) \longrightarrow \mathrm{O}(n)$ and $t : \mathbb{R}^n \longrightarrow E(n)$ such that $r(g) = A$ and $t(a) = (a, Id)$. Cocompact lattices Γ of $E(n)$ are called crystallographic groups. The next classical theorem due to Bieberbach describes the structure of crystallographic groups (see [LR10, Chapter 8], [Dek06, §2.1] or [Szc12, Chapter 2]).

Theorem 1.3.52. *Let Γ be a crystallographic group. We have:*

1. *(1st Bieberbach theorem) The group $\Lambda = \Gamma \cap \mathbb{R}^n$ is a lattice of $\mathbb{R}^n$ (in particular, $\Lambda \cong \mathbb{Z}^n$), Λ is the unique maximal normal abelian subgroup of Γ and Γ/Λ is a finite subgroup of $\mathrm{O}(n)$, which is called the holonomy of Γ.*
2. *(2nd Bieberbach theorem) Assume that Γ' is crystallographic group isomorphic to Γ. Then any isomorphism $\Gamma \longrightarrow \Gamma'$ is given by a conjugation by an element of $\mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$.*
3. *(3rd Bieberbach theorem) In each dimension n, there are only finitely many crystallographic groups up to isomorphism.*

There is also an algebraic characterization of crystallographic groups due to Zassenhaus.

Theorem 1.3.53. *[Dek06, Theorem 2.1.4] Let Γ be an abstract group which contains a normal, maximal abelian subgroup of finite index isomorphic to $\mathbb{Z}^n$. Then there exists an injective morphism $\phi : \Gamma \longrightarrow E(n)$ such that $\phi(\Gamma)$ is crystallographic.*

Let $\Gamma \leq E(n)$ be a crystallographic subgroup. The group Γ acts on $\mathbb{R}^n$ by restricting the action of $E(n)$ on $\mathbb{R}^n$. If Γ is torsion-free (in this case we say that Γ is a Bieberbach group) then the action is free and the quotient $\mathbb{R}^n/\Gamma$ is a closed manifold of dimension n. Every closed flat manifold M is isometric to $\mathbb{R}^n/\Gamma$ for some Bieberbach group Γ. By the first Bieberbach theorem we can take the maximal abelian normal subgroup $\Lambda \trianglelefteq \Gamma$ to obtain a finite regular covering $T^n = \mathbb{R}^n/\Lambda \longrightarrow \mathbb{R}^n/\Gamma$. In addition, if M_1 and M_2 are homotopically equivalent closed flat manifolds, then they are affinely diffeomorphic, which means that there exists a diffeomorphism which lifts to an affine transformation in the universal cover $\mathbb{R}^n$. Finally, by the third Bieberbach theorem, there are finitely many closed flat manifold up to diffeomorphism in each dimension.

Theorem 1.3.54. *[Cha12, Chapter III.5] For every finite group G there exists a closed connected flat manifold with holonomy group G.*

Note that if an n-dimensional flat manifold M is a nilmanifold then $M \cong T^n$. This is because $\pi_1(M)$ and $\mathbb{Z}^n$ are commensurable, which implies that they have the same rational

Mal'cev completion. This implies that $\pi_1(M)$ is abelian and hence $M \cong T^n$. On the other hand, there exist flat manifolds which are also solvmanifolds (for example the Klein bottle). For more details of flat solvmanifolds we refer to [AA58, Tol20].

Let N be a simply connected nilpotent Lie group and C be a maximal compact subgroup of $\mathrm{Aut}(N)$. A discrete cocompact subgroup Γ of $N \rtimes C$ is called an almost-crystallographic group (abbreviated as AC-group). Moreover, Γ is said to be almost-Bieberbach if it is torsion-free. Bieberbach theorems have been generalized to almost-crystallographic groups. Before announcing them we need to introduce some more notation.

Definition 1.3.55. *Let Λ be a finitely generated torsion-free nilpotent group. An extension $1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ in which G is finite is said to be essential if Λ is a maximal nilpotent subgroup of Γ.*

Theorem 1.3.56. *[Dek06, Chapter 2] Let N be a simply connected nilpotent Lie group and C a maximal compact subgroup of $\mathrm{Aut}(N)$.*

1. *(Generalized 1st Bieberbach theorem) Let $\Gamma \leq N \rtimes C$ be an AC-group of N. The subgroup $N \cap \Gamma = \Lambda$ is a lattice of N, Λ is the unique normal maximal nilpotent subgroup of Γ and Γ/Λ is a finite group.*
2. *(Generalized 2nd Bieberbach theorem) Let Γ and Γ' be AC-groups of N. Assume that there exists an isomorphism $f : \Gamma \longrightarrow \Gamma'$. Then f can be realized as a conjugation by an element of $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$.*
3. *(Generalized 3rd Bieberbach theorem) Let Λ be a lattice of N. There are only finitely many essential extensions $1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$.*

If Γ is an almost-Bieberbach group of N then Γ acts freely on N and the orbit space N/Γ is a closed connected aspherical manifold which is finitely covered by the nilmanifold N/Λ. These manifolds are known as infranilmanifolds, and they also have a geometric characterization.

Definition 1.3.57. *A closed manifold M is said to be almost-flat if for any $\epsilon > 0$ there exists a Riemannian metric g_ϵ such that $|K_\epsilon| diam(M, g_\epsilon)^2 < \epsilon$, where K_ϵ is the sectional curvature and $diam(M, g_\epsilon)$ is the diameter of M using the metric g_ϵ.*

Theorem 1.3.58. *[Gro78, Ruh82] A closed manifold M is almost flat if and only if M is an infranilmanifold.*

The algebraic characterization of crystallographic groups can also be generalized to AC-groups, but we need some further definitions to state it.

Definition 1.3.59. *Let Γ be a virtually polycyclic group. The Fitting subgroup $\mathrm{Fitt}(\Gamma)$ is the unique maximal normal nilpotent subgroup of Γ.*

Lemma 1.3.60. *[Dek06, Definition 2.3.3] The Fitting subgroup* $\mathrm{Fitt}(\Gamma)$ *is unique.*

Theorem 1.3.61. *[Dek06, Theorem 3.4.6] Let* Γ *be a virtually polycyclic group. The following are equivalent:*

1. Γ *is an AC-group.*

2. $\mathrm{Fitt}(\Gamma)$ *is torsion-free, maximal nilpotent and of finite index in* Γ*.*

3. Γ *contains a torsion-free nilpotent subgroup* Λ *of finite index such that* $C_\Gamma(\Lambda)$ *is torsion-free.*

4. Γ *contains a nilpotent subgroup of finite index and* Γ *does not contain any non-trivial finite normal subgroup.*

Corollary 1.3.62. *If* Γ *is a torsion-free virtually polycyclic group then* Γ *is an AC-group if and only if it contains a nilpotent subgroup of finite index.*

The proof of theorem 1.3.61 uses the next proposition, which will be also used in the following chapters.

Proposition 1.3.63. *Let* Λ *be a finitely generated torsion-free nilpotent group and* $1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ *a group extension where G is finite. Then* Λ *is maximal nilpotent in* Γ *if and only if the induced map* $\phi_{\mathbb{Q}} : G \longrightarrow \mathrm{Out}(\Lambda_{\mathbb{Q}})$ *is injective.*

This proposition is proved in [Dek06, Lemma 3.1.1] for the real Mal'cev completion, and the same argument can be used to prove proposition 1.3.63 with the rational Mal'cev completion.

The proposition can be also reformulated in the following way:

Proposition 1.3.64. *Let* Λ *be a finitely generated torsion-free nilpotent group and* $1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ *a group extension where G is finite and* Γ *is torsion-free. Then* Γ *is nilpotent if and only if the induced map* $\phi_{\mathbb{Q}} : G \longrightarrow \mathrm{Out}(\Lambda_{\mathbb{Q}})$ *is trivial.*

1.3.5 Lattices on semisimple Lie groups and aspherical locally homogeneous spaces

This section has 3 parts. The first one is devoted to introduce the main results of the theory of lattices in semisimple Lie groups. In the second part we give a brief introduction to the theory of relatively hyperbolic groups. Finally, in the third part we study lattices on connected Lie groups, without the extra assumptions that the group is either solvable or semisimple. Each of these three parts provides key ingredients to prove theorem 10.

Lattices on semisimple Lie groups

Most of the following results are well-known, as they are the cornerstone of the theory of discrete subgroups on semisimple Lie groups. Our main reference is the introductory book [Mor01b], although we also refer to [Wan72, Mar91, Rag12].

We have seen that not all nilpotent Lie groups have lattices. Therefore, it is a natural question to ask whether semisimple Lie groups always have lattices. This was answered affirmatively by Borel and Harish-Chandra.

Theorem 1.3.65. *[Rag12, Theorem 14.1] Let G be a connected non-compact semisimple Lie group. Then G has both cocompact and non-cocompact lattices.*

Note that, unlike the nilpotent and solvable case, there exist lattices which are not cocompact. Another difference with the solvable case is the following result by Kazdan and Margulis:

Theorem 1.3.66. *[Rag12, 11.9 Corollary] Let G be a connected semisimple Lie group without compact factors and μ a Haar measure on G. Then there exists a constant $C > 0$ with the following property. For any lattice Γ of G, we have $\mu(\mathcal{F}) > C$, where $\mathcal{F}$ is a strict fundamental domain for G/Γ on G.*

If a lattice in a semisimple Lie group is the fundamental group of an aspherical manifold then it needs to be torsion-free. Consequently, it is important to understand torsion elements in lattices of semisimple Lie groups. The main result is known as Selberg's lemma:

Theorem 1.3.67. *(Selberg's lemma, [Mor01b, (4.2.8)Theorem]) Let G be a connected semisimple linear group without compact factors and Γ a lattice of G. Then Γ contains a normal torsion-free lattice Γ' of finite index in Γ.*

Recall that a Lie group G is linear if there exists an injective morphism $G \longrightarrow \mathrm{GL}(l, \mathbb{R})$ for some l. The condition of G being linear is essential in theorem 1.3.67. In [Del78], Deligne constructed lattices in the universal cover of $\mathrm{Sp}(2n, \mathbb{R})$ which are not virtually torsion-free. The first example of cocompact lattice which is not virtually torsion-free was constructed by Raghunathan in [Rag84] by using the universal cover of $\mathrm{Spin}(2, n)$. Note that if G is a connected semisimple Lie group then $G/\mathcal{Z}G$ is a connected semisimple linear Lie group, by taking the adjoint representation $\mathrm{Ad} : G/\mathcal{Z}G \longrightarrow \mathrm{GL}(l, \mathbb{R})$ with $l = \dim G$.

The proof of Selberg's lemma uses Borel's density theorem, which has far-reaching consequences.

Theorem 1.3.68. *(Borel's density theorem,[Mor01b, (4.5.1) Theorem]) Assume that G is a connected semisimple linear group without compact factors, Γ is a lattice of G and V is a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ with a representation $\rho : G \longrightarrow \mathrm{GL}(V)$. Then any $\rho(\Gamma)$-invariant vector or subspace of V is $\rho(G)$-invariant.*

We are interested in the next two consequences of Borel's density theorem:

Corollary 1.3.69. *[Mor01b, (4.5.3) Corollary] Let G be a connected semisimple linear group without compact factors and Γ a lattice of G. The group $N_G(\Gamma)/\Gamma$ is finite and $C_G(\Gamma) = \mathcal{Z}G$ (equivalently, $\mathcal{Z}\Gamma = \Gamma \cap \mathcal{Z}G$).*

Corollary 1.3.70. *[Mor01b, (4.5.2) Corollary] Let G be a connected semisimple linear group without compact factors, Γ a lattice of G and H be a connected closed subgroup of G. Assume that Γ normalizes H, then H is normal in G.*

We can deduce the next corollary from corollary 1.3.70

Corollary 1.3.71. *Let G be a connected semisimple linear group without compact factors and Γ a torsion-free lattice of G. Then Γ does not contain normal solvable subgroups.*

Proof. Assume on the contrary, that Γ contains a solvable normal subgroup Λ. Then, the connected component of the identity of the Zariski closure $\overline{\Lambda}^o$ is a connected solvable Lie subgroup of G. Since Λ is normal in Γ then $\overline{\Lambda}^o$ is normalized by Γ. By corollary 1.3.70, $\overline{\Lambda}^o$ is a connected solvable normal subgroup of G, which contradicts the semisimplicity of G. □

Definition 1.3.72. *Let G be a connected semisimple Lie group. A lattice Γ is said to be irreducible if ΓH is dense in G for every non-compact closed normal subgroup H of G. A lattice that it is not irreducible will be called reducible.*

For other conditions which are equivalent to irreducibility we refer to [Rag12, 5.21. Corollary].

Remark 1.3.73. *If G is a simple Lie group then all lattices of G are irreducible.*

If Γ_i is a lattice of a connected semisimple Lie group G_i for $i = 1, 2$, then $\Gamma_1 \times \Gamma_2$ is a reducible lattice for $G_1 \times G_2$.

Proposition 1.3.74. *[Mor01b, (4.3.3) Proposition] Let G be a centreless connected semisimple Lie group without compact factors and let Γ be a lattice of G. Then there exist semisimple subgroups $G_1, \dots, G_r$ of G and lattices $\Gamma_i \leq G_i$ for all $1 \leq i \leq r$ such that $G = G_1 \times \cdots G_r$ and $\Gamma_1 \times \cdots \times \Gamma_r$ is a normal finite index subgroup of Γ.*

Another important concept to study lattices in semisimple Lie groups is the real rank of a semisimple group.

Definition 1.3.75. *Let G be a semisimple linear connected Lie group. A subgroup A is a $\mathbb{R}$-split torus if there exists an element $g \in \mathrm{GL}(n, \mathbb{R})$ such that gAg^{-1} consists of diagonal matrices. The* $\mathrm{rank}_{\mathbb{R}}$ *G is the dimension of a maximal $\mathbb{R}$-split torus A in G. It is independent of the choice of A and it only depends on G as a Lie group (that is, it is independent of the choice of embedding of G in a linear group).*

The real rank $\operatorname{rank}_{\mathbb{R}} G$ has a geometric interpretation. Given a semisimple linear connected Lie group G and a maximal compact subgroup $K \leq G$, a flat is a connected totally geodesic flat submanifold of the symmetric space G/K. Then $\operatorname{rank}_{\mathbb{R}} G$ is the dimension of the largest closed (in the sense that it contains all its accumulation points), simply connected flat of G/K. The next proposition summarises some useful properties of $\operatorname{rank}_{\mathbb{R}} G$.

Proposition 1.3.76. *[Mor01b, §8.1] Let G be a semisimple connected Lie group. Then:*

1. *We have* $\operatorname{rank}_{\mathbb{R}}(G) = 0$ *if and only if G is compact.*
2. *If* $\operatorname{rank}_{\mathbb{R}}(G) = n$*, then* $\mathbb{Z}^n \leq \Gamma$ *for any lattice of* Γ *of G.*
3. *Let* G' *be another semisimple connected Lie group. Then* $\operatorname{rank}_{\mathbb{R}}(G \times G') = \operatorname{rank}_{\mathbb{R}}(G) + \operatorname{rank}_{\mathbb{R}}(G')$.

The behaviour of lattices of a connected semisimple Lie group G is really different depending on whether $\operatorname{rank}_{\mathbb{R}} G = 1$ or $\operatorname{rank}_{\mathbb{R}} G > 1$. For example:

Proposition 1.3.77. *[Mor01b, §10.2] Assume that* Γ *is a cocompact lattice in G. Then* Γ *is hyperbolic (in the sense of remark 1.3.85) if and only if* $\operatorname{rank}_{\mathbb{R}} G = 1$.

Our aim is to study the outer automorphism group of a lattice Γ. In order to do so, we will need rigidity results which extend automorphisms of Γ to automorphisms of G.

Theorem 1.3.78. *(Mostow-Prasad-Margulis rigidity theorem, [Mor01b, (15.1.2) Theorem]) Let G and* G' *be connected semisimple linear Lie groups without compact factors, and let* Γ *and* Γ' *be lattices of G and* G' *respectively. Assume that G and* G' *have trivial center. Finally, assume that there does not exists any simple factor H of G such that* $H \cong \mathrm{PSL}(2,\mathbb{R})$ *and* $H \cap \Gamma_1$ *is a lattice in H. Then any isomorphism from* Γ *to* Γ' *extends to a unique continuous isomorphism from G to* G'.

Theorem 1.3.79. *(Margulis superrigidity, [Mor01b, §16.1]) Let G be a connected semisimple linear group without compact factors and* Γ *a lattice of G. Assume that* $\operatorname{rank}_{\mathbb{R}} G \geq 2$ *and that* Γ *is an irreducible lattice. Given a representation* $\rho : \Gamma \longrightarrow \mathrm{GL}(n,\mathbb{R})$*, let* $\overline{\rho(\Gamma)}^{o}$ *be the identity component of the Zariski closure of* $\rho(\Gamma)$ *in* $\mathrm{GL}(n,\mathbb{R})$ *and set* $\Gamma_0 = \rho^{-1}(\overline{\rho(\Gamma)}^{o})$*. Then there exists a representation* $\tilde{\rho} : G \longrightarrow \overline{\rho(\Gamma)}^{o}$ *such that* $\tilde{\rho}_{|\Gamma_0} = \rho_{|\Gamma_0}$.

Theorem 1.3.80. *(Margulis normal theorem,[Mor01b, (17.1.1) Theorem]) Let G be a connected semisimple linear group without compact factors and* Γ *a lattice of G. Assume that G has* $\operatorname{rank}_{\mathbb{R}} G \geq 2$ *and finite center, and that* Γ *is an irreducible lattice. If N is a normal subgroup of* Γ*, then either* $N \leq \mathcal{Z}G$ *or* Γ/N *is finite.*

Finally, we recall a well-known fact about the outer automorphism group of semisimple Lie groups.

Lemma 1.3.81. *[Gor94, Theorem 3.3.1] Let G be a connected semisimple Lie group. Then* $\mathrm{Out}(G)$ *is finite.*

Relatively hyperbolic groups

We now introduce the results we need from the theory of relatively hyperbolic groups. For an introduction to relatively hyperbolic groups we refer to [Bow97, Osi06].

Let us recall the definition of relative hyperbolicity from [Osi06, MO10]. Given a group H and a collection of proper subgroups $\{H_i\}_{i\in I}$, a subset X of G is a relative generating set of H with respect to $\{H_i\}_{i\in I}$ if X together with the union of all H_i generates H. In this situation H can be written as a quotient group of a free group $\mathcal{F} = (*_{i\in I} H_i) * F(X)$, where $F(X)$ denotes the free group with basis X. If the kernel $\mathcal{F} \longrightarrow H$ is the normal closure of a subset R of $\mathcal{F}$ then we say that H has a relative presentation $\langle X, H_i, i \in I | R\rangle$. If X and R are finite sets then we say that H is finitely presented relative to $\{H_i\}_{i\in I}$.

Set $\mathcal{H} = \bigsqcup_{i\in I}(H_i \setminus \{e\})$. Given a word W in the alphabet $X^{\pm} \cup \mathcal{H}$ representing the trivial element e in H there exists an element of the form $\prod_{j=1}^{k} f_j R_j^{\pm 1} f_j^{-1}$ which is equal to W in the group $\mathcal{F}$, where $R_j \in R$ and $f_j \in \mathcal{F}$. The smallest possible k for which W is equal to an element of the form $\prod_{j=1}^{k} f_j R_j^{\pm 1} f_j^{-1}$ is called the relative area of W and denoted by $Area^{rel}(W)$. If $||W||$ denotes the length of W in the alphabet $X^{\pm} \cup \mathcal{H}$ then:

Definition 1.3.82. *[MO10, Definition 2.1] A group H is hyperbolic relative to a collection of proper subgroup $\{H_i\}_{i\in I}$ if H is finitely presented relative to $\{H_i\}_{i\in I}$ and there is a constant $C > 0$ such that any word W in $X^{\pm} \cup \mathcal{H}$ representing the identity in H satisfies that*

$$Area^{rel}(W) \leq C||W||.$$

The groups H_i are called peripheral subgroups. We say that a group H is relatively hyperbolic if there exists a collection of subgroups $\{H_i\}_{i\in I}$ such that H is hyperbolic relative to $\{H_i\}_{i\in I}$.

Remark 1.3.83. *The definition is independent of the choice of X and R (see [Osi06]).*

Remark 1.3.84. *There are other definitions of relative hyperbolicity of a group H with respect a collection of subgroups $\{H_i\}_{i\in I}$ in the literature. If H is torsion-free and finitely presented and H_i is finitely presented for all i, then all the definitions are equivalent (see [BSB08, Definition 1.1] and references therein). This will happen in our setting.*

Remark 1.3.85. *A group H is hyperbolic if it is hyperbolic relative to the collection of subgroups which only contains the trivial subgroup.*

We need to introduce relative hyperbolicity because of the next theorem (compare to proposition 1.3.77):

Theorem 1.3.86. *([Far98] and [Gro87, §0.2(F)]) A lattice Γ in a connected semisimple Lie group without compact factors G is relatively hyperbolic if and only if $\operatorname{rank}_{\mathbb{R}} G = 1$. If $\operatorname{rank}_{\mathbb{R}} G = 1$, then Γ is hyperbolic relative to the collection of all its cusp subgroups associated to the cusps of the symmetric space $H \setminus G/\Gamma$. The cusp subgroups are virtually nilpotent.*

Let H be a group relatively hyperbolic to $\{H_i\}_{i\in I}$. Recall that $h \in H$ is said to be parabolic if its is conjugate to an element of H_i for some i. An element $h \in H$ which is not parabolic is said to be hyperbolic.

Lemma 1.3.87. *Let H be a torsion-free group relative hyperbolic to $\{H_i\}_{i\in I}$. Then:*

1. *[MO10, Lemma 2.4] If $h \in H$ is hyperbolic then $C_H(h)$ is cyclic.*
2. *[MO10, Proposition 3.3] The set of hyperbolic elements generates H.*
3. *[MO10, Lemma 2.2] Given $i \in I$ and $h \in H \setminus H_i$, we have $H_i \cap hH_ih^{-1} = \{e\}$.*

We are interested in the following corollary of item 3.

Corollary 1.3.88. *Let h be a non-trivial element of H_i, then $C_H(h) \leq H_i$.*

Proof. Suppose that $h' \in C_H(h)$. Then $h'hh'^{-1} = h$, therefore $H_i \cap h'H_ih'^{-1} \neq \{e\}$. By lemma 1.3.87, $h' \in H_i$. Thus, $C_H(h) \leq H_i$. □

Lattices in connected Lie groups

The last part of this section focuses on lattices in connected Lie groups without the semisimplicity assumption. Given a connected Lie group G, a lattice Γ of G and a closed subgroup H of G, our goal is to understand when $\Gamma \cap H$ is a lattice of H.

Definition 1.3.89. *Let G be a connected Lie group and H a closed subgroup. Given a lattice Γ of G we say that H is Γ-hereditary if $H \cap \Gamma$ is a lattice of H. We say that H is lattice hereditary if it is Γ-hereditary for every lattice Γ of G.*

Lemma 1.3.90. *[Gen15, Theorem 2.6] Let Γ and H be a lattice and a closed subgroup of G respectively. If H is normal or Γ is cocompact the following are equivalent:*

1. *H is Γ-hereditary.*
2. *The image of Γ is discrete in G/H.*
3. *The image of Γ is a lattice in G/H (when H is normal).*

Theorem 1.3.91. *[Gen15, Corollary 1.3] Let G be a connected Lie group and let N and R be its nilpotent and solvable radical respectively. If the semisimple part $S = G/R$ of G has no compact factor acting trivially on R then the nilradical N is lattice hereditary. Moreover if no compact factor of S acts trivially on R/N then R is lattice hereditary.*

From the theorem one can deduce:

Corollary 1.3.92. *[Gen15, Corollary 1.4] Let G be a connected Lie group and let N and R be its nilpotent and solvable radical respectively, let $S = G/R$ be the semisimple part of G and let C and*

S_K be the maximal connected semisimple compact normal subgroups of G and S respectively. Then the following subgroups of G are lattice hereditary:

$$C \subseteq NC \subseteq NS_K \subseteq RS_K.$$

The solvable radical R is not in general lattice hereditary. For example, let $G = \mathbb{R} \times \mathrm{SO}(3)$ and Γ is the subgroup generated by $(1, a)$, where $a \in \mathrm{SO}(3)$ has infinite order. Since $\mathrm{SO}(3)$ is compact $[0, 1] \times \mathrm{SO}(3)$ is a fundamental domain and $\Gamma \setminus G$ is compact, so Γ is a lattice. In this case the solvable radical is $R = \mathbb{R}$ and $S = \mathrm{SO}(3)$. Then, $R \cap \Gamma = (0, Id)$ which it is not a lattice of R.

To solve this problem we use the concept of amenability:

Definition 1.3.93. *A compact convex G-space is a G-invariant compact and convex subset of any locally convex topological vector space on which G acts linearly.*

We say that G is amenable if every non-empty convex compact G-space has a G-fixed point.

There are many equivalent definitions of amenability (see [Mor01b, (12.3.1) Theorem]). We are interested in the following properties of amenable groups:

Lemma 1.3.94. *(See [Zim13, §4.1] or [Mor01b, Chapter 12])*

1. *Let $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$ be a short exact sequence of groups. Then G_2 is amenable if and only if G_1 and G_3 are amenable.*
2. *Solvable Lie groups and compact Lie groups are amenable. In particular, finite groups and $\mathbb{Z}$ are amenable.*
3. $\mathrm{SL}(2, \mathbb{R})$ *is not amenable.*
4. *A lattice Γ of a Lie group G is amenable if and only if G is amenable.*

Let G be a connected Lie group, let R be its solvable radical and let S_K be the maximal connected semisimple compact subgroups of S respectively. The group $A = RS_K$ is known as the amenable radical of G, since it is the connected maximal amenable normal subgroup of G. We have a decomposition $G = A \rtimes S_{nc}$ (where $S_{nc} = S/S_K$), analogous to the Levi decomposition $G = R \rtimes S$.

By corollary 1.3.92, given a lattice Γ of G, the subgroup $\Gamma \cap A$ is a lattice of A. This fact will be crucial to prove theorem 10.

1.3.6 Group actions on aspherical manifolds

Let M be a closed connected manifold and let $x_0 \in M$. Assume that we have a finite group G acting on M. Then for each $g \in G$ we have an isomorphism $g_* : \pi_1(M, x_0) \longrightarrow$

$\pi_1(M, gx_0)$. If x_0 is fixed by the action of G on M, then we can define a group morphism $G \longrightarrow \operatorname{Aut}(\pi_1(M, x_0))$.

If x_0 is not fixed by the action of G on M then above group morphism is not well defined. However, since $\pi_1(M, x_0)$ and $\pi_1(M, gx_0)$ are isomorphic and the isomorphism is given by a conjugation, we have a well-defined group morphisms $\psi : G \longrightarrow \operatorname{Out}(\pi_1(M, x_0)) = \operatorname{Aut}(\pi_1(M, x_0))/\operatorname{Inn}(\pi_1(M, x_0))$ given by $\psi(g) = [g_* : \pi_1(M, x_0) \longrightarrow \pi_1(M, gx_0)]$. We will say that the action is inner if $\psi : G \longrightarrow \operatorname{Out}(\pi_1(M, x_0))$ is trivial. We will omit the base point whenever it is not necessary for the discussion.

We can always lift the action of G on M to an action of a group $\tilde{G}$ on the universal cover $\tilde{M}$, where the group $\tilde{G}$ fits into the short exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

The abstract kernel of the group extension coincides with the group morphism ψ.

Lemma 1.3.95. *[LR10, Lemma 3.1.14] Let G be a finite group acting effectively on a closed manifold M. Then, there is a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{Z}\pi_1(M) & \longrightarrow & C_{\tilde{G}}(\pi_1(M)) & \longrightarrow & \operatorname{Ker}\psi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(M) & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \psi & & \\
1 & \longrightarrow & \operatorname{Inn}(\pi_1(M)) & \longrightarrow & \operatorname{Aut}(\pi_1(M)) & \longrightarrow & \operatorname{Out}(\pi_1(M)) & \longrightarrow & 1 \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

where $C_{\tilde{G}}(\pi_1(M))$ is the centralizer of $\pi_1(M)$ in $\tilde{G}$.

Moreover, if M is aspherical we can use the above commutative diagram and theorem 1.1.37 to obtain the following theorem:

Theorem 1.3.96. *[LR10, Theorem 3.1.16] Let G be a finite group acting effectively on a closed connected n-dimensional aspherical manifold M, then:*

1. *$C_{\tilde{G}}(\pi_1(M))$ is torsion free.*
2. $\operatorname{Ker}\psi$ *is abelian. Moreover, if $\mathcal{Z}\pi_1(M)$ is finitely generate of rank k, then* $\operatorname{Ker}\psi$ *is a subgroup of the torus T^k.*

Proof. Suppose that $C_{\tilde{G}}(\pi_1(M))$ has torsion. Thus, there exists a prime p such that $\mathbb{Z}/p \leq C_{\tilde{G}}(\pi_1(M))$. There is an effective action of $C_{\tilde{G}}(\pi_1(M))$ on $\tilde{M}$. Let $\tilde{F} = \tilde{M}^{\mathbb{Z}/p}$, which a proper non-empty $\mathbb{Z}/p$-acyclic subset of $\tilde{M}$ by theorem 1.1.37. Since $\pi_1(M)$ acts freely on $\tilde{M}$ and $\mathbb{Z}/p$ and $\pi_1(M)$ commute we have a free action of $\pi_1(M)$ on $\tilde{F}$. Thus, $F = \tilde{F}/\pi_1(M)$ is a proper subset of M.

Since $\tilde{F}$ is $\mathbb{Z}/p$-acyclic, we have $H^*(F,\mathbb{Z}/p) \cong H^*(\pi_1(M),\mathbb{Z}/p) \cong H^*(M,\mathbb{Z}/p)$. Assume that M is orientable (if not we can take the orientable 2-cover), then $H^n(M,\mathbb{Z}/p) = \mathbb{Z}/p$. But $H^n(F,\mathbb{Z}/p) = 0$ since F is a proper subset of M. We have reached a contradiction. In conclusion, $C_{\tilde{G}}(\pi_1(M))$ is torsion free.

To prove the second part we note that the first row of the diagram of lemma 1.3.95 is a central short exact sequence. Since $C_{\tilde{G}}(\pi_1(M))$ is torsion-free then it is abelian and $\operatorname{rank} C_{\tilde{G}}(\pi_1(M)) = k$. Therefore, $\operatorname{Ker}\psi$ is a subgroup of T^k □

We assume now that G is a compact Lie group but not necessarily finite. Suppose that G acts effectively on a closed aspherical manifold M. Given $x_0 \in M$, we can define the evaluation map $ev_{x_0} : G \longrightarrow M$ such that $ev_{x_0}(g) = gx_0$. Recall that the action is said to be injective if the map $ev_{x_0 *} : \pi_1(G,e) \longrightarrow \pi_1(M,x_0)$ is injective for all $x_0 \in M$. Then:

Theorem 1.3.97. *[LR10, Corollary 3.1.17, Corollary 3.1.12] Let G be a compact Lie group acting effectively on a closed aspherical manifold M. Then:*

1. *If G is connected, then G is a torus T^k and the action is injective. In particular, any action of a connected Lie group on a closed connected aspherical manifold is almost-free.*
2. *If x_0 is a fix point, then G is finite and the group morphism $G \longrightarrow \operatorname{Aut}(\pi_1(M,x_0))$ is injective.*
3. *If $\mathcal{Z}\pi_1(M,x_0) = \{e\}$, then G is finite and $\psi : G \longrightarrow \operatorname{Out}(\pi_1(M,x_0))$ is injective.*

Recall that an action of a connected Lie group on a manifold is said to be almost free if all the stabilizers are finite. The proof of theorem 1.3.97 is similar to the proof of theorem 1.3.96, hence we will omit it.

The next part of this section is devoted to explain the Seifert space construction, which can be used to construct torus actions on some aspherical manifolds. The main reference used is [LR10].

Definition 1.3.98. *Let P be a manifold and G a Lie group acting effectively on P. A weak G-equivalence is a homeomorphism $f : P \longrightarrow P$ such that there exists a group morphisms $\alpha_f : G \longrightarrow G$ satisfying $f(gx) = \alpha_f(g)f(x)$ for all $g \in G$ and $x \in X$. We denote the group of weak G-equivalence by* $\operatorname{Homeo}_G(P)$.

We define $\operatorname{Map}_G(P,G)$ *to be the group of G-equivariant continuous maps $f : P \longrightarrow G$, where the* *action of G is given by the conjugation on G.*

Note that $\mathrm{Homeo}_G(P)$ is the normalizer of G in $\mathrm{Homeo}(P)$. The group $\mathrm{Homeo}_G(P)$ has the following structure:

Lemma 1.3.99. *[LR10, Proposition 4.2.8, Corollary 4.2.10] Assume that G acts freely on P in such a way that the projection $P \longrightarrow P/G = W$ is a principal G-bundle. Then we have the exact sequence*

$$1 \longrightarrow G \times_{\mathcal{Z}G} \mathrm{Map}_G(P,G) \longrightarrow \mathrm{Homeo}_G(P) \longrightarrow \mathrm{Out}(G) \times \mathrm{Homeo}(W).$$

If the principal G-bundle $P \longrightarrow W$ is trivial, then we have a short exact sequence

$$1 \longrightarrow \mathrm{Map}(W,G) \rtimes G \longrightarrow \mathrm{Homeo}_G(P) \longrightarrow \mathrm{Out}(G) \times \mathrm{Homeo}(W) \longrightarrow 1.$$

Let $\Lambda \leq \mathrm{Homeo}_G(P)$ be a discrete subgroup with respect the compact-open topology on $\mathrm{Homeo}_G(P)$. Then Λ acts on W. If this action is proper then we say that the induced map $\tau : \Lambda \setminus P \longrightarrow \Lambda \setminus W$ is a Seifert fibering and $\Lambda \setminus P$ is a Seifert fibered space modeled on the principal bundle $P \longrightarrow W$. Note that $\Gamma = G \cap \Lambda$ is a closed normal subgroup of Λ which acts properly and freely on each fiber of $P \longrightarrow W$. We will say that $\Gamma \setminus G$ is a typical fiber of τ. Note also that $Q = \Lambda/\Gamma$ acts on W. With these notions we can introduce the Seifert construction.

Definition 1.3.100. *[LR10, §4.6] Let $P \longrightarrow W$ be a principal G-bundle and let H be a closed subgroup of $\mathrm{Homeo}_G(P)$. A Seifert construction with uniformizing group H for*

1. *A short exact sequence $1 \longrightarrow \Gamma \longrightarrow \Lambda \longrightarrow Q \longrightarrow 1$*
2. *A group morphism $i : \Gamma \longrightarrow G \times_{\mathcal{Z}G} \mathrm{Map}_G(P,G)$*
3. *A proper action $\rho : Q \longrightarrow \mathrm{Homeo}(W)$*

is a group morphism $\theta : \Lambda \longrightarrow H$ such that $\theta_{|\Gamma} = i$ such that the diagram

$$\begin{array}{ccccc}
\Lambda & \xrightarrow{\theta} & H & \longrightarrow & \mathrm{Homeo}_G(P) \\
\Big\downarrow{\scriptstyle id_\Lambda} & & & & \Big\downarrow \\
\Lambda & \longrightarrow & Q & \xrightarrow{\rho} & \mathrm{Homeo}(W)
\end{array}$$

commutes.

Definition 1.3.101. *A Lie group G has the the unique automorphism extension property (UAEP), if for every lattice $\Gamma \leq G$ and automorphism $f \in \mathrm{Aut}(\Gamma)$ there exists a unique automorphism $\tilde{f} \in \mathrm{Aut}(G)$ such that $\tilde{f}_{|\Gamma} = f$.*

For example, all simply connected nilpotent Lie groups has the UAEP.

Assume that G is a simply connected solvable Lie group with the UAEP. Let W be a manifold and consider the trivial principal G-bundle $G \times W \longrightarrow W$. Then:

Theorem 1.3.102. *[LR10, Theorem 7.3.2](Seifert construction theorem) Let $\Gamma \leq G$ be a lattice and $\rho : Q \longrightarrow \mathrm{Homeo}(W)$ a proper discrete action by a discrete subgroup Q. Then for any extension $1 \longrightarrow \Gamma \longrightarrow \Lambda \longrightarrow Q \longrightarrow 1$ (with abstract kernel $\psi : Q \longrightarrow \mathrm{Out}(\Gamma)$), the following are true:*

1. ***Existence:*** *There exists $\theta : \Lambda \longrightarrow \mathrm{Homeo}_G(G \times W)$ making the following diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Lambda & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \tilde{\psi}\times\rho} & & \\ 1 & \longrightarrow & \mathrm{Map}(W,G) \rtimes \mathrm{Inn}(G) & \longrightarrow & \mathrm{Homeo}_G(G \times W) & \longrightarrow & \mathrm{Out}(G) \rtimes \mathrm{Homeo}(W) & \longrightarrow & 1 \end{array}$$

   commute, where $\tilde{\psi} : Q \longrightarrow \mathrm{Out}(G)$ is obtained by using the UAEP of G and the abstract kernel ψ.

2. ***Uniqueness:*** *Suppose that $\theta_1, \theta_2 : \Lambda \longrightarrow \mathrm{Homeo}_G(G \times W)$ are two group morphisms fitting on the above diagram with fixed i and $\tilde{\psi} \times \rho$. Then there exists $f \in \mathrm{Map}(W, G)$ such that $\theta_2 = f\theta_1 f^{-1}$.*

3. ***Rigidity:*** *Suppose that $\theta_1, \theta_2 : \Lambda \longrightarrow \mathrm{Homeo}_G(G \times W)$ are two group morphisms fitting on the above diagram with possible distinct i and ρ. Then there exists $f \in \mathrm{Homeo}_G(G \times W)$ such that $\theta_2 = f\theta_1 f^{-1}$.*

We have the following application for closed aspherical manifolds.

Theorem 1.3.103. *[LR10, Theorem 11.1.2] With the same hypothesis as in theorem 1.3.102, assume that W is also contractible and W/Q is compact. If Γ is torsion-free then for any Seifert construction $\theta : \Lambda \longrightarrow \mathrm{Homeo}_G(G \times W)$ the quotient $(G \times W)/\theta(\Lambda)$ is a closed aspherical manifold with fundamental group Λ.*

Using the Mostow fibration, theorem 1.3.103 can be extended as follows:

Theorem 1.3.104. *[LR10, Theorem 11.1.4] Let Λ be a torsion-free group fitting in the short exact sequence $1 \longrightarrow \Gamma \longrightarrow \Lambda \longrightarrow Q \longrightarrow 1$ where Γ is virtually polycyclic and Q acts properly on a contractible manifold with compact quotient. Then there exists a closed aspherical manifold with fundamental group Λ.*

Using the Seifert construction we can study torus actions on aspherical manifolds, as showed in the next theorems.

Theorem 1.3.105. *[LR10, Theorem 11.7.29] Let M be a closed connected aspherical manifold such that there exists a short exact sequence $1 \longrightarrow \Gamma_R \longrightarrow \pi_1(M) \longrightarrow \Gamma_{nc} \longrightarrow 1$ where Γ_R is virtually polycyclic and Γ_{nc} is a centreless cocompact lattice in a semisimple connected Lie group S. Then* *there exists a closed connected aspherical manifold* M' *homotopically equivalent to* M *such that* $\text{tor-sym}(M') = \text{rank}\,\mathcal{Z}\pi_1(M')$.

Proof. Since Γ_R is a virtually polycyclic group it contains a characteristic finite index subgroup Γ' which is a Mostow-Wang group (see theorem 1.3.51). Then Γ_R contains a characteristic subgroup of finite index Γ'' which is predivisible. The discrete nilradicals of these groups satisfy ${}^n\Gamma_R ={}^n \Gamma' ={}^n \Gamma$. Thus, we will denote them by Δ. Moreover, $\Gamma''/\Delta \cong \mathbb{Z}^m$. Finally, let $Q = \Gamma/\Gamma''$.

Let K be a maximal compact subgroup of S. Then the action of Γ_{nc} on $K \setminus S$ induces a group action of Q on $K \setminus S$ via the group morphism $Q = \Gamma/\Gamma'' \longrightarrow \Gamma_{nc}$. Since Δ is characteristic in Γ'' then Δ is normal in Γ. Hence, there exist short exact sequence $1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \Gamma/\Delta \longrightarrow 1$ and $1 \longrightarrow \mathbb{Z}^m \longrightarrow \Gamma/\Delta \longrightarrow Q \longrightarrow 1$, where the second short exact sequence exists because Γ'' is predivisible. We use the Seifert space construction with the second short exact sequence and the action of Q on $K \setminus S$ to obtain an action of Γ/Δ on $\mathbb{R}^m \times K \setminus S$. We use again the Seifert space construction, now using the first exact sequence and the group action of Γ/Δ on $\mathbb{R}^m \times K \setminus S$ to obtain a group action of Γ on $\Delta_{\mathbb{R}} \times \mathbb{R}^m \times K \setminus S$, where $\Delta_{\mathbb{R}}$ denotes the real Mal'cev completion of Δ. We have $\mathcal{Z}\Gamma \leq \mathcal{Z}\Delta \leq \mathcal{Z}\Delta_{\mathbb{R}}$. If $k = \text{rank}(\mathcal{Z}\Gamma)$ then there exists a subgroup $\mathbb{R}^k \leq \Delta_{\mathbb{R}}$ such that $\Gamma \cap \mathbb{R}^k \cong \mathbb{Z}^k$. In consequence, the action by multiplication of $\mathbb{R}^k$ on $\Delta_{\mathbb{R}} \times \mathbb{R}^m \times K \setminus S$ descends to an action of T^k on $M' = (\Delta_{\mathbb{R}} \times \mathbb{R}^m \times K \setminus S)/\Gamma$. Since M' is a closed aspherical manifold with the same fundamental group as M then M and M' are homotopically equivalent. Finally, the bound $\text{tor-sym}(M') \leq k$ is reached, so $\text{tor-sym}(M') = k$, as we wanted to see. □

Theorem 1.3.106. *[LR10, Theorem 11.7.28] Let* $K \setminus G/\Gamma$ *be a closed connected aspherical locally homogeneous space and let* R *be its radical. Assume that* R *is lattice-hereditary, that* R *has the UAEP and that* $\exp : \mathcal{L}(R) \longrightarrow R$ *is surjective. Then* $\text{tor-sym}(M) = \text{rank}\,\mathcal{Z}\Gamma$.

Proof. Let $G = R \rtimes_{\phi} S$ be the Levi decomposition of G and let

$$A = \{a \in R : (a, u) \in \mathcal{Z}\Gamma \text{ for some } u \in S\}.$$

Firstly, we will see that A is commutative. Indeed, take $(a, u) \in \mathcal{Z}\Gamma$ and $(z, e) \in \Gamma_R = \Gamma \cap R$. Then $(z, 1)(a, u) = (a, u)(z, 1)$ which implies that $\phi(u)(z) = a^{-1}za$. Since R has the UAEP, both automorphisms $\phi(u)$ and $c_{a^{-1}}$ extend to the same automorphism of R. Therefore, $\phi(u)(x) = a^{-1}xa$ for all $x \in R$. In addition, $\phi(v)(a) = a$ for any $(a, v) \in \Gamma$, which implies that A is a commutative subgroup of R.

Assume that $\mathcal{Z}\Gamma \cong \mathbb{Z}^k$ and choose generators (a_i, u_i) for $i = 1, \ldots, k$. Since $\exp : \mathcal{L}(R) \longrightarrow R$ is surjective, we can choose elements $A_i \in \mathcal{L}(R)$ such that $\exp(A_i) = a_i$. Then we can define a map $\alpha_R : \mathbb{R}^k \to \mathcal{L}(R) \xrightarrow{\exp} R$, where the first map sends the standard basis of $\mathbb{R}^k$ to $\{A_1, \ldots, A_k\}$. Since $\{A_1, \ldots, A_k\}$ generates a commutative Lie subalgebra (because A is commutative), the exponential map restricted to this subalgebra is a group morphism. In consequence, α is also a group morphism.

Now, we construct a group morphism $\alpha_S : \mathbb{R}^k \longrightarrow S$. Let $S = S_1 \times \cdots \times S_r$, where S_i are simple Lie groups. The maximal compact subgroup of the adjoint form of S_i is either H_i or $H_i \times S^1$, where H has no circle factors, depending on whether S_i has infinite center or not. This determines a subgroup $\tilde{H}_i \times \mathbb{R}^{\epsilon_i} \subseteq S_i$ where $\tilde{H}_i$ is compact and ϵ_i is 1 if S_i has infinite center and 0 otherwise. Then $K = \prod \tilde{H}_i$ is a maximal compact subgroup of S. Now we take the map $\mathcal{Z}\Gamma \longrightarrow \prod(\tilde{H}_i \times \mathbb{R}^{\epsilon_i}) \longrightarrow \prod \mathbb{R}^{\epsilon_i} \subseteq S$, which can be extended to a group morphism $\alpha_S : \mathbb{R}^k \longrightarrow S$.

The group K commutes with $\prod \mathbb{R}^{\epsilon_i}$, hence we have an induced $\mathbb{R}^k$ action on $K \setminus S$ (which may not be effective). We can define a group action of $\mathbb{R}^k$ on $K \setminus G$ such that $(b, w)x = (b\alpha_R(x), w\alpha_S(x))$. This action commutes with the action of Γ on $K \setminus G$ and since $\mathbb{R}^k \cap \Gamma = \mathcal{Z}\Gamma$, the action descends to an action of T^k on $K \setminus G/\Gamma$. □

We end this section by introducing some generalizations to the concept of aspherical manifolds which possess similar properties.

Definition 1.3.107. *[LR10, Definition 3.2.7] Let M be a closed connected oriented n-dimensional manifold. Then:*

1. *M is hyper-aspherical if there exists an aspherical n-dimensional manifold N and a degree one map $f : M \longrightarrow N$.*
2. *M is said to be a K-manifold if there exists a torsion-free group Γ and a map $f^* : M \longrightarrow K(\Gamma, 1)$ such that $f^* : H^n(K(\Gamma, 1), \mathbb{Z}) \longrightarrow H^n(M, \mathbb{Z})$ is surjective.*
3. *M is admissible if the only periodic self-homeomorphisms of $\tilde{M}$ commuting with the deck transformations group $\pi_1(M)$ are elements of $\mathcal{Z}\pi_1(M)$.*

Remark 1.3.108. *Note that*

$$\textit{Aspherical} \implies \textit{Hyper-aspherical} \implies \textit{K-manifold} \implies \textit{Admissible}$$

but no implication can be reversed (see [LR10, 3.2.8] or [LR87] for a complete discussion)

The next result is a generalization of Theorem 1.3.96 for the class of admissible manifolds.

Theorem 1.3.109. *[LR10, Theorem 3.2.2] Let M be an admissible manifold and suppose that $\mathcal{Z}\pi_1(M)/Torsion(\mathcal{Z}\pi_1(M))$ has finite rank k. Then:*

1. *If G is a connected Lie group acting on M then $G \cong T^r$ with $r \leq k$.*
2. *If G is a finite group acting on M then $\mathrm{Ker}(\psi : G \longrightarrow \mathrm{Out}(\pi_1(M)))$ is a subgroup of T^k.*
3. *If G is a finite group acting on M with a fix point x then $\phi : G \longrightarrow \mathrm{Aut}(\pi_1(M, x))$ is injective.*

1.4 Noncommutative ring theory

The objective of this section is to introduce some tools of noncommutative ring theory which will be used to prove theorem 23. Most of the results of this section are generalizations of well-known facts of commutative algebra. This will enable us to generalize the commutative algebra results in [MiR24a] to its noncommutative counterparts in a straightforward way. Our main references are the books [GW04, Rob13, Lam91, McR1987] as well as the notes [Bel88].

Our interest in noncommutative ring theory stems from the group ring of noncommutative groups. Let R be a ring and let G be a group. Recall that the group ring RG is a ring whose elements are formal sums $\sum_{g\in G} a_g g$ where only finitely many $a_g \in R$ are non-zero. Given another element $\sum_{g'\in G} b_{g'} g' \in RG$ we define

$$(\sum_{g\in G} a_g g)(\sum_{g'\in G} b_{g'} g') = \sum_{h\in G} c_h h$$

where

$$c_h = \sum_{gg'=h} a_g b_{g'}.$$

If either R or G are noncommutative then RG will be a noncommutative ring. Let us study more carefully the structure of group rings of finitely generated torsion-free 2-step nilpotent groups. Given a ring R and an automorphism $\alpha : R \longrightarrow R$, a skew-Laurent ring $S = R[x^{\pm 1}; \alpha]$ is a ring satisfying that $R \subseteq S$ is a subring, S is a free R-module with basis $\{1, x^{\pm 1}, x^{\pm 2}, \dots\}$ and $xr = \alpha(r)x$. This construction can be repeated over S to obtain an iterated skew-Laurent ring, denoted by $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}; \alpha_1, \dots, \alpha_n] = (R[x_1^{\pm 1}; \alpha_1] \cdots)[x_n^{\pm 1}; \alpha_n]$.

Let $\Gamma = \mathbb{Z}^f \times_c \mathbb{Z}^b$ be a torsion-free 2-step nilpotent group where $c : \mathbb{Z}^b \times \mathbb{Z}^b \longrightarrow \mathbb{Z}^f$ is a normalized 2-cocycle. Take a set of generators $\{z_1, \dots, z_f, x_1, \dots, x_b\}$ of Γ. Then the elements of $R\Gamma$ are of the form $\sum z_1^{e_1} \cdots z_f^{e_f} x_1^{a_1} \cdots x_b^{a_b} r$. Recall the product of two elements $(z, x), (z', x') \in \Gamma$ is $(z + z' + c(x, x'), x + x')$ and that for any $x \in \mathbb{Z}^b$, the maps $c(\cdot, x) : \mathbb{Z}^b \longrightarrow \mathbb{Z}^f$ and $c(x, \cdot) : \mathbb{Z}^b \longrightarrow \mathbb{Z}^f$ are group morphisms.

Lemma 1.4.1. *The group ring $R\Gamma$ is an iterated skew-Laurent ring.*

Proof. We proceed by induction on the number of generators of Γ. If $\Gamma \cong \mathbb{Z}$ then $R\mathbb{Z} = R[z^{\pm 1}; id_R]$. To use the induction step we consider $\Lambda = \langle z_1, \dots, z_f \rangle \trianglelefteq \Gamma$. Since $(z, x)(0, x_b) = (z + c(x, x_b), x + x_b)$ and $(0, x_b)(z, x) = (z + c(x_b, x), x + x_b)$ we can define an isomorphism $\alpha : \Lambda \longrightarrow \Lambda$ such that $\alpha(z, x) = (z + c(x_b, x) - c(x, x_b), x)$ which extends to an automorphism $\alpha : R\Lambda \longrightarrow R\Lambda$. Then $(R\Lambda)[x_b^{\pm 1}; \alpha] = R\Gamma$. □

More in general, group rings over virtually polycyclic groups are iterated skew-Laurent rings (see [GW04, pg. xvii]).

Our main goal is to understand the properties of $R\Gamma$. We start by recalling some basic definitions of noncommutative ring theory.

From now on R will denote a ring with unit. A right R-module is an abelian group $(M, +)$ together with an operation $\cdot : M \times R \longrightarrow M$ such that

1. $(x + y) \cdot r = x \cdot r + y \cdot r$ for all $x, y \in M$ and $r \in R$.
2. $x \cdot (r + s) = x \cdot r + x \cdot s$ for all $x \in M$ and $r, s \in R$.
3. $(x \cdot r) \cdot s = x \cdot (rs)$ for all $x \in M$ and $r, s \in R$.
4. $x \cdot 1 = x$ for all $x \in M$.

We can also define left R-modules analogously. We will assume that all the modules are right R-modules unless stated the contrary and we will write xr to denote $x \cdot r$. Note that R has structure of right R-module and of left R-module.

Let $\mathcal{A}$ be a collection of subsets of a set A. We say that $\mathcal{A}$ satisfies the ascending chain condition (or ACC) if for every ascending infinite chain $A_1 \subseteq A_2 \subseteq \cdots$ of elements of $\mathcal{A}$ there exists a number i_0 such that $A_i = A_{i_0}$ for all $i \geq i_0$. Recall that $B \in \mathcal{A}$ is a maximal element if for any $A \in \mathcal{A}$ such that $B \subseteq A$ we have $B = A$.

Proposition 1.4.2. *[GW04, Proposition 1.1] For a module M, the following are equivalent:*

1. *M has the ACC property on the set of its submodules.*
2. *Every nonempty family of submodules of M has a maximal element.*
3. *Every submodule of M is finitely generated.*

If M satisfies these equivalent conditions we say that M is Noetherian.

Definition 1.4.3. *A ring R is right (left) Notherian if it is Noetherian as a right (left) R-module. If R is both right and left Noetherian we say that R is a Noetherian ring.*

Lemma 1.4.4. *[GW04, Proposition 1.2] Let N be a submodule of M. Then M is Noetherian if and only if N and M/N are Noetherian. In particular, if R is a right Noetherian ring, all finitely generated right R-modules are Noetherian.*

The first important theorem we need is a generalization of the Hilbert basis theorem. Recall that the Hilbert basis theorem asserts that if R is a commutative Noetherian ring then the polynomial ring $R[x]$ is also Noetherian. The generalization to the noncommutative case is

Theorem 1.4.5. *[GW04, Theorem 1.14, Corollary 1.15] Let R be a right (left) Noetherian ring and $\alpha : R \longrightarrow R$ an automorphism. Then, the skew-Laurent ring $R[x^{\pm 1}; \alpha]$ is right (left) Noetherian.*

Theorem 1.4.5 can be used to prove by induction the next theorem due to P.Hall.

Theorem 1.4.6. *[GW04, Theorem 1.16] Let R be a right (left) Noetherian ring and G a virtually polycyclic group. Then, the group ring RG is right (left) Noetherian.*

In particular, if Γ is a torsion-free finitely generated nilpotent group then $\mathbb{Z}\Gamma$ is Noetherian.

The next part of the section is devoted to composition series. Let $\mathcal{A}$ be a collection of subsets of a set A. We say that $\mathcal{A}$ satisfies the descending chain condition (or DCC) if for every descending infinite chain $A_1 \supseteq A_2 \supseteq \cdots$ of elements of $\mathcal{A}$ there exists a number i_0 such that $A_i = A_{i_0}$ for all $i \geq i_0$. Recall that $B \in \mathcal{A}$ is a minimal element if for any $A \in \mathcal{A}$ such that $B \supseteq A$ we have $B = A$.

Proposition 1.4.7. *[GW04, Chapter 4, Artinian modules] For a module M, the following are equivalent:*

1. *M has the DCC property on the set of its submodules.*
2. *Every nonempty family of submodules of M has a minimal element.*

If M satisfies these equivalent conditions we say that M is Artinian.

Definition 1.4.8. *A ring R is right (left) Artinian if it is Artinian as a right (left) R-module. If R is both right and left Artinian we say that R is an Artinian ring.*

Lemma 1.4.9. *[GW04, Proposition 4.5] Let N be a submodule of M. Then M is Artinian if and only if N and M/N are Artinian. In particular, if R is a right Artinian ring, all finitely generated right R-modules are Artinian.*

Definition 1.4.10. *A ring is simple if its only ideals are* 0 *and R.*

In commutative algebra, the only simple ring are fields. However, in non commutative algebra there exist simple rings which are not division rings. For example, given a field k and an infinite order automorphism $\alpha : k \longrightarrow k$ the skew Laurent ring $k[x^{\pm 1}; \alpha]$ is simple (see [GW04, Example 1S]). Simple Artinian rings will play the role of fields in noncommutative ring theory.

Definition 1.4.11. *A composition series for a module M is a chain of submodules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that each of the factors M_i/M_{i-1} is a simple module. The number n is called the length of the composition series. We say that a module M has finite length if M has a composition series. By convention the module 0 *has length* 0*.*

Proposition 1.4.12. *[GW04, Proposition 4.8] A module M has finite length if and only if M is both Noetherian and Artinian.*

A finite length module can have different composition series. The Jordan-Hölder theorem also holds for modules over noncommutative rings:

Theorem 1.4.13. *[GW04, Theorem 4.11] Let M be a finite length module. Then all composition series of M have the same length.*

In particular, given a finite length module M we can define $\text{lenght}(M)$ to be the length of any of its composition series.

Lemma 1.4.14. *[GW04, Proposition 4.12] Let N be a submodule of a finite length module M. Then*

$$\text{lenght}(M) = \text{lenght}(N) + \text{lenght}(M/N).$$

Our next goal is to study prime ideals of Noetherian noncommutative rings. Recall that a proper ideal $\mathfrak{p}$ of R is prime if whenever I and J are ideals such that $IJ \subseteq \mathfrak{p}$ then $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$. Equivalently, $\mathfrak{p}$ is a prime ideal if and only if for any $x, y \in R$ satisfying $xRy \subseteq \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ (see [GW04, Proposition 3.1]). A prime ideal is said to be minimal if it does not properly contain any prime ideal.

Given a R-module M, the annihilator of M is

$$\text{ann}_R(M) = \{r \in R : xr = 0 \text{ for all } x \in M\}.$$

Note that $\text{ann}_R(M)$ is an ideal of R.

Definition 1.4.15. *A module M is faithful if $\text{ann}_R(M) = 0$. A module is fully faithful if all its non-zero submodules are faithful. A module M is said to be prime if it is fully faithful as a $R/\text{ann}_R(M)$-module.*

Theorem 1.4.16. *[GW04, Proposition 3.13, Proposition 3.14] Let M be a finitely generated module over a Noetherian ring R. Then there exists a series*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that:

1. *M_i/M_{i-1} is a prime module for each i.*
2. *$\text{ann}_R(M_i/M_{i-1}) = \mathfrak{p}_i$ is a prime ideal for each i.*
3. *If $\mathfrak{p}$ is a minimal prime over $\text{ann}_R(M)$ then there exists a number i such that $\mathfrak{p} = \mathfrak{p}_i$.*

Definition 1.4.17. *A ring R is prime if 0 is a prime ideal.*

Theorem 1.4.18. *[Lam91, (A) Connell's Theorem] Let R be a ring and G a group. Then the group ring RG is prime if and only if R is prime and G has no finite proper normal subgroups.*

In particular, if Γ is a finitely generated torsion-free nilpotent group then $\mathbb{Z}\Gamma$ is prime.

The results presented until now point are straightforward generalizations of the theory of commutative algebra. The next part of the section explains the theory of localization of prime ideals in noncommutative Noetherian rings, which present more difficulties than its commutative counterparts.

Given a ring R and a multiplicatively closed subset C, a right localization of R with respect C is a ring S together with a morphism $i : R \longrightarrow S$ such that:

1. $i(c)$ is a unit of S for each $c \in C$.

2. Every element of S is of the form $i(r)i(c)^{-1}$ for some $r \in R$ and $c \in C$.

3. $i(r)i(c)^{-1} = i(r')i(c)^{-1}$ if and only if there exists $d \in C$ such that $rd = r'd$.

Given another ring S' and a morphism $i' : R \longrightarrow S$ such that $i'(c)$ is a unit of S' for every $c \in C$ there exists a morphism $f : S \longrightarrow S'$ such that $f \circ i = i'$. Thus, S is unique and hence we will denote S by RC^{-1}. Notice that $\operatorname{Ker} i = \{r \in R : rc = 0 \text{ for some } c \in C\}$. Thus, if all elements of C are regular (they are not zero divisors) then condition 3 does not need the element d.

In commutative algebra, given a multiplicatively closed subset C we can always construct the localization RC^{-1}. This is not the case in noncommutative algebra:

Theorem 1.4.19. *[Bel88, Theorem 1.1] Let C be a multiplicatively closed set in R. The right localization RC^{-1} exists if and only if C satisfies the following 2 conditions:*

1. *(right Ore condition) For all $r \in R$ and $c \in C$ there exist $s \in R$ and $d \in C$ such that $rd = cs$.*

2. *(right reversible) For all $r \in R$ and $c \in C$ such that $cr = 0$ there exists $d \in C$ such that $rd = 0$.*

Lemma 1.4.20. *[Bel88] Let C be a multiplicatively closed subset of a right Noetherian ring R satisfying the right Ore condition. Then C is right reversible.*

Given a right R-module M, we can localize $MC^{-1} = M \otimes_R RC^{-1}$.

Lemma 1.4.21. *[Bel88, Lemma 1.3] Localizations of right R-modules is an exact functor and RC^{-1} is flat as left R-module.*

The next theorem shows that we can localize on the set of regular elements.

Theorem 1.4.22. *[Bel88, Theorem 1.10](Goldie theorem) Suppose that R is a right Noetherian ring and C is the set of regular elements of R. Then the localization RC^{-1} exists. Moreover, RC^{-1} is Artinian simple if and only if R is prime.*

We turn our attention to the localization of prime ideals. Given an ideal I of R we define $\mathcal{C}_R(I) = \{r \in R : r + I \text{ is regular in } R/I\}$. A prime ideal $\mathfrak{p}$ is right localizable if we can localize R with respect $\mathcal{C}_R(\mathfrak{p})$. Like in the commutative case, we will denote $R\mathcal{C}_R(\mathfrak{p})^{-1}$ by $R_{\mathfrak{p}}$.

Lemma 1.4.23. *[Bel88, Corollary 2.2] Let R be a right Noetherian ring and C a multiplicatively closed subset satisfying the right Ore condition. Then there exists a bijective inclusion-preserving correspondence between primes of R disjoint from C and primes of RC^{-1} given by $\mathfrak{p} \mapsto \mathfrak{p}C^{-1}$. If $C \subseteq \mathcal{C}_R(0)$, then the inverse is given by $\mathfrak{q} \mapsto \mathfrak{q} \cap R$.*

Corollary 1.4.24. *Let R be a right Noetherian ring and $\mathfrak{p}$ a right localizable prime ideal of R. Then $\mathfrak{p}R_{\mathfrak{p}}$ is the unique maximal ideal of $R_{\mathfrak{p}}$ and $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is Artinian. If R is prime then $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is simple Artinian.*

For our purpose, the main difference with commutative algebra is that in general the localization on a prime ideal may not exist. Therefore, the last result that we need is the following:

Theorem 1.4.25. *[Bel88, pg. 17](see also [Poo90, Ros71]) Let R be a commutative Noetherian ring and Γ a finitely generated torsion-free nilpotent group. Then every prime ideal of $R\Gamma$ is right localizable.*

1.5 Orbifolds

The aim of this section is to introduce the basic concepts of the theory of orbifolds, which will become relevant when studying iterated group actions which are not free. In particular, orbifolds will be used to prove theorem 25 and proposition 27. We will use [CJ19], [Thu22], [BG07, Chapter 4] and [ALR07] as references for generalities on orbifolds. We refer to [GGKRW18, §2] for an introduction to group actions on orbifolds. We will follow the classical theory of orbifolds, using local charts, instead of using Lie grupoids (for an introduction to orbifolds using Lie grupoids we refer to [Moe02, Ler10]).

Definition 1.5.1. *A local model is a pair $(\tilde{U}, \Gamma)$, where $\tilde{U}$ is an open connected set of $\mathbb{R}^n$ and Γ is a finite group acting effectively on $\tilde{U}$. A map between models $(\tilde{U}_1, \Gamma_1)$ and $(\tilde{U}_2, \Gamma_2)$ is a pair $(\phi_*, \tilde{\phi})$, where $\phi_* : \Gamma_1 \longrightarrow \Gamma_2$ is a group morphism and $\tilde{\phi} : \tilde{U}_1 \longrightarrow \tilde{U}_2$ is a ϕ_*-equivariant map.*

If $\tilde{\phi} : \tilde{U}_1 \longrightarrow \tilde{U}_2$ is an embedding then we say that $\tilde{\phi}$ is an embedding of models. In this situation, ϕ_* is injective.

Definition 1.5.2. *Let X be a topological space and $p \in X$. A local chart around p is a tuple $\mathcal{U} = (U, \tilde{U}, \Gamma, \pi)$ such that:*

1. *U is an open neighbourhood of p.*

2. *$(\tilde{U}, \Gamma)$ is a local model.*

3. *$\pi : \tilde{U} \longrightarrow U$ is a Γ-invariant map ($\pi(\gamma\tilde{u}) = \pi(\tilde{u})$ for all $\tilde{u} \in \tilde{U}$ and $\gamma \in \Gamma$) such that the*

following diagram commutes

$$\begin{array}{ccc} \tilde{U} & & \\ \downarrow & \searrow^{\pi} & \\ \tilde{U}/\Gamma & \xrightarrow{\cong} & U \end{array}$$

An orbifold atlas is a collection of local charts $\mathcal{A} = \{\mathcal{U}_\alpha\}_{\alpha \in A}$ *satisfying that* $\bigcup_{\alpha \in A} U_\alpha = X$ *and that for any* $p \in U_\alpha \cap U_\beta$ *there exists* $\mathcal{U}_\gamma \in \mathcal{A}$ *such that:*

(i) $p \in U_\gamma \subseteq U_\alpha \cap U_\beta$.

(ii) We have embeddings $(\tilde{U}_\gamma, \Gamma_\gamma) \longrightarrow (\tilde{U}_\alpha, \Gamma_\alpha)$ *and* $(\tilde{U}_\gamma, \Gamma_\gamma) \longrightarrow (\tilde{U}_\beta, \Gamma_\beta)$.

An atlas $\mathcal{A}$ *refines an atlas* $\mathcal{B}$ *if every chart in* $\mathcal{A}$ *admits and embedding in a chart in* $\mathcal{B}$. *Two atlases are equivalent if they have a common refinement.*

Definition 1.5.3. *A n-dimensional orbifold, denoted by* $\mathcal{O}$ *is a second-countable Hausdorff topological space* $|\mathcal{O}|$ *(called the underlying topological space) together with an equivalence class of orbifold atlas.*

Remark 1.5.4. *Usually the word orbifold refers to smooth orbifold, where we require the actions and the maps in definition 1.5.1 to be smooth. Our notion of orbifold is known as topological orbifold or orbispace (see [Che01]). Since the results we need hold for topological orbifolds as well as smooth orbifolds we will use the term orbifold without further adjective for the sake of brevity.*

Remark 1.5.5. *Given* $p \in \mathcal{O}$, *there always exists a local chart* $(U_p, \tilde{U}_p, \Gamma_p, \pi_p)$ *such that the preimage of* p *by* π_p *is a single point,* $\pi_p^{-1}(p) = \{\tilde{p}\}$. *Consequently,* $\tilde{p}$ *is fixed by* Γ_p. *Moreover,* Γ_p *only depends on* p *and not on the local chart. This type of local chart will be called good chart and denoted by* $\mathcal{U}_p$.

Given an orbifold $\mathcal{O}$ and a group Γ, we denote by $\Sigma_\Gamma = \{p \in \mathcal{O} : \Gamma_p \cong \Gamma\}$. In particular, $\Sigma_{\{e\}}$ is a manifold and a dense subset of $|\mathcal{O}|$. This follows from theorem 1.1.15.

Definition 1.5.6. *A map* $f : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ *between orbifolds is a continuous orbifold map if the map* $f : |\mathcal{O}_1| \longrightarrow |\mathcal{O}_2|$ *is continuous and for any* $p \in \mathcal{O}_1$ *and good charts* $\mathcal{U}_p$ *and* $\mathcal{U}_{f(p)}$ *such that* $f(U) \subseteq U_{f(p)}$ *there exists a (possibly non-unique) map of local models between* $(\tilde{U}_p, \Gamma_p)$ *and* $(\tilde{U}_{f(p)}, \Gamma_{f(p)})$ *such that the following diagram commutes*

$$\begin{array}{ccc} \tilde{U}_p & \xrightarrow{\tilde{f}_p} & \tilde{U}_{f(p)} \\ \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_{f(p)}} \\ U_p & \xrightarrow{f} & U_{f(p)} \end{array}$$

We say that $f : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ *is an orbifold homeomorphism if* f *has an inverse map* $f^{-1} : \mathcal{O}_2 \longrightarrow \mathcal{O}_1$ *which is continuous in the orbifold sense.*

Note that when $f : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ is a homeomorphism then $\Gamma_p \cong \Gamma_{f(p)}$ for all $p \in \mathcal{O}_1$ and given two lifts $\tilde{f}_1, \tilde{f}_2 : \tilde{U}_p \longrightarrow \tilde{U}_{f(p)}$ then there exists $\gamma \in \Gamma_{f(p)}$ such that $\tilde{f}_2 = \gamma \tilde{f}_1$.

Example 1.5.7. *A manifold M is an orbifold with $\Gamma_p = \{e\}$ for all $p \in M$. More in general, if a Lie group G acts effectively, properly and almost-freely on a manifold M, then the quotient M/G can be endowed with a structure of orbifold. If $\mathcal{O}$ is an orbifold homeomorphic to M/G with G is a discrete group then we will say that $\mathcal{O}$ is good. If G is finite then we will say that $\mathcal{O}$ is very good.*

If an orbifold $\mathcal{O}$ is not good then we will say that $\mathcal{O}$ is bad. For example, we consider the subsets of S^2, $B_N = S^2 \setminus S$ and $B_S = S^2 \setminus N$ where N and S are the north and south pole respectively. Thus, $S^2 = B_N \cup B_S$. Let $\tilde{B}_i = \mathbb{R}^2$ for $i = N, P$. Suppose that a_N and a_S are two natural numbers. We consider the local models $(\tilde{B}_i, \mathbb{Z}/a_i)$, where the $\mathbb{Z}/a_i$ acts by rotations of order a_i. We consider local charts $(B_i, \tilde{B}_i, \mathbb{Z}/a_i, \pi_i)$, where $\pi_i : \tilde{B}_i \longrightarrow B_i$ is obtained by composing the quotient map $\tilde{B}_i \longrightarrow \tilde{B}_i/(\mathbb{Z}/a_i)$ with the corresponding stereographic projection. It is straightforward to prove that $\{(B_N, \tilde{B}_N, \mathbb{Z}/a_N, \pi_N), (B_S, \tilde{B}_S, \mathbb{Z}/a_S, \pi_S)\}$ forms an orbifold atlas. Thus, we obtain an orbifold, whose underlying topological space is the sphere S^2, known as the (a_N, a_S)-football. If $a_S = 1$ then the south pole becomes a regular point and the corresponding orbifold is known as the a_N-teardrop.

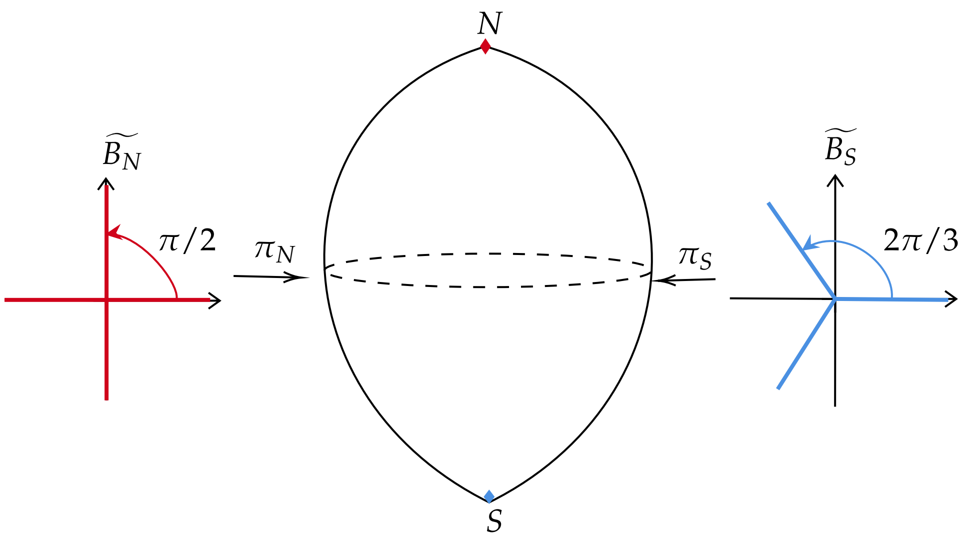


Figure 1.4: The $(4, 3)$-football orbifold.

It can be proved that the (a_N, a_S)-football orbifold is good if and only if $a_N = a_S$. In particular, the (a, a)-football orbifold is the orbit space of the action of $\mathbb{Z}/a$ by rotations around the axis of S^2 containing the north and south pole.

Remark 1.5.8. *Let $\mathcal{O}$ be an orbifold. The underlying space $|\mathcal{O}|$ is not necessarily a manifold. For example, consider the $\mathbb{Z}/2$ action on $\mathbb{R}^3$ given by the reflection with center the origin. Then $\mathcal{O} = \mathbb{R}^3/(\mathbb{Z}/2)$ is the cone of the projective space $\mathbb{R}P^2$. If $|\mathcal{O}|$ was a manifold then removing a point would not change the fundamental group of $|\mathcal{O}|$. But removing the cone of the vertex* *yields a space which is homotopically equivalent to $\mathbb{R}P^2$, whose fundamental group is not trivial. Consequently, $|\mathcal{O}|$ is not a manifold.*

We are ready to extend the notion of a covering map of manifolds to a covering map of orbifolds.

Definition 1.5.9. *A covering map of orbifolds is a continuous orbifold map $\rho : \tilde{\mathcal{O}} \longrightarrow \mathcal{O}$ such that:*

1. *For any $p \in \mathcal{O}$ there exists a local chart $\mathcal{U} = (U, \tilde{U}, \Gamma, \pi)$ such that $\rho^{-1}(U)$ is a disjoint union of open subsets V_i.*
2. *For each V_i we have a local chart $\mathcal{V}_i = (V_i, \tilde{V}_i, \Gamma_i, \pi_i)$ which makes the next diagram commutative*

$$\begin{array}{ccc} \tilde{V}_i & \xrightarrow{\cong} & \tilde{U} \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi} \\ V_i & \xrightarrow{\rho} & U \end{array}$$

and the induced map $(\rho_i)_* : \Gamma_i \longrightarrow \Gamma$ is an injective group morphism.

Example 1.5.10. *A covering of manifolds $\tilde{M} \longrightarrow M$ is a covering of orbifolds. If G is a finite group acting effectively on a manifold M, then $M \longrightarrow M/G$ is a covering of orbifolds.*

Definition 1.5.11. *A covering $\rho : \tilde{\mathcal{O}} \longrightarrow \mathcal{O}$ is a universal covering if for any other covering $\rho' : \mathcal{O}' \longrightarrow \mathcal{O}$ there exists a covering $\tau : \tilde{\mathcal{O}} \longrightarrow \mathcal{O}'$ such that $\rho = \rho' \circ \tau$.*

Theorem 1.5.12. *[Thu22, Proposition 13.2.4] Any connected orbifold has a universal cover.*

We will denote the universal cover of an orbifold $\mathcal{O}$ by $\tilde{\mathcal{O}}$.

Definition 1.5.13. *Let $\rho : \overline{\mathcal{O}} \longrightarrow \mathcal{O}$ be a covering, then*

$$\mathrm{Aut}(\rho) = \{f : \overline{\mathcal{O}} \longrightarrow \overline{\mathcal{O}} : \ f \textit{ is an orbifold homeomorphism and } \rho \circ f = \rho\}.$$

If $\mathcal{O}$ is connected and $\rho : \tilde{\mathcal{O}} \longrightarrow \mathcal{O}$ is the universal cover then $\mathrm{Aut}(\rho) = \pi_1^{orb}(\mathcal{O})$ (see [CJ19, Chapter 2]).

Proposition 1.5.14. *[CJ19, Proposition 2.3.5] The set of isomorphism classes of coverings of $\mathcal{O}$ is in bijection with the conjugacy classes of $\pi_1^{orb}(\mathcal{O})$.*

Remark 1.5.15. *If M is a manifold then $\pi_1^{orb}(M) \cong \pi_1(M)$.*

Any covering $\rho : \overline{\mathcal{O}} \longrightarrow \mathcal{O}$ induces an injection $\pi_1^{orb}(\overline{\mathcal{O}}) \longrightarrow \pi_1^{orb}(\mathcal{O})$. We say that the covering is regular if $\pi_1^{orb}(\overline{\mathcal{O}}) \trianglelefteq \pi_1^{orb}(\mathcal{O})$.

Lemma 1.5.16. *[ALR07, Example 2.20] Let G be a finite group acting on effectively on a manifold M, then we have a short exact sequence*

$$1 \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(M/G) \longrightarrow G \longrightarrow 1.$$

Lemma 1.5.17. *[CJ19, Example 2.3.2] Let G be a discrete group acting effectively and properly on M and $G' \leq G$. Then $M/G' \longrightarrow M/G$ is an orbifold covering.*

Proposition 1.5.18. *[ALR07, Lemma 2.22] Let $\mathcal{O}$ be an orbifold. The following statements are equivalent:*

1. *$\mathcal{O}$ is good.*
2. *$\tilde{\mathcal{O}}$ is a manifold.*
3. *We have an injection $\Gamma_p \longrightarrow \pi_1^{orb}(\mathcal{O})$ for all $p \in \mathcal{O}$.*

Definition 1.5.19. *An orbifold $\mathcal{O}$ is said to be aspherical if its universal cover $\tilde{\mathcal{O}}$ is a contractible manifold.*

With the introduction of regular covering of orbifolds, it is natural to ask which is the best way to define the concept of (finite) group action on an orbifold.

Definition 1.5.20. *Let G be a Lie group and $\mathcal{O}$ an orbifold. An orbifold action of G on $\mathcal{O}$ is a continuous orbifold map $\phi : G \times \mathcal{O} \longrightarrow \mathcal{O}$ such that $\phi(g_1 g_2, x) = \phi(g_1, \phi(g_2, x))$ and $\phi(e, x) = x$ for all $g_1, g_2 \in G$ and $x \in \mathcal{O}$.*

In particular, we have a continuous action of G on the underlying topological space $|\mathcal{O}|$. Moreover, a finite group G acts on an orbifold $\mathcal{O}$ if and only if G acts continuously on $|\mathcal{O}|$ and the map $\phi(g, \cdot) : \mathcal{O} \longrightarrow \mathcal{O}$ is an orbifold homeomorphism for all $g \in G$.

The next lemmas summarize some of the properties of group actions on orbifolds.

Lemma 1.5.21. *[GGKRW18, Lemma 2.11] Let G be a compact Lie group acting on an orbifold $\mathcal{O}$. The orbits are homogeneous manifolds.*

Lemma 1.5.22. *[GGKRW18, Lemma 2.12] Let G be a compact Lie group acting on an orbifold $\mathcal{O}$. Pick $x \in \mathcal{O}$. Let G_x denote the isotropy subgroup of x and let U_x be a G_x-invariant local chart. Then there exists a compact Lie group $\tilde{G}_x$ acting effectively on $\tilde{U}_x$ such that $\tilde{U}_x/\tilde{G}_x = U_x/G_x$ and such that $\tilde{G}_x$ fits on a short exact sequence*

$$1 \longrightarrow \Gamma_x \longrightarrow \tilde{G}_x \longrightarrow G_x \longrightarrow 1.$$

Moreover, there exists a G_x-invariant good local chart (which we also denote by U_x) such that any $y \in U_x$ has a G_y-invariant good local chart $U_y \subseteq U_x$ such that we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_y & \longrightarrow & \tilde{G}_y & \longrightarrow & G_y & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Gamma_x & \longrightarrow & \tilde{G}_x & \longrightarrow & G_x & \longrightarrow & 1 \end{array}$$

where the vertical arrows are inclusions.

Lemma 1.5.23. *[BG07, Theorem 4.3.18] Let G be a finite group acting effectively on an orbifold $\mathcal{O}$. Then the orbit space $|\mathcal{O}|/G$ has an orbifold structure (denoted by $\mathcal{O}/G$) and the orbit map $p : \mathcal{O} \longrightarrow \mathcal{O}/G$ is a regular orbifold covering. There exists a short exact sequence*

$$1 \longrightarrow \pi_1^{orb}(\mathcal{O}) \longrightarrow \pi_1^{orb}(\mathcal{O}/G) \longrightarrow G \longrightarrow 1.$$

In particular, given $x \in \mathcal{O}$, we have a short exact sequence

$$1 \longrightarrow \Gamma_x \longrightarrow \Gamma_{p(x)} \longrightarrow G_x \longrightarrow 1.$$

Finally, we generalize theorem 1.3.96. If $\mathcal{O}$ is a closed connected aspherical orbifold and G is a finite group acting effectively on $\mathcal{O}$ then there exists a short exact sequence

$$1 \longrightarrow \pi_1^{orb}(\mathcal{O}) \longrightarrow \pi_1^{orb}(\mathcal{O}/G) \longrightarrow G \longrightarrow 1.$$

Consequently, we have an abstract kernel $\psi : G \longrightarrow \mathrm{Out}(\pi_1^{orb}(\mathcal{O}))$.

Theorem 1.5.24. *Let $\mathcal{O}$ be a closed connected aspherical very good orbifold and let G be a finite group acting smoothly on $\mathcal{O}$. Assume that $\mathcal{Z}\pi_1^{orb}(\mathcal{O})$ is finitely generated of rank k. Then:*

1. $\mathrm{Ker}\,\psi$ *is a subgroup of T^k.*
2. *If the action has a fix point x, then the morphism $G \longrightarrow \mathrm{Aut}(\pi_1^{orb}(\mathcal{O}, x))$ is injective.*

Proof. Since $\mathcal{O}$ is very good, it is finitely covered by a manifold. In consequence, there exists a torsion-free normal finite index subgroup $\Gamma \trianglelefteq \pi_1^{orb}(\mathcal{O})$ which acts freely on the manifold $\tilde{\mathcal{O}}$. Note that $\mathcal{Z}\pi_1^{orb}(\mathcal{O})$ is torsion-free, since $C_{\pi_1^{orb}(\mathcal{O})}(\Gamma)$ is torsion-free by theorem 1.3.96 and $\mathcal{Z}\pi_1^{orb}(\mathcal{O}) \leq C_{\pi_1^{orb}(\mathcal{O})}(\Gamma)$.

Like in theorem 1.3.96, we will prove that $C_{\pi_1^{orb}(\mathcal{O}/G)}(\pi_1^{orb}(\mathcal{O}))$ is torsion-free. Assume on the contrary, then there exists a prime p such that $\mathbb{Z}/p \leq C_{\pi_1^{orb}(\mathcal{O}/G)}(\pi_1^{orb}(\mathcal{O}))$. Let $\tilde{F}$ be the proper $\mathbb{Z}/p$-acyclic subset $\tilde{F} = \tilde{\mathcal{O}}^{\mathbb{Z}/p}$. Since Γ and $\mathbb{Z}/p$ commute, we have a free group action of Γ on $\tilde{F}$. The same arguments from the proof of theorem 1.3.96 imply that this is not possible. Thus, $C_{\pi_1^{orb}(\mathcal{O}/G)}(\pi_1^{orb}(\mathcal{O}))$ is torsion-free.

We are ready to prove the first part. We have a central short exact sequence

$$1 \longrightarrow \mathcal{Z}\pi_1^{orb}(\mathcal{O}) \longrightarrow C_{\pi_1^{orb}(\mathcal{O}/G)}(\pi_1^{orb}(\mathcal{O})) \longrightarrow \mathrm{Ker}\,\psi \longrightarrow 1.$$

Since the first two group are torsion-free, we have $\mathrm{Ker}\,\psi \leq T^{\mathrm{rank}\,\mathcal{Z}\pi_1^{orb}(\mathcal{O})}$, as we wanted to see.

For the second part, since the action has a fix point we have a group morphism section $G \longrightarrow \pi_1^{orb}(\mathcal{O}/G)$. We can use the commutative diagram from lemma 1.3.95 together with the fact that $C_{\pi_1^{orb}(\mathcal{O}/G)}(\pi_1^{orb}(\mathcal{O}))$ is torsion-free to deduce that the group morphism $G \longrightarrow \mathrm{Aut}(\pi_1^{orb}(\mathcal{O}, x))$ is injective. □

There exist closed connected orbifolds which are good but not very good (see [Lan24]). However, it is not known whether all closed connected aspherical orbifolds are very good (see [Lan24, Question 2.2]). From Selberg's lemma (theorem 1.3.67), we have:

Proposition 1.5.25. *[Lan24, Proposition 2.1] A nonpostively curved compact locally symmetric orbifold is very good.*

Chapter 2

Large finite group actions on aspherical manifolds

In this section we study finite group actions on aspherical manifolds, proving theorem 7, theorem 10, theorem 11, as well as proposition 8, proposition 9 and proposition 12. Let us recall the statement of these results. First, we prove a general theorem for large finite group actions on closed connected aspherical manifolds:

Theorem 2.0.1. *Let M be a closed connected n-dimensional aspherical manifold such that $\mathcal{Z}\pi_1(M)$ is finitely generated and* $\mathrm{Out}(\pi_1(M))$ *is Minkowski. Then:*

1. $\mathrm{Homeo}(M)$ *is Jordan.*
2. $\text{disc-sym}(M) \leq \operatorname{rank} \mathcal{Z}\pi_1(M) \leq n$*, and* $\text{disc-sym}(M) = n$ *if and only if M is homeomorphic to T^n.*
3. *If $\chi(M) \neq 0$ then M is almost-asymmetric.*
4. *If* $\mathrm{Aut}(\pi_1(M))$ *is Minkowski, then M has the small stabilizers property and the few stabilizers property.*

It is important to know when the hypothesis of theorem 2.0.1 are satisfied. Thus, we prove:

Theorem 2.0.2. *Let Γ be a lattice in a connected Lie group G. Then* $\mathrm{Out}(\Gamma)$ *and* $\mathrm{Aut}(\Gamma)$ *are Minkowski.*

Proposition 2.0.3. *Let $M = M_1 \times \cdots \times M_m$, where M_i are a closed aspherical manifolds such that $\pi_1(M_i)$ is hyperbolic and* $\dim(M_i) \geq 3$*. Then* $\mathrm{Out}(\pi_1(M))$ *is finite and* $\mathrm{Aut}(\pi_1(M))$ *is Minkowski.*

In particular, we can use theorem 2.0.1 on closed connected aspherical locally homogeneous spaces $H \setminus G/\Gamma$, where G is a connected Lie group, H is a maximal compact subgroup of G and Γ is a torsion-free cocompact lattice of G. Note that part 2 of theorem 2.0.1 only gives an upper bound on the discrete degree of symmetry. We prove that this bound is reached for closed connected aspherical locally homogeneous manifolds.

Theorem 2.0.4. *Let $H \setminus G/\Gamma$ be a closed aspherical locally homogeneous space. Then* $\text{disc-sym}(H \setminus G/\Gamma) = \text{rank}\, \mathcal{Z}\Gamma$.

Finally, we also prove two further results about large finite group actions on aspherical manifolds.

Proposition 2.0.5. *There exists a closed connected aspherical manifold M such that* $\text{Homeo}(M)$ *is Jordan and $H^*(M) \cong H^*(T^2 \times S^3)$.*

Proposition 2.0.6. *Let M be a closed connected n-dimensional aspherical manifold such that $\mathcal{Z}\pi_1(M)$ is finitely generated,* $\text{Out}(\pi_1(M))$ *is Minkowski and* $\text{Inn}\,\pi_1(M)$ *has an element of infinite order. If* $\text{disc-sym}(M) = n-1$ *then $M \cong T^{n-2} \times K$ or $M \cong T^{n-3} \times SK$*

This chapter is divided in six sections. In the first section is devoted to prove theorem 2.0.1 and we also prove some results about the relation between the discrete degree of symmetry and finite coverings. The second and third section are devoted to prove theorem 2.0.2 in the case of solvmanifolds and locally homogeneous spaces. In section 4, we prove theorem 2.0.2 in full generality and we also prove proposition 2.0.3 and theorem 2.0.4. Finally, in section 5 we prove proposition 2.0.5 and in section 6 we prove proposition 2.0.6.

2.1 Finite group actions on aspherical manifolds: proof of theorem 2.0.1

We start this section by proving theorem 2.0.1. The main tool is theorem 1.3.96.

Proof of part 1. of theorem 2.0.1. Let C be the Minkowski constant of $\text{Out}(\pi_1(M))$. If G is a finite group acting effectively on M, then $\text{Ker}\,\psi$ is an abelian subgroup of G by theorem 1.3.96 and $[G : \text{Ker}\,\psi] = |G/\text{Ker}\,\psi| \leq C$ since $G/\text{Ker}\,\psi \leq \text{Out}(\pi_1(M))$. Thus $\text{Homeo}(M)$ is Jordan. □

To prove the second part we need the following group theoretic results (the second one due to Schur).

Lemma 2.1.1. *[MiR24a, Lemma 2.1] Let a, b, C be natural numbers and suppose that G is a subgroup of $(\mathbb{Z}/a)^b$ such that $[(\mathbb{Z}/a)^b : G] \leq C$. Then there exists a subgroup $G' \leq G$ isomorphic to $(\mathbb{Z}/a')^b$ such that $C!a' \geq a$.*

Lemma 2.1.2. *[Rob13, Theorem 4.12] Let Γ be a finitely generated group such that $[\Gamma : \mathcal{Z}\Gamma] < \infty$. Then the commutator subgroup $[\Gamma, \Gamma]$ is finite.*

Proof of part 2. of theorem 2.0.1. We start proving the inequality disc-sym$(M) \leq k$. Suppose that $\{a_i\}_{i\in\mathbb{N}}$ is a sequence of natural numbers such that $a_i \longrightarrow \infty$ and such that $(\mathbb{Z}/a_i)^b$ acts effectively on M for some $b \in \mathbb{N}$. We have induced group morphisms $\psi_i : (\mathbb{Z}/a_i)^b \longrightarrow \mathrm{Out}(\pi_1(M))$ for each i such that $[(\mathbb{Z}/a_i)^b : \mathrm{Ker}\,\psi_i] \leq C$. By lemma 2.1.1, there exists a sequence $\{a_i'\}_{i\in\mathbb{N}}$ such that $(\mathbb{Z}/a_i')^b \leq \mathrm{Ker}\,\psi_i$. Since $a_i'C! \geq a_i$, we have $a_i' \longrightarrow \infty$. Moreover, $(\mathbb{Z}/a_i')^b$ is a subgroup of T^k for any i, so we can conclude that $b \leq k$. In consequence disc-sym$(M) \leq k$. To prove the inequality $k \leq n$ we take the n-dimensional manifold $\tilde{M}/\mathcal{Z}\pi_1(M)$, where $\tilde{M}$ is the universal cover of M. Since $\tilde{M}$ is contractible, we have that $H^*(\tilde{M}/\mathcal{Z}\pi_1(M)) \cong H^*(T^k, \mathbb{Z})$. The fact that $\tilde{M}/\mathcal{Z}\pi_1(M)$ has dimension n implies that $H^i(\tilde{M}/\mathcal{Z}\pi_1(M), \mathbb{Z}) = 0$ for $i > n$, hence $k \leq n$.

Finally, we prove that disc-sym$(M) = n$ if and only if $M \cong T^n$. Clearly, disc-sym$(T^n) = n$ since $\mathrm{Out}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Z})$ is Minkowski. Conversely, assume that disc-sym$(M) = n$. Since the top cohomology $H^n(\tilde{M}/\mathcal{Z}\pi_1(M), \mathbb{Z})$ is non-zero, we can conclude that $\tilde{M}/\mathcal{Z}\pi_1(M)$ is a closed connected manifold and that the map $\tilde{M}/\mathcal{Z}\pi_1(M) \longrightarrow M$ is a regular finite cover. In consequence, $[\pi_1(M) : \mathcal{Z}\pi_1(M)] < \infty$ and $[\pi_1(M), \pi_1(M)]$ is trivial, by lemma 2.1.2 and the fact that $\pi_1(M)$ is torsion-free. Thus $\pi_1(M) \cong \mathbb{Z}^n$ and $M \cong T^n$, because the Borel conjecture holds for T^n. □

For the third part we need the following result:

Lemma 2.1.3. *[Got65, Theorem IV.1.] Let X be a topological space with the homotopy type of a compact connected aspherical CW-complex. If $\chi(X) \neq 0$, then* $\mathrm{rank}\,\mathcal{Z}\pi_1(X) = 0$.

Proof of part 3. of theorem 2.0.1. The manifold M has the homotopy type of a compact connected aspherical CW-complex, since M is a closed connected aspherical manifold. If $\chi(M) \neq 0$ then disc-sym$(M) \leq \mathrm{rank}\,\mathcal{Z}\pi_1(M) = 0$. In consequence disc-sym$(M) = 0$ and M is almost asymmetric. □

Finally, to prove the last part is a consequence of lemma 1.1.65:

Proof of part 4. of theorem 2.0.1. We have already seen that $\mathrm{Homeo}(M)$ is Jordan. Since a finite group G acting on M with a fix point is a subgroup of $\mathrm{Aut}(\pi_1(M))$ and $\mathrm{Aut}(\pi_1(M))$ is Minkowski, M has the small stabilizers property. Thus, part 4 is a consequence of lemma 1.1.65. □

This ends the proof of theorem 2.0.1. There are some interesting and natural questions that we summarise in the next remarks.

Remark 2.1.4. *(The hypothesis on the fundamental group) When are the two hypothesis on the fundamental group satisfied? No closed aspherical manifold with $\mathcal{Z}\pi_1(M)$ not finitely generated is* *known (see [LR10, Remark 3.1.19.]). However, there exists a finitely presented group G such that* $\mathcal{Z}G \cong (\mathbb{Q}, +)$*, which is not finitely generated (see [Hou07, Theorem II]).*

It is an interesting problem to find which closed connected aspherical manifolds M satisfy that $\mathrm{Out}(\pi_1(M))$ *is Minkowski (see [Gol23, Remark 7.1]). The next examples show some case where* $\mathrm{Out}(\pi_1(M))$ *is Minkowski:*

1. *In this thesis we prove that aspherical locally homogeneous spaces (or classical aspherical manifolds following the terminology on [FJ90]) satisfy that* $\mathrm{Out}(\pi_1(M))$ *is Minkowski.*

2. *Another source of closed connected aspherical manifolds is the strict hyperbolization processes (see [CD95] and references therein). Given a closed oriented manifold* M' *of dimension* $n \geq 3$ *we can construct a closed oriented aspherical manifold M and a non-zero degree map* $f : M \longrightarrow M'$ *such that* $\pi_1(M)$ *is a hyperbolic group. These groups satisfy that* $\mathrm{Out}(\pi_1(M))$ *is finite (see [Gro87, 5.4.A] or [Pau91]). Moreover* $\mathcal{Z}\pi_1(M) = 0$ *and therefore* M' *is almost-asymmetric.*

3. *If M is a closed connected aspherical 3-dimensional manifold, then* $\mathrm{Out}(\pi_1(M))$ *is Minkowski (see [Koj84]). This fact was used in [Zim14] to prove that* $\mathrm{Diff}(M)$ *is Jordan when M is a closed smooth 3-manifold.*

4. *If* $\mathrm{Out}(\pi_1(M))$ *is a finitely generated virtually abelian group then it is Minkowski. This is the case for piecewise linear locally symmetric spaces, a new type of closed aspherical manifold defined in [TNP11] by compactifying non-compact symmetric spaces and using the reflection trick on its corners. If M is a piecewise linear locally symmetric space then* $\mathrm{Out}(\pi_1(M))$ *is finitely generated and virtually abelian by [TNP11, Theorem 4]. Moreover,* $\mathcal{Z}\pi_1(M) = \{e\}$ *by [TNP11, Lemma 7], hence piecewise linear locally symmetric spaces are almost asymmetric.*

Remark 2.1.5. *(Groups whose (outer) automorphism group is not Minkowski) There exist groups* Γ *such that* $B\Gamma$ *is a finite CW-complex and* $\mathrm{Out}(\Gamma)$ *is not Minkowski. For example, let* Γ *be the Baumslag-Solitar group* $B(m, ml) = \langle a, b | ba^m b^{-1} = a^{ml} \rangle$ *with* $m, l \geq 2$*. The space* $BB(m, ml)$ *is a finite aspherical 2-dimensional CW-complex since it is a torsion-free one relator group (see [LSLS77]). Moreover* $\mathrm{Out}(B(m, ml))$ *and* $\mathrm{Aut}(B(m, ml))$ *have elements of order* $l^t(l-1)$ *for arbitrarily large t (see [CL83, Lemma 3.8] or [Lev07]) and thus they are not Minkowski.*

Remark 2.1.6. *(The discrete degree of symmetry versus the toral degree of symmetry) The question of when* $\text{tor-sym}(M)$ *is equal to* $\mathrm{rank}\,\mathcal{Z}\pi_1(M)$ *has been extensively studied for closed connected aspherical manifolds M. We also note that any effective torus action on a closed aspherical manifold M is almost-free (see [LR10, Corollary 3.1.12]), therefore* $\text{tor-sym}(M) = \mathrm{rank}(M)$*.*

If $\mathrm{Out}(\pi_1(M))$ *is Minkowski and* $\text{tor-sym}(M) = \mathrm{rank}\,\mathcal{Z}\pi_1(M)$ *then we have* $\text{disc-sym}(M) = \mathrm{rank}\,\mathcal{Z}\pi_1(M)$*, since* $\text{tor-sym}(M) \leq \text{disc-sym}(M) \leq \mathrm{rank}\,\mathcal{Z}\pi_1(M)$*. The equality is known to hold for infra-solvmanifolds or some aspherical locally homogeneous spaces (see [LR10, Section 11.7]). In theorem 2.0.4 we prove that the equality is valid for closed aspherical locally homogeneous* *spaces. On the other hand, there exist closed aspherical manifolds such that* $\text{tor-sym}(M) = 0$ *and* $\text{disc-sym}(M) \geq 1$ *(see [CWY13, MiR24a]). It is an interesting question whether all closed connected aspherical manifolds satisfy* $\text{disc-sym}(M) = \text{rank}\, \mathcal{Z}\pi_1(M)$.

Remark 2.1.7. *(Euler characteristic and asymmetry) The converse of part 3 of theorem 2.0.1 is not true. There exist asymmetric flat manifolds (see [Szc12]). The fundamental group of a closed flat manifold satisfies the hypothesis of theorem 2.0.1 and the manifold Euler characteristic is* 0 *since they are finitely covered by a torus.*

Remark 2.1.8. *(Non-compact aspherical manifolds) If M is a non-compact connected aspherical manifold then* $\text{Homeo}(M)$ *is not necessarily Jordan, even when* $\text{Out}(\pi_1(M))$ *is Minkowski. For example, since* $\mathbb{R}^3$ *admits effective actions by* $\text{SO}(3)$ *then* $\text{Homeo}(T^2 \times \mathbb{R}^3)$ *is not Jordan by [MiR17, Theorem 1].*

Remark 2.1.9. *(Large finite group actions on very good aspherical orbifolds) Note that we can use theorem 1.5.24 to replace closed connected aspherical manifold on theorem 2.0.1 by closed connected very good aspherical orbifold in parts 1,2 and 4.*

We end this section by exposing some facts on the relation between large finite group actions on manifolds and covering maps.

Lemma 2.1.10. *Let M and M' be closed connected manifolds and $p : M' \longrightarrow M$ be a finite covering. Then:*

1. *If* $\text{Homeo}(M')$ *is Jordan, then* $\text{Homeo}(M)$ *is Jordan.*
2. $\text{disc-sym}(M') \geq \text{disc-sym}(M)$.
3. *Assume that there exists a constant D' such that any finite group G' acting effectively on M' with a fix point satisfies $|G'| \leq D'$. Then there exists a constant D such that any finite group G acting effectively on M with a fix point satisfies $|G| \leq D$*

Proof. The first two parts are proven in [MiR10, §2.3] and [MiR24a, Theorem 1.12]. The proof of the third part follows the same arguments as the proofs of the first two parts.

Assume that $p : M' \longrightarrow M$ is a n-sheeted covering and G is a finite group acting effectively on M. Then G also acts on $\text{Cov}_n(M)$, the set of n-sheeted coverings of M, by pull-backs. On the other hand $\text{Cov}_n(M) \cong \text{Hom}(\pi_1(M), S_n)/\sim$ where S_n is the n-th symmetric group and the equivalence relation is given by conjugation of elements of S_n. Therefore $\text{Cov}_n(M)$ is finite, which implies that there exists a constant C only depending on M and n such that any finite group G acting effectively on M has a subgroup G_0 which acts trivially on $\text{Cov}_n(M)$ and $[G : G_0] \leq C$. Then there exists a finite group G_0' acting effectively on M' and a surjective group morphism $\pi : G_0' \longrightarrow G_0$ which makes the covering map $p : M' \longrightarrow M$ π-equivariant and $|\operatorname{Ker} \pi| \leq n!$.

Let G be a finite group acting effectively on M with a fix point $x \in M$. Then x is also fixed by the action of G_0 and G'_0 acts on $p^{-1}(x)$. Given $x' \in p^{-1}(x)$, the orbit of x' by G'_0 is $G'_0/G'_{0x'} \subseteq p^{-1}(x)$, which implies that $|G'_0/G'_{0x'}| \leq n$. Finally, $G'_{0x'}$ acts effectively on M' with a fixed point, therefore $|G'_{0x'}| \leq D'$. Since π is surjective, we can take $D = C \cdot D' \cdot n$. $\square$

In consequence, if $p : M' \longrightarrow M$ is a covering of closed connected aspherical manifolds and M' satisfies the hypothesis of theorem 2.0.1 then all the conclusions of theorem 2.0.1 also hold for M. This fact can also be deduced for regular coverings using lemma 1.2.24.

Given a regular covering $p : M' \longrightarrow M$, it is an interesting question to determine when $\text{disc-sym}(M) = \text{disc-sym}(M')$. The next proposition will be used to give a partial answer to this question for the case of closed connected aspherical manifolds, which will be given in corollary 2.1.12.

Proposition 2.1.11. *Let M be a closed connected aspherical manifold such that $\mathcal{Z}\pi_1(M)$ is finitely generated. Assume that G is a finite group acting freely on M. Then* $\text{rank}(\mathcal{Z}\pi_1(M))$ *and* $\text{rank}(\mathcal{Z}\pi_1(M/G))$ *are equal if and only if the map $\psi' : G \longrightarrow \text{Out}(\mathcal{Z}\pi_1(M))$ is trivial.*

Proof. Recall that the free action of G on M induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{Z}\pi_1(M) & \longrightarrow & C_{\tilde{G}}(\pi_1(M)) & \longrightarrow & \operatorname{Ker}\psi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(M) & \longrightarrow & \tilde{G} & \xrightarrow{\;p\;} & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \tilde{\psi}} & & \downarrow{\scriptstyle \psi} & & \\
1 & \longrightarrow & \operatorname{Inn}\pi_1(M) & \longrightarrow & \tilde{\psi}(\tilde{G}) & \longrightarrow & \psi(G) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $\tilde{\psi}(\tilde{G}) \leq \text{Aut}(\pi_1(M))$ and $\psi(G) \leq \text{Out}(\pi_1(M))$. We also note that $\mathcal{Z}\tilde{G} \trianglelefteq C_{\tilde{G}}(\pi_1(M))$ and $\tilde{G} = \pi_1(M/G)$ is torsion-free. Recall also that $\tilde{\psi}(\tilde{g}) = c_{\tilde{g}|\pi_1(M)}$ and that $\psi(g) = [c_{\sigma g|\pi_1(M)}]$ where $\sigma : G \longrightarrow \tilde{G}$ is a set-theoretic section of p.

Let $k = \text{rank}(\mathcal{Z}\pi_1(M))$. Since $\mathcal{Z}\pi_1(M)$ is a characteristic subgroup of $\pi_1(M)$, we have maps $\tilde{\psi}' : \tilde{G} \longrightarrow \text{GL}(k, \mathbb{Z})$ and $\psi' : G \longrightarrow \text{GL}(k, \mathbb{Z})$ by restricting (outer) automorphisms to $\mathcal{Z}\pi_1(M)$. Moreover, $\tilde{\psi}' = \psi' \circ p$ and since p is surjective we can conclude that $\tilde{\psi}'$ is trivial if and only if ψ' is trivial.

Assume now that ψ' is trivial and hence $\tilde{\psi}'$ is trivial too. This implies that $[\tilde{g}, z] = e$ for any $\tilde{g} \in \tilde{G}$ and any $z \in \mathcal{Z}\pi_1(M)$. Thus $\mathcal{Z}\pi_1(M) \trianglelefteq \mathcal{Z}\tilde{G} \trianglelefteq C_{\tilde{G}}(\pi_1(M))$. But $\text{rank}(\mathcal{Z}\pi_1(M)) = \text{rank}(C_{\tilde{G}}(\pi_1(M)))$ since the first row of the commutative diagram is a central exact sequence and $C_{\tilde{G}}(\pi_1(M))$ is torsion-free. In consequence, $\text{rank}(\mathcal{Z}\pi_1(M)) = \text{rank}(\mathcal{Z}\tilde{G})$.

If $\text{rank}(\mathcal{Z}\pi_1(M)) = \text{rank}(\mathcal{Z}\tilde{G})$ then $[C_{\tilde{G}}(\pi_1(M)) : \mathcal{Z}\tilde{G}] < \infty$. This implies that $\tilde{\psi}'(\tilde{G}) \leq \text{GL}(k, \mathbb{Z})$ fixes a sublattice of $\mathcal{Z}\pi_1(M)$ and therefore $\tilde{\psi}'$ is trivial. Thus ψ' is also trivial, as desired. □

Corollary 2.1.12. *Let M be a closed connected aspherical manifold such that $\mathcal{Z}\pi_1(M)$ is finitely generated,* $\text{Out}(\pi_1(M))$ *is Minkowski and* $\text{disc-sym}(M) = \text{rank}\,\mathcal{Z}\pi_1(M)$. *Assume that G is a finite group acting freely on M such that ψ' is not trivial. Then* $\text{disc-sym}(M/G) < \text{disc-sym}(M)$.

See remark 2.1.6 for examples where the hypothesis of the corollary holds true.

2.2 Nilmanifolds and solvmanifolds

In this section we study finite group actions on nilmanifolds and solvmanifolds.

Theorem 2.2.1. *[Weh94] Let Γ be a virtually polycyclic group, then* $\text{Out}(\Gamma)$ *is isomorphic to a subgroup of* $\text{GL}(n, \mathbb{Z})$ *for some n.*

The proof of this theorem uses the analogue statement for the automorphism group.

Theorem 2.2.2. *[Mer70] Let Γ be a virtually polycyclic group, then* $\text{Aut}(\Gamma)$ *is isomorphic to a subgroup of* $\text{GL}(n, \mathbb{Z})$ *for some n.*

Hence, if Γ is virtually polycyclic, then $\text{Out}(\Gamma)$ and $\text{Aut}(\Gamma)$ are Minkowski. In consequence, by using theorem 2.0.1 we can conclude that a closed solvmanifold M has Jordan homeomorphism group, $\text{disc-sym}(M) \leq \text{rank}(\mathcal{Z}\pi_1(M))$ (and indeed $\text{disc-sym}(M) = \text{rank}(\mathcal{Z}\pi_1(M))$ by the results in [LR10, Section 11.7]) and M has few stabilizers. These properties are also satisfied by any manifold finitely covered by a solvmanifold. This includes flat manifolds, almost-flat manifolds and infra-solvmanifolds.

Example 2.2.3. *Let us show a low dimensional example in order to illustrate some consequences of theorem 2.0.1 on nilmanifolds. Recall that the 3-dimensional Heisenberg group*

$$H = \{(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R}\}$$

, which is a simply connected nilpotent Lie group. Any lattice of H is isomorphic to a lattice of the

form

$$\Gamma_k = \{(x,y,z) = \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{Z}\}$$

where k is a positive integer. Recall that $\mathcal{Z}\Gamma_k = \langle(0,0,1)\rangle$ *and two lattices* $\Gamma_k \cong \Gamma_l$ *are isomorphic if and only if* $k = l$*. A possible presentation of these lattices is* $\Gamma_k = \langle a,b,c | [c,a] = [c,b] = 1, [a,b] = c^k\rangle$ *where* $a = (1,0,0)$*,* $b = (0,1,0)$ *and* $c = (0,0,1)$*. Thus* $\mathcal{Z}\Gamma_k = \langle c\rangle \cong \mathbb{Z}$ *for all k.*

The automorphism group $\mathrm{Aut}(\Gamma_k)$ *has been studied in [CR06, §8] and in [Osi15, LL20]. We briefly recall the structure of* $\mathrm{Aut}(\Gamma_k)$*. First, since H is a simply connected nilpotent Lie group the exponential map* $\exp : \mathfrak{h} \longrightarrow H$ *is bijective, so* $\mathrm{Aut}(H) \cong \mathrm{Aut}(\mathfrak{h})$*. More explicitly, if*

$$\mathfrak{h} = \{[x,y,z] = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x,y,z \in \mathbb{R}\}.$$

and we choose a basis $e_1 = [1,0,0]$*,* $e_2 = [0,1,0]$ *and* $e_3 = [0,0,1]$ *then the automorphism group of* $\mathfrak{h}$ *is*

$$\mathrm{Aut}(\mathfrak{h}) = \{\phi = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & ad-bc \end{pmatrix} : a,b,c,d,u,v \in \mathbb{R},\ ad-bc \neq 0\}.$$

If $f \in \mathrm{Aut}(H)$ *with differential is* $df = \phi$ *then*

$$f(x,y,z) = (ax+by, cx+dy, (ad-bc)z + \tfrac{1}{2}acy^2 + uy + bcxy + vx + \tfrac{1}{2}bdx^2).$$

An element $f \in \mathrm{Aut}(H)$ *is in* $\mathrm{Aut}(\Gamma_k)$ *if and only if it preserves* Γ_k*. A straightforward computation shows that*

$$\mathrm{Aut}(\Gamma_k) = \{f \in \mathrm{Aut}(H) : df = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & ad-bc \end{pmatrix}, a,b,c,d \in \mathbb{Z},\ \tfrac{1}{2}ac+u, \tfrac{1}{2}bd+v \in \tfrac{1}{k}\mathbb{Z}\}.$$

Note that $\Gamma_k/\mathcal{Z}\Gamma_k \cong \mathbb{Z}^2$*. Since* $\mathcal{Z}\Gamma_k$ *is a characteristic subgroup there exists a group morphism* $\theta : \mathrm{Aut}(\Gamma_k) \longrightarrow \mathrm{Aut}(\mathbb{Z}^2) = \mathrm{GL}(2,\mathbb{Z})$ *such that*

$$\theta(f) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is clear that θ *is surjective and that*

$$\mathrm{Ker}\,\theta = \{f \in \mathrm{Aut}(H) : df = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix},\ u,v \in \tfrac{1}{k}\mathbb{Z}\}.$$

In addition θ *has a section* $\mathrm{GL}(2,\mathbb{Z}) \longrightarrow \mathrm{Aut}(\Gamma_k)$*. In conclusion:*

Proposition 2.2.4. *[LL20, §2] The group morphism $\theta : \mathrm{Aut}(\Gamma_k) \longrightarrow \mathrm{GL}(2,\mathbb{Z})$ is surjective, $\mathrm{Ker}\,\theta \cong \mathbb{Z}^2$ and it has a section. In consequence,*

$$\mathrm{Aut}(\Gamma_k) \cong \mathbb{Z}^2 \rtimes \mathrm{GL}(2,\mathbb{Z})$$

where $\mathrm{GL}(2,\mathbb{Z})$ acts on $\mathbb{Z}^2$ in the usual way.

In particular, $\mathrm{Aut}(\Gamma_k)$ does not depend on k. Now we want to describe inner automorphisms. Given $(x,y,z),(u,v,w) \in H$, then $(u,v,w)(x,y,z)(u,v,w)^{-1} = (x,y,z+xv+uy)$. Thus the conjugation morphism $c_{(a,b,c)} : H \longrightarrow H$ satisfies that

$$dc_{(u,v,w)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix}.$$

If $(u,v,w) \in \Gamma_k$ then

$$\mathrm{Inn}\,\Gamma_k = \{f \in \mathrm{Aut}(\Gamma_k) : df = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix},\ u,v \in \mathbb{Z}\}.$$

Note that $\mathrm{Inn}\,\Gamma_k \trianglelefteq \mathrm{Ker}\,\theta$ and $\mathrm{Ker}\,\theta/\,\mathrm{Inn}\,\Gamma_k \cong (\mathbb{Z}/k)^2$. Moreover, the non-trivial elements of $\mathrm{Ker}\,\theta/\,\mathrm{Inn}\,\Gamma_k$ represent the automorphisms $f \in \mathrm{Aut}(\Gamma_k)$ such that $f \notin \mathrm{Inn}\,\Gamma_k$ but $f \in \mathrm{Inn}\,H$. This discussion leads to the following conclusion.

Lemma 2.2.5. *The outer automorphism group of Γ_k is $\mathrm{Out}(\Gamma_k) \cong (\mathbb{Z}/k)^2 \rtimes \mathrm{GL}(2,\mathbb{Z})$.*

Note that $\mathrm{Out}(\Gamma_k)$ does depend on k. Both of them are Minkowski and therefore the conclusions of theorem 2.0.1 are valid for H/Γ_k. In particular, $\mathrm{Homeo}(H/\Gamma_k)$ is Jordan and $\text{disc-sym}(H/\Gamma_k) = 1$. However, not all finite subgroups of $\mathrm{Out}(\Gamma_k)$ can be realized by a group action on H/Γ_k (see [RS77]). See also [LL20] for more information on the structure of the automorphism group of lattices of simply connected nilpotent groups of dimension 3 and 4.

If a n-dimensional flat manifold such that $\text{disc-sym}(M) = n = \text{disc-sym}(T^n)$ then M is homeomorphic to T^n. On the other hand, an almost-flat manifold M finitely covered by a nilmanifold N/Γ satisfying that $\text{disc-sym}(M) = \text{disc-sym}(N/\Gamma)$ is not necessarily isomorphic to N/Γ. The 3-dimensional Heisenberg manifold can be used to construct an almost-flat manifold M which is finitely covered by H/Γ_2 and $\text{disc-sym}(M) = \text{disc-sym}(H/\Gamma_2) = 1$ but M is not homeomorphic to H/Γ_2 (M is not even a nilmanifold). There is a free action of $\mathbb{Z}/2$ on H/Γ_2 such that its orbit space M is an almost-flat manifold with fundamental group $\pi_1(M) = \langle a,b,c,\alpha | [c,\alpha] = [c,a] = [c,b] = 1, [a,b] = c^2, \alpha a = a^{-1}\alpha, \alpha b = b^{-1}\alpha, \alpha^2 = c\rangle$ (see [Dek06, pg. 160]). It is clear that $\langle c\rangle \leq \mathcal{Z}\pi_1(M)$ and therefore $1 \leq \text{disc-sym}(M) \leq \text{disc-sym}(H/\Gamma_2) = 1$. In consequence, we have an equality $\text{disc-sym}(M) = \text{disc-sym}(H/\Gamma_2) = 1$. Note that $\alpha c \alpha^{-1} = c$, hence the morphism $\psi' : \mathbb{Z}/2 \longrightarrow \mathrm{Out}(\mathcal{Z}\Gamma_2)$ is trivial, as we expected from corollary 2.1.12.

Let M be a closed aspherical manifold satisfying the hypothesis of theorem 2.0.1. The fact that $\text{disc-sym}(M) = \text{rank}(\mathcal{Z}\pi_1(M))$ *does not imply that there are no effective actions of* $(\mathbb{Z}/k)^r$ *with* $r >$ disc-sym(M) *for some k. Indeed, if M is a compact solvmanifold R/Γ then [JL10, Corollary 3.3] asserts that if $(\mathbb{Z}/p)^r$ acts freely on R/Γ then $r \leq \dim R/\Gamma$. This bound is sharp even when R/Γ is not a torus. For an integer $k \leq 2$ we consider the subgroup*

$$\Gamma'_k = \{(x,y,z) = \begin{pmatrix} 1 & \frac{1}{k}x & \frac{1}{k^3}z \\ 0 & 1 & \frac{1}{k}y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{Z}\}$$

which is a lattice of H isomorphic to Γ_k. An straightforward computation shows that Γ_{k^2} is a normal subgroup of Γ'_k and $\Gamma'_k/\Gamma_{k^2} \cong (\mathbb{Z}/k)^3$. Therefore $(\mathbb{Z}/k)^3$ acts freely on H/Γ_{k^2} even though $\text{disc-sym}(H/\Gamma_{k^2}) = 1$ *(see [CS05] for a classification of all finite abelian group actions on Heisenberg manifolds). Therefore the bound of [JL10, Corollary 3.3] is sharp.*

2.3 Locally symmetric spaces

The aim of this section is to prove the following proposition.

Proposition 2.3.1. *Let Γ be a lattice of a connected semisimple Lie group G without compact factors. Then* $\text{Out}(\Gamma)$ *Minkowski.*

First, let us state the following well-known result and give a proof of it for the sake of completeness.

Lemma 2.3.2. *Let G be connected linear semisimple Lie groups with trivial center and no compact factors, let Γ be a lattice of G and assume that there does not exist any simple factor H of G such that $H \cong$* $\text{PSL}(2,\mathbb{R})$ *and $H \cap \Gamma_1$ is a lattice in H. Then the group* $\text{Out}(\Gamma)$ *is finite.*

Proof. Let $F : \text{Aut}(\Gamma) \longrightarrow \text{Aut}(G)$ be the morphism sending an automorphism of Γ to its unique extension on G by theorem 1.3.78. Clearly, $F(\text{Inn}(\Gamma)) \leq \text{Inn}(G)$, so F descends to a group morphism $f : \text{Out}(\Gamma) \longrightarrow \text{Out}(G)$. Then, the claim follows from the fact that $\text{Out}(G)$ is finite (see lemma 1.3.81) and that $\text{Ker} f = N_G(\Gamma)/\Gamma$ is also finite by corollary 1.3.69. □

Lemma 2.3.3. *Let G be a connected semisimple Lie groups without compact factors and let Γ be a lattice of G. If* $\text{Out}(\Gamma/\mathcal{Z}\Gamma)$ *is Minkowski then* $\text{Out}(\Gamma)$ *is Minkowski.*

Proof. We consider the central short exact sequence $1 \longrightarrow \mathcal{Z}\Gamma \longrightarrow \Gamma \longrightarrow \Gamma/\mathcal{Z}\Gamma \longrightarrow 1$. The center is a characteristic subgroup, hence $\text{Out}(\Gamma) = \text{Out}(\Gamma, \mathcal{Z}\Gamma)$.

On the other hand, $\text{Out}(\Gamma/\mathcal{Z}\Gamma)$ is Minkowski by hypothesis, $\text{Out}(\mathcal{Z}\Gamma)$ is Minkowski since $\mathcal{Z}\Gamma$ is a finitely generated abelian group and $\overline{H}^1(\Gamma/\mathcal{Z}\Gamma, \mathcal{Z}\Gamma)$ is Minkowski since it is finitely generated and abelian. Therefore by theorem 1.2.21 we obtain that $\mathrm{Out}(\Gamma)$ is Minkowski. $\square$

In view of the preceding lemma, to prove proposition 2.3.1 it only remains to show that $\mathrm{Out}(\Gamma/\mathcal{Z}\Gamma)$ is Minkowski. Note that $\Gamma/\mathcal{Z}\Gamma$ is a lattice in $G/\mathcal{Z}G$ which is a centreless connected semisimple linear Lie group without compact factors. By theorem 1.3.67 and proposition 1.3.74, $G/\mathcal{Z}G = G_1 \times \cdots \times G_n$ and $\Gamma/\mathcal{Z}\Gamma$ has a normal torsion-free finite index subgroup Λ of the form $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ where each Λ_i is an irreducible lattice of G_i. Note that each of this groups is centreless. We will use the decomposition $\Lambda = \Lambda^{(1)} \times \Lambda^{(\geq 2)}$ where $\Lambda^{(1)}$ is the product of all lattices of real rank 1 and $\Lambda^{(\geq 2)}$ is the product of all lattices of real rank greater or equal than 2. After reordering we can assume that $\Lambda^{(1)} = \Lambda_1 \times \cdots \Lambda_m$ and $\Lambda^{(\geq 2)} = \Lambda_{m+1} \times \cdots \Lambda_n$.

Lemma 2.3.4. *The group $\Lambda^{(\geq 2)}$ is characteristic in Λ.*

Proof. We will prove that given any $f \in \mathrm{Aut}(\Lambda)$ and any $m+1 \leq j \leq n$ we have $f(\Lambda_j) \leq \Lambda^{(\geq 2)}$.

Let $\pi_i : \Lambda \longrightarrow \Lambda_i$ and $\iota_i : \Lambda_i \longrightarrow \Lambda$ denote the natural projection and inclusion morphisms. We take the group morphism $\pi_i \circ f \circ \iota_j : \Lambda_j \longrightarrow \Lambda_i$ with $1 \leq i \leq m$ and $m+1 \leq j \leq n$. By Margulis normal subgroup theorem (see theorem 1.3.80), $N = \mathrm{Ker}(\pi_i \circ f \circ \iota_j) \trianglelefteq \Lambda_j$ is either finite or has finite index in Λ_j. If N is finite then it is trivial since Λ_j is torsion-free and $\pi_i \circ f \circ \iota_j$ is injective. If it has finite index then Λ_j/N is a finite subgroup of Λ_i and therefore it is trivial since Λ_i is torsion-free. In this case, $\pi_i \circ f \circ \iota_j$ is trivial.

The morphisms $\pi_i \circ f \circ \iota_j : \Lambda_j \longrightarrow \Lambda_i$ cannot be injective for any i and j. Indeed, since G_i is centreless (hence $G_i \leq \mathrm{GL}(n_i, \mathbb{R})$ for some i), we can construct a representation $\rho : \Lambda_j \longrightarrow \mathrm{GL}(n_i, \mathbb{R})$ given by the composition $\pi_i \circ f \circ \iota_j$ with the inclusions $\Lambda_i \leq G_i \leq \mathrm{GL}(n_i, \mathbb{R})$. By Margulis superrigidity (see theorem 1.3.79) there exists a finite index subgroup $\Lambda_{j0} \leq \Lambda_j$ and a representation $\tilde{\rho} : G_j \longrightarrow \overline{\rho(\Lambda_j)}^o$ such that $\tilde{\rho}_{|\Lambda_{j0}} = \rho_{|\Lambda_{j0}}$. Since $\overline{\rho(\Lambda_j)}^o \leq G_i$, $\tilde{\rho}$ induces a group morphism $G_j \longrightarrow G_i$, which is trivial since G_i and G_j are simple, $\mathrm{rank}_{\mathbb{R}}\, G_j \geq 2$ and $\mathrm{rank}_{\mathbb{R}}\, G_i = 1$. Therefore $\Lambda_{j0} \leq \mathrm{Ker}\, \pi_i \circ f \circ \iota_j$. Since $\pi_i \circ f \circ \iota_j$ is not injective then $\pi_i \circ f \circ \iota_j$ is trivial, as desired. $\square$

Note that $\mathrm{Out}(\Lambda^{(\geq 2)})$ is finite by lemma 2.3.2.

Lemma 2.3.5. *The group $\mathrm{Out}(\Lambda^{(1)})$ is Minkowski.*

Proof. Let $H = \{f \in \mathrm{Aut}(\Lambda^{(1)}) : f(\Lambda_i) = \Lambda_i \text{ for all } i\}$. Firstly, we will show that H has finite index in $\mathrm{Aut}(\Lambda^{(1)})$. Since $\mathrm{Inn}(\Lambda^{(1)}) \trianglelefteq H$, this will imply that $[\mathrm{Out}(\Lambda^{(1)}) : H/\mathrm{Inn}(\Lambda^{(1)})] < \infty$.

We denote by $r(\lambda)$ the number of non-trivial entries of an element $\lambda \in \Lambda^{(1)}$. Let S be the set of elements of $\Lambda^{(1)}$ whose centralizer $C_{\Lambda^{(1)}}(\lambda)$ is isomorphic as an abstract group to $\mathbb{Z} \times \Lambda_\lambda$, where Λ_λ is a product of lattices in $\mathbb{R}$-rank one semisimple Lie groups (which depend on the element $\lambda \in \Lambda$).

Firstly, we note that if $\lambda \in S$ then $r(\lambda) = 1$. Given $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda^{(1)}$, its centralizer is $C_{\Lambda^{(1)}}(\lambda) = \prod_{i=1}^m C_{\Lambda_i}(\lambda_i)$. In addition, $C_{\Lambda_i}(\lambda_i) = \Lambda_i$ if and only if $\lambda_i = e_i$. If λ_i is not trivial then $C_{\Lambda_i}(\lambda_i)$ is virtually nilpotent. Indeed, if λ_i is hyperbolic then $C_{\Lambda_i}(\lambda_i) \cong \mathbb{Z}$ by lemma 1.3.87(1) and if λ_i is parabolic then $C_{\Lambda_i}(\lambda_i)$ is virtually nilpotent by corollary 1.3.88 and theorem 1.3.86. If $r(\lambda) \neq 1$ and $\lambda \in S$ then $C_{\Lambda^{(1)}}(\lambda)$ would contain at least two virtually nilpotent factors and one of this factors would be a normal subgroup of a lattice in a centreless semisimple Lie group. This contradicts corollary 1.3.71. Thus, we can conclude that λ is not in S. However, note that it is possible for an element λ with $r(\lambda) = 1$ to not be in S. If $\lambda = (e_1, \ldots, \lambda_i, \ldots, e_m)$, then $C_{\Lambda^{(1)}}(\lambda) \cong C_{\Lambda_i}(\lambda_i) \times \Lambda_\lambda$ and $C_{\Lambda_i}(\lambda_i)$ is not necessarily isomorphic to $\mathbb{Z}$ if λ_i is not hyperbolic.

Clearly S is preserved by automorphisms of $\Lambda^{(1)}$. Since S contains all elements whose only non-trivial entry is hyperbolic, the set S generates $\Lambda^{(1)}$ by lemma 1.3.87(2). Consequently, any $f \in \mathrm{Aut}(\Lambda^{(1)})$ permutes the factors of $\Lambda^{(1)}$ and we can construct a group morphism $\phi : \mathrm{Aut}(\Lambda^{(1)}) \longrightarrow S_m$ such that $H = \mathrm{Ker}\, \phi$. Consequently, $[\mathrm{Aut}(\Lambda^{(1)}) : H] < m!$.

We are ready to prove that $\mathrm{Out}(\Lambda^{(1)})$ is Minkowski. We proceed by induction on the number of factors m. If $m = 1$ then $\Lambda^{(1)}$ is an irreducible lattice in a centreless semisimple Lie group G_1. If $G_1 \not\cong \mathrm{PSL}(2, \mathbb{R})$ then we can use lemma 2.3.2 to conclude that $\mathrm{Out}(\Lambda^{(1)})$ is finite. If $G_1 \cong \mathrm{PSL}(2, \mathbb{R})$ then $\Lambda^{(1)}$ is a Fuchsian group and therefore $\mathrm{Out}(\Lambda^{(1)})$ is virtually torsion-free [MS06, Corollary 2.6] and hence Minkowski.

Assume now that $\mathrm{Out}(\Lambda_1 \times \cdots \times \Lambda_{m-1})$ is Minkowski. By theorem 1.2.21, if $\mathrm{Out}(\Lambda_m)$, $\mathrm{Out}(\Lambda_1 \times \cdots \times \Lambda_{m-1})$ and $H^1(\Lambda_1 \times \cdots \times \Lambda_{m-1}, \mathcal{Z}\Lambda_m)$ are Minkowski then $\mathrm{Out}(\Lambda^{(1)}, \Lambda_m)$ is Minkowski. They are Minkowski by induction hypothesis and the fact that $\mathcal{Z}\Lambda_m$ is trivial. Since $H \leq \mathrm{Aut}(\Lambda^{(1)}, \Lambda_m) \leq \mathrm{Aut}(\Lambda^{(1)})$ and $[\mathrm{Aut}(\Lambda^{(1)}) : H] < \infty$ we have that $[\mathrm{Out}(\Lambda^{(1)}) : \mathrm{Out}(\Lambda^{(1)}, \Lambda_m)] < \infty$, which implies that $\mathrm{Out}(\Lambda^{(1)})$ is Minkowski. □

We are ready to prove that $\mathrm{Out}(\Gamma/\mathcal{Z}\Gamma)$ is Minkowski and finish the proof of proposition 2.3.1.

Lemma 2.3.6. *The group* $\mathrm{Out}(\Gamma/\mathcal{Z}\Gamma)$ *is Minkowski.*

Proof. Since $[\Gamma/\mathcal{Z}\Gamma : \Lambda] < \infty$, by lemma 1.2.24 it is enough to prove that $\mathrm{Out}(\Lambda)$ is Minkowski. We have seen that the groups $\mathrm{Out}(\Lambda^{(\geq 2)})$, $\mathrm{Out}(\Lambda^{(1)})$ and $H^1(\Lambda^{(1)}, \mathcal{Z}\Lambda^{(\geq 2)})$ are Minkowski. Thus, by theorem 1.2.21 $\mathrm{Out}(\Lambda)$ is Minkowski, as we wanted to see. □

The rest of this section is devoted to prove proposition 2.0.3. Recall that if H is a torsion-free hyperbolic group, then $C_H(h)$ is cyclic for all non-trivial $h \in H$. In addition, if H contains a normal abelian subgroup then H is cyclic (see [BH13, Part III Γ.3]). Finally, if M is a closed connected aspherical manifold of dimension $n \geq 3$ and $\pi_1(M)$ is hyperbolic then $\mathrm{Out}(\pi_1(M))$ is finite (see remark 2.1.4(2)).

The arguments used in lemma 2.3.5 can be used to prove the next statement.

Proposition 2.3.7. *Let $M = M_1 \times \cdots \times M_m$, where M_i are closed aspherical manifolds such that $\pi_1(M_i)$ is hyperbolic and $\dim(M_i) \geq 3$. Then $\mathrm{Out}(\pi_1(M))$ is finite.*

Proof. Note that $\pi_1(M) = \pi_1(M_1) \times \cdots \times \pi_1(M_m)$. Like in lemma 2.3.5, let $H = \{f \in \mathrm{Aut}(\pi_1(M)) : f(\pi_1(M_i)) = \pi_1(M_i) \text{ for all } i\}$. We will show that H has finite index in $\mathrm{Aut}(\pi_1(M))$. Since $\mathrm{Inn}(\pi_1(M)) \trianglelefteq H$, this will imply that $[\mathrm{Out}(\pi_1(M)) : H/\mathrm{Inn}(\pi_1(M))] < \infty$.

Let e_i denote the trivial element of $\pi_1(M_i)$. We know that for every $\lambda = (\lambda_1, \dots, \lambda_m) \in \pi_1(M)$ we have $C_{\pi_1(M)}(\lambda) = \prod_{i=1}^m C_{\pi_1(M_i)}(\lambda_i)$. In addition, $C_{\pi_1(M_i)}(\lambda_i) = \pi_1(M_i)$ if $\lambda_i = e_i$ and $C_{\pi_1(M_i)}(\lambda_i) = \mathbb{Z}$ otherwise. As before, let $r(\lambda)$ denote the number of non-trivial entries of λ. We claim that that if $r(\lambda) = 1$ and $f \in \mathrm{Aut}(\pi_1(M))$ then $r(f(\lambda)) = 1$.

Assume on the contrary, that $\lambda = (e_1, \dots, \lambda_i, \dots, e_m)$ and that $r(f(\lambda)) > 1$. Since we have $f(C_{\pi_1(M)}(\lambda)) = C_{\pi_1(M)}(f(\lambda))$, we can take the inverse morphism $f^{-1} : C_{\pi_1(M)}(f(\lambda)) \longrightarrow C_{\pi_1(M)}(\lambda)$ and restrict it to $\mathbb{Z}^{r(f(\lambda))} \leq C_{\pi_1(M)}(f(\lambda))$. The map $\pi_i \circ f^{-1}_{|\mathbb{Z}^{r(f(\lambda))}} : \mathbb{Z}^{r(f(\lambda))} \longrightarrow \mathbb{Z}$ cannot be injective. If a is a non trivial element of $\mathrm{Ker}\, \pi_i \circ f^{-1}_{|\mathbb{Z}^{r(f(\lambda))}}$, then there exists a $j \neq i$ such that $\pi_j \circ f^{-1}_{|\langle a \rangle} : \langle a \rangle \longrightarrow \Lambda_j$ is injective. Since $\langle a \rangle \trianglelefteq C_{\pi_1(M)}(f(\lambda))$ and $\pi_j : C_{\pi_1(M)}(\lambda) \longrightarrow \pi_1(M_j)$ is surjective we can conclude that $\mathbb{Z} \cong \pi_j \circ f^{-1}_{|\langle a \rangle}(\langle a \rangle) \trianglelefteq \pi_1(M_j)$. But $\pi_1(M_j)$ is hyperbolic, so from the fact that it contains an abelian normal subgroup we can conclude $\pi_1(M_j) \cong \mathbb{Z}$. This is a contradiction with the fact that $\pi_1(M_j)$ is the fundamental group of a closed aspherical manifold of dimension $\dim(M_j) \geq 3$.

In conclusion, any $f \in \mathrm{Aut}(\pi_1(M))$ permutes the factors of $\pi_1(M)$ and thus we can construct a group morphism to the permutation group of m letters, $\phi : \mathrm{Aut}(\pi_1(M)) \longrightarrow S_m$ such that $H = \mathrm{Ker}\, \phi$. Consequently, $[\mathrm{Aut}(\pi_1(M)) : H] < m!$.

We are now ready to prove that $\mathrm{Out}(\pi_1(M))$ is finite. We proceed by induction on the number of factors m. If $m = 1$ then $\mathrm{Out}(\pi_1(M))$ is finite (remark 2.1.4(2)). Assume now that $\mathrm{Out}(\pi_1(M_1) \times \cdots \times \pi_1(M_{m-1}))$ is finite. By theorem 1.2.21, if $\mathrm{Out}(\pi_1(M_m))$, $\mathrm{Out}(\pi_1(M_1) \times \cdots \times \pi_1(M_{m-1}))$ and $H^1(\pi_1(M_1) \times \cdots \times \pi_1(M_{m-1}), \mathcal{Z}\pi_1(M_m))$ are finite then $\mathrm{Out}(\pi_1(M), \pi_1(M_m))$ is finite. They are finite by induction hypothesis and the fact that $\mathcal{Z}\pi_1(M_m)$ is trivial. Since $H \leq \mathrm{Aut}(\pi_1(M), \pi_1(M_m)) \leq \mathrm{Aut}(\pi_1(M))$ and $[\mathrm{Aut}(\pi_1(M)) : H] < \infty$, we have $[\mathrm{Out}(\pi_1(M)) : \mathrm{Out}(\pi_1(M), \pi_1(M_m))] < \infty$. Therefore, $\mathrm{Out}(\pi_1(M))$ is finite. □

Note that torsion-free non-cyclic hyperbolic groups are centreless. Consequently, $\mathcal{Z}\pi_1(M)$ is trivial, which implies that M is almost asymmetric by theorem 2.0.1 and proposition 2.3.7. In addition:

Corollary 2.3.8. *The group* $\mathrm{Aut}(\pi_1(M))$ *is Minkowski.*

Proof. Note that $\mathcal{Z}\pi_1(M)$ is trivial, therefore there is a short exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow \mathrm{Aut}(\pi_1(M)) \longrightarrow \mathrm{Out}(\pi_1(M)) \longrightarrow 1.$$

Since $\mathrm{Out}(\pi_1(M))$ is finite and $\pi_1(M)$ is torsion-free, we conclude that $\mathrm{Aut}(\pi_1(M))$ is virtually torsion-free and hence Minkowski. □

2.4 Combining the two cases: Aspherical locally homogeneous spaces

The aim of this section is to finish the proof of theorem 2.0.2 and to prove theorem 2.0.4. For the first task, the strategy is to combine the results on solvable Lie groups and semisimple Lie groups obtained in the previous sections in a similar way we proved lemma 2.3.3.

We have the short exact sequence

$$1 \longrightarrow \Gamma \cap A \longrightarrow \Gamma \longrightarrow \Gamma / \Gamma \cap A \longrightarrow 1,$$

where the group $\Gamma \cap A$ is a lattice in the amenable radical A and $\Gamma/\Gamma \cap A$ is a lattice in the semisimple Lie group without compact factors S_{nc}. We write $\Gamma \cap A = \Gamma_A$ and $\Gamma/\Gamma \cap A = \Gamma_{nc}$.

Our first goal is to see that Γ_A is virtually polycyclic. Lemma 2.4.1 is probably well-known to experts, but it is difficult to find a proof in the literature. Hence, we provide a proof for the sake of completeness.

Lemma 2.4.1. *Let* Γ *be a lattice in an amenable group* A*. Then* Γ *is virtually polycyclic.*

Proof. Denote by $\pi_{A/R} : A \longrightarrow A/R$ the quotient map and define $L = \overline{\pi_{A/R}(\Gamma)}^{o}$. It is connected solvable Lie group by [Rag12, 8.24]. Moreover, L is abelian since it is a connected solvable Lie subgroup of the compact Lie group A/R. Then, $\tilde{R} = \pi_{A/R}^{-1}(L)$ is a connected solvable group since $\tilde{R}/R$ is abelian and R is connected.

$\tilde{R} \cap \Gamma$ is a lattice in $\tilde{R}$ (see [GS20, Claim 2.2]). We claim that $\tilde{R} \cap \Gamma$ is polycyclic. Indeed, $\tilde{R}/\tilde{R} \cap \Gamma$ is a compact solvmanifold. Then $\tilde{R}/\tilde{R} \cap \Gamma \cong R'/\Gamma'$, where R' is the universal cover of $\tilde{R}$ and Γ' is a lattice in a simply connected solvable Lie group thus it is polycyclic.

Moreover, using the long exact sequence of homotopy groups for a fibration we obtain the short exact sequence

$$1 \longrightarrow \pi_1(\tilde{R}) \cong \mathbb{Z}^n \longrightarrow \Gamma' \longrightarrow \tilde{R} \cap \Gamma \longrightarrow 1.$$

Since $\tilde{R} \cap \Gamma \cong \Gamma'/\mathbb{Z}^n$ we can conclude that $\tilde{R} \cap \Gamma$ is polycyclic.

We want to show that $[\Gamma : \tilde{R} \cap \Gamma] < \infty$. Let $H = N_A(\tilde{R})$ and let H^o be its connected component (then $R \leq \tilde{R} \leq H^o \leq H \leq A$). Thus $\Gamma \leq H$ and $|H/H^o|$ is bounded by [GS20, Corollary 2.6]. In addition, $H^o/\tilde{R}$ is a compact Lie group.

In consequence, $[\Gamma : \Gamma \cap \tilde{R}] = [\Gamma : \Gamma \cap H^o][\Gamma \cap H^o : \Gamma \cap \tilde{R}]$. The first term is finite because $\Gamma \leq H$ and $|H/H^o|$ is bounded. The second term is finite because $\Gamma/\Gamma \cap \tilde{R} \cong \Gamma\tilde{R}/\tilde{R} = \pi_{H^o/\tilde{R}}(\Gamma)$ is a discrete subgroup in a compact Lie group and thus it is finite. □

Now that we know the structure of Γ_A we are interested in the relation between Γ_A and Γ. Since we want to study the automorphisms and outer automorphisms of Γ, it is natural to study whether Γ_A is a characteristic subgroup of Γ.

If $\mathcal{Z}S_{nc} \neq \{e\}$ then we consider the subgroup $\pi^{-1}(\mathcal{Z}\Gamma_{nc}) = \Gamma'_A$, which is a virtually polycyclic group since it fits in the short exact sequence $1 \longrightarrow \Gamma_A \longrightarrow \Gamma'_A \longrightarrow \mathcal{Z}\Gamma_{nc} \longrightarrow 1$. In consequence, there is a short exact sequence $1 \longrightarrow \Gamma'_A \longrightarrow \Gamma \longrightarrow \overline{\Gamma}_{nc} \longrightarrow 1$, where $\overline{\Gamma}_{nc}$ is a lattice of the centreless connected semisimple Lie group $S_{nc}/\mathcal{Z}S_{nc}$. By theorem 1.3.67, there exists a normal finite index torsion-free lattice $\Gamma'_{nc} \leq \overline{\Gamma}_{nc}$. Then we can consider the short exact sequence $1 \longrightarrow \Gamma'_A \longrightarrow \Gamma' \longrightarrow \Gamma'_{nc} \longrightarrow 1$. We will later prove that $\mathrm{Out}(\Gamma')$ is Minkowski. This fact, together with the previous short exact sequence, will imply that $\mathrm{Out}(\Gamma)$ is Minkowski by lemma 1.2.24.

Lemma 2.4.2. Γ'_A *is a characteristic subgroup of* Γ'*.*

Proof. Assume that Γ'_A is not characteristic. Thus, there exists $f \in \mathrm{Aut}(\Gamma')$ such that $f(\Gamma'_A)$ has a non trivial projection Λ in Γ'_{nc}. The group Λ is a torsion-free virtually polycyclic subgroup and hence it has a characteristic subgroup Λ' which is polycyclic (see [LR10, §9.5]). In particular, Λ' is solvable and normal in the semisimple centreless lattice Γ'_{nc}, contradicting corollary 1.3.71. Consequently, Γ'_A is characteristic. □

In consequence, $\mathrm{Out}(\Gamma') = \mathrm{Out}(\Gamma', \Gamma'_A)$. Theorem 1.2.21 shows that to prove that $\mathrm{Aut}(\Gamma')$ and $\mathrm{Out}(\Gamma')$ are Minkowski it is enough to prove that $\mathrm{Aut}(\Gamma'_A)$, $\mathrm{Aut}(\Gamma'_{nc})$, $Z^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$, $\mathrm{Out}(\Gamma'_A)$, $\mathrm{Out}(\Gamma'_{nc})$ and $\overline{H}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ are Minkowski. We already know that $\mathrm{Aut}(\Gamma'_A)$ and $\mathrm{Out}(\Gamma'_A)$ are Minkowski because Γ'_A is virtually polycyclic (theorem 2.2.1), and $\mathrm{Out}(\Gamma'_{nc})$ is Minkowski by lemma 2.3.3. Thus we only need to check that $\mathrm{Aut}(\Gamma'_{nc})$, $Z^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$, $\mathrm{Out}(\Gamma'_A)$ and $\overline{H}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ are Minkowski.

Lemma 2.4.3. *The group* $\mathrm{Aut}(\Gamma'_{nc})$ *is Minkowski.*

Proof. Since Γ'_{nc} is centreless, we have a short exact sequence

$$1 \longrightarrow \Gamma'_{nc} \longrightarrow \mathrm{Aut}(\Gamma'_{nc}) \longrightarrow \mathrm{Out}(\Gamma'_{nc}) \longrightarrow 1.$$

Since Γ'_{nc} is virtually torsion-free and $\mathrm{Out}(\Gamma'_{nc})$ is Minkowski, then $\mathrm{Aut}(\Gamma'_{nc})$ is Minkowski. □

Lemma 2.4.4. *The group $\overline{H}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ is Minkowski.*

Proof. Since $H^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ is a finitely generated abelian group and there is a surjection $H^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A) \longrightarrow \overline{H}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$, the group $\overline{H}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ is finitely generated and abelian, hence it is Minkowski. □

Lemma 2.4.5. *The group $Z^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ is Minkowski.*

Proof. There is a short exact sequence

$$1 \longrightarrow \overline{B}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A) \longrightarrow Z^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A) \longrightarrow \overline{H}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A) \longrightarrow 1.$$

We have $\overline{B}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A) \cong \mathcal{Z}\Gamma'_A / \mathcal{Z}\Gamma'$ (see theorem 1.2.21), therefore $\overline{B}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ is a finitely generated abelian group and hence it is Minkowski. Since $\overline{H}^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ is also a finitely generated abelian group, we use lemma 1.1.54 to conclude that $Z^1_\psi(\Gamma'_{nc}, \mathcal{Z}\Gamma'_A)$ is Minkowski. □

These three lemmas complete the proof of theorem 2.0.2.

Remark 2.4.6. *Note that we can use the proof of theorem 2.0.2 to conclude that if Γ is a group which fits in a short exact sequence $1 \longrightarrow \Gamma_A \longrightarrow \Gamma \longrightarrow \Gamma_{nc} \longrightarrow 1$ where Γ_A is virtually polycyclic and Γ_{nc} is a lattice in a centreless semisimple Lie group then $\mathrm{Out}(\Gamma)$ is Minkowski. Not all groups of this form are lattices in connected Lie groups, see [BK23a, Theorem 7.5]. Baues and Kamishima introduced in [BK23a] the notion of the closed aspherical manifolds with large symmetry. From the definition of these manifolds [BK23a, Definition 1.1] one can deduce that their fundamental group satisfies the hypothesis of this remark and hence the outer automorphism group of their fundamental group is Minkowski.*

Remark 2.4.7. *The homeomorphism group of a coset space G/Γ where G is a connected Lie group and Γ is a cocompact lattice is not necessarily Jordan. Indeed, we can take a non-compact semisimple Lie group G' and a cocompact lattice Γ' such that a connected compact maximal subgroup K' containing $\mathrm{SU}(2)$ acts effectively on G'/Γ'. By [MiR17] we know that $\mathrm{Homeo}(G'/\Gamma' \times T^2)$ is not Jordan. Hence, taking $G = G' \times \mathbb{R}^2$ and $\Gamma = \Gamma' \times \mathbb{Z}^2$ we obtain a coset space G/Γ whose homeomorphism group is not Jordan. On the other hand, A. Golota proved in [Gol23] that if G is a complex Lie group and Γ is a cocompact lattice, then the group of automorphisms $\mathrm{Aut}(G/\Gamma)$ is Jordan.*

The end of this section is devoted to prove theorem 2.0.4.

Proof of theorem 2.0.4. Note that $\mathrm{Out}(\Gamma)$ is Minkowski by theorem 2.0.2, hence disc-sym$(H \setminus G/\Gamma) \leq \operatorname{rank} \mathcal{Z}\Gamma$ by theorem 2.0.1. On the other hand, it is clear that tor-sym$(H \setminus G/\Gamma) \leq$ disc-sym$(H \setminus G/\Gamma)$. Therefore, it suffices to show that tor-sym$(H \setminus G/\Gamma) = \operatorname{rank} \mathcal{Z}\Gamma$ to conclude that disc-sym$(H \setminus G/\Gamma) = \operatorname{rank} \mathcal{Z}\Gamma$.

By lemma 2.4.1 Γ satisfies the conditions of theorem 1.3.105, hence there exists a closed connected aspherical manifold M' which is homotopically equivalent to $H \setminus G/\Gamma$ and satisfies that tor-sym$(M') = \operatorname{rank} \mathcal{Z}\Gamma$. On the other hand, the Farrel-Jones conjecture is true for cocompact lattices in connected Lie groups (see [BL12, KLR16]), which implies the Borel conjecture for closed aspherical locally homogeneous space of dimension equal or greater than 5 (see [BL12, Proposition 0.3 (ii)]). In consequence, if $\dim(H \setminus G/\Gamma) \geq 5$ then $H \setminus G/\Gamma \cong M'$ and hence tor-sym$(H \setminus G/\Gamma) = \operatorname{rank} \mathcal{Z}\Gamma$. Thus, it remains to study the cases where $\dim(H \setminus G/\Gamma) \leq 4$.

If $\dim(H \setminus G/\Gamma) = 1$ or 2 then tor-sym$(H \setminus G/\Gamma) = \operatorname{rank} \mathcal{Z}\Gamma$ by classical results and if $\dim(H \setminus G/\Gamma) = 3$ then tor-sym$(H \setminus G/\Gamma) = \operatorname{rank} \mathcal{Z}\Gamma$ by [Gab92, Corollary 8.3] and [CJ94, Theorem 1.1]. Thus, it only remains to study the case where $\dim(H \setminus G/\Gamma) = 4$.

First, assume that the noncompact semisimple part S_{nc} of G is trivial. Then Γ is virtually polycyclic. Virtually polycyclic groups are Freedman good (this property was called "good" and it was introduced in [Fre83]). If the fundamental group is Freedman good, the Farrel-Jones conjecture implies the Borel conjecture in dimension 4 (see [BL12, Proposition 0.3 (ii)]) and hence tor-sym$(H \setminus G/\Gamma) = \operatorname{rank} \mathcal{Z}\Gamma$.

In consequence, it only remains to study the case where $S_{nc} \neq \{e\}$. We can assume two extra hypothesis without losing generality. Firstly, let $q : \tilde{G} \longrightarrow G$ be the universal cover of G. Let $\tilde{H}$ be a maximal compact connected subgroup of $\tilde{G}$ inside $q^{-1}(H)$. Then $q^{-1}(\Gamma) = \tilde{\Gamma}$ is a cocompact lattice of $\tilde{G}$ and $H \setminus G/\Gamma \cong \tilde{H} \setminus \tilde{G}/\tilde{\Gamma}$. Thus, we can assume that G is simply connected and hence its solvable radical R and its semisimple part S are also simply connected. Secondly, note that if K is a compact connected normal subgroup of G and $p : G \longrightarrow G/K$ is the quotient map then $\Gamma \cap K = \{e\}$ since Γ is torsion-free. This implies that $H \setminus G/\Gamma \cong p(H) \setminus p(G)/\Gamma$. Therefore, we can assume that G has no compact factors.

Note that $\dim(R) + \dim(S) - \dim(H) = 4$ and since $S_{nc} \neq \{e\}$ then $\dim(S) - \dim(H) \geq 2$. Hence, $\dim(R) \leq 2$. There are four simply connected solvable groups of dimension less or equal than 2: $\dim(R) = 0$ and $R = \{e\}$, $\dim(R) = 1$ and $R \cong \mathbb{R}$, and $\dim(R) = 2$ and $R \cong \mathbb{R}^2$ or $R \cong \mathrm{Aff}(\mathbb{R})^0 \cong \mathbb{R} \rtimes \mathbb{R}$. To construct the desired torus action on $H \setminus G/\Gamma$ with R belonging to one of this four cases we will use theorem 1.3.106. We need to check that the hypothesis of theorem 1.3.106 are satisfied in these four cases. Note that in all the cases the exponential map is surjective and any lattice automorphism extends uniquely to an automorphism of the group. This is because R is either abelian or solvable of type (R) (see [LR10, §6.3]). It only remains to see that $\Gamma \cap R$ is a lattice in R.

Assume that R is abelian, then if C is a compact factor of S acting trivially on R then C is normal on G and therefore $C = \{e\}$. Therefore no compact factor of S acts trivially on R. This implies by [Gen15, Theorem 1.3 (i)] that $R \cap \Gamma$ is a lattice in R. Thus, tor-sym$(H \backslash G/\Gamma) = \operatorname{rank} \mathcal{Z}\Gamma$ by theorem 1.3.106.

If $R \cong \mathrm{Aff}(\mathbb{R})^0$ then $\mathrm{Aut}(R) \cong R$ and hence any compact factor C of S needs to act trivially on R. Consequently, $C \trianglelefteq G$ and therefore $C = \{e\}$. By [Gen15, Theorem 1.3 (i)] if Γ were a lattice of G then $R \cap \Gamma$ would be a lattice of R. But R does not admit any lattice (see [Boc16, pg. 82]). Thus, a closed aspherical locally homogeneous 4-manifold with solvable radical R does not exist. □

Recall that if M is a closed connected aspherical manifold, then $\operatorname{rank}(M) = $ tor-sym(M). We finish the section with the following remark on the toral rank conjecture and the Carlsson conjecture on closed aspherical manifolds.

Lemma 2.4.8. *Let M be a closed aspherical connected manifold such that $\mathcal{Z}\pi_1(M)$ is finitely generated,* $\mathrm{Out}(\pi_1(M))$ *is Minkowski and* tor-sym$(M) = \operatorname{rank} \mathcal{Z}\pi_1(M)$. *Then the toral rank conjecture holds for M if and only if the stable Carlsson conjecture holds for M.*

Proof. Since $H^*(M, \mathbb{Z})$ is finitely generated, we can use the universal coefficients theorem to conclude that there exists a constant C_1 such that for all prime $p > C_1$ we have $\dim H^*(M, \mathbb{Q}) = \dim H^*(M, \mathbb{Z}/p)$. Let C_2 be the Minkowski constant of $\mathrm{Out}(\pi_1(M))$.

Set $C = \max\{C_1, C_2\}$. Moreover, assume that the toral rank conjecture holds for M, then $\dim H^*(M, \mathbb{Q}) \geq 2^{\operatorname{rank}(M)} = 2^{\operatorname{rank}(\mathcal{Z}\pi_1(M))}$. On the other hand, for all $p > C$ we have $\operatorname{rank}_p(M) \leq \operatorname{rank} \mathcal{Z}\pi_1(M)$. Consequently, $\dim H^*(M, \mathbb{Z}/p) = \dim H^*(M, \mathbb{Q}) \geq 2^{\operatorname{rank}(M)} = 2^{\operatorname{rank}(\mathcal{Z}\pi_1(M))} \geq 2^{\operatorname{rank}_p(M)}$ and the Carlsson conjecture holds for $p > C$.

Conversely, we have $\operatorname{rank}_p(M) \geq \operatorname{rank}(M)$ for all primes p. By hypothesis, we have $\dim H^*(M, \mathbb{Z}/p) \geq 2^{\operatorname{rank}_p(M)}$ for all primes $p > C$. Thus, by choosing a prime $p > C$ we have $\dim H^*(M, \mathbb{Q}) = \dim H^*(M, \mathbb{Z}/p) \geq 2^{\operatorname{rank}_p(M)} \geq 2^{\operatorname{rank}(M)}$ and hence the toral rank conjecture holds for M. Note that this implication does not need the asphericity assumption or any hypothesis on the fundamental group of M. □

The toral rank conjectures has been proved for closed flat manifolds in [KN12]. Thus, lemma 2.4.8, theorem 2.0.2 and theorem 2.0.4 imply:

Corollary 2.4.9. *Let M be a closed flat manifold, then there exists a constant C such that the inequality* $\dim H^*(M, \mathbb{Z}/p) \geq 2^{\operatorname{rank}_p(M)}$ *holds for all $p > C$.*

2.5 Jordan property and cohomology

This section is devoted to proving proposition 2.0.5. We start with the following general lemma.

Lemma 2.5.1. *Let Γ be a finitely generated group and let $\mathcal{Z}_i\Gamma$ be the i-th term of the upper central series. Assume that $\Gamma/\mathcal{Z}_i\Gamma$ is finitely generated, centreless and that* $\mathrm{Out}(\Gamma/\mathcal{Z}_i\Gamma)$ *is Minkowski for some i. Then* $\mathrm{Out}(\Gamma)$ *is Minkowski.*

Proof. Recall that the upper central series $\{e\} = \mathcal{Z}_0\Gamma \trianglelefteq \mathcal{Z}_1\Gamma \trianglelefteq \mathcal{Z}_2\Gamma \trianglelefteq \cdots$ of a group Γ is a series where $\mathcal{Z}_i\Gamma/\mathcal{Z}_{i-1}\Gamma \cong \mathcal{Z}(\Gamma/\mathcal{Z}_{i-1})$ for all $i \geq 0$. The group Γ is nilpotent if and only if $\Gamma \cong \mathcal{Z}_i\Gamma$ for some i. Moreover, the groups $\mathcal{Z}_i\Gamma$ are characteristic and each group morphism $\psi_i : \Gamma/\mathcal{Z}_i\Gamma \longrightarrow \mathrm{Out}(\mathcal{Z}_i\Gamma)$ is trivial.

Consequently, $\mathrm{Out}(\Gamma)$ is Minkowski if $\mathrm{Out}(\mathcal{Z}_i\Gamma)$, $\mathrm{Out}(\Gamma/\mathcal{Z}_i\Gamma)$ and $H^1(\Gamma/\mathcal{Z}_i\Gamma, \mathcal{Z}\Gamma_i/\mathcal{Z}\Gamma_{i-1})$ are Minkowski. But they are Minkowski by hypothesis and theorem 2.2.1, obtaining the desired conclusion. □

If we assume that Γ is torsion-free and $\Gamma/\mathcal{Z}_i\Gamma$ acts properly on a contractible manifold $\tilde{X}$ then we can use the Seifert fiber construction in [LR10, Chapter 7] to construct closed connected aspherical manifolds M such that $\pi_1(M) \cong \Gamma$. In addition, if $\tilde{X}/(\Gamma/\mathcal{Z}_i\Gamma)$ is a closed aspherical manifold, then M can be seen as an iterated principal torus bundle over $\tilde{X}/(\Gamma/\mathcal{Z}_i\Gamma)$. A special case of lemma 2.5.1 and theorem 2.0.1 is the next corollary.

Corollary 2.5.2. *If M is a closed connected aspherical manifold such that $\mathcal{Z}\pi_1(M)$ is trivial and* $\mathrm{Out}(\pi_1(M))$ *is Minkowski, then* $\mathrm{Homeo}(M \times T^n)$ *is Jordan.*

Finally, there exists closed hyperbolic 3-manifolds N which are integral homology spheres [BP92, Thu22]. The group $\mathcal{Z}\pi_1(N)$ is trivial and $\mathrm{Out}(\pi_1(N))$ is finite by [Gro87, 5.4 A]. Thus, $H^*(N \times T^2) \cong H^*(S^3 \times T^2)$, $\mathrm{Homeo}(N \times T^2)$ is Jordan by corollary 2.5.2 and $\mathrm{Homeo}(S^3 \times T^2)$ is not Jordan by [MiR17]. This completes the proof of proposition 2.0.5.

Another example can be produced using Brieskorn manifolds $\Sigma(p,q,r)$. Recall that

$$\Sigma(p,q,r) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^p + z_2^q + z_3^r = 0, \sum_{i=1}^{3} |z_i|^2 = 1\}.$$

If p, q and r are relatively prime and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ then $\Sigma(p,q,r)$ is a closed connected aspherical integral homology 3-sphere (see [Mil75]). Moreover, $\Sigma(p,q,r)$ are Seifert manifolds with $\mathcal{Z}\pi_1(\Sigma(p,q,r)) \cong \mathbb{Z}$ (see [LR10, §14.11]), hence there exists a S^1-action on $\Sigma(p,q,r)$ inducing the short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\Sigma(p,q,r)) \longrightarrow \mathrm{Inn}\,\pi_1(\Sigma(p,q,r)) \longrightarrow 1$$

where $\operatorname{Inn}\pi_1(M)$ is centreless. In addition, $\operatorname{Inn}\pi_1(M)$ is a subgroup of isometries of the hyperbolic plane. Hence, it contains a centreless torsion-free Fuchsian subgroup Q of finite index. Since $\operatorname{Out}(Q)$ is virtually torsion-free then $\operatorname{Out}(\operatorname{Inn}\pi_1(M))$ is also virtually torsion-free (see [MS06, Lemma 2.4, Corollary 2.6]) and therefore $\operatorname{Out}(\operatorname{Inn}\pi_1(M))$ is Minkowski.

The manifold $\Sigma(p,q,r) \times T^2$ is a closed connected aspherical manifold and its cohomology satisfies that $H^*(\Sigma(p,q,r) \times T^2) \cong H^*(S^3 \times T^2)$. Moreover, since $\mathcal{Z}\pi_1(\Sigma(p,q,r) \times T^2) \cong \mathbb{Z}^3$ we have a central extension

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow \pi_1(T^2 \times \Sigma(p,q,r)) \longrightarrow \operatorname{Inn}\pi_1(\Sigma(p,q,r)) \longrightarrow 1.$$

By lemma 2.5.1, $\operatorname{Out}(\pi_1(T^2 \times \Sigma(p,q,r)))$ is Minkowski. Therefore, theorem 2.0.1 implies that $\operatorname{Homeo}(T^2 \times \Sigma(p,q,r)))$ is Jordan.

Remark 2.5.3. *A lot of the results about the Jordan property on* $\operatorname{Homeo}(M)$ *of a closed connected manifold M rely on the cohomology of M, for example in [MiR19] its is proven that* $\operatorname{Diff}(M)$ *is Jordan if M is an integral homology sphere or $\chi(M) \neq 0$. However, proposition 2.0.5 shows that the Jordan property on* $\operatorname{Homeo}(M)$ *and* $\operatorname{Diff}(M)$ *does not only depend on the cohomology of M in general.*

Remark 2.5.4. *We also note that all three manifolds have different discrete degree of symmetry.*

We have $\text{disc-sym}(N \times T^2) = 2$*, since* $\text{disc-sym}(N \times T^2) \leq 2$ *by theorem 2.0.1 and $N \times T^2$ admits a T^2 free action. Similarly,* $\text{disc-sym}(\Sigma(p,q,r) \times T^2) = 3$*, since* $\text{disc-sym}(\Sigma(p,q,r) \times T^2) \leq 3$ *by theorem 2.0.1 and $\Sigma(p,q,r) \times T^2$ admits an action of T^3. Finally, T^4 acts effectively on $S^3 \times T^2$, hence* $\text{disc-sym}(S^3 \times T^2) \geq 4$*. Consequently* $\text{disc-sym}(N \times T^2) \neq \text{disc-sym}(S^3 \times T^2)$.

The two examples from above are in dimension 5, and they can be generalized for dimension $n \geq 5$ if we consider the aspherical manifolds $T^{n-3} \times N$ or $T^{n-3} \times \Sigma(p,q,r)$. Nevertheless, there are closed connected 4-manifolds whose homeomorphism group is not Jordan, like $T^2 \times S^2$. We will construct closed aspherical manifolds whose homeomorphism group is Jordan and their rational cohomology is isomorphic to the rational cohomology of $T^2 \times S^2$.

Let C_g be a hyperelliptic curve of genus g. The product $T^2 \times C_g$ is a closed connected aspherical manifold and $\operatorname{Homeo}(M)$ is Jordan by corollary 2.5.2. Consider the free action $\mathbb{Z}/2$ on $T^2 \times C_g$ given by a rotation on a S^1-direction in T^2 and the hyperelliptic involution in C_g. Then the quotient $(T^2 \times C_g)/(\mathbb{Z}/2)$ is a closed connected aspherical manifold and $\operatorname{Homeo}((T^2 \times C_g)/(\mathbb{Z}/2))$ is Jordan. We claim that $H^*((T^2 \times C_g)/(\mathbb{Z}/2), \mathbb{Q}) \cong H^*(T^2 \times S^2, \mathbb{Q})$. Indeed, $H^*((T^2 \times C_g)/(\mathbb{Z}/2), \mathbb{Q}) \cong H^*(T^2 \times C_g, \mathbb{Q})^{\mathbb{Z}/2} \cong (H^*(T^2, \mathbb{Q}) \otimes H^*(C_g, \mathbb{Q}))^{\mathbb{Z}/2}$. The action of $\mathbb{Z}/2$ on $H^*(T^2, \mathbb{Q}) \otimes H^*(C_g, \mathbb{Q}))$ is the diagonal action induced by the actions of $\mathbb{Z}/2$ on $H^*(T^2, \mathbb{Q})$ and $H^*(C_g, \mathbb{Q})$ respectively. The action of $\mathbb{Z}/2$ on $H^*(T^2, \mathbb{Q})$ is trivial and the action of $\mathbb{Z}/2$ on $H^*(C_g, \mathbb{Q})$ is trivial on $H^0(C_g, \mathbb{Q})$ and

$H^2(C_g,\mathbb{Q})$ and the multiplication by -1 in $H^1(C_g,\mathbb{Q})$. Thus, a straightforward computation shows that $H^*((T^2 \times C_g)/(\mathbb{Z}/2),\mathbb{Q}) \cong H^*(T^2 \times S^2,\mathbb{Q})$.

2.6 When the discrete degree of symmetry is close to the dimension of the aspherical manifold

We have seen that if M is a closed n-dimensional aspherical manifold with $\mathcal{Z}\pi_1(M)$ finitely generated and $\mathrm{Out}(\pi_1(M))$ Minkowski then disc-sym$(M) = n$ if and only if $M \cong T^n$. An interesting question is whether there exist similar rigidity results when disc-sym(M) is close to n, for example disc-sym$(M) = n-1$. The aim of this section is to answer this question provided we have an additional hypothesis on $\pi_1(M)$.

We start by recalling what happens in low dimensions. Let M be a 2-dimensional closed connected aspherical manifold. If M is orientable then M is either a torus T^2 and therefore tor-sym$(M) = 2$, or M is a surface Σ_g of genus $g \geq 2$ and tor-sym$(M) = 0$. If M is not orientable, then M is either the Klein bottle K and tor-sym$(M) = 1$ or M has a surface Σ_g of genus $g \geq 2$ as an orientable 2-cover, hence tor-sym$(M) = 0$.

If M is a closed connected 3-dimensional aspherical manifold with an effective S^1 action, then M falls in one of the following 4 cases (see [LR10, §14.4]):

1. $M \cong T^3$.
2. M is homeomorphic to $K \times S^1$ or SK, the non-trivial principal S^1-bundle over K.
3. $M \cong H/\Gamma$, where H is the 3-dimensional Heisenberg group and Γ is a lattice of H.
4. $\mathcal{Z}\pi_1(M) \cong \mathbb{Z}$ and $\mathrm{Inn}\,\pi_1(M) \cong \pi_1(M)/\mathcal{Z}\pi_1(M)$ is centreless.

Note that in all cases, we have a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow Q \longrightarrow 1$$

where Q acts effectively, properly and cocompactly on $\mathbb{R}^2$.

In the first case tor-sym$(M) = 3$ and in the third and fourth cases tor-sym$(M) = 1$. In the second case, it is clear that tor-sym$(K \times S^1) = 2$, so we focus on SK. One can see that $\pi_1(SK) \cong \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, where $\phi : \mathbb{Z} \longrightarrow \mathrm{GL}(2,\mathbb{Z})$ satisfies that

$$\phi(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $\phi(1)^2 = Id$, SK is a flat solvmanifold and therefore tor-sym$(SK) = \mathrm{rank}\,\mathcal{Z}\pi_1(SK)$.

Lemma 2.6.1. *Let $\phi : \mathbb{Z} \longrightarrow \mathrm{GL}(n, \mathbb{Z})$ be group morphism such that $\phi(1)$ has finite order a. Then $\mathcal{Z}(\mathbb{Z}^n \rtimes_\phi \mathbb{Z}) = \mathrm{Fix}(\phi) \times a\mathbb{Z}$, where $\mathrm{Fix}(\phi) = \{v \in \mathbb{Z}^n : \phi(1)v = v\}$.*

Proof. A straightforward computation shows that given $(v,t), (w,s) \in \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$ we have $(v,t)(w,s) = (w,s)(v,t)$ if and only if $(Id - \phi(s))v = (Id - \phi(t))w$. If $(v,t) \in \mathcal{Z}(\mathbb{Z}^n \rtimes_\phi \mathbb{Z})$ then we have $(Id - \phi(t))w = 0$ for all $w \in \mathbb{Z}^n$, by taking $s = 0$. Thus, $\phi(t) = Id$ and hence $t \in a\mathbb{Z}$.

If we assume that $w = 0$, then $(Id - \phi(s))v = 0$ for all $s \in \mathbb{Z}$. Thus, $v \in \mathrm{Fix}(\phi)$. Consequently, $\mathcal{Z}(\mathbb{Z}^n \rtimes_\phi \mathbb{Z}) \subseteq \mathrm{Fix}(\phi) \times a\mathbb{Z}$. The other inclusion follows from the fact that if $(v,t) \in \mathrm{Fix}(\phi) \times a\mathbb{Z}$ then $(Id - \phi(s))v = 0 = (Id - \phi(t))w$ for any $s \in \mathbb{Z}$ and $w \in \mathbb{Z}^n$. □

Applying the previous lemma to $\pi_1(SK)$, we obtain $\mathrm{Fix}(\phi) = \langle (1,1) \rangle$, hence $\mathcal{Z}\pi_1(SK) \cong \mathbb{Z}^2$ and $\text{tor-sym}(SK) = 2$. Thus, K and SK are the two only 3-dimensional aspherical manifolds such that where $\text{tor-sym}(M) = \dim(M) - 1$. The next proposition generalizes the previous fact to arbitrary dimension. It is probably well-known to experts but we could not find a proof in the literature, so we provide it for the sake of completeness.

Proposition 2.6.2. *Let M be a closed aspherical manifold of dimension n. If $\text{tor-sym}(M) = n-1$ then $M \cong K \times T^{n-2}$ or $M \cong T^{n-3} \times SK$. In particular, M is always a non-orientable flat solvmanifold.*

Proof. Let $H \leq T^{n-1}$ be the isotropy subgroup of the principal orbit of the action. Since T^{n-1} is abelian and its action on M is effective we can conclude that H is trivial. Therefore, the principal orbits of the action have dimension $n-1$. In this case we say that we have a cohomogenity one action. The next theorem describes the cohomogenity one actions.

Theorem 2.6.3. *[GGZ18, Theorem A] Let M be a n-dimensional closed connected manifold with a cohomogenity one action of a compact Lie group G. Let H be the isotropy subgroup of a principal orbit. Then we have one of these two options:*

1. *The quotient $M/G \cong S^1$. Then M is equivariantly homeomorphic to the total space of a fiber bundle $\pi : M \longrightarrow S^1$ is a fiber bundle with fiber G/H. The action does not have exceptional orbits.*
2. *The quotient $M/G \cong [-1,1]$. Then M is the union of two fiber bundles over the two singular orbits with isotropy subgroups K_+ and K_- whose fibers are cones over spheres or the Poincaré homology sphere. More explicitly,*

$$M = G \times_{K_-} C(K_-/H) \cup_{G/H} G \times_{K_+} C(K_+/H)$$

where $C(K_\pm/H)$ denotes the cone over $K_\pm/H$, which are spheres or Poincaré homology spheres. The exceptional orbits $G/K_\pm$ correspond to the preimages of $\pm 1 \in [-1,1]$.

If the cohomogenity one action of T^{n-1} on M does not have exceptional orbits, then the action is free and the orbit map $\pi : M \longrightarrow S^1$ is a principal T^{n-1}-bundle. Since the base is S^1 the principal bundle is trivial and therefore $M \cong T^n$ and tor-sym$(M) = n$, which is not possible. Therefore the action has exceptional orbits.

Since M is aspherical, the evaluation map $ev_x : T^{n-1} \longrightarrow M$ such that $ev_x(g) = gx$ for all $g \in T^{n-1}$ induces an injective group morphism $ev_{x*} : \pi_1(T^{n-1}) \longrightarrow \pi_1(M)$ for any $x \in M$ ([LR10, Lemma 3.1.11]). Consequently, all the isotropy subgroups of the action are discrete. The quotients $K_\pm/H \cong K_\pm$ are homeomorphic to a sphere or a Poincaré homology sphere. Since they are also discrete we obtain that $K_\pm \cong S^0 \cong \mathbb{Z}/2$. Then

$$M = T^{n-1} \times_{\mathbb{Z}/2} I_+ \cup_{T^{n-1}} T^{n-1} \times_{\mathbb{Z}/2} I_-,$$

where $I_+ = [0,1]$ and $I_- = [-1,0]$. Let $Z_+ = \pi_1(T^{n-1} \times_{\mathbb{Z}/2} I_+)$ and $Z_- = \pi_1(T^{n-1} \times_{\mathbb{Z}/2} I_-)$. The principal $\mathbb{Z}/2$-bundles $T^{n-1} \times I_\pm \longrightarrow T^{n-1} \times_{\mathbb{Z}/2} I_\pm$ induce two short exact sequences of fundamental groups

$$1 \to \mathbb{Z}^{n-1} \xrightarrow{i_\pm} Z_\pm \to \mathbb{Z}/2 \to 1.$$

Note that Z_+ and Z_- are isomorphic to $\mathbb{Z}^{n-1}$. By the Seifert-van Kampen theorem, $\pi_1(M) = Z_+ *_{\mathbb{Z}^{n-1}} Z_-$ where the amalgamated product is induced by the inclusions $i_\pm : \mathbb{Z}^{n-1} \longrightarrow Z_\pm$, which we are going to describe explicitly.

Firstly, there exist primitive elements $\alpha_\pm \in Z_\pm$ such that $\alpha_\pm \notin i_\pm(\mathbb{Z}^{n-1})$. Furthermore, there exist two $x_\pm \in \mathbb{Z}^{n-1}$ such that $i_\pm(x_\pm) = 2\alpha_\pm$. We have two possibilities, that $x_+ = x_-$ or that $x_+ \neq x_-$.

Firstly, assume that $x_+ = x_-$. If we denote $x_+ = x_- = x$ then we can choose a generator set $\{x, y_1, \ldots, y_{n-2}\}$ of $\mathbb{Z}^{n-1}$ such that $\{\alpha_+, i_+(y_1), \ldots, i_+(y_{n-2})\}$ generates Z_+ and $\{\alpha_-, i_-(y_1), \ldots, i_-(y_{n-2})\}$ generates $\mathbb{Z}_-$. Thus,

$$\pi_1(M) \cong \langle \alpha_+, \alpha_- | \alpha_+^2 = \alpha_-^2 \rangle \times \mathbb{Z}^{n-2} \cong \pi_1(K) \times \mathbb{Z}^{n-2} = \pi_1(K \times T^{n-2}).$$

Since $\pi_1(M)$ is the fundamental group of a flat manifold and the Borel conjecture holds for these groups (see [BL12]) we conclude that $M \cong K \times T^{n-2}$.

We now assume that $x_+ \neq x_-$. We can choose a generator set $\{x_+, x_-, y_1, \ldots, y_{n-3}\}$ of $\mathbb{Z}^{n-1}$ such that $\{\alpha_+, i_+(x_-), i_+(y_1), \ldots, i_+(y_{n-3})\}$ and $\{i_-(x_+), \alpha_-, i_-(y_1), \ldots, i_-(y_{n-3})\}$ generate $\mathbb{Z}_+$ and Z_- respectively. Thus,

$$\begin{aligned}\pi_1(M) \cong \langle &\alpha_+, i_+(x_-), i_-(x_+), \alpha_- | \\ &\alpha_+^2 = i_-(x_+), \alpha_-^2 = i_+(x_-), [\alpha_+, i_+(x_-)] = 1, [\alpha_-, i_-(x_+)] = 1\rangle \times \mathbb{Z}^{n-3}.\end{aligned}$$

The presentation of the first factor Λ can be rearranged to obtain a new presentation

$$\begin{aligned}\Lambda \cong&\langle a,b|[a,b^2]=1,[a^2,b]=1\rangle\\ \cong&\langle a,b,c,d|c=ab,d=ba,ac=bd,cb=da\rangle\\ \cong&\langle a,c,d|[c,d]=1,aca^{-1}=d,ada^{-1}=c\rangle.\end{aligned}$$

With the last presentation, we can define an isomorphism $f:\Lambda\longrightarrow\mathbb{Z}^2\rtimes_\phi\mathbb{Z}$ such that $f(a)=(0,0,1)$, $f(b)=(1,0,0)$ and $f(c)=(0,1,0)$. Consequently, $\pi_1(M)\cong\pi_1(SK\times T^{n-3})$. Since $\pi_1(M)$ is the fundamental group of a flat manifold and the Borel conjecture holds for these groups we conclude that $M\cong SK\times T^{n-3}$.

□

Corollary 2.6.4. *Let M be a closed connected aspherical manifold of dimension n. If $M\ncong T^n$ and M is orientable then* tor-sym$(M)\leq n-2$.

Remark 2.6.5. *The hypothesis of M being aspherical is essential. For example* tor-sym$(S^2)=1$ *and $S^2\ncong K\times T^{n-2},T^{n-3}\times SK$.*

Theorem 2.6.6. *Let M be a closed connected aspherical manifold such that $\mathcal{Z}\pi_1(M)$ is finitely generated and* Out$(\pi_1(M))$ *is Minkowski. Assume that* Inn$(\pi_1(M))$ *has an element of infinite order. Then* disc-sym$(M)=n-1$ *if and only if $M\cong K\times T^{n-2}$ or $M\cong T^{n-3}\times SK$.*

Proof. Because of the hypothesis on $\pi_1(M)$ we know that $\mathcal{Z}\pi_1(M)\cong\mathbb{Z}^{n-1}$. Let $x\in\operatorname{Inn}\pi_1(M)$ be an element of infinite order and consider the commutative diagram

$$\begin{array}{ccccccccc}1&\longrightarrow&\mathcal{Z}\pi_1(M)&\longrightarrow&p^{-1}(\langle x\rangle)&\longrightarrow&\langle x\rangle&\longrightarrow&1\\ &&\downarrow{\scriptstyle Id}&&\downarrow&&\downarrow&&\\ 1&\longrightarrow&\mathcal{Z}\pi_1(M)&\longrightarrow&\pi_1(M)&\xrightarrow{p}&\operatorname{Inn}\pi_1(M)&\longrightarrow&1\end{array}$$

where the vertical arrows are inclusion maps. The upper short exact sequence is also central and therefore it is classified by $H^2(\mathbb{Z},\mathbb{Z}^{n-1})=0$. In consequence, $p^{-1}(\langle x\rangle)\cong\mathbb{Z}^n$. We consider now the covering $q:\tilde{M}/p^{-1}(\langle x\rangle)\longrightarrow M$. Note that $H^*(\tilde{M}/p^{-1}(\langle x\rangle),\mathbb{Z})\cong H^*(\mathbb{Z}^n,\mathbb{Z})$ and therefore $H^n(\tilde{M}/p^{-1}(\langle x\rangle),\mathbb{Z})$ is not trivial. Consequently, $\tilde{M}/p^{-1}(\langle x\rangle)$ is a closed connected aspherical manifold and $\tilde{M}/p^{-1}(\langle x\rangle)\cong T^n$. Since M and $\tilde{M}/p^{-1}(\langle x\rangle)$ are compact, we obtain that q is a finite covering map and therefore $p^{-1}(\langle x\rangle)\leq\pi_1(M)$ has finite index and $\pi_1(M)$ is the fundamental group of a flat manifold. Thus, disc-sym$(M)=$ tor-sym(M) by theorem 2.0.4 and the result follows from proposition 2.6.2. □

It is an interesting question to know whether $\operatorname{Inn}\pi_1(M)$ always has an element of infinite order when M is closed aspherical manifold not homeomorphic to a torus and such that $\mathcal{Z}\pi_1(M)$ is finitely generated. Note that $\operatorname{Inn}\pi_1(M)$ cannot be finite by lemma 2.1.2 unless

M is homeomorphic to a torus. Thus, if $\operatorname{Inn}\pi_1(M)$ does not contain elements of infinite order then $\pi_1(M)$ is abelian or $\operatorname{Inn}\pi_1(M)$ is infinite periodic.

We present some evidences that support that the answer to the question is affirmative. Firstly, let Γ and Λ be two groups. Assume that we have a surjective group morphism $p : \Gamma \longrightarrow \Lambda$. Then there is an induced surjective group morphism $p' : \operatorname{Inn}\Gamma \longrightarrow \operatorname{Inn}\Lambda$ which sends a conjugation c_γ to $c_{p(\gamma)}$. Thus, if $\operatorname{Inn}\Lambda$ has an element of infinite order, then $\operatorname{Inn}\Gamma$ also has an element of infinite order.

Assume now that we have an inclusion $i : \Gamma \longrightarrow \Lambda$ instead. Then there is an induced injective group morphism $i' : \operatorname{Inn}\Gamma \longrightarrow \operatorname{Inn}\Lambda$ which sends a conjugation c_γ to $c_{i(\gamma)}$. In particular, if $\operatorname{Inn}\Gamma$ has an element of infinite order, then $\operatorname{Inn}\Gamma$ also has an element of infinite order. We can deduce the following corollary from these observations:

Corollary 2.6.7. *Let M be a closed aspherical manifold, then:*

(1) If $M' \longrightarrow M$ is a covering and $\operatorname{Inn}\pi_1(M')$ has an element of infinite order then $\operatorname{Inn}\pi_1(M)$ has an element of infinite order.

(2) Suppose that we have a fibration of closed connected aspherical manifolds $M' \longrightarrow M \longrightarrow M''$. If $\operatorname{Inn}\pi_1(M')$ or $\operatorname{Inn}\pi_1(M'')$ have an element of infinite order, then $\operatorname{Inn}\pi_1(M)$ has an element of infinite order.

Proof. In the first case, we use that $\pi_1(M') \leq \pi_1(M)$ and in the second case the short exact sequence $1 \longrightarrow \pi_1(M') \longrightarrow \pi_1(M) \longrightarrow \pi_1(M'') \longrightarrow 1$ together with the observations above. □

Note that if Γ is a non-abelian polycyclic group or Γ is torsion-free centreless then $\operatorname{Inn}\Gamma$ has elements of infinite order. Thus, closed aspherical locally homogeneous spaces and closed aspherical manifolds whose fundamental group is hyperbolic have an infinite order element in the inner automorphism group of their fundamental group. Moreover, any closed aspherical manifold M not homeomorphic to a torus constructed using fibrations of these two classes of aspherical manifolds will have an element of infinite order in $\operatorname{Inn}\pi_1(M)$.

There are cohomological restrictions to $\operatorname{Inn}\pi_1(M)$ being infinite periodic. Assume that M is a closed connected aspherical manifold such that $\operatorname{Inn}\pi_1(M)$ is infinite periodic. Then $H^1(\operatorname{Inn}\pi_1(M),\mathbb{Z}) = \operatorname{Hom}(\operatorname{Inn}\pi_1(M),\mathbb{Z})$ is trivial and the inflation-restriction exact sequence becomes

$$1 \longrightarrow H^1(\pi_1(M),\mathbb{Z}) \longrightarrow H^1(\mathcal{Z}\pi_1(M),\mathbb{Z}) \longrightarrow H^2(\operatorname{Inn}\pi_1(M),\mathbb{Z}) \longrightarrow H^2(\pi_1(M),\mathbb{Z})$$

where we see $\mathbb{Z}$ as a trivial $\pi_1(M)$-module. In particular, if $\operatorname{rank} H^1(M,\mathbb{Z}) > \operatorname{rank}\mathcal{Z}\pi_1(M)$ then $\operatorname{Inn}\pi_1(M)$ has elements of infinite order.

Moreover, since $H^2(\pi_1(M), \mathbb{Z}) \cong H^2(M, \mathbb{Z})$ and $H^1(\mathcal{Z}\pi_1(M), \mathbb{Z}) \cong H^1(T^{\operatorname{rank} \mathcal{Z}\pi_1(M)}, \mathbb{Z})$, the group $H^2(\operatorname{Inn} \pi_1(M), \mathbb{Z})$ needs to be finitely generated. Using the results in [AA18], we can conclude that $\operatorname{Inn} \pi_1(M)$ cannot be isomorphic to a free Burnside group $B(a, b)$ with $b \geq 665$ odd. The results in [Che21] imply that $\operatorname{Inn} \pi_1(M)$ cannot be an infinite periodic 2-group of bounded exponent. Finally, we note that $\operatorname{Inn} \pi_1(M)$ is a finitely presented group and it is an open question if there exists finitely presented infinite periodic groups (this question is the Burnside problem for finitely presented groups, see [Sap07, pg. 3])

Chapter 3

Large finite group actions and non-zero degree maps

The aim of this chapter is to generalize the following theorem:

Theorem 3.0.1. *[MiR24a, Theorem 1.3, Theorem 1.14] Let M be a closed connected orientable n-dimensional manifold which admits a non-zero degree map $f : M \longrightarrow T^n$. Then:*

1. $\mathrm{Homeo}(M)$ *is Jordan.*
2. *If* $\text{disc-sym}(M) = n$ *then there is an isomorphism of rings* $H^*(M, \mathbb{Z}) \cong H^*(T^n, \mathbb{Z})$.
3. *If* $\chi(M) \neq 0$ *then M is almost asymmetric.*
4. *M has few stabilizers.*

We generalize this result by assuming that M admits a non-zero degree map to a nilmanifold.

Theorem 3.0.2. *Let M be a closed oriented connected n-dimensional manifold and $f : M \longrightarrow N/\Gamma$ a non-zero degree map to a closed nilmanifold. Then:*

1. $\mathrm{Homeo}(M)$ *is Jordan.*
2. $\text{disc-sym}(M) \leq \operatorname{rank} \mathcal{Z}\Gamma$ *and if* $\text{disc-sym}(M) = n$ *then* $H^*(M, \mathbb{Z}) \cong H^*(T^n, \mathbb{Z})$.
3. *If* $\chi(M) \neq 0$ *then M is almost asymmetric.*
4. *M has few stabilizers.*

However, we note that there exists a closed connected oriented manifold which admits a non-zero degree map to a nilmanifold N/Γ such that $\text{disc-sym}(M) = \text{disc-sym}(N/\Gamma)$ but $H^*(M, \mathbb{Q}) \ncong H^*(N/\Gamma, \mathbb{Q})$ (see proposition 3.3.13).

The proof of theorem 3.0.2 is divided in two parts, which will be explained in sections

2 and 3 of this chapter. In the second section of this chapter we introduce some new properties of non-zero degree maps between manifolds $f : M \longrightarrow N$ and we use them to study the relation between finite a groups acting on M and N. In the third section we prove theorem 3.0.2 using the results in section 3.2. In the first section we prove a generalization of theorem 2.0.1 for admissible manifolds and discuss some of its applications.

3.1 Large finite group actions on admissible manifolds

Note that closed connected oriented manifolds admitting a non-zero degree map to a torus or a nilmanifold are admissible (see definition 1.3.107). One could use the same arguments used in chapter 2 together with theorem 1.3.109 to obtain:

Theorem 3.1.1. *Let M be a closed connected admissible manifold. Assume that* $\mathrm{Out}(\pi_1(M))$ *is Minkowski and that $\mathcal{Z}\pi_1(M)$ is finitely generated. Then:*

1. $\mathrm{Homeo}(M)$ *is Jordan.*
2. $\text{disc-sym}(M) \leq \mathrm{rank}(\mathcal{Z}\pi_1(M)/\,\mathrm{Torsion}(\mathcal{Z}\pi_1(M)))$.
3. *If $\chi(M) \neq 0$ and* $\mathrm{Aut}(\pi_1(M))$ *is Minkowski then M is almost asymmetric.*
4. *If* $\mathrm{Aut}(\pi_1(M))$ *is Minkowski then M has few stabilizers.*

Proof. The proofs of items 1, 2 and 4 are the same as in theorem 2.0.1. Note that we cannot use lemma 2.1.3 to prove item 3 of theorem 3.1.1 in the same way as in theorem 2.0.1. If we have the extra assumption that $\mathrm{Aut}(\pi_1(M))$ is Minkowski then item 3 follows from theorem 1.1.63 and lemma 1.1.64. □

An application of theorem 3.1.1 is the following. Suppose that M is a closed connected oriented aspherical manifold satisfying the hypothesis of theorem 2.0.1 and M' is a closed simply-connected manifold of the same dimension as M, then $M\#M'$ is a closed admissible manifold such that $\mathrm{Homeo}(M\#M')$ is Jordan, $\text{disc-sym}(M\#M') \leq \mathrm{rank}\,\mathcal{Z}\pi_1(M)$ and $M\#M'$ has few stabilizers. The discrete degree of symmetry inequality is usually strict. For example, assume that M' is a closed simply-connected even dimensional manifold such that $\chi(M') \neq 2$. Then $\chi(T^n\#M') = \chi(M') - 2 \neq 0$ and hence $\text{disc-sym}(T^n\#M') = 0 < n = \mathrm{rank}\,\mathcal{Z}\pi_1(T^n\#M')$, by theorem 3.1.1(3).

Theorem 3.1.1 and theorem 3.0.2 have a non-trivial overlap, but neither is more general than the other. On one hand, not all admissible manifolds admit a non-zero degree maps to a nilmanifold. On the other hand, there exist closed connected orientable manifolds admitting anon-zero degree map to a torus such that the outer automorphism group of their fundamental group is not Minkowski. In [Blo75, Theorem 1.1.1] it is proved that if Γ and Λ are isomorphic neither to a non-trivial free product nor to a infinite cyclic group then $\mathrm{Out}(\Gamma * \Lambda) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)$ if $\Gamma \not\cong \Lambda$ and $\mathrm{Out}(\Gamma * \Lambda) \cong (\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)) \times \mathbb{Z}/2$ if $\Gamma \cong \Lambda$. In particular, if M is a closed connected 4-manifold with fundamental group the Baumslag-Solitar group $B(m, ml) = \langle a, b | ba^m b^{-1} = a^{ml} \rangle$ with $m, l \geq 2$. Note that $B(m, ml)$ is not a proper free product of groups (see [LSLS77, II.5.13] or) Then $\mathrm{Out}(\pi_1(M\#T^4)) \cong \mathrm{Aut}(B(m, ml)) \times \mathrm{GL}(4, \mathbb{Z})$ is not Minkowski (see remark 2.1.5), but $M\#T^4$ is hypertoral and hence $\mathrm{Homeo}(M\#T^4)$ is Jordan by theorem 3.0.1.

3.2 Exporting and importing maps

A key result to prove [MiR24a, Theorem 1.3] is the following theorem.

Theorem 3.2.1. *[MiR24a, Theorem 4.1] Let M be a closed oriented manifold of dimension n which admits a continuous map $f : M \longrightarrow T^n$ of non-zero degree and let G be a finite group acting effectively on M and trivially on $H^1(M, \mathbb{Z})$, then there exist a group action of G on T^n and a continuous map $f_G : M \longrightarrow T^n$ which is G-equivariant and homotopic to f.*

Remark 3.2.2. *The constructed action of G on T^n is not necessarily effective. If K denotes the kernel of ineffectiveness, then $|K| \leq \deg f$ and the effective action of G/K on T^n is free and by rotations. Thus the induced group morphism $G/K \longrightarrow \mathrm{GL}(n, \mathbb{Z})$ is trivial.*

Theorem 3.2.1 inspires the following definition:

Definition 3.2.3. *Let M and M' be closed oriented manifolds of the same dimension and let $f : M \longrightarrow M'$ be a continuous map. We say that f exports group actions, or f is an exporting map, if there exists a constant C such that every finite group G acting effectively on M (which we denote by $\phi : G \longrightarrow \mathrm{Homeo}(M)$) has a subgroup $H \leq G$ such that:*

1. $[G : H] \leq C$
2. *There exists an action H on M' (denoted by $\phi' : H \longrightarrow \mathrm{Homeo}(M')$).*
3. *There exists an H-equivariant map $f_H : M \longrightarrow M'$ homotopic to f.*

By Minkowski lemma, given a closed connected manifold M there exists a constant C such that any finite group G acting effectively on M has a finite index subgroup H of index $[G : H] \leq C$ such that H acts trivially on $H^1(M, \mathbb{Z})$. Thus, theorem 3.2.1 states that any non-zero degree map $f : M \longrightarrow T^n$ is an exporting map. Another example of this property in the smooth setting is provided by the following theorem of R.Schoen and S.T.Yau.

Theorem 3.2.4. *[SY79, Theorem 8] Let M and M' be closed connected orientable smooth manifolds of the same dimension. Assume that M' has a Riemannian metric of non-positive curvature and there is a non-zero degree smooth map $f : M \longrightarrow M'$ such that $f_* : \pi_1(M) \longrightarrow \pi_1(M')$ is surjective. Let $A(M')$ denote the group of affine transformations on M' (that is, the group of diffeomorphisms preserving the Levi-Civita connection) and let $\overline{A}(M')$ denote the subgroup of the identity component* *of $A(M')$ generated by the parallel vector fields of M'. Given a finite group G acting effectively and smoothly on M, suppose that for each $g \in G$ there exists an element $g' \in \overline{A}(M')$ such that $f \circ g$ is freely homotopic to $g' \circ f$ (g' is not necessarily unique or non-trivial). Then there exist a group morphism $\gamma : G \longrightarrow \overline{A}(M')$ and a γ-equivariant smooth map $f_G : M \longrightarrow M'$ homotopic to f. Moreover, $|\operatorname{Ker}\gamma| \leq \deg(f)$.*

If there exists a constant C such that every finite group G acting smoothly and effectively on M has a subgroup $H \leq G$ satisfying the conditions of theorem 3.2.4 and $[G : H] \leq C$ then f is an exporting map. Some examples can be found in [SY79, Theorem 11, Theorem 13].

Theorem 3.2.5. *[SY79, Theorem 11, Theorem 13] Let M and M' be closed connected orientable smooth manifolds of the same dimension. Assume that there is a degree one map $f : M \longrightarrow M'$ such that $f_* : \pi_1(M) \longrightarrow \pi_1(M')$ is surjective. Furthermore, assume that M' satisfy one of the following set of conditions:*

1. *M' is diffeomorphic to a locally symmetric space $\Gamma \setminus G/H$, where all the factors of G have real rank equal or greater than 2.*
2. *M' is flat and $\mathcal{Z}\pi_1(M')$ is trivial.*

Then for any finite group G acting effectively and smoothly on M there exist a smooth effective action of G on M' and a G-equivariant map $f_G : M \longrightarrow M'$ homotopic to f.

Note that in theorem 3.2.5 we do not need to replace the finite group by a suitable subgroup of bounded index.

The aim of this section is to study exporting maps.

Lemma 3.2.6. *Let $f : M \longrightarrow M'$ be a non-zero degree exporting map. With the notation as in definition 3.2.3, we have $|\operatorname{Ker}\phi'| \leq \deg(f)$.*

The proof of the lemma is a consequence of the next fact.

Lemma 3.2.7. *[MiR24a, Lemma 4.4] Let M be a closed oriented n-manifold and suppose that G is a finite group acting on M effectively and preserving the orientation. If we denote by $\pi : M \longrightarrow M/G$ the quotient map and by d the cardinal of G then $\pi^*(H^n(M/G, \mathbb{Z})) \subseteq dH^n(M, \mathbb{Z})$.*

Proof of lemma 3.2.6. Since f_H is H-equivariant, we can restrict the action of H on M to the subgroup $\operatorname{Ker}\phi' \leq H$, obtaining a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\;f_H\;} & M' \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \overline{f}_H} & \\ M/\operatorname{Ker}\phi' & & \end{array}$$

In consequence, $\pi^* \circ \overline{f}^*_H$. Furthermore, since f and f_H are homotopic, we have $f_H^* = f^* : H^n(M', \mathbb{Z}) \longrightarrow H^n(M, \mathbb{Z})$. By lemma 3.2.7, we obtain that $|\operatorname{Ker}\phi'|$ divides $\deg(f)$. In particular $|\operatorname{Ker}\phi'| \leq \deg(f)$. □

The next lemma studies the composition of two exporting maps.

Lemma 3.2.8. *Let M, M' and M'' be closed oriented manifolds and let $f : M \longrightarrow M'$ and $g : M' \longrightarrow M''$ be exporting non-zero degree maps with constants C and D respectively. Then $g \circ f : M \longrightarrow M''$ exports group actions with constant $C \cdot D$.*

Proof. Assume that G is a finite group acting effectively on M and take the subgroup $H \leq G$ of definition 3.2.3. Thus, there exists an action of H on M', denoted by $\phi' : H \longrightarrow \operatorname{Homeo}(M')$, a H-equivariant map $f_H : M \longrightarrow M'$ homotopic to f and $[G : H] \leq C$. Since $\phi'(H)$ acts effectively on M', there exist a subgroup $K \leq \phi'(H)$, an action $\phi'' : K \longrightarrow \operatorname{Homeo}(M'')$ satisfying $[\phi'(H) : K] \leq D$, and a K-equivariant map $g_K : M' \longrightarrow M''$ homotopic to g. We can consider the subgroup $H' = \phi'^{-1}(K) \leq H$ and the action of H' on M'' given by the group morphism $\phi'' \circ \phi' : H' \longrightarrow K \leq \operatorname{Homeo}(M'')$. The map $g_K \circ f_H$ is H'-equivariant and homotopic to $g \circ f$. In addition, $[G : H'] \leq C \cdot D$ and hence the claim is proved. □

The main result of this section is that properties introduced in section 1.1.3 behave well with respect exporting maps. It is one of the main ingredients to prove theorem 10.

Theorem 3.2.9. *Let M and M' closed oriented manifolds which admit a non-zero degree exporting map $f : M \longrightarrow M'$. Then:*

1. *If* $\operatorname{Homeo}(M')$ *is Jordan, then* $\operatorname{Homeo}(M)$ *is Jordan.*
2. $\text{disc-sym}(M) \leq \text{disc-sym}(M')$.
3. *If M' has the small stabilizer property then M has the small stabilizer property.*
4. *If M' is almost asymmetric, then M is almost asymmetric.*

In order to prove this theorem we need the three following group-theoretic lemmas.

Lemma 3.2.10. *[MiR10, Lemma 2.2] Let d and r be natural numbers. There exists a natural number $C(d, r)$ such that if we have a short exact sequence of groups*

$$1 \longrightarrow K \longrightarrow G \longrightarrow A \longrightarrow 1$$

where $|K| \leq d$ and A is abelian and generated by r elements, then G has an abelian subgroup of index at most $C(d, r)$.

Lemma 3.2.11. *[MiR24a, Lemma 2.1] Let a,b and C be natural numbers and suppose that G' is a subgroup of $(\mathbb{Z}/a)^b$ satisfying $[(\mathbb{Z}/a)^b : G'] \leq C$. Then there exist a number a' and a subgroup $G'' \leq G'$ such that $G'' \cong (\mathbb{Z}/a')^b$ and $C!a' \geq a$.*

To state the last lemma we need to introduce two new invariants:

Definition 3.2.12. *Let M be a manifold. We define*

$$P\,\text{disc-sym}(M) = \max\{\{0\} \cup \{r : (\mathbb{Z}/p)^r \textit{ acts effectively on } M \textit{ for arbitrarily large prime } p\}\}.$$

For a fixed prime p, we define

$$\text{disc-sym}_p(M) = \max\{\{0\} \cup \{r : (\mathbb{Z}/p^s)^r \textit{ acts effectively on } M \textit{ for arbitrarily large } s\}\}.$$

If M is a closed connected manifold then $P\,\text{disc-sym}(M)$ and $\text{disc-sym}_p(M)$ are bounded theorem 1.1.32.

Lemma 3.2.13. *Let M be a closed manifold, then*

$$\text{disc-sym}(M) = \max(\{P\,\text{disc-sym}(M)\} \cup \{\text{disc-sym}_p(M) : p \textit{ prime}\}).$$

Proof. Clearly, $P\,\text{disc-sym}(M) \leq \text{disc-sym}(M)$ and $\text{disc-sym}_p(M) \leq \text{disc-sym}(M)$ for every prime p. Conversely, assume that $\text{disc-sym}(M) = b$ and therefore there exists a sequence of natural numbers $\{a_i\}_{i\in\mathbb{N}}$ such that $(\mathbb{Z}/a_i)^b$ acts effectively on M for all i and $a_i \longrightarrow \infty$ when $i \longrightarrow \infty$. Let $\mathcal{P}$ be the set of primes which divide a_i for some i.

If $|\mathcal{P}| = \infty$ then there exists a subsequence $\{a_k\}_{k\in\mathbb{N}}$ of $\{a_i\}_{i\in\mathbb{N}}$ such that each a_k is divided by a prime p_k satisfying that $p_k < p_{k+1}$. Thus, by taking $(\mathbb{Z}/p_k)^b \leq (\mathbb{Z}/a_k)^b$ we have an effective action of $(\mathbb{Z}/p_k)^b$ on M. Consequently, $b \leq P\,\text{disc-sym}(M)$.

If $|\mathcal{P}| < \infty$ then there exist m primes $p_1,\ldots, p_m$ such that $a_i = p_1^{x_{1,i}} \ldots p_m^{x_{m,i}}$ for all i. Since $a_i \longrightarrow \infty$ when $i \longrightarrow \infty$, by the pigeonhole principle there exists a subsequence $\{a_k\}_{k\in\mathbb{N}}$ and a number $l \in \{1,\ldots,m\}$ such that $x_{l,k} \longrightarrow \infty$ when $k \longrightarrow \infty$. Thus, we have effective group actions of $(\mathbb{Z}/p_l^{x_{l,k}})^b$ on M. Consequently, $b \leq \text{disc-sym}_{p_l}(M)$.

By combining these two cases we obtain the desired result. □

We are almost ready to give the proof of theorem 3.2.9. The last result we need is a corollary of a theorem by L.N.Mann and J.C.Su (see theorem 1.1.32).

Corollary 3.2.14. *Let M be a closed manifold of dimension n. There exists a number r such that if A is a finite abelian group acting effectively on M then* $\text{rank}(A) \leq r$.

Proof of theorem 3.2.9. For the first statement of the theorem assume that G is a finite group acting effectively on M. Then there exist a subgroup H of G such that $[G : H] \leq C$, an action of H on M' and a continuous map $f_H : M \longrightarrow M'$ which is H-equivariant and homotopic to f. Let $H_0 = \text{Ker}\,\phi'$. By lemma 3.2.6, we known that $|H_0| \leq \deg(f)$. The group $\phi'(H)$ acts effectively on M'. Thus there exists an abelian subgroup $A' \leq \phi'(H)$ such that $[\phi'(H) : A'] \leq C'$, where C' is the Jordan constant of $\text{Homeo}(M')$. By corollary 3.2.14, there exists a constant r' such that any abelian finite group acting effectively on M' has rank at most r'.

We consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_0 & \longrightarrow & \phi'^{-1}(A') & \xrightarrow{\phi'} & A' & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle Id} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H_0 & \longrightarrow & H & \xrightarrow{\phi} & \phi(H) & \longrightarrow & 1 \end{array}$$

Note that $[H : \phi'^{-1}(A')] \leq C'$. We can use lemma 3.2.10 on the upper short exact sequence to find a constant D, which only depends on d and r', and an abelian subgroup A of $\phi'^{-1}(A')$ such that $[\phi'^{-1}(A') : A] \leq D$. In conclusion, we have found a subgroup A of G such that $[G : A] \leq C \cdot C' \cdot D$ and we can conclude that $\text{Homeo}(M)$ is Jordan.

We note that by lemma 3.2.11 we can assume that given an increasing sequence $\{a_i\}_{i\in\mathbb{N}}$ such that $a_i \longrightarrow \infty$ and groups $(\mathbb{Z}/a_i)^b$ acting effectively on M then $(\mathbb{Z}/a_i)^b$ also act on M' for all i and that we can replace f by an homotopic equivariant map for each i.

By lemma 3.2.13 we can divide the prove in two parts. Firstly assume that we have an increasing sequence of primes $\{p_k\}_{k\in\mathbb{N}}$ and groups $(\mathbb{Z}/p_k)^b$ acting effectively on M. Like in the first part of the proof, we consider the group action $\phi'_{p_k,b} : (\mathbb{Z}/p_k)^b \longrightarrow \text{Homeo}(M')$ and the exact sequence

$$1 \longrightarrow K_{(p_k,b)} \longrightarrow (\mathbb{Z}/p_k)^b \xrightarrow{\phi'_{p_k,b}} \phi'_{p_k,b}((\mathbb{Z}/p_k)^b) \longrightarrow 1$$

for each k, where $K_{(p_k,b)} = \text{Ker}\,\phi'_{p_k,b}$. Since $K_{(p_k,b)}$ is a subgroup of $(\mathbb{Z}/p_k)^b$, there exists $x(k) \in \mathbb{N}$ such that $|K_{(p_k,b)}| = p_k^{x(k)}$. On the other hand, $|K_{(p_k,b)}| \leq d$. Since $p_k \longrightarrow \infty$ when $k \longrightarrow \infty$, there exist k_0 such that $p_k > d$ for all $k \geq k_0$. This implies that $x(k) = 0$ for $k \geq k_0$ and that $(\mathbb{Z}/p_k)^b$ acts effectively on M' for $k \geq k_0$. Thus, $P\,\text{disc-sym}(M) \leq \text{disc-sym}(M')$.

We fix a prime number p and denote by $c \in \mathbb{N}$ the largest number such that $p^c \leq d$. Assume that we have an increasing sequence $\{a_k\}_{k\in\mathbb{N}}$ such that $(\mathbb{Z}/p^{a_k})^b$ acts effectively on M. Since $a(k) \longrightarrow \infty$ when $k \longrightarrow \infty$, there exists a k_0 such that $a_{k_0} > c$ for all $k \geq k_0$. Then $K_{(p^{a_k},b)}$ is a subgroup of $(\mathbb{Z}/p^c)^b \leq (\mathbb{Z}/p^{a_k})^b$ and

$$(\mathbb{Z}/p^{a_k-c})^b \cong (\mathbb{Z}/p^{a_k})^b/(\mathbb{Z}/p^c)^b \leq (\mathbb{Z}/p^{a_k})^b/K_{(p^{a_k},b)} \cong \phi'_{p^{a_k},b}((\mathbb{Z}/p^{a_k})^b)$$

for $k \geq k_0$. Hence, $(\mathbb{Z}/p^{a_k-c})^b$ acts effectively on M' for $k \geq k_0$. Thus, $\text{disc-sym}_p(M) \leq \text{disc-sym}(M')$.

Joining the two cases we obtain that disc-sym$(M) \leq$ disc-sym(M').

To prove the third part assume that G is a finite group acting effectively on M with a fix point x. Let C be the constant provided by the exporting map property and let H, $\phi' : H \longrightarrow \text{Homeo}(M')$ and f_H be respectively the subgroup of G, the action on M' and the continuous map homotopic to f given by the assumptions. Since x is a fixed point of the action of H on M, $f_H(x)$ is a fixed point of the action of the effective $\phi'(H)$ on M'. If C' is the small stabilizer constant on M', then $|\phi'(H)| \leq C'$. We obtain the exact sequence

$$1 \longrightarrow H_0 \longrightarrow H \xrightarrow{\phi'} \phi'(H) \longrightarrow 1$$

where $|H_0| \leq d = \deg(f)$ and $|\phi'(H)| \leq C'$. In consequence, $|H| \leq C' \cdot d$ and $|G| \leq C \cdot C' \cdot d$.

The proof of the fourth part is analogous to the proof of the third part. If C' denotes now the constant provided by the almost-asymmetric property of M', then we have $|\phi'(H)| \leq C'$ and hence $|G| \leq C \cdot C' \cdot d$. □

Since $\text{Homeo}(S^4)$ is Jordan and $\text{Homeo}(T^2 \times S^2)$ is not Jordan, we can deduce:

Corollary 3.2.15. *Any non-zero degree map $f : T^2 \times S^2 \longrightarrow S^4$ is not an exporting map.*

Note also that disc-sym$(T^2 \times S^2) \geq 3 >$ disc-sym$(S^4) = 2$.

The next definition is the converse of the exporting map property.

Definition 3.2.16. *Let M and M' be closed oriented manifolds of the same dimension and let $f : M \longrightarrow M'$ be continuous map. We say that f imports group actions, or it is an importing map, if there exists a constant C such that any finite group G acting on M' has a subgroup $H \leq G$ such that:*

1. *$[G : H] \leq C$.*
2. *There exists a finite group $\tilde{H}$ acting effectively on M and a surjective group morphism $\rho : \tilde{H} \longrightarrow H$.*
3. *There exists a ρ-equivariant map $f_H : M \longrightarrow M'$ homotopic to f.*

Remark 3.2.17. *The property of importing group actions is similar to the property of propagating of group actions (see [AD02, Definition 3.1]). Given a continuous map between closed manifolds $f : M \longrightarrow M'$ and a finite group G acting effectively on M', we that the action of G on M' propagates to M across f if there exists an effective action of G on M and an equivariant map $f_G : M \longrightarrow M'$ homotopic to f. Propagation of group actions was used to study group actions on homology spheres (see [AD02, §3.2.2] and references therein).*

Importing and exporting maps have similar properties. The next lemma has an analogous proof to lemma 3.2.8.

Lemma 3.2.18. *Let M, M' and M'' be closed oriented manifolds and let $f : M \longrightarrow M'$ and $g : M' \longrightarrow M''$ be importing maps with constants C and D respectively. Then $g \circ f : M \longrightarrow M''$ imports group actions with constant $C \cdot D$.*

Proof. Let G be a finite group acting effectively on M''. Then there exist a subgroup $H \leq G$ satisfying that $[G : H] \leq D$, a group $\tilde{H}$ acting effectively on M', a surjective group morphism $\rho' : \tilde{H} \longrightarrow H$ and a ρ-equivariant map $g_H : M' \longrightarrow M''$. Since $\tilde{H}$ acts effectively on M', there exist a subgroup $K \leq \tilde{H}$ satisfying that $[\tilde{H} : K] \leq C$, a group $\tilde{K}$ acting effectively on M, a surjective group morphism $\rho : \tilde{K} \longrightarrow K$ and a ρ-equivariant map $f_K : M \longrightarrow M'$.

We consider now the subgroup $\rho'(K) \leq G$. We note that $[G : \rho'(K)] \leq C \times D$, that $\rho'_{|K} \circ \rho : \tilde{K} \longrightarrow \rho'(K)$ is a surjective group morphism and that $g_H \circ f_K$ is $\rho'_{|K} \circ \rho$-equivariant and homotopic to $g \circ f$. Consequently, $g \circ f$ is an importing map. □

Lemma 3.2.19. *Let M and M' be closed oriented manifolds and $f : M \longrightarrow M'$ a non-zero degree importing map. Then $|\operatorname{Ker}\rho| \leq \deg(f)$.*

Proof. Since f_H is ρ-equivariant, we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f_H\ } & M' \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \overline{f}_H} & \\ M/\operatorname{Ker}\rho & & \end{array}$$

By lemma 3.2.7, $\pi^*(H^n(M/\operatorname{Ker}\rho, \mathbb{Z})) \subseteq \deg(f) H^n(M, \mathbb{Z})$. Therefore, $|\operatorname{Ker}\rho|$ divides $\deg(f)$. Thus, $|\operatorname{Ker}\rho| \leq \deg(f)$. □

The main example of importing maps are coverings of manifolds.

Lemma 3.2.20. *Let $p : M \longrightarrow M'$ be a finite covering between closed oriented manifolds. Then p imports group actions.*

Proof. Assume that $p : M \longrightarrow M'$ is a n-sheeted covering and G is a finite group acting effectively on M. Then G also acts on $\operatorname{Cov}_n(M')$, the set of n-sheeted coverings of M' by pull-backs. On the other hand $\operatorname{Cov}_n(M') \cong \operatorname{Hom}(\pi_1(M'), S_n)/\sim$ where S_n is the n-th symmetric group and the equivalence relation is given by conjugation of elements of S_n. Therefore $\operatorname{Cov}_n(M')$ is finite, which implies that there exists a constant C only depending on M' and n such that any finite group G acting effectively on M has a subgroup H which acts trivially on $\operatorname{Cov}_n(M')$ and $[G : H] \leq C$. Then we can lift the action of H on M' to an effective action of a group $\tilde{H}$ on M. In addition, there exists a surjective group morphism $\rho : \tilde{H} \longrightarrow H$ which makes the covering map $p : M \longrightarrow N$ ρ-equivariant. □

We have the analogous result to theorem 3.2.9 for importing maps.

Theorem 3.2.21. *Let M and M' closed oriented manifolds which admit an importing map $f : M \longrightarrow M'$. Then:*

1. *If* $\text{Homeo}(M)$ *is Jordan, then* $\text{Homeo}(M')$ *is Jordan.*
2. $\text{disc-sym}(M') \leq \text{disc-sym}(M)$.
3. *If M has the almost fixed point property, then M' has the almost fixed point property.*
4. *If M is almost asymmetric, then M' is almost asymmetric.*

Proof. The proof of items 1. and 2. are the same as in the case of finite coverings (see [MiR10, MiR24a]). We prove the third part in detail.

Let G be a group acting effectively on M'. Let C be the constant of the definition of the importing map f and $H \leq G$, $f_H : M \longrightarrow M'$ and $\rho : \tilde{H} \longrightarrow H$ the data provided by the definition. Recall that $[G : H] \leq C$ and ρ is surjective. The group $\tilde{H}$ acts effectively on M. Since M has the almost fixed point property with constant D there exists $x \in M$ such that $[\tilde{H} : \tilde{H}_x] \leq D$. We use now that f_H is ρ-equivariant, therefore $\phi(\tilde{H}_x) \leq H_{f_H(x)}$. Since ρ is surjective, $[H : H_{f_H(x)}] \leq [H : \rho(\tilde{H}_x)] \leq D$. Finally. we have $[G : G_{f_H(x)}] \leq [G : H_{f_H(x)}] \leq C \cdot D$. We have seen that M' has the almost fixed point property with constant $C \cdot D$.

The proof of the fourth part is analogous to the third part. □

With theorem 3.2.21 it is straightforward to find example of maps which are not importing. For example, since $\text{Homeo}(T^4)$ is Jordan and $\text{Homeo}(T^2 \times S^2)$ is not Jordan, we have:

Corollary 3.2.22. *Any non-zero degree map $f : T^4 \longrightarrow T^2 \times S^2$ is not an importing map.*

Corollary 3.2.23. *Let M and M' be closed oriented manifolds and $f : M \longrightarrow M'$ a map which exports and imports group actions. Then* $\text{disc-sym}(M) = \text{disc-sym}(M')$ *and* $\text{Homeo}(M)$ *is Jordan if and only if* $\text{Homeo}(M')$ *is Jordan.*

We will see that finite coverings between nilmanifolds are an example of importing and exporting map.

3.3 Large finite group actions on manifolds admitting a non-zero degree map to a nilmanifold

The goal of this section is to prove theorem 3.0.2. To do so, we will use the following theorem:

Theorem 3.3.1. *Let M be a closed oriented connected manifold and $f : M \longrightarrow N/\Gamma$ a non-zero degree map to a nilmanifold N/Γ. Then f is an exporting map.*

We divide the proof of theorem 3.3.1 in three parts.

Part 1: In the first part we reduce the proof of theorem 3.3.1 to the case where $f : M \longrightarrow N/\Gamma$ induces a surjective map between fundamental groups.

First, we note that $f_*(\pi_1(M))$ is a finite index subgroup of Γ. Indeed, consider the diagram

$$\begin{array}{ccc} & & X \\ & \overset{f'}{\nearrow} & \big\downarrow p \\ M & \xrightarrow{\ f\ } & N/\Gamma \end{array}$$

where X is the covering space of N/Γ associated to the subgroup $f_*(\pi_1(M)) \leq \Gamma$ and f' is a lift of f (which exists by general properties of covering spaces). Since $f = p \circ f'$, the non-zero map $f^* : H^n(N/\Gamma, \mathbb{Z}) \longrightarrow H^n(M, \mathbb{Z})$ factors through $H^n(X, \mathbb{Z})$. This implies that $H^n(X, \mathbb{Z})$ is not zero and therefore X is a compact manifold. Since $p : X \longrightarrow N/\Gamma$ is a covering between compact manifolds, p is a finite covering and $[\Gamma : f_*(\pi_1(M))]$ is finite.

Consequently, $f_*(\pi_1(M)) = \Gamma'$ is also a lattice of N and $X \cong N/\Gamma'$. By lemma 3.2.8, if we prove that f' and p are exporting maps then f will be an exporting map.

Lemma 3.3.2. *The covering p is an exporting map.*

Proof. By theorem 1.3.96 and theorem 2.0.2, there exists a constant C depending only on Γ' satisfying that, if G is a finite group acting effectively on N/Γ' and H is the kernel of the group morphism $\psi : G \longrightarrow \text{Out}(\Gamma')$, then $[G : H] \leq C$. Moreover, H is isomorphic to a subgroup of the torus $\mathcal{Z}N/\mathcal{Z}\Gamma'$, and hence it is abelian.

Let G be a finite group acting effectively on N/Γ' and $H = \text{Ker}(\psi : G \longrightarrow \text{Out}(\Gamma'))$. We claim that the action of H on N/Γ' is free. Given a point $x \in N/\Gamma'$, the isotropy subgroup H_x injects into $\text{Aut}(\Gamma')$. Since the action of H on N/Γ' is inner, H_x is a subgroup of $\text{Inn}(\Gamma') = \Gamma'/\mathcal{Z}\Gamma'$. Since Γ' is a finitely generated torsion-free nilpotent group, $\text{Inn}(\Gamma')$ is torsion-free and therefore H_x is trivial, as claimed.

The action of H can be conjugated to an action of $H \leq \mathcal{Z}N/\mathcal{Z}\Gamma'$ obtained by restricting the standard torus action of $\mathcal{Z}N/\mathcal{Z}\Gamma'$ on N/Γ' (see [LR10, Remark 11.7.18.2]). Thus, we can assume without loss of generality that $H \leq \mathcal{Z}N/\mathcal{Z}\Gamma'$ and the action of H on N/Γ' is induced by the restriction of the torus action $\mathcal{Z}N/\mathcal{Z}\Gamma'$ on N/Γ'.

The covering $p : N/\Gamma' \longrightarrow N/\Gamma$ is given by $p(n\Gamma') = n\Gamma$ for $n \in N$ and hence it is N-equivariant. Since $\mathcal{Z}\Gamma' = \mathcal{Z}N \cap \Gamma'$ and $\mathcal{Z}\Gamma = \mathcal{Z}N \cap \Gamma$, we have that $\mathcal{Z}\Gamma' \leq \mathcal{Z}\Gamma \leq \mathcal{Z}N$. Consequently, we have a group morphism $\rho : \mathcal{Z}N/\mathcal{Z}\Gamma' \longrightarrow \mathcal{Z}N/\mathcal{Z}\Gamma$ between tori of the same dimension.

We note that $p : N/\Gamma' \longrightarrow N/\Gamma$ is ρ-equivariant. Since $H \leq \mathcal{Z}N/\mathcal{Z}\Gamma'$, we have an action of H on N/Γ given by $h(n\Gamma) = \rho(h)(n\Gamma)$. This implies that p is an exporting map, as we wanted to prove. □

In particular, $p : N/\Gamma' \longrightarrow N/\Gamma$ is an exporting map. It remains to prove that f' is also an exporting map.

Thus, from now on we will assume that $f : M \longrightarrow N/\Gamma$ induces a surjective map $f_* : \pi_1(M) \longrightarrow \Gamma$.

Part 2: Let $i : \Gamma \hookrightarrow N$ denote the inclusion of the lattice Γ in N. For this part of the proof we consider the set of isomorphism classes N-local systems $X(\pi_1(M), N) = \mathrm{Hom}(\pi_1(M), N)/\sim$, where $\sim$ denotes the equivalence relation given by the conjugation by elements of N. An effective action of a finite group G on M induces an action of G on $X(\pi_1(M), N)$ (this action is described explicitly in the proof of lemma 3.3.4). Our goal is to prove that if G fixes the class $[i \circ f_*] \in X(\pi_1(M), N)$ then there exists an action of G on N/Γ and a G-equivariant map $f_G : M \longrightarrow N/\Gamma$ which is homotopic to f. We will use induction on the dimension of N.

We start with the following lemma, which is a generalization of [MiR24a, Lemma 4.2] to non-compact manifolds. The arguments used to prove lemma 3.3.3 and [MiR24a, Lemma 4.2] are essentially the same.

Lemma 3.3.3. *Let M be a connected manifold, let $f : M \longrightarrow S^1$ be a continuous map and let θ be a generator of $H^1(S^1, \mathbb{Z})$. Suppose that H is a finite group of cardinal r acting effectively on M fixing $f^*\theta$. Consider the group extension*

$$1 \longrightarrow \pi_1(M) \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1,$$

where the group $\tilde{H}$ acts effectively on the universal cover $\tilde{M}$. Then there exists a group morphism $\tilde{\mu} : \tilde{H} \longrightarrow \mathbb{R}$ and a $\tilde{\mu}$-equivariant map $\tilde{f}_H : \tilde{M} \longrightarrow \mathbb{R}$ such that $\tilde{\mu}_{|\pi_1(M)} = f_*$.

Proof. Define $F : M \to S^1$ by $\zeta(x) = \sum_{h \in H} f(h \cdot x)$ for every $x \in M$. Then F is continuous and constant on the orbits of the action of H. Let $\phi_h : M \to M$ be the homeomorphism induced by the action of $h \in H$. By assumption, $\phi_h^* f^*\theta = f^*\theta$ for every $h \in H$. We have $F^*\theta = \sum_{h \in H} \phi_h^*(f^*\theta) = rf^*\theta$.

Consider the $\mathbb{Z}/r$-covering $p_r : S^1 \longrightarrow S^1$ defined as $p_r(t) = rt$. Let $q_r : M_r \longrightarrow M$ the pull-back of p_r by F. Recall that $M_r = \{(x,t) \in M \times S^1 : F(x) = rt\}$ and $q_r : M_r \longrightarrow M$ satisfies $q_r(x,t) = x$ for $(x,t) \in M_r$. Moreover, $q_r : M_r \longrightarrow M$ has a structure of principal $\mathbb{Z}/r$ given by the action $(x,t) \cdot \alpha = (x, t\alpha)$ for $(x,t) \in M_r$ and $\alpha \in \mathbb{Z}/r$.

We claim that q_r is a trivial principal bundle. This is equivalent to the triviality of the monodromy of q_r, which we denote by $\nu : \pi_1(M, x_0) \to \mathbb{Z}/r$, where $x_0 \in M$ is an arbitrary base point. If there existed some $\gamma \in \pi_1(X, x_0)$ such that $\nu(\gamma) \neq 0$, then the pairing of $f^*\theta$ with $[\gamma] \in H_1(X, \mathbb{Z})$ would not be divisible by r, which contradicts $F^*\theta = rf^*\theta$. Hence ν is trivial and consequently, the bundle q_r is a trivial, so we may choose a section $\sigma : M \longrightarrow M_r$.

Define $f_H : M \longrightarrow S^1$ by the condition that $\sigma(x) = (x, f_H(x))$. Then f_H is continuous and we have$rf_H(x) = F(x)$ for every $x \in M$. For any $h \in H$, define $\chi_h : M \to S^1$ by

$$\chi_h(x) = f_H(h \cdot x) - f_H(x).$$

We have

$$r\chi_h(x) = rf_H(h \cdot x) - rf_H(x) = F(h \cdot x) - F(x) = 0$$

because F is H-invariant. Hence χ_h takes values in $\mathbb{Z}/r \leq S^1$. Consequently, since χ_h is continuous, it is a constant map. We may thus define a map $\mu : H \longrightarrow \mathbb{Z}/r$ by the condition that $\mu(h) = \chi_h(x)$ for every $x \in X$.

Let $h, h' \in H$ and let $x \in X$. We have

$$\chi_{hh'}(x) = f_H(hh' \cdot x) - f_H(x) = f_H(hh' \cdot x) - f_H(h' \cdot x) + f_H(h' \cdot x) - f_H(x) = \chi_h(h' \cdot x) + \chi_{h'}(x),$$

which proves that $\mu(hh') = \mu(h) + \mu(h')$, so μ is a morphism of groups. From the definition of μ, it is immediate that f_H is μ-equivariant.

To conclude the proof, note that $rf_H^*\theta = F^*\theta = rf^*\theta$. Since $H^1(X;\mathbb{Z})$ has no torsion, we conclude that $f_H^*\theta = f^*\theta$. Thus, $f_* : \pi_1(M) \longrightarrow \mathbb{Z}$ and $f_{H*} : \pi_1(M) \longrightarrow \mathbb{Z}$ are equal. We can lift f_H to a map between the universal coverings $\tilde{f}_H : \tilde{M} \longrightarrow \mathbb{R}$. Since f_H is μ-equivariant, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \tilde{H} & \longrightarrow & H & \longrightarrow & 1 \\ & & \downarrow f_{H*} & & \downarrow \tilde{\mu} & & \downarrow \mu & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \xrightarrow{\tilde{\pi}} & \mathbb{Z}/r & \longrightarrow & 1. \end{array}$$

□

Lemma 3.3.4. *Let M be a closed connected oriented manifold and let $f : M \longrightarrow N/\Gamma$ be a map such that $f_*(\pi_1(M)) = \Gamma$. Assume that H is a finite group acting effectively on M which fixes the local system $[i \circ f_*] \in X(\pi_1(M), N)$. Consider the group extension*

$$1 \longrightarrow \pi_1(M) \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1,$$

where the group $\tilde{H}$ acts effectively on the universal cover $\tilde{M}$. Then there exists a group morphism $\tilde{\mu} : \tilde{H} \longrightarrow N$ and a $\tilde{\mu}$-equivariant map $\tilde{f}_H : \tilde{M} \longrightarrow N$ such that $\tilde{\mu}_{|\pi_1(M)} = f_*$.

Before proving lemma 3.3.4, we deduce the next corollary.

Corollary 3.3.5. *Let M be a closed connected oriented manifold and $f : M \longrightarrow N/\Gamma$ be a non-zero degree map such that $f_*(\pi_1(M)) = \Gamma$. Assume that H is a finite group acting effectively on M which fixes the local system $[i \circ f_*] \in X(\pi_1(M), N)$. Then there exists a group action of H on N/Γ and an equivariant map $f_H : M \longrightarrow N/\Gamma$ homotopic to f.*

Proof. Since $\tilde{\mu}_{|\pi_1(M)} = i \circ f_*$, the map $\tilde{f}_H : \tilde{M} \longrightarrow N$ induces a map $f_H : M \longrightarrow N/\Gamma$. Moreover, $\tilde{\mu} : \tilde{H} \longrightarrow N$ induces a surjective group morphism $\mu' : \tilde{H}/\pi_1(M) = H \longrightarrow \tilde{\mu}(\tilde{H})/\Gamma$ such that f_H is μ'-equivariant. Thus, we have an action $H \longrightarrow \tilde{\mu}(\tilde{H})/\Gamma \longrightarrow \mathrm{Homeo}(N/\Gamma)$ and f_H is H-equivariant. Finally, since N/Γ is a model of the Eilenberg-MacLane space $K(\Gamma, 1)$ and $[\tilde{\mu}_{|\pi_1(M)}] = [i \circ f_*]$, the maps f and f_H are homotopic. □

Remark 3.3.6. *Note that the action of $\tilde{\mu}(\tilde{H})/\Gamma$ is free and the quotient $(N/\Gamma)/(\tilde{\mu}(\tilde{H})/\Gamma)$ is the nilmanifold $N/\tilde{\mu}(\tilde{H})$.*

Proof of lemma 3.3.4. We proceed by induction on the dimension of the nilmanifold N/Γ. If $\dim(N/\Gamma) = 1$ then $N/\Gamma \cong S^1$ and the statement is a consequence of lemma 3.3.3. For the induction step, using theorem 1.3.39, we can take central exact sequences

$$1 \longrightarrow \mathbb{R} \longrightarrow N \xrightarrow{\tilde{\pi}} N' \longrightarrow 1$$

and

$$1 \longrightarrow \mathbb{R} \cap \Gamma \cong \mathbb{Z} \longrightarrow \Gamma \xrightarrow{\pi_*} \Gamma' \longrightarrow 1,$$

where $\pi_* = \tilde{\pi}_{|\Gamma}$. They induce a principal S^1-bundle, $\pi : N/\Gamma \longrightarrow N'/\Gamma'$. Since the short exact sequences are central, we can choose a normalized 2-cocycle $c : N' \times N' \longrightarrow \mathbb{R}$ such that $N \cong \mathbb{R} \times_c N'$ and $\Gamma \cong \mathbb{Z} \times_{c_{|\Gamma' \times \Gamma'}} \Gamma'$. Note that $\dim(N'/\Gamma') = \dim(N/\Gamma) - 1$.

We denote $\mu = i \circ f_*$. Note the we have an action of $\tilde{H}$ on $X(\pi_1(M), N)$ satisfying that $\tilde{h}[\nu] = [\nu \circ c_{\tilde{h}|\pi_1(M)}]$ for all $\tilde{h} \in \tilde{H}$ and $[\nu] \in X(\pi_1(M), N)$. If the action of H on $X(\pi_1(M), N)$ fixes $[\mu]$ then the action of $\tilde{H}$ also fixes $[\mu]$. This is a consequence of the fact that the action of $\tilde{H}$ on $X(\pi_1(M), N)$ factors through the action of H on $X(\pi_1(M), N)$, since the restriction to $\pi_1(M)$ of the action of $\tilde{H}$ on $X(\pi_1(M), N)$ is trivial.

This condition implies that there exists a map (which is not a group morphism in general) $a : \tilde{H} \longrightarrow N$ such that $\mu \circ c_{\tilde{h}|\pi_1(m)} = c_{a(\tilde{h})} \circ \mu$. Note that $a : \tilde{H} \longrightarrow N$ is not unique. If we have another map $a' : \tilde{H} \longrightarrow N$ with the same property, then $c_{a'(\tilde{h})} \circ \mu = \mu \circ c_{\tilde{h}|\pi_1(M)} = c_{a(\tilde{h})} \circ \mu$ and hence $c_{a(\tilde{h})^{-1}a'(\tilde{h})} \circ \mu = \mu$. Therefore, $a(\tilde{h})^{-1}a'(\tilde{h})$ centralizes the lattice Γ and hence $a(\tilde{h})^{-1}a'(\tilde{h}) \in \mathcal{Z}N$ for all $\tilde{h} \in \tilde{H}$.

The action of $\tilde{H}$ fixes $[\tilde{\pi} \circ \mu] \in X(\pi_1(M), N')$. Since $\pi_* : \Gamma \longrightarrow \Gamma'$ is surjective, the group morphism $\pi_* \circ f_* : \pi_1(M) \longrightarrow \Gamma'$ is surjective and, by induction, there exists a group morphism $\tilde{\mu}_B : \tilde{H} \longrightarrow N'$ and a $\tilde{\mu}_B$-equivariant map $\tilde{f}'_B : \tilde{M} \longrightarrow N'$ such that $\tilde{\mu}_{B|\pi_1(M)} = \tilde{\pi} \circ \mu$.

Consequently, we have a surjective group morphism $\mu_B : \tilde{H}/\pi_1(M) \cong H \longrightarrow \tilde{\mu}_B(\tilde{H})/\Gamma'$. Thus, we can define an action of H on N'/Γ' (which may not be effective) given by $h(n\Gamma') = \mu_B(h)(n\Gamma')$. The map $\tilde{f}'_B : \tilde{M} \longrightarrow N'$ induces an H-equivariant map $f'_B : M \longrightarrow N'/\Gamma'$ homotopic to $\pi \circ f$. Since $f'_{B*} : \pi_1(M) \longrightarrow \Gamma'$ factors through $\pi_* : \Gamma \longrightarrow \Gamma'$, there exists a map $f_B : M \longrightarrow N/\Gamma$ homotopic to f such that $\pi \circ f_B = f'_B$. We can lift f_B to a map $\tilde{f}_B : \tilde{M} \longrightarrow N \cong \mathbb{R} \times_c N'$. Note that $\tilde{f}_B$ is μ-equivariant and $\tilde{\pi} \circ \tilde{f}_B$ is $\tilde{\mu}_B$-equivariant.

We now consider the group $\tilde{H}_0 = \operatorname{Ker} \tilde{\mu}_B$ and $Z_0 = \mu^{-1}(\operatorname{Ker} \tilde{\pi}) \leq \pi_1(M)$ (recall that $\operatorname{Ker} \tilde{\pi} \cong \mathbb{R}$). Notice that $Z_0 = \tilde{H}_0 \cap \pi_1(M)$. Let $g \in \tilde{H}_0 \cap \pi_1(M)$. Then $\tilde{\mu}_B(g) = \tilde{\pi}(\mu(g)) = 0$, which implies that $g \in Z_0$. Conversely, if $g \in Z_0$, then $g \in \pi_1(M)$. Therefore, $\tilde{\mu}_B(g) = \tilde{\pi}(\mu(g)) = 0$, which implies that $g \in \tilde{H}_0$. Consequently, $g \in \tilde{H}_0 \cap \pi_1(M)$. In conclusion, $Z_0 = \mu^{-1}(\operatorname{Ker} \tilde{\pi}) \leq \pi_1(M)$. Therefore Z_0 is a normal subgroup of $\tilde{H}_0$. Moreover, $H_0 = \tilde{H}_0/Z_0$ is a subgroup of H and hence Z_0 is a finite index subgroup of $\tilde{H}_0$.

Since $Z_0 \leq \pi_1(M)$, $\tilde{f}_B : \tilde{M} \longrightarrow N$ induces a map $\tilde{M}/Z_0 \longrightarrow N/\mathbb{Z} \cong S^1 \times_c N'$. We obtain a map $f_Z : \tilde{M}/Z_0 \longrightarrow S^1$ by composing it with the projection to the S^1 factor. We also get an effective action of the finite group H_0 on $\tilde{M}/Z_0$. We consider $\mu_Z : Z_0 \longrightarrow \mathbb{R}$, which is the composition $f_{Z*} : Z_0 \longrightarrow \mathbb{Z}$ and the inclusion $i_Z = i_{|\mathbb{Z}} : \mathbb{Z} \longrightarrow \mathbb{R} \leq \mathcal{Z}N$. The group morphism μ_Z is the restriction of the map μ to Z_0, $\mu_Z = \mu_{|Z_0}$.

The action of H_0 on $X(Z_0, \mathbb{R}) = \operatorname{Hom}(Z_0, \mathbb{R})$ induces an action of $\tilde{H}_0$ on $X(Z_0, \mathbb{R})$ satisfying $\tilde{h} \cdot \nu = \nu \circ c_{\tilde{h}|Z_0}$ for $\nu \in X(Z_0, \mathbb{R})$ and $\tilde{h} \in \tilde{H}_0$. This action satisfies

$$\tilde{h} \cdot \mu_Z = \mu_{|Z_0} \circ c_{\tilde{h}|Z_0} = c_{a(\tilde{h})} \circ \mu_{|Z_0} = \mu_{|Z_0}$$

where the last equality holds because the image of $\mu_{|Z_0}$ lies on the center of N. Consequently, the group $\tilde{H}_0$ (and hence H_0) fixes $\mu_Z \in X(Z_0, \mathbb{R})$. We can use lemma 3.3.3 to conclude that there exists a group morphism $\tilde{\mu}_Z : \tilde{H}_0 \longrightarrow \mathbb{R}$ satisfying $\tilde{\mu}_{Z|Z_0} = \mu_Z$ and a $\tilde{\mu}_Z$-equivariant map $\tilde{f}_Z : \tilde{M} \longrightarrow \mathbb{R}$.

Summarizing the proof until this point, we have obtained two group morphisms $\tilde{\mu}_B$ and $\tilde{\mu}_Z$ satisfying

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{H}_0 & \longrightarrow & \tilde{H} & \xrightarrow{\tilde{\pi}} & \tilde{H}/\tilde{H}_0 & \longrightarrow & 1 \\
 & & \downarrow{\tilde{\mu}_Z} & & & & \downarrow{\overline{\mu}_B} & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & N & \xrightarrow{\tilde{\pi}} & N' & \longrightarrow & 1
\end{array}$$

where $\overline{\mu}_B : \tilde{H}/\tilde{H}_0 \longrightarrow N'$ is an injective group morphism induced by $\tilde{\mu}_B$. We also have a map $\tilde{f} : \tilde{M} \longrightarrow N$ of the form $\tilde{f} = (\tilde{f}_Z, \tilde{f}'_B)$. Our goal is to construct a group morphism $\tilde{\mu} : \tilde{H} \longrightarrow N$ making the above diagram commutative.

Firstly, we note that $\operatorname{Ker} \mu$ is a normal subgroup of $\tilde{H}$, since the action of $\tilde{H}$ fixes $[\mu]$. Let $\tilde{\Gamma} = \tilde{H}/\operatorname{Ker} \mu$, let $\tilde{\Gamma}_0 = \tilde{H}_0/\operatorname{Ker} \mu$ and let $\tilde{\nu}_Z : \tilde{\Gamma}_0 \longrightarrow \mathbb{R}$ be the group morphism induced by

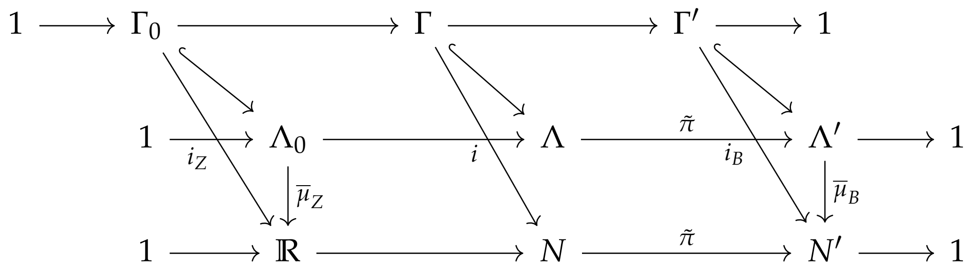
$\tilde{\mu}_Z$. We have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{\Gamma}_0 & \longrightarrow & \tilde{\Gamma} & \xrightarrow{\tilde{\pi}} & \tilde{H}/\tilde{H}_0 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \tilde{\nu}_Z} & & & & \downarrow{\scriptstyle \overline{\mu}_B} & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & N & \xrightarrow{\tilde{\pi}} & N' & \longrightarrow & 1
\end{array}$$

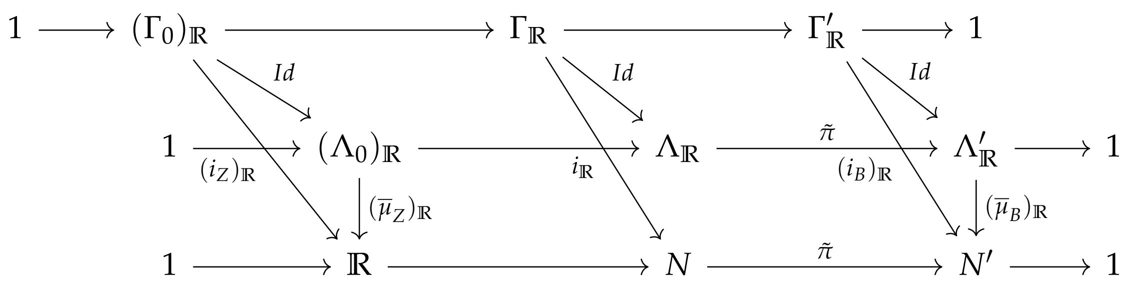
Let K denote $\operatorname{Ker}\tilde{\nu}_Z = \operatorname{Ker}\tilde{\mu}_Z/\operatorname{Ker}\mu$. We note that K injects in H_0 and therefore it is finite. Since $\tilde{\nu}_Z(\tilde{\Gamma}_0)$ is torsion-free, the subgroup K is characteristic in $\tilde{\Gamma}_0$. Thus, K is a normal subgroup of $\tilde{\Gamma}$. Let $\Lambda = \tilde{\Gamma}/K$, $\Lambda_0 = \tilde{\Gamma}_0/K \cong \mathbb{Z}$ and $\Lambda' = \tilde{H}/\tilde{H}_0$. We also denote by $\overline{\mu}_B : \Lambda_0 \longrightarrow \mathbb{R}$ the group morphism induced by $\tilde{\nu}_Z$. We have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Lambda_0 & \longrightarrow & \Lambda & \xrightarrow{\tilde{\pi}} & \Lambda' & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \overline{\mu}_Z} & & & & \downarrow{\scriptstyle \overline{\mu}_B} & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & N & \xrightarrow{\tilde{\pi}} & N' & \longrightarrow & 1
\end{array}$$

Moreover, note that $\Gamma \cap K$ is the identity element, therefore Γ is a finite index subgroup of Λ. Note also that Λ is torsion-free, finitely generated and nilpotent, hence Λ is a lattice of N. Using the inclusion $i : \Gamma \longrightarrow N$, we can define inclusions $i_B : \Gamma' \longrightarrow N'$ and $i_Z : \Gamma_0 \cong \mathbb{Z} \longrightarrow \mathbb{R}$, obtaining the following commutative diagram



The inclusion $\Gamma \leq \Lambda$ induces the identity map between the real Mal'cev completions $\Gamma_{\mathbb{R}}$ and $\Lambda_{\mathbb{R}}$. Consequently, we have a commutative diagram



Therefore, if we set $\overline{\mu} = i_{\mathbb{R}|\Lambda} : \Lambda \longrightarrow N$, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Lambda_0 & \longrightarrow & \Lambda & \xrightarrow{\tilde{\pi}} & \Lambda' & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \overline{\mu}_Z} & & \downarrow{\scriptstyle \overline{\mu}} & & \downarrow{\scriptstyle \overline{\mu}_B} & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & N & \xrightarrow{\tilde{\pi}} & N' & \longrightarrow & 1
\end{array}$$

The desired group morphism $\tilde{\mu} : \tilde{H} \longrightarrow N$ is obtained by composing the $\overline{\mu}$ with the projection $\tilde{H} \longrightarrow \Lambda$. By construction, the map $\tilde{f} : \tilde{M} \longrightarrow N$ is $\tilde{\mu}$-equivariant and $\tilde{\mu}_{|\pi_1(M)} = \mu$, as we wanted to see. □

Part 3: The last step of the proof of [MiR24a, Theorem 4.1] is a consequence of Minkowski's lemma.

Lemma 3.3.7. *Let M be a closed connected oriented manifold and let $f : M \longrightarrow N/\Gamma$ be a non-zero degree map such that $f_*(\pi_1(M)) = \Gamma$. There exists a constant C such that any finite group G acting effectively on M has a finite subgroup $H \leq G$ such that $[G : H] \leq C$ and H acts trivially on $X(\pi_1(M), N)$.*

Proof. As usual, we denote by $\pi_1(M)^j$ the j-th element of the lower central series (see definition 1.3.19). Assume that the nilpotency class of N is c, then for any $\nu \in \mathrm{Hom}(\pi_1(M), N)$ we have $\pi_1(M)^c \leq \mathrm{Ker}\,\nu$. In consequence, the projection $\pi_1(M) \longrightarrow \pi_1(M)/\pi_1(M)^c$ induces a bijection $\mathrm{Hom}(\pi_1(M)/\pi_1(M)^c, N) \longrightarrow \mathrm{Hom}(\pi_1(M), N)$. This bijection descends to a bijection between $X(\pi_1(M), N)$ and $X(\pi_1(M)/\pi_1(M)^c, N)$.

Since $\pi_1(M)^c$ is a characteristic subgroup of $\pi_1(M)$, any automorphism $\phi : \pi_1(M) \longrightarrow \pi_1(M)$ induces and automorphism $\overline{\phi} : \pi_1(M)/\pi_1(M)^c \longrightarrow \pi_1(M)/\pi_1(M)^c$ given by $\overline{\phi}(\gamma\pi_1(M)^c) = \phi(\gamma)\pi_1(M)^c$. Hence there is a group morphism $\rho : \mathrm{Out}(\pi_1(M)) \longrightarrow \mathrm{Out}(\pi_1(M)/\pi_1(M)^c)$ which sends $[\phi]$ to $[\overline{\phi}]$.

Notice that the bijection between $X(\pi_1(M), N)$ and $X(\pi_1(M)/\pi_1(M)^c, N)$ is ρ-equivariant. If $\psi : G \longrightarrow \mathrm{Out}(\pi_1(M))$ is the map induced by the action of G on M, then we have an action of G on $X(\pi_1(M)/\pi_1(M)^c, N)$ given by $[f]g = [f](\rho \circ \psi)(g)$. Since $\pi_1(M)/\pi_1(M)^c$ is finitely generated and nilpotent, we can use theorem 2.2.1 to conclude that there exists a constant C such that any finite subgroup of $\mathrm{Out}(\pi_1(M)/\pi_1(M)^c)$ is at most of order C. Thus, $H = \mathrm{Ker}(\rho \circ \psi)$ is a subgroup of G such that $[G : H] \leq C$ which acts trivially on $X(\pi_1(M)/\pi_1(M)^c, N)$ and hence it also acts trivially on $X(\pi_1(M), N)$. □

By combining lemma 3.3.4, lemma 3.3.7 and lemma 3.3.2 we complete the proof of theorem 3.3.1.

Finally, we prove theorem 3.0.2.

Proof of theorem 3.0.2. Recall that, by theorem 2.0.1 and theorem 2.0.2, the nilmanifold N/Γ satisfies:

1. $\mathrm{Homeo}(N/\Gamma)$ is Jordan.
2. $\text{disc-sym}(N/\Gamma) \leq \mathrm{rank}\, \mathcal{Z}\Gamma$.
3. N/Γ has small stabilizers.

Thus, the first part of theorem 3.0.2 is a direct consequence of theorem 3.2.9 and theorem 3.3.1. For the second part, the bound of the discrete degree of symmetry is also a consequence of theorem 3.2.9 and theorem 3.3.1. If we assume disc-sym$(M) = n$, then rank $\mathcal{Z}\Gamma \geq n$. This implies that $\Gamma \cong \mathbb{Z}^n$ and $N/\Gamma \cong T^n$. In consequence, the theorem follows from theorem 3.0.1.

For the third part, note that M has the almost fixed point property by theorem 1.1.63. Since M has the small stabilizers property (by theorem 3.2.9 and theorem 3.3.1) and the almost fixed point property, M is almost asymmetric, by lemma 1.1.64.

The fourth part is a consequence of M having the small stabilizers property and Homeo(M) being Jordan, as seen in lemma 1.1.65. □

Corollary 3.3.8. *Let M be a closed connected oriented manifold and $f : M \longrightarrow N/\Gamma$ a non-zero degree map to a nilmanifold. Assume that the toral rank conjecture holds for N/Γ. Then the toral rank conjecture and the stable Carlsson conjecture holds for M.*

Proof. Note that $\mathrm{rank}(M) \leq$ disc-sym$(M) \leq \mathrm{rank}\,\mathcal{Z}\Gamma$. Since f has non-zero degree, the induced morphism $f^* : H^*(N/\Gamma, \mathbb{Q}) \longrightarrow H^*(M, \mathbb{Q})$ is injective and therefore

$$\dim H^*(M, \mathbb{Q}) \geq \dim H^*(N/\Gamma, \mathbb{Q}) \geq 2^{\mathrm{rank}(\mathcal{Z}\Gamma)} \geq 2^{\mathrm{rank}(M)},$$

as we wanted to see.

Similarly, note that if $(\mathbb{Z}/p)^r$ acts freely on M with $p \geq \max\{D, \deg(f)\}$, where D is the exporting map constant of f, then $(\mathbb{Z}/p)^r$ acts freely on N/Γ. In particular, $\mathrm{rank}_p(M) \leq \mathrm{rank}_p(N/\Gamma)$ for $p > \max\{D, \deg(f)\}$. Let D' be constant given by the stable Carlsson conjecture of N/Γ, which exists since the toral rank conjecture and the stable Carlsson conjecture are equivalent for nilmanifolds, see lemma 2.4.8. Moreover, $\mathrm{rank}_p(N/\Gamma) = \mathrm{rank}\,\mathcal{Z}\Gamma$ for all $p > D'$. Let $C = \max\{D, \deg(f), D'\}$. Then for all $p > C$ we have

$$\dim H^*(M, \mathbb{Z}/p) \geq \dim H^*(N/\Gamma, \mathbb{Z}/p) \geq 2^{\mathrm{rank}(\mathcal{Z}\Gamma)} \geq 2^{\mathrm{rank}_p(M)},$$

as we wanted to see. □

Corollary 3.3.8 can be used when N/Γ is a 2-step nilmanifold, by [DS88]. For other nilmanifolds where corollary 3.3.8 applies see [CJP97, CJ97].

The conclusion of [MiR24a, Corollary 1.6] also holds for closed connected oriented manifolds admitting a non-zero degree map to a nilmanifold.

Corollary 3.3.9. *Let M be a closed connected oriented n-dimensional manifold admitting a non-zero degree map to a nilmanifold $f : M \longrightarrow N/\Gamma$. If* disc-sym$(M) = n$ *and $\pi_1(M)$ is virtually solvable then M is homeomorphic to T^n.*

Lemma 3.3.2 is false if we replace nilmanifold by solvmanifold. In order to give an example of a non-zero degree map between solvmanifolds which does not export group actions we need the following elementary group theoretic lemma.

Lemma 3.3.10. *Let G_1 and G_2 be groups and let $\phi : G_2 \longrightarrow \mathrm{Aut}(G_1)$ be a group morphism. Then*

$$\mathcal{Z}(G_1 \rtimes_\phi G_2) = \{(g_1, g_2) \in G_1 \rtimes_\phi G_2 : g_2 \in \mathcal{Z}G_2, g_1 \in \mathrm{Fix}(\phi) \text{ and } \phi(g_2) = c_{g_1}\},$$

where $\mathrm{Fix}(\phi) = \{g_1 \in G_1 : \phi(g_2)(g_1) = g_1 \text{ for all } g_2 \in G_2\}$.

Proof. Let $(g_1, g_2) \in \mathcal{Z}(G_1 \rtimes_\phi G_2)$. For any $(g_1', g_2') \in G_1 \rtimes_\phi G_2$ we have $(g_1, g_2)(g_1' g_2') = (g_1\phi(g_2)(g_1'), g_2 g_2')$ is equal to $(g_1', g_2')(g_1 g_2) = (g_1'\phi(g_2')(g_1), g_2' g_2)$. Hence $g_1\phi(g_2)(g_1') = g_1'\phi(g_2')(g_1)$ and $g_2 g_2' = g_2' g_2$ for all $g_1' \in G_1$ and $g_2' \in G_2$.

By the second condition, $g_2 \in \mathcal{Z}G_2$. If we set $g_1' = e$, then $g_1 = \phi(g_2')(g_1)$ for all $g_2' \in G_2$ and therefore $g_1 \in \mathrm{Fix}(\phi)$. Finally, by using that $g_1 = \phi(g_2')(g_1)$, we obtain that $g_1\phi(g_2)(g_1') = g_1' g_1$, which implies that $\phi(g_2)(g_1') = g_1^{-1} g_1' g_1$ and $\phi(g_2) = c_{g_1}$. □

We consider a non-trivial group morphism $\phi : \mathbb{Z} \longrightarrow \mathrm{GL}(n, \mathbb{Z})$ such that $\phi(1)$ and has finite order a. The group morphism $\phi(1)$ induces a finite order homeomorphism $f : T^n \longrightarrow T^n$ which we use to construct the mapping torus T^n_f. We claim that:

1. T^n_f is a compact solvmanifold. This is a consequence of the fact that T^n_f is the total space of a fibration $T^n \longrightarrow T^n_f \longrightarrow S^1$. In particular, $\pi_1(T^n_f) = \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$ is polycyclic.
2. T^n_f is finitely covered by the torus T^{n+1}. This is because the subgroup $\mathbb{Z}^n \rtimes_\phi a\mathbb{Z} \cong \mathbb{Z}^{n+1}$ is a normal subgroup of index a. Therefore, we have a regular covering $p : T^{n+1} \longrightarrow T^n_f$. In particular T^n_f is a flat manifold.

Proposition 3.3.11. *We have* $\text{disc-sym}(T^n_f) \leq n$.

Proof. Since T^n_f is a flat manifold, we know that $\text{disc-sym}(T^n_f) = \operatorname{rank} \mathcal{Z}(\mathbb{Z}^n \rtimes_\phi \mathbb{Z})$ by lemma 3.2.13. By lemma 3.3.10, $\mathcal{Z}(\mathbb{Z}^n \rtimes_\phi \mathbb{Z}) = \mathrm{Fix}(\phi) \times a\mathbb{Z}$. Since $\phi(1) \neq Id$, we obtain that $\operatorname{rank} \mathrm{Fix}(\phi) \leq n-1$ and therefore $\text{disc-sym}(T^n_f) = \operatorname{rank} \mathcal{Z}(\mathbb{Z}^n \rtimes_\phi \mathbb{Z}) \leq n$. □

Corollary 3.3.12. *The map $p : T^{n+1} \longrightarrow T^n_f$ does not export group actions.*

Now we present examples of closed orientable manifolds M admitting a non-zero degree map to a nilmanifold $f : M \longrightarrow N/\Gamma$ such that $\text{disc-sym}(M) = \text{disc-sym}(N/\Gamma)$ but $H^*(M, \mathbb{Z}) \ncong H^*(N/\Gamma, \mathbb{Z})$.

Firstly, let H be the Heisenberg group and Γ_1 and Γ_2 lattices of H (see example 1.3.26). Note that the covering map $p : H/\Gamma_1 \longrightarrow H/\Gamma_2$ is a non-zero degree map between and $\text{disc-sym}(H/\Gamma_1) = \text{disc-sym}(H/\Gamma_2) = 1$. However, $H_1(H/\Gamma, \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_1(H/\Gamma_2, \mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$.

Note that $H^*(H/\Gamma_1,\mathbb{Q}) \cong H^*(H/\Gamma_2,\mathbb{Q})$. Our next goal is to prove the next proposition:

Proposition 3.3.13. *There exists a closed orientable manifold M admitting a map to a nilmanifold $f : M \longrightarrow N/\Gamma$ satisfying* $\deg(f) = 1$, $\text{disc-sym}(M) = \text{disc-sym}(N/\Gamma) = 1$ *and* $H^*(M,\mathbb{Q}) \ncong H^*(N/\Gamma,\mathbb{Q})$.

Firstly, we recall some facts on principal S^1-bundles.

Lemma 3.3.14. *Let $p : E \longrightarrow B$ be a principal S^1-bundle, then $p^* : H^1(B,\mathbb{Q}) \longrightarrow H^1(E,\mathbb{Q})$ is injective. It is an isomorphism if and only if the first Chern class $c_1(E) \neq 0$.*

Lemma 3.3.15. *Let M be a closed manifold of dimension $n \geq 4$ and let $D \subseteq M$ be a disk. Then the inclusion $i : M \setminus D \longrightarrow M$ induces an isomorphism $i^* : H^2(M,\mathbb{Q}) \longrightarrow H^2(M \setminus D,\mathbb{Q})$.*

The first lemma is a consequence of the Gysin exact sequence and the second lemma is a consequence of the Mayer-Vietoris sequence.

Given a principal bundle $p : E \longrightarrow B$, then we have a principal S^1-bundle $p' : E \setminus p^{-1}(D) \longrightarrow B \setminus D$. Then $c_1(E \setminus p^{-1}(D)) = i^*c_1(E)$.

Note that $\partial(E \setminus p^{-1}(D)) \cong S^{n-1} \times S^1$ has a right S^1 action induced by the action of the principal S^1-bundle.

Given principal S^1-bundles $p_i : E_i \longrightarrow B_i$ with $i = 1,2$, we can construct a S^1-equivariant homeomorphism $f : \partial(E_1 \setminus p_1^{-1}(D_1)) \longrightarrow \partial(E_2 \setminus p_2^{-1}(D_2))$. There is a principal S^1-bundle $p : E_1 \setminus p_1^{-1}(D_1) \cup_f E_2 \setminus p_2^{-1}(D_2) \longrightarrow B_1 \# B_2$.

Lemma 3.3.16. *If* $\dim(B_i) \geq 4$ *then*

$$c_1(E_1 \setminus p_1^{-1}(D_1) \cup_f E_2 \setminus p_2^{-1}(D_2) \longrightarrow B_1 \# B_2) = (c_1(E_1), c_1(E_2)) \in H^2(B_1,\mathbb{Q}) \oplus H^2(B_2,\mathbb{Q})$$

.

Finally note that we have S^1-equivariant maps of degree one, $f_i : E \longrightarrow E_i$ for $i = 1,2$. We are ready to prove proposition 3.3.13.

Proof of proposition 3.3.13. Now we consider the Heisenberg manifold of dimension 5, H_5/Γ, which is the total space of a principal S^1-bundle over a torus T^4. We also consider the filiform nilmanifold of dimension 5, F_5/Λ, which is the total space of a principal S^1-bundle over a filiform nilmanifold of dimension 4, F_4/Λ' (see example 1.3.26). Note that the discrete degree of symmetry of both nilmanifolds is 1. Let $E = H_5/\Gamma \setminus p_1^{-1}(D_1) \cup_f F_5/\Lambda \setminus p_2^{-1}(D_2)$ and consider the degree one maps $f_1 : E \longrightarrow H_5/\Gamma$ and $f_2 : E \longrightarrow F_5/\Lambda$. Observe that $p : E \longrightarrow T^4 \# F_4/\Lambda'$ is a principal S^1-bundle and therefore $\text{disc-sym}(E) \geq 1$. On the other hand, $\text{disc-sym}(E) \leq 1$, since the maps f_i export group actions. In conclusion, $\text{disc-sym}(E) = 1$. However, since $c_1(E) \neq 0$, one can compute that $H^1(E,\mathbb{Q}) \cong H^1(T^4 \# F_4/\Lambda',\mathbb{Q}) \cong \mathbb{Q}^5$. On the other hand, we have $H^1(H_5/\Gamma,\mathbb{Q}) \cong H^1(T^4,\mathbb{Q}) \cong \mathbb{Q}^4$ and

$H^1(F_5/\Lambda, \mathbb{Q}) \cong H^1(F_4/\Lambda', \mathbb{Q}) \cong \mathbb{Q}^2$, therefore $H^*(E, \mathbb{Q}) \ncong H^*(H_5/\Gamma, \mathbb{Q})$ and $H^*(E, \mathbb{Q}) \ncong H^*(F_5/\Lambda, \mathbb{Q})$. □

Chapter 4

Iterated group actions

To establish a rigidity result for closed oriented manifolds admitting non-zero degree maps to nilmanifolds, we must refine the discrete degree of symmetry of a manifold. Since nilmanifolds arise as iterated principal S^1-bundles (see theorem 1.3.39), this naturally leads us to study iterated group actions.

Definition 4.0.1. *Let $\mathcal{G} = \{G_i\}_{i=1,\ldots,n}$ be a collection of groups and let X be a topological space. An iterated action of $\mathcal{G}$ on X (denoted by $\mathcal{G} \curvearrowright X$) is:*

1. *A sequence of surjections of topological spaces*

$$X = X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_2} X_2 \xrightarrow{p_3} \cdots \xrightarrow{p_n} X_n,$$

2. *and a collection of group actions $\{\Phi_i : G_i \longrightarrow \mathrm{Homeo}(X_{i-1})\}_{i=1,\ldots,n}$,*

satisfying the property that the maps $p_i : X_{i-1} \longrightarrow X_i$ are the orbit maps of the action of G_i on X_{i-1}.

Notation 4.0.2. *Assume that X has an iterated action of $\mathcal{G} = \{G_i\}_{i=1,\ldots,n}$. Then:*

1.

1. *We define the length of $\mathcal{G}$ to be $l(\mathcal{G}) = n$.*
2. *We denote the composition of all orbit maps by $p : X \longrightarrow X/\mathcal{G}$.*
3. *Given $x \in X$, the iterated image of x is the collection of points $\{x_i\}_{i=0,\ldots,n}$, where $x_0 = x$ and $x_i = p_i(x_{i-1})$ for all $1 \leq i \leq n$.*
4. *The action of G_i on X_{i-1} will be called the i-th step of the iterated action.*

From now on, unless stated the contrary, all iterated actions will be assumed to be of finite groups.

Given an action of a finite group G on a topological space X, we can construct an iterated action as follows.

Example 4.0.3. *Let G be a finite group. We consider a normal series $G^0 = \{e\} \trianglelefteq G^1 \trianglelefteq \cdots \trianglelefteq G^n = G$. Define $G_i = G^i/G^{i-1}$ for all $1 \leq i \leq n$ and let $\mathcal{G} = \{G_i\}_{i=1,\dots,n}$.*

Let X be a topological space and suppose that G acts on X. Then we can construct an iterated action $\mathcal{G} \curvearrowright X$ as follows. Set $X_0 = X$ and $X_i = X/G^i$ for $1 \leq i \leq n$. Given an orbit $G^{i-1}(x) \in X_i$, we define the surjective maps $p_i : X_{i-1} \longrightarrow X_i$ as $p_i(G^{i-1}(x)) = G^i(x)$. We define an action of G_i on X_{i-1} satisfying $(gG^{i-1})(G^{i-1}(x)) = G^{i-1}(gx)$ for all $gG^{i-1} \in G^i$ and $G^{i-1}(x) \in X_{i-1}$. Note that these actions are well defined because G^{i-1} is normal in G^i for all i. The orbit maps of these actions are precisely $p_i : X_{i-1} \longrightarrow X_i$. Thus, we have an iterated action $\mathcal{G} \curvearrowright X$. Note that the composition of all the surjections p_i is the orbit map $p : X \longrightarrow X/G$.

Recall that if $f : X \longrightarrow Y$ is an equivariant map between G-spaces, then we can induce a continuous map $\overline{f} : X/G \longrightarrow Y/G$ which satisfies $p_Y \circ f = \overline{f} \circ p_X$, where $p_X : X \longrightarrow X/G$ and $p_Y : Y \longrightarrow Y/G$ are the orbit maps. This observation leads to the definition of equivariant maps of iterated actions.

Definition 4.0.4. *Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous map. Assume that we have iterated actions of $\mathcal{G} = \{G_i\}_{i=1,\dots,n}$ on X and Y. Then f is $\mathcal{G}$-equivariant if the maps $f_i : X_i \longrightarrow Y_i$ are G_{i+1}-equivariant for all $0 \leq i \leq n-1$, where $f_0 = f$ and $f_i : X_i \longrightarrow Y_i$ is the map induced by the G_i-equivariant map $f_{i-1} : X_{i-1} \longrightarrow Y_{i-1}$.*

Definition 4.0.5. *Let $\mathcal{G} = \{G_i\}_{i=1,\dots,n}$ be a collection of finite groups and let $\mathcal{G} \curvearrowright X$ be an iterated action. We say that $\mathcal{G} \curvearrowright X$ is simplifiable if there exist:*

1. *A group G with a normal series of subgroups $G^0 = \{e\} \trianglelefteq G^1 \trianglelefteq \cdots \trianglelefteq G^n = G$ such that $G^i/G^{i-1} \cong G_i$ for all $1 \leq i \leq n$.*
2. *An action of G on X. This action induces an iterated action of $\mathcal{G}$ on X as seen in example 4.0.3. We denote this iterated action by $\mathcal{G}_G \curvearrowright X$.*
3. *A $\mathcal{G}$-equivariant homeomorphism $f : X \longrightarrow X$ between the iterated action $\mathcal{G} \curvearrowright X$ and $\mathcal{G}_G \curvearrowright X$.*

In this case, we will say that $\mathcal{G} \curvearrowright X$ is simplifiable by an action of G on X.

In particular, if the iterated action $\mathcal{G} \curvearrowright X$ is simplifiable, then $X/G \cong X/\mathcal{G}$.

Recall that if M is a manifold and G is a finite group acting freely on M then the quotient map M/G is a manifold. Hence, we will start developing a theory of iterated actions where each action is free. Our first goal is to study free finite group actions on manifolds. To do so, we introduce the following definition:

Definition 4.0.6. *Let $\mathcal{G} = \{G_i\}_{i=1,\dots,n}$ and $\mathcal{G}' = \{G'_i\}_{i=1,\dots,n'}$ be two collections of finite groups* *which act freely on M. Let* $p = p_n \circ \cdots \circ p_1$ *and* $p' = p'_{n'} \circ \cdots \circ p'_1 = p'$. *We say that the iterated actions* $\mathcal{G} \curvearrowright M$ *and* $\mathcal{G}' \curvearrowright M$ *are equivalent (and we denote it by* $\mathcal{G} \curvearrowright M \sim \mathcal{G}' \curvearrowright M$*) if:*

1. *There exists a homeomorphism* $f : M_n \longrightarrow M_{n'}$.
2. *The coverings* $p : M \longrightarrow M/\mathcal{G}$ *and* $f^*p' : M \longrightarrow M/\mathcal{G}$ *are isomorphic. That is, there exists a homeomorphism* $\overline{f} : M \longrightarrow M$ *satisfying* $p' \circ \overline{f} = f \circ p$.

The equivalence class will be denoted by $[\mathcal{G} \curvearrowright M]$.

Remark 4.0.7. *Following the notation in definition 4.0.5, if* $\mathcal{G} \curvearrowright M$ *is simplifiable by an action of* G *on* M, *then* $\mathcal{G} \curvearrowright M \sim \mathcal{G}_G \curvearrowright M$.

Our first result describes the structure of free finite iterated actions on nilmanifolds.

Theorem 4.0.8. *Let* N/Γ *be a closed nilmanifold. There exists a constant* C *only depending on* Γ *such that any free iterated action* $\mathcal{G} \curvearrowright N/\Gamma$ *is equivalent to a free iterated action* $\mathcal{G}' \curvearrowright N/\Gamma$ *where* $\mathcal{G}' = \{A_1, \ldots, A_c, G'\}$, A_i *are finite abelian groups and* $|G'| \leq C$.

To study rigidity using free iterated actions we develop two new invariants, the length of free iterated actions on manifolds and the iterated discrete degree of symmetry.

Definition 4.0.9. *Given a free iterated action* $\mathcal{G} \curvearrowright M$, *the length of the iterated action is*

$$l(\mathcal{G} \curvearrowright M) = \min\{l(\mathcal{G}') : \mathcal{G}' \curvearrowright M \in [\mathcal{G} \curvearrowright M]\}.$$

The iterated length of a manifold M is

$$l(M) = \max\{l(\mathcal{G} \curvearrowright M) : \textit{ free iterated action } \mathcal{G} \curvearrowright M\}.$$

Given a free iterated action $\mathcal{A} \curvearrowright M$ *of finite abelian groups, the rank of the iterated action is*

$$\mathrm{rank}_{ab}(\mathcal{A} \curvearrowright M) = \min\{\sum_{i=1}^{n} \mathrm{rank}\, A'_i : \{A'_1, \cdots, A'_n\} \curvearrowright M \in [\mathcal{A} \curvearrowright M]\ A'_i \textit{ abelian for all } i\}.$$

We define $\mu_2(M)$ *as the set of all pairs* $(f, b) \in \mathbb{N}^2$ *which satisfy:*

1. *There exist an increasing sequence of prime numbers* $\{p_i\}$, *a sequence of natural numbers* $\{a_i\}$ *and a collection of free iterated actions* $\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M$ *for each* $i \in \mathbb{N}$.
2. $\mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M) = f + b$ *for each* $i \in \mathbb{N}$.

Consider the lexicographic order in $\mathbb{N}^2$, *that is* $(a, b) \geq (c, d)$ *if* $a > c$, *or* $a = c$ *and* $b \geq d$. *Define the iterated discrete degree of symmetry of M as*

$$\text{disc-sym}_2(M) = \max\{(0,0) \cup \mu_2(M)\}.$$

A first natural question is whether we can bound $l(M)$ and $\text{disc-sym}_2(M)$ when M is a closed connected manifold. For $l(M)$, we have the following two results:

Theorem 4.0.10. *(Cases where $l(M)$ is bounded)*

1. *If N/Γ is a c-step nilmanifold, then $l(N/\Gamma) \leq c+1$.*
2. *Given a locally symmetric space $H \setminus G/\Gamma$, there exists C depending on Γ such that $l(H \setminus G/\Gamma) \leq C$.*

Theorem 4.0.11. *(Cases where $l(M)$ is not bounded)*

1. *There exists a closed solvmanifold M such that $l(M) = \infty$.*
2. *There exists a closed connected aspherical locally homogeneous space $H \setminus G/\Gamma$ such that the solvable radical of G is abelian and $l(H \setminus G/\Gamma) = \infty$.*

We can generalize theorem 1.1.32 to free iterated actions of p-groups on closed connected manifolds M. As a consequence, we can bound $\text{disc-sym}_2(M)$ as follows:

Theorem 4.0.12. *Let M a closed connected manifold, then there exists a constant C such that if* $\text{disc-sym}_2(M) = (d_1, d_2)$ *then $d_1 \leq C$ and $d_2 \leq C$.*

We want to generalize part 2 of theorem 2.0.1 and theorem 3.0.2 using the iterated discrete degree of symmetry.

Theorem 4.0.13. *Let M a closed connected aspherical manifold of dimension n such that $\mathcal{Z}\pi_1(M)$ and $\mathcal{Z}(\operatorname{Inn}\pi_1(M))$ are finitely generated are finitely generated and the groups* $\operatorname{Aut}(\operatorname{Inn}\pi_1(M))$ *and* $\operatorname{Out}(\operatorname{Inn}\pi_1(M))$ *are Minkowski. If* $\text{disc-sym}_2(M) = (f, b)$ *with $f + b = n$ then $M \cong N/\Gamma$, where N/Γ is a 2-step nilmanifold which is a principal T^f-bundle over T^b.*

Theorem 4.0.14. *Let M be a closed connected manifold admitting a non-zero degree map $f : M \longrightarrow N/\Gamma$ to a 2-step nilmanifold, which is the total space of a principal T^f-bundle over T^b. Then* $\text{disc-sym}_2(M) \leq (f, b)$ *and if* $\text{disc-sym}_2(M) = (f, b)$ *then $H^*(M, \mathbb{Q}) \cong H^*(N/\Gamma, \mathbb{Q})$.*

We also compute the iterated discrete degree of symmetry for closed connected aspherical 3-manifolds. For the next result K will denote the Klein bottle and SK will denote the non-trivial principal S^1-bundle over K.

Theorem 4.0.15. *Let M be a 3-dimensional closed connected aspherical manifold. Then:*

1. $\text{disc-sym}_2(M) = (3, 0)$ *if $M \cong T^3$.*
2. $\text{disc-sym}_2(M) = (2, 0)$ *if $M \cong K \times S^1$ or $M \cong SK$.*
3. $\text{disc-sym}_2(M) = (1, 2)$ *if $M \cong H/\Gamma$.*
4. $\text{disc-sym}_2(M) = (1, 0)$ *if $\mathcal{Z}\pi_1(M) \cong \mathbb{Z}$ and $\operatorname{Inn}\pi_1(M)$ is centreless.*
5. $\text{disc-sym}_2(M) = (0, 0)$ *if M does not belong to one of the previous 4 cases.*

Lastly, we introduce a new property of finite iterated actions on actions with the aim to replace the freeness hypothesis with a weaker condition.

Definition 4.0.16. *Assume that we have an iterated action of a collection of finite groups $\mathcal{G}$ on a topological space X. An open subset $U \subseteq X$ is said to be $\mathcal{G}$-invariant if there exists a connected open subset $V \subseteq X/\mathcal{G}$ such that $p^{-1}(V) = U$, where $p : X \longrightarrow X/\mathcal{G}$ is the orbit map.*

An iterated action of $\mathcal{G} \curvearrowright X$ is said to be locally simplifiable if for every $x \in X$ there exists an open $\mathcal{G}$-invariant neighbourhood U of x such that the iterated action of $\mathcal{G}$ on U is simplifiable.

We need to use orbifolds to obtain results on locally simplifiable actions.

Theorem 4.0.17. *Assume that we have a locally simplifiable iterated action of $\mathcal{G}$ on a manifold M. Then $M/\mathcal{G}$ is an orbifold and $p : M \longrightarrow M/\mathcal{G}$ is an orbifold covering.*

Theorem 4.0.17 enables us to work as if the iterated action were free, but using the orbifold fundamental group. Thus, we can define the iterated discrete degree of symmetry for locally simplifiable actions.

Definition 4.0.18. *We define $\mu_2^{ls}(M)$ as the set of all pairs $(f, b) \in \mathbb{N}^2$ which satisfy:*

1. *There exist an increasing sequence of prime numbers $\{p_i\}$, a sequence of natural numbers $\{a_i\}$ and a collection of locally simplifiable iterated actions $\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M$ for each $i \in \mathbb{N}$.*
2. $\mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M) = f + b$ *for each $i \in \mathbb{N}$.*

Consider the lexicographic order in $\mathbb{N}^2$. Define the locally simplifiable iterated discrete degree of symmetry of M as

$$\text{disc-sym}_2^{ls}(M) = \max\{(0,0) \cup \mu_2^{ls}(M)\}.$$

Theorem 4.0.13, theorem 4.0.15 and theorem 4.0.14 also hold for locally simplifiable actions, since all manifolds appearing in the theorems satisfy that if a finite p-group acts on them for p a prime large enough, then the action is free. An example where $\text{disc-sym}_2^{ls}(M) \neq \text{disc-sym}_2(M)$ is the following:

Proposition 4.0.19. *We have* $\text{disc-sym}_2^{ls}(S^n) = ([\frac{n+1}{2}], 0)$ *and* $\text{disc-sym}_2(S^n) = (\frac{(-1)^{n+1}+1}{2}, 0)$, *where $[x]$ denotes the integer part of x.*

This chapter is divided as follows. In the first section we give some definitions and results which are straightforward generalizations of some concepts of the theory of compact transformation groups. In the second section we start the study of free finite iterated actions. Third section is devoted to the length of iterated actions on a manifold. We prove theorem 4.0.10, theorem 4.0.11 and theorem 4.0.8. We introduce the iterated discrete degree of symmetry and prove theorem 4.0.12 in section five. Sections six, seven and eight are devoted to prove theorem 4.0.13, theorem 4.0.15 and theorem 4.0.14 respectively. In section eight we introduce locally simplifiable actions and we prove proposition 4.0.19.

4.1 Basic definitions and properties

Most of the basic definitions of the theory of transformation groups have straightforward generalizations to iterated group actions.

Definition 4.1.1. *Assume that we have an iterated action of finite groups* $\mathcal{G} = \{G_i\}_{i=1,\ldots,n}$ *on* X *and let* $x \in X$*. Then:*

1. *The cardinal of* $\mathcal{G}$ *is* $|\mathcal{G}| = \prod_{i=1}^{n} |G_i|$.
2. *The iterated stabilizer of* x *is the collection of subgroups* $\mathcal{G}_x = \{(G_i)_{x_{i-1}}\}_{i=1,\ldots,n}$.
3. *The iterated orbit of* x *is the collection of points* $\mathcal{G}(x) = \{y \in X : p(y) = p(x)\} = p^{-1}(p(x))$.
4. *We say that the the iterated action is effective if all the actions* G_i *on* X_{i-1} *are effective. The iterated action is free if all the actions* G_i *on* X_{i-1} *are free.*
5. *A point* $x \in X$ *is fixed by* $\mathcal{G}$ *if* $\mathcal{G}(x) = \{x\}$*. We denote the set of fix points by* $X^{\mathcal{G}}$.

We will implicitly assume that all iterated actions are effective. The existence of a $\mathcal{G}$-equivariant homeomorphism in the definition of an iterated action being simplifiable implies:

Lemma 4.1.2. *Let* $\mathcal{G} \curvearrowright X$ *be a simplifiable iterated action by an action of* G *on* X*. Then:*

1. $\mathcal{G} \curvearrowright X$ *is free if and only if the action of* G *on* X *is free.*
2. $X^{\mathcal{G}} \cong X^{G}$.

Proof. Assume that $\mathcal{G} \curvearrowright X$ is simplifiable and let $f : X \longrightarrow X$ and G be as in definition 4.0.5. By definition $|G| = \prod_{i=1}^{n} |G_i| = |\mathcal{G}|$. Since f is a $\mathcal{G}$-equivariant homeomorphism, the cardinal of the orbit $\mathcal{G}(x)$ is equal to the cardinal of the orbit $G(f(x))$ for any $x \in X$.

If $\mathcal{G} \curvearrowright X$ is free then we have $|\mathcal{G}(x)| = |\mathcal{G}| = |G| = |G(f(x))|$ for all $x \in X$. In particular, the isotropy subgroup $G_{f(x)}$ is trivial for all $x \in X$. This implies that the action of G on X is free.

Similarly, $x \in X^{\mathcal{G}}$ if and only if $f(x) \in X^{G}$. Thus, $f_{|X^{\mathcal{G}}}$ is a homeomorphism between $X^{\mathcal{G}}$ and X^{G}. □

Most of the basic results of the theory of finite transformation groups also generalize to the context of iterated actions.

Lemma 4.1.3. *Assume that we have an iterated action of finite groups* $\mathcal{G} \curvearrowright X$*. Then:*

1. *If* X *is Hausdorff, then* X_i *is Hausdorff for all* i.

2. *The maps $p_i : X_{i-1} \longrightarrow X_i$ are open, closed and proper. In particular, all of the compositions $X_i \longrightarrow X_j$ are open, closed and proper.*

3. *X is compact if and only if $X/\mathcal{G}$ is compact.*

Proof. The lemma is true for group actions, see proposition 1.1.9. In the general case, we only need to apply recursively proposition 1.1.9 at each step of the iterated action. □

The slice theorem for finite groups can also be generalized.

Definition 4.1.4. *Let X be a Hausdorff topological space with an iterated action of finite groups $\mathcal{G} \curvearrowright X$. Given $x \in X$, recall that we denote $x_i = p_i(x_i)$ for $0 \leq i \leq n-1$, where $x = x_0$. An iterated slice S at $x \in X$ is a collection of subspaces $S = \{S_i\}_{i=0,\ldots,n-1}$ satisfying:*

1. *$S_i \subseteq X_i$ is a slice at x_i of the action of G_{i+1} on X_i for all $0 \leq i \leq n-1$.*

2. *$p_{i+1}(S_i) = S_{i+1}$ for all $0 \leq i \leq n-1$.*

The set $p^{-1}(p(S_0))$ will be called a tube around the orbit $\mathcal{G}(x)$.

Theorem 4.1.5. *(Slice theorem for iterated action) Let X be a Hausdorff topological space with an iterated action of finite groups $\mathcal{G} \curvearrowright X$. There exists an iterated slice S at every point $x \in X$.*

Proof. We construct the slice recursively. For the first iteration, the classical slice theorem for finite group actions (theorem 1.1.18) gives us a slice S_0' at x_0 for the action of G_1. Since X_1 is Hausdorff and $p_1(S_0')$ is open, we can find a slice $S_1 \subseteq X_1$ at x_1 for the action of G_2 such that $S_1 \subseteq p_1(S_0')$. Thus, $S_0' \cap p_1^{-1}(S_1) = S_0$ is a slice at x_0 for the action of G_1 such that $p_1(S_0) = S_1$. Thus we have constructed an iterated slice for the action of $\{G_1, G_2\} \curvearrowright X$. We can repeat this process recursively to obtain an iterated slice for the iterated action of $\mathcal{G}$ on X of any length. □

Remark 4.1.6. *Let X be a topological space with an iterated action of finite group $\mathcal{G} \curvearrowright X$ and let S be an iterated slice at $x \in X$. For each i, the isotropy subgroup $G_{i x_{i-1}}$ acts on S_{i-1} and $p_i(S_{i-1}) = S_{i-1}/G_{x_i} = S_i$. Thus, we have an iterated action of $\mathcal{G}_x$ on S_0 satisfying $S_i = S_{i-1}/G_{i x_{i-1}}$.*

4.2 Free iterated actions

Recall that if G is a finite group acting freely on a manifold M then M/G is also a manifold and $p : M \longrightarrow M/G$ is a regular covering. Moreover, we have a short exact sequence $1 \longrightarrow \pi_1(M) \longrightarrow \pi_1(M/G) \longrightarrow G \longrightarrow 1$. Therefore, if we have an iterated free action of $\mathcal{G} = \{G_i\}_{i=1,..,n}$ on a manifold M, then every M_i is a manifold, $p_i : M_{i-1} \longrightarrow M_i$ is a regular cover and the map $p : M \longrightarrow M_n$ is a covering on M. A free iterated action of

$\mathcal{G} = \{G_i\}_{i=1,\dots,n}$ on M induces a series of groups

$$\pi_1(M) = \pi_1(M_0) \trianglelefteq \pi_1(M_1) \trianglelefteq \cdots \trianglelefteq \pi_1(M_n)$$

where $\pi_1(M_i)/\pi_1(M_{i-1}) \cong G_i$.

Lemma 4.2.1. *Let $\mathcal{G} \curvearrowright M$ and $\mathcal{G}' \curvearrowright M$ be a free iterated actions of finite groups inducing coverings $p : M \longrightarrow M/\mathcal{G}$ and $p' : M \longrightarrow M/\mathcal{G}'$ respectively. If $\mathcal{G} \curvearrowright M \sim \mathcal{G}' \curvearrowright M$ then there exists an isomorphism $\phi : \pi_1(M/\mathcal{G}) \longrightarrow \pi_1(M/\mathcal{G}')$ satisfying $\phi(p_*(\pi_1(M))) = p'_*(\pi_1(M))$.*

Proof. Let $f : M/\mathcal{G} \longrightarrow M/\mathcal{G}'$ be the homeomorphism provided by definition 4.0.6. By construction, the coverings $f \circ p : M \longrightarrow M/\mathcal{G}'$ and $p' : M \longrightarrow M/\mathcal{G}'$ are isomorphic. Therefore, there exists $\gamma \in \pi_1(M/\mathcal{G}')$ such that $f_*(p_*(\pi_1(M)))) = c_\gamma(p'_*(\pi_1(M)))$. We can write $\phi(p_*(\pi_1(M))) = p'_*(\pi_1(M))$, where $\phi = c_{\gamma^{-1}} \circ f_*$. □

Lemma 4.2.1 shows that to study free iterated actions up to equivalence, we can focus on the inclusion $p_* : \pi_1(M) \longrightarrow \pi_1(M/\mathcal{G})$ induced by the covering. For example:

Corollary 4.2.2. *Let $\mathcal{G} \curvearrowright M$ and $\mathcal{G}' \curvearrowright M$ be equivalent free iterated actions of finite groups inducing coverings $p : M \longrightarrow M/\mathcal{G}$ and $p' : M \longrightarrow M/\mathcal{G}'$ respectively. Then $p_*(\pi_1(M)) \trianglelefteq \pi_1(M/\mathcal{G})$ if and only if $p'_*(\pi_1(M)) \trianglelefteq \pi_1(M/\mathcal{G}')$.*

From definition 4.0.6 and definition 4.0.5, a free iterated action $\mathcal{G} \curvearrowright M$ is simplifiable if and only if the covering $p : M \longrightarrow M/\mathcal{G}$ is regular. Therefore:

Lemma 4.2.3. *A free iterated action of a collection of finite groups $\mathcal{G} = \{G_i\}_{i=1,..,n}$ on M is simplifiable if and only if $\pi_1(M) \trianglelefteq \pi_1(M/\mathcal{G})$.*

Corollary 4.2.4. *A free iterated action on a simply connected manifold M is simplifiable.*

However, M being simply connected is not a necessary condition to have simplifiability of all free iterated actions on M.

Lemma 4.2.5. *Any free iterated action on S^1 or T^2 is simplifiable.*

Proof. Let $\mathcal{G} \curvearrowright S^1$ be a free iterated group action. Then $S^1/\mathcal{G}$ is a closed 1-dimensional manifold and hence $S^1/\mathcal{G} \cong S^1$. This implies that $\pi_1(M) \cong \mathbb{Z} \trianglelefteq \pi_1(S^1/\mathcal{G}) \cong \mathbb{Z}$ and hence the free iterated group action is simplifiable. All the groups of $\mathcal{G}$ are cyclic and the simplification is given by group action of a cyclic group of order $|\mathcal{G}|$.

The proof for the second case is similar. Assume that we have a free iterated group action $\mathcal{G} \curvearrowright T^2$. Then $T^2/\mathcal{G}$ is homeomorphic to T^2 or the Klein bottle K. Then the result follows from the fact that any subgroup of $\pi_1(T^2) \cong \mathbb{Z}^2$ or $\pi_1(K) \cong \mathbb{Z} \rtimes \mathbb{Z}$ isomorphic to $\mathbb{Z}^2$ is normal, hence the action is simplifiable. □

Remark 4.2.6. *Lemma 4.2.5 cannot be extended to T^n for $n \geq 3$. Consider the Bieberbach group with presentation*

$$\Gamma = \langle t_1, t_2, t_3, \alpha | [t_i, t_j] = e \,\forall i, j, \alpha^3 = t_1, \alpha t_2 \alpha^{-1} = t_3, \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1} \rangle.$$

The group generated by t_1, t_2 and t_3 is normal and isomorphic to $\mathbb{Z}^3$, which we denote by Z. Note that $\Gamma/Z \cong \mathbb{Z}/3$. Let Z' be the subgroup generated by t_1, t_2 and t_3^2, which is also isomorphic to $\mathbb{Z}^3$. We have a normal series $Z' \trianglelefteq Z \trianglelefteq \Gamma$ with $\Gamma/Z \cong \mathbb{Z}/3$ and $Z/Z' \cong \mathbb{Z}/2$. On the other hand, Z' is not normal in Γ, since $\alpha t_2 \alpha^{-1} = t_3 \notin Z'$. Now we can define a free iterated action $\{\mathbb{Z}/2, \mathbb{Z}/3\} \curvearrowright T^3$ such that $\pi_1(T^3) \cong Z'$, $\pi_1(T^3/\mathbb{Z}/2) \cong Z$ and $\pi_1((T^3/\mathbb{Z}/2)/\mathbb{Z}/3) \cong \Gamma$. This free iterated action is not simplifiable.

We have seen that free iterated actions produce a covering map that is not necessarily regular. Conversely, given a finite covering map $q : M \longrightarrow M'$ we can ask whether there exists an iterated action $\mathcal{G}$ on M such that $p : M \longrightarrow M/\mathcal{G}$ is isomorphic to $q : M \longrightarrow M'$. The next example shows that this does not happen in general.

Example 4.2.7. *Let M be a closed flat manifold with holonomy group the alternate group A_5 (it exists by theorem 1.3.54) and let $n = \dim M$. We take the short exact sequence*

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \pi_1(M) \xrightarrow{\rho} A_5 \longrightarrow 1$$

Take a non-trivial subgroup $G \leq A_5$ and consider the finite covering of closed flat manifold $q : M' \longrightarrow M$, where $\pi_1(M') = \rho^{-1}(G)$. If q were induced by a free iterated action, then there would exist a group Γ such that $\pi_1(M') \leq \Gamma \trianglelefteq \pi_1(M)$. This would imply that $G \leq \Gamma/\mathbb{Z}^n \trianglelefteq A_5$, which is not possible since A_5 is simple.

Let k be a natural number and let $\mathrm{Cov}_k(M)$ be the set of all coverings of M of k-sheets up to equivalence of coverings. Let $p : \tilde{M} \longrightarrow M$ be a k-covering and pick $x \in M$. We can enumerate the points of the fiber $p^{-1}(x) = \{x_1, \ldots, x_k\}$. Given $\alpha \in \pi_1(M, x)$ there is a unique lift $a_i : I \longrightarrow \tilde{M}$ such that $a_i(0) = x_i$. Thus, we can define an element in the group of permutations of k letters σ_α such that x_i goes to $a_i(1)$. If we remove the choice of the base point x, then $\mathrm{Cov}_k(M) \cong \mathrm{Hom}(\pi_1(M), S_k)/\sim$, where S_k is the permutation group of k elements acting by conjugations on $\mathrm{Hom}(\pi_1(M), S_k)$.

Assume that a finite group G acts effectively on M, then we have an action of G on $\mathrm{Cov}_k(M)$ given by the pull-back of each element of G. Explicitly, $g[\tilde{M} \longrightarrow M] = [g^*\tilde{M} \longrightarrow M]$ for all $g \in G$ and $[\tilde{M} \longrightarrow M] \in \mathrm{Cov}(M)_k$. The induced action of G on $\mathrm{Hom}(\pi, S_k)/S_k$ is given by $g[f : \pi \longrightarrow S_k] = [f \circ g_* : \pi \longrightarrow S_k]$, where $g_* : \pi_1(M) \longrightarrow \pi_1(M)$ is the group morphism induced by g on the fundamental group.

We also recall the lifting condition of continuous maps. Assume that we have $f : M' \longrightarrow M$ a continuous map such that $f(y) = x$ for some $y \in M'$. Then, there exists $\tilde{f} : M' \longrightarrow \tilde{M}$

such that $f = p \circ \tilde{f}$ if and only if $f_*(\pi_1(M', y)) \leq p_*(\pi_1(\tilde{M}, \tilde{x}))$. In particular, a homeomorphism $f : M \longrightarrow M$ can be lifted to a homeomorphism $\tilde{f} : \tilde{M} \longrightarrow \tilde{M}$ if and only if $f_*(p_*(\pi_1(\tilde{M}, \tilde{y}))) \leq p_*(\pi_1(\tilde{M}, \tilde{x}))$.

Lemma 4.2.8. *Assume that we have a free iterated action of $\mathcal{G} = \{G_1, G_2\}$ on a manifold M. Then, the following statements are equivalent:*

(1) $\mathcal{G}$ is simplifiable

(2) We can lift $g_2 : M/G_1 \longrightarrow M/G_1$ to a homeomorphism $\tilde{g}_2 : M \longrightarrow M$ for all $g_2 \in G_2$.

(3) The action of G_2 on $\mathrm{Cov}_{|G_1|}(M/G_1)$ fixes $[p_1 : M \longrightarrow M/G_1]$.

Proof. We will prove the chain of implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (1). If the action of $\mathcal{G}$ is simplifiable then there exists a group G fitting in the exact sequence $1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$ which acts freely on M and $p : M \longrightarrow M/G = p_2 \circ p_1$. Given any $g_2 \in G_2$ we choose an element $\tilde{g}_2 \in G$ which inside the preimage of $G \longrightarrow G_2$. The induced homeomorphism $\tilde{g}_2 : M \longrightarrow M$ is a lift of $g_2 : M/G_1 \longrightarrow M/G_1$.

We now prove the second implication. Note that if we can lift $g_2 : M \longrightarrow M$, then $p_1 : M \longrightarrow M/G_1$ and the pullback $g_2^* M \longrightarrow M/G_1$ are isomorphic as regular G_1-coverings. If $[p_1] \in \mathrm{Cov}_{|G_1|}(M/G_1)$ denotes the class of the covering $p_1 : M \longrightarrow M/G_1$ then $g_2[p_1] = [p_1]$ for all $g_2 \in G_2$.

Finally, if G_2 fixes $[p_1]$ then there exists a group G of the form $1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$ acting freely on M. This action extends the action of G_1 on M and it also covers the action of G_2 on M/G_1. Therefore $p : M \longrightarrow M/G = p_2 \circ p_1$ and the free iterated action is simplifiable. □

4.3 The length of a free iterated action

Recall the following definition (see definition 4.0.9).

Definition 4.3.1. *Given a free iterated action $\mathcal{G} \curvearrowright X$, the length of the iterated action is*

$$l(\mathcal{G} \curvearrowright X) = \min\{l(\mathcal{G}') : \mathcal{G}' \curvearrowright X \in [\mathcal{G} \curvearrowright X]\}.$$

The iterated length of the space X is

$$l(X) = \max\{l(\mathcal{G} \curvearrowright X) : \text{ free iterated action } \mathcal{G} \curvearrowright X\}.$$

With this notation, a free iterated action $\mathcal{G} \curvearrowright X$ is simplifiable if and only if $l(\mathcal{G} \curvearrowright X) = 1$ and all free iterated actions on X are simplifiable if and only if $l(X) = 1$.

A natural question is whether for a closed manifold M, there exists a constant C such that $l(M) \leq C$. For example:

Lemma 4.3.2. *Let M be a closed manifold with $\chi(M) \neq 0$. Then $l(M) \leq \log_2 \chi(M)$.*

Proof. Recall that if G is a finite group acting freely on M then $\chi(M) = |G|\chi(M/G)$. If we have an iterated free action $\mathcal{G} \curvearrowright M$ then $\chi(M) = \prod_{i=1}^{n} |G_i|\chi(M/\mathcal{G})$. Since $|G_i| \geq 2$ for all i, then $n \leq \log_2 \chi(M)$. □

The first main result of this section bounds the iterated length of nilmanifolds. Let N/Γ be a c-step nilmanifold and let $\{0\} = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots \trianglelefteq Z_c = \Gamma$ be the upper central series. The groups Z_{i+1}/Z_i are finitely generated torsion-free and abelian, so we denote by $b_i = \mathrm{rank}(Z_{i+1}/Z_i)$. Then:

Theorem 4.3.3. *There exists a constant C only depending on Γ such that any free iterated action $\mathcal{G} \curvearrowright N/\Gamma$ is equivalent to a free iterated action $\mathcal{G}' \curvearrowright N/\Gamma$ where $\mathcal{G}' = \{A_1, \ldots, A_c, G'\}$, A_i are finite abelian groups such that $\mathrm{rank}(A_i) \leq b_i$ and $|G'| \leq C$.*

Corollary 4.3.4. *If N/Γ is a c-step nilmanifold then $l(N/\Gamma) \leq c + 1$.*

We need some preliminary lemmas before proving theorem 4.3.3. Firstly, we study the case where $(N/\Gamma)/\mathcal{G}$ is a nilmanifold.

Lemma 4.3.5. *Let N be a c-step nilpotent Lie group and let Γ and Λ be lattices of N such that $\Gamma \leq \Lambda$. There exist subgroups $\Lambda_0, \Lambda_1, \ldots, \Lambda_c$ such that $\Lambda_0 = \Gamma$, $\Lambda_c = \Lambda$ and $\Lambda_i \trianglelefteq \Lambda_{i+1}$ for all i. Moreover, $A_i = \Lambda_{i+1}/\Lambda_i$ is abelian and $\mathrm{rank}(A_i) \leq b_i$ for all i.*

Proof. We prove the claim by induction on c. If $c = 1$ then N is abelian and therefore $\Gamma = \Lambda_0 \trianglelefteq \Lambda_1 = \Lambda$.

Assume that N is c-step nilpotent and let $\pi : N \longrightarrow N/\mathcal{Z}N$ be the quotient map to the $(c-1)$-step nilpotent Lie group $N/\mathcal{Z}N$. Then $\pi(\Lambda)$ and $\pi(\Gamma)$ are lattices of $N/\mathcal{Z}N$. Since $\mathcal{Z}\Lambda = \Lambda \cap \mathcal{Z}N$ and $\mathcal{Z}\Gamma = \Gamma \cap \mathcal{Z}N$, there are short exact sequences

$$1 \longrightarrow \mathcal{Z}\Lambda \longrightarrow \Lambda \xrightarrow{\pi_{|\Lambda}} \pi(\Lambda) \longrightarrow 1$$

and

$$1 \longrightarrow \mathcal{Z}\Gamma \longrightarrow \Gamma \xrightarrow{\pi_{|\Gamma}} \pi(\Gamma) \longrightarrow 1.$$

By induction hypothesis, there exists $\Lambda'_0 = \pi(\Gamma) \trianglelefteq \Lambda'_1 \trianglelefteq \cdots \trianglelefteq \Lambda'_{c-1} = \pi(\Lambda)$. Then we define $\Lambda_{i+1} = \pi_{|\Lambda}^{-1}(\Lambda'_i)$ for $0 \leq i \leq c-1$, which satisfy that $\Lambda_i \trianglelefteq \Lambda_{i+1}$ for all $0 \leq i \leq c-1$.

Lastly, we take $\Lambda_0 = \Gamma$. We have the commutative diagram where the rows are central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{Z}\Lambda & \longrightarrow & \Lambda_1 & \xrightarrow{\pi_{|\Lambda}} & \pi(\Gamma) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle id} & & \\ 1 & \longrightarrow & \mathcal{Z}\Gamma & \longrightarrow & \Gamma & \xrightarrow{\pi_{|\Gamma}} & \pi(\Gamma) & \longrightarrow & 1. \end{array}$$

Consequently, $\Gamma \trianglelefteq \Lambda_1$. □

Corollary 4.3.6. *A free iterated action $\mathcal{G} \curvearrowright N/\Gamma$ such that $(N/\Gamma)/\mathcal{G}$ is a nilmanifold is equivalent to a free iterated action $\mathcal{G}' \curvearrowright N/\Gamma$ where $\mathcal{G}' = \{A_1, \dots, A_c\}$ and A_i are finite abelian groups such that* $\text{rank}(A_i) \leq b_i$.

For the general case we need theorem 1.3.61, proposition 1.3.64 to prove the following two lemmas.

Lemma 4.3.7. *Let $\mathcal{G} \curvearrowright N/\Gamma$ be a free iterated action. Then $\pi_1((N/\Gamma)/\mathcal{G}) = E$ is an AC-group and* $\Gamma \leq \text{Fitt}(E)$.

Proof. Since $(N/\Gamma)/\mathcal{G}$ is a closed aspherical manifold, then E is torsion-free and contains Γ as a finite index subgroup. Using theorem 1.3.61, E is an AC-group. It only remains to prove that $\Gamma \leq \text{Fitt}(E)$.

Consider the exact sequence

$$1 \longrightarrow \text{Fitt}(E) \longrightarrow E \xrightarrow{p} F \longrightarrow 1$$

If $G = p(\Gamma)$ and $\Lambda = p^{-1}(G)$ we obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \text{Fitt}(E) \cap \Gamma & \longrightarrow & \Gamma & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \\ 1 & \longrightarrow & \text{Fitt}(E) & \longrightarrow & \Lambda & \xrightarrow{p} & G & \longrightarrow & 1 \end{array}$$

where the vertical arrows are inclusions. Let $\sigma : G \longrightarrow \Gamma$ be a set-theoretic section of p and $\psi : G \longrightarrow \text{Out}(\text{Fitt}(E) \cap \Gamma)$ the induced group morphism such that $\psi(g) = [c_{\sigma(g)|\,\text{Fitt}(E)\cap\Gamma}]$. The map $\overline{\sigma} = i \circ \sigma$ is a section for the second short exact sequence and it induces a group morphism $\bar{\psi} : G \longrightarrow \text{Out}(\text{Fitt}(E))$ such that $\tilde{\psi}(g) = [c_{\sigma(g)|\,\text{Fitt}(E)}]$.

Since Γ and $\text{Fitt}(E)$ have finite index inside E, then $\text{Fitt}(E) \cap \Gamma$ has finite index inside $\text{Fitt}(E)$ and Γ and therefore $\Gamma_{\mathbb{Q}} = \text{Fitt}(E)_{\mathbb{Q}} = (\text{Fitt}(E) \cap \Gamma)_{\mathbb{Q}}$. This implies that the maps $\psi' : G \longrightarrow \text{Out}(\Gamma_{\mathbb{Q}})$ and $\bar{\psi}' : G \longrightarrow \text{Out}(\Gamma_{\mathbb{Q}})$ are the same. By proposition 1.3.64 the morphism ψ' is trivial since Γ is nilpotent. On the other hand $\overline{\psi}'$ is injective by proposition 1.3.64 and part 2. of theorem 1.3.61. Therefore, the only option is that G is trivial and $\Gamma \leq \text{Fitt}(E)$. □

Lemma 4.3.8. *There exists a constant C such that if G is a finite subgroup of* $\text{Out}(\Gamma_{\mathbb{Q}})$ *then* $|G| \leq C$.

Proof. Let $\psi : G \longrightarrow \mathrm{Out}(\Gamma_{\mathbb{Q}})$ denote the inclusion of the finite group. We have $H^2_\psi(G, \mathbb{Q}^n) = 0$, since G is finite and $\mathbb{Q}^n$ is divisible and therefore there is an injective lift $\tilde{\psi} : G \longrightarrow \mathrm{Aut}(\Gamma_{\mathbb{Q}})$ of ψ. We use the exponential map to identify $\mathrm{Aut}(\Gamma_{\mathbb{Q}})$ with the group of automorphisms of the associated rational Lie algebra $\mathrm{Aut}(\mathcal{L}(\Gamma_{\mathbb{Q}}))$, which is a subgroup of $\mathrm{GL}(m, \mathbb{Q})$ for some m. In consequence, G is conjugated to a finite subgroup of $\mathrm{GL}(m, \mathbb{Z})$ and therefore the bound is a consequence of Minkowski lemma. □

Proof of theorem 4.3.3. Assume that we have a free iterated action of $\mathcal{G}$ on a nilmanifold N/Γ. Then $(N/\Gamma)/\mathcal{G} = M$ is a closed almost-flat manifold and $\pi_1(M)$ is an almost-Bieberbach group. In consequence, $\pi_1(M)$ contains a maximal normal nilpotent group Λ, which has Γ as a subgroup by maximality by lemma 4.3.7. Since $[\pi_1(M) : \Gamma] < \infty$, $[\Lambda : \Gamma] < \infty$ and therefore Λ is a lattice of N. By lemma 4.3.5, there exists a subnormal series $\Gamma = \Lambda_0 \trianglelefteq \Lambda_1 \trianglelefteq \cdots \trianglelefteq \Lambda_c = \Lambda \trianglelefteq \pi_1(M)$. We have a free iterated action of $\{\Lambda_i/\Lambda_{i-1}, \pi_1(M)/\Lambda\}_{i=1,\ldots,n}$ on N/Γ equivalent to $\mathcal{G} \curvearrowright N/\Gamma$. By lemma 4.3.5, $\mathrm{rank}(\Lambda_i/\Lambda_{i-1}) \leq b_i$. Moreover, $\pi_1(M)/\Lambda \leq \mathrm{Out}(\Lambda_{\mathbb{Q}})$ and $\Lambda_{\mathbb{Q}} \cong \Gamma_{\mathbb{Q}}$. Therefore, by proposition 1.3.64 $|\pi_1(M)/\Lambda| \leq C$, where C is a constant depending on $\Gamma_{\mathbb{Q}}$. □

Remark 4.3.9. *The bound of corollary 4.3.4 is sharp. For example, the free iterated action on T^3 from remark 4.2.6 shows that $2 \geq l(T^3)$ and therefore $l(T^3) = 2$. Consequently, $l(T^n) = 2$ for all $n \geq 3$. On the other hand, in lemma 4.2.5 we show that all free iterated actions on S^1 and T^2 are simplifiable, which implies that $l(T^n) = 1$ for $n = 1, 2$. Thus, corollary 4.3.4 is not an equality in general.*

On the other hand, a bound for $l(M)$ when M is a solvmanifold does not always exist.

Theorem 4.3.10. *There exists a 3-dimensional solvmanifold M such that $l(M) = \infty$.*

Proof. Consider the 3-dimensional solvable Lie group $R = \mathrm{Sol}^3$. Explicitly, $R \cong \mathbb{R}^2 \rtimes_\psi \mathbb{R}$, where

$$\psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}.$$

Any lattice of R is isomorphic to a semi-direct $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, where $\phi : \mathbb{Z} \longrightarrow \mathrm{SL}(2, \mathbb{Z})$ satisfies $\mathrm{tr}(\phi(1)) > 2$ (see [LT15, §2]). Two of these lattices Γ and Γ' are isomorphic if and only if $\phi'(1)$ is conjugate to $\phi(1)$ or $\phi(1)^t$.

We take the matrix

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

and the lattice $\Gamma = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ of R where $\phi(1) = A$. We are going to see that the solvmanifold $M = R/\Gamma$ satisfies $l(M) = \infty$. We start by studying the lattice Γ and some of its sublattices.

For an integer $k \geq 0$ we define the group $\Gamma_k = 2^k(\mathbb{Z})^2 \rtimes_\phi \mathbb{Z}$. We have a series $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \dots$. Firstly, we note that the group morphism $f_k : \Gamma \longrightarrow \Gamma_k$ such that $f_k(v,t) = (2^k v, t)$ is an isomorphism for all $k \geq 0$, which implies that R/Γ and R/Γ_k are diffeomorphic for all $k \geq 0$.

We claim now that the normalizer $N_\Gamma(\Gamma_k) = \Gamma_{k-1}$. Firstly, a computation shows that:

1. If $(v,t) \in \Gamma$ then $(v,t)^{-1} = (-A^{-t}v, -t)$.
2. If $(v,t),(w,s) \in \Gamma$ then $(v,t)(w,s)(v,t)^{-1} = ((Id - A^s)v + A^t w, s)$.

Therefore, the normalizer takes the form

$$N_\Gamma(\Gamma_k) = \{(v,t) \in \Gamma : (Id - A^s)v \in 2^k\mathbb{Z}^2 \text{ for all } s\}.$$

The matrix A is of the form $A = Id + B$ with

$$B = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix} \in M_{2\times 2}(2\mathbb{Z})$$

which implies that $Id - A^s = -\sum_{i=1}^{s} \binom{s}{i} B^s \in M_{2\times 2}(2\mathbb{Z})$ for all $s \geq 1$ (and we have a similar form for A^{-s}). Let $(v,t) \in N_\Gamma(\Gamma_k)$ with $v = (v_1, v_2) \in \mathbb{Z}^2$. Since $(Id - A^s)v \in 2^k\mathbb{Z}^2$ for all s, in particular $-Bv = (Id - A)v \in 2^k\mathbb{Z}^2$. We obtain that $2v_1, 2v_2 \in 2^k\mathbb{Z}$ and therefore $v_1, v_2 \in 2^{k-1}\mathbb{Z}$. Moreover, if $v \in 2^{k-1}\mathbb{Z}^2$ then $(Id - A^s)v \in 2^k\mathbb{Z}^2$ for all s. Thus, we have seen that

$$N_\Gamma(\Gamma_k) = \{(v,t) \in \Gamma : v \in 2^{k-1}\mathbb{Z}^2\} = \Gamma_{k-1}.$$

In consequence, $\Gamma_k \trianglelefteq \Gamma_{k-1}$ and $\Gamma_k \ntrianglelefteq \Gamma_{k-i}$ for $i > 1$. In addition, $\Gamma_{k-1}/\Gamma_k \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. If $\pi_k : \Gamma_{k-1} \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$ denotes the quotient map, then we can define new lattices

$$\Gamma_k^{(1,0)} = \pi_k^{-1}(\langle(1,0)\rangle) = (2^k\mathbb{Z} \times 2^{k-1}\mathbb{Z}) \rtimes_\phi \mathbb{Z},$$

$$\Gamma_k^{(0,1)} = \pi_k^{-1}(\langle(0,1)\rangle) = (2^{k-1}\mathbb{Z} \times 2^k\mathbb{Z}) \rtimes_\phi \mathbb{Z},$$

$$\Gamma_k^{(1,1)} = \pi_k^{-1}(\langle(1,1)\rangle) = \{((v_1,v_2),t) \in \Gamma_{k-1} : v_1 + v_2 \in 2^k\mathbb{Z}\}.$$

Analogous computations show that $N_\Gamma(\Gamma_k^{(i,j)}) = \Gamma_{k-1}^{(i,j)}$ and $\Gamma_k^{(i,j)}/\Gamma_{k-1}^{(i,j)} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ for all $(i,j) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$, where we set $\Gamma_k^{(0,0)} = \Gamma_{k-1}$.

We consider the regular $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ self-covering $p : R/\Gamma_1 \cong M \longrightarrow R/\Gamma \cong M$. We can obtain a tower of regular self-coverings $p^k : R/\Gamma_k \cong M \longrightarrow R/\Gamma \cong M$, which has associated a free iterated action of $\mathcal{G}_k = \{\mathbb{Z}/2 \oplus \mathbb{Z}/2, \dots, \mathbb{Z}/2 \oplus \mathbb{Z}/2\}$ with $l(\mathcal{G}_k) = k$ on M.

Let $\mathcal{G}' = \{G_1', \dots, G_m'\} \curvearrowright M$ be a free iterated action equivalent to $\mathcal{G}_k \curvearrowright M$. Then we have a subnormal series $\Gamma_k = \Lambda_0 \trianglelefteq \Lambda_1 \trianglelefteq \dots \trianglelefteq \Lambda_m = \Gamma$ such that $\Lambda_i/\Lambda_{i-1} \cong G_i'$. We have $\Lambda_1 \leq N_\Gamma(\Gamma_k)$ and therefore $\Lambda_1 = \Gamma_k^{(i,j)}$ for some $(i,j) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In consequence, $|G_1'| \leq 4$.

We can repeat the same process with $\Lambda_1 = \Gamma_k^{(i,j)}$, since $N_\Gamma(\Gamma_k^{(i,j)}) = \Gamma_{k-1}^{(i,j)}$. Repeating this process we find that all the subgroups of the subnormal series are of the form $\Gamma_a^{(i,j)}$. This implies that $|G_i'| \leq 4$ for all $1 \leq i \leq m$.

Finally, if $\mathcal{G}' \curvearrowright M$ satisfies that $m = l(\mathcal{G}_k \curvearrowright M)$ then $[\Gamma : \Gamma_k] = 4^k = \prod_{i=1}^m |G_i| \leq 4^m$, which implies that $l(\mathcal{G}_k \curvearrowright M) = k$. Since k can be chosen to be arbitrarily large, we obtain that $l(M) = \infty$. □

Remark 4.3.11. *The solvmanifold M of theorem 4.3.10 satisfies* disc-sym$(M) = 0$. *Indeed, since M is aspherical and Γ is polycyclic, we have* disc-sym$(M) \leq$ rank $\mathcal{Z}\Gamma$. *Since Γ is isomorphic to the semi-direct product $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, we can use lemma 3.3.10 to conclude that $\mathcal{Z}\Gamma$ is trivial. Consequently,* disc-sym$(M) \leq$ rank $\mathcal{Z}\Gamma = 0$ *and therefore* disc-sym$(M) = 0$.

We study now the iterated length of locally symmetric spaces.

Lemma 4.3.12. *Let $K \setminus G/\Gamma$ be a locally symmetric space where G is a connected semisimple Lie group without compact factors, K is a maximal compact subgroup and Γ is a lattice. Then $l(K \setminus G/\Gamma)$ is bounded by a constant C depending on Γ.*

Proof. Recall that if μ is the Haar measure of G then $\mathrm{vol}(G/\Gamma) = \mu(F)$, where F is a fundamental domain of Γ in G. By theorem 1.3.66, there exists a constant A such that $\mathrm{vol}(G/\Gamma) > A$ for all lattices of G. Moreover, if Γ' is another lattice containing Γ as a finite index subgroup then $\mathrm{vol}(G/\Gamma) = [\Gamma' : \Gamma]\,\mathrm{vol}(G/\Gamma')$.

Let $\mathcal{G} \curvearrowright K \setminus G/\Gamma$ be a free iterated action with $l(\mathcal{G}) = l(\mathcal{G} \curvearrowright K \setminus G/\Gamma) = l$. Then $(K \setminus G/\Gamma)/\mathcal{G} \cong K \setminus G/\Gamma'$ where Γ' is a lattice of G. Then

$$\mathrm{vol}(G/\Gamma) = [\Gamma' : \Gamma]\,\mathrm{vol}(G/\Gamma') = \prod_{i=1}^{l} |G_i|\,\mathrm{vol}(G/\Gamma') \geq 2^n A.$$

In consequence $l \leq \log_2(\frac{\mathrm{vol}(G/\Gamma)}{A})$. The proof is finished by taking $C = \log_2(\frac{\mathrm{vol}(G/\Gamma)}{A})$. □

The dependence on the lattice cannot be removed. In order to give an example we need the following result:

Proposition 4.3.13. *[BBS01, Proposition 2.3] Let $N_0 \longrightarrow N_1 \longrightarrow \cdots \longrightarrow N_s$ be a tower of coverings of closed 3-manifolds. There exists a tower of coverings $M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_s$ of closed hyperbolic 3-manifolds and maps $f_i : M_i \longrightarrow N_i$ such that* $\deg(f_i) = 1$ *and the covering $M_{i-1} \longrightarrow M_i$ is the pullback of $N_{i-1} \longrightarrow N_i$ by f_i for all i.*

Corollary 4.3.14. *Assume that we are in the setting of proposition 4.3.13, $\pi_1(M_i) \trianglelefteq \pi_1(M_j)$ if and only if $\pi_1(N_i) \trianglelefteq \pi_1(N_j)$ for all $i < j$.*

Now, we take $N_i = M$ the solvmanifold from theorem 4.3.10 for all i and the covering $N_{i-1} \longrightarrow N_i$ the self-covering of theorem 4.3.10. By proposition 4.3.13, for each action $\mathcal{G}_k = \{\mathbb{Z}/2 \oplus \mathbb{Z}/2, \cdots \mathbb{Z}/2 \oplus \mathbb{Z}/2\} \curvearrowright M$, there exists a closed hyperbolic 3-manifold $M_{0,k}$ (which depends on k) such that we have a free iterated action $\mathcal{G}_k \curvearrowright M_{0,k}$. By corollary 4.3.14, we have $l(\mathcal{G}_k \curvearrowright M_{0,k}) = k$. Since $M_{0,k}$ are hyperbolic manifolds for all k, the bound of lemma 4.3.12 needs to depend on the lattice and not only the Lie group (in this example $\mathrm{SO}^+(3,1)$).

Question 4.3.15. *Let $K \setminus G/\Gamma$ be a locally symmetric space where G is connected semisimple without compact factors and* $\mathrm{rank}_{\mathbb{R}}\, G \geq 2$*, K is a maximal compact subgroup and Γ is an irreducible lattice. Does there exist a constant C only depending on G such that $l(K \setminus G/\Gamma) \leq C$?*

The solvmanifold of theorem 4.3.10 can be used to construct other closed aspherical locally homogeneous space $K \setminus G/\Gamma$ such that $l(K \setminus G/\Gamma) = \infty$.

Proposition 4.3.16. *There exists a closed aspherical locally homogeneous space $K \setminus G/\Gamma$ such that the solvable radical of G is abelian and $l(K \setminus G/\Gamma) = \infty$.*

Proof. Let Λ be the fundamental group of a closed hyperbolic manifold of dimension $n \geq 3$ such that there exists an epimorphism $f : \Lambda \longrightarrow \mathbb{Z}$ and let $\phi : \mathbb{Z} \longrightarrow \mathrm{GL}(2, \mathbb{Z})$ be as in the proof of theorem 4.3.10. Then we can define $\Gamma_k = 2^k(\mathbb{Z}^2) \rtimes_{\phi \circ f} \Lambda$. Using the Seifert construction (see [LR10, Theorem 11.7.29]) we can construct a closed aspherical locally homogeneous space with fundamental group Γ_k for all k, which we denote by M_k.

The same arguments as in theorem 4.3.10 show that all Γ_k are isomorphic and since the Borel conjecture is true for lattices in connected Lie groups (see [BL12, KLR16]) we can conclude that $M_k \cong M$ for all k. The inclusion $\Gamma_k \longrightarrow \Gamma_{k-1}$ induces a regular self-covering $M_k \longrightarrow M_{k-1}$. Finally, the tower of self-coverings $M_k \longrightarrow M_{k-1} \longrightarrow \cdots \longrightarrow M_0$ induces a free iterated group action $\mathcal{G}_k = \{\mathbb{Z}/2 \oplus \mathbb{Z}/2, \cdots, \mathbb{Z}/2 \oplus \mathbb{Z}/2\} \curvearrowright M$ such that $l(\mathcal{G}_k \curvearrowright M) = k$. Thus $l(M) = \infty$. □

Question 4.3.17. *Let $K \setminus G/\Gamma$ be a closed aspherical locally homogeneous space where the solvable radical R of G is nilpotent and G/R is semisimple without compact factors and* $\mathrm{rank}_{\mathbb{R}}\, G/R \geq 2$*. Does there exist a constant C such that $l(K \setminus G/\Gamma) \leq C$?*

4.4 The iterated discrete degree of symmetry

Recall that if G is a finite group, then $\mathrm{rank}\, G$ is the minimum number of elements which are needed to generate G. We want to extend this notion to iterated group action.

Definition 4.4.1. *Given a free iterated action $\mathcal{G} \curvearrowright X$, the rank of the iterated action is*

$$\operatorname{rank}(\mathcal{G} \curvearrowright X) = \min\{\sum_{i=1}^{n} \operatorname{rank} G_i' : \mathcal{G}' = \{G_1', \dots, G_n'\} \curvearrowright X \in [\mathcal{G} \curvearrowright X]\}.$$

The iterated rank of the space X is

$$\operatorname{rank}(X) = \max\{\operatorname{rank}(\mathcal{G} \curvearrowright X) : \text{ free iterated action } \mathcal{G} \curvearrowright X\}.$$

Lemma 4.4.2. *Assume that we have a free iterated action of $\mathcal{G} = \{G_1, \dots, G_n\}$ on X. If $\mathcal{G} \curvearrowright X$ is simplifiable with a group G, then* $\operatorname{rank}(\mathcal{G} \curvearrowright X) = \operatorname{rank} G$.

Proof. By definition, $\operatorname{rank}(\mathcal{G} \curvearrowright X) \leq \operatorname{rank} G$. Since $\mathcal{G} \curvearrowright X$ is simplifiable there exists a subnormal series $G^0 = \{e\} \trianglelefteq G^1 \trianglelefteq \cdots \trianglelefteq G^n = G$ such that $G^i/G^{i-1} \cong G_i$. In particular, $\operatorname{rank} G^i \leq \operatorname{rank} G^{i-1} + \operatorname{rank} G_i$ for all $1 \leq i \leq n$. This implies that $\operatorname{rank} G = \operatorname{rank} G^n \leq \sum_{i=1}^{n} \operatorname{rank} G_i + \operatorname{rank} G^0 = \operatorname{rank}(\mathcal{G} \curvearrowright X)$. ☐

Note that $\operatorname{rank}(\mathcal{G} \curvearrowright X)$ depends on the free iterated group action, see remark 4.4.7 below.

We also note that $l(X) \leq \operatorname{rank}(X)$ and therefore we cannot bound the iterated rank of a closed manifold. To generalize the discrete degree of symmetry we will need a notion of rank which only uses abelian groups.

Definition 4.4.3. *Let $\mathcal{G} = \{G_1, \dots, G_n\}$ act freely on X and assume that G_i is solvable for all i. The abelian rank of the iterated action is*

$$\operatorname{rank}_{ab}(\mathcal{G} \curvearrowright X) = \min\{\sum_{i=1}^{m} \operatorname{rank} A_i' : \{A_1', \dots, A_m'\} \curvearrowright X \in [\mathcal{G} \curvearrowright X],\ A_i' \text{ abelian for all } i\}.$$

Note that $\operatorname{rank}(\mathcal{G} \curvearrowright X) \leq \operatorname{rank}_{ab}(\mathcal{G} \curvearrowright X)$. We would like to define an invariant of free iterated actions with similar properties to the discrete degree of symmetry. An essential property of the discrete degree of symmetry is that if M is a closed manifold, then disc-sym$(M) < \infty$. The proof of this fact is a direct consequence of theorem 1.1.32. We can generalize this theorem to the context of free iterated actions. Recall that given a closed manifold M of dimension n, we define $b(M) = \sum_{i=1}^{n} \operatorname{rank} H_i(M, \mathbb{Z})$, where rank is understood to be the minimum number of generators needed to generate $H_i(M, \mathbb{Z})$ (note that $b(M)$ is not the Betti number of $H_*(M, \mathbb{Z})$, since we also count the torsion part of $H_*(M, \mathbb{Z})$). If p is a prime number, then $b_p(M) = \sum_{i=1}^{n} \dim H_i(M, \mathbb{Z}/p)$.

Theorem 4.4.4. *Let M be a closed connected n-dimensional manifold. There exists a sequence of numbers $\{f_i\}_{i \in \mathbb{N}}$ depending only on n and $b(M)$ such that for any prime p and any free iterated action $\{(\mathbb{Z}/p^{k_i})^{a_i}\}_{i=1,\dots,r} \curvearrowright M$, where k_i are arbitrary positive integers, the numbers a_i satisfy $a_i \leq f_i$ for all i.*

Proof. We construct the sequence f_i recursively. For the first step, if $(\mathbb{Z}/p^{k_1})^{a_1}$ acts freely on M then so it does the group $(\mathbb{Z}/p)^{a_1}$. Thus, we can take $f_1 = f(n, b_p(M))$, where f is the function defined in theorem 1.1.32.

By lemma 1.1.36, if $\mathbb{Z}/p^{k_1}$ acts freely on M, then

$$\dim H_k(M/(\mathbb{Z}/p^{k_1}), \mathbb{Z}/p) \leq \sum_{i+j=k} \dim H_i(B\mathbb{Z}/p^{k_1}, \mathbb{Z}/p) \dim H_j(M, \mathbb{Z}/p)$$

for any k. Moreover, $H_i(B\mathbb{Z}/p^{k_1}, \mathbb{Z}/p) \cong \mathbb{Z}/p$ for all $i \geq 0$ (see [Bro12, Chapter III, §1]). Thus, $\dim H_k(M/(\mathbb{Z}/p^{k_1}), \mathbb{Z}/p) \leq b_p(M)$ for all k and therefore $b_p(M/(\mathbb{Z}/p^{k_1})) \leq nb_p(M)$. Using this inequality recursively, we obtain that $b_p(M/(\mathbb{Z}/p^{k_1})^{a_1}) \leq n^{a_1} b_p(M) \leq n^{f_1} b_p(M)$.

We can use theorem 1.1.32 on $M/(\mathbb{Z}/p^{k_1})^{a_1}$ to deduce that $a_2 \leq f(n, b_p(M/(\mathbb{Z}/p^{k_1})^{a_1})) \leq f(n, n^{f_1} b_p(M))$. We set $f_2 = f(n, n^{f_1} b_p(M))$. Repeating the same argument as above, we obtain $f_i = f(n, n^{f_1+\cdots+f_{i-1}} b_p(M))$ for all $1 \leq i \leq r$.

By the universal coefficients theorem, $b_p(M) \leq 2b(M)$. Thus, by replacing $b_p(M)$ with $2b(M)$ we obtain a bound not depending on the prime p. □

We are ready to define an iterated discrete degree of symmetry for free iterated actions of length 2. In $\mathbb{N}^2$ we have a partial order relation called the lexicographic order defined as follows: If $(n,m), (n',m') \in \mathbb{N}^2$ then $(n,m) \geq (n',m')$ if and only if $n > n'$ or $n = n'$ and $m \geq m'$.

Definition 4.4.5. *We define $\mu_2(M)$ as the set of all pairs $(f,b) \in \mathbb{N}^2$ which satisfy:*

1. *There exist an increasing sequence of prime numbers $\{p_i\}$, a sequence of natural numbers $\{a_i\}$ and a collection of free iterated actions $\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M$ for each $i \in \mathbb{N}$.*
2. $\mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M) = f + b$ *for each $i \in \mathbb{N}$.*

We define the iterated discrete degree of symmetry of M as

$$\text{disc-sym}_2(M) = \max\{(0,0) \cup \mu_2(M)\}.$$

To understand better the definition of the iterated discrete degree of symmetry, let us compute it for tori.

Lemma 4.4.6. *We have* $\text{disc-sym}_2(T^n) = (n, 0)$.

Proof. Let C be the Minkowski constant of $\mathrm{GL}(n,\mathbb{Z})$. Assume that $\text{disc-sym}_2(T^n) = (d_1, d_2)$. Since T^n admits actions of $(\mathbb{Z}/p)^n$ for any prime p and $\text{disc-sym}(T^n) = n$, we have $d_1 = n$. By hypothesis there exists an increasing sequence of prime numbers $\{p_i\}_{i\in\mathbb{N}}$

with $p_i > C$ and free iterated group actions of $\{(\mathbb{Z}/p_i^{a_i})^n, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright T^n$ such that $\mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^n, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright T^n) = n + d_2$.

The action of $(\mathbb{Z}/p_i^{a_i})^n$ on T^n is by rotations for all i, therefore $T^n/(\mathbb{Z}/p_i^{a_i})^n \cong T^n$. Since $p_i > C$ the group $(\mathbb{Z}/p_i)^{d_2}$ also acts by rotations on T^n, hence $T^n/(\{(\mathbb{Z}/p_i^{a_i})^n, (\mathbb{Z}/p_i)^{d_2}\})$ is homeomorphic to T^n. In consequence, the free iterated action $\{(\mathbb{Z}/p_i^{a_i})^n, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright T^n$ is simplifiable for all i. The simplification gives a group G_i and a short exact sequence

$$1 \longrightarrow (\mathbb{Z}/p_i^{a_i})^n \longrightarrow G_i \longrightarrow (\mathbb{Z}/p_i)^{d_2} \longrightarrow 1.$$

Moreover, we also have that a short exact sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow G_i \longrightarrow 1$$

induced by the regular covering $p : T^n \longrightarrow T^n/\{(\mathbb{Z}/p_i^{a_i})^n, (\mathbb{Z}/p_i)^{d_2}\}$. This implies that G_i is abelian and $\mathrm{rank}\, G_i \leq n$. On the other hand,

$$\mathrm{rank}\, G_i = \mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^n, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright T^n) = n + d_2 \leq n,$$

which implies that $d_2 = 0$. □

Remark 4.4.7. *As an example, we consider two different free iterated actions of $\{\mathbb{Z}/p, \mathbb{Z}/p\} \curvearrowright T^2$, as shown in the Figure 4.1.*

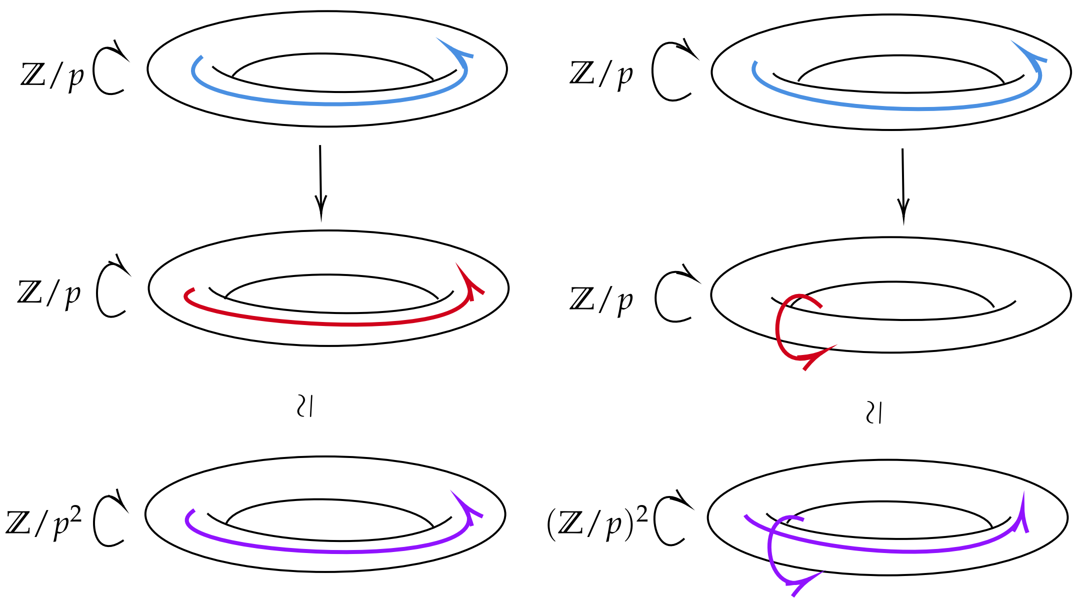


Figure 4.1: Free iterated actions on T^2

Let us describe explicitly these two free iterated actions. The first step of both free iterated actions is by rotations on the horizontal plane. More explicitly, given $g \in \mathbb{Z}/p$ and $[x, y] \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$, we have $g[x, y] = [x + \frac{g}{p}, y]$. The quotient is $T^2/(\mathbb{Z}/p) \cong T^2$. The second step of the first *iterated action is by rotations on the horizontal plane. The free iterated action is simplifiable by an action of* $\mathbb{Z}/p^2$ *on* T^2 *such that* $g[x,y] = [x + \frac{g}{p^2}, y]$ *for* $g \in \mathbb{Z}/p^2$ *and* $[x,y] \in T^2$. *Note that* $\mathrm{rank}_{ab}(\{\mathbb{Z}/p, \mathbb{Z}/p\} \curvearrowright T^2) = 1 < 2$. *Hence, this free iterated action is not considered in the definition of the iterated discrete degree of symmetry.*

The second step of the second iterated actions is by rotations such that $g[x,y] = [x, y + \frac{g}{p}]$ *for* $g \in \mathbb{Z}/p$ *and* $[x,y] \in T^2$. *The free iterated action is simplifiable by an action of* $(\mathbb{Z}/p)^2$ *on* T^2 *such that* $(g,h)[x,y] = [x + \frac{g}{p}, y + \frac{h}{p}]$ *for* $(g,h) \in (\mathbb{Z}/p)^2$ *and* $[x,y] \in T^2$. *In this case* $\mathrm{rank}_{ab}(\{\mathbb{Z}/p, \mathbb{Z}/p\} \curvearrowright T^2) = 2$, *but it does not satisfy the maximality condition on the definition of the iterated discrete degree of symmetry, since we have a free iterated action* $\{(\mathbb{Z}/p)^2, \{e\}\} \curvearrowright T^2$.

Corollary 4.4.8. *Let* M *be a closed manifold. There exists* $(f,b) \in \mathbb{N}^2$ *such that* $\text{disc-sym}_2(M) \leq (f,b)$.

Proof. We can choose $f = f_1$ and $b = f_2$ from theorem 4.4.4. □

Note that if $\text{disc-sym}_2(M) = (d_1, d_2)$ then $d_1 \leq \text{disc-sym}(M)$. We want to investigate the value d_2.

Remark 4.4.9. *If* M *is a manifold such that* $\text{disc-sym}(M) = 0$ *then* $\text{disc-sym}_2(M) = (0,0)$.

The next lemma shows a case where $d_2 = 0$, so we do not have large iterated group actions.

Lemma 4.4.10. *Let* M *be a closed manifold. Assume that* $l(M) = 1$ *and* $\mathrm{Homeo}(M)$ *is Jordan. Then* $\text{disc-sym}_2(M) = (d, 0)$.

Proof. Assume that C is the Jordan constant of $\mathrm{Homeo}(M)$ and that $\text{disc-sym}_2(M) = (d_1, d_2)$. We have an increasing sequence of prime numbers $\{p_i\}_{i \in \mathbb{N}}$ and free iterated group actions of $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\}$ on M such that $\mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M) = d_1 + d_2$. We may assume without loss of generality that $p_i > C$. Since all actions are simplifiable, for each i there exists a p_i-group G_i acting freely on M which fits in the short exact sequence

$$1 \longrightarrow (\mathbb{Z}/p_i^{a_i})^{d_1} \longrightarrow G_i \longrightarrow (\mathbb{Z}/p_i)^{d_2} \longrightarrow 1.$$

Any proper subgroup H of G_i has index $[G : H] > p_i > C$. Since $\mathrm{Homeo}(M)$ is Jordan of constant C, we can conclude that G_i is abelian. Consequently, we have $\mathrm{rank}\, G_i = \mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M) = d_1 + d_2$. Therefore, $G_i \cong (\mathbb{Z}/p_i^{a_i})^{d_1} \oplus (\mathbb{Z}/p_i)^{d_2}$. We can take a subgroup $(\mathbb{Z}/p_i)^{d_1+d_2} \leq G_i$, which acts freely on M for all i. This implies that $(d_1 + d_2, 0) \leq (d_1, d_2)$ and since $d_1, d_2 \geq 0$, $d_2 = 0$. □

Proposition 4.4.11. *Let* $p : M' \longrightarrow M$ *be a regular finite covering. Then* $\text{disc-sym}_2(M') \geq \text{disc-sym}_2(M)$.

Proof. Firstly, we need to prove the following group theoretic fact. Let G be a finite group and p a prime such that $p > |G|!$. If G' is a group which fits in the short exact sequence $1 \longrightarrow G \longrightarrow G' \longrightarrow (\mathbb{Z}/p^r)^b \longrightarrow 1$ then $G' \cong G \times (\mathbb{Z}/p^r)^b$.

The abstract kernel $\psi : (\mathbb{Z}/p^r)^b \longrightarrow \operatorname{Out}(G)$ is trivial since $|\operatorname{Out}(G)| \leq |G|! < p$. This leads to the following diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathcal{Z}G & \longrightarrow & H & \longrightarrow & (\mathbb{Z}/p^r)^b & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Id} & & \\
1 & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & (\mathbb{Z}/p^r)^b & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Inn} G & \xrightarrow{Id} & \operatorname{Inn} G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

where the above short exact sequence is central. Since $(\mathbb{Z}/p^r)^b$ is abelian, there is a cocycle $\beta : (\mathbb{Z}/p^r)^b \times (\mathbb{Z}/p^r)^b \longrightarrow \mathcal{Z}G$ which is a group morphism. More explicitly, $\beta(x, y) = [\tilde{x}, \tilde{y}]$, where $\tilde{x}, \tilde{y} \in H$ are preimages of x and y respectively. We use that $|\mathcal{Z}G| < p$ to conclude that β is trivial. This implies that the central short exact sequence is trivial and $H \cong \mathcal{Z}G \times (\mathbb{Z}/p^r)^b$.

We can construct a group morphism $(\mathbb{Z}/p^r)^b \longrightarrow H \longrightarrow G$ which makes the short exact sequence split. Therefore $G' \cong G \rtimes_\phi (\mathbb{Z}/p^r)^b$. The group morphism $\phi : (\mathbb{Z}/p^r)^b \longrightarrow \operatorname{Aut}(G)$ is trivial since $|\operatorname{Aut}(G)| \leq |G|! < p$. In conclusion, $G' \cong G \times (\mathbb{Z}/p^r)^b$.

Assume now that we have a regular G-covering $p : M' \longrightarrow M$ and $\operatorname{disc-sym}_2(M) = (d_1, d_2)$. There is an increasing sequence of prime numbers $\{p_i\}_{i \in \mathbb{N}}$ such that $p_i > |G|!$ for all i and free iterated group actions of $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M$. They induce free iterated actions of $\{G, (\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M'$.

Since $p_i > |G|!$, the actions of $(\mathbb{Z}/p_i^{a_i})^{d_1}$ on $\operatorname{Cov}_{|G|}(M)$ are trivial. By lemma 4.2.8 the free iterated action $\{G, (\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M'$ is equivalent to a free iterated action of $\{G'_i, (\mathbb{Z}/p_i^{k_2})^{d_2}\} \curvearrowright M'$ for all i. For each i there exists a short exact sequence

$$1 \longrightarrow G \longrightarrow G'_i \longrightarrow (\mathbb{Z}/p_i^{a_i})^{d_1} \longrightarrow 1.$$

Since $p_i > |G|!$, $G'_i \cong G \times (\mathbb{Z}/p_i^{a_i})^{d_1}$. Thus, $\{G'_i, (\mathbb{Z}/p_i^{k_2})^{d_2}\} \curvearrowright M'$ is equivalent to $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, G, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M'$ for all i. Repeating the same argument we can show that $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, G, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M'$ is equivalent to $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}, G\} \curvearrowright M'$ for all i.

In conclusion, we have free iterated actions $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M'$ for all i, thus $\text{disc-sym}_2(M) = (d_1, d_2) \leq \text{disc-sym}_2(M')$. $\square$

We start now the study of the iterated discrete degree of symmetry on nilmanifolds. We start with 2 preliminary lemmas, which will also be used in section 4.5 and section 4.6.

Lemma 4.4.12. *[LR10, Proposition 3.1.21] Let M be a closed aspherical manifold and let $\mathcal{Z}\pi_1(M)$ be finitely generated. Assume that we have a free action of an abelian group A on M such that $\psi : A \longrightarrow \text{Out}(\pi_1(M))$ is trivial. Then $\mathcal{Z}\pi_1(M/A) = C_{\pi_1(M/A)}(\pi_1(M))$ and it is an extension of $\mathcal{Z}\pi_1(M)$ by A. In particular,* $\text{rank}(\mathcal{Z}\pi_1(M/A)) = \text{rank}(\mathcal{Z}\pi_1(M))$.

Lemma 4.4.13. *Let $\{A, A'\} \curvearrowright M$ be a free iterated action of abelian groups on a closed connected aspherical manifold such that $\psi : A \longrightarrow \text{Out}(\pi_1(M))$ and $\psi' : A' \longrightarrow \text{Out}(\pi_1(M/A))$ are trivial. Then $\{A, A'\} \curvearrowright M$ is simplifiable by an abelian group.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z}\pi_1(M) & \longrightarrow & \mathcal{Z}\pi_1(M/A) & \longrightarrow & \mathcal{Z}\pi_1((M/A)/A') \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(M) & \longrightarrow & \pi_1(M/A) & \longrightarrow & \pi_1((M/A)/A') \\
\downarrow p & & \downarrow q & & \downarrow q' \\
\text{Inn}\,\pi_1(M) & \xrightarrow{Id} & \text{Inn}\,\pi_1(M) & \xrightarrow{Id} & \text{Inn}\,\pi_1(M) \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}$$

Given $\gamma \in \pi_1(M)$ and $g' \in \pi_1((M/A)/A')$, we want to see that $g'\gamma g'^{-1} \in \pi_1(M)$. Let $\gamma' \in \pi_1(M)$ such that $p(\gamma') = q'(g')$, then $g'\gamma g'^{-1} = (g'\gamma'^{-1})\gamma''(g'\gamma'^{-1})^{-1}$ with $\gamma'' \in \pi_1(M)$. Since $q'(g'\gamma'^{-1})$ is trivial then $g'\gamma'^{-1} \in \mathcal{Z}\pi_1((M/A)/A')$ and therefore $g'\gamma g'^{-1} = \gamma'' \in \pi_1(M)$, as we wanted to see.

Note that $\pi_1((M/A)/A')/\pi_1(M) \cong \mathcal{Z}\pi_1((M/A)/A')/\mathcal{Z}\pi_1(M)$, which is abelian of rank at most rank $\mathcal{Z}\pi_1(M)$. $\square$

The next result shows that the definition of the iterated degree of symmetry is suitable to study 2-step nilmanifolds.

Theorem 4.4.14. *Let N/Γ be a nilmanifold of dimension n, $f = \text{rank}\,\mathcal{Z}\Gamma$ and $\text{disc-sym}_2(N/\Gamma) = (d_1, d_2)$. Then $d_1 = f$ and $d_2 \leq n - f$. If $d_2 = n - f$ then N/Γ is a 2-step nilmanifold. Conversely, if N/Γ is a 2-step nilmanifold then* $\text{disc-sym}_2(N/\Gamma) = (f, n - f)$.

Proof. Let $\{p_i\}_{i\in\mathbb{N}}$ be the increasing sequence of primes and $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright N/\Gamma$ be the free iterated actions provided by the condition $\text{disc-sym}_2(N/\Gamma) = (d_1, d_2)$. Let C be the Minkowski constant of $\text{Out}(\Gamma)$. Assume that all $p_i > C$ and hence the group morphism $\psi_i : (\mathbb{Z}/p_i^{a_i})^{d_1} \longrightarrow \text{Out}(\Gamma)$ is trivial for all i. By lemma 1.3.95 and lemma 4.4.12, each action induces a commutative diagram of fundamental groups

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathcal{Z}\Gamma \cong \mathbb{Z}^f & \longrightarrow & \mathcal{Z}\Gamma_i \cong \mathbb{Z}^f & \longrightarrow & (\mathbb{Z}/p_i^{a_i})^f & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Id} & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Gamma_i & \longrightarrow & (\mathbb{Z}/p_i^{a_i})^f & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Gamma' & \xrightarrow{Id} & \Gamma' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where Γ_i is the fundamental group of the nilmanifold $(N/\Gamma)/(\mathbb{Z}/p_i^{a_i})^f$ and the identity map of the third row is induced by the inclusion $\text{Inn}(\Gamma) \longrightarrow \text{Aut}(\Gamma)$. Moreover, notice that $\text{rank}\, \mathcal{Z}\Gamma_i = f$ for all i.

We claim that the group morphism $\psi_i' : (\mathbb{Z}/p_i)^{d_2} \longrightarrow \text{Out}(\Gamma_i)$ is injective for big enough i. If not, there exists $\mathbb{Z}/p_i \leq \text{Ker}\, \psi_i'$ such that the action $\mathbb{Z}/p_i \curvearrowright N/\Gamma_i$ is inner. The first step of the iterated action is also inner. Thus, by lemma 4.4.13, the free iterated action $\{(\mathbb{Z}/p_i^{a_i})^f, \mathbb{Z}/p_i\} \curvearrowright N/\Gamma$ is equivalent to a free action of an abelian p_i-group G_i of $\text{rank}\, G_i = f$. The free iterated action $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright N/\Gamma$ is equivalent to $\{G_i, (\mathbb{Z}/p_i)^{d_2-1}\} \curvearrowright N/\Gamma$, which contradicts the fact that $\text{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright N/\Gamma) = f + d_2$.

In consequence, we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^f & \xrightarrow{Id} & \mathbb{Z}^f & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma_i & \longrightarrow & \Gamma_{i_2} & \longrightarrow & (\mathbb{Z}/p_i)^{d_2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \phi & & \\
1 & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'_i & \longrightarrow & (\mathbb{Z}/p_i)^{d_2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where Γ_{i_2} is the fundamental group of the quotient $(N/\Gamma)/\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\}$. We claim that Γ'_i is torsion-free for i large enough. Firstly, $\Gamma'_i \leq \operatorname{Aut}(\Gamma_i)$. For each i we have an injective morphism $\operatorname{Aut}(\Gamma_i) \longrightarrow \operatorname{Aut}(\Gamma_{\mathbb{Q}})$, since all lattices $\Gamma_i \leq N$ are commensurable to Γ. Moreover, $\operatorname{Aut}(\Gamma_{\mathbb{Q}}) = \operatorname{Aut}(\mathcal{L}(\Gamma_{\mathbb{Q}})) \leq \operatorname{GL}(m, \mathbb{Q})$ for some m, where $\mathcal{L}(\Gamma_{\mathbb{Q}})$ denotes the rational Lie algebra of $\Gamma_{\mathbb{Q}}$. Since any finite subgroup of $\operatorname{GL}(m, \mathbb{Q})$ is conjugated to a finite subgroup of $\operatorname{GL}(m, \mathbb{Z})$, we can conclude that each Γ'_i is Minkowski with a constant C' which does not depend on i. Finally, since Γ' is torsion-free the torsion of Γ'_i injects in $(\mathbb{Z}/p_i)^{d_2}$. Thus, Γ'_i is torsion-free if $p_i > C$.

Since $\operatorname{Out}(\Gamma')$ is Minkowski with a constant not depending on i, we have a commutative diagram for i large enough.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathcal{Z}\Gamma' & \longrightarrow & C_{\Gamma'_i}(\Gamma') & \longrightarrow & (\mathbb{Z}/p_i)^{d_2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow Id & & \\
1 & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'_i & \longrightarrow & (\mathbb{Z}/p_i)^{d_2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Inn}\Gamma' & \xrightarrow{Id} & \operatorname{Inn}\Gamma' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Since Γ'_i is torsion-free, the group $C_{\Gamma'_i}(\Gamma')$ is torsion-free. The first row of the diagram is central, hence $C_{\Gamma'_i}(\Gamma')$ is abelian and $d_2 \leq \operatorname{rank} \mathcal{Z}\Gamma'$. Furthermore, Γ' is the fundamental group of the nilmanifold N'/Γ', obtained as the orbit space of the free action of $\mathcal{Z}N/\mathcal{Z}\Gamma \cong T^f$ on N/Γ. Therefore, $\operatorname{rank} \mathcal{Z}\Gamma' \leq \dim(N'/\Gamma') = n - f$. In conclusion, $d_2 \leq n - f$. If $d_2 = n - f$ then $N'/\Gamma' \cong T^{n-f}$ and N/Γ is a 2-step nilmanifold with $\operatorname{rank} \mathcal{Z}\Gamma = f$, as we wanted to see.

Let us now prove the converse implication. Suppose that N/Γ is a 2-step nilmanifold with $f = \operatorname{rank} \mathcal{Z}\Gamma$ and let $b = n - f$. Then $\Gamma \cong \mathbb{Z}^f \times_c \mathbb{Z}^b$, where $c : \mathbb{Z}^b \times \mathbb{Z}^b \longrightarrow \mathbb{Z}^f$ is a 2-cocycle representing a cohomology class $[c] \in H^2(\mathbb{Z}^b, \mathbb{Z}^f) = H^2(T^b, \mathbb{Z}^f)$ which determines the principal T^f-bundle $\pi : N/\Gamma \longrightarrow T^b$. The cocycle can be chosen to be of the form

$$c((x_1, \dots, x_b), (y_1, \dots, y_b)) = (\sum_{1 \leq i < j \leq b} c^1_{i,j} x_i y_j, \dots, \sum_{1 \leq i < j \leq b} c^f_{i,j} x_i y_j)$$

where $c^k_{i,j} \in \mathbb{Z}$ for all $1 \leq i, j \leq f$ and $1 \leq k \leq b$.

For any prime p we can define a free iterated action $\{(\mathbb{Z}/p^2)^f, (\mathbb{Z}/p)^b\} \curvearrowright N/\Gamma$. The first action is by right multiplication on the fiber T^f. The orbit space is a nilmanifold with fundamental group $\Gamma_p \cong (\frac{1}{p^2}\mathbb{Z})^f \times_c \mathbb{Z}^b$, where $c : \mathbb{Z}^b \times \mathbb{Z}^b \longrightarrow \mathbb{Z}^f \subseteq (\frac{1}{p^2}\mathbb{Z})^f$. The second iterated is by rotations on the torus of the basis T^b. More explicitly, we define the lattice $\Gamma'_p \cong (\frac{1}{p^2}\mathbb{Z})^f \times_{c_p} (\frac{1}{p}\mathbb{Z})^b$, where $c_p : (\frac{1}{p}\mathbb{Z})^b \times (\frac{1}{p}\mathbb{Z})^b \longrightarrow \mathbb{Z}^f \subseteq (\frac{1}{p^2}\mathbb{Z})^f$ is of the form

$$c((\tfrac{1}{p}x_1, \dots, \tfrac{1}{p}x_b), (\tfrac{1}{p}y_1, \dots, \tfrac{1}{p}y_b)) = (\sum_{1 \leq i < j \leq b} \frac{c^1_{i,j}}{p^2} x_i y_j, \dots, \sum_{1 \leq i < j \leq b} \frac{c^f_{i,j}}{p^2} x_i y_j).$$

We have that $\Gamma_p \trianglelefteq \Gamma'_p$ and $\Gamma'_p/\Gamma_p \cong (\mathbb{Z}/p)^b$, which defines a free action of $(\mathbb{Z}/p)^b$ on N/Γ_p.

Moreover, $\operatorname{rank}_{ab}(\{(\mathbb{Z}/p^2)^f, (\mathbb{Z}/p)^b\} \curvearrowright N/\Gamma) = n$, therefore $(f, b) \leq \text{disc-sym}_2(N/\Gamma)$. But the first part of the theorem implies that $\text{disc-sym}_2(N/\Gamma) \leq (f, b)$. In consequence $\text{disc-sym}_2(N/\Gamma) = (f, b)$. □

4.5 Free iterated actions on closed aspherical 3-dimensional manifolds

If M is a closed 3-dimensional aspherical manifold with an effective S^1 action, then M can be one of the following four cases (see [LR10, §14.4]):

1. $M \cong T^3$.
2. M is homeomorphic to $K \times S^1$ or SK, where K denotes the Klein bottle and SK the non-trivial principal S^1-bundle over K.
3. $M \cong H/\Gamma$, where H is the 3-dimensional Heisenberg group and Γ is a lattice of H (see example 1.3.26)

4. $\mathcal{Z}\pi_1(M) \cong \mathbb{Z}$, and $\operatorname{Inn}\pi_1(M) \cong \pi_1(M)/\mathcal{Z}\pi_1(M)$ is centreless.

Note that in all cases, we have a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow Q \longrightarrow 1$$

where Q acts effectively, properly and cocompactly on $\mathbb{R}^2$. In section 2.6 we computed the toral degree of symmetry in these four cases. We proved that tor-sym$(T^3) = 3$, tor-sym$(K \times S^1) =$ tor-sym$(SK) = 2$, tor-sym$(H/\Gamma) = 1$, and tor-sym$(M) = 1$ if M belongs to the case 4. Moreover, if M is a closed connected aspherical 3-manifold then $\operatorname{Out}(\pi_1(M))$ is Minkowski by [Koj84] and tor-sym$(M) =$ disc-sym$(M) = \operatorname{rank}\mathcal{Z}\pi_1(M)$ by [Gab92, Corollary 8.3] and [CJ94, Theorem 1.1]. Thus, if M is a closed connected aspherical 3-manifold which does not belong to one of the four cases above, then tor-sym$(M) = \operatorname{rank}\mathcal{Z}\pi_1(M) = 0$ and hence disc-sym$(M) = 0$.

We will compute the iterated discrete degree of symmetry and show that it can be used to distinguish the four cases of the classification (see theorem 4.0.15). We start by providing a different proof that $\operatorname{Out}(\pi_1(M))$ is Minkowski if M belongs to one of these four cases.

Lemma 4.5.1. *Let M be a closed 3-dimensional aspherical manifold with an effective S^1-action. Then $\mathcal{Z}\pi_1(M)$ is finitely generated and $\operatorname{Out}(\pi_1(M))$ is Minkowski.*

Proof. In the first three cases the fundamental group is polycyclic and therefore $\operatorname{Out}(\pi_1(M))$ is Minkowski (see theorem 2.2.1). We only need to check the case where we have a short exact sequence $1 \longrightarrow \mathcal{Z}\pi_1(M) \cong \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \operatorname{Inn}(\pi_1(M)) \longrightarrow 1$ and $\operatorname{Inn}(\pi_1(M))$ is centreless. Since $\mathcal{Z}\pi_1(M)$ is a characteristic subgroup of $\pi_1(M)$ then there are short exact sequences

$$1 \longrightarrow K \longrightarrow \operatorname{Out}(\pi_1(M)) \longrightarrow \operatorname{Out}(\operatorname{Inn}\pi_1(M)) \longrightarrow 1$$

and

$$1 \longrightarrow \overline{H}^1(\operatorname{Inn}\pi_1(M), \mathbb{Z}) \longrightarrow K \longrightarrow \operatorname{GL}(1, \mathbb{Z}) \longrightarrow 1.$$

Recall that $\overline{H}^1(\operatorname{Inn}\pi_1(M), \mathbb{Z}) \cong Z^1(\operatorname{Inn}\pi_1(M), \mathbb{Z})/\overline{B}^1(\operatorname{Inn}\pi_1(M), \mathbb{Z})$ and

$$\overline{B}^1(\operatorname{Inn}\pi_1(M), \mathbb{Z}) \cong p^{-1}(\mathcal{Z}\operatorname{Inn}\pi_1(M) \cap C_{\pi_1(M)}(\mathcal{Z}\pi_1(M)))/\mathcal{Z}\pi_1(M)$$

where $p : \pi_1(M) \longrightarrow \operatorname{Inn}\pi_1(M)$ is the quotient map. Since $\mathcal{Z}\operatorname{Inn}\pi_1(M)$ is trivial, the group $\overline{B}^1(\operatorname{Inn}\pi_1(M), \mathbb{Z})$ is also trivial and

$$\overline{H}^1(\operatorname{Inn}\pi_1(M), \mathbb{Z}) \cong Z^1(\operatorname{Inn}\pi_1(M), \mathbb{Z}) \cong H^1(\operatorname{Inn}\pi_1(M), \mathbb{Z}).$$

The group $\operatorname{Out}(\pi_1(M))$ will be Minkowski if $\operatorname{Out}(\operatorname{Inn}\pi_1(M))$ and $H^1(\operatorname{Inn}\pi_1(M), \mathbb{Z})$ are Minkowski. Since $\operatorname{Inn}\pi_1(M)$ acts effectively, properly and cocompactly on $\mathbb{R}^2$, $\operatorname{Inn}\pi_1(M)$ is a subgroup of isometries of the Euclidean plane or the hyperbolic plane. In both cases $H^1(\operatorname{Inn}\pi_1(M), \mathbb{Z})$ is a finitely generated abelian group and therefore Minkowski.

If $\operatorname{Inn}\pi_1(M)$ is a subgroup of isometries of the Euclidean plane then it is virtually abelian and therefore $\operatorname{Out}(\operatorname{Inn}\pi_1(M))$ is Minkowski. If $\operatorname{Inn}\pi_1(M)$ is a subgroup of isometries of the hyperbolic plane, then it contains a centreless torsion-free Fuchsian subgroup Q of finite index. Since $\operatorname{Out}(Q)$ is virtually torsion-free, $\operatorname{Out}(\operatorname{Inn}\pi_1(M))$ is also virtually torsion-free (see [MS06, Lemma 2.4, Corollary 2.6]) and therefore Minkowski. □

The following is theorem 4.0.15.

Theorem 4.5.2. *Let M be a 3-dimensional closed connected aspherical manifold. Suppose that there exists an effective S^1 action on M. Then:*

1. $\text{disc-sym}_2(M) = (3,0)$ *if and only if* $M \cong T^3$.
2. $\text{disc-sym}_2(M) = (2,0)$ *if and only if* $M \cong K \times S^1$ *or* $M \cong SK$.
3. $\text{disc-sym}_2(M) = (1,2)$ *if and only if* $M \cong H/\Gamma$.
4. $\text{disc-sym}_2(M) = (1,0)$ *if and only if* $\mathcal{Z}\pi_1(M) \cong \mathbb{Z}$ *and* $\operatorname{Inn}\pi_1(M)$ *is centreless.*

The proof of theorem 4.5.2 is mainly a combination of lemma 4.4.13, lemma 4.4.12 and the following lemma.

Lemma 4.5.3. *Let M be a closed aspherical manifold. Assume that $\mathcal{Z}\pi_1(M)$ is finitely generated, that $\operatorname{Out}(\pi_1(M))$ is Minkowski and that there exists a constant C satisfying that for every free inner action of an abelian group A on M, $\operatorname{Out}(\pi_1(M/A))$ is Minkowski with a constant less or equal than C. Then* $\text{disc-sym}_2(M) = (d_1, 0)$.

Proof. Assume that $\text{disc-sym}_2(M) = (d_1, d_2)$. We have an increasing sequence of primes $\{p_i\}_{i\in\mathbb{N}}$, a sequence of positive integers $\{a_i\}_{i\in\mathbb{N}}$ and a collection of free iterated actions $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M$ satisfying that

$$\operatorname{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M) = (d_1, d_2).$$

We can assume that $p_i > C$ for all i. In this case, $\psi_i : (\mathbb{Z}/p_i^{a_i})^{d_1} \longrightarrow \operatorname{Out}(\pi_1(M))$ and $\psi_i' : (\mathbb{Z}/p_i)^{d_2} \longrightarrow \operatorname{Out}(\pi_1(M))$ are trivial for all i. By lemma 4.4.13, this iterated action is simplifiable, which implies that there exists a free group action of $(\mathbb{Z}/p_i)^{d_1+d_2}$ on M. But then $d_1 + d_2 \leq d_1$, which implies that $d_2 = 0$. □

From lemma 4.5.3 we can deduce the following corollary, which proves part (2) of theorem 4.5.2.

Corollary 4.5.4. *Let M be a closed flat manifold. Then* $\text{disc-sym}_2(M) = (\operatorname{rank}\mathcal{Z}\pi_1(M), 0)$.

Proof. Given any abelian group A acting freely on a n-dimensional closed flat manifold M such that $\psi : A \longrightarrow \operatorname{Out}(\pi_1(M))$, the quotient space M/A is a closed flat manifold of the same dimension and hence $\pi_1(M/A)$ is a Bieberbach group. Because there is a finite number of isomorphism classes of Bieberbach groups in each dimension and their outer automorphisms group is Minkowski, we can use lemma 4.5.3 by taking C to be the maximum of all Minkowski constants of the outer automorphism group of Bieberbach groups of flat manifolds of dimension n. □

Proof of theorem 4.5.2. We have already proved that $\text{disc-sym}_2(M) = (3,0)$ if $M \cong T^3$ (see lemma 4.4.6) and that $\text{disc-sym}_2(M) = (1,2)$ if $M \cong H/\Gamma$ (see theorem 4.4.14). Moreover, SK and $K \times S^1$ are closed flat manifolds, which implies that $\text{disc-sym}_2(K \times S^1) = \text{disc-sym}_2(SK) = (2,0)$ (see corollary 4.5.4). Therefore, we only need to check the last case.

Let M be a 3-aspherical manifold of the case 4 and assume that the Minkowski constant of $\text{Out}(\pi_1(M))$ is C. Since $\text{disc-sym}(M) = \text{tor-sym}(M) = \text{rank}\, \mathcal{Z}\pi_1(M) = 1$, there exist an increasing sequence of primes $\{p_i\}_{i\in\mathbb{N}}$ and a sequence of positive integers $\{a_i\}_{i\in\mathbb{N}}$ such that $\mathbb{Z}/p_i^{a_i}$ acts freely on M and $p_i > C$ for all i. Therefore each action induces the trivial group morphism $\psi_i : \mathbb{Z}/p_i^{a_i} \longrightarrow \text{Out}(\pi_1(M))$ and a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathcal{Z}\pi_1(M) \cong \mathbb{Z} & \longrightarrow & C_{G_i}(\pi_1(M)) = \mathcal{Z}G_i & \longrightarrow & \mathbb{Z}/p_i^{a_i} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Id} & & \\
1 & \longrightarrow & \pi_1(M) & \longrightarrow & G_i & \longrightarrow & \mathbb{Z}/p_i^{a_i} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \text{Inn}\,\pi_1(M) & \xrightarrow{Id} & \text{Inn}\,\pi_1(M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where $G_i = \pi_1(M/(\mathbb{Z}/p_i^{a_i}))$. Note that $\text{Inn}\,\pi_1(M)$ is centreless.

We claim that $\text{Out}(G_i)$ is Minkowski with a constant that does not depend on i. Since $\mathcal{Z}G_i$ is a characteristic subgroup of G_i and $\text{Inn}\,\pi_1(M)$ is centreless there exist two short exact sequence (see lemma 4.5.1)

$$1 \longrightarrow K \longrightarrow \text{Out}(G_i) \longrightarrow \text{Out}(\text{Inn}\,\pi_1) \longrightarrow 1$$

and

$$1 \longrightarrow H^1(\text{Inn}\,\pi_1(M), \mathbb{Z}) \longrightarrow K \longrightarrow \text{GL}(1, \mathbb{Z}) \longrightarrow 1.$$

The Minkowski constant of the group $\text{Out}(G_i)$ can be bounded by the Minkowski constants of $\text{Out}(\text{Inn}\,\pi_1(M))$, $\text{GL}(1, \mathbb{Z})$ and $H^1(\text{Inn}\,\pi_1(M), \mathbb{Z})$, which do not depend on p_i. Thus, we can use lemma 4.5.3 to conclude that $\text{disc-sym}_2(M) = (1,0)$. □

Remark 4.5.5. *The key observation of the proof of theorem 4.5.2 is that if* $\operatorname{Inn}\pi_1(M)$ *is centreless then* $\bar{H}^1(\operatorname{Inn}\pi_1(M),\mathbb{Z}) = H^1(\operatorname{Inn}\pi_1(M),\mathbb{Z})$, *which does not depend on the action of* $(\mathbb{Z}/p_i^{a_i})$ *on M. In general,* $\bar{H}^1(\operatorname{Inn}\pi_1(M),\mathbb{Z})$ *does depend on the action of* $(\mathbb{Z}/p_i^{a_i})$ *on M.*

4.6 Free iterated actions on closed aspherical manifolds

The aim of this section is to prove theorem 4.0.13. We will use the following definition:

Definition 4.6.1. *If* Γ *is a finitely generated group satisfying that* $\mathcal{Z}\Gamma$ *and* $\mathcal{Z}\operatorname{Inn}\Gamma$ *are finitely generated and that* $\operatorname{Out}(\operatorname{Inn}\Gamma)$ *and* $\operatorname{Aut}(\operatorname{Inn}\Gamma)$ *are Minkowski then we say that* Γ *has the 2-step Minkowski property.*

Remark 4.6.2. *Assume that* Γ *is 2-step Minkowski. Then* $\mathcal{Z}\Gamma$ *and* $\operatorname{Inn}\Gamma$ *are finitely generated and therefore the group of closed 1-cocycles* $Z^1(\operatorname{Inn}\Gamma,\mathcal{Z}\Gamma)$ *is a finitely generated abelian group, and hence* $Z^1(\operatorname{Inn}\Gamma,\mathcal{Z}\Gamma)$ *is Minkowski.*

Lemma 4.6.3. *Assume that* Γ *is torsion-free and 2-step Minkowski. Then* $\operatorname{Out}(\Gamma)$ *and* $\operatorname{Aut}(\Gamma)$ *are Minkowski.*

Proof. We consider the central short exact sequence $1 \longrightarrow \mathcal{Z}\Gamma \longrightarrow \Gamma \longrightarrow \operatorname{Inn}\Gamma \longrightarrow 1$. We know that $\operatorname{Out}(\mathcal{Z}\Gamma) = \operatorname{Aut}(\mathcal{Z}\Gamma) = \mathrm{GL}(n,\mathbb{Z})$ for some n, hence $\operatorname{Out}(\mathcal{Z}\Gamma)$ is Minkowski. Moreover, $H^1(\operatorname{Inn}\Gamma,\mathcal{Z}\Gamma)$ is Minkowski because $\operatorname{Inn}\Gamma$ and $\mathcal{Z}\Gamma$ are finitely generated, and $\operatorname{Out}(\operatorname{Inn}\Gamma)$ is Minkowski by hypothesis. By theorem 1.2.21 and lemma 1.1.54, we can conclude that $\operatorname{Out}(\Gamma)$ is Minkowski.

The group $Z^1(\operatorname{Inn}\Gamma,\mathcal{Z}\Gamma)$ is Minkowski by remark 4.6.2 and $\operatorname{Aut}(\operatorname{Inn}\Gamma)$ is Minkowski by hypothesis. Therefore $\operatorname{Aut}(\Gamma)$ is Minkowski by theorem 1.2.22 and lemma 1.1.54. □

Remark 4.6.4. *Note that if* Γ *is a centreless finitely generated group, then* $\operatorname{Inn}\Gamma \cong \Gamma$. *Therefore* Γ *is 2-step Minkowski if and only if* $\operatorname{Out}(\Gamma)$ *and* $\operatorname{Aut}(\Gamma)$ *are Minkowski.*

Proposition 4.6.5. *Let* Γ *be lattice of a connected Lie group, then* Γ *has the 2-step Minkowski property.*

Proof. Recall that there is a short exact sequence $1 \longrightarrow \Gamma_A \longrightarrow \Gamma \longrightarrow \Gamma_{nc} \longrightarrow 1$ where Γ_A is virtually polycyclic and Γ_{nc} is a centreless lattice in a semisimple Lie group (see section 2.4). In consequence, $\mathcal{Z}\Gamma \trianglelefteq \Gamma_A \trianglelefteq \Gamma$. Thus, we have a short exact sequence $1 \longrightarrow \Gamma_A/\mathcal{Z}\Gamma \longrightarrow \operatorname{Inn}\Gamma \longrightarrow \Gamma_{nc} \longrightarrow 1$. Moreover $\Gamma_A/\mathcal{Z}\Gamma$ is virtually polycyclic.

Since Γ_{nc} is centreless, the center $\mathcal{Z}\operatorname{Inn}\Gamma$ is a subgroup of $\Gamma_A/\mathcal{Z}\Gamma$. Since $\Gamma_A/\mathcal{Z}\Gamma$ is virtually polycyclic, $\mathcal{Z}\operatorname{Inn}\Gamma$ is finitely generated. Moreover, using again that $\Gamma_A/\mathcal{Z}\Gamma$ is virtually polycyclic and Γ_{nc} is a centreless lattice in a semisimple Lie group and that $\Gamma_A/\mathcal{Z}\Gamma$ is virtually polycyclic we can conclude that $\mathrm{Out}(\mathrm{Inn}\,\Gamma)$ and $\mathrm{Aut}(\mathrm{Inn}\,\Gamma)$ are Minkowski (see section 2.4 and in particular, remark 2.4.6).

□

As we said, our main goal in this section is to prove theorem 4.0.13, which we recall:

Theorem 4.6.6. *Let M be a closed connected n-dimensional aspherical manifold such that $\pi_1(M)$ is 2-step Minkowski. If* $\text{disc-sym}_2(M) = (f, b)$ *with $f + b = n$ then $M \cong N/\Gamma$, where N/Γ is the total space of a principal T^f-bundle over T^b.*

Proof of theorem 4.6.6. The idea of the proof is to reduce the general case to the case where M is an infranilmanifold finitely covered by a 2-step nilmanifold. We divide the proof in 4 parts.

Part 1. Study of the first step of the iterated actions: Since $\text{disc-sym}_2(M) = (f, b)$ there exist a strictly increasing sequence of prime numbers $\{p_i\}_{i\in\mathbb{N}}$ and a sequence of numbers $\{a_i\}_{i\in\mathbb{N}}$ such that $\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M$ freely. Since $\mathrm{Out}(\pi_1(M))$ is Minkowski, there exists i_0 such that for all $i \geq i_0$ the induced group morphism $\psi_i : (\mathbb{Z}/p_i^{a_i})^f \longrightarrow \mathrm{Out}(\pi_1(M))$ is trivial. Thus, we can assume without loss of generality that all ψ_i are trivial.

Let $\tilde{G}_i = \pi_1(M/(\mathbb{Z}/p_i^{a_i})^f)$. By lemma 1.3.95 and lemma 4.4.12, we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathcal{Z}\pi_1(M) & \longrightarrow & C_{\tilde{G}}(\pi_1(M)) = \mathcal{Z}\tilde{G}_i & \longrightarrow & (\mathbb{Z}/p_i^{a_i})^f & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & \pi_1(M) & \longrightarrow & \tilde{G}_i & \xrightarrow{p} & (\mathbb{Z}/p_i^{a_i})^f & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Inn}\,\pi_1(M) & \xrightarrow{\cong} & \mathrm{Inn}\,\tilde{G}_i & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where the isomorphism of the last row is induced by the inclusion morphism $\mathrm{Inn}\,\pi_1(M) \longrightarrow \mathrm{Aut}(\pi_1(M))$. Since $\pi_1(M)$ is 2-step Minkowski and $\mathrm{Inn}\,\pi_1(M) \cong \mathrm{Inn}\,\tilde{G}_i$ for all i, the groups $\mathrm{Aut}(\mathrm{Inn}\,\tilde{G}_i)$ and $\mathrm{Out}(\mathrm{Inn}\,\tilde{G}_i)$ are Minkowski for all i. The key observation is that Minkowski constants of $\mathrm{Aut}(\mathrm{Inn}\,\tilde{G}_i)$ and $\mathrm{Out}(\mathrm{Inn}\,\tilde{G}_i)$ do not depend on i.

Part 2. Study of the second step of the iterated actions: Note that $M/(\mathbb{Z}/p_i^{a_i})^f$ is a closed aspherical manifold. Therefore the second step of each iterated action induces a group morphism $\psi_i' : (\mathbb{Z}/p_i)^b \longrightarrow \operatorname{Out}(\tilde{G}_i)$. We claim that ψ_i' is injective for all i. Assume the contrary. Then, each iterated action is equivalent to the iterated action in 3 steps $\{(\mathbb{Z}/p_i^{a_i})^b, \operatorname{Ker}\psi_i', (\mathbb{Z}/p_i)^b/\operatorname{Ker}\psi_i'\} \curvearrowright M$, where the first 2 steps of the iterated action are inner actions.

By lemma 4.4.13, the iterated action $\{(\mathbb{Z}/p_i^{a_i})^b, \operatorname{Ker}\psi_i'\} \curvearrowright M$ is simplifiable by an abelian group A_i for all i. This implies that $\{(\mathbb{Z}/p_i^{a_i})^b, (\mathbb{Z}/p_i)^b\} \curvearrowright M$ and $\{A_i, (\mathbb{Z}/p_i)^b/\operatorname{Ker}\psi_i\} \curvearrowright M$ are equivalent. Since $\text{disc-sym}_2(M) = (f, b)$ we obtain that $\operatorname{rank} A_i = f$. Moreover, we have $\operatorname{rank}((\mathbb{Z}/p_i)^b/\operatorname{Ker}\psi_i < b)$. Consequently, $\operatorname{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^b, (\mathbb{Z}/p_i)^b\} \curvearrowright M) < f + b$, which contradicts the fact that $\text{disc-sym}_2(M) = (f, b)$. Thus, the only possibility is that $\operatorname{Ker}\psi_i'$ is trivial for all i.

Like in the first step of the proof, we can use lemma 1.3.95 and lemma 4.4.12 to obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{Z}\tilde{G}_i & \xrightarrow{id} & C_{\tilde{G}_i'}(\tilde{G}_i) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \tilde{G}_i & \longrightarrow & \tilde{G}_i' & \xrightarrow{p} & (\mathbb{Z}/p_i)^b & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & \operatorname{Inn}\tilde{G}_i & \longrightarrow & G_i & \longrightarrow & (\mathbb{Z}/p_i)^b & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where $\tilde{G}_i'$ is the fundamental group of $M/\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\}$ and $G_i \leq \operatorname{Aut}(\tilde{G}_i)$. The key observation in this case is the following lemma:

Lemma 4.6.7. *The group* $\operatorname{Aut}(\tilde{G}_i)$ *is Minkowski with a constant that does not depend on* i*.*

Proof. Consider the short exact sequence $1 \longrightarrow \mathcal{Z}\tilde{G}_i \longrightarrow \tilde{G}_i \longrightarrow \operatorname{Inn}\tilde{G}_i \longrightarrow 1$. Since $\mathcal{Z}\tilde{G}_i$ is a characteristic subgroup there exist short exact sequences

$$1 \longrightarrow K_i \longrightarrow \operatorname{Aut}(\tilde{G}_i) \longrightarrow L_i \longrightarrow 1$$

and

$$1 \longrightarrow Z^1(\operatorname{Inn}\tilde{G}_i, \mathcal{Z}\tilde{G}_i) \longrightarrow K_i \longrightarrow L_i' \longrightarrow 1$$

such that $L_i \leq \operatorname{Aut}(\operatorname{Inn}\tilde{G}_i)$ and $L_i' \leq \operatorname{Aut}(\mathcal{Z}\tilde{G}_i)$. Since $\operatorname{Inn}\tilde{G}_i \cong \operatorname{Inn}\pi_1(M)$ and $\mathcal{Z}\tilde{G}_i \cong \mathcal{Z}\pi_1(M)$ we obtain that $\operatorname{Aut}(\tilde{G}_i)$ is Minkowski with a constant not depending on i. □

Part 3. M **is an infra-nilmanifold:** Consider the abstract kernel of the extension

$$1 \longrightarrow \operatorname{Inn}\tilde{G}_i \longrightarrow G_i \longrightarrow (\mathbb{Z}/p_i)^b \longrightarrow 1$$

which we denote by $\phi_i : (\mathbb{Z}/p_i)^b \longrightarrow \operatorname{Out}(\operatorname{Inn}\tilde{G}_i)$. Since $\operatorname{Out}(\tilde{G}_i)$ is Minkowski with a constant not depending on i, ϕ_i is trivial for i large enough. Thus, we will assume that ϕ_i is trivial. We obtain the following diagram where the first row is a central extension and the columns are inclusions.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{Z}\operatorname{Inn}\pi_1(M) & \longrightarrow & C_{G_i}(\operatorname{Inn}\pi_1(M)) & \longrightarrow & (\mathbb{Z}/p_i)^b & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & \operatorname{Inn}\pi_1(M) & \longrightarrow & G_i & \longrightarrow & (\mathbb{Z}/p_i)^b & \longrightarrow & 1
\end{array}$$

Note that we cannot assume that $C_{G_i}(\operatorname{Inn}\pi_1(M))$ or $\mathcal{Z}\operatorname{Inn}\pi_1(M)$ are torsion-free. Nevertheless, the group $\mathcal{Z}\operatorname{Inn}\pi_1(M)$ is finitely generated. Consequently, $\mathcal{Z}\operatorname{Inn}\pi_1(M) \cong \mathbb{Z}^r \oplus T$, where T is the torsion subgroup of $\mathcal{Z}\operatorname{Inn}\pi_1(M)$. Moreover, $C_{G_i}(\operatorname{Inn}\pi_1(M)) \leq G_i$, which is Minkowski with a constant which does not depend on i. Therefore, for i large enough we can assume that the order of torsion elements of G_i is smaller than p_i. In this setting we can use the following lemma:

Lemma 4.6.8. *Let*

$$1 \longrightarrow \mathbb{Z}^r \oplus T \longrightarrow G \longrightarrow (\mathbb{Z}/p)^b \longrightarrow 1$$

be a central group extension such that the order of the torsion elements of G is smaller than the prime p. Then $r \geq b$.

Proof. We have a central short exact sequence

$$1 \longrightarrow \mathbb{Z}^r \longrightarrow G/T \longrightarrow (\mathbb{Z}/p)^b \longrightarrow 1$$

Note that G/T is torsion-free. If not, there would exist an element $g \in G$ such that $g^p \in T$ and therefore its order $o(g) \geq p$, contradicting the fact that the order of the torsion elements of G is smaller than the prime p. Since the extension is central and G/T is torsion-free this implies that $r \geq b$. □

In consequence $\mathbb{Z}^b \leq \operatorname{Inn}\pi_1(M)$ and we have the following commutative diagram where the rows are central extensions and the columns are inclusions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^f & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^b & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}^f & \longrightarrow & \pi_1(M) & \longrightarrow & \operatorname{Inn}\pi_1(M) & \longrightarrow & 1 \end{array}$$

The group Γ is a finitely generated torsion-free 2-step nilpotent group and hence it is a lattice in a 2-step nilpotent Lie group N. The nilmanifold N/Γ is a the total space of a principal T^f-fibration over T^b.

Let $\tilde{M}$ denote the universal covering of M. We claim that the covering $\tilde{M}/\Gamma \longrightarrow M$ is a finite covering. Indeed, since $\tilde{M}$ is contractible, we have $H^*(\tilde{M}/\Gamma) \cong H^*(\Gamma, \mathbb{Z}) = H^*(N/\Gamma, \mathbb{Z})$. Therefore, $H^n(\tilde{M}/\Gamma) \neq 0$, which implies that $\tilde{M}/\Gamma$ is a closed connected manifold. The map $\tilde{M}/\Gamma \longrightarrow M$ is a covering between closed manifolds and hence a finite covering.

We reach the conclusion that $[\pi_1(M) : \Gamma] < \infty$. Since $\pi_1(M)$ is torsion-free, $\pi_1(M)$ is an almost-Bieberbach group (see theorem 1.3.61). Since the Borel conjecture holds for almost-Bieberbach groups (see section 1.3), we obtain that M is an infra-nilmanifold.

Remark 4.6.9. *The central short exact sequence*

$$1 \longrightarrow \mathbb{Z}^f \longrightarrow \Gamma \longrightarrow \mathbb{Z}^b \longrightarrow 1$$

can be trivial and therefore $\Gamma \cong \mathbb{Z}^{f+b} = \mathbb{Z}^n$. *This implies that* M *is a flat manifold and hence* $\text{disc-sym}_2(M) = (\operatorname{rank}\pi_1(M), 0)$ *by corollary 4.5.4. Thus,* $b = 0$ *and* $\operatorname{rank}\mathcal{Z}\pi_1(M) = f = n$. *By theorem 2.0.1, we can conclude that* $M \cong T^n$, *which completes the proof of theorem 4.6.6 in this particular case. Thus, from now on we will assume that the above central exact sequence is not trivial and hence* Γ *will be a 2-step nilpotent torsion-free group.*

Part 4. If M is an infra-nilmanifold with $\text{disc-sym}_2(M) = (f, b)$ and $f + b = n$ then M is a nilmanifold: To prove the bolded claim and finish the proof of theorem 4.6.6 we need some preliminary results of free group actions on 2-step nilmanifolds.

Let N/Γ be a 2-step nilmanifold of dimension n. Assume that $\operatorname{rank}\mathcal{Z}\Gamma = f$ and let $b = n - f$. Thus, N/Γ is the total space of principal T^f-bundle over T^b and Γ is a finitely generated torsion-free 2-step nilpotent group fitting in the short exact sequence $1 \longrightarrow \mathcal{Z}\Gamma \cong \mathbb{Z}^f \longrightarrow \Gamma \longrightarrow \mathbb{Z}^b \longrightarrow 1$. Then we have short exact sequences

$$1 \longrightarrow K \longrightarrow \operatorname{Out}(\Gamma) \longrightarrow \operatorname{Out}(\mathbb{Z}^b) = \operatorname{GL}(\mathbb{Z}, b) \longrightarrow 1$$

and

$$1 \longrightarrow \overline{H}^1(\mathbb{Z}^b, \mathbb{Z}^f) \longrightarrow K \longrightarrow K' \longrightarrow 1,$$

where $K' \leq \mathrm{Out}(\mathbb{Z}^b) = \mathrm{GL}(b,\mathbb{Z})$ and $\overline{H}^1(\mathbb{Z}^b,\mathbb{Z}^f) = \{[g] \in \mathrm{Out}(\Gamma) : g_{|\mathbb{Z}^f} = id_{\mathbb{Z}^f}, \overline{g} : \mathbb{Z}^b \longrightarrow \mathbb{Z}^b = id_{\mathbb{Z}^b}\}$.

Let G be a finite group acting freely on N/Γ and let $\tilde{G}$ be the fundamental group of the manifold $(N/\Gamma)/G$. There exists a group morphism $\phi : G \longrightarrow \mathrm{Out}(\Gamma)$. Using proposition 1.3.32, ϕ induces a morphism to the outer automorphism group of the rational Mal'cev completion of Γ, $\phi_{\mathbb{Q}} : G \longrightarrow \mathrm{Out}(\Gamma_{\mathbb{Q}})$. The next lemmas provide a relation between $\mathrm{Ker}(\phi_{\mathbb{Q}} : G \longrightarrow \mathrm{Out}(\Gamma_{\mathbb{Q}}))$ and the image $\phi(G) \leq \mathrm{Out}(\Gamma)$.

The first result is proposition 2.1.11. Recall that since $\mathcal{Z}\Gamma$ is a characteristic subgroup of Γ, the restriction of outer automorphisms to $\mathcal{Z}\Gamma$ induces a group morphism $\phi' : G \longrightarrow \mathrm{Aut}(\mathcal{Z}\Gamma)$.

Lemma 4.6.10. *We have* $\mathrm{rank}\,\mathcal{Z}\tilde{G} = \mathrm{rank}\,\mathcal{Z}\Gamma$ *if and only if* $\phi' : G \longrightarrow \mathrm{Aut}(\mathcal{Z}\Gamma)$ *is trivial. In this situation,* $\mathcal{Z}\tilde{G} = C_{\tilde{G}}(\Gamma)$.

Recall that we can define the isolator of the commutator $[\Gamma,\Gamma]$ as $\sqrt{[\Gamma,\Gamma]} = \{\gamma \in \Gamma : \gamma^r \in [\Gamma,\Gamma] \text{ for some } r\}$. It is a characteristic subgroup of Γ and $\Gamma/\sqrt{[\Gamma,\Gamma]}$ is torsion-free (see [Dek06, Lemma 1.1.2]).

Lemma 4.6.11. *[Dek06, Proposition 2.4.1] Given a positive integer a, consider the extension*

$$1 \longrightarrow \Gamma \longrightarrow \tilde{\Gamma} \longrightarrow \mathbb{Z}/a \longrightarrow 1.$$

The group $\tilde{\Gamma}$ is nilpotent if and only if the induced map $\overline{\phi} : \mathbb{Z}/a \longrightarrow \mathrm{Aut}(\Gamma/\sqrt{[\Gamma,\Gamma]})$ is trivial.

Since Γ is 2-step nilpotent, $\sqrt{[\Gamma,\Gamma]} \subseteq \mathcal{Z}\Gamma$ and there exists a number l such that $\mathcal{Z}\Gamma = \sqrt{[\Gamma,\Gamma]} \oplus \mathbb{Z}^l$. Thus, $\Gamma/\sqrt{[\Gamma,\Gamma]} \cong \Gamma/\mathcal{Z}\Gamma \oplus \mathbb{Z}^l$.

Lemma 4.6.12. *We have* $\phi(\mathrm{Ker}\,\phi_{\mathbb{Q}}) \leq \overline{H}^1(\mathbb{Z}^b,\mathbb{Z}^f)$. *Conversely, if $g \in G$ such that $\phi(g) \in \overline{H}^1(\mathbb{Z}^b,\mathbb{Z}^f)$, then $g \in \mathrm{Ker}\,\phi_{\mathbb{Q}}$.*

Proof. Let $g \in \mathrm{Ker}\,\phi_{\mathbb{Q}}$. Then there exists $x \in \Gamma_{\mathbb{Q}}$ such that $\phi(g)(\gamma) = x\gamma x^{-1}$ for all $\gamma \in \Gamma$. Thus, $\phi(g)_{|\mathbb{Z}^f} = id_{\mathbb{Z}^f}$ since $\mathbb{Z}^f$ is in the center of $\Gamma_{\mathbb{Q}}$ and $\overline{\phi(g)} = id_{\mathbb{Z}^b}$ since $\overline{\phi(g)}$ is a conjugation on an abelian group.

Conversely, given $g \in G$, we consider the extension

$$1 \longrightarrow \Gamma \longrightarrow \tilde{\Gamma} \longrightarrow \langle g \rangle \longrightarrow 1.$$

If $\phi(g) \in \overline{H}^1(\mathbb{Z}^b,\mathbb{Z}^f)$, then $\overline{\phi(g)} = id_{\mathbb{Z}^b}$ and $\phi(g)_{\mathbb{Z}^f} = id_{\mathbb{Z}^f}$. In particular, $\phi(g)$ fixes all elements in $\mathbb{Z}^l$. Thus, since $\langle g \rangle$ is a finite group, we have $\overline{\phi} : \langle g \rangle \longrightarrow \mathrm{Aut}(\Gamma/\sqrt{[\Gamma,\Gamma]}) = \mathrm{Aut}(\mathbb{Z}^b \oplus \mathbb{Z}^l)$ is trivial. In consequence, $\tilde{\Gamma}$ is a torsion-free nilpotent group and $g \in \mathrm{Ker}\,\phi_{\mathbb{Q}}$ by proposition 1.3.64. $\square$

We are ready to finish the proof of theorem 4.6.6. Recall that if $\text{disc-sym}_2(M) = (f, b)$ with $f + b = n$ then there exists a finite index subgroup $\Gamma \leq \pi_1(M)$ which is the fundamental group of a 2-step nilmanifold N/Γ, which is the total space of a principal T^f-bundle over T^b.

We consider the Fitting subgroup $\operatorname{Fitt}(\pi_1(M))$, which is torsion-free, nilpotent and of finite index in $\pi_1(M)$ (see theorem 1.3.61). Since Γ has finite index on $\pi_1(M)$, the groups $\operatorname{Fitt}(\pi_1(M))$ and Γ are commensurable, and therefore we have a central short exact sequence

$$1 \longrightarrow \mathbb{Z}^f \longrightarrow \operatorname{Fitt}\pi_1(M) \longrightarrow \mathbb{Z}^b \longrightarrow 1.$$

In particular, $\text{disc-sym}_2(N/\operatorname{Fitt}(\pi_1(M))) = (f, b)$.

We also have a short exact sequence

$$1 \longrightarrow \operatorname{Fitt}(\pi_1(M)) \longrightarrow \pi_1(M) \longrightarrow G \longrightarrow 1$$

where G is a finite group. If $\phi : G \longrightarrow \operatorname{Out}(\operatorname{Fitt}(\pi_1(M)))$ denotes the abstract kernel of the group extension, then the induced map $\phi_{\mathbb{Q}} : G \longrightarrow \operatorname{Out}(\operatorname{Fitt}(\pi_1(M))_{\mathbb{Q}})$ on the rational Mal'cev completion is injective (see proposition 1.3.63).

Since M is an infranilmanifold, we have $\text{disc-sym}(M) = \operatorname{rank} \mathcal{Z}\pi_1(M)$ (see theorem 2.0.4 or [LR10, Theorem 11.7.7]). In consequence, $\operatorname{rank} \mathcal{Z}\pi_1(M) = \operatorname{rank} \mathcal{Z}\operatorname{Fitt}(\pi_1(M)) = f$ and ϕ' is trivial by lemma 4.6.10. The injectivity of $\phi_{\mathbb{Q}}$ together with lemma 4.6.12 implies that $\overline{\phi} : G \longrightarrow \operatorname{Out}(\operatorname{Fitt}(\pi_1(M))/\mathcal{Z}\operatorname{Fitt}(\pi_1(M))) \cong \operatorname{GL}(b, \mathbb{Z})$ is injective. We obtain the commutative diagram (see lemma 1.3.95)

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{Z}\operatorname{Fitt}(\pi_1(M)) = \mathbb{Z}^f & \xrightarrow{id} & \mathcal{Z}\pi_1(M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \operatorname{Fitt}(\pi_1(M)) & \longrightarrow & \pi_1(M) & \xrightarrow{p} & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & \operatorname{Inn}\operatorname{Fitt}(\pi_1(M)) = \mathbb{Z}^b & \longrightarrow & \operatorname{Inn}\pi_1(M) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where the identity map in the first row is induced by the inclusion $\mathcal{Z}\operatorname{Fitt}(\pi_1(M)) \longrightarrow C_{\pi_1(M)}(\operatorname{Fitt}(\pi_1(M))) = \mathcal{Z}\pi_1(M)$. The abstract kernel of the third row is $\overline{\phi}$. Since $\overline{\phi}$ is injective, $\operatorname{Inn}\pi_1(M)$ is a crystallographic group. Moreover, $\text{disc-sym}_2(M) = (f, b)$ implies that $\mathbb{Z}^b \leq \mathcal{Z}\operatorname{Inn}\pi_1(M)$ and hence $[\operatorname{Inn}\pi_1(M) : \mathcal{Z}\operatorname{Inn}\pi_1(M)] < \infty$. Therefore, $[\operatorname{Inn}\pi_1(M), \operatorname{Inn}\pi_1(M)]$ is a finite normal subgroup of a crystallographic group $\operatorname{Inn}\pi_1(M)$, which means that $[\operatorname{Inn}\pi_1(M), \operatorname{Inn}\pi_1(M)]$ is trivial (see theorem 1.3.61). Hence $\operatorname{Inn}\pi_1(M)$ is abelian and $\pi_1(M)$ is a finitely generated torsion-free 2-step nilpotent group. This implies that M is a nilmanifold, as desired. □

The same proof can be used to obtain the following bound on the iterated discrete degree of symmetry.

Corollary 4.6.13. *Let M be a closed connected aspherical manifold such that $\pi_1(M)$ is 2-step Minkowski. Then* $\text{disc-sym}_2(M) \leq (\operatorname{rank}\mathcal{Z}\pi_1(M), \operatorname{rank}\mathcal{Z}\operatorname{Inn}\pi_1(M))$ *and* $\operatorname{rank}\mathcal{Z}\pi_1(M) + \operatorname{rank}\mathcal{Z}\operatorname{Inn}\pi_1(M) \leq \dim(M)$.

Combining proposition 4.6.5 and corollary 4.6.13, we obtain:

Corollary 4.6.14. *Let G be a connected Lie group, K a maximal subgroup of G and Γ a torsion-free cocomapct lattice of G. The closed aspherical locally homogeneous space $\Gamma \setminus G/K$ satisfies* $\text{disc-sym}_2(M) \leq (\operatorname{rank}\mathcal{Z}\Gamma, \operatorname{rank}\mathcal{Z}(\operatorname{Inn}\Gamma))$.

Corollary 4.6.13 can be used in the following situation:

Corollary 4.6.15. *Let M be a closed connected aspherical manifold. Assume that* $\operatorname{Out}(\pi_1(M))$ *and* $\operatorname{Aut}(\pi_1(M))$ *are Minkowski and that $\mathcal{Z}\pi_1(M)$ is trivial. Let E be the total space of a principal T^f-bundle over M. Then* $\text{disc-sym}_2(E) = (f, 0)$.

Proof. Consider the central short exact sequence

$$1 \longrightarrow \mathcal{Z}\pi_1(E) \cong \mathbb{Z}^f \longrightarrow \pi_1(E) \longrightarrow \pi_1(M) \longrightarrow 1.$$

Consequently, $\operatorname{Inn}\pi_1(E) \cong \pi_1(M)$ and $\pi_1(E)$ is 2-step Minkowski. Hence, by corollary 4.6.13, $\text{disc-sym}_2(M) \leq (f, 0)$.

On the other hand, since E has a free action of T^f, E also admits free actions of $(\mathbb{Z}/p^a)^f$ for any prime p and positive integer a. Consequently, $\text{disc-sym}_2(E) \geq (f, 0)$. Thus, $\text{disc-sym}_2(E) = (f, 0)$. □

4.7 Free iterated group actions on manifolds admitting a non-zero degree map to a nilmanifold

The aim of this section is to study the relation between the iterated discrete degree of symmetry and rigidity on manifolds which admit a non-zero degree map to a nilmanifold. In particular, we prove theorem 4.0.14. We start with a result on closed connected oriented manifolds admitting a non-zero degree map to a torus.

Proposition 4.7.1. *Let M be a closed oriented connected manifold of dimension n and $f : M \longrightarrow T^n$ a non-zero degree map. Then* $\text{disc-sym}_2(M) \leq (n, 0)$.

Proof. Suppose that $\text{disc-sym}_2(M) = (d_1, d_2)$. Let $\{p_i\}_{i\in\mathbb{N}}$ be the increasing sequence of primes and let $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M$ be the collection of free iterated actions given by the definition of the iterated discrete degree of symmetry. Note that we can assume without loss of generality that $p_i > \max\{\deg(f), C\}$ for all i, where C is the Minkowski constant of $\mathrm{GL}(r, \mathbb{Z})$ with $r = \operatorname{rank} H^1(M, \mathbb{Z})$. Consequently, the induced action of $(\mathbb{Z}/p_i^{a_i})^{d_1}$ on $H^1(M, \mathbb{Z})$ is trivial for all i.

By theorem 3.2.1, for each i there exists a group morphism $\eta_i : (\mathbb{Z}/p_i^{a_i})^{d_1} \longrightarrow T^n$, which is injective since $|\operatorname{Ker}\eta_i| \leq d < p_i$. Moreover, for each i there exists a η_i-equivariant map $f_i : M \longrightarrow T^n$ homotopic to f. We consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f_i \quad} & T^n \\ \downarrow & & \downarrow \\ M_i \cong M/(\mathbb{Z}/p_i^{a_i})^{d_1} & \xrightarrow{f_i'} & T^n/(\mathbb{Z}/p_i^{a_i})^{d_1} \cong T^n \end{array}$$

Note that $\deg(f_i') = \deg(f)$ for all i. The first cohomology group does not have torsion, hence $\operatorname{rank} H^1(M_i, \mathbb{Z}) = \operatorname{rank} H^1(M_i, \mathbb{Q}) = \operatorname{rank} H^1(M, \mathbb{Q})^{(\mathbb{Z}/p_i^{a_i})^{d_1}} = \operatorname{rank} H^1(M, \mathbb{Q}) = r$, where the last equality holds because the action of $(\mathbb{Z}/p_i^{a_i})^{d_1}$ on $H^1(M, \mathbb{Q})$ is trivial. Since we have assumed that $p_i > C$ for all i then the action of $(\mathbb{Z}/p_i)^{d_2}$ on $H^1(M_i, \mathbb{Z})$ is trivial for all i.

Consequently, by theorem 3.2.1, for each i there exists an injective group morphism $\eta_i' : (\mathbb{Z}/p_i)^{d_2} \longrightarrow T^n$ and a η_i'-equivariant map $h_i' : M_i \longrightarrow T^n$ homotopic to f_i'. Since $T^n \longrightarrow T^n/(\mathbb{Z}/p_i^{a_i})^{d_1}$ is a principal $(\mathbb{Z}/p_i^{a_i})^{d_1}$-bundle, for each i there exists a $(\mathbb{Z}/p_i^{a_i})^{d_1}$-equivariant homeomorphism between the total space of the pull-back $T^n \longrightarrow T^n/(\mathbb{Z}/p_i^{a_i})^{d_1}$ by h_i' and M, which we denote by $\phi_i : (h_i')^*T^n \longrightarrow M$. It induces a commutative diagram

$$\begin{array}{ccc} (h_i')^*T^n & & \\ \downarrow \phi_i & \searrow^{h_i} & \\ M & \xrightarrow{f_i} & T^n \end{array}$$

Note that each h_i is a $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\}$-equivariant map of degree $\deg(f)$. We have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad h_i \quad} & T^n \\ \downarrow & & \downarrow \\ M_i & \xrightarrow{\quad h'_i \quad} & T^n/(\mathbb{Z}/p_i^{a_i})^{d_1} \cong T^n \\ \downarrow & & \downarrow \\ M_i/(\mathbb{Z}/p_i)^{d_2} & \longrightarrow & T^n/(\mathbb{Z}/p_i)^{d_2} \cong T^n \end{array}$$

The induced iterated action $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright T^n$ is simplifiable by an abelian p_i-group G_i for each i.

The map h_i maps bijectively iterated orbits of $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright T^n$ to orbits of the action of G_i on T^n. In consequence, we can simplify $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M$ with a free action of G_i on M. This implies that $d_1 + d_2 = \text{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\}) = \text{rank}(G_i) \leq n$. If $d_1 < n$ then $\text{disc-sym}_2(M) < (n,0) = \text{disc-sym}_2(T^n)$. If $d_1 = n$ then $d_2 = 0$ and therefore $\text{disc-sym}_2(M) = (n,0) = \text{disc-sym}_2(T^n)$.

□

Theorem 4.7.2. *Let M be a closed oriented connected n-dimensional manifold M and $f : M \longrightarrow N/\Gamma$ a non-zero degree map where N/Γ is a 2-step nilmanifold. Then:*

1. $\text{disc-sym}_2(M) \leq \text{disc-sym}_2(N/\Gamma)$.
2. *If the map $f_* : \pi_1(M) \longrightarrow \Gamma$ is surjective and* $\text{disc-sym}_2(M) = \text{disc-sym}_2(N/\Gamma)$ *then* $H^*(M,\mathbb{Z}) \cong H^*(N/\Gamma,\mathbb{Z})$.

Before proving the theorem, let us show what happens if we remove the hypothesis of $f_* : \pi_1(M) \longrightarrow \Gamma$ being surjective from theorem 4.7.2.2.

Corollary 4.7.3. *Let M be a closed oriented connected n-dimensional manifold M and $f : M \longrightarrow N/\Gamma$ a non-zero degree map where N/Γ is a 2-step nilmanifold. If we have* $\text{disc-sym}_2(M) = \text{disc-sym}_2(N/\Gamma)$ *then $H^*(M,\mathbb{Q}) \cong H^*(N/\Gamma,\mathbb{Q})$.*

Proof. The condition $d = \deg(f) \neq 0$ implies that $[\Gamma : f_*\pi_1(M)] < \infty$. Therefore $f_*\pi_1(M)$ is a lattice of N and we have a finite covering $N/f_*\pi_1(M) \longrightarrow N/\Gamma$. We can lift the map f to a map f', obtaining a commutative diagram

$$\begin{array}{ccc} & & N/f_*\pi_1(M) \\ & \overset{f'}{\nearrow} & \downarrow \\ M & \xrightarrow{\quad f \quad} & N/\Gamma \end{array}$$

Note that $\deg(f') \neq 0$ and $f'_* : \pi_1(M) \longrightarrow f_*\pi_1(M)$ is surjective. Thus, we have a chain of isomorphisms

$$H^*(M, \mathbb{Q}) \cong H^*(N/f_*\pi_1(M), \mathbb{Q}) \cong H^*(\mathcal{L}(N), \mathbb{Q}) \cong H^*(N/\Gamma, \mathbb{Q}).$$

□

The proof of theorem 4.7.2 is a generalization of the arguments used to prove theorem 3.0.1.2. in [MiR24a] and it follows the same outline as the proof of theorem 4.6.6. We will divide the proof in six parts. The first two parts are analogous to proposition 4.7.1 and will prove the first part of theorem 4.7.2. We denote by $\tilde{M}$ the total space of the pull-back of $\rho : N \longrightarrow N/\Gamma$ by f. In the third and fourth part of the proof, we assume that $\text{disc-sym}_2(M) = \text{disc-sym}_2(N/\Gamma)$ and we discuss the structure of $H^*(\tilde{M}, \mathbb{Z})$ as $\mathbb{Z}\Gamma$-module. In part 5, we use non-commutative ring theory to prove that $\tilde{M}$ is an acyclic manifold. From this fact we deduce that $H^*(M, \mathbb{Z}) \cong H^*(N/\Gamma, \mathbb{Z})$ in part 6.

Before starting the proof we fix some notation. We set $a = \dim \mathcal{Z}N$ and $b = \dim N/\mathcal{Z}N$. We have a projection map $\pi : N \longrightarrow \mathbb{R}^b$ with $\operatorname{Ker} \pi = \mathcal{Z}N$. thus, we have a short exact sequence

$$1 \longrightarrow \mathbb{R}^a \longrightarrow N \xrightarrow{\pi} \mathbb{R}^b \longrightarrow 1$$

Since $\mathcal{Z}N$ is lattice hereditary, we have $\mathcal{Z}\Gamma = \mathcal{Z}N \cap \Gamma \cong \mathbb{Z}^a$ and $\pi(\Gamma) \cong \mathbb{Z}^b$. We have a normalized 2-cocycle $c : \mathbb{Z}^b \times \mathbb{Z}^b \longrightarrow \mathbb{Z}^a$ such that $\Gamma \cong \mathbb{Z}^a \times_c \mathbb{Z}^b$ and $N \cong \mathbb{R}^a \times_c \mathbb{R}^b$. We can write the 2-cocycle c

$$c((x_1, \dots, x_b), (y_1, \dots, y_b)) = \Big(\sum_{1 \leq j,k \leq b} c^1_{jk} x_j y_k, \dots, \sum_{1 \leq j,k \leq b} c^a_{jk} x_j y_k \Big)$$

where $c^l_{jk} \in \mathbb{Q}$ and $c^l_{jk} = -c^l_{kj}$ for all j, k, l and $c(\pi(\Gamma) \times \pi(\Gamma)) \subseteq \mathcal{Z}\Gamma$.

We denote $e_i = ((0, \dots, 1, \dots, 0), 0) \in \mathbb{Z}^a \times_c \mathbb{Z}^b$, where the 1 is in the i-th position. Similarly, we denote $e'_i = (0, (0, \dots, 1, \dots, 0)) \in \mathbb{Z}^a \times_c \mathbb{Z}^b$, where the 1 is in the i-th position. Note that the set $\{e_1, \dots, e_a, e'_1, \dots, e'_b\}$ generates Γ.

Denote by $\tilde{M}$ the pull-back by f of the universal cover $N \longrightarrow N/\Gamma$.

Part 1. First step of the iterated actions: Assume that $\text{disc-sym}_2(M) = (d_1, d_2)$ and $\text{disc-sym}_2(N/\Gamma) = (a, b)$.

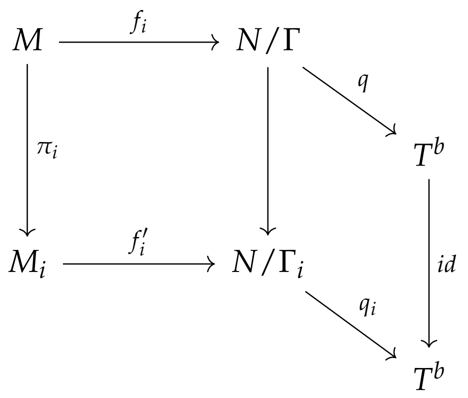
Let $\{p_i\}$ be the sequence of increasing primes and let $\{a_i\}$ be a sequence of natural numbers such that we have a free action $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M$ for each i. We can assume without loss of generality that $p_i > \max\{\deg(f), C\}$, where C is the exporting map constant of f (see definition 3.2.3 and theorem 3.3.1). In consequence, there exists an inner free action $(\mathbb{Z}/p_i^{a_i})^{d_1}$ on N/Γ and a $(\mathbb{Z}/p_i^{a_i})^{d_1}$-equivariant map $f_i : M \longrightarrow N/\Gamma$ which is homotopic to

f for each i. In particular, $d_1 \leq a$. If the inequality is strict then the first part of the theorem is proven. Thus, we will assume that $d_1 = a$.

Part 2. Second step of the iterated action: Before we continue with the proof we introduce some notation. We denote by M_i the orbit space $M/(\mathbb{Z}/p_i^{a_i})^{d_1}$ and by $\pi_i : M \longrightarrow M_i$ the orbit map. The quotient $(N/\Gamma)/(\mathbb{Z}/p_i^{a_i})^{d_1}$ is a nilmanifold with the same simply connected nilpotent Lie group N and lattice Γ_i, which fits into the short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Gamma_i \longrightarrow (\mathbb{Z}/p_i^{a_i})^{d_1} \longrightarrow 1.$$

The map f_i induces a map $f_i' : M_i \longrightarrow N/\Gamma_i$ for each i. We have a commutative diagram



Note that $q_i \circ f_i' \circ \pi_i$ is homotopic to $q \circ f$ and $\deg(f_i') = \deg(f)$.

By Minkowski's lemma, there exists a constant C' such that $(\mathbb{Z}/p_i^{a_i})^{d_1}$ acts trivially on $H^*(M, \mathbb{Q})$ for all $p_i \geq C'$. The map π_i induces an isomorphism in cohomology $H^*(M_i, \mathbb{Q}) \cong H^*(M, \mathbb{Q})^{(\mathbb{Z}/p_i^{a_i})^{d_1}} = H^*(M, \mathbb{Q})$. Since the first cohomology group has no torsion, we have $H^1(M_i, \mathbb{Z}) \cong H^1(M, \mathbb{Z})$. The action of $(\mathbb{Z}/p_i)^{d_2}$ on M_i induces an action of $(\mathbb{Z}/p_i)^{d_2}$ on $H^1(M_i, \mathbb{Z})$. Since $H^1(M_i, \mathbb{Z})$ does not depend on i up to isomorphism, we can use Minkowski's lemma again to conclude that there exists a constant $C'' > C'$ such that the action of $(\mathbb{Z}/p_i)^{d_2}$ on $H^1(M_i, \mathbb{Z})$ is trivial for $p_i \geq C''$. Thus, we can assume without loss of generality that the action of $(\mathbb{Z}/p_i)^{d_2}$ on $H^1(M_i, \mathbb{Z})$ is trivial.

We consider the map $q_i \circ f_i' : M_i \longrightarrow T^b$, for each i. Since $(\mathbb{Z}/p_i)^{d_2}$ acts trivially on $H^1(M_i, \mathbb{Z})$, there exist a group morphism $\eta_i' : (\mathbb{Z}/p_i)^{d_2} \longrightarrow T^b$ and a η_i'-equivariant map $F_i : M_i \longrightarrow T^b$ homotopically equivalent to $q_i \circ f_i'$. If η_i' is injective then disc-sym$_2(M) = (a, d_2) \leq (a, b) =$ disc-sym$_2(N/\Gamma)$ and the first part of the theorem would be proved.

Our next goal is to prove that η_i' is injective. We divide the proof in two lemmas, since the first lemma will be also useful in other steps of the proof. Recall that $X(\pi_1(M_i), N)$ is the set of isomorphism classes of N-locla systems, $X(\pi_1(M_i), N) = \mathrm{Hom}(\pi_1(M), N)/\sim$.

Lemma 4.7.4. *There exists a constant D such that the action of $(\mathbb{Z}/p_i)^{d_2}$ on $X(\pi_1(M_i), N)$ is trivial for $p_i \geq D$.*

Proof. We will prove that the restriction of the action of $(\mathbb{Z}/p_i)^{d_2}$ on $X(\pi_1(M_i), N)$ to any cyclic subgroup of $\mathbb{Z}/p_i$ is trivial. Thus, we will study actions of $\mathbb{Z}/p_i$ on $X(\pi_1(M_i), N)$. Recall that we also assume that $p_i > C''$ and therefore $\mathbb{Z}/p_i$ acts trivially on $H^1(M_i, \mathbb{Z})$.

Let us fix some more notation. Let $\phi_i \in \text{Homeo}(M_i)$ be the homeomorphism induced by the action of $\mathbb{Z}/p_i$ on M_i corresponding to $\overline{1} \in \mathbb{Z}/p_i$, $\pi : N \longrightarrow \mathbb{R}^b$ is the projection such that $\text{Ker}\,\pi = \mathcal{Z}N \cong \mathbb{R}^a$ and $\iota_i : \Gamma_i \longrightarrow N$ is an injective morphism. Set $\mu_i = \iota_i \circ f'_{i*}$ and consider the representative $[\mu_i] \in X(\pi_1(M_i), N)$. We consider the morphism $q_i \circ \mu_i \in X(\pi_1(M_i), \mathbb{R}^b) = \text{Hom}(\pi_1(M_i), \mathbb{R}^b)$ (note that we do not have the conjugation equivalence relation since $\mathbb{R}^b$ is abelian). Since $\mathbb{Z}/p_i$ acts trivially on $H^1(M_i, \mathbb{Z})$, we have $q_i \circ \mu_i \circ \phi_{i*} = q_i \circ \mu_i$. Therefore, we can define a map $\zeta_i : \pi_1(M_i) \longrightarrow \mathbb{R}^f$ such that

$$\zeta_i(\alpha) = \mu_i(\phi_{i*}(\alpha))\mu_i(\alpha)^{-1}$$

for $\alpha \in \pi_1(M_i)$. It is well defined since the image is inside $\text{Ker}\,\pi = \mathcal{Z}N \cong \mathbb{R}^a$ and it is also a group morphism. Indeed, we have

$$\begin{aligned}\zeta_i(\alpha\beta) &= \mu_i(\phi_{i*}(\alpha))\mu_i(\phi_{i*}(\beta))\mu_i(\beta)^{-1}\mu_i(\alpha)^{-1} \\ &= \mu_i(\phi_{i*}(\alpha))\mu_i(\alpha)^{-1}\mu_i(\phi_{i*}(\beta))\mu_i(\beta)^{-1} = \zeta_i(\alpha)\zeta_i(\beta)\end{aligned}$$

for $\alpha, \beta \in \pi_1(M_i)$, where we use that $\mu_i(\phi_{i*}(\beta))\mu_i(\beta)^{-1} \in \mathcal{Z}N$. Consequently, $\zeta_i \in \text{Hom}(\pi_1(M_i), \mathbb{R}^a)$. Note that $\mathbb{Z}/p_i$ acts on $\text{Hom}(\pi_1(M_i), \mathbb{R}^a)$ by precomposition, therefore it fixes a lattice of $\mathbb{R}^a$. Thus, by Minkowski lemma, there exists a constant $D > C''$ such that $\mathbb{Z}/p_i$ acts trivially on $\text{Hom}(\pi_1(M_i), \mathbb{R}^a)$. From now on we assume that $p_i \geq D$. This implies that $\zeta_i \circ \phi_{i*} = \zeta_i$.

On the other hand $[\mu_i \circ \phi^p_{i*}] = [\mu_i]$ and therefore there exists $n \in N$ such that $n\mu_i(\phi^p_{i*}(\alpha)) = \mu_i(\alpha)n$ for all $\alpha \in \pi_1(M_i)$. A computation shows that

$$[\mu_i(\alpha), n] = \sum_{j=0}^{p_i-1} \zeta_i(\phi^j_{i*}(\alpha)) = \zeta_i(\alpha)^{p_i}.$$

Recall that $N = \mathbb{R}^a \times_c \mathbb{R}^b$ where $c : \mathbb{R}^b \times \mathbb{R}^b \longrightarrow \mathbb{R}^a$ is a normalized 2-cocycle such that $c(\pi(\Gamma) \times \pi(\Gamma)) \subseteq \mathcal{Z}\Gamma$. Since $\pi(\Gamma) = \pi(\Gamma_i)$ for all i, we have $c(\pi(\Gamma_i) \times \pi(\Gamma_i)) \subseteq \mathcal{Z}\Gamma \subseteq \mathcal{Z}\Gamma_i$ for all i. Using $N \cong \mathbb{R}^a \times_c \mathbb{R}^b$, we can write $n = (x, y)$ and $\mu_i(\alpha) = (\mu'_i(\alpha), \pi(\mu_i(\alpha)))$.

A straightforward computation shows that the conjugation by n takes the form

$$c_n(\mu_i(\alpha)) = (\mu'_i(\alpha) + c(y, \pi(\mu_i(\alpha))) - c(\pi(\mu_i(\alpha)), y), \pi(\mu_i(\alpha))).$$

Hence

$$\zeta_i(\alpha)^{p_i} = c(y, \pi(\mu_i(\alpha))) - c(\pi(\mu_i(\alpha)), y).$$

We define $m = (x, \frac{y}{p}) \in \mathbb{R}^a \times_c \mathbb{R}^b$, which satisfies

$$\zeta_i(\alpha) = c(\tfrac{y}{p}, \pi(\mu_i(\alpha))) - c(\pi(\mu_i(\alpha)), \tfrac{y}{p}).$$

Summarising the previous computations, we showed that $\mu_i \circ \phi_{i*} = c_m \circ \mu_i$ which implies that $[\mu_i \circ \phi_{i*}] = [\mu_i] \in X(\pi_1(M_i), N)$, as we wanted to prove. □

Lemma 4.7.5. *The morphism η_i' is injective for all $p_i \geq D$.*

Proof. Assume that η_i' is not injective and take $\mathbb{Z}/p_i \leq \operatorname{Ker}\eta_i'$. The iterated action of $\{(\mathbb{Z}/p_i^{a_i})^a, \mathbb{Z}/p_i, (\mathbb{Z}/p_i)^{d_2-1}\} \curvearrowright M$ is equivalent to $\{(\mathbb{Z}/p_i^{a_i})^a, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M$. We will prove that $\{(\mathbb{Z}/p_i^{a_i})^a, \mathbb{Z}/p_i\} \curvearrowright M$ is simplifiable by an abelian group of rank a. This will imply that $\operatorname{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^a, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M) < a + d_2$, which is a contradiction with the assumption that $\text{disc-sym}_2(M) = (a, d_2)$.

Firstly, note that $F_i(gx) = \eta_i'(g)F_i(x) = F_i(x)$ for all $g \in \mathbb{Z}/p$ and $x \in M_i$. Using the homotopy lifting property, we can replace f_i' by a homotopic map $f_i'' : M_i \longrightarrow N/\Gamma_i$ such that $f_i'' = q_i \circ F_i$. Note that the orbits of the action of $\mathbb{Z}/p_i$ are inside the fibers of $q_i : N/\Gamma_i \longrightarrow T^b$.

By lemma 4.7.4 and lemma 3.3.7, there exists a $\mathbb{Z}/p_i$ action on N/Γ_i and a $\mathbb{Z}/p_i$-equivariant map $h_i' : M_i \longrightarrow N/\Gamma_i$ homotopic to f_i'', and hence homotopic to f_i'. The orbit space $(N/\Gamma_i)/(\mathbb{Z}/p_i)$ is a nilmanifold N/Γ_i'. The equality $\eta_i'(\mathbb{Z}/p_i) = 0$ implies that $\pi(\Gamma_i) = \pi(\Gamma_i')$ which leads to the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathcal{Z}\Gamma_i & \longrightarrow & \mathcal{Z}\Gamma_i & \longrightarrow & \mathbb{Z}/p_i & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & \Gamma_i & \longrightarrow & \Gamma_i' & \xrightarrow{p} & \mathbb{Z}/p_i & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \pi(\Gamma_i) & \xrightarrow{id} & \pi(\Gamma_i') & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The action of $\mathbb{Z}/p_i$ on N/Γ_i is inner, hence $\{(\mathbb{Z}/p_i^{a_i})^a, \mathbb{Z}/p_i\} \curvearrowright N/\Gamma$ is simplifiable by an abelian group A_i (see lemma 4.4.13). Thus, $\operatorname{rank}_{ab}(A_i) \leq a = \operatorname{rank} \mathcal{Z}\Gamma$. In conclusion $\{(\mathbb{Z}/p_i^{a_i})^a, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M$ is equivalent to $\{A_i, (\mathbb{Z}/p_i)^{d_2-1}\} \curvearrowright M$ and therefore $\operatorname{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^a, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M) \leq a + d_2 - 1$, as we wanted to see. □

Since $\eta_i' : (\mathbb{Z}/p_i)^{d_2} \longrightarrow T^b$ is injective, $d_2 \leq b$. Thus, the proof of item 1 of theorem 4.7.2 is completed.

If $\eta_i' : (\mathbb{Z}/p_i)^{d_2} \longrightarrow T^b$ is injective then the action of $(\mathbb{Z}/p_i)^{d_2}$ on $\mathrm{Hom}(\pi_1(M_i), \mathbb{R}^b)$ is trivial. By lemma 4.7.4, the action of $(\mathbb{Z}/p_i)^{d_2}$ on $\mathrm{Hom}(\pi_1(M_i), N)$ is trivial. Thus, we can construct a $(\mathbb{Z}/p_i)^{d_2}$ action on N/Γ_i and a $(\mathbb{Z}/p_i)^{d_2}$-equivariant map $h_i' : M_i \longrightarrow N/\Gamma_i$ such that $q_i \circ h_i' : M_i \longrightarrow T^b$ is η_i'-equivariant.

Part 3. $H^*(\tilde{M}, \mathbb{Z})$ **as a** $\mathbb{Z}\Gamma$**-module:** The action of Γ on $\tilde{M}$ induces an action on $H^*(\tilde{M}, \mathbb{Z})$, which we denote by $\Phi : \Gamma \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}, \mathbb{Z}))$. Thus, $H^*(\tilde{M}, \mathbb{Z})$ has a structure of $\mathbb{Z}\Gamma$-module. To prove the following lemma we use the same argument as in [MiR24a, Lemma 8.2]

Lemma 4.7.6. *$H^*(\tilde{M}, \mathbb{Z})$ is finitely generated as a $\mathbb{Z}\Gamma$-module.*

Proof. Recall that compact manifolds are Euclidean Neighbourhood Retracts (see, for example, [Hat02, Corollary A.9]) and therefore we can identify M with a closed subset of $\mathbb{R}^m$ for some m large enough such that there exist an open neighbourhood $U \subseteq \mathbb{R}^m$ of M and a retraction $r : U \longrightarrow M$. Given $x \in M$ let B_x denote an open ball centred at x and contained in U. By the compactness of M there exists a collection of point $x_1, \dots, x_s$ in M such that $M \subseteq \bigcup_{i=1}^s B_{x_i} = B$. Let $F = f \circ r : B \longrightarrow N/\Gamma$, $B_i = B_{x_i}$ and $F_i = f \circ r : B_i \longrightarrow N/\Gamma$.

Since B_i is contractible, the principal Γ-bundle $F_i^* \pi : F_i^* N \longrightarrow B_i$ obtained by pulling back the universal covering $\pi : N \longrightarrow N/\Gamma$ by F_i is trivial. This implies that for every subset $S \subseteq B_i$, we have $H^*((F_i^* \pi)^{-1}(S), \mathbb{Z}) \cong H^*(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma$. So if $H^*(S, \mathbb{Z})$ is finitely generated then $H^*((F_i^* \pi)^{-1}(S), \mathbb{Z})$ is finitely generated as $\mathbb{Z}\Gamma$-module.

Let $B_{\leq j} = B_1 \cup \dots \cup B_j$ and $F_{\leq j} = f \circ r_{|B_{\leq j}} : B_{\leq j} \longrightarrow N/\Gamma$. To ease notation, we set $X_i = (F_i^* \pi)^{-1}(B_i)$ and $X_{\leq i} = (F_{\leq i}^* \pi)^{-1}(B_{\leq i})$. We will prove using the Mayer-Vietoris long exact sequence that $H^*(X_{\leq j}, \mathbb{Z})$ is finitely generated as $\mathbb{Z}\Gamma$-module.

If $j = 1$ then $H^*(X_{\leq 1}, \mathbb{Z}) \cong H^*(X_1, \mathbb{Z}) \cong \mathbb{Z}\Gamma$ and hence it is finitely generated. Assume now that $H^*(X_{\leq j-1}, \mathbb{Z})$ is a finitely generated $\mathbb{Z}\Gamma$-module. By using that $B_{\leq j} = B_{\leq j-1} \cup B_j$, we have a long exact sequence of $\mathbb{Z}\Gamma$-modules:

$$\dots \longrightarrow H^k(X_{\leq j}, \mathbb{Z}) \longrightarrow H^k(X_{\leq j-1}, \mathbb{Z}) \oplus H^k(X_j, \mathbb{Z}) \longrightarrow H^k(X_{\leq j} \cap X_j, \mathbb{Z}) \longrightarrow \dots$$

Since $H^*(B_{\leq j-1} \cap B_j, \mathbb{Z})$ is finitely generated and it is a subset of B_j, the cohomology group $H^k(X_{\leq j} \cap X_j, \mathbb{Z})$ is a finitely generated $\mathbb{Z}\Gamma$-module. By induction hypothesis, we have that $H^k(X_{\leq j-1}, \mathbb{Z}) \oplus H^k(X_j, \mathbb{Z})$ is finitely generated. We can conclude that $H^k(X_{\leq j}, \mathbb{Z})$ is finitely generated. Finally, $H^k(X_{\leq j}, \mathbb{Z}) = 0$ for $k > m$ implies that $H^*(X_{\leq j}, \mathbb{Z})$ is a finitely generated $\mathbb{Z}\Gamma$-module.

Since we have an inclusion $i : M \longrightarrow B$ and a retraction $r : B \longrightarrow M$, the $\mathbb{Z}\Gamma$-module $H^*(\tilde{M}, \mathbb{Z})$ is a $\mathbb{Z}\Gamma$-submodule of $H^*((F^*\pi)^{-1}(B), \mathbb{Z})$. Since $H^*((F^*\pi)^{-1}(B), \mathbb{Z})$ is finitely generated and $\mathbb{Z}\Gamma$ is Noetherian, $H^*(\tilde{M}, \mathbb{Z})$ is finitely generated as $\mathbb{Z}\Gamma$-module. □

Part 4. Setting for the proof of the second part of the theorem: We suppose now that $(d_1, d_2) = (a, b)$. By theorem 3.3.1, there exists a $\{(\mathbb{Z}/p_i^{a_i})^a, (\mathbb{Z}/p_i)^b\}$-equivariant map $h_i : M \longrightarrow N/\Gamma$ homotopic to f for all i. Hence, for each i there exist non-zero degree maps $h'_i : M_i \longrightarrow N/\Gamma_i$ and $h''_i : M_i/(\mathbb{Z}/p_i)^b = M'_i \longrightarrow N/\Gamma'_i$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{h_i} & N/\Gamma \\ \downarrow & & \downarrow \\ M_i & \xrightarrow{h'_i} & N/\Gamma_i \\ \downarrow & & \downarrow \\ M'_i & \xrightarrow{h''_i} & N/\Gamma'_i \end{array}$$

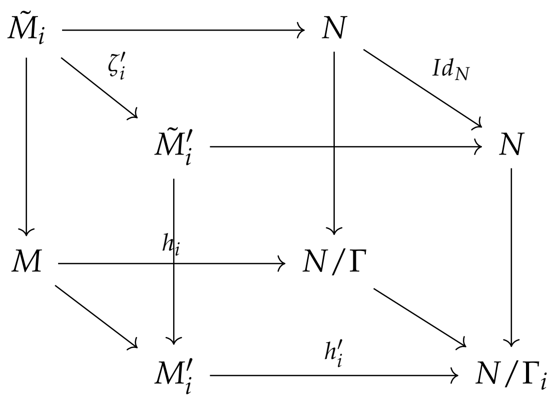
The vertical arrows are the orbit maps of the iterated group actions on M and N/Γ.

In this part of the proof we show that for each i there exists group morphism $\Phi'_i : \Gamma'_i \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}, \mathbb{Z}))$ such that $\Phi'_{i|\Gamma} = \Phi$. We will construct Φ' in two steps. First, we construct a group morphism $\Phi_i : \Gamma_i \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}, \mathbb{Z}))$ such that $\Phi_{i|\Gamma} = \Phi$. Thereafter, we construct a group morphism $\Phi'_i : \Gamma'_i \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}, \mathbb{Z}))$ such that $\Phi'_{i|\Gamma_i} = \Phi_i$.

Lemma 4.7.7. *The action of* Γ_i *on* $H^*(\tilde{M}, \mathbb{Z})$ *induces a morphism* $\Phi_i : \Gamma_i \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}, \mathbb{Z}))$ *such that* $\Phi_{i|\Gamma} = \Phi$ *for all* i.

Proof. We denote by $\tilde{M}_i$ the total space of the pull-back of $N \longrightarrow N/\Gamma$ by $h_i : M \longrightarrow N/\Gamma$. Similarly, we denote by $\tilde{M}'_i$ the pull-back of $N \longrightarrow N/\Gamma_i$ by $h'_i : M_i \longrightarrow N/\Gamma_i$.

We also have a free action of Γ'_i on $\tilde{M}'_i$. We consider the commutative diagram



The maps $\tilde{M}'_i \longrightarrow M_i$ and $\tilde{M}_i \longrightarrow M_i$ are isomorphic coverings. Thus, $\tilde{M}_i$ admits a free action of Γ_i and the map $\zeta'_i : \tilde{M}_i \longrightarrow \tilde{M}'_i$ is a Γ_i equivariant homeomorphism. By construction, the restriction to Γ of the action of Γ_i on $\tilde{M}_i$ is the action induced by the pull-back of $N \longrightarrow N/\Gamma$ by $h_i : M \longrightarrow N/\Gamma$. Thus, the action of Γ_i on $\tilde{M}_i$ induces a group morphism $\Psi'_i : \Gamma_i \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}_i, \mathbb{Z}))$ such that $\Psi'_{i|\Gamma} = \Psi_i$.

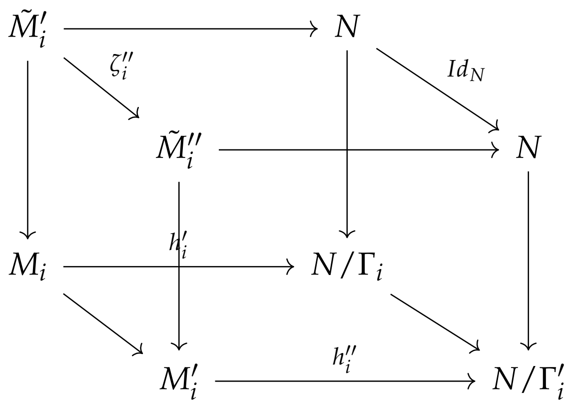
Since, h_i is homotopic to f, there exists a Γ-equivariant homeomorphism $\zeta_i : \tilde{M} \longrightarrow \tilde{M}_i$ which induces an isomorphism of $\mathbb{Z}\Gamma$-modules $\zeta_i^* : H^*(\tilde{M}_i, \mathbb{Z}) \longrightarrow H^*(\tilde{M}, \mathbb{Z})$. We define $\Phi_i : \Gamma_i \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}, \mathbb{Z}))$ as $\Phi_i(\gamma) = \zeta_i^* \circ \Psi_i(\gamma) \circ (\zeta_i^*)^{-1}$. Since ζ_i^* is an isomorphism of $\mathbb{Z}\Gamma$, we have $\Phi_{i|\Gamma} = \Phi$. □

The proof of the next lemma is analogous to the proof of lemma 4.7.7.

Lemma 4.7.8. *The action of* Γ'_i *on* $H^*(\tilde{M}, \mathbb{Z})$ *induces a morphism* $\Phi'_i : \Gamma'_i \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}, \mathbb{Z}))$ *such that* $\Phi'_{i|\Gamma} = \Phi$.

Proof. We continue using the same notation introduced in lemma 4.7.7. We denote by $\tilde{M}''_i$ the pull-back of $N \longrightarrow N/\Gamma'_i$ by $h''_i : M_i \longrightarrow N/\Gamma'_i$.

We also have a free action of Γ'_i on $\tilde{M}'_i$. We consider the commutative diagram



The maps $\tilde{M}'_i \longrightarrow M'_i$ and $\tilde{M}''_i \longrightarrow M'_i$ are isomorphic coverings. Thus, $\tilde{M}'_i$ admits a free action of Γ'_i and the map $\zeta''_i : \tilde{M}_i \longrightarrow \tilde{M}'_i$ is a Γ'_i-equivariant homeomorphism. By construction, the restriction to Γ_i of the action of Γ'_i on $\tilde{M}_i$ is the action induced by the pull-back of $N \longrightarrow N/\Gamma$ by $h_i : M \longrightarrow N/\Gamma$. Thus, the action of Γ'_i on $\tilde{M}'_i$ induces a group morphism $\Psi''_i : \Gamma'_i \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^*(\tilde{M}'_i, \mathbb{Z}))$ satisfying $(\zeta'^*_i \circ \Psi''_i \circ (\zeta'^*_i)^{-1})_{|\Gamma_i} = \Psi'_i$, where $\zeta'^*_i : H^*(\tilde{M}'_i, \mathbb{Z}) \longrightarrow H^*(\tilde{M}_i, \mathbb{Z})$ is the isomorphism of $\mathbb{Z}\Gamma_i$-modules induced by ζ'_i defined in lemma 4.7.7.

Finally, we can define Φ''_i as $\Phi''_i(\gamma) = (\zeta_i^* \circ \zeta'^*_i) \circ \Psi''_i(\gamma) \circ (\zeta_i^* \circ \zeta'^*_i)^{-1}$. Since ζ'^*_i is an isomorphism of $\mathbb{Z}\Gamma_i$-modules, we have $\Phi''_{i|\Gamma_i} = \Phi_i$. □

Part 5. $H^*(\tilde{M}, \mathbb{Z})$ **is a finitely generated** $\mathbb{Z}$**-module:** Our objective in part 5 of the proof will be to prove that $H^*(\tilde{M}, \mathbb{Z})$ is finitely generated as a $\mathbb{Z}$-module.

Recall that $\Gamma_i/\Gamma \cong \mathcal{Z}\Gamma_i/\mathcal{Z}\Gamma \cong (\mathbb{Z}/p_i^{a_i})^a$ and $\Gamma'_i/\Gamma_i \cong q(\Gamma_i)/q(\Gamma) \cong (\mathbb{Z}/p_i)^b$. Note that $\Gamma_i \cong (\frac{1}{p_i^{a_i}}\mathbb{Z})^f \times_c \mathbb{Z}^b$ and $\Gamma'_i \cong (\frac{1}{p_i^{a_i}}\mathbb{Z})^f \times_c (\frac{1}{p_i}\mathbb{Z})^b$. In particular, the lattice Γ'_i is generated $\{\frac{1}{p_i^{a_i}}e_1, \ldots, \frac{1}{p_i^{a_i}}e_a, e'_1, \ldots, e'_b\}$ and Γ''_i is generated by $\{\frac{1}{p_i^{a_i}}e_1, \ldots, \frac{1}{p_i^{a_i}}e_a, \frac{1}{p_i}e'_1, \ldots, \frac{1}{p_i}e'_b\}$.

By lemma 4.7.7 and lemma 4.7.8, for each $1 \leq j \leq a$ there exists a collection $\{w_{j,i}\}_{i\in\mathbb{N}}$ of automorphisms of $H^*(M,\mathbb{Z})$ satisfying that $(w_{j,i})^{p_i^{a_i}} = \Phi(e_j)$. Similarly, for each $1 \leq j \leq b$ there exists a collection $\{w'_{j,i}\}_{i\in\mathbb{N}}$ of automorphisms of $H^*(M,\mathbb{Z})$ satisfying that $(w'_{j,i})^{p_i} = \Phi(e'_j)$.

Recall that $\mathbb{Z}\Gamma$ is an iterated skew-Laurent ring generated by $\{e_1^{\pm 1}, \ldots, e_a^{\pm 1}, e_1'^{\pm 1}, \ldots, e_b'^{\pm 1}\}$ (see lemma 1.4.1). Explicitly, $\mathbb{Z}\Gamma \cong \mathbb{Z}[e_1^{\pm 1}, \ldots, e_a^{\pm 1}, e_1'^{\pm 1}, \ldots, e_b'^{\pm 1}; \alpha_1, \cdots, \alpha_{a+b}]$ where the automorphism are defined using the cocycle c.

The main theorem of this part is the generalization of [MiR24a, Theorem 6.1]. Be aware that the ring involved R involved in the theorem below is in general not commutative. See section 1.4 for some comments on the situation.

Theorem 4.7.9. *Let R be a prime Noetherian ring such that any prime ideal $\mathfrak{p}$ is right localizable. Given an automorphism $\alpha : R \longrightarrow R$, we consider the skew-polynomial ring $R[z;\alpha]$. Suppose that X is a finitely generated $R[z;\alpha]$-module and that there exists a sequence of positive integers $r_j \longrightarrow \infty$ and a collection of automorphisms $w_j : X \longrightarrow X$ such that $w_j^{r_j}$ coincides with the multiplication by z on the right. Then X is finitely generated as R-module.*

Proof. Let $S = \{x_1, \ldots, x_s\}$ be a generating set of X as a $R[z;\alpha]$-module and let $X_0 \subseteq X$ be the R-module generated by S. Consider the increasing sequence of finitely generated R-modules $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ defined by the condition $X_i = X_{i-1} + X_{i-1}z$. Multiplication by z induces surjective morphisms of R-modules $\mu_i : X_{i-1}/X_{i-2} \longrightarrow X_i/X_{i-1}$ and thus we can define surjective maps $\nu_i = \mu_i \circ \cdots \circ \mu_1 : X_0 \longrightarrow X_i/X_{i-1}$. We have an increasing sequence $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ of submodules of X_0 where $K_i = \operatorname{Ker} \nu_i$. Since R is Noetherian and X_0 is finitely generated there exists a i_0 such that $K_i = K_{i_0}$ for all $i \geq i_0$. In particular, μ_i is an isomorphism for all i_0.

Let $Y = X_{i_0}/X_{i_0-1}$. If $Y = 0$ then $X = X_{i_0-1}$ and we are done. Thus we will assume that $Y \neq 0$ and reach a contradiction. Since Y is a finitely generated R-module and R is Noetherian, there exists an increasing series of submodules $0 = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_r = Y$ such that Y_i/Y_{i-1} is a prime module (see theorem 1.4.16). Let $\mathfrak{p}$ denote a minimal prime ideal of the collection of the associated prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. We can now consider the right localization $R_\mathfrak{p}$ and we can also consider the localization $Y_\mathfrak{p}$. Since localizing is an exact functor (see lemma 1.4.21), $(Y_{i\mathfrak{p}}/Y_{i-1\mathfrak{p}}) = (Y_i/Y_{i-1})_\mathfrak{p}$. Moreover, $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is simple Artinian (see corollary 1.4.24), hence the length lenght$(Y_\mathfrak{p}) = \lambda$ of the composition series is finite.

We will now prove that there cannot exists any $R[z;\alpha]$-module automorphism w such that $w^r = z$ with $r > \lambda$ by reaching a contradiction. Thus, assume that there exists an automorphism $w : X \longrightarrow X$ such that $w^r = z$ and $r > \lambda$. Consider the following R-modules defined recursively as $X'_0 = X_0$ and $X'_i = X'_{i-1} + w(X'_{i-1})$ for $i \geq 1$. Using the same arguments as above, there exists a i'_0 such that $\mu'_i : X'_{i-1}/X'_{i-2} \longrightarrow X'_i/X'_{i-1}$ is an isomorphism for all $i \geq i'_0$. Let $Y' = X'_{i'_0}/X'_{i'_0-1}$.

It is clear that $X_i \subseteq X'_{ri}$ for all i. On the other hand, since X is finitely generated as a $R[z;\alpha]$-module we can write

$$w^j(x_k) = x_1 P_{jk1} + \cdots + x_s P_{jks}$$

for $j = 1, \ldots, r-1$ and $k = 1, \ldots, s$, where P_{jkl} are polynomials in z with coefficients in R. Let $e = \max \deg P_{jkl}$. Then $w^j(x_k) \in X_e$ for $j = 1, \ldots r-1$ and $k = 1, \ldots, s$. This implies that $X'_i \subseteq X_{[i/r]+e}$ for any i. Thus, we have inclusions $X_i \subseteq X'_{ir} \subseteq X_{i+e}$.

The next step is to localize at $\mathfrak{p}$ in order to use the length of the composition series. Firstly, we consider the inclusions $X_{i+L} \subseteq X'_{(i+L)r} \subseteq X_{i+L+e}$ where L is large number which we will determine below. If we fix i such that $i \geq i_0$ and $i \geq i'_0 r$ these inclusions imply that

$$0 \leq \text{lenght}(X_{i+L+e,\mathfrak{p}}/X'_{(i+L)r,\mathfrak{p}}) \leq \text{lenght}(X_{i+L+e,\mathfrak{p}}/X_{(i+L)r,\mathfrak{p}}) = e\lambda.$$

Moreover, the chain of inclusion $X_{i,\mathfrak{p}} \subseteq X'_{ir,\mathfrak{p}} \subseteq X'_{(i+1)r,\mathfrak{p}} \subseteq \cdots \subseteq X'_{(i+L)r,\mathfrak{p}} \subseteq X_{i+L+e,\mathfrak{p}}$ imply that

$$(i+L)\lambda = \text{lenght}(X'_{(i+L)r,\mathfrak{p}}/X'_{ir,\mathfrak{p}}) + \text{lenght}(X_{i+L+e,\mathfrak{p}}/X'_{(i+L)r,\mathfrak{p}})$$

and

$$\text{lenght}(X'_{(i+L)r,\mathfrak{p}}/X'_{ir,\mathfrak{p}}) = rL\,\text{lenght}(Y'_{\mathfrak{p}}).$$

In conclusion,

$$\lambda/r \leq \text{lenght}(Y'_{\mathfrak{p}}) \leq \tfrac{L+e}{Lr}\lambda.$$

Note that the lower bound is inside the interval $(0,1)$ and if L is big enough then the upper bound also is inside the interval $(0,1)$, contradicting the fact that lenght$(Y'_{\mathfrak{p}})$ is an integer. □

We want now to extend Theorem 4.7.9 to skew-Laurent rings. Thus, we consider the skew-Laurent ring $R[z^{\pm 1};\alpha]$, where $\alpha \in \text{Aut}(R)$. We now construct an iterated skew-polynomial ring as follows (see [GW04, Exercise 1R]). First, we consider the skew-polynomial ring $R[t_+;\alpha_+]$ where $\alpha_+ = \alpha$. We now define a map $\alpha_- : R[t_+;\alpha_+] \longrightarrow R[t_+;\alpha_+]$ satisfying that $\alpha_-(\sum t^i r_i) = \sum t^i \alpha^{-1}(r_i)$. Using that $rt = t\alpha^{-1}(r)$ and that α is an automorphism, it is straightforward to prove that α_- is an automorphism of $R[t_+;\alpha_+]$. Thus, we can consider the iterated skew-polynomial ring $(R[t_+;\alpha_+])[t_-;\alpha_-]$. Consider the map

$\mu : (R[t_+;\alpha_+])[t_-;\alpha_-] \longrightarrow R[z^{\pm 1};\alpha]$ defined as $\mu(\sum t_-^i(\sum t_+^j r_{ij})) = \sum z^{j-i} r_{ij}$. As before, a routine check shows that μ is a surjective ring morphism. Thus, if X is a finitely generated $R[z_{\pm 1};\alpha]$-module then X is finitely generated as a $(R[t_+;\alpha_+])[t_-;\alpha_-]$-module. Assume that there exists a sequence of positive integers $r_i \longrightarrow \infty$ and a collection of $R[z^{\pm 1};\alpha]$-automorphisms $w_i : X \longrightarrow X$ such that $w_i^{r_i}$ coincides with the multiplication by z on the right. We define automorphisms $w_{i+} : X \longrightarrow X$ and $w_{i-} : X \longrightarrow X$ satisfying that $w_{i+} = w_i$ and $w_{i-} = w_i^{-1}$ for all i. By construction, $w_{i+}^{r_i}$ coincides with the right multiplication by t_+ and $w_{i+}^{r_i}$ coincides with the right multiplication by t_-. Thus, we have:

Corollary 4.7.10. *Let R be a prime Noetherian ring such that any prime ideal $\mathfrak{p}$ is right localizable. Suppose that M is a finitely generated $R[z^{\pm 1};\alpha]$-module and that there exists a sequence of positive integers $r_j \longrightarrow \infty$ and a collection of automorphisms $w_j : M \longrightarrow M$ such that $w_j^{r_j}$ coincides with the right multiplication by z. Finally, assume that $R[t_+;\alpha_+]$ defined as above is also a prime Noetherian ring such that any prime ideal $\mathfrak{p}$ is right localizable. Then M is finitely generated as R-module.*

Recall that $\tilde{M}$ denotes the total space of the pull-back of the covering $N \longrightarrow N/\Gamma$ by $f : M \longrightarrow N/\Gamma$. Since the group ring $\mathbb{Z}\Gamma$ satisfies the conditions of corollary 4.7.10 (see theorem 1.4.18 and theorem 1.4.25) and they are iterated skew-Laurent rings, we can use downward induction to prove:

Corollary 4.7.11. *$H^*(\tilde{M},\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module.*

Proof. By lemma 4.7.6, $H^*(\tilde{M},\mathbb{Z})$ is a finitely generated as $\mathbb{Z}\Gamma$-module. Recall also that $\mathbb{Z}\Gamma \cong \mathbb{Z}[e_1^{\pm 1},\ldots,e_a^{\pm 1},e_1'^{\pm 1},\ldots,e_b'^{\pm 1};\alpha_1,\cdots,\alpha_{a+b}]$. By lemma 4.7.7 and lemma 4.7.8, there exists a collection of $\{w'_{b,i}\}_{i\in\mathbb{N}}$ of automorphisms satisfying that $(w'_{b,i})^{p_i}$ coincides with the multiplication of e'_b. Thus, by corollary 4.7.10, $H^*(\tilde{M},\mathbb{Z})$ is a finitely generated as $\mathbb{Z}\Gamma \cong \mathbb{Z}[e_1^{\pm 1},\ldots,e_a^{\pm 1},e_1'^{\pm 1},\ldots,e'^{\pm 1}_{b-1};\alpha_1,\cdots,\alpha_{a+b-1}]$. We can repeat this process with each generator to conclude that $H^*(\tilde{M},\mathbb{Z})$ is a finitely generated as $\mathbb{Z}$-module. □

Part 6. Concluding the proof The last step of the proof theorem 4.7.2 is to prove that $H^*(\tilde{M},\mathbb{Z})$ is acyclic (that is, $H^0(\tilde{M},\mathbb{Z}) \cong \mathbb{Z}$ and $H^i(\tilde{M},\mathbb{Z}) = 0$ for all $i > 0$). We will prove this statement by contradiction.

Assume that $\tilde{M}$ is not $\mathbb{Z}$-acyclic. Note that $f_*(\pi_1(M)) = \Gamma$ implies $H^0(\tilde{M},\mathbb{Z}) \cong \mathbb{Z}$. Hence if $\tilde{M}$ is not $\mathbb{Z}$-acyclic there exists $j > 0$ such that $H^j(\tilde{M},\mathbb{Z}) \neq 0$.

By the universal coefficients theorem, there exists a prime l such that $\tilde{M}$ is not $\mathbb{Z}/l$-acyclic. Let $k = \max\{j : H^j(\tilde{M},\mathbb{Z}/l) \neq 0\} > 0$. Since $H^*(\tilde{M},\mathbb{Z})$ is a finitely generated abelian group, then $\mathrm{Aut}(H^*(\tilde{M},\mathbb{Z}/l))$ is finite. Let $\Phi_{(l)} : \Gamma \longrightarrow \mathrm{Aut}(H^*(\tilde{M},\mathbb{Z}/l))$ be the group morphism induced by the action of Γ on $\tilde{M}$. The kernel of the morphism $\Phi_{(l)} : \Gamma \longrightarrow \mathrm{Aut}(H^*(\tilde{M},\mathbb{Z}/l))$, which we denote by Λ, has finite index in Γ. Hence Λ is a lattice of N.

Consider the diagram

$$\begin{array}{ccc} \tilde{M} \times_\Lambda N & \xrightarrow{\Theta} & \tilde{M}/\Lambda \\ \downarrow{\scriptstyle \pi} & & \\ N/\Lambda & & \end{array}$$

where Θ and π are the natural projections. Since N is contractible the map Θ is a homotopy equivalence and $H^j(\tilde{M} \times_\Lambda N, \mathbb{Z}/l) \cong H^j(\tilde{M}/\Lambda, \mathbb{Z}/l) = 0$ for $j > n$.

To compute the cohomology of the other fibration we need to use the Serre spectral sequence. The monodromy action of Λ on $H^j(\tilde{M}, \mathbb{Z}/l)$ is trivial, thus

$$E_2^{r,s} = H^r(N/\Lambda, H^s(\tilde{M}, \mathbb{Z}/l)) \cong H^r(N/\Lambda, \mathbb{Z}/l) \otimes H^s(\tilde{M}, \mathbb{Z}/l) \implies H^{r+s}(\tilde{M} \times_\Lambda N, \mathbb{Z}/l).$$

We have $E_2^{k,n} \neq 0$, but for dimensional reasons $E_2^{k,n}$ does not belong to the image of any differential. Furthermore, $E_2^{k,n}$ is not inside the kernel of any differential. Consequently, $H^{k+n}(\tilde{M} \times_\Lambda N, \mathbb{Z}/l) \neq 0$ which is a contradiction because $\tilde{M} \times_\Lambda N \cong \tilde{M}/\Lambda$ and $\tilde{M}/\Lambda$ has dimension n.

Finally, we take the fibration

$$\begin{array}{c} \tilde{M} \times_\Gamma N \\ \downarrow{\scriptstyle \pi} \\ N/\Gamma \end{array}$$

with fiber $\tilde{M}$. Since $\tilde{M}$ is acyclic the Serre spectral sequence collapses on the second page. This implies that $H^*(M, \mathbb{Z}) \cong H^*(\tilde{M} \times_\Gamma N, \mathbb{Z}) \cong H^*(N/\Gamma, \mathbb{Z})$ as we wanted to see.

Corollary 4.7.12. *With the same assumptions as in theorem 4.7.2.2, suppose also that $\pi_1(M)$ is virtually solvable. Then M is homeomorphic to N/Γ.*

Proof. Firstly, recall that $f : M \longrightarrow N/\Gamma$ induces a surjective map $f_* : \pi_1(M) \longrightarrow \Gamma$. Since $\pi_1(M)$ is virtually solvable there exists a finite covering $q : M' \longrightarrow M$ such that $\pi_1(M')$ is solvable. We have a commutative diagram

$$\begin{array}{ccc} M' & \xrightarrow{f'} & N/\Gamma' \\ \downarrow{\scriptstyle r} & & \downarrow \\ M & \xrightarrow{f} & N/\Gamma \end{array}$$

where $f'_* : \pi_1(M') \longrightarrow \Gamma'$ is surjective. Notice that $\text{disc-sym}_2(N/\Gamma') = \text{disc-sym}_2(N/\Gamma)$ and that $\text{disc-sym}_2(M') \geq \text{disc-sym}_2(M)$. Consequently, we have $\text{disc-sym}_2(N/\Gamma') = \text{disc-sym}_2(M')$. The next step is to prove that M' is a nilmanifold.

The acyclic manifold $\tilde{M}'$ has solvable fundamental group. This implies, since $H_1(\tilde{M}', \mathbb{Z})$ is trivial, that $\tilde{M}'$ is simply connected. Consequently, $\tilde{M}'$ is contractible by Hurewicz theorem (see [Hat02, Corollary 4.33]) and M' is a closed connected aspherical manifold. The map $\pi : \tilde{M}' \times_{\Gamma'} N \longrightarrow N/\Gamma'$ is a homotopy equivalence. Since Θ is also a homotopy equivalence we can construct a homotopy equivalence $M' \longrightarrow N/\Gamma'$. Since the Borel conjecture is true for nilmanifolds, we can conclude that M' is homeomorphic to N/Γ.

Finally, M is also a closed connected aspherical manifold, since the finite covering $r : M' \longrightarrow M$ is obtained via pullback by f of the covering of nilmanifolds $N/\Gamma' \longrightarrow N/\Gamma$, we have $\pi_1(M) \cong \Gamma$. Thus, M is homeomorphic to N/Γ, as we wanted to see. □

Remark 4.7.13. *Using the same argument as in corollary 4.7.3, we can remove the assumption that* $f_* : \pi_1(M) \longrightarrow \Gamma$ *is surjective. In this case, if* $\pi_1(M)$ *is virtually solvable then* M *is homeomorphic to a nilmanifold and* $\pi_1(M)$ *is commensurable to* Γ.

4.8 Locally simplifiable iterated actions

The aim of this section is to replace the freeness condition used through section 4.2 to section 4.7 with a weaker hypothesis but still retaining the conclusions of previous sections.

Definition 4.8.1. *Assume that we have an iterated action of a collection* $\mathcal{G}$ *of finite groups on a topological space* X. *Let* $p : X \longrightarrow X/\mathcal{G}$ *denote the orbit map. An open subset* $U \subseteq X$ *is said to be* $\mathcal{G}$*-invariant if* $p^{-1}(p(U)) = U$.

An iterated action of $\mathcal{G} \curvearrowright X$ *is said to be locally simplifiable if for every* $x \in X$ *there exists an open* $\mathcal{G}$*-invariant neighbourhood* U *of* x *such that the iterated action of* $\mathcal{G}$ *on* U *is simplifiable.*

Remark 4.8.2. *A* $\mathcal{G}$*-invariant open neighbourhood of an orbit always exists by theorem 4.1.5.*

Remark 4.8.3. *Assume that we have a locally simplifiable action* $\{G_1, \dots, G_n\} \curvearrowright M$. *Then for all* $1 \leq i \leq n$ *the iterated action* $\{G_1, \dots, G_i\} \curvearrowright M$ *is locally simplifiable. Indeed, pick* $x \in M$ *and let* U *be a* $\mathcal{G}$*-invariant neighbourhood of* x *where the iterated action is simplifiable. There exists a group* G *and a normal series* $\{e\} = G^0 \trianglelefteq G^1 \trianglelefteq \cdots \trianglelefteq G^n = G$ *such that* $G^i/G^{i-1} \cong G_i$ *simplifying the iterated action* $\mathcal{G} \curvearrowright U$. *In particular,* $G^i \curvearrowright U$ *simplifies* $\{G_1, \dots, G_i\} \curvearrowright M$. *Notice also that free iterated group actions are locally simplifiable.*

Our objective is to prove that locally simplifiable actions behave in a similar way to free iterated actions. To do so, we will use the theory of orbifolds explained in section 1.5.

Proposition 4.8.4. *Assume that we have a locally simplifiable iterated action of* $\mathcal{G}$ *on a manifold* M. *Then* $M/\mathcal{G}$ *supports a structure of orbifold such that the orbit map* $p : M \longrightarrow M/\mathcal{G}$ *is an orbifold covering.*

Proof. Let $x' \in M/\mathcal{G}$ and $x \in M$ such that $p(x) = x'$. Let $U' \subseteq M/\mathcal{G}$ be a neighbourhood of x' small enough such that U, the connected component of $p^{-1}(U')$ containing x, is contained in a chart of M and that $\mathcal{G} \curvearrowright p^{-1}(U')$ is simplifiable. Let G be the group which simplifies the iterated action $\mathcal{G} \curvearrowright p^{-1}(U')$. The isotropy group G_x acts on U and $U/G_x = U'$. Then $(U, U', G_x, p_{|U})$ is a local chart around x.

Note that if $y \in M$ such that $p(y) = x'$ then there exists $g \in G$ such that $gx = y$. Then we have a homeomorphism $\phi : U \longrightarrow V = gU$ satisfying $\phi(z) = gz$ and a group morphism $\phi_* : G_x \longrightarrow G_y$ such that $\phi_*(h) = ghg^{-1}$. In consequence, $(U, U', G_x, p_{|U})$ and $(V, U', G_y, p_{|V})$ are equivalent local models.

We now consider two local charts $(U, U', G_x, p_{|U})$ and $(V, V', G_y, p_{|V})$ such that $U' \cap V' \neq \emptyset$. Then there exists $W' \subseteq U' \cap V'$ with such that G simplifies the action $\mathcal{G} \curvearrowright p^{-1}(W')$. Pick a connected component W of $p^{-1}(W')$ and $z \in W$ such that $(W, W', G_z, p_{|W})$ is a local chart. We have embeddings $(W, G_z) \hookrightarrow (U, G_x)$ and $(W, G_z) \hookrightarrow (V, G_y)$, given by the inclusions on M and conjugations by elements of G. Therefore $M/\mathcal{G}$ is an orbifold.

Finally, we need to prove that $p : M \longrightarrow M/\mathcal{G}$ is an orbifold covering, where we assume that M has the trivial orbifold structure given by the manifold structure. Then $p^{-1}(U') = \bigcup_i U_i$. If we pick $(U_i, U_i, \{e\}, id)$ as a local model, then we have a diagram

$$\begin{array}{ccc} U_i & \xrightarrow{id} & U_i \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle p} \\ U_i & \xrightarrow{p} & U' \end{array}$$

which makes p into an orbifold covering. □

Corollary 4.8.5. *The locally simplifiable iterated action of $\mathcal{G}$ on M is simplifiable if and only if $\pi_1(M) \trianglelefteq \pi_1^{orb}(M/\mathcal{G})$. In particular, if M is simply connected then any locally simplifiable iterated action is simplifiable.*

Proof. Let $q : \tilde{M} \longrightarrow M$ be the universal covering of M. Then $p \circ q : \tilde{M} \longrightarrow M/\mathcal{G}$ is the universal cover of $M/\mathcal{G}$. In particular, $M/\mathcal{G}$ is good and $M/\mathcal{G} \cong \tilde{M}/\pi_1^{orb}(M/\mathcal{G})$. The orbifold covering $p : \tilde{M}/\pi_1(M) \longrightarrow \tilde{M}/\pi_1^{orb}(M/\mathcal{G})$ induces an inclusion $\pi_1(M) \leq \pi_1^{orb}(M/\mathcal{G})$.

By lemma 1.5.16 and lemma 1.5.23, if the iterated action $\mathcal{G} \curvearrowright M$ is simplifiable by a group G, then $\pi_1^{orb}(M/\mathcal{G})/\pi_1(M) \cong G$ and $\pi_1(M) \trianglelefteq \pi_1^{orb}(M/\mathcal{G})$. Conversely, assume that $\pi_1(M) \trianglelefteq \pi_1^{orb}(M/\mathcal{G})$ and denote $\pi_1^{orb}(M/\mathcal{G})/\pi_1(M) \cong G$. We can define a group action of G on M equivalent to $\mathcal{G} \curvearrowright M$. If M is simply connected then the action of $\pi_1^{orb}(M/G)$ on M simplifies $\mathcal{G} \curvearrowright M$. □

Like in the case of free iterated actions, being simply connected is not a necessary condition for the simplificability of locally simplifiable actions.

Proposition 4.8.6. *A locally simplifiable iterated action on S^1 is simplifiable. There exists locally simplifiable iterated actions on T^2 which are not simplifiable.*

Proof. Let $\mathcal{G} \curvearrowright S^1$ be a locally simplifiable action. Then $\pi_1^{orb}(S^1/\mathcal{G})$ is a 1-dimensional crystallographic group and hence $\pi_1^{orb}(S^1/\mathcal{G}) \cong \mathbb{Z}$ or $\pi_1^{orb}(S^1/\mathcal{G}) \cong D_\infty$, the infinite dihedral group. In both cases, any subgroup of $\pi_1^{orb}(S^1/\mathcal{G})$ isomorphic to $\mathbb{Z}$ is normal, and therefore the action is simplifiable.

For the second part of the proposition, we first note that if we have a locally simplifiable iterated action $\mathcal{G} \curvearrowright T^2$, then $T^2/\mathcal{G}$ is a flat 2-orbifold and hence $\pi_1^{orb}(T^2/\mathcal{G})$ is a crystallographic group.

We now construct a locally simplifiable iterated action on T^2 which is not simplifiable. Consider the 2-dimensional crystallographic group with presentation $\Gamma = \langle x, y, \alpha | [x, y] = 1, \alpha^2 = 1, \alpha x \alpha^{-1} = y \rangle$ and the normal series $\langle x, 2y \rangle \cong \mathbb{Z}^2 \trianglelefteq \langle x, y \rangle \cong \mathbb{Z}^2 \trianglelefteq \Gamma$. This normal series induces an iterated group action $\{\mathbb{Z}/2, \mathbb{Z}/2\} \curvearrowright T^2$. The first step of the iterated action is free, it corresponds to the short exact sequence $1 \longrightarrow \langle x, 2y \rangle \cong \mathbb{Z}^2 \longrightarrow \langle x, y \rangle \cong \mathbb{Z}^2 \longrightarrow \mathbb{Z}/2 \longrightarrow 1$, and $T^2/(\mathbb{Z}/2) \cong T^2$. The second step of the iterated action corresponds to the short exact sequence $1 \longrightarrow \langle x, y \rangle \cong \mathbb{Z}^2 \longrightarrow \Gamma \longrightarrow \mathbb{Z}/2 \longrightarrow 1$ and the quotient $T^2/(\mathbb{Z}/2)$ has the structure of an orbifold with orbifold fundamental group Γ. Since the first action is free, the orbit map $p : T^2 \longrightarrow T^2/\{\mathbb{Z}/2, \mathbb{Z}/2\}$ is an orbifold covering and hence $\{\mathbb{Z}/2, \mathbb{Z}/2\} \curvearrowright T^2$ is locally simplifiable. Note however that $\langle x, 2y \rangle \not\trianglelefteq \Gamma$. Thus, $\{\mathbb{Z}/2, \mathbb{Z}/2\} \curvearrowright T^2$ is not simplifiable. □

Remark 4.8.7. *Note that if we have a locally simplifiable action on a closed manifold $\mathcal{G} \curvearrowright M$, then the quotient orbifold $M/\mathcal{G}$ is a very good orbifold. Indeed, the covering $M \longrightarrow M/\mathcal{G}$ induces an inclusion of fundamental groups $\pi_1(M) \leq \pi_1^{orb}(M/\mathcal{G})$. Let S be a finite set of representatives of the coset $\pi_1^{orb}(M/\mathcal{G})/\pi_1(M)$. Then $\Gamma = \bigcap_{g \in S} g\pi_1(M)g^{-1}$ is a finite index normal subgroup of $\pi_1^{orb}(M/\mathcal{G})$. We have $\Gamma \trianglelefteq \pi_1(M)$, hence we can consider the covering $\overline{M} \longrightarrow M$ associated to $\Gamma \trianglelefteq \pi_1(M)$. Consequently, $\overline{M} \longrightarrow M/\mathcal{G}$ is a regular orbifold finite covering. Since $\overline{M}$ is a manifold, $M/\mathcal{G}$ is a very good orbifold.*

We also generalize the principal orbit theorem theorem 1.1.15 for locally simplifiable iterated actions.

Theorem 4.8.8. *Assume that we have a locally simplifiable iterated action of $\mathcal{G}$ on a manifold M. Let $M' = \{x \in M : \mathcal{G}_x = \{e_1, \ldots, e_n\}\}$. Then M' is open and dense in M.*

The proof requires the following lemmas. The first one is an elementary point-set topology result.

Lemma 4.8.9. *1. Let $Z \subseteq Y \subseteq X$ be topological spaces. If Y is open and dense in X and Z is open and dense in Y then Z is open and dense in X.*

2. *Let $f : X \longrightarrow Y$ be a continuous open map. If $Z \subseteq Y$ is open and dense in Y, then $f^{-1}(Z)$ is open and dense in X.*

Proof. For the first part, let U be a non-empty open subset of X. Since Y is open and dense in X, we have $U \cap Y$ is a not empty open subset of Y. This implies that $U \cap Z = Z \cap (U \cap Y)$ is not empty. Consequently Z is open and dense in X.

For the second part, let U be an non-empty open set of X, then $f(U)$ is a non-empty open subset of Y and hence $f(U) \cap Z$ is not empty. Consequently, $U \cap f^{-1}(Z)$ is not empty and $f^{-1}(Z)$ is dense in X. □

The second lemma is a key result for locally simplifiable actions. We will use the following notation: Given a locally simplifiable action $\mathcal{G} \curvearrowright M$, we define M'_{i-1} to be the set of points of M whose iterated stabilizers are trivial for the first i steps, $M'_{i-1} = \{x \in M : \{(G_1)_x, \ldots, (G_{i-1})_{x_{i-2}}\} = \{e_1, \ldots, e_{i-1}\}\}$. Then:

Lemma 4.8.10. *Given a locally simplifiable action $\mathcal{G} \curvearrowright M$, the group G_i acts on $p_{i-1}(M'_{i-1}) \subseteq M_{i-1}$ for all $1 \leq i \leq n$.*

Proof. We need to check that given $g_i \in G_i$ then the image of any point of $p_{i-1}(M'_{i-1})$ by g_i lies in $p_{i-1}(M'_{i-1})$. Assume on the contrary, that there exist points $x' \in p_{i-1}(M'_{i-1})$ and $x \in M_{i-1} \setminus p_{i-1}(M'_{i-1})$ and $g_i \in G_i$ such that $g_i x' = x$. We have that $|p_{i-1}^{-1}(x)| < \prod_{j=1}^{i-1} |G_i|$. On the other hand, we have $|p_{i-1}^{-1}(x')| = \prod_{j=1}^{i-1} |G_i|$. Since the action $\{G_1, \ldots, G_i\} \curvearrowright M$ is locally simplifiable, there exists a group G^i simplifying the action on a neighbourhood of the orbit containing $p_{i-1}^{-1}(x)$ and $p_{i-1}^{-1}(x')$. There also exists $g \in G^i$ which is send to g_i by the projection $G^i \longrightarrow G_i$. In particular, the image of $p_{i-1}^{-1}(x')$ by g is $p_{i-1}^{-1}(x)$, which is not possible since g is bijective. □

Proof of theorem 4.8.8. We proceed by induction on $l(\mathcal{G})$. If $l(\mathcal{G}) = 1$ then the assertion is a consequence of theorem 1.1.15. Suppose that now that theorem 4.8.8 holds for all locally simplifiable iterated actions of length $n-1$ and suppose that we have a locally simplifiable iterated action $\mathcal{G} = \{G_1, \ldots, G_n\} \curvearrowright M$. By induction hypothesis M'_{n-1} is a dense open subset of M. Note that $\{G_1, \ldots, G_{n-1}\}$ acts freely on M'_{n-1}, hence $p_{n-1}(M'_{n-1})$ is a manifold, which is also an open and dense subset of M_{n-1}. Since the iterated action $\mathcal{G} \curvearrowright M$ is locally simplifiable, the action of G_n on M_{n-1} restricts to an action of G_n on $p_{n-1}(M'_{n-1})$ by lemma 4.8.10. Consequently, the set $\{x_{n-1} \in p_{n-1}(M'_{n-1}) : (G_n)_{x_{n-1}} = \{e_n\}\}$, which is equal to $p_{n-1}(M')$, is open and dense in $p_{n-1}(M'_{n-1})$ by theorem 1.1.15. Therefore $p_{n-1}(M')$ is open and dense in M_{n-1}, which implies that $M' = p_{n-1}^{-1}(p_{n-1}(M'))$ is open and dense in M by lemma 4.8.9. □

The notion of length and rank of locally simplifiable iterated actions can be defined in an analogous way, by using the orbifold fundamental group instead of the usual fundamental group.

Definition 4.8.11. *We define $\mu_2^{ls}(M)$ as the set of all pairs $(f,b) \in (\mathbb{N})^2$ which satisfy:*

1. *There exist an increasing sequence of prime numbers $\{p_i\}$, a sequence of natural numbers $\{a_i\}$ and a collection of locally simplifiable iterated actions $\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M$ for each $i \in \mathbb{N}$.*
2. $\mathrm{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^f, (\mathbb{Z}/p_i)^b\} \curvearrowright M) = f + b$ *for each $i \in \mathbb{N}$.*

Consider the lexicographic order in $\mathbb{N}^2$ ($(a,b) \geq (c,d)$ if $a > c$, or $a = c$ and $b \geq d$). Define the locally simplifiable iterated discrete degree of symmetry of M as

$$\text{disc-sym}_2^{ls}(M) = \max\{(0,0) \cup \mu_2^{ls}(M)\}.$$

The next result is a generalization of theorem 1.1.32 for locally simplifiable actions. Unlike the free version theorem 4.4.4, the next result only works for large enough primes.

Theorem 4.8.12. *Let M be a closed connected n-dimensional manifold. There exists a constant C and a sequence of numbers $\{f_i\}_{i\in\mathbb{N}}$ depending only on n and $b(M)$ such that any locally simplifiable iterated action $\{(\mathbb{Z}/p^{k_i})^{a_i}\}_{i=1,\dots,r} \curvearrowright M$ with $p > C$, where k_i are arbitrary positive integers, satisfies $a_i \leq f_i$ for all i.*

Proof. We follow the same notation introduced in the proof of theorem 4.8.8. We have an effective action of $(\mathbb{Z}/p^{k_1})^{a_1}$ on M, thus by theorem 1.1.32 there exists f_1 such that $a_1 \leq f_1$ for all prime p. On the other hand, by corollary 1.1.61 there exists a constant C_1 such that for all primes $p > C_1$ the Betti number $b_p(M_0') \leq C_1$. Since $(\mathbb{Z}/p^{k_1})^{a_1}$ acts freely on M_0', its quotient is a manifold $(M_0')/(\mathbb{Z}/p^{k_1})^{a_1}$ such that $b_p((M_0')/(\mathbb{Z}/p^{k_1})^{a_1}) \leq n^{f_1}C_1$, as in theorem 4.4.4.

We now consider the second step of the locally simplifiable iterated action $(\mathbb{Z}/p^{k_2})^{a_2} \curvearrowright M_1$. By lemma 4.8.10, we have can restrict the action of $(\mathbb{Z}/p^{k_2})^{a_2}$ on M_1 to an action of $(\mathbb{Z}/p^{k_2})^{a_2}$ on $M_0'/(\mathbb{Z}/p^{k_1})^{a_1}$. Since the cohomology of $M_0'/(\mathbb{Z}/p^{k_1})^{a_1}$ is finitely generated we can use theorem 1.1.32 again to conclude that there exists a number f_2 such that if $(\mathbb{Z}/p^{k_2})^{a_2}$ acts effectively on $M_0'/(\mathbb{Z}/p^{k_1})^{a_1}$ then $a_2 \leq f_2$. Since f_2 only depends on n and $b_p(M_0'/(\mathbb{Z}/p^{k_1})^{a_1})$, we can make f_2 independent of the action of $(\mathbb{Z}/p^{k_1})^{a_1}$ on M_0' if we use $n^{f_1}C_1$ instead of $b_p(M_0'/(\mathbb{Z}/p^{k_1})^{a_1})$ to compute f_2.

We can use corollary 1.1.61 again. There exists a constant C_2 such that for all primes $p > C_2$ the Betti number $b_p(p_1(M_1')) \leq C_2$. Again, C_2 only depends on the Betti number of $M_0'/(\mathbb{Z}/p^{k_1})^{a_1}$, thus it can be made independent of the manifold $M_0'/(\mathbb{Z}/p^{k_1})^{a_1}$ if we use the bound $b_p(M_0'/(\mathbb{Z}/p^{k_1})^{a_1}) \leq n^{f_1}C_1$. Like before, we have $b_p(p_1(M_1')/(\mathbb{Z}/p^{k_2})^{a_2}) \leq n^{f_2}C_2$.

Set $C_0 = 1$. Repeating the same process the required steps we obtain numbers $C_0, \dots, C_{r-1}$ and $f_1, \dots, f_r$ which satisfy that for $1 \leq i \leq r$, $a_i \leq f_i$ if $p \geq C_{i-1}$. By taking $C = \max\{C_0, \dots, C_{n-1}\}$ we obtain the desired conclusion. □

Corollary 4.8.13. *Let M be a closed manifold. There exists $(f,b) \in \mathbb{N}^2$ such that* $\text{disc-sym}_2^{ls}(M) \leq (f,b)$.

Theorem 21, theorem 22 and theorem 23 also hold vacuously for locally simplifiable actions, since all manifolds appearing in the theorems satisfy that if a finite p-group acts on them for a prime p large enough, then the action is free. This is a consequence of the small stabilizers property (see theorem 2.0.1, theorem 3.0.2 and remark 1.1.66). An example where $\text{disc-sym}_2^{ls}(M) \neq \text{disc-sym}_2(M)$ is the following:

Proposition 4.8.14. *We have* $\text{disc-sym}_2^{ls}(S^n) = ([\frac{n+1}{2}], 0)$ *and* $\text{disc-sym}_2(S^n) = (\frac{(-1)^{n+1}+1}{2}, 0)$, *where $[x]$ denotes the integer part of x.*

Proof. Recall that $\text{Homeo}(S^n)$ is Jordan (see theorem 1.1.42.2). Let C denote the Jordan constant of $\text{Homeo}(S^n)$ and suppose that $\text{disc-sym}_2^{ls}(S^n) = (d_1, d_2)$. There exist an increasing sequence of prime numbers $\{p_i\}_{i \in \mathbb{N}}$ and locally simplifiable iterated group actions of $\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\}$ on M satisfying

$$\text{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M) = d_1 + d_2.$$

We may assume without loss of generality that $p_i > C$. Since S^n is simply connected, all the locally simplifiable actions are simplifiable (see corollary 4.8.5). Consequently, for each i there exists a p_i-group G_i acting freely on M which fits in the short exact sequence

$$1 \longrightarrow (\mathbb{Z}/p_i^{a_i})^{d_1} \longrightarrow G_i \longrightarrow (\mathbb{Z}/p_i)^{d_2} \longrightarrow 1.$$

Any proper subgroup H of G_i has index $[G_i : H] > p_i > C$. Since $\text{Homeo}(S^n)$ is Jordan of constant C, the group G_i is abelian. Thus, $\text{rank}\, G_i = \text{rank}_{ab}(\{(\mathbb{Z}/p_i^{a_i})^{d_1}, (\mathbb{Z}/p_i)^{d_2}\} \curvearrowright M) = d_1 + d_2$. Therefore $G_i \cong (\mathbb{Z}/p_i^{a_i})^{d_1} \oplus (\mathbb{Z}/p_i)^{d_2}$. We can take a subgroup $(\mathbb{Z}/p_i)^{d_1+d_2} \leq G_i$, which acts effectively on M for all i. This implies that $(d_1 + d_2, 0) \leq (d_1, d_2)$ and, since $d_1, d_2 \geq 0$, $d_2 = 0$. Finally, using that $\text{disc-sym}(S^n) = [\frac{n+1}{2}]$ (see theorem 1.1.50) we can conclude that $\text{disc-sym}_2^{ls}(S^n) = ([\frac{n+1}{2}], 0)$.

By lemma 4.4.10, $\text{disc-sym}_2(S^n) = (d_1, 0)$. Since we are now considering free actions of abelian p-groups of the form $(\mathbb{Z}/p^a)^{d_1}$ on spheres, we have $d_1 = 0$ if n is even and $d_1 = 1$ if n is odd. In conclusion, $\text{disc-sym}_2(S^n) = (\frac{(-1)^{n+1}+1}{2}, 0)$. □

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