Argumentación Práctica y Criterios de Idoneidad Didáctica en la Reflexión del Profesorado de Matemáticas en Diferentes Contextos de Formación Docente

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TESIS DOCTORAL

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Reconocimientos

Este trabajo se desarrolló dentro de los siguientes proyectos de investigación:

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Dedicatoria

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Listado de Abreviaturas / List of Abbreviations

Abreviaturas

CID	:	Criterios de Idoneidad Didáctica.
EC	:	Estudio de Clases.
EOS	:	Enfoque Onto-Semiótico.
GEAEM	:	Grupo de Estudio de Argumentación Educativa Matemática.
TAC	:	Teoría de la Acción Comunicativa.

Abbreviations

DMKC	:	Didactic-Mathematical Knowledge and Competences model.
DSC	:	Didactic Suitability Criteria.
ECE	:	Early Childhood Education.
LS	:	Lesson Study.
MFP	:	Master's Degree Final Project (pluralised as MFPs).
NCTM	:	National Council of Teachers of Mathematics.
OSA	:	Onto-Semiotic Approach.
PT	:	Pythagorean theorem.

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Resumen

La reflexión sobre la práctica docente es un aspecto clave en la formación de profesores, ya que favorece la mejora de los procesos de enseñanza, especialmente en matemáticas. Esta tesis se centra en el uso de la argumentación práctica para analizar las razones que los profesores de matemáticas (en formación y en servicio) y los formadores de profesores dan al justificar sus decisiones en los procesos de enseñanza y aprendizaje, en diferentes contextos formativos. Dado que la argumentación práctica se desarrolla a partir de la reflexión, esta puede orientarse mediante los Criterios de Idoneidad Didáctica (CID). El objetivo general de esta tesis es investigar la argumentación práctica de los profesores de matemáticas, tanto en formación como en servicio, y de los formadores de profesores, durante su reflexión sobre el diseño, la implementación y el rediseño de secuencias didácticas, y su relación con los CID. Este objetivo general se concreta en cuatro objetivos específicos: 1) Investigar la argumentación práctica de formadores de profesores de matemáticas, expertos en el uso de los CID, durante la reflexión sobre el diseño de una unidad didáctica en un ciclo de *Lesson Study* (LS). 2) Investigar la argumentación práctica de futuros profesores de matemáticas, conocedores de los CID, durante la reflexión sistematizada sobre una unidad didáctica implementada en su práctica educativa. 3) Investigar la argumentación práctica de profesores de matemáticas en servicio durante su reflexión en cursos formativos que combina el LS con los CID. 4) Investigar los CID emergentes en la argumentación práctica de futuros profesores durante su reflexión en un ciclo de LS. Metodológicamente, se utilizó un análisis detallado de la reflexión docente, integrando los CID y otros referentes teóricos de la argumentación. Para el primer objetivo, se aplicó el modelo de Toulmin para representar los argumentos prácticos de formadores de profesores durante un ciclo de LS sobre funciones. Para el segundo objetivo, se utilizó la técnica de diagramación y el modelo de la Pragma-dialéctica para representar los argumentos prácticos en las discusiones grupales de futuros profesores, estudiantes de un máster de formación de profesores de secundaria de matemáticas en Cataluña, sobre el diseño de una tarea de modelización implementada en el Trabajo de Fin de Máster de uno de los participantes. El tercer objetivo empleó el modelo de Toulmin, que incorporó los CID en la representación de los argumentos prácticos, primero en la descripción de la trayectoria argumentativa de dos profesoras mexicanas en un curso formativo sobre la introducción a la derivada direccionado a estudiantes del bachillerato mexicano y después en un estudio de caso de profesores brasileños en servicio que diseñaron y rediseñaron una clase sobre el Teorema de Pitágoras. Finalmente, para el cuarto objetivo, se usaron las herramientas de la Pragma-dialéctica y del modelo de Toulmin para representar los argumentos prácticos en las reflexiones de futuros profesores brasileños sobre una clase de geometría plana dirigida a alumnos de la educación infantil. Los resultados muestran que, en el primer objetivo, los argumentos prácticos representados con el modelo de Toulmin se corresponden con los CID. En el segundo objetivo, se constató que el uso de los CID promueve la argumentación práctica, permitiendo identificar creencias y conocimientos de los futuros docentes. El tercer objetivo destacó que valorar clases implementadas con los CID y proponer mejoras favorece la reflexión y la argumentación práctica. En el cuarto objetivo, se observó que los CID emergen implícitamente en las reflexiones grupales, y que los criterios cognitivo, mediacional e interaccional tienen una mayor influencia en la toma de decisiones didácticas de los participantes. De los resultados se concluye que la argumentación práctica y los CID emergen de la reflexión docente en distintos contextos, y que el conocimiento y uso de los CID favorecen la argumentación práctica. Sin embargo, la enseñanza de los CID en los cursos de formación no es suficiente para su interiorización, siendo necesario aumentar la experiencia en su uso y adaptar teorías para facilitar su aplicación en la reflexión docente.

Palabras clave: Argumentación práctica; Criterios de idoneidad didáctica; Formación del profesorado de matemáticas; Reflexión sobre la práctica.

Abstract

Title: Practical Argumentation and Didactic Suitability Criteria in the Reflection of Mathematics Teachers in Different Teacher Education Contexts.

Abstract: Reflection on teaching practice is a key aspect in teacher education, as it favours the improvement of teaching processes, especially in mathematics. This thesis focuses on the use of practical argumentation to analyse the reasons that mathematics teachers (prospective and practising teachers) and teacher educators given when justifying their decisions in teaching and learning processes, in different training contexts. Since practical argumentation is developed from reflection, it can be guided by the Didactic Suitability Criteria (DSC). The general objective of this thesis is to investigate the practical argumentation of mathematics teachers (prospective and practising teachers) and teacher educators during their reflection on the design, implementation, and redesign of didactic sequences and their relationship with the DSC. This general objective is specified in four specific objectives: 1) To investigate the practical argumentation of teacher educators, experts in the use of the Didactic Suitability Criteria, during the reflection on the design of a didactic unit in a Lesson Study (LS) cycle. 2) To investigate the practical argumentation of prospective mathematics teachers, familiar with the DSC, during the systematised reflection on a didactic unit implemented in their educational internship experiences. 3) To investigate the practical argumentation of practising mathematics teachers during their reflection in training courses on LS and DSC. 4) To investigate the emerging DSC in the practical argumentation of prospective teachers during their reflection in a LS cycle. Methodologically, a detailed analysis of teaching reflection was conducted, integrating the DSC and other theoretical references of argumentation. For the first objective, Toulmin's model was applied to represent the practical arguments of teacher educators during a LS on functions. For the second objective, the diagramming technique and the Pragma-dialectics model were used to represent the practical arguments in group discussions of prospective teachers, students of a master's programme in secondary education mathematics teacher education in Catalonia (Spain), on the design of a modelling task implemented in the master's degree final project of one of the participants. The third objective used Toulmin's model, which incorporated DSC in the representation of practical arguments, first in the description of the argumentative trajectory of two Mexican teachers in a training course on the introduction of derivatives aimed at Mexican baccalaureate education students and then in a case study of Brazilian practising teachers who designed and redesigned a lesson on the Pythagorean theorem. Finally, for the fourth objective, the tools of Pragma-dialectics and Toulmin's model were used to represent the practical arguments in the reflections of Brazilian prospective teachers on a flat-geometry lesson aimed at early childhood education students. The results show that, in the first objective, the practical arguments represented with Toulmin's model correspond with DSC. In the second objective, it was found that the use of DSC promotes practical argumentation, allowing the identification of beliefs and knowledge of prospective teachers. The third objective highlighted that assessing lessons implemented with DSC and proposing improvements favours reflection and practical argumentation. In the fourth objective, it was observed that DSC emerge implicitly in group reflections, and that cognitive, mediational, and interactional criteria have a greater influence on the didactic decision-making of the participants. The results show that practical argumentation and DSC emerge from teacher reflection in different contexts, and that knowledge and use of DSC favour practical argumentation. However, teaching DSC in training courses is not enough for their internalisation, making it necessary to increase experience in their use and adapt theories to facilitate their application in teaching reflection.

Keywords: Didactic suitability criteria; Mathematics teacher education; Practical argumentation; Reflection on practice.

Resum

Títol: Argumentació Pràctica i Criteris d'Idoneïtat Didàctica en la Reflexió del Professorat de Matemàtiques en Diferents Contextos de Formació Docent.

Resum: La reflexió sobre la pràctica docent és un aspecte clau en la formació de professors, ja que afavoreix la millora dels processos d'ensenyament, especialment en matemàtiques. Aquesta tesi es centra en l'ús de l'argumentació pràctica per a analitzar les raons que els professors de matemàtiques (en formació i en servei) i els formadors de professors donen al justificar les seves decisions en els processos d'ensenyament i aprenentatge, en diferents contextos formatius. Atès que l'argumentació pràctica es desenvolupa a partir de la reflexió, aquesta pot orientar-se per mitjà dels Criteris d'Idoneïtat Didàctica (CID). L'objectiu general d'aquesta tesi és investigar l'argumentació pràctica dels professors de matemàtiques, tant en formació com en servei, i dels formadors de professors, durant la seva reflexió sobre el disseny, la implementació i el redisseny de seqüències didàctiques, i la seva relació amb els CID. Aquest objectiu general es concreta en quatre objectius específics: 1. Investigar l'argumentació pràctica de formadors de professors de matemàtiques, experts en l'ús dels CID, durant la reflexió sobre el disseny d'una unitat didàctica en un cicle de *Lesson Study* (LS). 2. Investigar l'argumentació pràctica de futurs professors de matemàtiques, coneixedors dels CID, durant la reflexió sistematitzada sobre una unitat didàctica implementada en la seva pràctica educativa. 3. Investigar l'argumentació pràctica de professors de matemàtiques en servei durant la seva reflexió en cursos formatius que combinen els LS i CID. 4. Investigar els CID emergents en l'argumentació pràctica de futurs professors durant la seva reflexió en un cicle de LS. Metodològicament, es va utilitzar una anàlisi detallada de la reflexió docent, en la qual es va integrar els CID i altres referents teòrics de l'argumentació. Per al primer objectiu, es va aplicar el model de Toulmin per a representar els arguments pràctics de formadors de professors durant un cicle de LS sobre funcions. Per al segon objectiu, es va utilitzar la tècnica de diagramació i el model de la Pragma-dialèctica per a representar els arguments pràctics en les discussions grupals de futurs professors, estudiants d'un màster de formació de professors de secundària de matemàtiques a Catalunya, sobre el disseny d'una tasca de modelització implementada en el Treball de Fi de Màster d'un dels participants. El tercer objectiu va emprar el model de Toulmin, que va incorporar els CID en la representació dels arguments pràctics, primer en la descripció de la trajectòria argumentativa de dues professores en un curs formatiu sobre la introducció a la derivada adreçat a estudiants del batxillerat mexicà i després en un estudi de cas de professors brasilers en servei que van dissenyar i van redissenyar una classe sobre el Teorema de Pitàgores. Finalment, per al quart objectiu, es van usar les eines de la Pragma-dialèctica i del model de Toulmin per a representar els arguments pràctics en les reflexions de futurs professors brasilers sobre una classe de geometria plana dirigida a alumnes de l'Educació Infantil. Els resultats mostren que, en el primer objectiu, els arguments pràctics representats amb el model de Toulmin es corresponen amb els CID. En el segon objectiu, es va constatar que l'ús dels CID promou l'argumentació pràctica, cosa que permet identificar creences i coneixements dels futurs docents. El tercer objectiu va destacar que valorar classes implementades amb els CID i proposar millores afavoreix la reflexió i l'argumentació pràctica. En el quart objectiu, es va observar que els CID emergeixen implícitament en les reflexions grupals, i que els criteris cognitiu, de mitjans i d'interacció tenen una major influència en la presa de decisions didàctiques dels participants. Dels resultats es conclou que l'argumentació pràctica i els CID emergeixen de la reflexió docent en diferents contextos, i que el coneixement i ús dels CID afavoreixen l'argumentació pràctica. No obstant això, l'ensenyament dels CID en els cursos de formació no és suficient per a la seva interiorització, sent necessari augmentar l'experiència en el seu ús i adaptar teories per a facilitar la seva aplicació en la reflexió docent.

Paraules-clau: Argumentació pràctica; formació de professorat de matemàtiques; criteris d'idoneïtat didàctica; reflexió sobre la pràctica.

Chapter 1: Research Justification, Problem, and Objectives

The first section of this chapter explains the motivations underlying the development of this doctoral research; the second section explains the theoretical references that support it; the third section describes the implementation contexts; the fourth section presents the research problem and objectives; and the fifth section explains the structure of this thesis.

1.1. Research Motivation

This research raises from my interest in mathematical proof. As a mathematics student, I found this practice to be highly difficult, and as a mathematics teacher, I have experienced the complexity of teaching it effectively. With this concern, during my master's studies, I analysed the treatment of mathematical proof in textbooks, focusing on the topic of function limits. This first analysis led me to explore the Onto-Semiotic Approach to Mathematical Cognition and Instruction (Godino et al., 2019), using the *epistemic configuration* construct to examine the treatment of proof in the topic of function limits within calculus textbooks.

As a result of my master's thesis, I began to participate in the monthly meetings of the Study Group on Mathematical Educational Argumentation (GEAEM, its acronym in Spanish language), created in Colombia. In this group, I learned about various research approaches to mathematical educational argumentation, covering topics such as student learning, promoting argumentation, the use of technology, and teacher education. Some works within the GEAEM explored the relationship between proof and argumentation and showed different ways of analysing them.

With this experience and my interests in argumentation in Mathematics Education, I joined the research project: *Use of lesson study and notion of didactic suitability for the development of the competence in analysis and didactic intervention in mathematics teachers' education* (reference code PGC2018-098603-B-I00), focused on the development of the competence in analysis and didactic intervention, specifically, on the sub-competence in analysis and assessment of didactic suitability of mathematical teaching and learning processes. The project had several objectives, one of which was to design and implement teacher training courses based on the Lesson Study methodology (Hart et al., 2011) and the Didactic Suitability Criteria construct (Godino, 2013), in order to develop the competence in analysis and didactic intervention.

Hummes's (2022) doctoral work is part of the project mentioned above and, in her research, the author analysed how a training course for practising mathematics teachers, based on Didactic Suitability Criteria and Lesson Study, promoted the development of reflection in a group of participating teachers. This research was an initial reference for the approach of my own research within the project, with the aim of integrating educational argumentation into the teacher training courses proposed within the framework of the project.

Thus, taking into consideration aspects such as the need to promote reflection on teaching practice; that such reflection is a common interest in Lesson Study, Didactic Suitability Criteria, and practical argumentation; and that there was no previous research that considered these three references simultaneously to work on this type of reflection, this doctoral research was initially proposed as part of the project with the aim of designing and implementing teacher training courses based on Lesson Study and Didactic Suitability Criteria, investigating the practical argumentation within the reflection on teaching practice as a central axis.

From the first analyses on the practical argumentation of mathematics teachers in the training courses included in the project, it was observed that these analyses could be applied to other contexts. In this way, the analysis of practical argumentation was extended to courses other than those initially proposed, as well as to master's degree final projects of prospective teachers, using data from courses in Brazil, Spain, and Mexico, and including prospective and practising mathematics teachers and teacher educators. Thus, the relationship between Didactic Suitability Criteria and the analysis of practical argumentation was explored, highlighting how both references can enrich teaching reflection and didactic decision-making. This change led to this doctoral work becoming part of the research project: *Teacher's practical argumentation when reflecting on interdisciplinary didactic sequences that incorporate trends in the teaching of mathematics* (reference code PID2021-127104NB-I00).

Therefore, the following are the motivations that have given rise to this study:

1) The interest in the study of argumentation in the field of Mathematics Education, which was strengthened by my participation in GEAEM, where the importance of researching argumentation in this area was highlighted.

2) My incorporation into the research project with reference code PGC2018-098603-B-I00 allowed me to recognise the need to promote reflection on teaching practice in mathematics teachers' education.
3) The work of Hummes (2022) has been fundamental for the development of two aspects of this research. First, for an initial analysis of teaching practical argumentation in a course based on Lesson Study and Didactic Suitability Criteria; and second, for the adaptation of said course to the analysis of practical argumentation in mathematics teachers' education.
4) In the research agenda of the Onto-Semiotic Approach, the Didactic Suitability Criteria construct is presented as a tool to guide reflection on teaching practice. Along these lines, the following phenomenon has been identified: when prospective or practising teachers are asked to comment on a situation or didactic resource, such as a video-recorded class or an activity for students, their comments are usually evaluative, descriptive, or explanatory (Breda et al., 2018; Morales-López & Font, 2017; Ramos, 2006). From these comments, it is possible to infer general criteria that guide their teaching practice and, due to the consensual nature of the Didactic Suitability Criteria, these comments can be reinterpreted in terms of their components and indicators, thus facilitating a more structured analysis of teaching practice. This phenomenon was considered to define the problem of this doctoral research and, in this way, to analyse the practical argumentation that emerges in the situation mentioned before.

Thus, the articulated use of theoretical references focused on reflection on teaching practice, together with others related to educational argumentation, and their application in different teacher training contexts, has allowed to identify and structure the relationships between Didactic Suitability Criteria and practical argumentation. Since this relationship had not been previously studied, it is considered that its analysis can contribute to a better understanding of the reflection on teaching practice, allowing to identify the reasons that justify the choice of certain actions in a mathematical teaching and learning process. Once these reasons have been identified, it is possible to evaluate them and then propose changes with the aim of improving that teaching and learning process.

1.2. Theoretical References of this Doctoral Research

Taking as a basis that reflection on teaching practice is the central axis of this research, it study was approached through the use of two fundamental theoretical references: on one hand, practical argumentation (Fenstermacher & Richardson, 1993; Gómez, 2017; Lewiński, 2017) and, on the other hand, the Didactic Suitability Criteria construct (Godino, 2013). The choice of both references is justified by the fact that, on one hand, practical argumentation, in the study of reflection on teaching practice, seeks to identify the reasons taken by teachers to decide on their actions, evaluates them, and strengthens them, with the purpose of seeking their improvement in a teaching and learning process; on the other hand, Didactic Suitability Criteria are a construct capable of covering the different facets of a mathematical teaching and learning process that also allows the teacher to reflect on these facets in the different stages involved in the didactic design (preliminary study, planning, implementation, and reflection). Thus, these two references were used in the different analyses carried out throughout this doctoral research, adapting them to different contexts of mathematics teacher training, always focusing on reflection on practice and placing special emphasis on the relationships that could be established between both.

This section begins by addressing the importance of teaching reflection in education, continues with the relationship between this type of reflection and the Didactic Suitability Criteria and practical argumentation, respectively, and ends with an outline of the relationship between both theoretical references.

1.2.1. Teaching reflection in education

A first aspect to highlight is that Dewey (1933) defines reflective thinking as the "active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (p. 9). Reflection on teaching practice is a key strategy for improving teaching and learning processes that also allows for the relationship between theory and practice (Elliott, 1991; Hart et al., 2011; Schön, 1983). Along these lines, Schön (1987) states that reflection on teaching practice is essential in teacher education, as it fosters professional development. Similarly, Schoenfeld and Kilpatrick (2008) suggest that reflection is key to teacher professional growth, pointing out that, once incorporated as a habit, it can become the main mechanism or improving one's own practice. For his part, Elliott (1991) considers action research in teaching as a form of professional development that promotes reflective teaching practice.

The are different types of teaching reflection: *reflection for action*, *reflection in action*, and *reflection on action*. Similarly, there are different phases in the reflection process: a) understanding what has happened in practice and being able to describe it; b) interpreting what has happened and, if appropriate, being able to give some explanation; c) assessing what has happened; and d) guiding future practice in such a way that the aspects identified in the assessment as being improvable are improved. In the literature, different models of reflective cycles are also proposed, and authors characterise reflective competence and other competences that include various aspects related to teaching reflection (for example, those described below). On the other hand, research on teaching reflection has shown that the dilemma between unguided reflection versus guided reflection has to be decided in favour of guided reflection, so some guidelines have been proposed to organise reflection – among them, the Didactic Suitability Criteria with their components and indicators – which can be used to observe and assess both practice of other teachers and one's own.

Given the importance of teaching reflection in education, various frameworks have been proposed in Mathematics Education to promote its development, among which the following stand out:

- *Lesson Study* (Hart et al., 2011), in which a group of teachers collaborate to design, implement, and reflect on a lesson.
- *Professional noticing* (Llinares, 2013; Mason, 2002), an approach based on observation and reflection on the teaching and learning of mathematics.
- *Didactic Suitability Criteria* (Breda et al., 2017; Godino, 2013), used as a tool to guide teaching reflection.
- *Practical argumentation* (Fenstermacher & Richardson, 1993), which allows to analyse the reasons underlying the actions made in the teaching and learning process, among other frameworks.

As stated above, Didactic Suitability Criteria and practical argumentation are the main theoretical references of this doctoral research.

1.2.2. Teaching reflection and Didactic Suitability Criteria

In Mathematics Education, the Onto-Semiotic Approach (Godino et al., 2019) proposes the Didactic Suitability Criteria construct (Breda et al., 2017; Godino, 2013), which consists of a multidimensional construct that materialises in a guidelines of criteria, components, and indicators, and which is used both to design and assess the mathematical teaching and learning processes and to guide teaching reflection on practice.

In the last 10 years, various investigations have used this construct in teacher education contexts, highlighting its relevance as a key tool to guide reflection on practice in teacher education programmes (both initial and continuing) in different Ibero American countries (Spain, Brazil, Mexico, etc.). For example, Ledezma, Breda, and Font (2024) use these Criteria to study the aspects of the teaching and learning process that prospective teachers prioritise in their reflections on the implementation of mathematical modelling during their educational internship experiences. For their part, Hummes, Seckel, and da Silva (2023) study the teaching reflection in teacher training courses that combine Didactic Suitability Criteria and Lesson Study; while Burgos and Castillo (2021) highlight the importance of teaching reflection in the assessment of educational videos using this construct in their analyses. Also, Esqué and Breda (2021) study a teacher's reflection on his practice when using the Didactic Suitability Criteria to assess and redesign a didactic unit; while Morales-Maure et al. (2019) use this construct in the education of mathematics teachers to develop teaching competences. Finally, Pochulu et al. (2016) analyse the use of these Criteria by mathematics teachers' educators in the design of sequences of tasks.

As one of the main theoretical references of this research, the Didactic Suitability Criteria construct is explained in the theoretical framework sections of the publications that make up this doctoral thesis.

1.2.3. Teaching reflection and practical argumentation

Fenstermacher and Richardson (1993) highlight the importance of promoting teaching reflection on practice in teacher education processes and point out the need for specific proposals to facilitate such reflection. To this end, they define *practical rationality*, which they structure through practical argumentation as a means to promote reflection on teaching practice. In this context, reasons are offered that allow to understand what is tried to be achieved, why it is chosen to act in a certain way, and how that action fits into a desired objective. For their part, Herbst and Chazan (2023) state that an important objective of *practical rationality* is to explain the decisions that teachers make and the actions they take during the educational process.

Various authors have addressed the topic of practical argumentation and its importance in teacher education. For example, Lewiński (2017, 2018) studies practical argumentation and highlights two key aspects of practical arguments. The first is that these culminate in an act related to the action, whether it is advice, a call to action, a recommendation, among others. The second is that, when comparing two possible actions in practical argumentation, it is necessary to establish a clear criterion for comparison that justifies the choice of one over the other. For his part, Gómez (2017) comments that, with practical argumentation, the aim is to answer the question: What should *a* do in a situation *x*? where *a* represents an agent and *x* described a situation. Furthermore, Pendlebury (1990) highlights that understanding practical argumentation is essential for both teachers and their educators; while Boody et al. (1998) use practical argumentation in teacher meetings to reflect on their practice as a way of understanding and explaining actions. Herbst et al. (2011) explore the practical reasoning of mathematics teachers, identifying shared principles or values to justify possible actions in teaching processes. Finally, Vesterinen et al. (2014) analyse practical argumentation in the education of prospective teachers and its role in the development of professional knowledge.

1.2.4. Relations between Didactic Suitability Criteria and practical argumentation

As stated above, Didactic Suitability Criteria, with their components and indicators, guide the teaching reflection to design, evaluate, and redesign mathematical teaching and learning processes, while practical argumentation allows to identify and represent the reasons that guide the choice of certain actions in said processes. Thus, in this doctoral research, both references have been used in different contexts, each of which has allowed to investigate their relationship in terms of reflection on teaching practice.

Finally, it is important to highlight the relationships between Didactic Suitability Criteria and practical argumentation. The first is that both references seek to promote reflection on teaching practice; the second is that they present both a descriptive and normative approach; and the third is that they have been used in various investigations on teacher education to facilitate reflection on practice. In other words, while Didactic Suitability Criteria allow to reinterpret the general principles that guide teaching practice, as well as guide or interpret the reflection on said practice, practical argumentation allows to delve deeper into the rationality of teachers when choosing certain actions within this reflection. These characteristics mean that, when coordinated, both references complement each other to analyse different teacher education contexts.

1.3. Research Context

This doctoral research was initially framed within the project with reference code PGC2018-098603-B-I00 (described in section 1.1). For the purpose of incorporating this project, it was proposed to develop the research only in teacher training course contexts that combined Lesson Study methodology with the teaching of Didactic Suitability Criteria. However, as the research progressed, the focus was broadened to include not only practical argumentation in the originally planned courses, but also in other instances of mathematics teacher education. This increase in the contexts analysed led to the doctoral research being reoriented within the project with reference code PID2021-127104NB-I00 (described in section 1.1).

The main changes in the research contexts included the extension of the analysis to consider different subjects, namely prospective and practising mathematics teachers and teacher educators. In addition, the type of data analysed was expanded, incorporating not only dialogues from teacher training courses, but also master’s degree final projects, in which the Didactic Suitability Criteria are used to assess and reflect on teaching practice. Finally, different types of Lesson Study cycles were considered, including those in which participants did not know or use the Didactic Suitability Criteria, and others in which this construct was applied throughout the cycle. In this way, some of the contexts analysed did not consider either the Lesson Study cycle or the Didactic Suitability Criteria, characteristics that were fundamental in the original research project.

As a result of all these adjustments, the central axis of the research became practical argumentation and its relationship with the Didactic Suitability Criteria in the education of mathematics teachers in different contexts of reflection on teaching practice. Although these contexts share some common characteristics, they also have several differences, some of which are:

- Instances of development: In a master's programme for mathematics teachers; in a Lesson Study cycle.
- Profile of participants: Prospective and practising teachers, and teacher educators.
- Country of implementation: Brazil, Spain, Mexico.
- Knowledge and use of Didactic Suitability Criteria: Participants with extensive knowledge and experience in the (explicit) use of this construct; participants with limited knowledge of this construct and little experience in its use; participants who do not know this construct or its use.
- Educational level of the participants' students: Early childhood (aged 3–6) or Secondary education (aged 12–15) in Brazil; Compulsory Secondary Education (aged 12 – 16) in Spain; Baccalaureate Education (aged 15–18) in Mexico.
- Mathematical content: Functions, Pythagorean theorem, derivatives, flat geometry.

In the following subsections, the contexts in which the different analyses that are part of this doctoral research were developed are described. To this end, the particular characteristics of each of the contexts are described, three of which consist of Lesson Study cycles – presented according to the knowledge and use of the Didactic Suitability Criteria by the participants (not in chronological order) – and the fourth is a master's programme.

1.3.1. Lesson Study training cycles

A common characteristic of mathematics teacher education contexts is the presence of a Lesson Study cycle, each with its own adaptations and particular characteristics, such as the differences described above. Although there are different models for the development of Lesson Study cycles, they all include the same stages, which were followed with some modifications, depending on the particular context of each cycle (Hurd & Lewis, 2011; Lim-Ratnam, 2013):

- Problem definition, choice of mathematical content, study of its curricular approach, and establishment of learning objectives.
- Lesson planning.
- Implementation and observation of the lesson, recording the impact of planning on student learning and collecting data through observation.

- Joint reflection on the recorded data.

1.3.1.1. Cycle with teacher educators on functions

This Lesson Study cycle was implemented during the period October/2022 – March/2023 at the Department of Linguistic, Scientific, and Mathematics Education of the Faculty of Education of the University of Barcelona (Spain). Like this doctoral research, this cycle was part of the research project with reference code PID2021-127104NB-I00.

The mathematical content chosen for this Lesson Study cycle was that of functions for fourth-year students of Compulsory Secondary Education (aged 15–16), and the curricular approach was based on the guidelines of the Department of Education of Catalonia (Spain). The research objectives of this Lesson Study cycle were as follows:

1) To identify the use of Didactic Suitability Criteria in the practical argumentation of the participants in the different stages of the cycle for the design of a didactic unit on functions for fourth-year secondary education students.
2) To analyse the epistemic dimension of knowledge and the meta-didactic-mathematical competence of the participant who implemented the didactic unit with her group of students during the planning, reflection, and redesign of the didactic unit on functions (this objective is part of the doctoral work of another participant in this cycle).

This Lesson Study cycle consisted of 17 sessions, each last approximately two hours. Although all the sessions of the cycle were held in person, it was also decided to record them using the Microsoft Teams® platform provided by the university. On this platform, all the participants connected to a meeting, and in this way, their interventions were videorecorded. The participants in this cycle were eight mathematics teachers' educators and their composition was as follows:

- Three full-time university professors who are expert in Didactic Suitability Criteria:
    - AB: Bachelor's degree in mathematics; bachelor's degree in actuarial sciences; master and Doctor of Education in science and mathematics; 15 years of teaching experience.

    - GS: Primary Education Teacher; bachelor's degree in educational psychology; master's degree in specific didactics; Doctor of Mathematics education; 11 years of teaching experience.
    - VF: Bachelor's degree in science (mathematics); Doctor of Philosophy and Education Sciences; 45 years of teaching experience.
- One visiting professor expert in Lesson Study:
    - AR: Doctor of Mathematics education; 24 years of teaching experience.
- Three doctoral students / associate professors:
    - AC: Bachelor's degree in statistics; master's degree in teacher education for compulsory secondary and baccalaureate education, vocational training, and language teaching (specialising in mathematics); one year of teaching experience.
    - NI: Bachelor's degree in mathematics; 26 years of teaching experience.
    - TS: Bachelor's degree in mathematics; master's degree in mathematics education; 8 years of teaching experience.
- One associate professor / secondary education mathematics teacher:
    - AS: Bachelor's degree in mathematics; master's degree in teacher education for compulsory secondary and baccalaureate education, vocational training, and language teaching (specialising in mathematics); Doctor of Mathematics education; 6.5 years of teaching experience.

All participants were already familiar with the use of Didactic Suitability Criteria before starting the Lesson Study cycle. Furthermore, except for the visiting professor, all of them were affiliated with the University of Barcelona. It is worth mentioning that, for the analyses reported in the article in Chapter 2, the dialogues of the reflections carried out during sessions 15 and 16 of this cycle were considered. Table 1 synthesises the structure of this Lesson Study cycle.

Table 1. Structure of the Lesson Study cycle for the design of a didactic unit on functions.

Stage of the Lesson Study cycle	Session	Activities
Problem definition	1) Presentación.	– Study of what Lesson Study is and experiences carried out (taught by AR). – Presentation of the project, initial definition of the research objectives, contextualisation of the group of students in which the implementation will be carried out and definition of the mathematical topic to be developed.
	2) Contents of the didactic unit.	– Presentation of the Didactic Suitability Criteria. – Detailed study of the didactic material (textbooks) and definition of the aspects to be addressed in the topic of functions. – Review of the ambiguities that usually occur in the topic of functions.
	3) Prior knowledge and introductory activities.	– Definition of the research objectives. – Proposal for the assessment of prior knowledge using Socrative or Kahoot! – Definition of the prior knowledge required and what students are expected to have. – Definition of the students’ learning objectives. – First proposal of problems to introduce the concept of function. – Definition of the didactic unit with a constructivist approach.

Table 1. Structure of the Lesson Study cycle for the design of a didactic unit on functions. (cont.)

Stage of the Lesson Study cycle	Session	Activities
Design of the didactic unit	4) Planning the first lesson session.	– Review of learning objectives. – Review of questions to assess previous knowledge. – Planning of the session on prior knowledge. – Review of some experimentation activities.
	5) Mathematical competences in the new curriculum.	– Definition of research objectives. – Review of the questions proposed for prior knowledge. – Identification of the mathematical competences of the new curriculum with which the didactic unit will be worked on. – Discussion on the self-regulation competence. – Identification of the knowledge that will be worked on in the didactic unit. – Proposal of activities to introduce other types of functions (periodic functions, etc.). – Definition of tools to carry out the initial evaluation.
	6) Planning the first, second, and third lesson sessions.	– Definition of tools to carry out the initial evaluation. – Planning the problems session (“Ill’s temperature”, “Parachutist”, and “Boxes”). – Possibility of using GeoGebra. – Definition of characteristics of functions to be taught (domain-range, discontinuity, growth-decrease intervals).
	7) Planning the following lesson sessions.	– Discussion and definition of the actions to be taken if students do not have the necessary prior knowledge (explaining the content in a previous session or in future sessions). – Definition of the domain as the maximum possible domain. – Definition of the order in which the contents will be presented.

Table 1. Structure of the Lesson Study cycle for the design of a didactic unit on functions. (cont.)

Stage of the Lesson Study cycle	Session	Activities
Design of the didactic unit	8) Review of the wording of the tasks.	– Review of the joint document describing the planned didactic unit, defining the time allocated for each activity in one session. – Concretisation of the activity “Parachutist” using GeoGebra (drawing the graph in the software) and concretisation of the questions posed to the students regarding that task.
	9) Evaluation.	– Review of the schedule of sessions and the organisation of the implementation, as well as the presentation of the project to the educational centre. – Concretisation of the explanation of the concept of Cartesian plane. – Review of the previously defined tasks.
	10) Review of the student material.	– Review and adjustment of the tasks in the dossier for students, seeking to promote argumentation processes and high cognitive demand. – Preparation of the first draft of the evaluation rubric for some tasks. – Analysis of the competences of the new curriculum for secondary education present in each of the proposed tasks.
	11) Adjustment of the tasks to the curriculum competences and the evaluation rubric	– Final review of the didactic unit based on the assessment of the tasks with the competences of the new curriculum and assessment of the didactic unit using the Didactic Suitability Criteria. – Adjustments to the evaluation rubric. – Discussion on the education of secondary education teachers in relation to the new curriculum.
	12) Review of the implementation calendar.	– Evaluation rubric for the tasks “I think” and “Docks”. – Review of the implementation schedule.

Table 1. Structure of the Lesson Study cycle for the design of a didactic unit on functions. (cont.)

Stage of the Lesson Study cycle	**Session**	**Activities**
Implementation and observation	13) Implementation and observation.	– Evaluation of prior knowledge. – Correction of the task “Samir’s friends” (Cartesian coordinates). – Implementation of the “Ill’s temperature” task. – Implementation of the “Parachutist” task.
	14) Implementation and observation.	– Correction of the “Ill’s temperature” task. – Implementation of the “Parachutist” task.
Joint reflection	15) Reflection on the design of the didactic unit.	– Use of the Didactic Suitability Criteria (epistemic and cognitive) to guide joint reflection on the design of the didactic unit.
	15) Reflection on the design of the didactic unit.	– Use of the Didactic Suitability Criteria (interactional, mediational, affective, and ecological) to guide joint reflection on the design of the didactic unit.
	17) Reflection on the implemented lessons.	– Use of the Didactic Suitability Criteria to guide joint reflection on the implemented lessons.

1.3.1.2. Cycle with practising teachers on the Pythagorean theorem

This Lesson Study cycle was part of the training course developed by Hummes (2022) in her doctoral work, which was linked to the research project with reference code PGC2018-098603-B-I00. The course was presented under the continuing education modality, entitled «Lesson Study and Didactic Suitability Criteria: Continuing education or teachers who teach mathematics». This course was aimed at practising mathematics teachers from Brazil, and its general objective was to promote the development of reflection among the participating teachers through a training course that combined the use of Lesson Study and Didactic Suitability Criteria.

The participants in this cycle were eight mathematics teachers, divided into two groups, each of which developed a complete Lesson Study cycle. The participants in the first group worked in Secondary Education (students aged 12–15):

- T1: Bachelor's degree in mathematics; master's degree in science and mathematics education; 15 years of teaching experience.
- T2: Bachelor's degree in mathematics; 3 years of teaching experience.
- T3: Bachelor's degree in mathematics, education specialisation; 5 years of teaching experience.
- T4: Bachelor's degree in mathematics; 5 years of teaching experience.

Some of the participants in the second group were in Baccalaureate (students aged 16–18) and others in secondary education:

- T5: Bachelor's degree in mathematics; master's degree in mathematics teaching; baccalaureate education teacher with 6 years of teaching experience.
- T6: Bachelor's degree in mathematics; master's degree in mathematics teaching; secondary education teacher with 5 years of teaching experience.
- T7: Bachelor's degree in mathematics, education specialisation; secondary education teacher with 5 years of teaching experience.
- T8: Bachelor's degree in mathematics; secondary education teacher with 3 years of teaching experience.

The training course consisted of 16 sessions of 2.5 hours each, held from March to July 2020. Due to the COVID-19 pandemic, only the first session of the training course could be held in person and the others were held online. The course was taught by Hummes (2022) and its structure consisted of three phases:

- In the first phase, with the teachers divided into two groups, two complete Lesson Study cycles were carried out in parallel. In both cycles, the typical stages of a Lesson Study cycle were followed and the mathematical content chosen by both groups was the Pythagorean theorem.
- The second phase focused on the study of Didactic Suitability Criteria. For teaching purposes, it was taken into consideration that, during the first stage of the Lesson Study cycle, the participating teachers reflected on how the sequence of tasks they had proposed in this stage should be structured. These reflections were used as evidence of the implicit use of some components and indicators of these Criteria, which allowed to explain the trends in the teaching of mathematics and to study this construct. To conclude this phase, a reflection guided by the Didactic Suitability Criteria was carried out on the lesson implemented during the Lesson Study cycle of the first phase.
- In the third phase, a new analysis was carried out on the lesson implemented in the Lesson Study cycle of the first group using the Didactic Suitability Criteria as a tool to guide the reflection and redesign of the lesson. In this phase, the teachers from both groups participated and each teacher analysed the lesson implemented by the first group from the perspective of one of the six criteria of this construct. Subsequently, each participant presented his/her analysis to the rest of the group and a joint discussion was held to decide which aspects of the lesson should be kept or modified.

The results of the analyses conducted in this context are reported in the manuscript in Chapter 4.

1.3.1.3. Cycle with practising teachers on derivatives

This Lesson Study cycle was implemented in 2021, in a context of restrictions due to the COVID-19 pandemic. A group of baccalaureate education mathematics teachers (students aged 15–18) in Mexico were invited to participate in a training course, the aim of which was to promote the development of reflection on teaching practice and to analyse their practical argumentation. Although five teachers initially accepted the invitation, in the end only two teachers participated, colleagues from the same educational centre. The participants of this cycle were the following:

- ML: Civil engineer; master's degree in mathematics education; 6 years of teaching experience; free-time teacher.
- NL: Mechatronics engineer; master's degree in mathematics education; doctoral student of mathematics education; 9 years of teaching experience; full-time teacher.

To carry out this training course, the author of this thesis taught the online sessions to the two participants. The course consisted of 17 sessions held every two weeks, each lasting two hours, from August 2021 to April 2022. The structure of the course was an adaptation of Hummes' (2022) proposal and was developed in three phases:

1) In the first phase of the course, a Lesson Study cycle was implemented. Firstly, the participating teachers decided to work on the topic of introduction to derivatives in a Calculus course, which was chosen due to the difficulties they identified in learning this content and because they both taught classes in the same course, which allowed them to implement the same activities in their respective groups of students. Secondly, the lesson was planned to include an introductory video on derivatives and a questionnaire on the Kahoot! platform to evaluate the understanding of the video, the definition of derivatives, and the use of formulas to calculate them. Thirdly, each participating teacher implemented the lesson in three sessions with their respective groups of students, which were videorecorded for later analysis. And fourthly, a joint reflection on the data was carried out, where the participating teachers commented on the implementation of the lessons, ending with a session to redesign the implemented activities.
2) In the second phase of the course, Didactic Suitability Criteria were taught as follows: firstly, the trainer presented evidence of the implicit use of some components and indicators of this construct that the participating teachers had used during their reflections in the first phase of the course; and secondly, this evidence served to introduce and explain the notion of didactic suitability, as well as the complete guideline of Didactic Suitability Criteria (criteria, components, and indicators).
3) In the third phase of the course, each participating teacher shared the assessment of her implemented lesson using all the components of the Didactic Suitability Criteria to guide the reflection. In this process, both the elements that favoured

the didactic suitability of the lesson and those that could be improved were discussed, proposing modifications for a new implementation of the lesson.

The results of the analyses conducted in this context are reported in the communication of Chapter 5.

1.3.1.4. Cycle with prospective teachers on flat geometric figures

This Lesson Study cycle was implemented in a public university in southern Brazil, in the Bachelor of Pedagogy degree, and was coordinated by a trainer specialising in Lesson Study methodology and professor at that university. The participants were nine prospective teachers who were taking the Mathematical Literacy course in the tenth semester of their degree. This Lesson Study cycle was carried out in ten sessions, where the prospective teachers designed a lesson on flat geometrical figures aimed at Early Childhood Education (students aged 3–6). The structure of the cycle was as follows:

- Two sessions were dedicated to defining the grade level, the mathematical topic, and the research question (3 hours for each session).
- Five sessions were dedicated to lesson planning, studying the Early Childhood Education curricular guidelines, discussing the construction of mathematical notions, and preparing materials for the lesson (3 hours for each session).
- Two sessions were dedicated to implementing the lesson (80 minutes for each session).
- One session was dedicated to reflection after the end of the cycle (3 hours for the session).

The lesson was taught by one of the prospective teachers on two consecutive days to a group of 14 students, the aim of which was to identify and analyse what criteria the students used to classify geometric shapes when creating an artwork with logical blocks. Unlike the Lesson Study cycles described above, the Didactic Suitability Criteria were not taught here, nor did they guide the participants' reflection (their comments were interpreted in terms of the components and indicators of this construct). The results of the analyses conducted in this context are reported in the article in Chapter 6.

1.3.2. Professionalising master's programme for teacher education

One of the analyses was conducted in the context of the Interuniversity Master's Programme in Teacher Training for Compulsory Secondary and Baccalaureate Education, Vocational Training, and Language Teaching (specialising in Mathematics), taught by the public universities of Catalonia (Spain). This subsection firstly provides an overview of the context of the master's programme and then the context of the analysis is specifically described.

The aim of this master's programme is to provide initial professional training that qualifies prospective teachers to teach mathematics and, at the same time, to act as educators with exercise of the tutoring function. The entry profile for the programme is made up of professionals from different areas related to mathematics (mathematicians, architects, engineers, economists, etc.) who wish to teach. The study plan is organised into the following modules: Complements of disciplinary training in mathematics; Teaching and learning of mathematics; Educational-psychological and social training; Innovation and introduction to research in Mathematics Education; Educational internships; and Master's degree final project. Three of these modules are related to the analyses presented in this thesis, which are described below.

In the «Educational internships» module, prospective teachers must carry out educational internship experiences in collaboration with institutions established through agreements with universities and recognised as internship centres. The internship period consists of two phases: one of observation (during two weeks in November) and one of intervention (during six weeks from February), both carried out under the supervision of a mentor teacher from the internship centre. In the intervention phase, prospective teachers must implement a didactic unit that they have previously designed, which is determined by the internship centre, the educational level of the students, and the time of the school year in which they carry out their intervention. The general objectives of this module are:

- To facilitate forms of relationship and contrast with the educational reality of one's own ideas and professional training option.
- To test the prospective teachers' competences to design, implement, and evaluate educational activities.

- To promote professional attitudes useful for working with secondary/baccalaureate education students, as well as with teaching teams, external professionals, and the community.

In the «Innovation and introduction to research in Mathematics Education» module, reflection on the teaching and learning processes is considered a key element to introduce innovations that improve future implementations. In addition, the need for tools for planning, managing, and improving teaching and learning processes is taken into consideration. The module is made up of three blocks: Material resources and technologies for learning and knowledge in mathematics education; Introduction to research; and Evaluation. In the «Introduction to research» block, prospective teachers are taught the Didactic Suitability Criteria, along with the guidelines for their application (Breda et al., 2017). The general objectives of this module are:

- To know and critically analyse teaching resources and manipulative materials, in order to develop the curricular contents through appropriate teaching resources that allow to assist the diversity of students and overcome the difficulties of mathematical learning.
- To know and critically analyse material resources and technologies for learning and knowledge to improve teaching methodologies and, consequently, learning processes.
- To know evaluation techniques (formative, diagnostic, etc.) to plan, apply, and analyse evaluation strategies and instruments adapted to the characteristics of the mathematical competences to be developed.
- To know criteria to analyse and assess the quality of teaching and learning processes in order to formulate didactic questions and problems and make proposals for improvement justified by the knowledge that has been developed by research in Mathematics Education.

In the «Master's Degree Final Project (MFP)» module, prospective teachers must prepare an MFP, which consists of an original, autonomous, and individual work. This master's programme offers prospective teachers two modalities for preparing an MFP: the first modality consists of a written reflection on educational internship experience, with a professional orientation, and where the level of research abilities required is lower than for a research-oriented master's thesis; the second modality is oriented towards educational research (master's thesis). At the beginning of the master's programme, prospective teachers must decide which of the two modalities they want to develop during the course, in agreement with their tutor professor. For this doctoral research, an MFP prepared in the first modality mentioned above was considered, in which prospective teachers must use the Didactic Suitability Criteria to assess the didactic unit they implemented during their educational internship experiences and propose changes that aim to improve the didactic suitability of the mathematical teaching and learning process implemented. The general objectives of this module are:

- To allow for cross-curricular work on many of the competences provided for in the master's programme.
- To allow students to demonstrate that they have acquired the competences of the master's programme.

In this doctoral research, although there was no direct intervention in the master's programme described above, there was work with a prospective teacher, a student of this programme. In this context, the study subject was the prospective teacher AD, who knew the Didactic Suitability Criteria because he learned them in the «Introduction to research» block of the «Innovation and introduction to research in mathematics education» module; however, he was not fully familiar with this construct. AD has a bachelor's degree in industrial engineering and a doctoral degree in this field, with 15 years of experience in university teaching and academic publications derived from his research work in this field.

Within the «Complements of disciplinary training in mathematics» module, the study plan of this master's programme includes a submodule on mathematical modelling (for further information on this process, see Ledezma, 2024). As a study in the master's programme, during the 2020–2021 academic year, AD expressed his interest in including work with modelling in the didactic unit he was to design and implement during his educational internship experience. This led to the study subject beginning to prepare his MFP after this experience under the supervision of two tutor professors. In this context, the analysis of the practical argumentation of the study subject was conducted to delve into the reasons to justify the incorporation (or not) of this mathematical process in his didactic unit. The results of the analyses conducted in this context are reported in the article in Chapter 2.

1.4. Research Problem and Objectives

This section presents the problem and objectives of this doctoral research. The problem statement is based on the need to promote reflection on teaching reflection in training processes, an importance that has been highlighted in section 1.2. In this doctoral research, research that addresses the topic of reflection on teaching practice from the perspective of Didactic Suitability Criteria and practical argumentation was considered. Both approaches, although treated independently, provide valuable perspectives on the research problem. In order to formulate the research objectives, in addition to considering the need to promote reflection on teaching practice, the characteristics of the different contexts of mathematics teacher training described in section 1.3 were considered.

1.4.1. Research problem statement

In this doctoral research, practical argumentation is used to analyse the reasons that prospective and practising mathematics teachers and teacher educators provide when justifying their decisions in teaching and learning processes, in different training contexts. Since practical argumentation is developed within the reflection process, which can be guided or reinterpreted through the Didactic Suitability Criteria, it is proposed to analyse practical argumentation based on its relationship with these Criteria.

Font and Godino (2011) point out that Mathematics Education faces two fundamental demands. The first demand is the creation of theoretical constructs that facilitate the understanding of teaching and learning processes, allowing their description, interpretation, and explanation. From this perspective, Mathematics Education adopts a descriptive and explanatory approach, aimed at answering key questions about a teaching and learning process: What has happened? How has it happened? Why has it happened? The second demand involves the generation of theoretical constructs aimed at improving teaching and learning processes, which includes their assessment. In this case, Mathematics Education adopts an evaluative perspective, aimed at answering the question: What could be improved? Breda et al. (2018) relate these demands to two statements regarding the improvement of mathematics teaching: a) the greater the capability to describe, understand, and explain the teaching and learning processes, the better the conditions for improving these processes; b) the knowledge and results generated by describing, understanding, and explaining the teaching and learning processes influence the generation of values and norms that guide improvement.

The notion of Didactic Suitability proposed by the Onto-Semiotic Approach was designed to facilitate reflection on teaching practice and guide its improvement (Breda et al., 2018). The guidelines of Didactic Suitability Criteria can be applied to a mathematics teaching and learning process in two ways: *a priori*, as a guide for the design of the process, and *a posteriori*, as a tool for its evaluation. Likewise, this construct allows a transition from descriptive-explanatory didactics to normative didactic (Godino, 2013).

Considering the importance of developing teacher education that encourages teaching reflection, Fenstermacher and Richardson (1993) point out that it is not enough to affirm the relevance of reflective teaching practice or to say that it should be promoted, but that it is necessary to define specific procedures for its development. In this regard, these authors pose the following questions: How is reflection carried out in teaching practice? Is it possible to improve the capability or ability to reflect in said practice? To address these issues, they propose practical argumentation as a means to understand and explain specific actions or sets of actions.

There are two perspectives on practical arguments: a descriptive and a normative perspective. From a descriptive perspective, practical arguments are accounts of actions that explain or justify why a teacher acted in a certain way; they are not an exact description of the teacher’s mental process when deciding on an action. The normative perspective, on the other hand, focuses on improving practical reasoning, identifying possible improvements in said reasoning and seeking to develop more sophisticated and better-founded arguments, which increases the teacher’s capability to reflect more deeply and effectively on his/her action.

Through practical argumentation, teachers reflect on the reasons that underpin their decisions in the teaching and learning process, adopting a descriptive perspective when bringing these arguments to the surface. Subsequently, once these reasons have been identified, they can be evaluated and adjusted in order to improve the teaching and learning processes, thus assuming a normative function.

Considering the importance of promoting reflection on teaching practice in specific training contexts; the similarity of interests between the notion of didactic suitability and practical argumentation, together with the questions raised from both references; and given that the literature that addresses reflection on teaching practice has studied it using both the Didactic Suitability Criteria and practical argumentation, but not with both references together, this doctoral research proposes to analyse the reflection on teaching practice in training processes using both theoretical references, with the aim of identifying possible relationships between them.

1.4.2. General objective

To address the research problem raised above, the following general objective is proposed:

> To investigate the practical argumentation of mathematics teachers (prospective and practising teachers) and teacher educators during their reflection on the design, implementation, and redesign of didactic sequences and their relationship with the Didactic Suitability Criteria.

1.4.3. Specific objectives

In order to achieve the general objective, the following specific objectives are proposed, each of which considers different training contexts:

- First specific objective: To investigate the practical argumentation of teacher educators, experts in the use of the Didactic Suitability Criteria, during the reflection on the design of a didactic unit in a Lesson Study cycle.
- Second specific objective: To investigate the practical argumentation of prospective mathematics teachers, familiar with the Didactic Suitability Criteria, during the systematised reflection on a didactic unit implemented in their educational internship experiences.
- Third specific objective: To investigate the practical argumentation of practising mathematics teachers during their reflection in training courses on Lesson Study and Didactic Suitability Criteria.
- Fourth specific objective: To investigate the emerging Didactic Suitability Criteria in the practical argumentation of prospective teachers during their reflection in a Lesson Study cycle.

1.5. Thesis Structure

This doctoral thesis was prepared as a compendium of publications, which means that this research was developed through publications whose central axis is the study of practical argumentation and its relationship with Didactic Suitability Criteria. For this reason, its structure differs from a traditional dissertation in the form: Introduction, Theoretical framework, Methodology, Results, and Discussion/Conclusions. Another differentiating element is the language used throughout the doctoral thesis, as each chapter is written in Spanish or English, depending on the language used in each publication, to comply with the requirements of the international mention. This has an impact on the numbering of the Figuras and Tablas in Spanish, and of the Figures and Tables in English, which is independent according to the language. Thus, this doctoral thesis is written as follows:

- Chapter 1: Written entirely in English.
- Chapters 2 and 5: Written entirely in Spanish.
- Chapters 3, 4, and 6: Publications written in English with additional information in Spanish.
- Chapter 7: Written primarily in English with the final section in Spanish.

The order in which the publications are presented (Chapters 2 – 6) is based on the level of knowledge and use of the Didactic Suitability Criteria in the reflection on teaching practice carried out by the participating teachers according to the context in which the analyses of their practical argumentation were conducted. Therefore, after this first chapter, in which the motivations, backgrounds, and the research problem and objectives were presented, the structure of this doctoral thesis is as follows:

- Chapter 2 presents the article entitled «Criterios de idoneidad didáctica en la argumentación de formadores de profesores de matemática», accepted for publication in *Uniciencia* journal. The objective of this study was to analyse the practical arguments of a group of mathematics teachers' educators during the assessment of the design of a didactic unit on functions for secondary education students (aged 15–16). With this article, the first specific objective of this doctoral research is fulfilled.
- Chapter 3 presents the article entitled «Knowledge and beliefs on mathematical modelling inferred in the argumentation of a prospective teacher when reflecting

on the incorporation of this process in his lessons», published in *Mathematics* journal. The objective of this study was to infer, based on the practical argumentation made in his teaching reflection on the design, implementation, and redesign of a modelling task, the knowledge, beliefs, values, and guidelines for action on mathematical modelling of a prospective teacher. With this article, the second specific objective of this doctoral research is fulfilled.

- Chapter 4 presents the manuscript entitled «Teachers' practical argumentation on the teaching of the Pythagorean theorem», submitted for evaluation to *International Journal of Science and Mathematics Education*. The objective of this study was to analyse two episodes of practical argumentation in which a group of practising teachers, involved in a training course based on Lesson Study, reflected on the design, implementation, and redesign of a lesson with the Pythagorean theorem. Although this manuscript is not intended to fulfil any specific objective, its inclusion is justified because it complements the development of the third specific objective of this doctoral research.
- Chapter 5 presents the book chapter entitled «Reflexión docente sobre la enseñanza de la introducción a la derivada en un proceso de instrucción», published in *Actas del Primer Congreso Internacional de Didáctica de la Matemática*. The objective of this study was to identify the teaching practical argumentation emerging from the teaching reflection guided by the didactic suitability criteria in the analysis and teaching assessment of a lesson implemented in a teaching training course that combines the Lesson Study methodology and the Didactic Suitability Criteria. With this book chapter, the third specific objective of this doctoral research is fulfilled.
- Chapter 6 presents the article entitled «Didactic suitability criteria that emerge in the future teachers' practical argumentation», published in *Zetetiké* journal. The objective of this study was to analyse the didactic suitability criteria that emerge in the practical argumentation of future teachers who participated in a Lesson Study cycle, in which they planned a lesson on plane geometry for pupils in Early Childhood Education. With this article, the fourth specific objective of this doctoral research is fulfilled.

The presentation of the publications attempted to achieve a balance between their original design and the formal requirements of this doctoral thesis. For example, consecutive numbering is followed for Figuras/Figures and Tablas/Tables in Spanish/English, in addition to maximising their sizes for better reading and understanding, and all references were placed at the end of this document (not after each publication). Finally, Chapter 7 presents the conclusions of this doctoral research.

Capítulo 2: Primer Artículo Científico

El artículo científico presentado en este capítulo tiene por objetivo analizar los argumentos prácticos de un grupo de formadores de profesores de matemática durante la valoración del diseño de una unidad didáctica sobre funciones para estudiantes de educación secundaria (15–16 años), atendiendo así al cumplimiento del primer objetivo específico de esta investigación doctoral.

Los avances del desarrollo de este objetivo específico fueron presentados en diferentes eventos académicos, como la *Conference of the International Group for the Psychology of Mathematics Education* (PME 46), la *Reunión Latinoamericana de Matemática Educativa* (RELME 36), el *Simposio de la Sociedad Española de Investigación en Educación Matemática* (SEIEM 36 y SEIEM 37), el *Seminário Internacional de Lesson Study no Ensino de Matemática* (II SILSEM), el *Simposio Internacional de Educación Matemática y Secundaria*, el *Seminario Latinoamericano de Colaboración sobre el Enfoque Onto-Semiótico*, el *Congreso Internacional Multidisciplinar de Investigación Educativa* (CIMIE24), y el *Seminário Internacional de Educação em Ciências, Educação Matemática e Tecnologias Educativas* (II SIECEMTE).

En la Tabla 1, se presentan los datos de identificación de este primer artículo.

Tabla 1. Datos de identificación del primer artículo científico.

Criterios	Datos
Título	Criterios de idoneidad didáctica en la argumentación práctica de formadores de profesores de matemática.
Autores	Telesforo Sol, Carlos Ledezma, Adriana Breda, Vicenç Font.
Afiliaciones	Departamento de Educación Lingüística, Científica, y Matemática, Facultad de Educación, Universidad de Barcelona (España).
Revista	Uniciencia.
Fecha de envío	7 de julio de 2024.
Fecha de aceptación	30 de septiembre de 2024.
Referencia	Sol, T., Ledezma, C., Breda, A., y Font, V. (en prensa). Criterios de idoneidad didáctica en la argumentación práctica de formadores de profesores de matemática. *Uniciencia*.

Fuente: Elaboración del autor.

Esta revista está indexada en más de 30 bases de datos, incluyendo Dialnet, MIAR, REDIB, Scopus, Google Académico, entre otras.

De acuerdo con la información disponible en Scopus, esta revista está indexada en las categorías Matemática/Matemática general (rango #156/399, 61º percentil), Ciencias

Sociales/Ciencias sociales generales (rango #112/275, 59º percentil), entre otras, con un CiteScore en 2023 de 1,6 y un CiteScore Tracker en 2024 de 1,6.
De acuerdo con la información disponible en Scimago Journal & Country Rank (SJR), esta revista está indexada en las categorías Matemática/Misceláneo, Ciencias Sociales/Misceláneo, entre otras, su Índice H es 8 y el SJR 2023 es 0,27 (Q2 en Ciencias Sociales).
De acuerdo con la información disponible en el Master Journal List de Web of Science (WoS), esta revista está indexada en el Emerging Sources Citation Index (ESCI) en la categoría Ciencias Multidisciplinarias, con un Indicador de Citas de Revista de 0,22 en 2023.

Resumen y Palabras Clave / Abstract and Keywords

Resumen: [Objetivo] El objetivo de este estudio fue analizar los argumentos prácticos de un grupo de formadores de profesores de matemática durante la valoración del diseño de una unidad didáctica sobre funciones para estudiantes de educación secundaria (15–16 años). [Metodología] Se siguió una metodología cualitativa que consistió en un estudio de caso intrínseco, donde ocho formadores de profesores de matemática, que conocían el constructo Criterios de Idoneidad Didáctica, participaron en un ciclo de Estudio de Clases. En la etapa de reflexión de este ciclo, los participantes utilizaron este constructo para valorar el diseño de la unidad didáctica (planificación del profesor y dosier para los estudiantes). Los argumentos prácticos emergentes se estructuraron utilizando el modelo de Toulmin. [Resultados] Se destacan tres resultados: primero, la valoración de la unidad didáctica con los componentes de los Criterios de Idoneidad Didáctica permitió identificar los aspectos que favorecían y que requerían mejoras en el proceso de enseñanza y aprendizaje de las funciones; segundo, estos componentes se utilizaron como datos, respaldos, y garantías en los argumentos prácticos elaborados por los participantes, debido a su conocimiento de este constructo; y tercero, los participantes utilizaron todos estos componentes para valorar la unidad didáctica y argumentaron sus propuestas de cambio basándose en tales componentes. [Conclusiones] El papel de los Criterios de Idoneidad Didáctica en la argumentación práctica de los participantes fue de generador de argumentos prácticos y de elemento central en la estructuración de estos argumentos.

Finalmente, se proponen algunas consideraciones para futuras implementaciones de ciclos de Estudio de Clases.

Palabras clave: Argumentación práctica; Criterios de idoneidad didáctica; Educación matemática; Estudio de clases; Funciones; Reflexión docente.

Abstract: [Objective] The objective of this study was to analyse the practical arguments of a group of mathematics teachers' educators during the assessment of the design of a didactic unit on functions for secondary education students (aged 15–16). [Methodology] A qualitative methodology was followed, which consisted of an intrinsic case study, where eight mathematics teachers' educators, who knew the Didactic Suitability Criteria construct, were involved in a Lesson Study cycle. In the reflection stage of this cycle, the participants used this construct to assess the design of the didactic unit (teacher's plan and students' dossier). The emerging practical arguments were structured using Toulmin's model. [Results] Three results are highlighted: first, the assessment of the didactic unit with the components of Didactic Suitability Criteria allowed it to identify the aspects that favoured and that required improvement in the teaching and learning process of functions; second, these components were used as data, backings, and warrants in the practical arguments elaborated by the participants, due to their knowledge of this construct; and third, the participants used all these components to assess the didactic unit and argued their proposals of change based on such components. [Conclusion] The role played by Didactic Suitability Criteria in the participants' practical argumentation was as a generator of practical arguments and as a central element in the structuring of these arguments. Finally, some considerations for future implementations of Lesson Study cycles are proposed.

Keywords: Didactic suitability criteria; Functions; Lesson study; Mathematics education; Practical argumentation; Teacher reflection.

2.1. Introducción

El desarrollo de la reflexión docente sobre la práctica es un aspecto clave en la formación de profesores (Schön, 1987) pues cuando esta reflexión se hace habitual, se puede convertir en un importante mecanismo para mejorar la práctica educativa (Schoenfeld y Kilpatrick, 2008). No obstante, el desarrollo de esta reflexión requiere adoptar marcos conceptuales y metodológicos que permitan estructurarla y guiarla, donde una instancia conveniente es, por ejemplo, cuando se trabaja con el Estudio de Clases (Huang et al., 2019). Este enfoque se basa en que un grupo de profesores y expertos, con un interés compartido en el aprendizaje de los estudiantes, se reúnen, planifican una clase y, finalmente, examinan y discuten sus observaciones durante la implementación de dicha clase (Fernández y Yoshida, 2004; Hart et al., 2011; Lewis, 2002).

En España, en línea con otras investigaciones desarrolladas en el contexto iberoamericano, se ha combinado la metodología del Estudio de Clases con el constructo Criterios de Idoneidad Didáctica (Godino, 2013). Esto se debe a que, si bien el Estudio de Clases es reconocido como una potente estrategia para el desarrollo profesional docente, aun así, se trata de una metodología muy general y con un bajo nivel teórico (Elliott, 2012). De este modo, el uso de constructos de algún modelo teórico, como los Criterios de Idoneidad Didáctica propuestos por el Enfoque Onto-Semiótico (Godino et al., 2007), son contenidos por el Estudio de Clases como un envoltorio o infraestructura, implementándose en los procesos formativos de profesores a nivel inicial y continuo (véase Font et al., 2023; Franzen, 2022; Sol, Lujambio et al., 2023; entre otros). En estas investigaciones, donde se implementa un ciclo de Estudio de Clases y se enseña el constructo Criterios de Idoneidad Didáctica, se ha puesto de manifiesto la importancia que tiene la argumentación práctica, es decir, aquélla en la que se producen a) argumentos cuya conclusión es una acción que se realiza, o b) argumentos sobre acciones consideradas en la discusión, pero que no se realizan (Gómez, 2017). En un ciclo de Estudio de Clases emergen consensos en la reflexión sobre la práctica (Erbilgin y Arikan, 2021), donde los profesores justifican acciones que acuerdan realizar mediante argumentos prácticos, los cuales se respaldan en principios sobre lo que consideran es la opción más conveniente para mejorar dicha práctica (Hummes, Breda, y Font, 2022). Así, el estudio de la argumentación práctica aporta herramientas para analizar cómo se alcanzan los consensos en el diálogo entre los participantes, donde se dan razones a favor y en contra para realizar o no ciertas acciones (Kinach, 2002). Por lo tanto, este tipo de estudios son un buen contexto para investigar sobre el papel que juegan los Criterios de Idoneidad Didáctica en la argumentación práctica del profesor para justificar la realización (o no) de una acción, en que primero se identifican episodios de argumentación práctica (Lewiński, 2018) en la discusión del grupo y, luego, estos episodios se analizan utilizando modelos de argumentación como referencia teórica (véase Sol, Sánchez et al., 2022; Sol, Breda et al., 2023; entre otros).

Por lo tanto, dada la importancia de la reflexión docente sobre la práctica y de la argumentación práctica en dicha reflexión, en este estudio se planteó la siguiente pregunta de investigación: ¿Cuál es el papel de los Criterios de Idoneidad Didáctica en la argumentación práctica de un grupo de formadores de profesores durante la valoración del diseño de una unidad didáctica sobre funciones? Para ello, se reportan los análisis de los argumentos prácticos elaborados por un grupo de formadores de profesores de matemática, quienes participaron de un ciclo de Estudio de Clases, durante la valoración – utilizando el constructo Criterios de Idoneidad Didáctica – del diseño de una unidad didáctica sobre funciones para estudiantes de educación secundaria (15–16 años). Estos argumentos prácticos emergentes se estructuraron utilizando el modelo de Toulmin (1958/2003) para así caracterizar el papel de este constructo en las estructuras obtenidas de los argumentos.

2.2. Referentes Teóricos

En esta sección, se describen los referentes teóricos consideradas en este estudio.

2.2.1. Criterios de idoneidad didáctica

En el Enfoque Onto-Semiótico (EOS), la noción de idoneidad didáctica se considera una herramienta esencial para valorar los procesos de enseñanza y aprendizaje matemáticos (Godino, 2013), la cual corresponde al grado en que las características de estos procesos se pueden considerar como óptimas o adecuadas para lograr una superposición entre los *significados personales* alcanzados por los estudiantes (aprendizaje) y los *significados institucionales* pretendidos o implementados por un profesor (enseñanza), considerando las circunstancias y recursos disponibles (entorno) (Godino et al., 2019). De este modo, dado un tema específico para enseñar en un determinado contexto educativo, la noción de idoneidad didáctica permite responder, entre otras, a la pregunta: ¿Qué cambios se deberían realizar en el diseño e implementación de un proceso de enseñanza y aprendizaje matemático para aumentar su idoneidad didáctica en el futuro? La respuesta a esta pregunta, algunas veces, incluye acciones que el profesor debe o propone hacer, las cuales forman parte del interés de este estudio.

Así, esta noción se materializa en un constructo multidimensional que consiste en seis criterios de idoneidad didáctica (CID), cuya operatividad se concretiza en componentes para cada criterio y en indicadores para cada componente. De este modo, un proceso de enseñanza y aprendizaje matemático alcanzará un alto grado de idoneidad didáctica si es capaz de articular, de manera coherente y sistemática, los seis CID, referidos a cada una de las dimensiones que intervienen en estos procesos. En este estudio, se utiliza la pauta de CID propuesta por Breda et al. (2017) que, a su vez, es una adaptación de la pauta propuesta por Godino (2013) para usar en los procesos formativos de futuros profesores de matemática de educación secundaria (por ejemplo, Ledezma, Sánchez, e Hidalgo-Moncada., 2024; Sánchez et al., 2022; entre otros), así como también en el caso que se reporta en esta investigación. En la Tabla 2, se presenta la descripción y los componentes de cada CID.

Tabla 2. Criterios de idoneidad didáctica y sus componentes.

Criterios	**Descripción**	**Componentes**
Epistémico	Para valorar si la matemática que se enseña es una *matemática de buena calidad*.	– Errores. – Ambigüedades. – Riqueza de procesos. – Representatividad de la complejidad del objeto matemático.
Cognitivo	Para valorar, antes de iniciar el proceso de enseñanza y aprendizaje, si el tema a enseñar está adaptado a lo que los estudiantes ya saben y, después de este proceso, si el aprendizaje alcanzado está cerca de lo que se pretendía.	– Conocimientos previos. – Adaptaciones curriculares a las diferencias individuales. – Aprendizaje – Alta demanda cognitiva.
Interaccional	Para valorar si la interacción resuelve las dudas o dificultades de los estudiantes.	– Interacción docente-discente. – Interacción entre discentes. – Autonomía. – Evaluación formativa.
Mediacional	Para valorar la adecuación de los recursos materiales y temporales usados en el proceso de enseñanza y aprendizaje.	– Recursos materiales. – Número de estudiantes, horario y condiciones de aula. – Tiempo.
Afectivo	Para valorar el involucramiento de los estudiantes (intereses y motivación) durante el proceso de enseñanza y aprendizaje.	– Intereses y necesidades. – Actitudes. – Emociones.

Criterios	Descripción	Componentes
Ecológico	Para valorar la adaptación del proceso de enseñanza y aprendizaje al proyecto educativo escolar, las directrices curriculares, y las condiciones del entorno social y profesional, etc.	– Adaptación al currículo. – Conexiones intra- e interdisciplinarias. – Utilidad sociolaboral. – Innovación didáctica.

Fuente: Adaptado desde Ledezma, Sánchez e Hidalgo-Moncada (2024, p. 7).

Breda et al. (2017) presentan rúbricas con indicadores para cada uno de los componentes de la Tabla 2 como, por ejemplo, los indicadores de los componentes del criterio *epistémico* que se presentan en la Tabla 3.

Tabla 3. Componentes del criterio epistémico y sus indicadores.

Componentes	Indicadores
Errores	– No se observan prácticas que se consideren no válidas desde el punto de vista matemático.
Ambigüedades	– No se observan ambigüedades que puedan llevar a confusión a los estudiantes; definiciones y procedimientos clara y correctamente enunciados, adaptados al nivel educativo al que se dirigen; adecuación de las explicaciones, comprobaciones, y demostraciones al nivel educativo al que se dirigen; uso controlado de metáforas, etc.
Riqueza de procesos	– La secuencia de tareas contempla la realización de procesos relevantes en la actividad matemática (modelización, argumentación, conexiones, etc.).
Representatividad de la complejidad del objeto matemático	– Los significados parciales (definiciones, propiedades, procedimientos, etc.) son una muestra representativa de la complejidad de la noción matemática que se quiere enseñar teniendo en cuenta el nivel educativo. – Para uno o varios significados parciales, se ofrece una muestra representativa de problemas. – Para uno o varios significados parciales, se usan diferentes modos de expresión (verbal, gráfico, simbólico…) y conversiones entre los mismos.

Fuente: Adaptado desde Breda et al. (2017, p. 1903, traducción de los autores).

Los CID representan una pauta (con criterios, componentes, e indicadores) que se desarrolló considerando las tendencias contemporáneas en la enseñanza de la matemática, los principios del *National Council of Teachers of Mathematics* (NCTM, 2000), y los hallazgos de la investigación en Didáctica de la Matemática que son ampliamente aceptados por la comunidad profesional (Breda et al., 2018). Por ejemplo, el criterio *epistémico* considera el siguiente hallazgo: los objetos matemáticos emergen de las prácticas matemáticas, lo que conlleva su complejidad (véase Font et al., 2013). A partir de este hallazgo, se deriva el componente *Representatividad de la complejidad del objeto* *matemático* e implica guiar a los profesores para que consideren la complejidad matemática de la noción a enseñar cuando diseñen secuencias didácticas.

2.2.2. Argumentación práctica

La argumentación práctica es definida por Lewiński (2018) como "la argumentación destinada a decidir un curso de acción" (p. 219, traducción de los autores) y, de manera similar, Gómez (2017) la entiende como aquello que sucede en contextos sociales y que se orienta hacia escoger una acción para resolver un problema práctico. La argumentación práctica emerge de las prácticas didácticas, generando acciones planificadas o ejecutadas en el aula, y permite responder preguntas como '¿cómo justifica *a* la elección de la acción *b* ante una situación *c*?', donde *a* es el nombre o la descripción de un agente, *b* es la acción, y *c* es la descripción de una situación problemática donde puede haber varias opciones posibles para realizar. Lewiński (2018) y Gómez (2017) coinciden en que debe de existir la elección de una acción y la argumentación explícita o implícita para esta elección, lo cual no sucede cuando sólo se afirma si algo es verdadero o falso (como en la argumentación matemática).

El modelo de Toulmin (1958/2003), aunque no fue pensado específicamente para el análisis de la argumentación práctica, también se puede aplicar a este tipo de argumentación. De acuerdo con este autor, un argumento se compone de los siguientes elementos:

- Premisas: Enunciados que se buscan validar.
- Datos: Información en la cual se basa la premisa o conclusión.
- Garantías: Proposiciones (reglas, principios, o enunciados) que permiten realizar inferencias y que conectan los datos con la premisa.
- Refutaciones: Circunstancias en las que la garantía es inválida.
- Calificadores modales: Enunciados que indican el grado de fuerza de la garantía en la relación entre datos y premisa, con palabras como 'necesariamente', 'probablemente', 'generalmente', 'casi', 'quizás', etc.
- Respaldos: Justificaciones que respaldan las garantías. Estos respaldos pueden ser: 1) conocimiento generado mediante la propia práctica docente, lo cual Fenstermacher y Richardson (1993) llaman *premisa empírica* o 2) conocimiento

científico, lo cual Fenstermacher y Richardson (1993) llaman *premisa estipulativa*.

2.3. Metodología

En este estudio, se siguió una metodología de investigación cualitativa desde un paradigma interpretativo (Cohen et al., 2018), que consiste en un estudio de caso intrínseco (Stake, 2005), donde se analizaron los episodios de argumentación práctica de un grupo de formadores de profesores de matemática durante un ciclo de Estudio de Clases (EC).

2.3.1. Contexto de la investigación

Esta investigación se desarrolló en el contexto de un ciclo de EC, donde uno de sus objetivos fue identificar evidencias del uso de los CID en la argumentación práctica de los participantes. Si bien existen diferentes modelos para el enfoque EC, todos incluyen las siguientes etapas, las cuales se siguieron – con ligeras modificaciones – en esta investigación (Hurd y Lewis, 2011: Lim-Ratnam, 2013):

- Definición del problema, elección de un contenido matemático, estudio de su abordaje curricular, y establecimiento de los objetivos de aprendizaje.
- Planificación de la clase.
- Implementación y observación de la clase, registrando el impacto de la planificación sobre el aprendizaje de los estudiantes y recolectando datos mediante la observación.
- Reflexión conjunta sobre los datos registrados.

En este ciclo de EC, el contenido matemático elegido fue el de funciones para estudiantes de cuarto año de Educación Secundaria Obligatoria (ESO) (15–16 años), y el abordaje curricular se basó en las directrices del Departamento de Educación de Cataluña (España).

Este ciclo se implementó durante el periodo octubre/2022 – marzo/2023 y constó de 17 sesiones, con una duración aproximada de dos horas cada una. Si bien todas las sesiones del ciclo de EC se desarrollaron de manera presencial, igualmente, se decidió grabarlas mediante la plataforma Microsoft Teams® proporcionada por la universidad. En esta plataforma, todos los participantes se conectaron a una reunión y, de este modo, se registraron sus intervenciones como videograbaciones. Todos los formadores de profesores participaron voluntariamente en este ciclo de EC y en esta investigación, previo consentimiento informado.

2.3.2. Participantes del Estudio de Clases

Los participantes de este ciclo de EC fueron ocho formadores de profesores de matemática y su composición (incluidas sus iniciales) fue la siguiente:

- Tres profesores universitarios a tiempo completo expertos en los CID (AB, GS, VF).
- Una profesora visitante experta en EC (AR).
- Tres estudiantes de doctorado/profesores asociados (AC, NI, TS).
- Una profesora asociada/profesora de matemática de ESO (AS).

Todos los participantes ya estaban familiarizados con el uso de los CID antes de iniciar el ciclo de EC; además, con excepción de la profesora visitante, todos se encontraban vinculados al Departamento de Educación Lingüística, Científica, y Matemática de la Universidad de Barcelona (España); la experiencia en docencia de la matemática de los participantes fluctuaba entre algunos meses y los 25 años; y cuatro de ellos ya habían participado previamente en otros ciclos de EC.

2.3.3. Diseño y valoración de la unidad didáctica

La unidad didáctica (UD) se diseñó para implementarse en un instituto de educación secundaria de la provincia de Barcelona (España), con estudiantes de cuarto año de ESO (15–16 años), a cargo de la profesora AS (participante del ciclo de EC). De este modo, durante la primera etapa del ciclo de EC, se planificó conjuntamente la UD, estructurándola en cuatro sesiones, de una hora cada una, destinadas a implementarse en una semana, como se muestra a continuación:

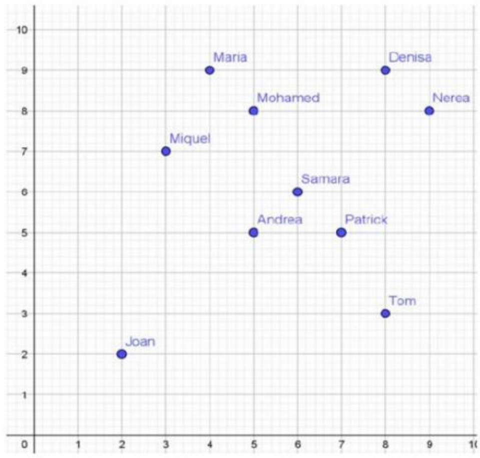
- Sesión 1: Repaso de los conocimientos previos de los estudiantes.
- Sesión 2: Definición del concepto de función.
- Sesión 3: Características de las funciones.
- Sesión 4: Actividad de experimentación y cierre.

Los contenidos para trabajar fueron la definición de función, variable dependiente e independiente, domino y recorrido, tratamiento y conversión entre representaciones de una función, crecimiento y decrecimiento, etc. Todo lo anterior, se encontraba supeditado por los objetivos de aprendizaje planteados por las directrices curriculares del Departamento de Educación de Cataluña (España).

Junto con la UD antes descrita, también se diseñó un dosier para los estudiantes que incluyó una actividad inicial, enfocada en revisar los conocimientos previos de los estudiantes mediante la ubicación de puntos en el plano cartesiano (ver Figura 1), y nueve actividades para trabajar durante las sesiones.

Actividad inicial: Las notas de los amigos de Samara

Samara y sus amigos están comentando las notas que han sacado en las asignaturas de Ciencias y Lenguas.



Si el eje de las abscisas representa las notas de Lenguas y el eje de las ordenadas las notas de Ciencias, completa el gráfico con la información que falta y responde a las siguientes preguntas:

a) ¿Qué asignatura aprueban más los alumnos?

b) ¿Hay alumnos que sacan la misma nota de una asignatura que de la otra? ¿Cuáles son?

c) ¿Qué alumnos sacan mejor nota en Ciencias que en Lenguas?

d) ¿Y qué alumnos sacan mejor nota en Lenguas que en Ciencias?

e) Añade al gráfico a Pedro, quien ha sacado un 4 en Lenguas y un 7 de Ciencias.

f) Rebecca estuvo enferma en el examen de Lenguas, pero en Ciencias ha sacado un 9, ¿podemos situarla en el gráfico? Si es posible, añádela al gráfico.

g) Raza siempre saca la misma nota de las dos asignaturas, ¿podrías acotar los puntos donde estarán sus notas? Si se puede, indícalo en el gráfico.

Figura 1. Actividad inicial de la UD.
Fuente: Elaboración del grupo de EC.

En este artículo, se reporta la valoración realizada por los participantes, utilizando el constructo CID, del diseño de la UD antes descrita. Este diseño incluyó la planificación de la profesora y las actividades del dosier para los estudiantes, cuya valoración se realizó durante dos sesiones del ciclo de EC (15 y 16), como se detalla a continuación (Breda et al., 2023, p. 195):

- Sesión 15: Valoración de la UD utilizando los componentes e indicadores de los criterios *epistémico* y *cognitivo*.
- Sesión 16: Valoración de la UD utilizando los componentes e indicadores de los criterios *interaccional*, *mediacional*, *afectivo*, y *ecológico*.

Para la valoración de la UD, se pidió a los participantes que, previo a la sesión 15, la valoraran individualmente utilizando los criterios y componentes de los CID. Así, durante las sesiones 15 y 16, cada participante presentó sus valoraciones de la UD y los demás las comentaron, expresando su acuerdo o desacuerdo y, de ser necesario, añadiendo información adicional. Para ello, se elaboró una tabla en la cual se escribió la valoración consensuada por todos los participantes del ciclo de EC, según la siguiente estructura: las primeras dos columnas incluyeron los criterios y componentes de los CID, respectivamente; en la tercera columna, se indicaron los elementos que favorecían la idoneidad didáctica de la UD; en la cuarta columna, los elementos que la disminuían; y en la última columna, se propusieron elementos para mejorar la idoneidad didáctica de la UD. A modo de ejemplo, se presenta un fragmento de la valoración consensuada con los componentes de los criterios *mediacional* (Tabla 4) y *cognitivo* (Tabla 5).

Tabla 4. Fragmento de la valoración consensuada de la UD con los componentes del criterio mediacional.

Componentes	Elementos que favorecen la idoneidad didáctica	Elementos que disminuyen la idoneidad didáctica	Propuestas de mejora para la UD
Recursos materiales	Durante la etapa de diseño, se realizó una sesión para revisar los recursos utilizados. (GeoGebra para graficar puntos, uso de calculadora, interMatia, muelles, etc.).	No se identifican elementos que disminuyen el grado de idoneidad de este componente.	La valoración de este componente es positiva y no se proponen cambios.
Número de estudiantes, horario y condiciones de aula	En el diseño, se consideró que el número de estudiantes, las condiciones del aula y el horario eran adecuados. También,	Se sabe que la clase es después del descanso de los estudiantes y que se invierte entre 5 y 10 minutos de la clase para que los	La valoración de este componente es positiva y no se proponen cambios.

Componentes	Elementos que favorecen la idoneidad didáctica	Elementos que disminuyen la idoneidad didáctica	Propuestas de mejora para la UD
	la participante AS hizo una descripción del contexto.	estudiantes se sienten.	
Tiempo	La UD se distribuye en cuatro sesiones. El tiempo no tiene una distribución más allá de estas sesiones y se dedica exclusivamente al tema de función.	No se observa una asignación de tiempo aproximado para las actividades; tampoco, una distribución aproximada de las sesiones en inicio-desarrollo-cierre.	Para aprovechar mejor el tiempo en el aula, se deben asignar tiempos aproximados a las actividades que se realizarán en cada sesión y estructurar las sesiones en inicio-desarrollo-cierre, aunque los tiempos no se vayan a cumplir en la implementación.

Fuente: Elaboración del grupo de EC.

Tabla 5. Fragmento de la valoración consensuada de la UD con los componentes del criterio cognitivo.

Componentes	Elementos que favorecen la idoneidad didáctica	Elementos que disminuyen la idoneidad didáctica	Propuestas de mejora para la UD
Conocimientos previos	La UD contiene actividades para valorar y trabajar los conocimientos previos de los estudiantes.	No se identifican elementos que disminuyen el grado de idoneidad de este componente.	La valoración de este componente es positiva y no se proponen cambios.
Adaptación curricular a las diferencias individuales	En el diseño, se consideraron las características de los estudiantes y se decidió que la UD fuera prácticamente una unidad de refuerzo.	No hay actividades que trabajen la diversidad en el aula.	Para aumentar el grado de idoneidad de este componente, la profesora puede organizar los grupos de trabajo en la clase para que sean heterogéneos.
Aprendizaje	La UD contiene varios elementos que permiten valorar el aprendizaje de los estudiantes.	No se identifican elementos que disminuyen el grado de idoneidad de este componente.	La valoración de este componente es positiva y no se proponen cambios.
Alta demanda cognitiva	Las tareas propuestas en la UD exigían una	No se exigía un nivel máximo de demanda cognitiva, al no	La demanda cognitiva de la UD es razonable para

Componentes	Elementos que favorecen la idoneidad didáctica	Elementos que disminuyen la idoneidad didáctica	Propuestas de mejora para la UD
	demanda cognitiva elevada.	desarrollarse generalizaciones ni procesos metacognitivos.	este grupo y no es necesario realizar modificaciones para este componente.

Fuente: Elaboración del grupo de EC.

2.3.4. Análisis de datos

El análisis de los datos se desarrolló considerando las videograbaciones de las dos sesiones de valoración (15 y 16) del diseño de la UD. Para ello, se transcribieron los diálogos de estas videograbaciones y se dividieron en segmentos; cada segmento correspondió a las valoraciones de la UD que hicieron los participantes del ciclo de EC a partir de cada uno de los componentes de los CID (ver Tabla 2). De este modo, se generaron 22 segmentos entre las dos videograbaciones. Luego, en cada uno de estos segmentos, se identificaron los episodios de argumentación práctica donde los participantes propusieron acciones para aumentar el grado de idoneidad de la UD con respecto al componente en el que se basaba dicho segmento. Los argumentos prácticos de otro tipo, como aquéllos relacionados con decidir cómo realizar la valoración de la UD, no fueron considerados. De este modo, se identificaron ocho argumentos prácticos en los 22 segmentos.

La identificación y el análisis de los argumentos prácticos se realizaron a partir del siguiente procedimiento:

- Primero, se buscaron los argumentos prácticos en la discusión, donde se identificaron el dato, la garantía, y la premisa de estos argumentos. Para ello, se tuvo en cuenta la presencia de palabras como, por ejemplo, 'por lo tanto', 'entonces', 'si', 'a menos que', entre otras, que indicaran la existencia de argumentos. Además, la premisa debía incluir la sugerencia de una acción como, por ejemplo, qué cambios hacer en la UD, alguna recomendación para la profesora durante la clase, o cómo actuar con los estudiantes ante una determinada situación.
- Segundo, estos argumentos prácticos se estructuraron según el modelo de Toulmin de la siguiente manera: en la discusión, se identificó el dato (algo que se puede mejorar de la UD o alguna información del contexto) y la premisa (acción que se propone para mejorar el proceso de enseñanza y aprendizaje); luego, se

planteó una garantía (muchas veces, implícita en el discurso) con sus calificadores modales (palabras como 'posiblemente', 'seguramente', o 'quizás'); después, se buscaron los respaldos para esta garantía; y finalmente, se buscaron las refutaciones.

- Tercero, cada argumento se analizó para establecer su relación con los CID, pues algunos argumentos generados en la valoración del componente de un determinado criterio evocaron otros componentes que no pertenecían, necesariamente, al mismo criterio que se estaba valorando.

Tomando como ejemplo el fragmento de diálogo presentado y analizado en la siguiente sección, se acordó que la intervención de VF en la línea 04 daba cuenta de un argumento práctico al identificar una acción para modificar el dosier de los estudiantes, siendo esta acción la premisa del argumento práctico: cambiar el texto donde se define el sistema cartesiano por el texto donde se da un procedimiento para dibujar el sistema cartesiano. Para llegar a esta premisa, los datos se obtuvieron de la línea 01, cuando NI identificó una ambigüedad en el dosier para los estudiantes (dato 1) y consideró que se debió haber puesto otro ejemplo (dato 2), y de la línea 02, cuando VF dijo que faltaba en el dosier la palabra 'normalmente' o 'habitualmente' cuando se definía el sistema de coordenadas cartesianas (dato 3). Para proponer la garantía de este argumento, se tuvo en cuenta que la propuesta de modificar el texto donde se define el sistema cartesiano tuvo por objetivo evitar la ambigüedad identificada; por lo tanto, la garantía que se propuso fue: 'Debemos intentar controlar la ambigüedad'. Para esta garantía, se identificaron dos respaldos: 1) la experiencia del participante VF, que es una *premisa empírica*, y 2) la valoración dentro del componente *Ambigüedades* del criterio *epistémico*, que es una *premisa estipulativa*.

2.4. Análisis y Resultados

En esta sección, se presentan y analizan las valoraciones de la UD realizadas por los participantes del ciclo de EC utilizando los componentes de los CID. Para ello, en cada subsección, se presentan los argumentos prácticos identificados en el diálogo de los participantes y se estructuran utilizando el modelo de Toulmin. Por cuestiones de espacio, sólo se presenta un extracto del diálogo para el primer argumento práctico identificado en la valoración de la UD con el componente *Ambigüedades* del criterio *epistémico* y, para las valoraciones con los demás componentes, sólo se describe y muestra la estructura del argumento práctico identificado.

2.4.1. Valoración de la UD con los componentes del criterio epistémico

La valoración con el componente *Errores* fue positiva pues, al revisar el contenido matemático, los participantes comentaron que no encontraron errores. Un participante comentó "están bien cuidados los errores" y los demás estuvieron de acuerdo. Al ser el primer componente con el que se valoró la UD, los participantes acordaron sobre cómo utilizar los CID para desarrollar esta valoración.

La valoración con el componente *Ambigüedades* también fue positiva. Si bien un participante comentó "en la planificación, procuramos evitar ambigüedades", sólo identificaron una ambigüedad en la UD, ante lo cual propusieron una modificación para evitarla. La ambigüedad que los participantes observaron tuvo relación con la identificación de los ejes de coordenadas con un eje horizontal y vertical, y la propuesta para evitarla fue cambiar la definición del sistema de ejes de coordenadas en la UD por la descripción del procedimiento para dibujar un sistema de ejes de coordenadas (véase Sol, Breda, Font, y Sala-Sebastià, 2024). A continuación, se presenta un extracto del diálogo donde los participantes, mientras valoraban la UD con el componente *Ambigüedades*, identificaron la ambigüedad antes descrita y propusieron una acción para controlarla.

01. NI: Quizás, sí que hemos cometido una ambigüedad, porque en el texto [*dosier para los estudiantes*] dice "el sistema de coordenadas cartesianas está formado por dos ejes perpendiculares entre sí formando un plano, el plano *XY*. El eje de abscisas es el eje horizontal […]" que, igual, hace referencia al gráfico [*que está en el dosier*], pero, entonces, yo creo que deberíamos haber puesto, al menos, otro [*gráfico*] en el que los ejes no sean horizontal y vertical. O sea, que no coincida para que [*los estudiantes*] puedan visualizar, al menos, un caso en que la perpendicularidad no sea de esta forma. […]. Yo pondría "normalmente", como en esta figura, pero yo sí que, de alguna manera, y más en el dosier [*para los estudiantes*] que está por escrito, diría que hay posibilidad de que no sea así, sólo por si algún día, de mayores, [*los estudiantes*] se lo miran.

02. VF: Sí, pero no veo problema que diga vertical y horizontal, puedes poner "normalmente" o "habitualmente", lo que noto a faltar, porque también luego todos van a ser horizontales y todos van a ser verticales, simplemente es esto. Creo que hemos hecho una discusión interesante con estos matices. Entonces, podríamos refinar un poco el texto, en el rediseño, podemos refinar este texto, [*el sistema cartesiano de ejes de coordenadas*] está formado por dos ejes perpendiculares formando un plano. El eje de abscisas es normalmente el eje horizontal, o algo así.

03. AB: Sí y, por ejemplo, en el documento anterior, estaba escrito "el procedimiento para dibujar un sistema de ejes cartesiano", no se definía el sistema de ejes cartesianos, se proponía un procedimiento para dibujar el sistema de ejes cartesianos y que había que tener varios pasos […].

04. VF: Bueno, entonces, una conclusión sería cambiar el texto que hay por este texto [*el del procedimiento para construir un sistema de coordenadas*].

En la Figura 2, se presenta la estructura del argumento práctico obtenido del diálogo anterior.

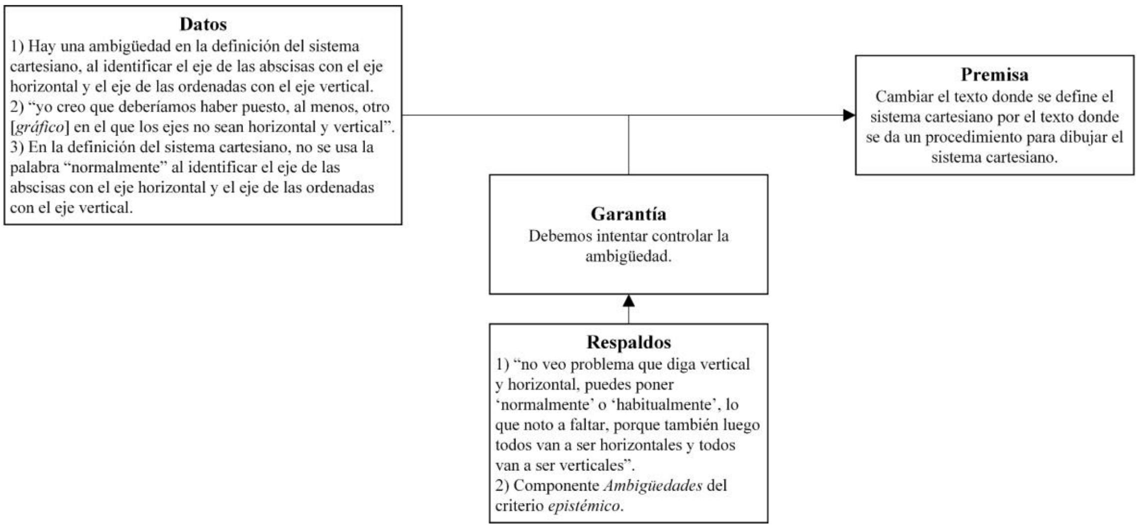


Figura 2. Argumento práctico en la valoración con el componente Ambigüedades.
Fuente: Elaboración de los autores.

Con relación al argumento práctico de la Figura 2, el respaldo 1 es de tipo *experiencial* (*premisa empírica*) y el respaldo 2 es de tipo *conocimiento* (*premisa estipulativa*) que se aplica de dos formas: primero, para generar el dato 1, identificando una ambigüedad, y segundo, para generar la garantía del tipo *se debe hacer*. De lo anterior, se tiene que el componente *Ambigüedades* tiene tres funciones en este argumento: la primera es que se utiliza como respaldo de tipo conocimiento, la segunda es que se utiliza para identificar un dato, y la tercera es que se genera una garantía específica para proponer una acción en la premisa.

Durante la valoración con el componente *Ambigüedades*, los participantes también hicieron otros comentarios como, por ejemplo, las ambigüedades generadas por los enunciados de las tareas. Uno de los participantes afirmó: “Todos los enunciados tienen una cierta imprecisión porque, si no quieres que sea impreciso, vas a tener que hacer un enunciado larguísimo”, con la finalidad de señalar que es difícil evitar las ambigüedades en los enunciados de las tareas. Otro participante argumentó a favor de plantear tareas abiertas con una cierta imprecisión para facilitar la realización de procesos matemáticos relevantes. En este diálogo, se observó que los participantes interpretaron de manera diferente la noción de ambigüedad, pero la discusión no continuó en esta dirección.

En la valoración con el componente *Riqueza de procesos*, los participantes comentaron algunos elementos de los ejercicios propuestos en la UD que favorecieron el grado de idoneidad de este componente, como la argumentación, la contextualización, y las conversiones y el tratamiento de representaciones. Entre los comentarios de los participantes, se destacan los siguientes: “hay un nivel razonable de procesos”, “la riqueza de procesos, si la tuviera que valorar sobre diez, no pasaría de siete”, y “el nivel de riqueza de procesos es suficiente”. Considerando que la valoración de la idoneidad de este componente no fue alta, un participante planteó la posibilidad de hacer una modificación para un rediseño de la UD, a partir de la cual se estructuró el argumento práctico que se presenta en la Figura 3.

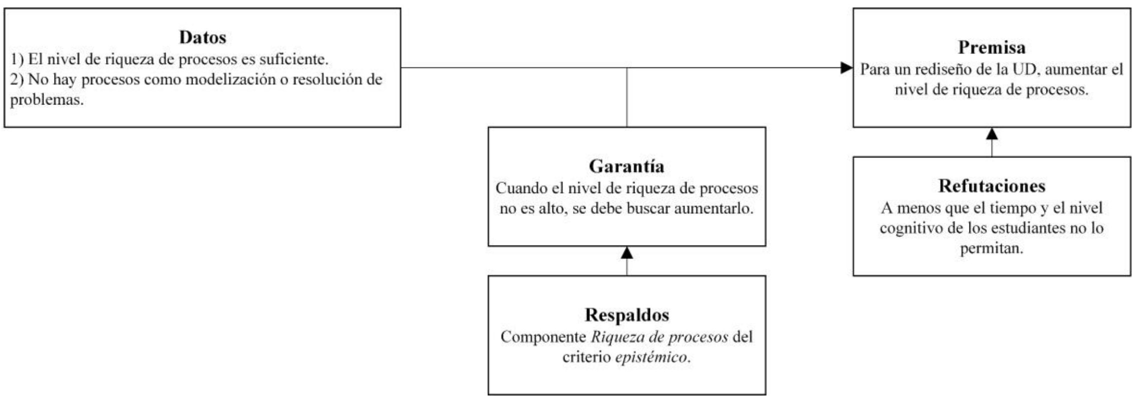


Figura 3. Argumento práctico en la valoración con el componente Riqueza de procesos.
Fuente: Elaboración de los autores.

En la refutación del argumento de la Figura 3, se puede observar cómo los componentes *Tiempo* y *Alta demanda cognitiva* de los criterios *mediacional* y *cognitivo*, respectivamente, restringen el grado de idoneidad del componente *Riqueza de procesos*. Esto se debe a que los CID se plantean como criterios dependientes e interrelacionados.

En la valoración con el componente *Representatividad de la complejidad del objeto matemático*, los participantes comentaron que este componente se cuidó en el diseño de la UD, al considerar aspectos como los diferentes significados de una función (relación entre magnitudes, relación entre variables, y subconjunto del producto cartesiano) y la buena representatividad en las representaciones de una función (lenguaje natural, simbólico, tabular, gráfico). Entre los comentarios de los participantes, se destacan los siguientes: "ésta sí que la cuidamos un poco, tuvimos en cuenta los diferentes significados que puede tener una función" y "también, cuidamos tener una representatividad de las representaciones". Dado que los participantes acordaron rápidamente una buena valoración para este componente, concluyeron que no era necesario realizar modificaciones en la UD con relación a este componente.

2.4.2. Valoración de la UD con los componentes del criterio cognitivo

En la valoración con el componente *Conocimientos previos*, los participantes se refirieron a tareas desarrolladas durante la segunda fase del ciclo de EC (diseño de la UD), como la reflexión realizada durante dos sesiones sobre los temas que ya se les habían enseñado a los estudiantes en clases anteriores y la elaboración de un cuestionario para realizar una evaluación diagnóstica de los conocimientos previos de los estudiantes en la primera sesión de la UD. Entre los comentarios de los participantes, se destacan los siguientes: "la actividad inicial sirve para revisar los conocimientos previos de una manera adecuada" y "los hemos cuidado bastante". Dado que los participantes acordaron rápidamente una buena valoración para este componente, concluyeron que no era necesario realizar modificaciones en la UD con relación a este componente, en particular, en el cuestionario previamente diseñado.

En la valoración con el componente *Adaptación curricular a las diferencias individuales*, una participante comentó que había una pregunta de ampliación en una actividad, ante lo cual, otro participante dijo que eso no era suficiente para afirmar que se hizo un tratamiento a la diversidad. Otros participantes comentaron que el tratamiento a la diversidad estaba dado por las características del grupo, ya que algunos de los estudiantes estaban en aulas de acogida y otros en grupos de refuerzo de matemática. Por lo tanto, los participantes concluyeron que no era necesario realizar modificaciones en la UD para mejorar el tratamiento a la diversidad, pero sí que podían hacerlas en la organización de los grupos de estudiantes para que su composición fuese heterogénea, atendiendo al máximo posible la diversidad presente en el aula (entre otros, género, perfil y actitud de los estudiantes). A partir de la reflexión anterior, surgió una propuesta para el tratamiento a la diversidad, la cual se estructuró en el argumento práctico de la Figura 4.

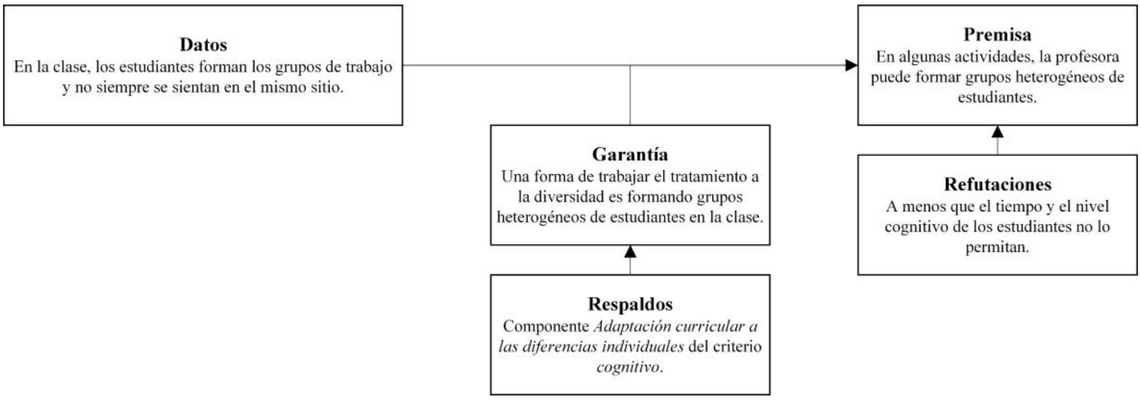


Figura 4. Argumento práctico en la valoración con el componente Adaptación curricular a las diferencias individuales.
Fuente: Elaboración de los autores.

En la valoración con el componente *Aprendizaje*, un participante comentó "se tienen bastantes elementos para ir valorando el aprendizaje de los estudiantes". Así, los participantes concluyeron que el diseño de la UD permitía tener evidencias del aprendizaje de los estudiantes durante su implementación, como las tareas con rúbrica de evaluación, el cuestionario de conocimientos previos, y el acceso a la aplicación interMatia (https://www.intermatia.com) para que los estudiantes resolvieran ejercicios de manera interactiva.

En la valoración con el componente *Alta demanda cognitiva*, los participantes comentaron que, si bien las tareas exigían una demanda cognitiva elevada, no se exigía a un nivel máximo, al no desarrollarse generalizaciones ni procesos metacognitivos en la mayoría de las tareas. Dado que los participantes valoraron que la demanda cognitiva de la UD era razonable para este grupo de estudiantes, pues algunas tareas promovieron la deducción de una fórmula y otras el establecimiento de conjeturas, concluyeron que no era necesario realizar modificaciones en la UD con relación a este componente.

2.4.3. Valoración de la UD con los componentes del criterio interaccional

En la valoración con el componente *Interacción docente-discente*, una participante comentó que la UD se diseñó para que los estudiantes trabajaran en equipo y que la profesora los invitara a opinar. Además, los participantes se refirieron a que, durante la segunda fase del ciclo de EC (diseño de la UD), la participante AS, quien había trabajado con los estudiantes, describió el contexto de la clase donde uno de los aspectos contemplados fue la interacción entre profesora y estudiantes.

En la valoración con el componente *Interacción entre discentes*, un participante comentó que en la UD se buscó fomentar el diálogo entre los estudiantes a través de actividades diseñadas para el trabajo en equipo. Además, los participantes se refirieron a que, como ya conocían el contexto de la clase, la participante AS mencionó que los estudiantes solían discutir las actividades entre ellos.

En la valoración con el componente *Autonomía*, una participante comentó que este componente se tuvo en cuenta al diseñar las actividades para que fuesen dirigidas por la profesora en lugar de no guiadas, porque sabían de la poca autonomía de los estudiantes.

En la valoración con el componente *Evaluación formativa*, los participantes comentaron que la UD incluye diferentes elementos que permitieron evaluar el aprendizaje de los estudiantes, como hojas de trabajo para las actividades de los estudiantes y preguntas planteadas por la profesora.

2.4.4. Valoración de la UD con los componentes del criterio mediacional

En la valoración con el componente *Recursos materiales*, un participante comentó que, al finalizar el diseño de la UD, se realizó una sesión para revisar los recursos considerados (GeoGebra, calculadora, interMatia, muelles, etc.). Este comentario, se interpreta como una valoración positiva para este componente.

En la valoración con el componente *Número de estudiantes, horario y condiciones de aula*, los participantes comentaron que los tres aspectos de este componente se tuvieron en cuenta durante el diseño de la UD. La participante AS, que conocía al grupo de estudiantes, explicó estos aspectos cuando contextualizó el curso, mencionando que la clase de matemática era después del descanso de los estudiantes y que esto provocaba una pérdida de tiempo al inicio, pues se debía esperar a que los estudiantes se sentaran.

En la valoración con el componente *Tiempo*, algunos participantes comentaron que no hubo una distribución del tiempo de la clase durante la etapa de diseño de la UD. Si bien se consideró distribuir cada sesión en inicio-desarrollo-cierre con un tiempo aproximado para cada parte, esta distribución no se llevó a cabo. A partir de la reflexión anterior, surgió una propuesta para ajustar los tiempos, la cual se estructuró en el argumento práctico de la Figura 5.

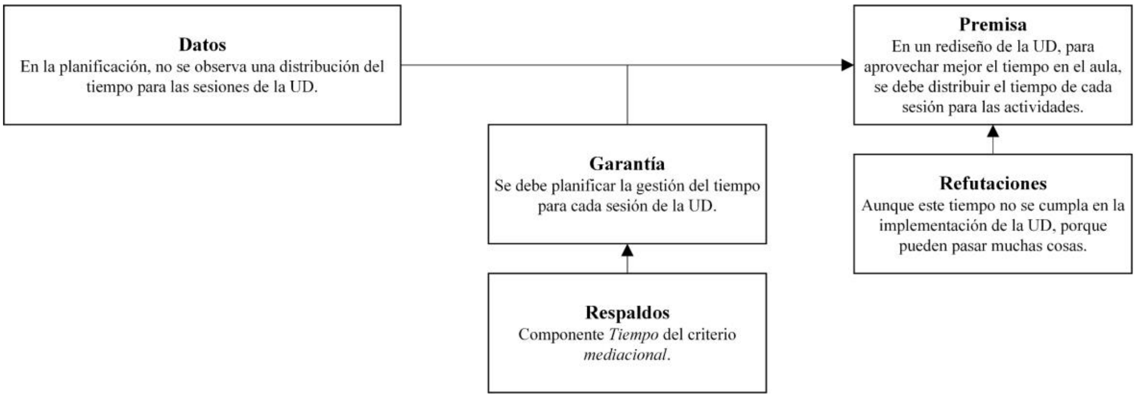


Figura 5. Argumento práctico en la valoración con el componente Tiempo.
Fuente: Elaboración de los autores.

2.4.5. Valoración de la UD con los componentes del criterio afectivo

En la valoración con el componente *Intereses y necesidades*, los participantes comentaron aspectos como que se dio importancia a presentar ejercicios contextualizados en la UD, que los ejercicios diseñados fueron resueltos antes de presentarse a los estudiantes, y que se supuso que serían interesantes para ellos. Además, los participantes recordaron que este componente se discutió en detalle durante la fase de diseño de la UD. Finalmente, los participantes comentaron que, un elemento que posibilitó alcanzar un alto grado de idoneidad didáctica en este componente fue que las actividades se diseñaron teniendo en cuenta los conocimientos previos de los estudiantes, como se explicó en la valoración de este componente en subsecciones anteriores.

En la valoración con el componente *Actitudes*, un participante comentó que las actividades de la UD buscaron favorecer la argumentación y la reflexión de los estudiantes.

En la valoración con el componente *Emociones*, un participante comentó que las actividades contemplaban ejercicios que los estudiantes pudieran resolver para que no se sintieran superados, esto con el objetivo de evitar el rechazo a la matemática.

2.4.6. Valoración de la UD con los componentes del criterio ecológico

En la valoración con el componente *Adaptación al currículo*, un participante comentó que este aspecto fue claro y que hubo un análisis profundo de las competencias del nuevo currículum de ESO. Para ello, los participantes realizaron sesiones donde analizaron todas las competencias y las relacionaron con las actividades del material para los estudiantes. Dado que los participantes acordaron rápidamente una buena valoración para este componente, concluyeron que no era necesario realizar modificaciones en la UD con relación a este componente.

En la valoración con el componente *Conexiones intra e interdisciplinarias*, un participante comentó que se hicieron pocas conexiones, señalando que hubo una con la vida cotidiana de los estudiantes, alguna con la física, y una conexión muy débil con otros conocimientos matemáticos. También, mencionó que, si bien las conexiones extra-matemáticas sí se trabajaron, las intra-matemáticas fueron muy pocas (relaciones entre las diferentes representaciones de una función) y de escasa profundidad. A pesar de esta valoración, los participantes no hicieron propuestas de cambios para la UD en este componente.

En la valoración con el componente *Utilidad sociolaboral*, un participante comentó que este aspecto estuvo contemplado porque los contenidos de matemática en la ESO son útiles para cualquier individuo, pero que para esta UD no hubo un especial énfasis en dicha utilidad. Además, otra participante comentó que el contexto fue una excusa para trabajar el contenido de funciones. Por último, otro participante comentó que, durante el diseño de la UD, habían atribuido más importancia a introducir la complejidad de las funciones para trabajar el componente *Representatividad de la complejidad del objeto matemático*, razón por la cual no se profundizó en otros componentes como el de *Utilidad sociolaboral*. A pesar de esta valoración, los participantes no hicieron propuestas de cambios para la UD en este componente.

En la valoración con el componente *Innovación didáctica*, los participantes comentaron que este aspecto se abordó, fundamentalmente, porque la UD se desarrolló en un ciclo de EC y hubo un intento de adaptarla a las competencias del nuevo currículo. A partir de la reflexión anterior, surgió una propuesta para trabajar con mayor profundidad las competencias del nuevo currículo, la cual se estructuró en el argumento práctico de la Figura 6.

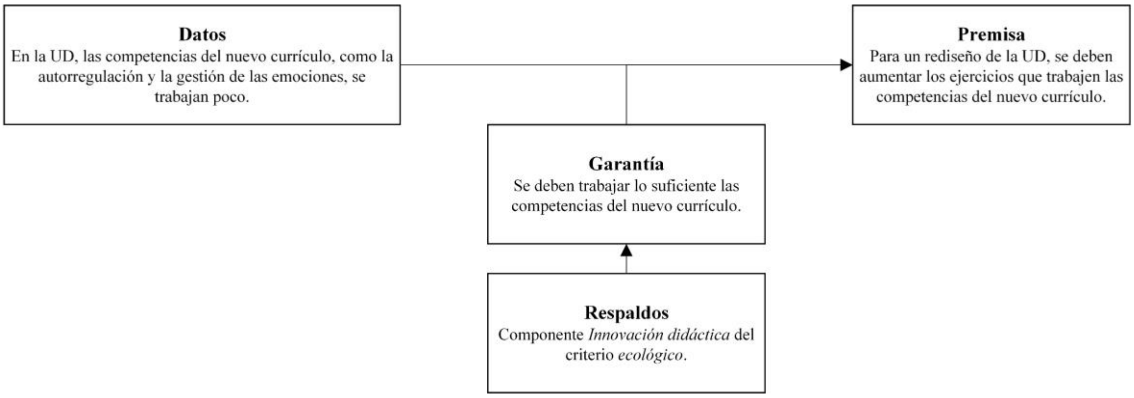


Figura 6. Argumento práctico en la valoración con el componente Innovación didáctica.
Fuente: Elaboración de los autores.

2.5. Discusión y Conclusiones

Este artículo se centró en el análisis de los argumentos prácticos elaborados por un grupo de formadores de profesores de matemática, quienes participaron de un ciclo de EC, durante la valoración – utilizando el constructo CID – del diseño de una UD sobre funciones para estudiantes de educación secundaria. En particular, se identificaron los argumentos prácticos emergente de la valoración de la UD con cada componente de los CID, y luego se estructuraron utilizando el modelo de Toulmin (1958/2003) para así caracterizar el papel de los CID en las estructuras obtenidas de los argumentos.

El primer resultado para destacar de este estudio se relacionó con la valoración de la UD guiada por los componentes de los CID. Dicha valoración, no sólo permitió identificar los aspectos que favorecían el proceso de enseñanza y aprendizaje de las funciones en el nivel de ESO, sino también, aquéllos que requerían mejoras. Cuando los participantes valoraron la UD con los componentes de los CID e identificaron posibles aspectos para mejorar, buscando aumentar la idoneidad didáctica de la UD, formularon argumentos prácticos para proponer dichos cambios. Este aspecto puede explicarse porque los CID y sus respectivos componentes, constructos para orientar la reflexión sobre la práctica docente (Breda et al., 2017, 2018), fueron el punto de partida y el elemento direccionador de la práctica reflexiva del grupo de formadores de profesores participantes del estudio. Así, el uso de los CID atiende a la demanda planteada por Burgos y Castillo (2021) sobre la necesidad de que la formación docente aporte herramientas para la valoración y el análisis crítico de los recursos educativos, lo cual se manifestó cuando estas autoras identificaron en su estudio que algunas de valoraciones de futuros profesores de educación primaria participantes en su estudio eran imprecisas o ambiguas.

El segundo resultado para destacar de este estudio se relacionó con la incidencia de los CID en la estructura de los argumentos prácticos según el modelo de Toulmin. En primer lugar, los componentes de los CID se utilizaron para identificar elementos de la UD que funcionaron como *datos* en los argumentos; en segundo lugar, se utilizaron como *respaldos* ya que, en el contexto de esta investigación, los participantes conocían los CID y sus componentes en profundidad; por último, permitieron generar *garantías* específicas para proponer una acción en la *premisa*. De manera similar a los hallazgos de Molina et al. (2023), quienes también utilizaron el modelo de Toulmin, este estudio permitió identificar explícitamente las razones de las propuestas de los participantes, las cuales se relacionaron con los componentes de los CID.

El tercer resultado para destacar de ese estudio es que los participantes utilizaron todos los componentes de cada CID para valorar la UD y argumentaron sus propuestas de cambio basándose en estos componentes. Además, todos los argumentos prácticos identificados contenían, al menos, cuatro de los elementos que propone el modelo de Toulmin: *dato*, *garantía*, *respaldo*, y *premisa*. Este último resultado difiere de otros estudios que se preocuparon por analizar la argumentación práctica de los participantes en el contexto de un ciclo de EC, donde los argumentos prácticos no siempre tuvieron los cuatro elementos antes mencionados, ya sea porque los participantes no conocían a los CID (Sol, Breda, Richit, y Sala-Sebastià, 2024), o porque el análisis de la argumentación práctica se hizo en una etapa del ciclo de EC diferente de la etapa de reflexión (Sol, Sánchez et al., 2023).

Una primera conclusión de ese estudio está relacionada con la práctica reflexiva del grupo de formadores de profesores. El conocimiento y operacionalización de los CID, por parte de los participantes, promovió una reflexión docente bastante amplia y profunda en todas las etapas del ciclo de EC. Más específicamente, elementos como conocimientos previos, evaluación, errores, ambigüedades, procesos matemáticos, currículo, tiempo y recursos didácticos, fueron centrales en el desarrollo de la reflexión de los participantes. Además, la valoración guiada por los componentes de los CID fomentó la reflexión sobre temas más generales, tales como el sistema educativo o el comportamiento de los estudiantes fuera del aula de clase. Por lo tanto, la valoración de la UD con los CID funciona como herramienta para ejercer una actividad reflexiva y, al realizar la reflexión grupal en un ciclo de EC, promueve la confrontación de ideas y la generación de argumentos prácticos, lo cual coincide con lo mencionado por Fourés (2011).

Una segunda conclusión de este estudio está relacionada con la dinámica del EC y la valoración de la UD con los CID. Se observó que la valoración permitió a los participantes aumentar su capacidad para identificar aspectos para mejorar y proponer cambios para atenderlos, ya que en este proceso compartieron conocimientos y experiencias. Además, reflexionaron sobre los componentes de los CID para aclarar su uso y significado, y discutieron sobre cómo utilizarlos en la valoración de la UD. Este resultado se encuentra en línea con los de Richit, da Ponte, y Quaresma (2021), que concluyen que el EC promueve habilidades y competencias profesionales. Por lo tanto, con este estudio, se muestra la ventaja de incorporar los CID y sus respectivos componentes en los ciclos de EC, ya sea para el diseño de la UD, su valoración, o la de la clase implementada.

Retomando la pregunta de investigación, sobre cuál es el papel de los CID en la argumentación práctica de un grupo de formadores de profesores durante la valoración del diseño de una unidad didáctica sobre funciones, se puede concluir que este papel es, en primer lugar, de generador de los argumentos prácticos y, en segundo lugar, de elemento central en la estructuración de estos argumentos. Como perspectiva futura, se pretende comparar la argumentación práctica del diseño de la UD, la valoración de este diseño, y la valoración de la clase implementada para, de este modo, poder analizar el papel que juegan los CID en diferentes momentos de reflexión durante un ciclo de EC.

Como recomendación, las futuras propuestas de formación docente deberían enfocarse en promover una reflexión sobre la práctica, en la cual se utilicen de manera equilibrada todos los componentes de los CID y fomenten la argumentación práctica docente. Para ello, se subraya la importancia de incorporar, en la etapa de reflexión de los ciclos de EC, la valoración del diseño de la UD y no solamente la de la implementación. Además, el moderador de las sesiones de los ciclos de EC podría fomentar la argumentación práctica, utilizando expresiones que permitan desencadenar justificaciones o explicaciones en los participantes que se puedan estructurar como argumentos prácticos.

Enunciados adicionales:

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- Agradecimiento: Agradecemos a los formadores de profesores de matemática por haber aceptado participar voluntariamente en este Estudio de Clases y en esta investigación.
- Consentimiento informado: Este estudio ha seguido el Código Ético de Integridad y Buenas Prácticas de la Universidad de Barcelona (disponible en http://hdl.handle.net/2445/137899) y ha sido aprobado por la comisión del programa de Doctorado en Didáctica de las Ciencias, las Lenguas, las Artes, y las Humanidades de la Universidad de Barcelona con fecha 25 de junio de 2024.
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- Declaración de la contribución de los autores: Todos los autores afirmamos que se leyó y aprobó la versión final de este artículo. Los roles de los autores según CRediT fueron: T.S. y C.L.: Conceptualización, Metodología, Supervisión. T.S., C.L., y A.B.: Curación de datos, Análisis formal; Software, Redacción – Borrador original, Redacción – Revisión y edición, Visualización. A.B. y V.F.: Validación, Investigación, Recursos. El porcentaje total de contribución para la conceptualización, preparación, y corrección de este artículo fue el siguiente: T.S. 35%, C.L. 30%, A.B. 25% y V.F. 10%.
- Declaración de disponibilidad de los datos: Los datos que respaldan los resultados de este estudio serán puestos a disposición por el autor correspondiente A.B., previa solicitud razonable.
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Capítulo 3: Segundo Artículo Científico

El artículo científico presentado en este capítulo tiene por objetivo inferir, a partir de la argumentación práctica en su reflexión docente sobre el diseño, implementación, y rediseño de una tarea de modelización, los conocimientos, creencias, valores, y orientaciones para la acción sobre modelización matemática de un futuro profesor, atendiendo así al cumplimiento del segundo objetivo específico de esta investigación doctoral.

Los avances del desarrollo de este objetivo específico fueron presentados en tres eventos académicos: la *Escuela de Invierno en Matemática Educativa* (EIME XXV), el *Encontro de Investigação em Educação Matemática* (EIEM 2022), y el *Simposio de la Sociedad Española de Investigación en Educación Matemática* (SEIEM XXV).

En la Tabla 6, se presentan los datos de identificación de este segundo artículo.

Tabla 6. Datos de identificación del segundo artículo científico.

Criterios	Datos
Título	Knowledge and beliefs on mathematical modelling inferred in the argumentation of a prospective teacher when reflecting on the incorporation of this process in his lessons.
Autores	Carlos Ledezma, Telesforo Sol, Gemma Sala-Sebastià, Vicenç Font.
Afiliaciones	Departamento de Educación Lingüística y Literaria y de Didáctica de las Ciencias Experimentales y de la Matemática, Facultad de Educación, Universidad de Barcelona (España).
Revista	Mathematics.
Volumen	10.
Número	18 (septiembre-2 2022).
Número de artículo	3339.
Páginas	1–38.
DOI	10.3390/math10183339
Fecha de publicación	15 de septiembre de 2022.
Fecha de envío	21 de agosto de 2022.
Fecha de aceptación	12 de septiembre de 2022.
Referencia	Ledezma, C., Sol, T., Sala-Sebastià, G., y Font, V. (2022). Knowledge and beliefs on mathematical modelling inferred in the argumentation of a prospective teacher when reflecting on the incorporation of this process in his lessons. *Mathematics, 10*(18), Artículo 3339. https://doi.org/10.3390/math10183339

Fuente: Elaboración del autor.

Esta revista está indexada en más de 30 bases de datos, incluyendo DOAJ, EBSCO, Scopus, ProQuest, Google Académico, entre otras.

De acuerdo con la información disponible en Scopus, esta revista está indexada en las categorías Matemática/Matemática general (rango #38/399, 90º percentil) e Ingeniería/Ingeniería miscelánea (rango #62/204, 69º percentil), con un CiteScore en 2023 de 4,0 y un CiteScore Tracker en 2024 de 4,3.

De acuerdo con la información disponible en Scimago Journal & Country Rank (SJR), esta revista está indexada en las categorías Ciencias Informáticas/Misceláneo, Ingeniería/Misceláneo, y Matemática/Misceláneo, su Índice H es 68 y el SJR 2023 es 0,48 (Q2 en Matemática/Misceláneo).

De acuerdo con la información disponible en el Master Journal List de Web of Science (WoS), esta revista está indexada en el Science Citation Index Expanded (SCIE), Physical, Chemical & Earth Sciences, y Essential Science Indicators, todas en la categoría Matemática, con un Indicador de Citas de Revista de 2,15 en 2023.

Abstract and Keywords / Resumen y Palabras Clave

Abstract: Mathematical modelling enjoys affirmed relevance in educational curricula worldwide, and teacher education programmes consider that this process should also be experienced during educational internships. The interest of this study focuses on the argumentation of a prospective teacher to justify the incorporation (or not) of mathematical modelling in his lessons. To do this, we analysed the reflection of a prospective mathematics teacher on the design of a modelling task during his educational internship. Methodologically, it is a case study in which, based on the Didactic Suitability Criteria tool, the study subject structured the reflection on his educational internship in his Master's Degree Final Project (MFP), whose central axis was modelling. We collected data from video recordings of group reflection sessions with the study subject and other prospective teachers of a professionalising master's programme, and from the analysis of his MFP. The results evidenced the prospective teacher's arguments to justify the design of a modelling task, his knowledge, and his beliefs about this process, among other aspects. We conclude that the specialised knowledge that is inferred from this argumentation is of different types and is part of a conglomerate formed, in addition, by values, beliefs, and guidelines for action.

Keywords: Argumentation; Didactic suitability criteria; Master's degree final project; Mathematical modelling.

Resumen: La modelización matemática se ha ido haciendo un espacio relevante en los currículos educativos a nivel mundial, y los programas de educación de profesores consideran que este proceso debe también experimentarse durante las prácticas educativas. El interés de este estudio se centra en la argumentación de un futuro profesor para justificar la incorporación (o no) de la modelización matemática en sus clases. Para ello se analizó la reflexión de un futuro profesor de matemática sobre el diseño de una tarea de modelización durante su práctica educativa. Metodológicamente, se trata de un estudio de caso en que, a partir de la herramienta Criterios de Idoneidad Didáctica, el sujeto de estudio estructuró la reflexión sobre su práctica educativa en su Trabajo Final de Máster (TFM), cuyo eje central fue la modelización. La recolección de datos se realizó a partir de videograbaciones de sesiones grupales de reflexión con el sujeto de estudio y otros futuros profesores de un programa de máster profesionalizante, y del análisis de su TFM. Los resultados evidenciaron los argumentos del futuro profesor para justificar el diseño de una tarea de modelización, sus conocimientos y creencias sobre este proceso, entre otros aspectos. Se concluye que el conocimiento especializado que se infiere a partir de esta argumentación es de diferentes tipos, y forma parte de un conglomerado formado, además, por valores, creencias, orientaciones para la acción, entre otros aspectos.
Palabras clave: Argumentación; Criterios de idoneidad didáctica; Modelización matemática; Trabajo final de máster.

3.1. Introduction

The study of knowledge and competences that a mathematics teacher should have to perform correctly in his/her professional practice is one of the topics that has been investigated by the Mathematics Education community. International research has led to conceptualisations (and models) about the components of knowledge that teachers use to teach mathematics (see Hill, Ball, & Schilling, 2008; Neubrand, 2018; Petrou & Goulding, 2011; Rowland, 2013; Rowland et al., 2005; Schoenfeld & Kilpatrick, 2008; Shulman, 1987). Some studies have focused on the characterisation and development of the professional competences necessary for the mathematics teacher (see Kunter et al., 2013; Leuders et al., 2018; Zaslavsky & Sullivan, 2011). Other studies explore the assessment of teachers' knowledge as one of the most critical parameters of teaching quality (see An & Wu, 2012; Bell et al., 2010; Blömeke & Delaney, 2012). However, researchers devoted to delving into teaching competences have given particular emphasis to the role of reflection on educational practice, perceiving the reflective component as a critical competence for professional development and improvement of teaching (see Davis, 2008; Fernández & Yoshida, 2004; Fortuny & Rodríguez, 2012; Gellert et al., 2013; Korthagen, 2010; Llinares, 2012).

The initial and continuing mathematics teachers' education is still one of the greatest challenges facing the research community in Mathematics Education. In this area, several research agendas have been proposed and developed, among which the characterisation and development of didactic and mathematical knowledge that allows teachers to favour the management of their lessons stands out.

There is currently a broad consensus regarding both the importance of including mathematical modelling in school curricula and the development of competences related to this process (Kaiser, 2020). In the same way, it is considered that the integration of modelling helps students to improve their understanding of mathematics, provides real contexts for their learning and, consequently, contributes to the development of different mathematical competences, among other benefits (Blum, 2011). Given the importance attributed to modelling, teachers are required to have the knowledge and teaching strategies to implement this process in the classroom (Blum & Borromeo Ferri, 2009).

Several studies have addressed the prospective mathematics teachers' knowledge and beliefs about modelling in different countries and contexts. In some cases, researchers follow a very common structure (for example, Breda, Pochulu et al., 2021), which consists of presenting hypothetical student resolutions to problems (in this case, modelling problems), and then asking prospective teachers what types of simulated interactions they would manage to address a certain student resolution. For example, in the Australian context, Stillman and Brown (2011) analyse the professional knowledge and beliefs of prospective secondary school teachers on the use of modelling tasks, and the influences of receiving parallel (at undergraduate level) versus specific (at postgraduate level) modelling preparation. In this study, knowledge and beliefs are analysed based on the prospective teachers' responses to student resolutions, and how they would manage potential errors and difficulties. In this way, findings are reported regarding the diagnostic and didactic-reflexive competencies, and the affinity with modelling at the educational level in which the prospective teachers will work.

In other cases, researchers analyse the teacher's practice in order to infer their knowledge and beliefs. For example, in the American context, Doerr (2007) discusses the results of two studies, one of which was developed with prospective teachers. This study was conducted during a modelling course, focusing on the knowledge and beliefs about this process, and on the changes that the participants experienced during their practice throughout the course. We highlight two results of this study: on one hand, the participants experienced changes in their conception of modelling, from a linear perspective to one of the 'non-linear cycle' type; and, on the other hand, the importance of individuals reflecting on their own modelling process in order to better understand its cyclical nature.

Several studies have been conducted in the German context in this regard, due to its long tradition of research in modelling. For example, Kaiser and Maaß (2007) present the results of two studies, one of which was developed with servicing teachers. This study focused on teachers' beliefs about mathematical modelling and applications tasks, within the framework of a teacher professional development programme promoted by the German government. Through the application of in-depth interviews, the participants manifested a weak relationship between the mathematical knowledge of the subject with the real world and considered that applications and modelling play a minor role, both in mathematics and in its teaching. For their part, Kaiser and collaborators (2013) study the prospective teachers' professional knowledge on modelling competences, from the perspectives of mathematical content knowledge, pedagogical content knowledge, and general pedagogical knowledge. Through the application of questionnaires on the areas 'modelling and real-world context' and 'argumentation and proof', and interviews with the participants, the following results were obtained: a) there is a direct relationship between knowledge about the modelling process and the modelling competencies, and between comprehension of modelling and understanding real-world situations; b) curricular times affect the way in which modelling can be implemented in the classroom; and c) the predominance of pedagogical content knowledge in developing professional content knowledge on modelling. Lastly, Wendt and collaborators (2020) conducted a case study about a teacher, who showed a broadening of her perception of the use of metacognition by students in mathematical modelling processes.

Finally, in the Colombian context, Villa-Ochoa (2015) identifies some arguments of servicing teachers about the use of modelling in their classes, observing a tendency of teachers to use word problem to work on this process. Through direct observations, interviews, questionnaires, and the study of lesson episodes, the need to prepare teachers to integrate more realistic situations in the use of modelling that considers everyday aspects and culture in school mathematics is stated conclusively.

In the Spanish context, prospective mathematics teachers must take a professionalising master's degree to teach at secondary and baccalaureate education (students aged 12 to 18). The admission profile for this master's degree programme is made up of professionals from different areas related to mathematics who are interested in teaching. To obtain the degree, prospective teachers must prepare a Master's Degree Final Project (MFP), which consists of an original, autonomous, and individual work that, on one hand, reflects the formative content received and the skills acquired during the master's programme in an integrated way and, on the other hand, contributes to the reflection, analysis, and improvement of the own practice. In terms of research with MFPs, studies such as the conducted by Ledezma and collaborators (Ledezma et al., 2021; Ledezma, Breda et al., 2022) have focused on the aspects of the teaching and learning process that prospective teachers relate to modelling, through the analysis of these documents.

Given this background, it is interesting to ask whether prospective teachers, on one hand, have relevant theoretical-methodological preparation in modelling and its teaching, or whether they only have vague notions to implement this process in the classroom; and, on the other hand, in the case they have specialised education in modelling and its teaching, if the context in which they implement their didactic proposals leads them to make decisions that privilege other aspects to the detriment of modelling (curricular times, sociocultural level of the students, etc.). In this line, for this study we posed the following research questions:

- What argumentation does a prospective teacher make to justify the incorporation (or not) of mathematical modelling in his lessons?
- What knowledge, values, beliefs, and guidelines for action, among other aspects, are inferred in this argumentation?

To answer them, we studied the reflection process of a prospective teacher on the design, implementation, and redesign of a mathematical modelling task, both during his educational internship and in the preparation of his MFP. This prospective teacher organised the reflection in his MFP using the Didactic Suitability Criteria tool, proposed by the Onto-Semiotic Approach (Godino et al., 2007). In methodological terms, it is a case study with a student of a professionalising master's programme for mathematics teacher education. In this way, we propose the following objective: to infer, based on the argumentation made in his teaching reflection on the design, implementation, and redesign of a modelling task, the knowledge, beliefs, values, and guidelines for action on mathematical modelling of a prospective teacher.

An innovative aspect of this research is that we inferred the knowledge and beliefs based on the argumentation of the prospective teacher studied, which we analysed from unusual theoretical references, namely, from the pragma-dialectical perspective (van Eemeren & Grootendorst, 2004) for the analysis of a critical discussion through four stages (confrontation, opening, argumentation, and concluding stages); and from the diagramming technique (Guevara, 2011), which allows, on one hand, to make a spatial map of the argumentation, considering the language and, on the other hand, to shape the written reasoning. In the literature review we have not found works that infer prospective teachers' knowledge and beliefs from the analysis of the structure of the argumentation – which they realise when they reflect on their practice – with models other than Toulmin's (1958/2003) (among others, the works of Solar et al., 2022; Tekin, 2019). Similarly, unlike the research commented in the literature review (Breda, Pochulu et al., 2021; Doerr, 2007; Kaiser & Maaß, 2007; Kaiser et al., 2013; Stillman & Brown, 2011; Villa-Ochoa, 2015; Wendt et al., 2020), where knowledge and beliefs are extracted through the application of questionnaires and interviews with items aimed at individuals clearly expressing their knowledge and beliefs; in this study we opted for argumentation as a means to infer these elements indirectly.

3.2. Theoretical References

In this section, we present the theoretical references considered for this study.

3.2.1. Mathematical modelling

Modelling can be understood, in general terms, as a process that transits from the «real world» to «mathematics» for the solving of a problem-situation taken from reality. From the theoretical plane, different cycles have been designed for its analysis (Borromeo Ferri, 2006) and different perspectives on its implementation have emerged (Abassian et al., 2020). This variety of approaches is mainly due to the theoretical diversity around modelling (Borromeo Ferri, 2013), although the proposed cycles tend to converge in similar phases (see Geiger et al., 2018). In this study, we adopted the cycle proposed by Blum & Leiß (2007), since it is adapted to our research context.

This modelling cycle, of a didactic-cognitive nature, has its roots in the theories by Reusser (1998) and Verschaffel et al. (2000), and is focused on the individual who solves this type of problems. In terms of what is described by Blum & Leiß (2007), the *real situation* is given by the wording of the problem, which must be understood to build a *situation model*. Then, this situation must be simplified and structured to form a *real model* of it. Product of the mathematisation (translation into mathematical language) of the *real model*, it will be transformed into a *mathematical model* that, by working with it, will allow obtaining *mathematical results*. These results must be interpreted in the context of the problem to obtain *real results* that must be validated in real world terms. Finally, the solver exposes the answer obtained for the problem. (An illustrative example on the use of this cycle can be found in Blum & Leiß, 2007, pp. 225–227). The transit between the phases of the cycle described above is carried out through transitions or, in terms of Maaß (2006), *modelling sub-competences* (numbered on the right of Figure 1).

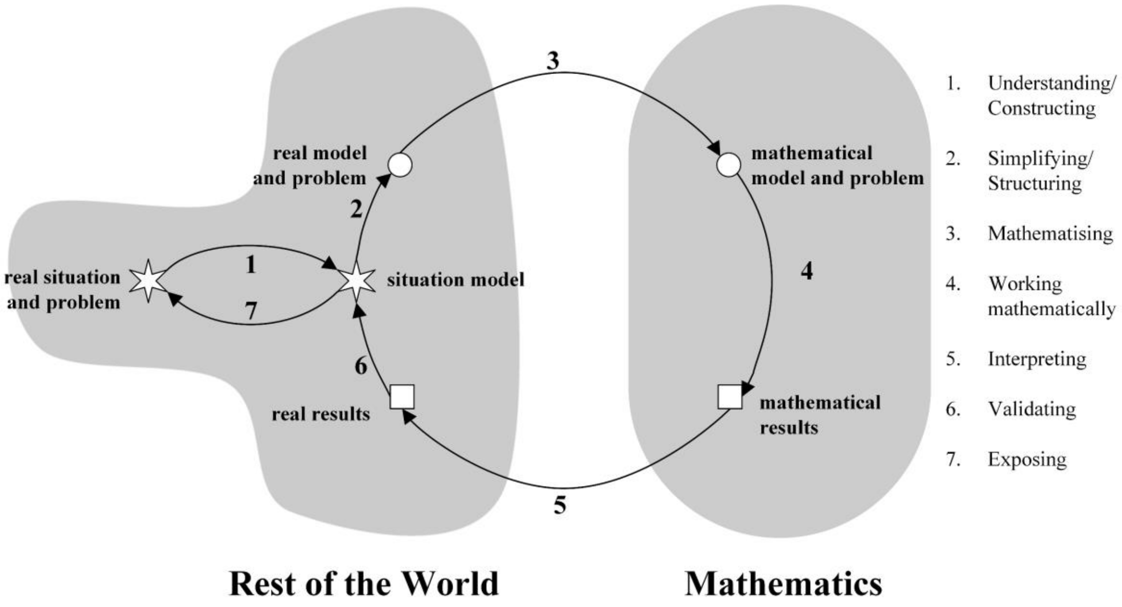

Figure 1. Modelling cycle.
Source: Adapted from Blum and Leiß (2007, p. 225).

Along with this cycle, we consider some attributes agreed upon by the mathematical modelling research community that characterise both the work with this process in the classroom and this type of task.

Modelling work in the classroom is usually carried out in small groups of students, who are presented with a real-world problem-situation that they must mathematise (Doerr & English, 2003; Shahbari & Tabach, 2019), that is, translate into mathematical language in order to work with it. Another way of working on modelling is through problem-oriented projects (Blomhøj & Kjeldsen, 2006, 2018), which is a methodology understood as "a didactical situation where a group of students are working together on a problem during a longer period of time" (Doerr & English, 2003, p. 167). In both contexts, the modelling task must meet the following characteristics (Borromeo Ferri, 2018):

- *Open*, that is, not limited to specific answers or procedures.
- *Complex*, that is, given the information in the wording of the task, students must search for the relevant data for its solving.
- *Realistic*, that is, incorporating elements taken from the real world.
- *Authentic*, that is, consistent with an event that has occurred or could occur in reality (in terms of Palm, 2007).
- A *problem* that should not be solved by applying known algorithms (in terms of Schoenfeld, 1994) and that requires strategies for its solving.

- *Solvable through a modelling cycle*, in which all the phases are passed through.

The modelling problem that is posed, in the form of a question or task, must be challenging for the students and be solved in a period of time that allows both its assimilation and the search for information from different sources.

3.2.2. Didactic-Mathematical Knowledge and Competences model (DMKC model)

As we mentioned at the beginning, there are various models regarding the knowledge that a mathematics teacher should have in order to properly manage their student learning (for example, Hill, Ball, & Schilling, 2008; Neubrand, 2018; Petrou & Goulding, 2011; Rowland, 2013; Rowland et al., 2005; Schoenfeld & Kilpatrick, 2008; Shulman, 1987). In the Onto-Semiotic Approach (OSA), a theoretical model called Didactic-Mathematical Knowledge and Competences model (DMKC model) of the mathematics teacher has been developed (Breda et al., 2017; Godino et al., 2017; Pino-Fan et al., 2015, 2017). In the DMKC model, it is considered that the two key competences of the mathematics teacher are the 'mathematical competence' and the 'analysis and didactic intervention competence'. The fundamental core of this last competence is (Breda et al., 2017): designing, applying, and assessing own learning sequences and those of others, through didactic analysis techniques and quality criteria, to establish cycles of planning, implementation, and evaluation, as well as to put forward improvement proposals. This general competence is made up of different sub-competences (Breda et al., 2017):

1) Analysis of mathematical activity: This sub-competence is made up of (a) analysis of global meanings competence and (b) onto-semiotic analysis of mathematical practices competence (Godino et al., 2017).
2) Analysis and management of the interaction and its effect on student learning.
3) Analysis of norms and meta-norms.
4) Assessment of the didactic suitability of teaching and learning processes.

3.2.2.1. Assessment of the didactic suitability of teaching and learning processes sub-competence: Didactic suitability criteria

For the assessment of teaching and learning processes, the OSA proposes the notion of didactic suitability (Font et al., 2010) as an essential tool. Given a specific topic in a certain educational context, the notion of didactic suitability leads to being able to answer questions such as (Breda et al., 2018; Godino, 2013): What is the degree of didactic suitability of the implemented teaching and learning process? What changes should be introduced in the design and implementation of the teaching and learning process to increase its didactic suitability in future implementations?

The didactic suitability of a teaching and learning process is defined as the degree to which such process (or a part of it) meets certain characteristics that allow it to be qualified as optimal or adequate to reach the adaptation between the *personal meanings* achieved by the students (learning) and the intended or implemented *institutional meanings* (teaching), considering the circumstances and available resources (environment). A teaching and learning process will achieve a high degree of didactic suitability if it is capable of articulating, in a coherent and systematic way, the following six partial criteria of didactic suitability, referring to each of the six facets involved in the teaching and learning process (Breda et al., 2017):

- *Epistemic criterion*: To assess whether the mathematics that is taught is 'good mathematics'.
- *Cognitive criterion*: To assess, before starting the teaching and learning process, whether what is intended to be taught is at a reasonable distance from what the students know.
- *Interactional criterion*: To assess whether the interaction solves students' doubts and difficulties.
- *Mediational criterion*: To assess the adequacy of resources and time used in the teaching and learning process.
- *Affective (or emotional) criterion*: To assess the students' involvement (interest, motivation) in the teaching and learning process.
- *Ecological criterion*: To assess the adaptation of the teaching and learning process to the educational project of the school, the curricular guidelines, the conditions of the social and professional environment, etc.

In turn, each didactic suitability criterion (DSC) has its respective components, and their utility requires defining a set of observable indicators that allow assessing the degree of suitability of each facet of the teaching and learning process. Table 2 presents the components of each DSC, based on the guidelines by Breda et al. (2017).

Table 2. Didactic suitability criteria and their components.

Criteria	Components
Epistemic	– Errors. – Ambiguities. – Richness of processes. – Representativeness of the complexity of the mathematical object.
Cognitive	– Prior knowledge. – Curricular adaptation to individual differences. – Learning. – High cognitive demand.
Interactional	– Teacher-student interaction. – Student interaction. – Autonomy. – Formative assessment.
Mediational	– Material resources. – Number of students, class schedule, and classroom conditions. – Time.
Affective	– Interests and needs. – Attitudes. – Emotions.
Ecological	– Curriculum adaptation. – Intra and interdisciplinary connections. – Social and labour usefulness. – Didactic innovation.

Source: Adapted from Breda et al. (2017).

In the context of OSA (Godino et al., 2007), modelling is considered a *hyper* or *mega process*, since it includes other relevant *processes* of mathematical activity (representation, simplification, idealisation, etc.). In addition, this framework has tools to analyse the mathematical activity underlying the modelling process (see Ledezma et al., 2023). Finally, the OSA considers that working on modelling in the classroom is an aspect that improves the didactic suitability of the teaching and learning process (Sala et al., 2017).

3.2.2.2. Knowledge, beliefs, and conceptions

The DMKC model considers that the teacher's knowledge is organised in three large dimensions: mathematical, didactic, and meta didactic-mathematical.

The mathematical dimension of the DMKC model refers to the knowledge that allows teachers to solve mathematical problems or tasks specific to the educational level in which they will teach (common knowledge), and to link the mathematical objects of such educational level with mathematical objects that will be studied at later levels (expanded knowledge).

The authors of the different models of mathematics teacher's knowledge agree that, in addition to the mathematical content, the teacher must have knowledge about the various factors that influence when the teaching of such mathematical content is planned and implemented. In this sense, the didactic dimension of the DMKC model proposes six subcategories of teacher's knowledge:

- Epistemic facet: Refers to the specialised knowledge of the mathematical dimension (use of various representations, arguments, problem-solving strategies, and partial meanings of a mathematical object), and incorporates notions such as 'knowing mathematics with depth and breadth' (Schoenfeld & Kilpatrick, 2008) and 'specialised content knowledge' (Hill, Ball, & Schilling, 2008).
- Cognitive facet: Refers to the knowledge about the cognitive aspects of students (difficulties, errors, conflicts, learning, etc.).
- Interactional facet: Refers to the knowledge about the interactions that arise in the classroom (teacher–students, student–student, student–resources, etc).
- Mediational facet: Refers to the knowledge about the resources and media that can enhance student learning, and about the times designated for teaching.
- Affective facet: Refers to the knowledge about the affective, emotional, and attitudinal aspects of the students.
- Ecological facet: Refers to the knowledge about the different aspects (curricular, contextual, social, political, economic, etc.) that influence the management of student learning.

The meta didactic-mathematical dimension of the DMKC model refers to the knowledge necessary to reflect on the own's practice (Schoenfeld & Kilpatrick, 2008; Schön, 1983), which allows the teacher to assess the teaching and learning process, and carry out a redesign that improves it in future implementations.

The three dimensions described above are present in the different phases of the teaching and learning process of a certain mathematical content, namely preliminary study, planning, implementation, and assessment (Pino-Fan et al., 2018).

In the DMKC model (see Figure 2), a belief ('I believe P'), according to Peirce (1877), is understood as a 'disposition for action', and a conception is understood as 'a set of beliefs'. Likewise, beliefs are based on knowledge, accompanied by assessments, and are evidenced in a disposition for action in accordance with certain principles/criteria that are positively valued.

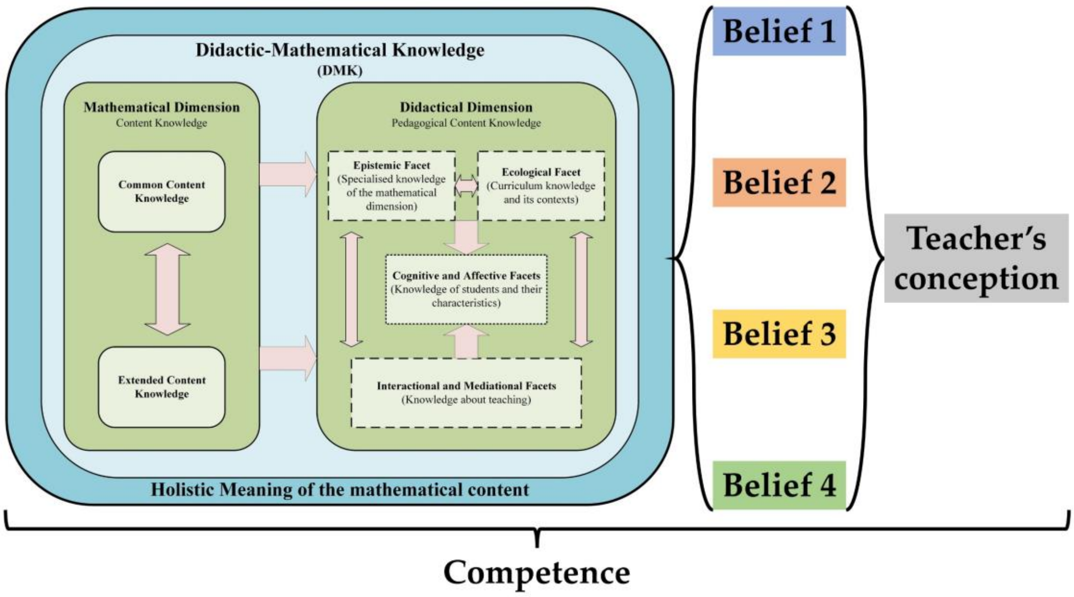

Figure 2. Didactic-Mathematical Knowledge and Competences model of the mathematics teacher.
Source: Adapted from Pino-Fan et al. (2018, p. 66).

3.2.3. Argumentation

We analysed a portion of the data of this study from the pragma-dialectical perspective (van Eemeren y Grootendorst, 2004) as a theoretical reference. Since pragma-dialectics proposes an ideal model for critical discussion, four stages can occur (or not) in this technique, namely:

1) Confrontation stage: Establishes the difference of opinion. In a non-mixed difference of opinion, this simply means that one party's point of view is not immediately accepted by the other, but instead is met with doubt or criticism. In a mixed difference of opinion, the other party advances its opposing point of view.
2) Opening stage: Refers to the starting points of the discussion and assigns the roles of protagonist and antagonist (in a mixed difference of opinion there are two protagonists and two antagonists). Moreover, the rules of the debate and the starting points are agreed upon.

3) Argumentation stage: The protagonist defends his/her point of view against the antagonist's persistent criticism, advancing arguments to meet the antagonist's objections or to remove the antagonist's doubts.
4) Concluding stage: The parties evaluate the extent to which the resolution of the difference of opinion reached and in favour of whom. If the protagonist withdraws the point of view, the difference of opinion is resolved in favour of the antagonist; if the antagonist abandons his/her doubts, it is resolved in favour of the protagonist.

We analysed other portion of the data of this study using the diagramming technique (Guevara, 2011). This technique allows, on one hand, to make a spatial map of the possible ways that premises are linked to support a conclusion, considering the language and, on the other hand, to visually represent the written reasoning. In this technique, four basic structures of arguments are defined:

- Convergent structure: Two or more premises independently support the conclusion.
- Dependent structure: The premises are united to support the conclusion, i.e., both premises (or all of them) need each other to infer the conclusion.
- Divergent structure: The same premise is supporting more than one conclusion (it can be said that there are two or more unitary arguments).
- Chained structure: One of the propositions is the conclusion of a premise and, in turn, it is the premise of another conclusion.

3.3. Methodological Aspects

In this study, we followed a qualitative research methodology from an interpretative paradigm (Cohen et al., 2018), which mainly consists of an intrinsic case study (Stake, 2005). In this section, we describe the methodological aspects of this work.

3.3.1. Research context

This research was developed in the context of a professionalising master's degree programme for the education of secondary and baccalaureate education mathematics teachers taught by the public universities of Catalonia (Spain), during the 2020–2021 academic year. The admission profile for this master's degree programme is made up of professionals from different areas related to mathematics (mathematicians, architects, engineers, economists, etc.) who are interested in teaching.

Within the Specific Formation module, the study programme of this master's course includes a submodule on mathematical modelling. In this submodule, the cycle proposed by Blum & Leiß (2007) is presented to the prospective teachers, along with some examples of modelling tasks. At the end of the submodule, the prospective teachers must expose a modelling problem (wording of the task, solving process, curricular location of the contents) as a final task.

Within the Practicum module, the study programme of this master's course states that the prospective teachers must carry out educational internships. These internships must be realised in educational institutions established through agreements with universities, since they must be recognised as internship centres; as well as the mentor teachers (classroom teachers) who must guide the prospective teachers. During the internship period, the prospective teachers must design and implement a teaching and learning sequence (known as 'didactic unit'), which is determined by: (a) the educational institution and the mentor teacher; (b) the educational level of the students; and (c) the time of the school year in which it is implemented. Due to these conditions, the prospective teachers cannot work exclusively on modelling in their didactic units. However, these restrictions are not considered in the redesign that they propose in their MFPs.

To obtain the master's degree, the prospective teachers must prepare an MFP, which consists of an original, autonomous, and individual work. For its preparation, the prospective teachers are presented with the DSC, along with the guidelines of components and descriptors that allow their application (see Breda et al., 2017). Finally, the prospective teachers must use the DSC to assess in their MFPs the didactic unit that they implemented during their educational internships and, in this way, propose changes that aim to improve the didactic suitability of the implemented teaching and learning process.

3.3.2. Study subject

In this work we constituted an intrinsic case study since our interest was focused on the study of the particular case (Stake, 2005). The study subject of this research was the prospective teacher AD, who has an undergraduate degree in industrial engineering and a doctoral degree in his area. In addition, the study subject has 15 years of experience in university teaching and academic publications derived from research work in his area. In this context, the study subject has worked with the modelling of phenomena typical of his study area. However, in his experience as a university professor, he has not taught modelling.

As a student of the master's programme described in the previous section, the study subject expressed his interest in including the work with modelling in the didactic unit that he had to design to implement during his educational internship. In this way, the selection of this particular case is justified, firstly, by the interest of the study subject in implementing modelling during his educational internship; secondly, and as a result of the above, the master's programme assigned him two tutor professors (the first and the fourth author) for his supervision, which allowed us to accompany the study subject throughout the development of his work; and thirdly, in the possible analysis that the study subject would carry out in his MFP on the implementation of modelling. We consider these three reasons sufficient to be able to study the argumentation of this prospective teacher to justify the incorporation (or not) of mathematical modelling in his lessons and, in this way, infer his knowledge, values, beliefs, and guidelines for action from this argumentation.

During his educational internship, the study subject designed and implemented a didactic unit, consisting of 18 session of 60 minutes each, for the teaching of linear and affine functions in the third grade of secondary education (students aged 14–15), whose central axis was modelling (Deltell, 2021). According to what is described in his MFP, the study subject carried out his internship in an educational centre under the supervision of a mentor teacher (teacher from the centre), who introduced him to the system of articulated work of the mathematics department, which did not give much space to educational innovation. The mathematics class had four lessons a week (Monday, Tuesday, Thursday, and Friday), in one of which (Thursday) a workshop called 'Món Matemàtic' ('Mathematical World' in Catalan language) was implemented. In this workshop, the students solved ad hoc problems distinct from the mathematical content worked on in the regular lessons on the other days (Monday, Tuesday, and Friday), and for which the study subject designed a modelling task to implement in the workshop.

After his educational internship, the study subject began to elaborate his MFP under the supervision of his two tutor professors. In that document, the study subject made a series of evaluative comments about his didactic unit, guiding his reflection based on the DSC components (see Table 2), most of which he related to modelling. These evaluative comments are analysed in subsection 3.4.2.

3.3.3. Data collection and analysis

For this research, we collected data from two sources: on one hand, the video recordings of group reflection sessions in which the study subject participated with his tutor professors and, on the other hand, the study subject's MFP. We applied specific analysis techniques for each of these sources.

3.3.3.1. Group reflection sessions

Regarding the group reflection sessions, since three prospective teachers from the master's programme (JL, RD, and the study subject AD) worked on the topic of functions at the same educational level (third grade of secondary education) during their internships, and their MFPs were under the supervision of one common tutor professor (the fourth author), six group reflection sessions were held with them. In these sessions, developed in a virtual format, each prospective teacher shared his experiences during two sessions by explaining the design, implementation, and redesign of his respective didactic unit, while the other participants (the remaining prospective teachers and the tutor professors) gave their opinions and suggested ideas. The study subject was the only one of the three prospective teachers who decided to include modelling work in his didactic unit, which is why he was under the supervision of two tutor professors (the first and the fourth author), to whom he shared the progress of his didactic unit prior to these sessions.
The dynamics of the two group reflection sessions in which he shared his experience was as follows:

- In the first session, the study subject described the context in which he was carrying out his educational internship and presented his didactic unit for the teaching of functions.
- In the second session, the content and considerations for the preparation of the MFP were discussed.

Although we reviewed the video recordings of the two group reflection sessions with the study subject, in this research we only considered the first session, since it was the design phase of the didactic unit, and its description allowed us to obtain a first reading of his knowledge, beliefs, and conception of modelling. On the other side, the second session was mainly focussed on the formal aspects for the preparation of his MFP.

The analysis of the video recordings was carried out from the pragma-dialectical perspective (van Eemeren & Grootendorst, 2004) since, being a dialogue performed in an academic context, it can be considered as a critical discussion. We analysed these data according to the four stages that pragma-dialectics proposes as an ideal model for a critical discussion: confrontation, opening, argumentation, and concluding stage (see subsection 3.2.3). The analysis of the arguments was carried out with the following steps:

1) We made a general review of the video recordings of the group reflection sessions, which allowed us to have a first appreciation of the development of this critical discussion and, at the same time, allowed us to make a timeless list of the ideas and positions of the participants.
2) We transcribed the dialogues from the video recordings of the group reflection sessions, labelling each participant and their respective interventions.
3) We identified the stages that appeared in the dialogue, according to the pragma-dialectics model. To do this, we justified each segment of dialogue with the definition of each stage of the model.

For the specific case of the argumentation stage, we identified the arguments taking into account two elements: on one hand, who defended the conclusion and, on the other hand, the conclusion and the premises that support it. Finally, we selected only the arguments related to modelling.

The unit of analysis was made up of segments of the dialogue between the study subject and his two tutor professors, excluding the interventions performed by the other two participating prospective teachers (JL and RD), since they were not of interest for this study.

3.3.3.2. Study subject's MFP

Regarding the study subject's MFP, an important chapter within this document is the one in which he analysed the implementation of his didactic unit and used the DSC tool to assess the didactic suitability of the implemented teaching and learning process. In the 'Implementation analysis' chapter, we found evaluative comments where the study subject related different DSC components with the modelling process, since he identified it as an aspect that would increase the degree of suitability for the redesign of the didactic unit implemented during his educational internship. Another element of interest for this study is the modelling task proposed by the study subject, which he described in his MFP, and which was part of the planning of his implemented didactic unit.

We analysed the study subject's MFP as follows:

- Firstly, we identified the criteria and components of the DSC with which the study subject related his reflections on modelling when he assessed the suitability of his didactic unit in the 'Implementation analysis' chapter. Various works have studied teacher reflection during the mathematics teachers' education from different points of view (see Breda, 2020, from didactic analysis; Hidalgo-Moncada et al., 2021, from self-regulation practices; Sánchez et al., 2022, from creativity; Ledezma et al., 2021, from mathematical modelling; among others), whose common denominator has been that, by using the DSC, the prospective teachers' knowledge can also be inferred. In this sense, we followed a content analysis methodology like that used by Ledezma, Breda et al. (2022), focused on the 'Implementation analysis' chapter of the MFP, which consisted of (1) carrying out a search for keywords related to modelling (*model*, *context*, *problem*, *real*); (2) identifying the criterion and component of the DSC in which we found the keywords and the comment that contained them.
- Secondly, to identify the different structures of the arguments in the assessment of the didactic suitability that the study subject made in his MFP, we used the diagramming technique (Guevara, 2011), which is more appropriate for an argumentative text, such as is the case of the MFP. The steps for diagramming are: (1) all propositions in the text are enclosed in curly brackets; (2) propositions are listed in order of appearance; (3) the argument is structured, spatially locating the place of the conclusion; (4) a way in which the premises are related is proposed. Finally, in the diagrams of the arguments we distinguished the propositions that referred to modelling from those that did not.
- Thirdly, we analysed the modelling task proposed in the MFP, using the characterisation for a problem of this type, also considering the phases of the modelling cycle that occurred (or could occur) for its solving.

In both data sources, we looked for arguments that would make evident, on one hand, the study subject's conception of the modelling process and, on the other hand, after his educational internship, how this experience allowed him to think – and rethink – modelling as an element to improve the didactic suitability of the implemented teaching and learning process.

3.4. Presentation and Analysis of Results

In this section, we present and analyse the results of this study. To do this, in the first subsection, we analyse the first group reflection session of the study subject with his tutor professors, from the perspective of pragma-dialectics. In the second subsection, we analyse the study subject's MFP using the DSC, the diagramming technique, and the theoretical framework on mathematical modelling. In this work we consider argumentation as a means to resolve a difference of opinion, which can be manifested in a discussion, a meeting, or an essay.

3.4.1. Analysis of the first group reflection session

In this subsection, we analyse the dialogues of the first group reflection session. To do this, first, we identified the four stages in a critical discussion, according to the pragma-dialectical model; and, secondly, we present the structures of the main arguments used by the study subject during the argumentation stage. The interlocutors mentioned in the following paragraphs are the two tutor professors (T1 and T2) and the study subject (AD). As mentioned above, we must take into account that, since pragma-dialectics proposes an ideal model, each of the stages may or may not appear within a critical discussion and not necessarily in the numbering order proposed by this model. Moreover, the interpretation of these stages is not unique, since some factors that influence it are part of the context in which the discussion took place. In this case, since it is an educational context, cordiality between the tutor professors and the students is different from, for instance, a political debate, so differences of opinion are not always shown in an explicit and impetuous way.

3.4.1.1. Confrontation stage

At this stage, the parties establish that they have a difference of opinion. In this group reflection session, the confrontation stage can be evidenced from the following excerpt from the dialogue, where T1 asked AD a general question about how to introduce modelling with the topic of functions, and it is the one that guided the dialogue of the session.

> T1: How to introduce the topic of modelling with a topic of functions? Of course, when you reflect on this, there are many aspects to take into account, it does not only depend on your willingness to model, but it also depends on the conditions of the [educational] centre, on the type of students you will have on the didactic unit… and that this problem is specific to each [educational] centre. We are going to take the opportunity to discuss the problem.

3.4.1.2. Opening stage

At this stage, the parties decide to try to resolve the difference of opinion, and the roles of protagonist and antagonist are assigned. This difference of opinion is identified by the fact that the tutor professors had the progress of the AD's didactic unit prior to the session. To solve it, the tutor professors decided to pose a series of questions to AD with the intention that he reflect on his definition of modelling and the way in which we would implement this process in his didactic unit. The role of protagonist was assigned to AD when T1 asked him to explain two things: the characteristics of the educational centre where he carried out his internship and how he intended to implement modelling. The rules for the dialogue are implicit as it is a session that is part of a master's programme.

> T1: What is modelling? How can functions be introduced? And how can they be introduced in the specific case of your [educational] centre? Which has characteristics that are unique compared to other [educational institutions.] The reflection on the problem of modelling would be, to what extent can it be applied or not? And if it cannot be incorporated, why could it not be incorporated? Now, what we will do is that AD will explain to us a little about the characteristics of the [educational] centre in which he implements his didactic unit, […] and, later, he will explain his proposal of how he intends to introduce modelling.

3.4.1.3. Argumentation stage

At this stage, the protagonist defends his/her point of view. For this stage, we present some arguments put forward by AD in which posed some reasons that he considered for the implementation of modelling. We stress that, during this stage, AD mentioned a particular modelling task that he designed for his didactic unit, contextualised in the fuel consumption of a car during a family trip to the beach. This task and its redesign proposal are analysed in subsection 3.4.2.3.

Therefore, as a starting point, we infer that AD had a belief that can be formulated, more or less, in the following terms: 'for the topic of functions, a modelling task can be implemented'. This belief disposed him to act in accordance with the principle that this process should be incorporated, which he valued positively. Furthermore, AD believed that the designed task was a modelling task, and he showed a common mathematical knowledge that allowed him to create and solve the task.

In the AD's first Argument (see Figure 3) reasons about the difficulties for implementing modelling are given. On one hand, there is the work dynamics of the institution, and, on the other hand, there is limited access to technological resources.

> AD: The whole mathematics department works […] like a gear, that is, they all do exactly the same thing, the same exams, they coordinate even the dates of the exams, and that makes it a bit difficult to introduce something new. […]. The only problem that I have seen is that, based on what they have told me, during normal hours [sic], they do not have access to the computer room. So, certain modelling problems, for example, with GeoGebra or a spreadsheet where they can work with these tools, […] they will have to do it on their own and, at best, project with the computer on one of these [interactive] boards that they do have. That is right, but it cannot be an interactive thing with the students.

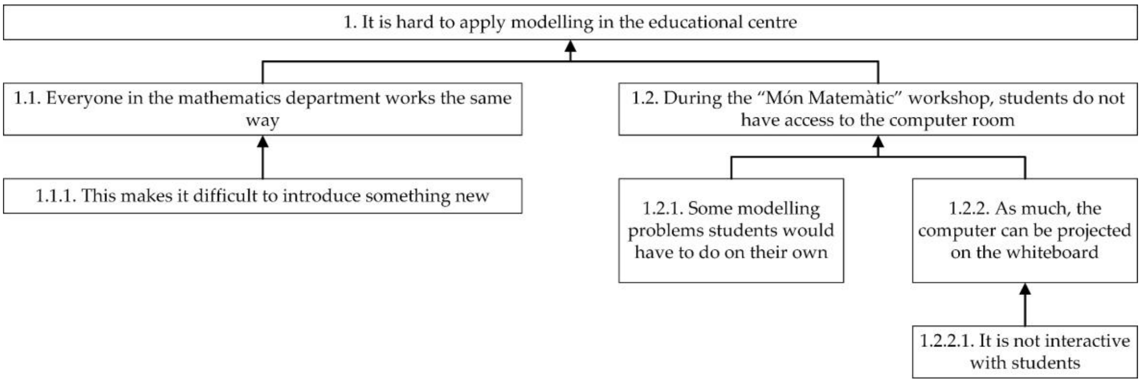

Figure 3. Structure of the AD's first Argument.
Source: Authors' interpretation.

In the AD's second Argument (see Figure 4) it is declared that an advantage that the educational centre where AD carried out his internship had was the 'Món Matemàtic' workshop, in which mathematical problems are posed in parallel to regular lessons of the subject. This space allowed AD, in principle, the implementation of modelling tasks with functions, particularly, that based on the family trip to the beach.

> AD: [In the mathematics department] They have a peculiarity and that is that, in parallel to the solving of problems in class and the usual dynamics, they pose some problems that they call 'Món Matemàtic', which is still a slightly broader plan, which accompanies all the topics. That is, each subject has its 'Món Matemàtic' and is like a problem a little more in context and is a little more extensive. This problem, in this case, my idea was to focus it towards modelling [continues in the next Argument].

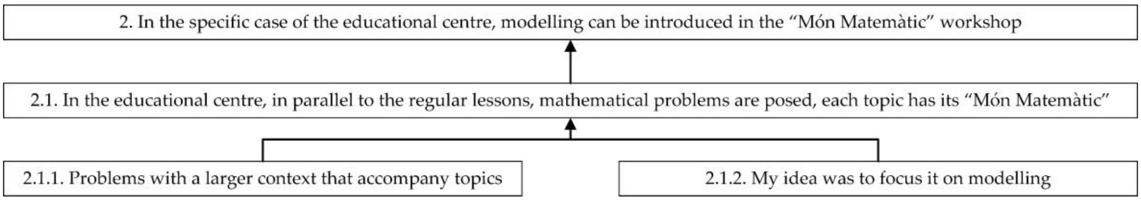

Figure 4. Structure of the AD's second Argument.
Source: Authors' interpretation.

In the AD's third Argument (see Figure 5), we observe the intention, on the part of the study subject, to build an increasingly complex mathematical model with the students, enriching the context of the problem-situation posed as a modelling task, that is, that based on the family trip to the beach.

AD: … and, therefore, with a modelling context with a specific topic, take advantage of a context that is a little richer and propose, because this was my idea, different problems or different questions that, step by step, address the entire topic of the functions section, so let them start with a single context: first, problems of graphical representation; next, we will begin with domain and range; to end up posing line equations and then solving something more complex using these tools. And that is more or less the idea I had.

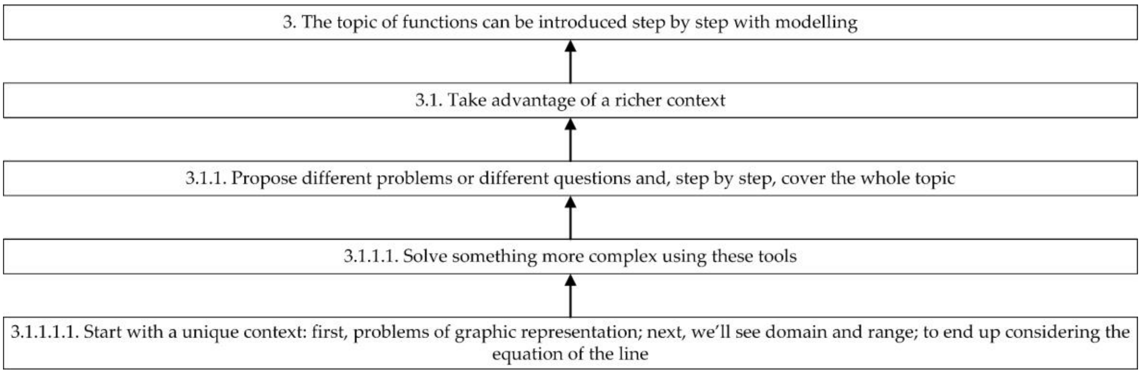

Figure 5. Structure of the AD's third Argument.
Source: Authors' interpretation.

The AD's fourth Argument (see Figure 6) emerges from a question posed by T2 to the study subject while the task of the family trip to the beach was shown on a shared screen 'How did you see the topic of working on modelling? Did you give them the model and they worked on it? Was that your idea?' In response to T2, AD gave the reasons why he gave the students a pre-set mathematical model and did not ask them to build it.

AD: Yes, the idea is that, since it is the first time that they [the students] do functions, they can give the model that algebraic structure. Let them make the model themselves or let them look for information to make a model, unless the case is very simple, I am not very sure that they can do it. But, by giving them the model, by giving them the context and so that they can understand its parts, then apply it, I think it can be easier for them. And, also, that is why I have put in the last section that, precisely, it is that idea: once I give them the context, it is not that I give them the data that they fill in the model and they get an equation that they

> can represent, but it is the other way around, I give them the points and they, through those points, find the equation from which they can deduce the starting data, that is, as in the opposite direction. That would be the last thing, so in a way, they would not build the model, that is true, because the model and the context are fixed, but, at that level of complexity, I think they could not do it on their own [continues in the next Argument].

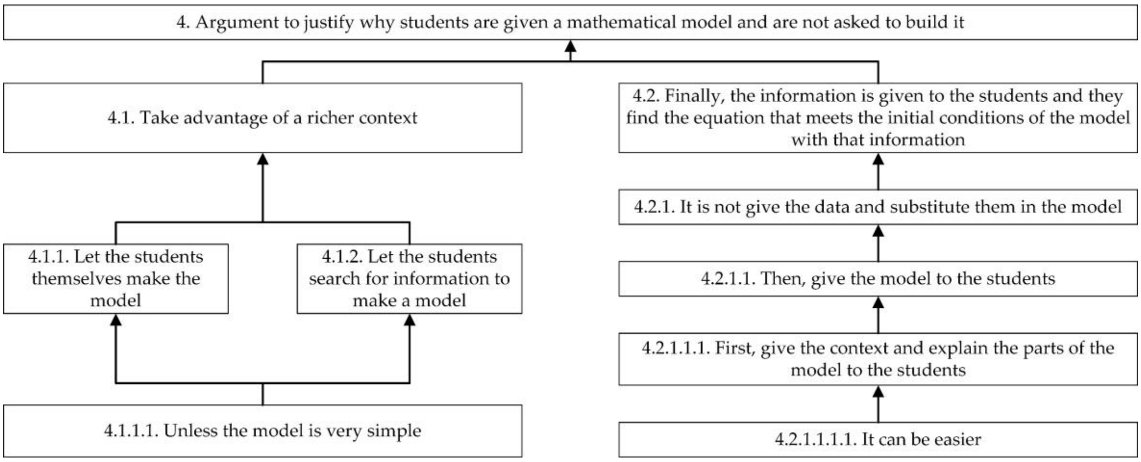

Figure 6. Structure of the AD's fourth Argument.
Source: Authors' interpretation.

In the AD's fifth Argument (see Figure 7) the study subject commented on some aspects that he considered as impediments to fully implement the modelling process (an entire cycle) in the institution where he carried out his educational internship.

> AD: In that sense, perhaps this problem is not very creative for them, because they do not generate their own model, but adhere to an existing model, but, perhaps, since it is the first time they do it, surely, perhaps it is easier, because generating the model already means that they have a somewhat more advanced knowledge of mathematics, but also physics, I do not know, it could also be tried. […]. But, also, if you leave it [the problem] so open to them, I get the impression, from what I have seen, that maybe one or two of them will come up with a better or worse solution. I am not saying that it is good, but the rest, perhaps, surely, will not amount to anything. So, as a first stone in this world of modelling, perhaps it is better to give them a fixed

> context, that they handle themselves algebraically, understand a little and, perhaps in the fourth [grade of secondary education] or baccalaureate [education], then they already generate their own models. […]. What is a little more complicated for me is to leave them with such an open model but let them do it on their own. If I have them [the students] in class, and I dedicate the whole lesson to working on this based on questions that help them to build something, I do see that as more feasible, but give them a brushstroke of ten minutes and let that they on their own can develop an open work, perhaps it is more complicated, I do not know.

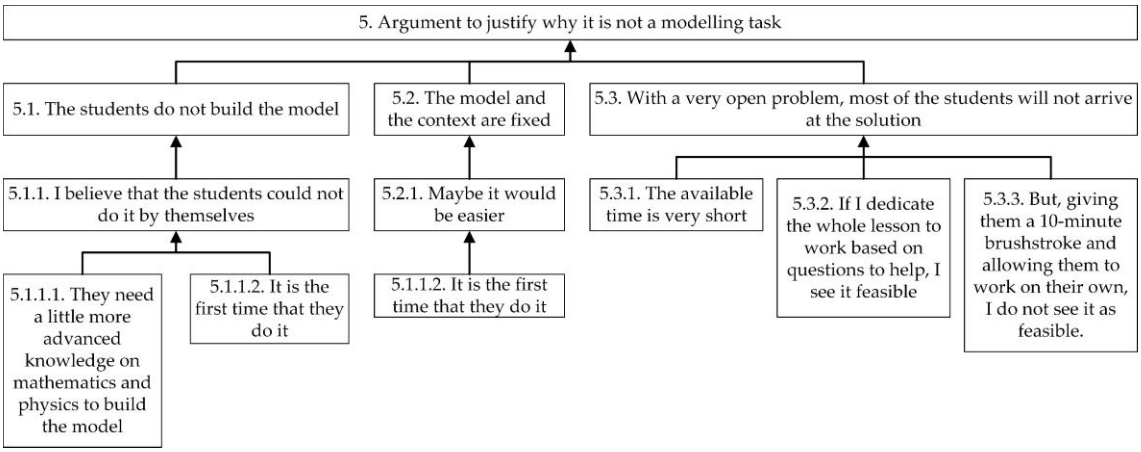

Figure 7. Structure of the AD's fifth Argument.
Source: Authors' interpretation.

In his argumentation, AD used knowledge from the different types proposed by the DMKC model. In this sense, in the first and second Arguments he used, above all, knowledge from the mediational facet; in the third Argument, knowledge from the epistemic facet is mainly inferred; in the fourth Argument, knowledge from the epistemic and cognitive facets is inferred; and in the fifth Argument, above all, they are knowledge from the cognitive facet.

3.4.1.4. Concluding stage

At this stage, the parties evaluate the extent to which the resolution of the difference of opinion was reached and in favour of whom. In the case of this discussion, the one who made the conclusion was T1, for which he considered the comments of all the participants (T2, AD, JL, RD) and, in summary, raised the following points:

- Considering the cognitive level of the students, it is not possible to pose an open problem for modelling.
- It was decided to reduce the mathematical richness by giving the model to the students (this task and its redesign are analysed in subsection 3.4.2.3).
- For the redesign proposal, it is necessary to: (a) propose working on the task in a group, considering the advantages of this way of working; and (b) modify the wording of the modelling task to customise the situation to the students' context.
- Considering the situation of the context in which the didactic unit was implemented, it be assumed that the proposed modelling task did not necessarily meet all the characteristics of a modelling problem.

3.4.2. Analysis of the study subject's MFP

In this section, we examine the 'Implementation analysis' chapter of the study subject's MFP. To do this, first, we identified the criteria and components of the DSC with which AD related modelling (subsection 3.4.2.1). Second, we analysed the comments found through the diagramming technique (subsection 3.4.2.2). Finally, we analysed the modelling task incorporated in the didactic unit (subsection 3.4.2.3).

Since, at the time of preparing his MFP, the study subject knew and used the DSC as a tool to guide his reflection, derived from his educative process in the master's programme, we can affirm that he mainly evidences knowledge from the meta didactic-mathematical dimension of the DMKC model. However, by studying his argumentation in detail, we also infer knowledge from the mathematical and didactic dimensions. In the diagramming that follows we detail such knowledge, which allows us to affirm, metaphorically, that mathematical, didactic, and meta didactic-mathematical knowledge are 'dense' in their reflection guided by the DSC tool.

3.4.2.1. DSC components related to modelling

From the search for keywords in the study subject's MFP, we were able to identify the criteria and components of the DSC in which AD made evaluative comments on modelling, as listed below:

- Epistemic criterion: ‘Richness of processes’ and ‘Representativeness of the complexity of the mathematical object’ components.
- Cognitive criterion: ‘Prior knowledge’, ‘Learning’, and ‘High cognitive demand’ components.
- Interactional criterion: ‘Student interaction’ and ‘Autonomy’ components.
- Mediational criterion: We did not find keywords or comments on modelling in any of the components of this criterion.
- Affective criterion: ‘Interest and needs’ component.
- Ecological criterion: We found keywords and comments on modelling in the four components of this criterion.

After identifying these criteria and components, we selected the evaluative comments that were directly related to modelling and the purposes of this study for their consequent diagramming, as described in the following paragraphs.

3.4.2.2. Diagramming

Based on the criteria and components of the DSC identified in the previous paragraphs, here, we diagram the text of the MFP to identify the reasoning constructed by the study subject. For this, we present the text portions of the MFP in which we identified and numbered the propositions for their diagramming and, later, we specifically deal with those related to the modelling process. In the diagrams of Figures 8–18, we represent in blue the propositions in the argumentation of the text that are not related to modelling, and in green those that are related to this process. After each diagram, we present an interpretation of the propositions about modelling to infer the study subject’s conceptions about this process.

The first criterion in which the suitability of the implemented didactic unit was assessed is the ‘epistemic criterion’. In this criterion, we found evaluative comments on modelling in the ‘Richness of processes’ and ‘Representativeness of the complexity of the mathematical object’ components.

The diagram of the analysis of the ‘Richness of processes’ component is as follows:

> 1{It is clear that the proposed activities must not only represent an exemplification of mathematical concepts but must also contribute to the work of competences that put into play the

students' abilities, knowledge, attitudes, and skills through cognitive processes}. 2{In this sense, the central axis of the didactic unit is the modelling process that allows many tools to work on different competences}. […]. Competence 1. Translate a problem into mathematical language or mathematical representation using appropriate variables, symbols, diagrams, and models. 3{This competence has been extensively worked on in all the modelling problems, especially in sessions 2, 7, and 8, as well as in the synthesis work, which in the end could not be taken to the classroom}. 4{Here the expression of the problems in algebraic, graphical, or table form stands out, which have been integrated into all the modelling problems}, 5{not only in order to see the different forms of expression of the problem}, but also 6{the student's abstraction capability to understand a situation in a 'real' context} such as 7{buying fruit in session 2, calculating the speed of a runner in session 7, or the choice of the best rate of mobile operators}. Related to this competence, 8{an intense work has been done on the processes problem solving and conversion between different representations of functions}, 9{in this case, between natural and mathematical languages (translation)}. In all cases 10{it has been insisted that the mathematical translation be formally correct}, 11{always trying that the student correctly assigns and defines the independent and dependent variable(s), considering the units and their compatibility and coherence, as well as unit conversion where appropriate}. Competence 5. Construct, express, and contrast arguments to justify and validate the statements made in mathematics. At the same time, the previous competence, and especially in the modelling problem posed in session 8, 12{the student is encouraged to make small conjectures} about the rate behaviour of a service or, at least, 13{to propose a thesis and mathematically check it}, 14{forcing the student to build a model}, 15{to express more or less real situations through the model}, and 16{to contrast information or check whether the statements are correct or not}. 17{In terms of processes, mathematical argumentation has been worked on}. 18{Modelling problems, in general, always propose not only to model, but also to apply this model to a situation, often very close to reality} 19{(for which I would also work intensively on competences 6 and 7 from the mathematical area, that is 'Use mathematical reasoning in non-mathematical environments' and 'Identify the mathematics involved in close and academic situations and look for situations that can be related to concrete mathematical ideas')}. 20{The process associated with these competences would be contextualisation}. (Deltell, 2021, pp. 4–5)

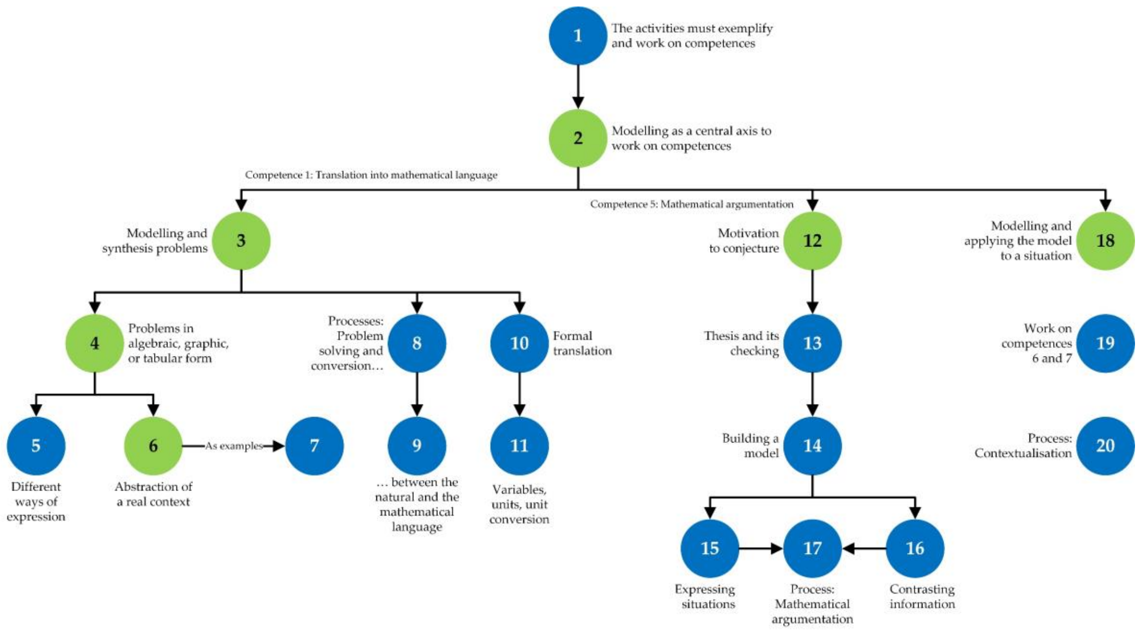

Figure 8. Diagramming of the assessment of the 'Richness of processes' component.
Source: Authors' interpretation.

The study subject made the assessment of the 'Richness of processes' component considering, furthermore, the competences established in the educational curriculum for mathematics (Departament d'Educació, 2019). In this line, proposition 3 shows that the study subject related competence 1 of the curriculum to work with modelling. More specifically, competence 1 is somewhat related to the definition of the 'mathematisation' transition, which includes the modelling cycle (see Figure 1, no. 3). In terms of Maaß

(2010), there are tasks to work on certain transitions of the modelling cycle, which does not imply that a modelling process is being carried out as such, since their intention is to focus on some specific sub-competence(s). However, from his writing, we can infer that the study subject considered the idea: 'by working on competence 1, in addition, I am working on modelling'.

For its part, proposition 6 shows that the study subject, in one of the beliefs that form his conception of modelling, considered the understanding of the task in its context as a starting point to develop the modelling process. In particular, this idea can be related to the 'understanding/constructing' transition that includes the modelling cycle (see Figure 1, no. 1). Likewise, from proposition 12 to 17, we can interpret that the study subject considered different transitions of the modelling cycle. For example, proposition 12 can be related to the 'simplifying/structuring' transition (see Figure 1, no. 2), and proposition 16 to the 'validating' transition (see Figure 1, no. 6).

Finally, proposition 18 evidences certain limited interpretations of the study subject on modelling. On one hand, we can infer an overlap between the 'mathematical modelling' and 'mathematical applications' processes. In terms of Blum (2002), although both processes denote all kinds of relationships between the «real world» and «mathematics», modelling starts from the «real world» towards «mathematics», focused on the process; while applications start in the opposite direction, focused on the mathematical object. On the other hand, the study subject considered that modelling also implies applying a mathematical model to a situation close to reality. Although this idea is not wrong, it is imprecise in the sense that, as stated in subsection 3.2.1, a modelling problem is characterised by coming from a 'realistic' and 'authentic' context, and its writing seems to show that calls into question – with the phrase 'many times' – the condition of closeness to reality, and not as a sine qua non condition to pose this type of problem.

For the next diagrams of the analysis of the DSC components, we only show the text portions extracted from the study subject's MFP that include numbered propositions related to modelling. However, the figure associated with each analysis includes the diagram of each component in its entirety.

The diagram of the analysis of the 'Representativeness of the complexity of the mathematical object' component is as follows:

11{In the contextualised examples and, above all, in the modelling problems, the change in the conception of function has been used as a tool for analysis and problem solving}, 12{starting from an abstract vision of function as a relation of variables}, 13{adding meaning to these variables, in order to identify the independent and dependent variables of a problem in its context} to, 14{finally, enable the algebraic representation of a function}. Beyond the conception of function as a relation of magnitudes or variables, 15{a vision of function as a subset of the cartesian product of two sets has been introduced in class, that is, a vision as a set}. For example, 16{in the modelling problem of buying oranges, when algebraically expressing the function, its domain and range as an abstract mathematical function was one}, but, 17{if we conceive function as a relation of magnitudes, that domain and range are restricted}, since 18{the purchase of a negative mass of oranges or obtaining negative costs of a purchase does not represent an easy interpretation}, (despite the fact that these could be interpreted as sales), so 19{the characteristics of domain and range of a function can differ for the same algebraic representation, defining in itself different functions depending on the domain considered, that is, different functions that, despite having the same algebraic expression, they differ depending on the subset of independent variables considered (vision of a function as a subset of the cartesian product)}. 20{To a lesser extent, the verbal expression of a function has been explicitly worked on, that is, it has not been asked as an objective to verbally express a function}, but 21{the oral expression of a function has been worked on in the context of the model development}, so 22{this representation has been worked on, albeit implicitly}. (Deltell, 2021, pp. 6–7)

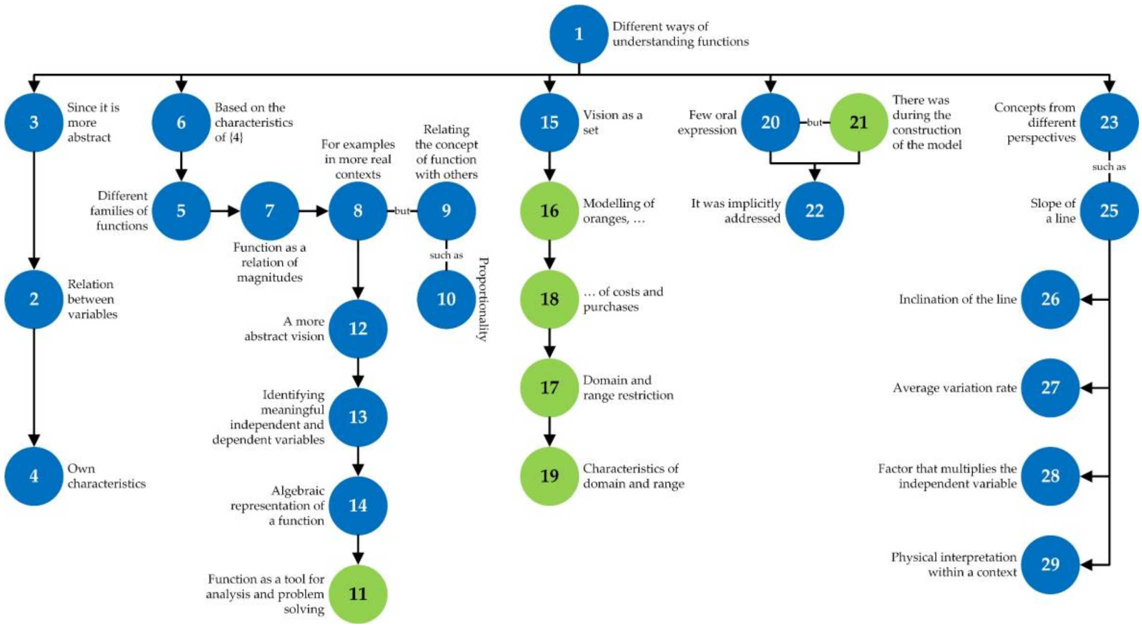

Figure 9. Diagramming of the assessment of the ‘Representativeness of the complexity of the mathematical object’ component.
Source: Authors’ interpretation.

In the assessment of this component, the study subject explicitly related the work with modelling to the treatment of the complexity of the mathematical object function, as evidenced in proposition 11, through the change of conception that one has of it (function as a ‘relation between variables’ and as a ‘subset of the cartesian product’). Likewise, from proposition 16 to 19, the study subject considered that, in a modelling problem, the context is what determines the behaviour of the mathematical object involved. This idea of the study subject is related to the position of Ledezma et al. (2023) on the epistemic (metamathematical) norms that regulate the modelling activity, in which the solution of the problem must make sense in the context of the problem-situation posed. One aspect that we stress is that, in the study subject’s reflection on this component, he repeatedly used the term ‘conception’. However, he did not specify what he means by such term. For this reason, we infer that sometimes he interprets its use as a ‘partial meaning’ (see Godino et al., 2007) of a mathematical object.

The second criterion in which the suitability of the implemented didactic unit was assessed is the ‘cognitive criterion’. In this criterion, we found evaluative comments on modelling in the ‘Prior knowledge’, ‘High cognitive demand’, and ‘Learning’ components; however, in the latter we did not find a direct relationship with the purposes of this study, which is why we discarded it for analysis.

The diagram of the analysis of the ‘Prior knowledge’ component is as follows:

3{The objective of this initial evaluation was focused on analysing the student knowledge of units of measurement and unit conversion}, 4{to see their mastery of the topic}, 5{which is essential for modelling and contextualised exercises and problems}. In addition, 6{it is where there may be more level of disparity between students}, since 7{units of measurement and unit conversion are not only addressed in mathematics}. 8{The assessed prior knowledge is the algebraic manipulation of linear equations}, 9{to see if students will be able to work with functions algebraically expressed with sufficient ease so that, in modelling problems, the algebraic manipulation not be an impediment to exercise solving}. In general, 10{it can be seen that, in terms of knowledge and unit conversion, 6 students have made 2 errors or more in the 3-unit conversion questions} and that, therefore, 11{may have problems when following the modelling problems}. (Deltell, 2021, p. 7)

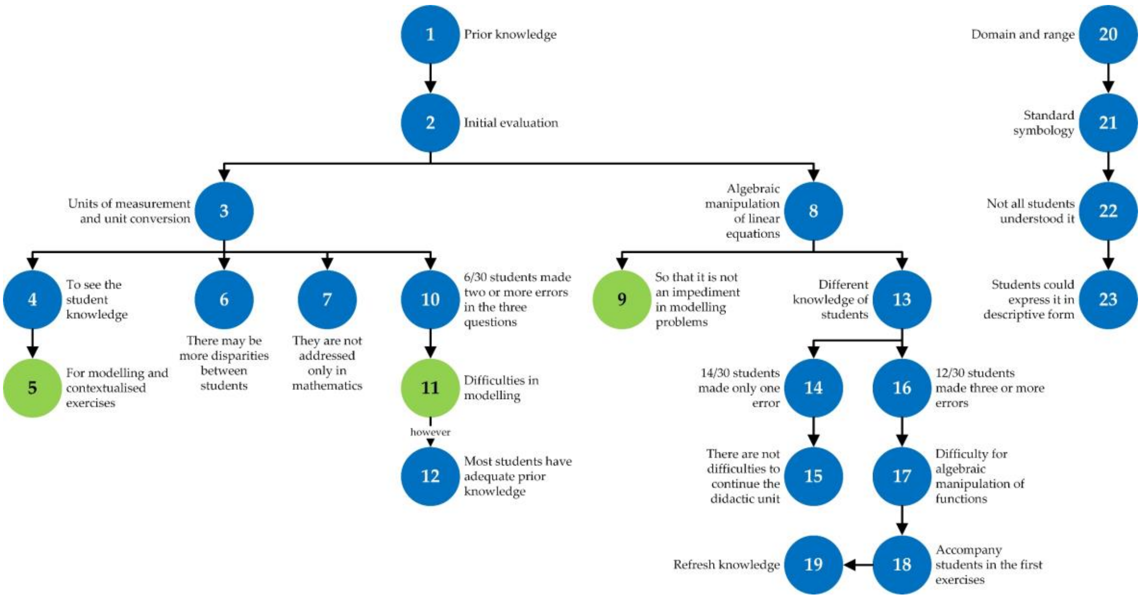

Figure 10. Diagramming of the assessment of the 'Prior knowledge' component.
Source: Authors' interpretation.

In the assessment of this component, from proposition 3 to 5, we can evidence that the study subject attributed great importance to prior knowledge about the mathematical object function to, later, work on modelling and avoid difficulties during the solving of this kind of tasks (there ideas are reinforced in propositions 9 and 11). As mentioned above, modelling primarily focuses on the transition process between the «real world» and «mathematics», rather than on the mathematical object involved (as in the case of applications) (Maaß, 2010). However, the study subject's position sheds light to the contrary. In the same way, we did not observe in his writing that he has considered the previous experiences of modelling that the students may have had as prior knowledge. The diagram of the analysis of the 'High cognitive demand' component is as follows:

> However, 5{the cognitive demand is conditioned} by the 6{structure of the didactic unit, with problems based on modelling processes}. 7{In these problems is where the cognitive demand on the student is higher}. First of all, 8{every modelling problem required a first phase of understanding the context (contextualisation)} 9{to define what the variables are {independent and dependent)} and 10{in what units they want to be represented (conversion from natural to mathematical language)}. 11{Here we already find a first degree of abstraction that, as verified in the final exam, many times modelling problems are solved}, but 12{it is difficult for students to formally determine the variables and, therefore, that they reach the maximum degree of abstraction of the problem, which will entail the representation of the problem in the form of a function, typically expressed in algebraic form, in order to be mathematically analysed}. 13{This conversion process is carried out in all the modelling problems proposed in this didactic unit that is analysed}. Beyond the conversion, 14{modelling problems allow the student to propose solving specific situations}, so that 15{they must propose different situations or theses} and 16{verify them from the model (mathematical argumentation)}, such as, for example, in the modelling problem of session 8, 17{where the student is asked to determine which mobile rate is more convenient}. 18{This problem can be solved by calculating the costs for each situation that the problem poses}, or 19{by performing a change of representation, plotting the costs of the

different companies (treatment) and, from the cut-off point of the functions, see from which mobile consumption one company or another pays better}. 20{In this last resolution model, the cognitive demand is higher}, not only 21{due to the higher degree of abstraction that the student needs}, but 22{due to changes in representation} and 22{the fact of correctly interpreting the cut-off points of the functions}, 23{to make deductions of what happens in reality with the cost of the mobile}. Also, it is important to note that 24{in the modelling process we have a first step of generalisation and abstraction} 25{when building and expressing the model}, but also, 26{a concrete exercise} 27{when the mathematical results must be interpreted} 28{to draw conclusions in the context of the problem (contextualisation)}. Many times, like, 29{for example, in the problem proposed in session 2 about the purchase of oranges}, 30{the purely mathematical analysis of the function represented by the model would have characteristics such as the domain and range} which, later, 31{its concretion to the real context of the problem makes both the domain and the range make sense in more restricted intervals} (since the purchase of a negative mass of oranges or obtaining negative costs of a purchase does not present an easy interpretation), 32{when passing from the interpretation of a function as a relation of variables to that of a relation of magnitudes}. 33{It is therefore thanks to the modelling problems that this richness of nuances can be presented in the definition of function and to be able to exemplify what it is for taking them into account}. 34{This process requires a high cognitive demand}. (Deltell, 2021, pp. 9–10)

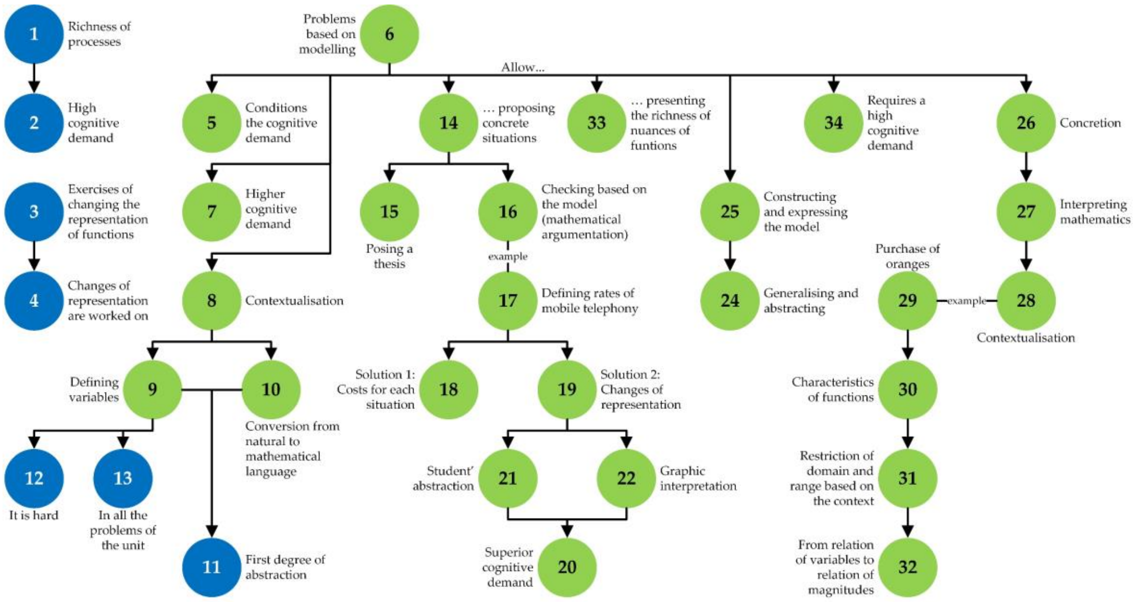

Figure 11. Diagramming of the assessment of the 'High cognitive demand' component.
Source: Authors' interpretation.

In the assessment of this component, we can observe that many propositions posed by the study subject are related to modelling. This is fundamentally since he considered this process as the central axis of his implemented didactic unit. Proposition 8 explicitly considers the 'understanding/constructing' transition that the modelling cycle includes (see Figure 1, no. 1). For its part, proposition 12 evidences the 'working mathematically' transition (see Figure 1, no. 4), when specifically referring to the analysis of the problem with the tools of mathematics. From proposition 17 to 23, the study subject focused his analysis on some particular problems, where he alluded to the semiotic aspects (in terms of Duval, 2017) that working with modelling implies. These problems only consist of exercises of conversion between registers. Propositions 27 and 28 are related to the 'interpreting' and 'validating' transitions (see Figure 1, nos. 5 and 6). From the assessment of this component, we can highlight two aspects: on one hand, the study subject referred to different transitions of the modelling cycle, which he exemplified with the modelling problems that he included in his didactic unit; and, on the other hand, he considered that this process requires a high cognitive demand for its development, including other processes of mathematical activity (in line with the position of Godino et al., 2007).

The third criterion in which the suitability of the implemented didactic unit was assessed is the 'interactional criterion'. In this criterion, we found evaluative comments on modelling in the 'Student interaction' and 'Autonomy' components.

The diagram of the analysis of the 'Student interaction' component is as follows:

> 3{Initially, I wanted that part of the synthesis work of the didactic unit (Annex 3) could be worked in virtual groups among themselves}, but 4{the lack of habit of the group in these work dynamics and the advice from the mentor teacher, made me change my mind}. 5{Unfortunately, the fact that some sessions were cancelled} 6{because of students' activities outside the classroom (outings and gymkhanas)}, 7{caused that, finally, the synthesis work had to be suspended}, which is where 8{it was expected to further enhance the student interaction}. In group meetings, 9{the students commented on how they thought the problems were solved and tried to explain why they tried one way or another}. Finally, 10{they reached consensus (or not) which consulted with the teacher}. (Deltell, 2021, pp. 11–12)

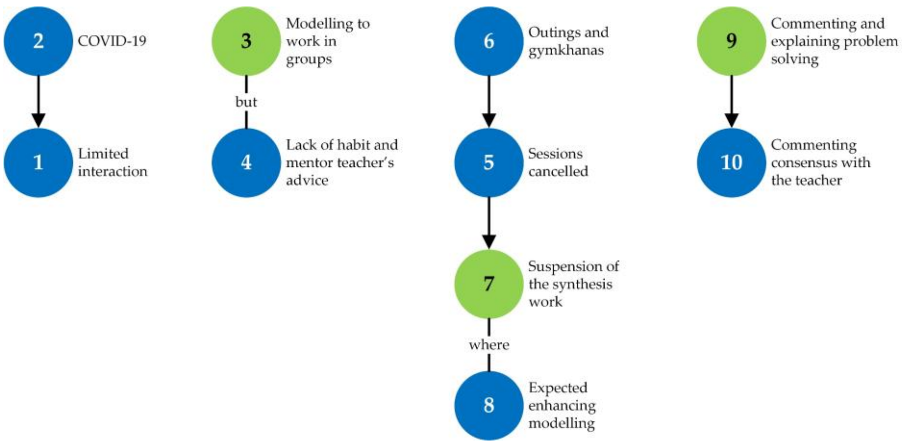

Figure 12. Diagramming of the assessment of the 'Student interaction' component.
Source: Authors' interpretation.

In the assessment of this component, the study subject made evident the greatest difficulty he had in implementing his didactic unit in the way he had planned it. Due to mismatches with the temporary organisation of the educational centre, there were two sessions (10 and 13) in which his class made extracurricular outings, and which were intended to develop the synthesis work of the didactic unit during the 'Món Matemàtic' workshop.

More specifically, it was in these two sessions that he would implement the modelling task on which he reflected the most within his MFP, which ended up being hypothetical reflections. On the other hand, propositions 3 and 9 show that the study subject considered that an important aspect to work on modelling is to promote interactions between students through group work with this type of tasks, which is in line with the position of Doerr and English (2003).

The diagram of the analysis of the 'Autonomy' component is as follows:

> 3{Given this lack of autonomy during the classes, I wanted to take advantage of the synthesis work to increase this autonomy}, 4{letting the students do part of this work on their own outside the classroom}. By not being able to introduce the synthesis work, it was not possible to enhance the autonomy of the group. 5{Modelling problems, by requiring the student to make decisions in various processes}, 6{tend to increase autonomy and, when reaching solutions, create a certain feeling of self-esteem for having solved a problem that, being contextualised, is perceived as something related to reality}. (Deltell, 2021, p. 12)

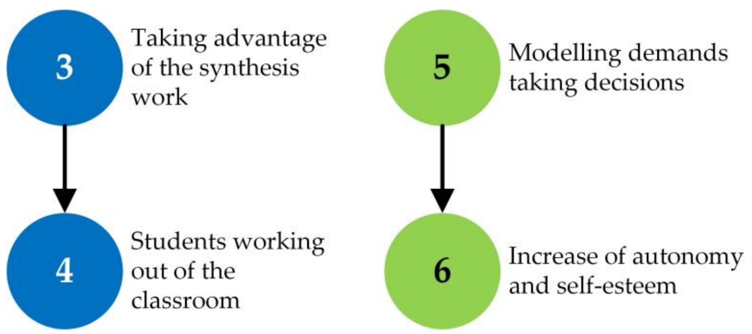

Figure 13. Diagramming of the assessment of the 'Autonomy' component.
Source: Authors' interpretation.

In the assessment of this component, the study subject considered that the work with modelling promotes decision-making, along with the improvement of the students' autonomy and self-esteem. Although this is a hypothetical reflection, i.e., it is based on his notion of the positive aspects of modelling work by students, the idea is in line with the position of Blum (2011).

The fourth criterion in which the suitability of the implemented didactic unit was assessed was the ‘mediational criterion’. However, we did not find evaluative comments on modelling in any of the components of this criterion.

The fifth criterion in which the suitability of the implemented didactic unit was assessed is the ‘affective criterion’. In this criterion, we found evaluative comments on modelling in the ‘Interests and needs’ component.

The diagram of the analysis of the ‘Interests and needs’ component is as follows:

> 1{The exercises in class, in general, were for practising}, 2{but the modelling problems, distributed throughout the didactic unit, were contextualised to appeal to everyday situations where the students felt identified}. 3{No typical contexts have been introduced to attract the attention of the students such as football, movies, videogames, or something like that}. Although 4{these topics connect with the interests of the vast majority of students}, 5{they end up provoking non-mathematical debates that do not contribute to improving the work climate in the classroom}, on the other hand, 6{everyday situations, where many times they identify themselves in adult situations, they also find them attractive and, above all, if they incorporate the economic factor}. This, 7{the contexts proposed by the 3 modelling problems defined for the didactic unit are the purchase of products in bulk (oranges), the analysis of the speed of a runner in a race, and the choice of a rate of mobile telephony among different options}. Likewise, 8{in the final exam, a contextualised modelling problem was included when looking for the most appropriate hiring rate for a gym service based on its use}. 9{The objective of this contextualisation of problems was also aimed at showing the students that mathematical knowledge is not only a tool for those who want to continue their studies in baccalaureate education}, but also 10{is found and useful in close and everyday environments}. (Deltell, 2021, pp. 14–15)

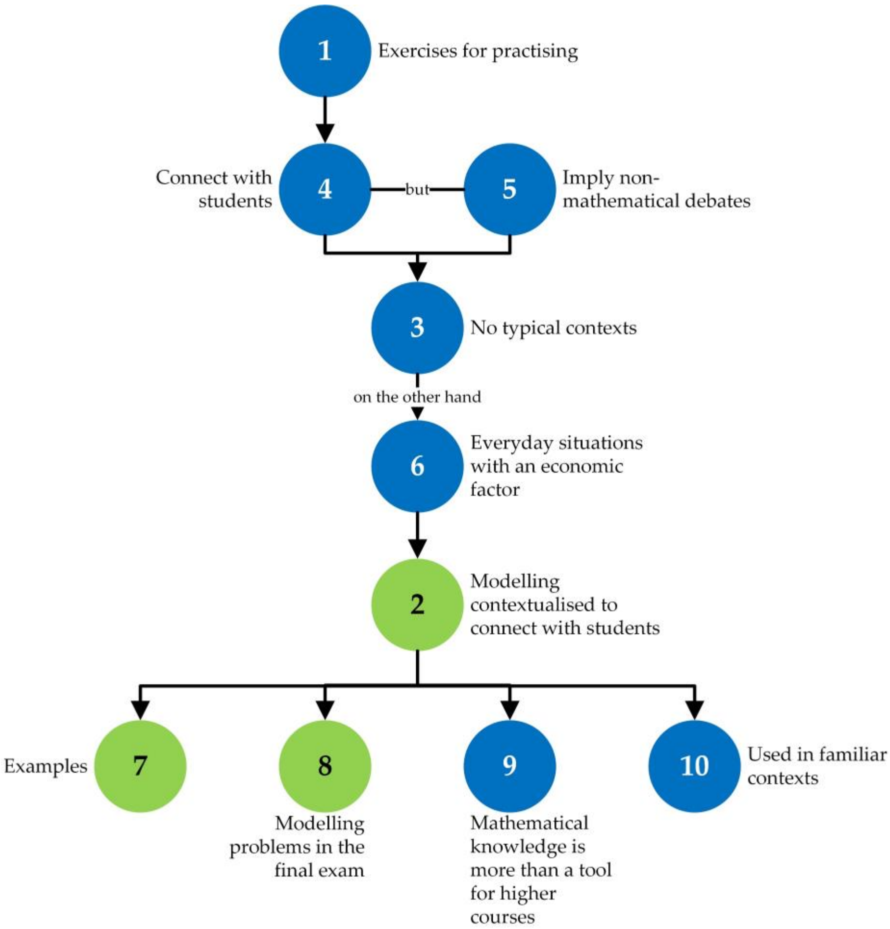

Figure 14. Diagramming of the assessment of the 'Interests and needs' component.
Source: Authors' interpretation.

In the assessment of this component, propositions 3 and 4 allow us to see some aspects that the study subject took into account to design the wording of their modelling problems. In these aspects, those focused on the students' daily life were privileged, avoiding context that could produce disparities of opinions and, in this way, focusing on the mathematical aspects of the solving.

The sixth (and final) criterion in which the suitability of the implemented didactic unit was assessed is the 'ecological criterion'. In this criterion, we found evaluative comments on modelling in its four components. We will make a general comment on this criterion at the end of the four diagrams, due to the fact that the assessments of the components are interrelated.

The diagram of the analysis of the 'Curriculum adaptation' component is as follows:

> 4{Finally, the point use of functions for solving problems in different contexts is fully covered with the use of modelling

> problems, which have been solved in the didactic unit}. (Deltell, 2021, p. 16)

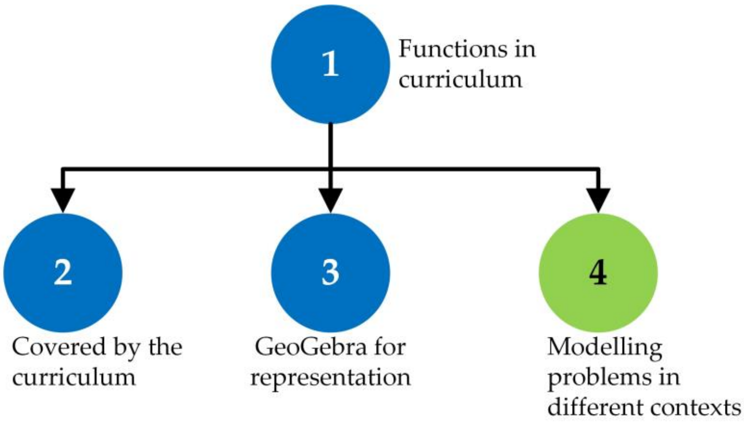

Figure 15. Diagramming of the assessment of the 'Curriculum adaptation' component.
Source: Authors' interpretation.

The diagram of the analysis of the 'Intra and interdisciplinary connections' component is as follows:

> 1{The intradisciplinary connections (within mathematics) of the didactic unit are continuous}. […]. 3{The interdisciplinary connections are also multiple}, 4{given the nature of the didactic unit of functions}, 5{there are obvious connections with the physics and technology subjects}, 6{specially, due to the use of modelling in problems, which is natural in these disciplines}. […]. It has not been formally established in the development of the didactic unit, but it could be easy, the connection with social sciences or economics subjects. (Deltell, 2021, p. 17)

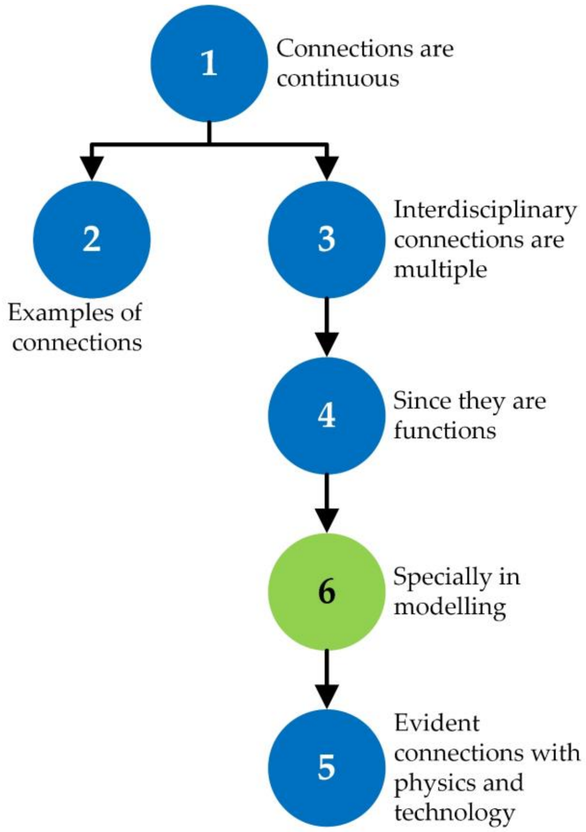

Figure 16. Diagramming of the assessment of the 'Intra and interdisciplinary connections' component. Source: Authors' interpretation.

The diagram of the analysis of the 'Social and labour usefulness' component is as follows:

> 1{The social and labour utility of functions has been exposed, mainly, from modelling problems, which have been contextualised in everyday situations}. 2{The labour utility, perhaps, is further away from 3rd grade of secondary education students} and, even so, it has been commented on. 3{The social utility, which I believe may be closer to 3rd grade of secondary education students}, 4{has been introduced from the contextualisation of the contexts proposed by the 3 modelling problems defined for the didactic unit}, which are the purchase of products in bulk (oranges), the analysis of the speed of a runner in a race, and the choice of a mobile phone rate among different options. 5{These are everyday situations, which avoid grandiloquent statements}, but which I believe 6{project the image of the usefulness of mathematics on a day-to-day basis and,

therefore, the need and convenience of learning them}. 7{The problems based on modelling processes are usually realistic and authentic}, which 8{shows their usefulness in everyday life and not only in special and professional situations}. (Deltell, 2021, p. 17)

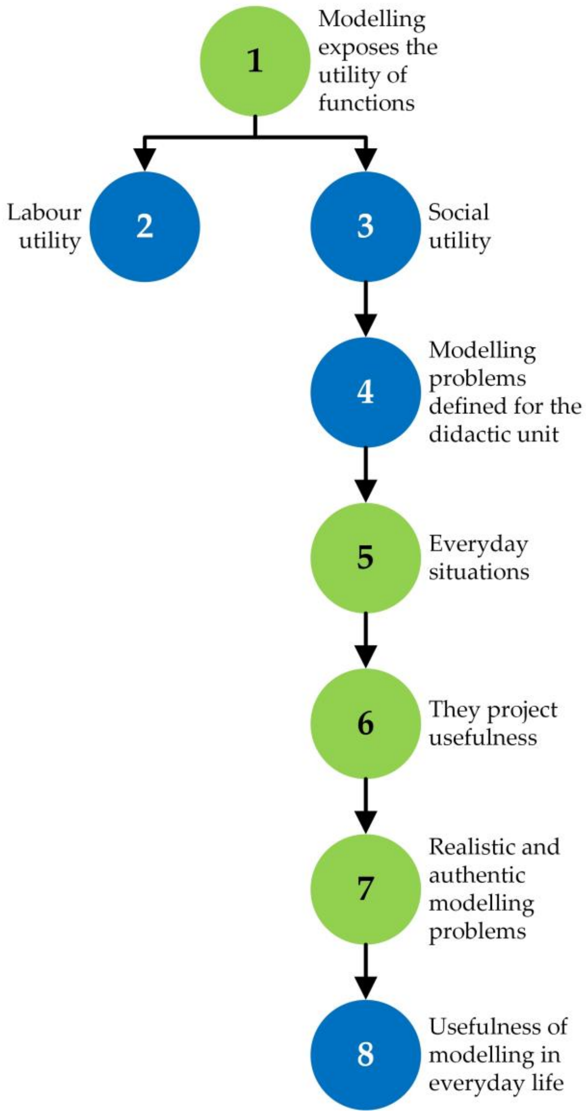

Figure 17. Diagramming of the assessment of the 'Social and labour usefulness' component. Source: Authors' interpretation.

The diagram of the analysis of the 'Didactic innovation' component is as follows:

1{The innovation in this didactic unit does not come from the content, which is the classic, but from two aspects}. 2{Firstly, the use of modelling as a central axis of the didactic unit}, 3{which allows to propose contextually rich problems to the students}, and

4{which allow different mathematical competences to be worked on}, such as those discussed in section 3.1.3 dedicated to the analysis of richness of processes. In this sense, 5{it is difficult for me to imaging the fact of proposing this didactic unit of functions without using modelling}. Also, during the didactic unit, 6{{software such as GeoGebra} 7{was used to immediately visualise the graphical representations of functions} and 8{encourage the change of representation} as a tool 9{to solve problems or understand concepts in class}. (Deltell, 2021, p. 17)

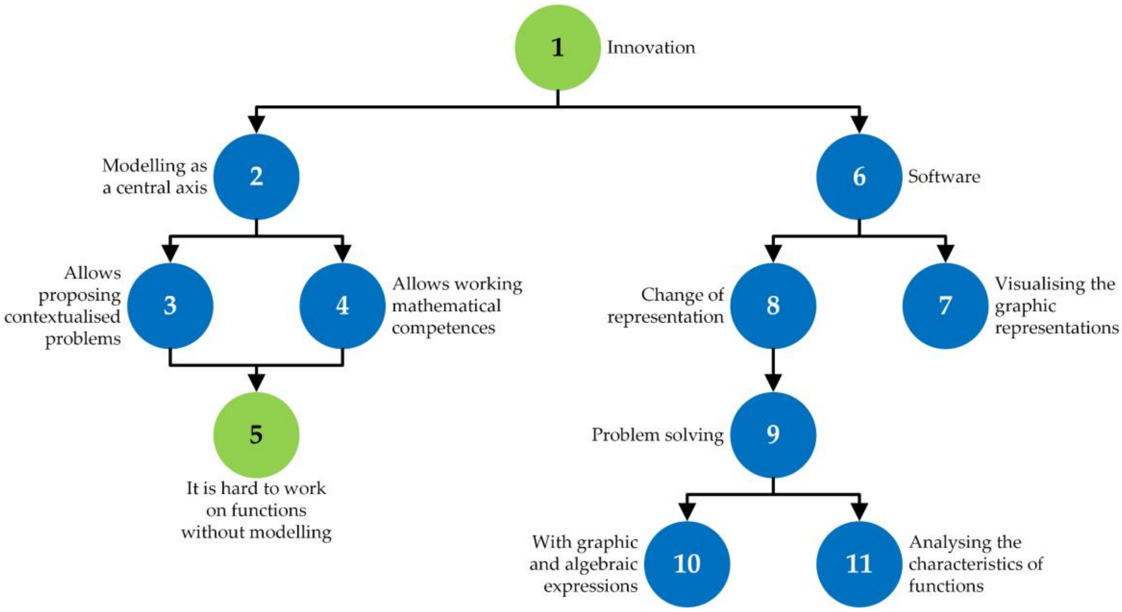

Figure 18. Diagramming of the assessment of the 'Didactic innovation' component.
Source: Authors' interpretation.

In the assessment of the 'Curriculum adaptation' component, proposition 4 in Figure 15 shows that the study subject considered functions as an appropriate mathematical object to work on modelling (in line with the position of Michelsen, 2006) and, in this way, addressing both the content and the competences that are established in the educational curriculum. Hand in hand with the above, in the assessment of the 'Intra and interdisciplinary connections' component, propositions 5 and 6 in Figure 16 show that the study subject considered that, although functions naturally connect with other disciplines included in the educational curriculum, it was through modelling that he tried to make such connections more evident, which he considered to be an advantage. Regarding the assessment of the 'Social and labour usefulness' component, propositions

6 to 9 in Figure 17 reinforce the previous ideas about the benefits of contextualisation as part of modelling, which highlights the social and labour utility of mathematics and, in particular, the concept of function. Finally, regarding the assessment of the 'Didactic innovation' component, proposition 2 in Figure 18 once again highlights that modelling, as a central axis of the didactic unit, allows the development of different mathematical competences, both those included in the curriculum (proposition 4) as those that are typical of working with this process.

3.4.2.3. Modelling task

As we have stated throughout this article, the study subject's didactic unit considered the modelling process as the central axis. To do this, in his MFP he repeatedly stated that he designed and implemented this type of tasks to work with his students. Moreover, during the first group reflection session, emphasis was placed on a task based on the fuel consumption of a car during a family trip to the beach. In the following paragraphs we present this task and its reformulation, which we analysed from the theoretical framework of modelling described in subsection 3.2.1.

The task initially proposed by the study subject was designed to be implemented during the 'Món Matemàtic' workshop and was part of a work guide for the students. The guide included an introductory text describing three formulas, considered as mathematical models, to calculate the fuel consumption of a car, explaining, in turn, the intervening variables. Table 3 presents these three formulas that, in terms of the study subject, correspond to 'simplified models'.

Table 3. Formulas for the calculation of fuel consumption.

Description	**Formulas**
Real distance consumption (CD_r)	$CD_r = CD_e \cdot \left(\frac{m_v + \sum_{i=n}^{i=0} m_{ci}}{m_v}\right)$
Consumption at idle (C_p)	$C_p = \left(\frac{3l}{h}\right) \cdot t$
Total consumption (C)	$C = C_p + L \cdot \frac{CD_r}{100}$

Source: Adapted from Deltell (2021, p. 80).

The variables involved in the three formulas in Table 3 and the magnitudes in which they are expressed are:

- CD_r: Fuel consumption per real distance (expressed in litres per 100 km).

- CD_e: Fuel consumption per standard distance (reported by the manufacturer; expressed in litres per 100 km).
- m_v: Mass of the vehicle (without additional load; expressed in kg).
- m_{ci}: Mass loaded by the vehicle (passengers, packages; expressed in kg).
- C_p: Fuel consumption at idle (expressed in litres; a constant of 3 litres per hour is estimated).
- t: Time that the vehicle is stopped (expressed in hours).
- C: Total fuel consumption (expressed in litres).
- L: Distance travelled (expressed in km).

The three statements after the introductory text (items (a) to (c)) consisted of tasks to mathematically work with the formulas and, in terms of the study subject, consisted of exercises with the 'models' previously presented. It is in item (d) where the modelling task designed by the study subject is introduced (see Figure 19), which is the same task that was discussed at length during the first group reflection session (see section 3.4.1).

> The Puigventós family has a house in the centre of Figueras. Mariona, the mother, is a tall woman who weighs 90 kg. She is married to Josep, who is a short man and weighs 75 kg. Mariona and Josep have a son named Rodolf, who is 24 years old and weighs 85 kg. Rodolf is a neighbour of his parents and lives with Tere, his girlfriend, who weighs 60 kg.
> Mariona has just bought a new car, which the manufacturer said has a standard consumption per distance (CD_s) of 5 litres per 100 km, and a mass (m_v) of 1000 kg.
> The Puigventós family wants to go to a beach which is 110 km from their home. Being midsummer, they know that there will be traffic jams and that they will be stopped for 3 hours with the engine running. Mariona filled the fuel tank, which cost her €1.2 per litre.
> Due to some strange reason, Tere is not sure if she will travel. Can you calculate how much money Mariona would spend if Tere stays at home? Model the cost of the Puigventós family's trip to the beach with and without Tere.

Figure 19. Modelling task designed for the didactic unit.
Source: Translated from Deltell (2021, p. 82).

The solving of the task basically consisted of interpreting the data of the statement and placing them in the respective variable of the 'mathematical models' of the introductory text of the work guide (described in Table 3). After performing the corresponding mathematical operations, the results for the problem question would be (in rounded values): €10.89 (with Tere) and €10.88 (without Tere).

In his MFP, the study subject mentioned that this task corresponded to a modelling problem. Based on the characteristics for this type of problem that were described in subsection 3.2.1, we can state the following:

- The situation is neither *open* nor *complex* because, on one hand, the solving procedure is determined by the application of the formulas – or 'mathematical models' – that were mentioned in Table 3 and, on the other hand, the wording of the task provides all the information necessary for its solving.
- Although the wording of the task is *realistic*, by adding elements from the real world (the Puigventós family, the car, the trip to the beach), the situation described contains numerical values that were defined to obtain certain results, so we cannot consider it like an *authentic* situation.
- Finally, and taking into account the two points raised above, this task cannot be considered as a modelling *problem*, which is why its solving cannot be carried out or explained in terms of *a modelling cycle*.

On this last point, the statement in Figure 19 is more consistent with a task to work on the 'work mathematically' sub-competence given a mathematical model and the data necessary for its use, in accordance with the position of Maaß (2010) and her proposed classification on modelling tasks.

The two tutor professors pointed out these observations to the study subject during the first group reflection session. However, due to the work dynamics of the educational centre in which he carried out his educational internship, the study subject could not modify his work guide, which had been previously reviewed and approved by his mentor teacher. Despite this situation, the study subject redesigned the wording of the task in the question (see Figure 20) for his MFP, taking into account the observations and suggestions made by the tutor professors.

> Let's imagine that you want to go from your house to the nearest beach to spend the weekend. See how far your house is from that beach and write it down, attaching a screenshot or picture of the map that allowed you to measure the distance. Assuming you encounter traffic jams that last three hours, what is the fuel consumption for you to go from your house to the beach and how much money will it cost to you?

Figure 20. Modelling task redesigned for the MFP.
Source: Translated from Deltell (2021, p. 79).

The redesign of the task kept the context of the original proposal, that is, the fuel consumption of a car during a family trip to the beach. However, the modifications to the wording of the task brought it closer to a modelling problem than in the initially designed proposal:

- The situation now is more *open* and *complex* since, although the procedure is still determined by the applications of the formulas in Table 3, now it is the students who must search for the necessary data to be able to solve the task.
- The wording of the task now includes a more *realistic* and *authentic* situation since, derived from the previous point, the opening of the problem now allows the information to be adapted to the context of each student, with variables, such as the price of a litre of fuel in their environment, the data of consumption and mass of the family vehicle, the distance from their home to a beach of free choice, etc.
- Finally, taking into account that this task was redesigned for a new hypothetical implementation of the didactic unit, it could be considered as a *problem*, as long as the students have not previously experimented with the originally designed task. In the same way, the solving of the redesigned task can be *solvable through a modelling cycle*.

On this last point, Figure 21 presents a hypothetical solving (elaborated by the authors) of the redesigned modelling task, which is explained from the modelling cycle of Blum and Leiß (2007), and using our own reference values for the variables involved in the 'mathematical models' in Table 3.

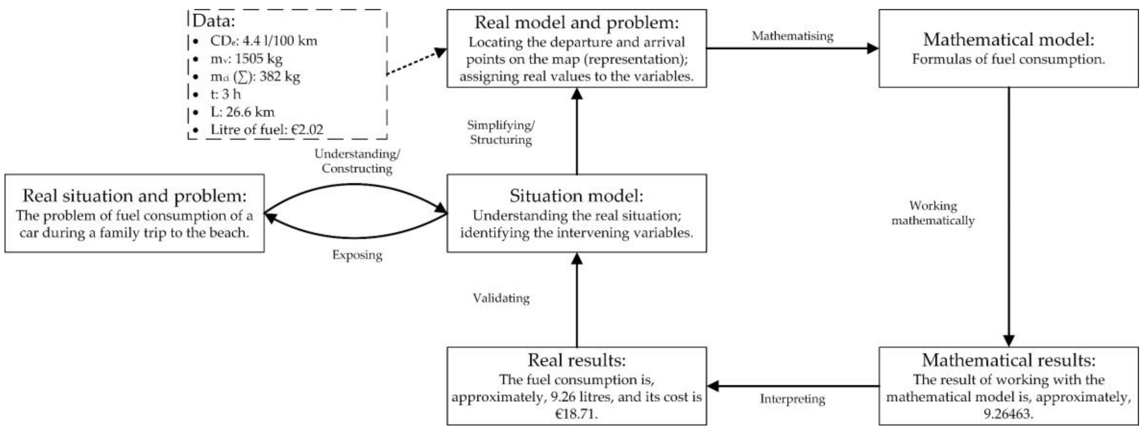

Figure 21. Solving of the redesigned modelling task from the cycle by Blum and Leiß (2007).
Source: Authors' interpretation.

The redesigned modelling task is much closer to a modelling problem, both in its wording and in its solving, than the originally designed task. For this hypothetical solving, we considered a hotel in the centre of the city of Figueras (Spain) as a starting point and Garvet Beach as a destination, tracing the route by vehicle with a map application. In turn, we obtained the data on fuel consumption per standard distance and vehicle mass based on the car of one of the authors.

However, when solving the task, we could identify certain inconsistencies with the real context in which the task is situated. Firstly, the city centre of Figueras is located at a linear distance of 15 km from the coast, with multiple road connections to get there; therefore, would traffic jams for 3 hours make sense? Secondly, in the event that the solver's vehicle was equipped with the 'start-stop' technology, would the calculation of the fuel consumption at idle make sense? Finally, in the case that the solver's vehicle was hybrid or electric, would mathematical models make sense to solve the problem?

Although these aspects reduce the authenticity of the redesigned modelling task, these questions could be raised in class in order to reflect on the importance of real context in modelling processes.

3.5. Discussion and Conclusions

In this section, we discuss the results presented in the previous section. In a first subsection we return to the arguments put forward by the study subject during the first group reflection session (see subsection 3.4.1), which were analysed from the pragma-dialectical perspective. In a second subsection, we return to the arguments obtained through the diagramming technique of the assessment of the didactic unit made by the study subject in his MFP. Finally, we conclude the results of the study, establishing relationships in the argumentation of both subsections. The analysis of the argumentation in the reflection sessions and in the assessment of the MFP allowed inferring some beliefs that constitute the study subject's conception regarding the modelling process, some of which coincide with the characteristics that this type of task should have and others that are not compatible.

3.5.1. On the group reflection sessions

During the group reflection sessions, some comments regarding the interaction with students were repeatedly mentioned by the study subject to justify the design and implementation of his didactic unit. These comments referred to the fact that, in his role as a teacher, he was the one who should guide the modelling process that his students would develop, which suggests a double reading. On one hand, we can infer that a belief of the study subject about working with modelling is that, when problems of this type are posed for the first time, it is the teacher who directly guides their students in solving the task. Although this position is not fully shared by the mathematical modelling research community, it can be justified, to a certain extent, in the duality of teaching this process, first, from an atomistic approach, to then work from a holistic approach (see Blomhøj & Jensen, 2007). In other words, 'teaching to model to be able to model'. On the other hand, this possible belief conflicts with the methodology suggested in the literature to work on the modelling process in the classroom where, among other aspects, a balance between maximum student autonomy and minimal teacher intervention must be achieved (see a discussion on these 'dilemmas' in Ng et al., 2015).

From the third Argument (Figure 5) we can infer that a belief of the study subject about modelling is that this process can be used to develop a topic such as, in this case, functions. That is, the modelling process can be used 'as a vehicle' to introduce a mathematical object, in line with the position of Julie and Mudaly (2007). In this particular argument, the study subject raised a proposal that he considered would allow him to work with modelling as a means to introduce functions. In propositions 3.1.1.1.–3.1. in the third Argument, the study subject stated that he wanted to address the topic of functions from elementary mathematical concepts (graphical representations, domain, range) to the equation of a line, and ended by presenting a situation where he considered that he would be using modelling. However, what he was actually doing was posing a situation of application of a mathematical model. By raising his motivation for naively including this process in his didactic unit, his tutor professors began to question him about this idealistic vision. In this sense, the third Argument represents a turning point within the group reflection session.

The tutor professors' questions about how to work modelling with the students were mainly based on the fact that there were aspects of this idealistic vision that were not compatible with the characteristics of a modelling problem, nor with the conditions for developing this process in the classroom. In other words, the students had no previous modelling experience, they did not know the heuristics of a cycle, there was not enough time to teach it (only the 'Món Matemàtic' workshop), etc. That said, in the fourth (Figure 6) and fifth (Figure 7) Argument, the study subject justified that the cognitive level of his students and the restrictions of the context would not allow him to develop modelling tasks with all these necessary characteristics and conditions. Although these two arguments showed that the study subject became more aware of the limitations to carry out a modelling process during his didactic unit, this did not imply making substantial changes in the modelling tasks proposed in his work program but had repercussion only in the redesign proposed in his MFP and in the assessment of his didactic unit.

From the fourth Argument (Figure 6), we can infer that the idea of the study subject to work on modelling was in phases (atomistic approach) and did not seek to develop a complete modelling process (holistic approach). This was justified by the fact that, on one hand, it was the first time that the students faced this type of problem and, on the other hand, that the model was not so simple that they could construct it on their own. For these reasons, the study subject decided to give the mathematical models to the students, which were described in the work guide and explained their intervening variables. On this last aspect we can infer another belief of the study subject, that a modelling process necessarily implies the construction of new mathematical models, beyond the knowledge acquired by students up to a certain educational level. This idea, somewhat voluntarist on the part of the study subject, is not fully consistent with the definition of 'mathematical model' that is considered in the cycle proposed by Blum and Leiß (2007), where this concept is understood as the set of mathematical objects that allow to explain the phenomenon to be modelled (see Abassian et al., 2020, p. 4) and that, not necessarily, imply the generation of new objects, since it is equally valid that those already known by the problem solvers are used. In the same way, if they are students who, on one hand, did not have previous experiences of working on modelling and, on the other hand, were studying a third grade of secondary education (students aged 14–15), this idea of the study subject would have no basis in the educational context in which he developed his internship.

Finally, his intention was reduced to the fact that, once the students understood the model, they would be able to apply it to situations that were adjusted to the model for its solving. In other words, the study subject taught his students to apply a function in different situations, through a diversification strategy, in order to work on a representative sample of problems with the same mathematical model. These results partially coincide with the findings of Villa-Ochoa (2015), as far as the modelling problems posed by teachers are reduced to statements to evaluate the students' skills to register, through a symbolic expression, a mathematical relationship covered in a, more or less, realistic word problem.

From the fifth Argument (Figure 7), we infer that one of the initial beliefs that form the conception of modelling of the study subject is that it is not necessary for this type of problem to be *open*, which is inconsistent with the characteristics of a modelling problem (as described in subsection 3.2.1).

Finally, we stress that, in his argumentation, the study subject used knowledge from the different types proposed by the DMKC model, namely epistemic (third and fourth Argument), cognitive (fourth and fifth Argument), and mediational (first and second Argument) knowledge.

3.5.2. On the assessment of the didactic unit in the MFP

The diagramming of the assessment of the didactic unit carried out in the MFP, guided by the DSC, allowed us to relate the arguments in the study subject's reflection with some elements of the modelling cycle; as well as identify some ideas, conceptions, or beliefs of AD regarding modelling that are (or are not) compatible with the characteristics of both a modelling problem and the work with this process.

In the diagramming of the assessment of the 'Richness of process' component (Figure 8) of the epistemic criterion, we were able to relate some propositions with the 'mathematisation', 'understanding/constructing', 'simplifying/structuring', and 'validating' transitions that include the modelling cycle (see Figure 1, nos. 1, 2, 3, 6). However, we evidenced certain limited interpretations of the study subject about this process. On one hand, we were able to infer an overlap between the 'mathematical modelling' and 'mathematical applications' processes. On the other hand, his conceptualisation of modelling gives a greater weight to mathematical aspects ('model' and 'mathematical work') than to the interpretation and validation of results with the content of the problem-situation posed, calling into question the closeness of this process with the «real world». This idea is reinforced in the diagramming of the 'Prior knowledge' component (Figure 10) of the cognitive criterion, where the study subject showed greater interest in the students' prior knowledge about the mathematical object function than in their knowledge about the modelling process. As this last aspect was not considered by the study subject, we infer that his conception of modelling does not focus its interest on the transition between «real world» and «mathematics» in a bidirectional way.

Regarding the aspects consistent with the literature about modelling, some propositions of the diagramming of the assessment of the ‘Representativeness of the complexity of the mathematical object’ component (Figure 9) of the epistemic criterion, are considered as a sign that the study subject considered that, in a modelling process, the mathematical object (model) is determined by the context of the problem-situation. Regarding the assessment of the ‘High cognitive demand’ component (Figure 11) of the cognitive criterion, we were able to identify that the study subject, in some way, considered the following transitions in a modelling process, at least within his reflection (not so in the design of the task originally proposed): ‘understanding/constructing’, ‘working mathematically’, ‘interpreting’, and ‘validating’. Similarly, the study subject considered that the modelling process implies a high cognitive demand, due to the multiplicity of mathematical processes involved.

Finally, in his MFP, the study subject made explicit some changes that he had from the conclusions of the first group reflection session, which were reflected both in the assessment of his didactic unit and in his redesign proposal. On one hand, in the diagramming of the assessment of the ‘Student interaction’ component (Figure 12) of the interactional criterion, the study subject considered as important that the modelling process promotes interactions between students through group work. On the other hand, in the diagramming of the assessment of the ‘Didactic innovation’ component (Figure 18) of the ecological criterion, the study subject downplayed the restrictions that he had in his implementation phase, e.g., the impossibility of using a computer room to work on the modelling task using GeoGebra software, since its reformulation made him dispense with these limitations. Now, first, this is innovating didactically in his context, such as in the educational centre where he undertook his internship, and second, it is innovating with support in the literature of the Didactics of Mathematics. That is, innovation is relative to the context. In addition to that, we must take into account that, in terms of Niss (2001), the implementation of modelling in regular mathematics lessons is not usual, due to different restrictions presented by the educational context, such as those that occurred in the institutions where the study subject carried out his educational internship.

Given that, at the time of preparing his MFP, the study subject already knew and used the DSC as a tool to guide his reflection, we can affirm that he shows knowledge of the meta didactic-mathematical dimension of the DMKC model. However, the detailed study of his argumentation also allowed inferring knowledge of the mathematical and didactic dimensions.

3.5.3. Conclusions

In relation to the first research question, regarding what argumentation a prospective teacher makes to justify the incorporation (or not) of mathematical modelling in his lessons, in section 3.5, we have described the arguments used by the study subject in two different phases.

In relation to the second research question, regarding what knowledge, values, beliefs, and guidelines for action are inferred in this argumentation, we can make several affirmations. In the first place, the detailed analysis of the arguments present in the study subject's reflection evidences the use of the three dimensions of knowledge proposed by the DMKC model (mathematical, didactic, and meta didactic-mathematical dimensions), while knowledge of the six facets of didactic knowledge (epistemic, cognitive, interactional, mediational, affective, and ecological facets) appear. In the second place, with respect to the study subject's beliefs, we were able to observe a change from his initial reflections to those contemplated in his MFP. Initially, he considered that it was enough to translate the wording of a task from natural to mathematical language to talk about modelling, while, in his final reflection, he considered that there are certain necessary characteristics to be able to consider a modelling problem as such. This last belief disposed the study subject to redesign the originally proposed task so that, in the improvement proposal of his MFP, it meets many of these characteristics.

An additional conclusion is that the study subject showed both good knowledge and competent use of the DSC construct. In his initial proposal, the study subject had given great importance to epistemic suitability, since he considered incorporating modelling in his didactic unit on the teaching of functions, but without taking into account the other criteria. Finally, in the reflection of his MFP, the study subject assigned a determining weight to different DSC, even though the planned modelling task could not be implemented. In summary, his reflection organised with the DSC, on one hand, improves epistemic suitability by ensuring that the redesigned modelling task meets the characteristics of a modelling problem and, on the other hand, considers the rest of the criteria in a more balanced way.

Our last conclusion is that, based on the analysis of the group reflection sessions and the study subject's MFP, we can suppose that AD's education as an engineer influenced part of his beliefs about modelling. However, we need a deeper analysis to fully affirm this conclusion, with a study that includes, for instance, interviews or questionnaires with the study subject that focus on this specific aspect. In any case, there are some elements that partially reinforce this conclusion, e.g., the fact that AD has reduced the modelling process to the application of a mathematical model in a specific situation, consistent with the view of Pollak (1979) on applied mathematics; in the design of a task contextualised in a problem from physics, where formulas, considered as 'mathematical models', were included for its analysis and solving; and, finally, that the interest of his tasks was that students understand and apply such formulas to a specific situation.

The results of this study may be useful for the master's programme in which we carried out this research, in the sense that a DSC guideline could be developed with more specific indicators to determine what will be understood by a modelling process or reconsider the way in which the incorporation of modelling is taught for secondary and baccalaureate education levels within the submodule dedicated to the teaching of such process. More specifically, the study subject, who had completed this submodule before designing his didactic unit, had the intention of working on modelling, but he was not aware that his students should have knowledge about the heuristics of the modelling process or, at least, having had previous modelling experiences, either with the complete cycle (holistic approach) or with some of its phases and transitions (atomistic approach). Because the study subject did not fully handle a modelling cycle either, what he ended up doing was a proposal to work on some of its elements (above all, 'mathematisation' and 'mathematical work') and say that he worked with modelling.

Finally, this study opens a research line to infer teachers' knowledge and beliefs from the argumentation made in their reflection, which differs from what is more common in this plane, such as the application of questionnaires or interviews, since we incorporate a new way of analysing the teachers' argumentation through pragma-dialectics and diagramming.

Additional statements

- Author contributions: Conceptualisation, C.L. and V.F.; methodology, C.L., T.S., G.; software, not applicable; validation, C.L., T.S., G.S-S., and V.F.; formal analysis, C.L., T.S., G.S-S., and V.F.; investigation, C.L. and T.S.; resources, V.F.; data curation, C.L., T.S., G.S-S., and V.F.; writing—original draft preparation, C.L. and T.S.; writing—review and editing, C.L., T.S., G.S-S., and V.F.; visualization, C.L. and T.S.; supervision, G.S-S. and V.F.; project administration, C.L. and V.F.; funding acquisition, C.L. and V.F. All authors have read and agreed to the published version of the manuscript.
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Capítulo 4: Manuscrito Científico

El manuscrito presentado en este capítulo tiene por objetivo analizar dos episodios de argumentación práctica en que un grupo de profesores en servicio, participantes de un curso formativo basado en el *Lesson Study*, reflexionaron sobre el diseño, implementación, y rediseño de una clase sobre el teorema de Pitágoras. Si bien con este manuscrito no se pretende dar cumplimiento a algún objetivo específico, su inclusión se justifica en que complementa el desarrollo del tercer objetivo específico. Cabe destacar que, tanto este manuscrito como el artículo presentado en el Capítulo 6 reportan los análisis de episodios de *Lesson Study* donde no se tuvo participación en el diseño e implementación del ciclo, sino que sólo se tuvo acceso a los datos derivados del ciclo para su posterior análisis. Esto se debe a que, si bien ambos ciclos de *Lesson Study* formaron parte del proyecto PGC2018-098603-B-I00, en un principio, esta investigación doctoral era independiente del desarrollo de estos ciclos.

En la Tabla 7, se presentan los datos de identificación de este manuscrito.

Tabla 7. Datos de identificación del manuscrito científico.

Criterios	Datos
Título	Teachers' practical argumentation on the teaching of the Pythagorean theorem.
Autores	Telesforo Sol, Carlos Ledezma, Alicia Sánchez, Vicenç Font.
Afiliaciones	Departamento de Educación Lingüística, Científica, y Matemática, Facultad de Educación, Universidad de Barcelona (España).
Revista	International Journal of Science and Mathematics Education.
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Referencia	Sol, T., Ledezma, C., Sánchez, A., y Font, V. (en evaluación). Teachers' practical argumentation on the teaching of the Pythagorean theorem. *International Journal of Science and Mathematics Education*.

Fuente: Elaboración del autor.

Esta revista está indexada en 25 bases de datos, incluyendo CLOCKSS, Sciences, EBSCO, SCImago, SCOPUS, entre otras.

De acuerdo con la información disponible en Scopus, esta revista está indexada en las categorías Matemática/Matemática general (rango #26/399, 93º percentil) y Ciencias

Sociales/Educación (rango #269/1543, 82º percentil), con un CiteScore en 2023 de 5,1 y un CiteScore Tracker en 2024 de 5,3.
De acuerdo con la información disponible en Scimago Journal & Country Rank (SJR), esta revista está indexada en las categorías Matemática/Misceláneo y Ciencias Sociales/Educación, su Índice H es 56 y el SJR 2023 es 1,04 (Q1 en Educación).
De acuerdo con la información disponible en el Master Journal List de Web of Science (WoS), esta revista está indexada en el Social Sciences Citation Index (SSCI), Social And Behavioral Sciences, y Essential Science Indicators, en las categorías Educación/Investigación educativa, Educación, y Ciencias Sociales/General, respectivamente, con un Indicador de Citas de Revista de 1,09 en 2023.

Abstract and Keywords / Resumen y Palabras Clave

Abstract: Delving into teaching reflection on practice and the underlying argumentation are key aspects for teacher professional development. Our interest in this study is focused on the role played by the Didactic Suitability Criteria, as a construct to guide teaching reflection on practice, in the practical argumentation of a group of practising teachers during their reflection on a mathematical teaching and learning process. Methodologically, it is a case study conducted in a training course to foster teacher reflection involving eight Brazilian practising mathematics teachers who designed, implemented, and redesigned a lesson with the Pythagorean theorem. In this article, we report two episodes of argumentation, in which the teacher reflection was analysed with the Didactic Suitability Criteria construct and the practical arguments were structured with a proposal of adaptation of Toulmin’s model with this construct. The main results are: (a) the teaching of the Criteria made the teachers’ reflection to be broader and better structured, where their practical argumentation considered more aspects; (b) in both episodes, when some of these Criteria conflicted among them, the teachers’ practical argumentation allowed them to reach consensus through seeking balances. Finally, we highlight that our proposal of adaptation allowed incorporating the Didactic Suitability Criteria to Toulmin’s model, generating a more nuanced tool to study mathematics teachers’ practical argumentation.
Keywords: Didactic suitability criteria; Lesson study; Practical argumentation; Pythagorean theorem; Teacher education.

Resumen: Profundizar en la reflexión docente sobre la práctica y en la argumentación subyacente son aspectos claves para el desarrollo profesional docente. El interés de este estudio se centra en el papel de los Criterios de Idoneidad Didáctica, como constructo para pautar la reflexión docente sobre la práctica, en la argumentación práctica de un grupo de profesores en servicio durante su reflexión sobre un proceso de enseñanza y aprendizaje matemático. Metodológicamente, se trata de un estudio de caso desarrollado en un curso formativo para fomentar la reflexión docente, en el cual participaron ocho profesores de matemática en servicio brasileños quienes diseñaron, implementaron, y rediseñaron una clase sobre el teorema de Pitágoras. En este artículo, se reportan dos episodios de argumentación, en que la reflexión docente se analizó con el constructo Criterios de Idoneidad Didáctica y los argumentos prácticos se estructuraron con una propuesta de adaptación del modelo de Toulmin con este constructo. Los principales resultados son: (a) la enseñanza de estos Criterios hizo más amplia y mejor estructurada la reflexión de los profesores con relación a la realizada antes de conocer este constructo, donde su argumentación práctica consideró más aspectos; (b) en ambos episodios, cuando algunos de estos Criterios conflictuaban entre sí, la argumentación práctica de los profesores permitió alcanzar consensos mediante la búsqueda de balances. Finalmente, se destaca que la propuesta de adaptación permitió incorporar los Criterios de Idoneidad Didáctica al modelo de Toulmin, generando una herramienta más matizada para estudiar la argumentación práctica de los profesores de matemática.
Palabras clave: Argumentación práctica; Criterios de idoneidad didáctica; Educación de profesores; Estudio de clases; Teorema de Pitágoras.

4.1. Introduction

Current trends in both initial and continuous teacher education stress the importance of delving into teaching reflection on practice and the underlying argumentation as key aspects for teacher professional development. An instance to develop both aspects is, for example, when working with Lesson Study (Huang et al., 2019). This approach is based on a group of teachers and experts, with a shared interest in student learning, meeting, planning a lesson, and finally examining and discussing their observations during the implementation of that lesson (Fernández & Yoshida, 2004; Hart et al., 2011; Lewis, 2002).

One of the interests of our work is teacher argumentation in reflection on practice. Within the study of argumentation is that of practical argumentation, that is, argumentation where a) arguments whose conclusion is an action to be performed are produced, or b) arguments on actions considered in the discussion but which are not performed are produced (Gómez, 2017). In a Lesson Study cycle, explicit or implicit consensus arise in the reflection on practice (Erbilgin & Arikan, 2021), where teachers justify actions that they agree upon to perform through practical arguments, which are supported by principles on what they consider is the best option to improve such practice (Hummes et al., 2019). Thus, the study of practical argumentation provides tools to analyse how this consensus is reached in the dialogue among participants, where reasons are given for and against performing or not certain actions (Kinach, 2002) (see further details in subsection 4.2.1).

Another interest of our work is teaching reflection on practice. Aiming at guiding a quality teaching practice and a quality reflection on it, some authors have compiled different criteria based on findings from Mathematics Education research (e.g., Hill, Blunk et al., 2008; Praetorius & Charalambous, 2018; Prediger et al., 2022), where the Didactic Suitability Criteria construct (Godino, 2013) is an example of this. Studies conducted with this construct (e.g., Breda, 2020; Breda et al., 2017, 2018) have shown that it can be used to reinterpret the guidelines for practice that appear in teachers' discussions when, for example, they assess a classroom episode or justify the improvement of a lesson plan, despite not having knowledge of this construct or its use to guide teaching reflection (see further details in subsection 4.2.4).

Therefore, because of the importance of teachers' practical argumentation in reflection on teaching practice, we posed the following research question for this study: What is the role played by the Didactic Suitability Criteria in the practical argumentation of a group of practising teachers during their reflection on the design, implementation, and redesign of a lesson with the Pythagorean theorem? To this end, we report the analyses of two episodes of practical argumentation in which a group of practising teachers, involved in a training course based on Lesson Study, reflected on the design, implementation, and redesign of a lesson with the Pythagorean theorem. We analysed these episodes using the Didactic Suitability Criteria construct (Godino, 2013) and structured the emerging practical arguments using a proposal of adaptation of Toulmin's (1958/2003) model that included this construct.

The relevance of our study lies in the fact that we address the need to foster reflection on teaching practice, focusing on the analysis of practical argumentation to improve it. Two major positions, among others, agree on this need. On one hand, Fenstermacher and Richardson (1993) consider that talking about the importance of reflection in teaching practice is not enough, but it is also necessary promoting specific ways to foster such a reflection. These authors mention that, although there are teacher education programmes that have attempted to help teachers become more reflective, the processes of change in reflective capability remain confusing and specific ways need to be found to enhance this capability in teachers. On the other hand, Font and Godino (2011) state that Mathematics Education has two major demands: the first is that its theoretical constructs are useful to understand mathematical teaching and learning processes (by answering the question, what has occurred here?); and the second is that these constructs guide the improvement of those processes (by answering the question, what could be improved?). Therefore, in this study, we intend to address both positions. Regarding the study of Fenstermacher and Richardson (1993), we present practical argumentation as a tool to analyse teaching reflection in two episodes of argumentation that occurred in the context of a training course. This analysis may help to better understand the teachers' pedagogical thinking and better address its development (Vesterinen et al., 2014) by fostering teaching reflection. In response to the second demand proposed by Font and Godino (2011), when we analyse the teaching reflection in these two episodes of argumentation, we add a theoretical-methodological tool, that is usually used to guide teaching reflection, to the structuring of the emerging practical arguments.

4.2. Theoretical Framework

In this section, we describe the theoretical references considered in this study.

4.2.1. Argumentation, practical argumentation, and collective argumentation

Researchers in Mathematics Education (e.g., Nussbaum, 2008; Singletary & Conner, 2015; Yackel, 2002) and international curricular documents (e.g., Ministério da Educação do Brasil, 2018; National Council of Teachers of Mathematics [NCTM], 2000, 2014) consider the development of argumentation as a crucial aspect for mathematics learning. The Theory of Communicative Action defines argumentation as “that type of speech in which participants thematize contested validity claims and attempt to vindicate or criticize them through arguments” (Habermas, 1981/1984, p. 18), where an argumentation contains reasons that are systematically linked to the claim to validate a problematic statement.

Within argumentation, we can find that of practical type, which has been increasingly studied in fields such as political sciences, critical discourse analysis, and education, among others (Macagno & Walton, 2018). Practical argumentation is defined by Lewiński (2018) as that “argumentation aimed at deciding on a course of action” (p. 219) and, similarly, Gómez (2017) understands it as what happens in social contexts, and which is oriented towards choosing an action to solve a practical problem. In both definitions, the existence of an action in the arguments is considered.

Fenstermacher and Richardson (1993) pose the following three questions: “How, precisely, does a teacher use research or practice reflexively? Are there ways to enhance one’s capacity or skill at using research or practising reflexively? If so, what are they?” (p. 101). To answer these questions, these authors base on *practical reasoning* (or *practical rationality*), which they consider as “the process of thought that ends in an action or an intention to act” (Fenstermacher & Richardson, 1993, p. 102), where a practical argument represents such reasoning. Thus, a practical argument can be structured with the following four types of premises and an action or intention to act:

- Value premise: A statement of the human benefit or good to be derived because of performing this action. This premise may be phrased as a declarative (‘My goal is to…’) or imperative (‘Someone must…’) statement.
- Stipulative premise: A statement (or set of statements) that defines, interprets, or establishes meaning. In some cases, these premises may arise from theory (e.g., when a teacher attempt to explain his/her action based on some learning theory).
- Empirical premise: A statement (or set of statements), subject to empirical scrutiny, to be tested. These statements may be confirmed or denied by using methods like those of science (e.g., ‘Students whose parents read to them when they are young will learn how to read faster that those whose parents do not do that’).

- Situational premise: A statement (or set of statements) that describes the context in which the action occurs (e.g., when designing a lesson taking into consideration the fact that it will be implemented in a virtual format, which may limit the interaction with the students).
- An action or intention to act: The conclusion of a practical argument.

These authors suggest that practical argumentation is valuable in actions as complex as those that occur in the classroom, since it allows the reasons and their relationships with the actions to be analysed, and teachers' practical argumentation must be fostered, both its elicitation and assessment, by someone who accompanies the teacher. They also claim that practical arguments have a descriptive and a normative sense: the first refers to the moments in which teachers explain their actions, while the second is related to the assessment and improvement of such arguments. These two senses of practical arguments are closely related to the second demand proposed by Font and Godino (2011) and, in a certain way, to the Didactic Suitability Criteria construct described in subsection 4.2.4.

Pendlebury (1990) argues that understanding practical argumentation is important, both for teachers and teacher educators. This author makes a clarification about a practical argument, and that is, when someone reaches a conclusion and is asked how he/she came to formulate it, the person begins to describe thought processes as if he/she progressed linearly, which is not a copy of the corresponding reasoning.

In this study, our interest was focused on the arguments constructed by a group of teachers during their reflection on the design, implementation, and redesign of a lesson with the Pythagorean theorem, which we analysed from the perspective of practical argumentation. Given this interest, it is necessary to distinguish the mathematical argumentation from the practical argumentation of these teachers, since the first one emerges from mathematical practices, generating inductive, abductive, analogical, or deductive arguments, whereas the second one emerges from didactic practices, generating planned actions or those performed in the classroom. Likewise, it is also necessary to distinguish theoretical argumentation from practical argumentation, since the first one allows answering questions such as 'is *p* true?', where *p* is a description of the world, whereas the second one allows answering questions such as 'what should *A* do in a situation x?', where A is the name or description of an agent and x is the description of a problem-situation (Gómez, 2017).

Since our interest in this study was focused on the arguments constructed by a group of teachers, we also considered collective argumentation which, in terms of Conner et al. (2014), is characterised by involving several individuals who come to agreement on a conclusion, often by consensus. However, unlike the work by these authors, who focused on mathematical argumentation in the classroom, we considered the collective practical argumentation of a group of teachers during their reflection on the design, implementation, and redesign of a lesson, where we identified teacher actions focused on favouring the different aspects of the mathematical teaching and learning process. Herbst et al (2011) observe that most of the investigations on the relationship between teacher's thoughts and actions have been centred on teachers as individuals and remark the importance of analysing the normative elements with which a collective could endorse or reject certain actions. Thus, these authors are interested in studying the principles that could justify or refute actions that deviate from what is expected within the discourse of a collective.

Hereinafter, our references to practical argumentation imply the participating teachers' collective practical argumentation in the context of this study.

4.2.2. Studies on teachers' practical argumentation

The topic of teachers' practical argumentation has been widely addressed not only by research in Mathematics Education – to which our study is circumscribed – but also by other areas of knowledge. Considering the importance of developing reflection on teaching practice as a key mechanism for its improvement (Schoenfeld & Kilpatrick, 2008), the central axis of our investigation is practical argumentation. As described in the previous subsection, this type of argumentation allows us to represent the practical reasoning underlying this reflection and, in this way, to infer the criteria that guide teacher's practice to decide his/her decisions in a mathematical teaching and learning process. In this subsection, we present some relevant studies for our investigation.

Fenstermacher and Richardson (1993) studied teaching practice and the reflection on this practice by considering practical reasoning and practical argument. With these elements, the authors sought to show how practical argumentation helps teachers to understand the practical reasoning behind their actions and to deliberate and evaluate their thinking and action, by offering a way to promote changes and improvements in their thinking, criteria, and decisions. Furthermore, in this study, the use of practical argumentation and its description in the reflection of a teacher when she worked reading comprehension with her students was exemplified, showing the elicitation and reconstruction of a practical argument, and stating that this process benefited the participants in terms of beliefs and practices.

Boody et al. (1998) agreed with Fenstermacher (1994) on the need to develop teaching reflection and considered that teacher education programmes emphasise more on theoretical and technical knowledge than on personal and shared practical meanings that support teacher behaviour. In this article, the use of practical argumentation was exemplified as a process where a multidisciplinary group of teachers met to reflect on teaching practice. To this end, the sessions were structured by using practical argumentation as a way of understanding and explaining particular actions or sets of actions. Some benefits that the authors concluded about the use of practical argumentation are as follows: the significant changes in the participants' thought and teaching; the relationship between these changes and the improvements in the students' thought and learning; and the general positive effects in the participants' teaching as a result of the process of joint reflection and structured with practical argumentation.

Herbst et al. (2011) proposed a way of studying the practical reasoning of a group of mathematics teachers when managing the teaching of geometric theorems in secondary education. In this study, the authors identified categories of perception and appreciation used by the teachers when presented with videos depicting certain teaching situations. These categories describe what the teachers perceive in the instruction and delimitate what is reasonable or usual that a teacher does, remarking the normative elements that regulate the practice of teaching geometry. Finally, these authors stated that practical reasoning also involves the principles or values shared by teachers, who use them to justify or discard possible actions in a certain teaching situation.

Vesterinen et al. (2014) investigated the practical reasoning and the development of professional knowledge of prospective teachers during their educational internship experiences in initial training. These authors argue that reflecting about why something was done generates new beliefs and even new knowledge, which is why they consider that practical argumentation about teaching practice must not be seen solely as a means to defend affirmations, but also as an opportunity for professional learning. Furthermore, these authors remarked that the analysis of practical arguments is a useful way to investigate teacher's learning and pedagogical thinking in the context of teacher education. The authors commented that this type of analysis requires the participants to have the capability to present practical arguments and to reflect on their practice; however, there were participants whose arguments and reflections were brief.

Finally, Sol, Breda, Richit, and Sala-Sebastià (2024) analysed the practical argumentation of prospective teachers, involved in a Lesson Study cycle in Brazil, who designed a lesson on flat geometric shapes for preschool education. Although the group reflection did not have specific guidelines during this cycle, the participants remarked the importance of the students' previous knowledge for selecting materials, the need to foster autonomy in the classroom, and the relationship between arts and mathematics.

Unlike the reviewed studied, in this article, we analyse the teachers' practical argumentation before and after implementing a lesson, while others only focus on what occurred during the lesson (c.f. Vesterinen et al., 2014) or in the reflection after the lesson (c.f. Herbst et al., 2011), in the context of a Lesson Study cycle. Furthermore, we propose the adaptation of a model to structure practical arguments and to represent the relationship among their elements, through the inclusion of a tool to foster and guide teaching reflection.

4.2.3. Toulmin's model

As mentioned in subsection 4.2.1, Fenstermacher and Richardson (1993) described the structure of a practical argument. Although these authors considered a structure of four premises and an action or intention to act, in this study, we propose the use of Toulmin's model (1958/2003) to structure practical arguments and to represent the relationship among their elements. According to this model, an argument is made up of the following elements:

- Claims: Statements seeking to be validated.
- Data: Information on which the claim or conclusion is based.
- Warrants: Propositions (rules, principles, or statements) that allow making inferences and that connect data to the claim.

- Rebuttals: Circumstances in which the warrant is invalid.
- Modal qualifiers: Statements that indicate the degree of strength of the warrant in the relation between data and claim, with words as 'necessarily', 'probably', 'generally', 'almost', 'perhaps', etc.
- Backings: Justifications that support the warrants.

Toulmin's model has been used in different investigations to study teachers' argumentation, both in mathematics (e.g., Tekin, 2019) and scientific education (e.g., Lee & Lin, 2005). Unlike those studies, in which the arguments constructed by teachers in different training contexts are structured with Toulmin's model, in this article, we add a theoretical-methodological tool to this model and thus enrich this structuring, redirecting it towards practical argumentation. This theoretical-methodological tool is presented in the following subsection.

4.2.4. Didactic suitability criteria

To show the relationship between the two demands of Mathematics Education posed by Font and Godino (2011), Breda et al. (2018) state that the greatest the capability to describe, understand, and explain the mathematical teaching and learning processes, the better the conditions for achieving improvements. Likewise, these authors declare that the knowledge and results generated from this understanding influence the generation of values and norms that guide these improvements. Thus, to address both demands, the Onto-Semiotic Approach (OSA; Godino et al., 2007) proposes the notion of didactic suitability, which is considered an essential tool for assessing mathematical teaching and learning processes (Godino, 2013).

This notion corresponds to the degree to which the characteristics of these processes can be considered as optimal or adequate for attaining overlap between the *personal meanings* achieved by students (learning) and the *institutional meanings* intended or implemented by teachers (teaching), considering the circumstances and available resources (environment) (Godino et al., 2019). In this way, given a specific topic to be taught in a certain educational context, the notion of didactic suitability allows answering, among others, the following question: What changes should be made in the design and implementation of a mathematical teaching and learning process to increase its didactic suitability in the future? The answer to this question, sometimes, includes actions that the teacher should or proposes to do, which are part of the interests of our study.

Thus, this notion is a multidimensional construct that consists of six didactic suitability criteria (DSC), whose functionality is concretised in components for each criterion and indicators for each component. In this study, we use the DSC guidelines proposed by Breda et al. (2017) which, in turn, is an adaptation of the guidelines proposed by Godino (2013) for using in educational processes for prospective secondary education mathematics teachers (e.g., Ledezma, Breda, & Font, 2024; Sánchez et al., 2022), as well as in the case reported in this investigation. Table 4 presents the description and components of each DSC.

Table 4. Didactic suitability criteria and their components.

Criteria	Description	Components
Epistemic	To assess whether the mathematics that is taught is *good quality mathematics*.	– Errors – Ambiguities – Richness of processes – Representativeness of the complexity of the mathematical object
Cognitive	To assess, before starting the teaching and learning process, whether the topic to be taught is adapted to what students already know and, after this process, whether the achieved learning is close to what was intended.	– Prior knowledge – Curricular adaptation to individual differences – Learning – High cognitive demand
Interactional	To assess whether interaction solves student doubts and difficulties.	– Teacher-student interaction – Student interaction – Autonomy – Formative assessment
Mediational	To assess the adequacy of material and temporal resources used in the teaching and learning process.	– Material resources – Number of students, class schedule and conditions – Time
Affective	To assess student involvement (interests and motivation) during the teaching and learning process.	– Interests and needs – Attitudes – Emotions
Ecological	To assess the adaptation of the teaching and learning process to the school educational project, curricular guidelines, and conditions of the social and professional environment, etc.	– Adaptation to curriculum – Intra- and interdisciplinary connections – Social and labour usefulness – Didactic innovation

Source: Adapted from Ledezma, Breda, and Font (2024, p. 1062).

As an example, the *Prior knowledge* component from the *cognitive* criterion is specified through two indicators: “a) Students have the necessary previous knowledge to study the topic […]; b) The intended meanings (reasonable difficulty) can be taught through its diverse components” (Breda et al., 2017, p. 1904).

DSC represent guidelines (with criteria, components, and indicators) that were developed considering contemporary trends in mathematics teaching, the principles of the NCTM (2000) and the findings in Mathematics Education research that are broadly accepted by the professional community. For example, the *epistemic* criterion considers the following finding: mathematical objects emerge from mathematical practices, which entails their complexity (see Font et al., 2013). From this finding, the *Representativeness of the complexity of the mathematical object* component is derived and implies guiding teachers to consider the mathematical complexity of the notion to be taught when they design didactic sequences.

The DSC guidelines is a consensual tool used to structure teaching reflection in different training programmes taught in the Ibero-American context (see Burgos & Castillo, 2021; Esqué & Breda, 2021; Morales-Maure et al., 2019; among others), as well as in the training course reported in our study. Hereinafter, our references to DSC imply both the six criteria and the complete guidelines of criteria, components, and indicators.

4.3. Methodology

In this study, we followed a qualitative research methodology from an interpretative paradigm (Cohen et al., 2018), which consists of an intrinsic case study (Stake, 2005) where we analysed the episodes of practical argumentation of a group of practising mathematics teachers during a training course.

4.3.1. Research context

We conducted this research in the context of a training course for Brazilian practising mathematics teachers, in which we adopted the Lesson Study (LS) approach. The objective of this course was to analyse to what extent a training course based on LS and DSC fostered the development of the reflection in a group of mathematics teachers (see Hummes, 2022). The training course consisted of 16 sessions of 2.5 hours each carried out from March to July 2020. Due to the COVID-19 pandemic, only the first session of the training course could be held in person and the rest were held online. In this section, we describe the participants and the three phases that made up this training course.

4.3.1.1. Participants

The participants of the training course were one teacher trainer and eight practising mathematics teachers who worked in schools from southern Brazil with students aged 11–18 years. All the teachers had a bachelor's degree in mathematics, three of them also had a master's degree in mathematics education, and their teaching experience ranged between 3–15 years. All the teachers participated voluntarily in this training course and research, with prior informed consent by the trainer (see Hummes, 2022).

4.3.1.2. First phase of the training course

In this first phase, a LS cycle made up by the following stages was implemented (Hurd & Lewis, 2011; Lim-Ratnam, 2013):

1) Choice of one mathematical content, study of its curricular approach, and establishment of learning objectives. In this LS, the Pythagorean theorem (PT) was the chosen content, and the curricular approach was based on the guidelines by Ministério da Educação do Brasil (2018).
2) Planification of a lesson.
3) Implementation and observation of the lesson, recording the impact of the planning on the student learning and collecting data through observation.
4) Joint reflection on the data.

In this first phase, the teachers were divided into two groups of four members each: the first group (teachers T1 to T4) planned one lesson with the PT and implemented it with 9th grade students (final level of elementary education in Brazil), whereas the second group (T5 to T8) only planned one lesson with the PT for a group of secondary education students, but without implementing it. T4 implemented the lesson in a virtual classroom with her students, on which the teachers of both groups performed a joint reflection. However, it was not until the third phase of the training course (see subsection 4.3.1.4) that the implemented lesson was redesigned.

4.3.1.3. Second phase of the training course

In this second phase, the trainer taught DSC to the teachers from both groups throughout a series of stages. Firstly, classroom episodes were analysed, based on the teachers' previous knowledge, beliefs, and values, without giving them any guidelines for this. Next, it was determined whether the criteria they used were based on individual or moral values transmitted by the Mathematics Education community. Secondly, it was sought to reach a group consensus among the teachers regarding their criteria for considering a mathematical teaching and learning process as good, appropriate, of quality, etc. This allowed it to reveal a relationship among the criteria that guide teaching practice towards improvement and the findings and theoretical frameworks of Mathematics Education. Thirdly, the trainer explained that some authors have compiled different criteria for guiding teaching practice and its reflection (e.g., Hill, Blunk et al., 2008; Praetorius & Charalambous, 2018; Prediger et al., 2022), reflecting on the fact that, although there is a broad consensus on these lists of criteria in Mathematics Education, they are located in a certain historical and cultural context, outside of which they may not generate consensus. Fourthly, the trainer referred to the OSA as an example of a theoretical framework that develops this line of research, which gave way to explaining the notion of didactic suitability and presenting DSC. Finally, DSC were operationalised considering their criteria, components, and indicators, by generating a rubric (like that of Breda et al., 2017) that can help the teachers assess their practice and guide their redesign proposal.

4.3.1.4. Third phase of the training course

In this third phase, DSC were used to reflect on and redesign the implemented lesson. To this end, each teacher assessed the implemented lesson using one specific criterion (with its respective components and indicators) of the DSC guidelines as follows: T1 and T2 used the epistemic criterion, T3 and T4 used the cognitive criterion, and from T5 to T8 used the mediational, interactional, affective, and ecological criteria, respectively. Afterwards, the teachers met again with the trainer to share their assessment and reflect on the didactic suitability of the implemented lesson. In other words, the teachers now had a tool to guide their reflection (DSC) and thus propose ways to redesign and improve the lesson plan (e.g., Breda & Lima, 2016; Ledezma, Sol et al., 2022; Sánchez et al., 2022; among others). Although DSC are principles, the trainer emphasised that they are not binary application rules, so it is important to relativise didactic suitability and shape it to the teacher's context and judgement. In this way, this LS cycle represented a useful way to introduce DSC to the teachers, where guidelines to organise teaching reflection were generated and, in addition, the structure of the LS cycle could be expanded through this training course.

4.3.2. Data analysis procedure

All sessions of the training course were videorecorded, so we conducted the data analysis based on the video recordings of the first (LS) and third (reflection with DSC) phases of the training course, following the procedure described below. Firstly, each author reviewed the videorecorded sessions to identify what constituted an episode of practical argumentation, guided by the following criterion: segment of video recordings in which the participating teachers opted by one or more alternatives of action after weighing them.

Secondly, the four authors reviewed the episodes selected by each and we reached agreement on which of them our analysis would focus, now guided by the following more specific criterion: episode that includes an agreement among the participating teachers on one or more actions that have influence on the planned/implemented lesson. In this way, our analysis was focused on 18 of the 25 episodes of practical argumentation initially selected.

Thirdly, we conducted a triangulation in the following way: 1) each author reviewed the dialogues of each of the 18 episodes to identify the interventions in which the teachers stated actions that should be done; 2) each author reinterpreted these interventions in terms of DSC, assigning them an identifying code; 3) we compared our codifications, achieving an agreement percentage of 93% among the four of us; 4) we discussed our differences of codifications and achieved a consensus, due to our experience in this type of analysis. Taking as an example the first episode presented and analysed in the following section, we agreed that T4's intervention in line 06 could be reinterpreted as an aspect of the *cognitive* criterion regarding the *Prior knowledge* component, since the teacher implicitly considered the importance of the students' prior knowledge for the design of a mathematical teaching and learning process, in the form of a positive guideline for practice.

Fourthly, in the selected episodes of practical argumentation, we also identified the elements that make up an argument according to Toulmin's model (described in subsection 4.2.3), which we used as an analytical framework in this study. To structure the practical arguments using this model, based on the dialogue among the teachers, we began by identifying the data (what could be improved in the planned/implemented lesson) and the claim (an action that was proposed to improve this lesson); then, we posed a warrant (often implicit in the discussion) with its modal qualifiers (words that gave strength to the warrant); next, we sought the backings for this warrant (coming from the teachers' knowledge and/or experiences); and finally, we sought the rebuttals (cases where the warrant cannot connect the data with the claim). In this study, we also adapted the structure of Toulmin's model oriented towards action by considering DSC, as we will detail in the next section.

4.4. Presentation and Analysis of Two Episodes of Practical Argumentation

In this section, we will present and analyse two episodes of practical argumentation among the teachers during the first and third phases of the training course, respectively. The choice of these two episodes is justified by the fact that both were the clearest evidence of practical argumentation, that is, where the teachers elaborated arguments to justify any change made in the planned lesson. To this end, we present the dialogues among the teachers (translated from Portuguese), indicating each intervention in numbered lines, and we analyse the episodes of practical argumentation (following the corresponding chronological order) from the perspective of DSC and Toulmin's model, as described in subsection 4.2.3.

4.4.1. First episode of practical argumentation

This episode occurred during the first phase of the training course, in the stage of planning a lesson, with the first group of teachers (T1–T4). The choice of this episode is justified by the fact that it is a relatively complex example of practical argumentation when DSC had not been taught yet. Here, the teachers considered the possibility of presenting examples of right-angled triangles with sides of irrational length to thus use the PT to calculate the length of the diagonal of a square with sides equal to 1.

01. T1: I think this will be easy for them [referring to the PT as a relationship between the sides of a right-angled triangle that form a Pythagorean triple]. Do not you think so, T4? They [the students] will have problems when she [referring to T4, the teacher who will implement the lesson] talks about the square root of 2, because, in my experience, and in yours too [referring to T2, T3, and T4], I guess, they find it really difficult to understand irrational numbers and, when we have square with side [equal to] 1, I think it causes some difficulty. Do not you think so, T4?

02. T2: But the idea was showing that it also works for the square root [of 2], was not it?

03. T1: Yes, that was the idea. But in this case, we should remind them that the square of the square root of 2 is 2, for example.

04. T4: They did not learn that.

05. T1: In that case, there should be an easy way to verify it [the PT]. Did not they learn the notion of the square root of 2 in the number line last year?

06. T4: Yes. For example, I liked working on irrational numbers, and I would say we made some progress. But you are right in what you said, they find it really difficult to accept that the square root of 2 is a number. However, what I really like teaching them with the [square] root of 2, or other roots and with calculator, is that they have an infinite decimal expansion. We have done some good work on this, although I am not sure they will remember it.

07. T1: Regarding what you said, I think we could talk about what a square of area 2 is. For example, you have a square with a side [equal to] 1, so the area is 1. So, how do you get an area of 2? If the side of the square is [equal to] 2, then the area will be [equal to] 4. So, they will have an idea that, to get an area [equals] 2, the length of the sides of the square must be a number between 1 and 2, one that is less than 2 and greater than 1. Because they have already seen the result of the [square] root of 2 last year, the aim now should be to remind them of this number. But how will we do it? Will they have access to a calculator or a mobile phone? If so, we could pose the question: Which numbers can we use for the side of this square so that it has an area [equal to]

2? And they will try with 1.1, 1.2, 1.3, etc. to see which one gets closest [to the answer]. What do you think, T4?

08. T4: I think this is a great idea if the lesson is face-to-face.

09. T3: Because often they see a root and get a mental block. They do not even wait to see what you are asking for. I think we should do an initial class to explore the content of roots in more depth.

10. T4: I think that, after teaching the Pythagorean theorem and showing that it works, we can have a triangle with two sides [equal to] 1 and get them to work out the third side. Then, I can mention that the square root of 2 is an irrational number and this length, which we have just found, is an irrational number.

11. T1: What about using a ruler to make, for example, a square of sides 1 and then using a compass to show the length of the diagonal on the number line? What do you think? [asking T4].

12. T4: It like it.

4.4.2. Analysis of the first episode of practical argumentation

In the first phase of the training course, DSC had not been taught to the teachers yet, so we could reinterpret the guidelines for action agreed upon by them (derived from their knowledge, beliefs, and values) in the different lines of dialogue in terms of DSC, as shown below.

Prior to this episode, the teachers chose the mathematical content (PT), studied its curricular approach, and established the learning objectives, which we could reinterpret as aspects from the *ecological* criterion. Also, when they considered the possibility of presenting examples of right-angled triangles with sides of irrational length, we could reinterpret aspects from the *epistemic* criterion, specifically, the indicator "For one or more partial meanings, a representative sample of problems is provided" (Breda et al., 2017, p. 1903) of the *Representativeness of the complexity of the mathematical object* component. (Action 1 that triggers the practical argumentation in this episode)

In line 01, we reinterpreted aspects from the *cognitive* criterion, specifically, the indicator "The intended meanings (reasonable difficulty) can be taught through its diverse components" (Breda et al., 2017, p. 1904) of the *Prior knowledge* component, since T1 emphasised the students' difficulties with irrational numbers. On their part, in line 02, we reinterpreted aspects from the *epistemic* criterion, specifically, the indicator "For one or more partial meanings, a representative sample of problems is provided" mentioned above, since T2 kept the idea of showing examples with irrational numbers. Thus, in line 03, we reinterpreted that T1 assumed the T2's aspects from the *epistemic* criterion, although making them compatible with the aspects from the *cognitive* criterion proposed by himself (in line 01). In other words, T1 sought a balance among the aspects from both criteria by loosening those of the *epistemic* criterion when he proposed fewer complex situations to the students.

In line 04, we again reinterpreted aspects from the *cognitive* criterion regarding the *Prior knowledge* component, since T4 went to implement the lesson, knew the students, and considered that they had not learned what is needed to apply the PT to a triangle with legs of length 1. We reinterpreted the intervention in line 05 as a new attempt to make the *cognitive* criterion (proposed by T1) compatible with the *epistemic* criterion (proposed by T2), and T1 sought to specify the information about the students' prior knowledge provided by T4. In response to T1's question, T4 provided more information about that prior knowledge in line 06, to which we reinterpreted T1's intervention in line 07 as a new attempt to make the *epistemic* criterion (proposed by T2) compatible with the *cognitive* criterion (proposed by T1). Thus, in this intervention, we also reinterpreted aspects from the *epistemic* criterion regarding the *Richness of processes* component, since T1 proposed a task in which the students had to explore and make conjectures.

In line 08, we reinterpreted aspects from the *interactional* and *mediational* criteria since, although T4 did not say it explicitly, she considered that the T1's proposal would be interesting if the lesson were face-to-face, hence we could infer that she did not consider the virtual teaching context as suitable for implementing this proposal. (Action 2: Not performing Action 1)

In line 09, we reinterpreted aspects from the *affective* criterion regarding the *Emotions* component, since T3 alluded to the students' mental blocks and proposed a previous lesson to address (in our reinterpretation) the *Prior knowledge* component. (Action 3: Preparing a lesson to understand the PT). Then, in line 10, we reinterpreted that T4 sought a balance among the aspects from the *epistemic* and *cognitive* criteria by loosening those of the former, since she made a new proposal that was the one that the group finally agreed upon by consensus. This proposal consisted of applying the PT, after teaching it using

Pythagorean triples, to a triangle with two sides equal to 1 to find the third side and then briefly introducing irrational numbers.

Finally, in line 11, we reinterpreted aspects from the *mediational* criterion regarding the *Material resources* component, since T1 proposed the use of ruler and compass to link the PT to the representation of irrational numbers in the number line. Similarly, we reinterpreted aspects from the *epistemic* criterion in two ways: regarding the *Representativeness of the complexity of the mathematical object* component (since T1 kept the idea of working with a representative sample of examples) and the *Richness of processes* component (since T1 implicitly fostered the process of connections between the PT and the representation of irrational numbers).

In general, we could reinterpret the teachers' interventions in this episode in terms of the six DSC, although the *epistemic*, *cognitive*, and *mediational* criteria were especially relevant. One reason that justifies the presence of the *interactional* and *mediational* criteria is the lack of experience of the teachers in teaching in virtual contexts.

The analysis of this episode using DSC was complemented with a proposal that adapts Toulmin's model to structure practical arguments. In this sense, the claim would become a specific action that, to be considered, requires the teacher to take into account a principle (in this case, DSC) and some data taken from the situation. This principle is relativised (depending on the situation) and is based on one or several warrants (certain knowledge, not necessarily formulated in terms of an action). In turn, these warrants are supported by some backings, which can be: 1) knowledge generated through teaching practice itself, which Fenstermacher and Richardson (1993) call *empirical claim* (e.g., when in line 01, T1 says "because, in my experience, and in yours too […], I guess, they find it really difficult to understand irrational numbers") or 2) scientific knowledge, which Fenstermacher and Richardson (1993) call *stipulative claim* (e.g., general scientific knowledge about the importance of prior knowledge for learning). In this proposal of adaptation, modal qualifiers and rebuttals function the same as in the original Toulmin's model.

Therefore, in our adaptation, we do not consider DSC as a typical warrant, since they are principles about how things should be done (guidelines for practice) and are based – among others – on findings from Mathematics Education research. On the contrary, although a warrant can be supported by knowledge derived from previous experiences or research findings, it does not have to be formulated in terms of actions to be performed. That said, this transition from knowledge to a norm about how things should be done is what leads us to identifying DSC as a category derived from typical warrants. Although other researchers may wish to subsume the Warrant and DSC boxes in Figure 22 under a single warrant category, we consider this to be like including backings into the warrant category.

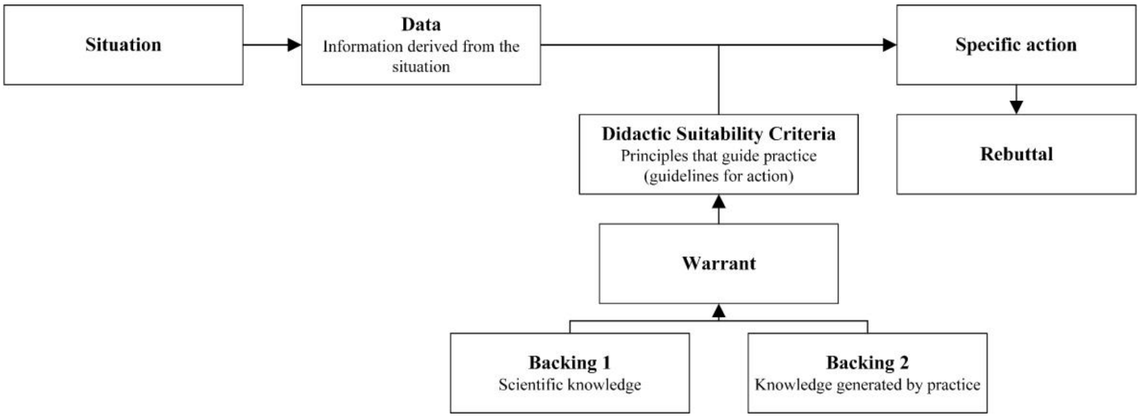

Figure 22. Adaptation of Toulmin's model including DSC to structure practical arguments.
Source: Authors' elaboration.

Therefore, instead of fitting DSC into Toulmin's model as synonymous with a warrant, we consider that it is better to incorporate them into the model as an independent element, thus generating a more nuanced tool to study mathematics teachers' practical argumentation. Figure 23 presents the scheme of practical argumentation for the first episode based on the adaptation of Toulmin's model proposed in Figure 22.

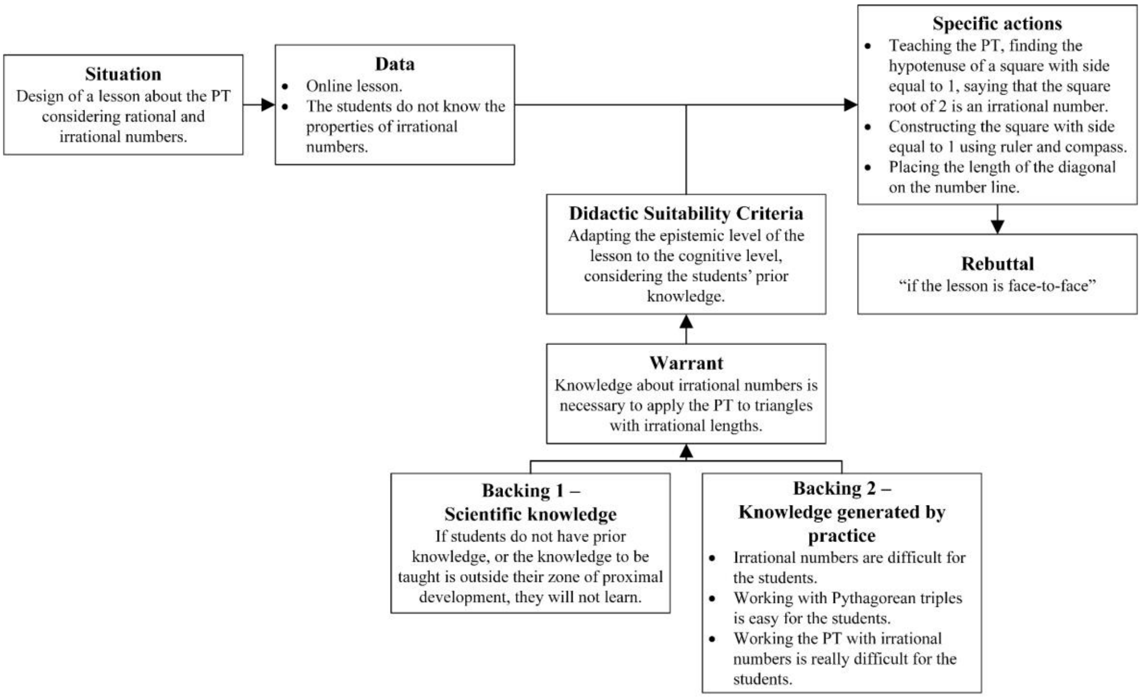

Figure 23. Scheme of practical argumentation for the first episode.
Source: Authors' elaboration.

One aspect to highlight about this scheme is why DSC, which appeared implicitly in the teachers' discussion, acted as a principle, considered that they had not yet been taught. For example, none of the teachers questioned the importance of student prior knowledge, which may be because they assumed that this knowledge plays a relevant role in learning (Backing 2 – Knowledge generated by practice). Furthermore, cognitive research on mathematics learning offers a finding that the teachers seemed to assume: if students do not have prior knowledge, or the knowledge to be taught is outside their zone of proximal development, they will not learn (see Gunduz & Hursen, 2015; Maxim, 2010; Mishra, 2015) (Backing 1 – Scientific knowledge).

4.4.3. Second episode of practical argumentation

This episode occurred during the third phase of the training course, in the stage of reflection on and redesign of the implemented lesson, in which the eight teachers participated. Since the trainer taught DSC to them – as a tool to guide reflection on practice – during the second phase of the training course, in this third phase, the teachers had to reflect on the different partial meanings of the PT, which of these meanings were implemented, how they were connected, and which of these meanings could be incorporated in a proposal of redesigned lesson. Along with that, the teachers had to reflect on how to improve the richness of mathematical processes in the proposed tasks.

This third phase began with each teacher assessing the implemented lesson using one specific criterion. Thus, the two teachers who used the *epistemic* criterion, after presenting their reflections to the entire group, did not question whether the PT could be taught only by proving a certain case. This is not surprising, since the teachers were not experts in using DSC. For her part, the trainer explained the complexity of the PT (in terms of its diversity of meanings) and pointed out that it is one of the few theorems that can be proven with a certain degree of rigour in elementary education. Therefore, although the trainer did not state what a proof is or show other possible proofs of the PT, she did point out the importance of considering the proof process, which could contribute to the richness of mathematical processes in the sequence of tasks proposed by the teachers. In this way, the trainer's intervention gave rise to a discussion among the teachers about the guidelines to consider redesigning the lesson and improving its *epistemic* aspects. This caused the teachers to question, for the first time, the idea that the PT could be taught only by proving a certain case, so they considered the possibility of introducing a more rigorous proof of the PT in the proposal of redesigned lesson.

The choice of this episode is justified by the fact that it is also a relatively complex example of practical argumentation once DSC had been taught. Here, the teachers discussed on the possibility of delving into a proof or exploring different experimental verifications of the PT.

13. T1: It was wondering if it would be interesting to watch a video with the proof. We could work on the video and the theorem afterwards, for example, by using the algebraic proof.
14. T4: But we should bear in mind that this is an online lesson. So how are they [the students] going to understand this proof? Maybe the experimental verification would be better, given that the lesson is virtual, and they are not used to proofs.
15. T2: Also, and because of what T6 said about us not being able to see the students, we will not know how they react when they see that proof. I am not sure whether the students would really understand the proof.

16. T5: The video of the proof would have to be chosen very carefully. There are some things they [the students] are not used to. That whole thing of letters moving from one place to another, and operations [referring to the equivalence of algebraical expressions]. We could do the geometrical proof using the same materials that we used in the lesson. Children understand much more when they can touch things than when they only watch what others do.

17. T6: More cases are essential because the idea of generalisation begins to emerge. However, especially with my students, I am not sure if we can show the proof in a way that they can really understand. Five of them will understand it, but the others only to some extent.

4.4.4. Analysis of the second episode of practical argumentation

In this episode, the situation began when T1 noticed that the lesson plan did not include the proof of the PT which, from his perspective, was important to add. Thus, in line 13, we reinterpreted aspects from the *epistemic* criterion regarding the *Richness of processes* component, since T1 proposed to provide the algebraic proof of the PT. In addition, we reinterpreted aspects from the *mediational* criterion regarding the *Material resources* component, since he also proposed teaching this proof through a video to facilitate the students' understanding. (Action 1: Teaching the algebraic proof of the PT through a video)

In line 14, we reinterpreted aspects from the *interactional* and *cognitive* criteria regarding the *Teacher-student interaction* and *Prior knowledge* components, respectively, since T4 argued against Action 1 considering as data that the lesson was virtual and as a backing that the students were not used to doing proofs. This led T4 to suggest doing an experimental verification as an alternative, which we reinterpreted as seeking a balance among the aspects from the *epistemic* and *interactional* criteria by loosening those of the former. (Action 2: Doing an experimental verification of the PT)

In line 15, we reinterpreted aspects from the *interactional* criterion regarding the *Teacher-student interaction* and *Formative assessment* components, since T2 also considered Action 1 as unfeasible. To do this, he posed the limitations of interaction in the virtual classroom as data, since the students' learning would be affected by not being able to see them. In line 16, we reinterpreted aspects from the *cognitive* criterion regarding the *Prior knowledge* component, since T5 also considered Action 1 as unfeasible because the students did not have the necessary prior knowledge. In addition, we reinterpreted aspects from the *cognitive* criterion regarding the *Learning* component, since T5 suggested doing the geometric proof of the PT using manipulative resources that, from her perspective, facilitate learning. (Action 3: Doing the geometric proof of the PT)

Finally, in line 17, we reinterpreted aspects from the *epistemic* criterion regarding the *Richness of processes* component, since T6 suggested doing the experimental verification of the PT considering various cases (specific action) to facilitate the generalisation process. However, T6 also reduced the degree of *epistemic* suitability when arguing against Action 1. In addition, we reinterpreted aspects from the *cognitive* criterion regarding the *Prior knowledge* component, since T6 considered that the proof was out of the students' zone of proximal development. Likewise, T6 made implicit reference to the complexity involved in proving a conjecture for students at that educational level, although not from the perspective of the norms that a teacher should promote in the classroom to enable such a proof (Molina et al., 2021) or from the question of whether what was intended to be carried out was actually a proof or just a verbal justification (Tabach et al., 2010), but rather she posed it in terms of the cognitive limitations of the students (DSC that guide practice to propose a specific action). We reinterpreted this proposal as T6 seeking a balance among the aspects from the *epistemic* and *cognitive* criteria, being the one that the group finally agreed upon by consensus.

After considering the points of conflict among the different criteria, the teachers chose the action on which there was the greatest consensus, that is, Action 2. Figure 24 presents the scheme of practical argumentation for the second episode based on the adaptation of Toulmin's model proposed in Figure 22.

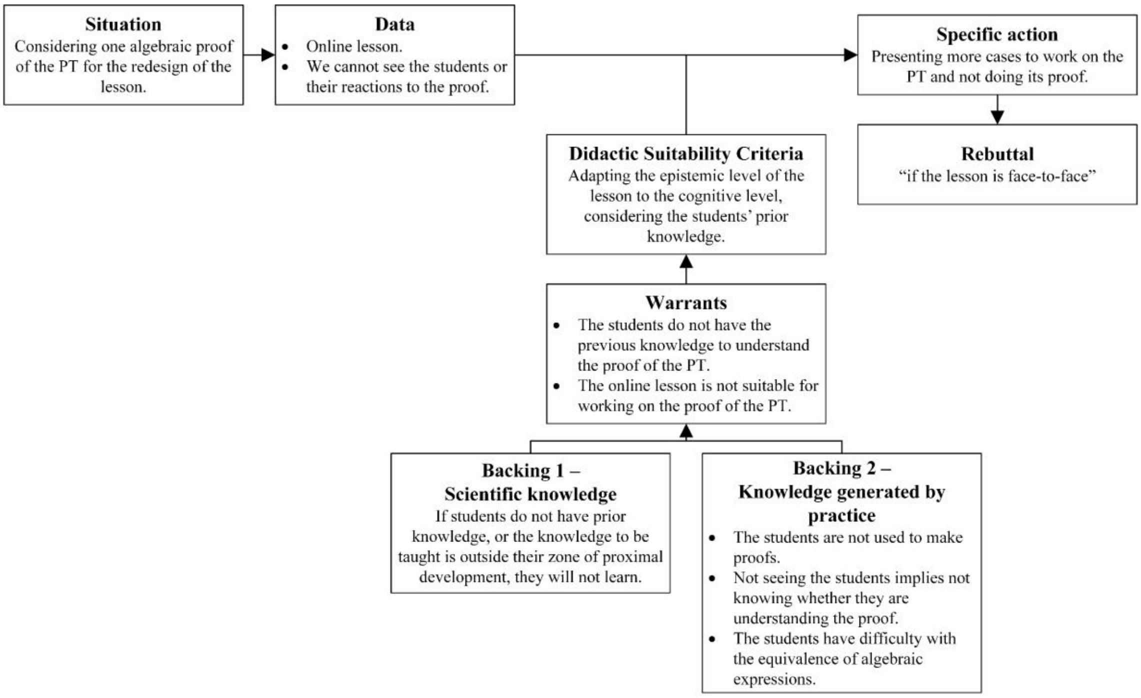

Figure 24. Scheme of practical argumentation for the second episode.
Source: Authors' elaboration.

4.5. Discussion and Conclusions

Our interest in this study was focused on the role played by DSC, as a construct to guide teaching reflection on practice, in the practical argumentation of a group of practising mathematics teachers in the context of a training course to foster the development of teaching reflection. In this article, we reported the analyses of two episodes of practical argumentation using a proposal of adaptation of Toulmin's model with DSC to structure practical arguments.

A first aspect to highlight is that, although the teachers formulated and used principles to guide their practice, we could reinterpret these guidelines for action in terms of some indicators and components of DSC. Regarding the LS, this approach could be considered as a broader but unguided phase of reflection, where the incorporation of DSC allows guiding this reflection to improving a mathematical teaching and learning process. Thus, during a LS session, implicit consensuses will emerge among the participating teachers on aspects that they see as positive and that, again, we could reinterpret in terms of DSC (Hummes et al., 2019). In a similar way to what was concluded by Robinson and Leikin (2012), in this LS cycle, we observed that the group reflection allowed the participating teachers to focus on their students' thought, difficulties, and abilities, by considering the inclusion of a proof for the PT.

A second aspect to highlight is that, from the analyses of both episodes of argumentation, we made evident that two principles – reinterpreted in terms of DSC – which, a priori, seemed suitable, sometimes conflicted with each other in a certain situation. For example, in the second episode, a conflict arose between the *cognitive* and *epistemic* aspects, since the students' prior knowledge (*Prior knowledge* component) conflicted with the inclusion of the proof process (*Richness of processes* component) in the redesign of the lesson. However, this conflict was resolved when the teachers (a) relied on different DSC to justify their decision, in the event that they were against performing a certain action; (b) adopted an intermediate position, which we could reinterpret as seeking a balance between two DSC; and (c) reached a consensus that depended on the context, attributing relative importance to the DSC that were in conflict. This situation of conflict between two principles was made evident in a similar way in Sol, Breda, Richit, and Sala-Sebastià (2024), when the participants had to decide the material to work on flat geometric shapes and, during their election, a conflict emerged between the *cognitive* and *mediational* aspects, since the students' previous knowledge limited the use of certain materials.

Resuming our research question in this study about what role is played by DSC in the practical argumentation of the group of participating teachers, we can conclude three aspects. Firstly, the participating teachers' practical argumentation included reasons that they considered to choose their actions in a mathematical teaching and learning process from different points of view, which was contained in their reflection and was promoted and guided by DSC when combined with a LS cycle. Secondly, DSC were related to the choice of a specific action in the teachers' practical arguments, and we used them to reinterpret the dialogues of the analysed episodes, describing and relating their components and indicators to the contexts of the discussions. And thirdly, in our proposal of adaptation of Toulmin's model (see Figure 23 and Figure 24), DSC nuanced the warrant to define a specific action in the claim, that is, they acted as principles to choose an action within the teachers' practical argumentation. This is mainly because of two reasons: on one hand, DSC are a construct proposed to guide teaching practice (Breda et al., 2017, 2018) and, on the other hand, practical reasoning consists of the principles or values shared by teachers to justify or discard possible actions in a certain teaching situation (Herbst et al., 2011).

Our proposal represents an evolution of previous studies that have analysed other episodes of practical argumentation in LS cycles, where DSC were related to warrants (see Sol, Breda, Richit, & Sala-Sebastià, 2024). In methodological terms, the incorporation of DSC in Toulmin's model (see Figure 22) generates a more nuanced tool to study mathematics teachers' practical argumentation. Likewise, the proposal presented in this article could help to better explain why some current trends in Mathematics Education that, in theory, are considered interesting and innovative, are adopted in teaching practice while others are not. When the teachers assessed the lesson plan during the first phase of the training course and DSC had not yet been taught to them, we made evident, in a similar way to Vesterinen et al. (2014), that their reflections were brief and considered few elements, being oriented towards improvement based on their common criteria. However, after teaching them this construct, we made evident that their reflections were broader and better organised, where their practical argumentation considered more aspects. This result is consistent with what was stated by Fenstermacher and Richardson (1993), who also identified benefits in terms of changes in beliefs and teaching practices. Even so, in this study, we did not make evident the explicit use of DSC in the teachers' discussion, which is because, for them to explicitly use this construct, they must be more familiar with its use in different contexts. This does not mean that teachers should use DSC in their discussion all the time, since in other LS cycles, where the participants have been experts in the knowledge and use of this construct, although they explicitly mention them, they do not always do so (see Sol, Ledezma et al., in press).

Additional statements:

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- Ethical statement: This study has followed the Code of Ethics on Integrity and Best Practices by the University of Barcelona (available at http://hdl.handle.net/2445/137937)) and it was approved by the committee of the

PhD Programme Didactics of Sciences, Languages, Arts, and Humanities of the University of Barcelona, on 26 April 2024.

- Conflicts of interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Capítulo 5: Capítulo de Libro

El capítulo de libro aquí presentado tiene por objetivo identificar la argumentación práctica docente que emerge de la reflexión docente guiada por los criterios de idoneidad didáctica en el análisis y valoración docente de una clase implementada en un curso de formación docente que combina la metodología *Lesson Study* y los Criterios de Idoneidad Didáctica, atendiendo así al cumplimiento del tercer objetivo específico de esta investigación doctoral.

Los avances del desarrollo de este objetivo específico fueron presentados en tres eventos académicos: el *Seminario Latinoamericano de Colaboración sobre el Enfoque Onto-Semiótico*, la *Reunión Latinoamericana de Matemática Educativa* (RELME 35), y el *Primer Congreso Internacional de Didáctica de la Matemática* (CIDIDMAT 2022).

En la Tabla 8, se presentan los datos de identificación de este capítulo de libro.

Tabla 8. Datos de identificación del capítulo de libro.

Criterios	Datos
Título	Reflexión docente sobre la enseñanza de la introducción a la derivada en un proceso de instrucción.
Autores	Telesforo Sol [1], Mariana Lujambio [2], Noemí Lara [2], Vicenç Font [1].
Afiliaciones	[1] Departamento de Educación Lingüística y Literaria y de Didáctica de las Ciencias Experimentales y de la Matemática, Facultad de Educación, Universidad de Barcelona (España). 2 Universidad Autónoma de Querétaro (México).
Libro	Actas del Primer Congreso Internacional de Didáctica de la Matemática.
Editores	Sofía Caviedes, Jesús Lugo-Armenta, Luis Pino-Fan, Alicia Sánchez.
Editorial	Universidad de Los Lagos.
ISBN	978-956-6043-86-7.
Páginas	197–205.
Fecha de publicación	Noviembre de 2023.
Referencia	Sol, T., Lujambio, M., Lara, N., y Font, V. (2023). Reflexión docente sobre la enseñanza de la introducción a la derivada en un proceso de instrucción. En S. Caviedes, J. G. Lugo-Armenta, L. R. Pino-Fan, y A. Sánchez (Eds.), *Actas del Primer Congreso Internacional de Didáctica de la Matemática* (pp. 197–205). Universidad de Los Lagos.

Fuente: Elaboración del autor.

Resumen y Palabras Clave / Abstract and Keywords

Resumen: En este trabajo se muestran los resultados obtenidos de la reflexión de dos profesoras de un bachillerato mexicano en un curso de formación docente, en el cual se articuló la metodología Lesson Study y los Criterios de Idoneidad Didáctica considerando la argumentación que emerge en las sesiones. En particular, se analizó una sesión donde las profesoras compartieron la valoración de su clase sobre la introducción a la derivada con los criterios de idoneidad didáctica y proponían acciones para un rediseño de la clase, buscando aumentar la idoneidad didáctica. El objetivo es identificar la argumentación práctica docente en la reflexión guiada por los criterios de idoneidad didáctica en el análisis y valoración de la propia clase implementada.

Palabras clave: Argumentación; Criterios de idoneidad didáctica; Derivada; Lesson study; Reflexión docente.

Abstract: This work presents the results obtained from the reflection of two female teachers at a Mexican high school in a teacher training course, in which the Lesson Study methodology and the Didactic Suitability Criteria were articulated considering the argumentation that emerges in the sessions. In particular, a session was analysed in which the teachers shared the evaluation of their class on the introduction to the derivative with the Didactic Suitability Criteria and proposed actions for a redesign of the class, seeking to increase the didactic suitability. The objective is to identify the practical teaching argumentation in the reflection guided by the didactic suitability criteria in the analysis and evaluation of the implemented class itself.

Keywords: Argumentation; Derivative; Didactic suitability criteria; Lesson study; Teaching reflection.

5.1. Introducción

El presente reporte de investigación muestra los resultados obtenidos en un curso de formación docente en el cual se articuló la metodología Lesson Study (LS) y los Criterios de Idoneidad Didáctica (CID) considerando la argumentación que emerge en las sesiones. En la revisión de literatura no se han ubicado trabajos sobre la enseñanza de la derivada y la reflexión docente a través del LS y los CID en el nivel medio superior. Los trabajos encontrados hacen referencia a otras temáticas y niveles de enseñanza, sin embargo, la estructura y puesta en marcha de estas investigaciones indican algunas características que son de utilidad para este trabajo de investigación. Además, permiten identificar la relación entre el LS, el mejoramiento de las planeaciones de clase y la reflexión docente al finalizar las investigaciones.

Un primer trabajo tiene que ver con los CID usados por un grupo de profesores para la preparación, ejecución y reflexión de una clase de matemática basada en la metodología de LS (Hummes et al., 2020). Los resultados de esta investigación mencionan que el proceso realizado consiste en la planeación, realización, observación y reflexión conjunta para mejorar una clase, perfeccionando de este modo las prácticas en la enseñanza.

Un estudio más es una investigación de caso donde el objetivo principal fue mejorar la práctica docente en el nivel superior mediante la reflexión y la formación de una comunidad en un curso de pedagogía en la Universidad de Oviedo (Hevia et al., 2019). El aporte de este estudio hace énfasis en el trabajo colaborativo de profesores en una comunidad de aprendizaje, así como la importancia de documentar los diálogos y la construcción del saber pedagógico a través de la reflexión.

Los CID han sido implementados en la reflexión de los docentes que participan en cursos de formación los cuales se han documentado en los trabajos de investigación de Font et al. (2017) sobre su uso en un proceso de formación de futuros profesores de matemática en distintos países (España, México, Brasil, entre otros) con la finalidad de organizar la reflexión de la práctica docente de una futura profesora, con la cual mejoró su competencia de análisis e intervención didáctica. Morales-Maure et al. (2019) muestran los resultados obtenidos en la implementación de un diplomado del que fueron parte profesores de primaria y preescolar para fortalecer las competencias de análisis e intervención didáctica, realizando un análisis de valoración con la idoneidad didáctica. Por último, como antecedente de reflexión e idoneidad didáctica se considera la investigación propuesta en Seckel y Font (2020) que presentan un estudio de caso sobre el desarrollo de la competencia reflexiva en una formadora en etapa inicial y los resultados obtenidos de la aplicación de los CID, así como su uso en el rediseño del proceso de enseñanza.

A partir de los antecedentes mencionados y del proyecto de investigación más amplio del que forma parte esta investigación se plantea el siguiente objetivo: identificar la argumentación práctica docente que emerge de la reflexión docente guiada por los criterios de idoneidad didáctica en el análisis y valoración docente de una clase implementada en un curso de formación docente que combina la metodología LS (Hart et al., 2011) y CID (Breda et al., 2018). Los argumentos identificados en esta etapa se analizan desde el punto de vista de la Teoría de la Acción Comunicativa (Habermas, 1981/1984) considerando la trayectoria argumentativa desarrollada por Ramos (2006).

5.2. Fundamentos Teóricos

5.2.1. Lesson study

El LS es un método japonés de mejoramiento de clases diseñado para aumentar su calidad (Isoda et al., 2007). La metodología LS se enmarca en un tipo de investigación-acción (Elliot, 1991) ya que es el propio docente quien ejerce el papel de investigador para mejorar su práctica educativa. El proceso del LS consiste en la preparación de una clase, implementación de la clase y sesiones de revisión de manera conjunta, el proceso se puede dividir en las siguientes fases:

- Fase 1: Revisión del currículo y preparación de la clase.
- Fase 2: Implementación y observación de la clase.
- Fase 3: Examinación conjunta de la clase implementada y de las observaciones obtenidas.
- Fase 4: Rediseño.

5.2.2. Idoneidad didáctica

Desde el Enfoque Onto-Semiótico (EOS) se define la noción de idoneidad didáctica como criterio sistémico de optimización de un proceso de instrucción matemática. Su definición se enuncia a continuación: grado en que el proceso, o una parte, reúne ciertas características para calificarlo como óptimo o adecuado y conseguir la adaptación entre significados personales logrados por los estudiantes (aprendizaje) y los significados institucionales pretendidos o implementados (enseñanza), teniendo en cuenta las circunstancias y recursos disponibles (entorno) (Godino et al., 2019).

La idoneidad didáctica considera la articulación de los siguientes criterios como guía de los procesos de instrucción matemáticos:

1) Idoneidad epistémica, se refiere al grado de representatividad de los significados institucionales implementados (o previstos), respecto de un significado de referencia.

2) Idoneidad cognitiva, expresa el grado en que los significados pretendidos/implementados estén en la zona de desarrollo potencial de los alumnos, así como la proximidad de los significados personales logrados a los significados pretendidos/implementados.
3) Idoneidad interaccional, se refiere al grado en que las configuraciones y trayectorias didácticas permiten, por una parte, identificar conflictos semióticos potenciales (que se puedan detectar a priori) y, por otra parte, resolver los conflictos que se producen durante el proceso de instrucción mediante la negociación de significados.
4) Idoneidad mediacional, se refiere al grado de disponibilidad y adecuación de los recursos materiales y temporales necesarios para el desarrollo del proceso de enseñanza-aprendizaje.
5) Idoneidad emocional, se refiere al grado de implicación (interés, motivación) del alumnado en el proceso de estudio.
6) Idoneidad ecológica, se refiere al grado de adaptación del proceso de estudio al proyecto educativo del centro, las directrices curriculares, las condiciones del entorno social, etc. (Godino et al., 2006).

5.2.3. Argumentación

Nos interesa la argumentación orientada a la acción basada en el consenso de un grupo de profesores, que en términos de la Teoría de la Acción Comunicativa (TAC) (Habermas, 1981/1984) se puede reinterpretar como racionalidad instrumental y comunicativa (que no se deben entender excluyentes sino complementarias en su uso a fin de desenvolverse adecuadamente en la realidad existente). En la TAC, se define la argumentación como: “tipo de habla en que los participantes tematizan las pretensiones de validez que se han vuelto dudosas y tratan de desempeñarlas o de recusarlas por medio de argumentos” (Habermas, 1981/1984, p. 37).

Para identificar la argumentación en este trabajo se hace una adaptación de la trayectoria argumentativa desarrollada por Ramos (2006) para analizar la argumentación, ya que dicha trayectoria permite identificar algunos elementos de la argumentación que suceden en los episodios del curso de formación.

Una argumentación contiene razones conectadas con la pretensión de validez de la manifestación o emisión problematizadas. La fuerza de una argumentación se mide en un contexto dado por la pertinencia de las razones. Esta se pone de manifiesto si es capaz de motivar a los participantes a la aceptación de la pretensión de validez en litigio.

5.3. Metodología

Se presenta una investigación de carácter interpretativo/cualitativo (Blumer, 1982), cuyos análisis forman parte de un estudio más amplio, que tiene como objetivo general analizar en qué medida un curso de formación basado en el LS y en los CID promueve el desarrollo del proceso de argumentación docente. En las valoraciones guiadas por los CID se identifica la argumentación práctica de las docentes y su reflexión.

5.3.1. Contexto y participantes

El curso de formación docente se impartió en el año 2021 y participaron dos profesoras de un bachillerato mexicano, que imparten la asignatura de cálculo. Por la pandemia derivada del COVID-19 las clases con sus estudiantes fueron impartidas por Zoom. El tema que seleccionaron las profesoras fue el de introducción a la derivada, algunas de las razones para tratar este tema fueron: complejidad del tema, es un tema de una materia que ambas profesoras imparten en sus cursos de manera regular, el tema se adaptó a los tiempos del dispositivo implementado y al programa de estudios de la materia. Las reuniones con las profesoras fueron virtuales por la plataforma Zoom, en un periodo quincenal y en sesiones de dos horas. Respecto al contexto de las profesoras una de ellas cuenta con 9 años de experiencia impartiendo clases, mientras que la otra docente 6 años. Ambas profesoras laboran en el mismo centro educativo y han compartido experiencias de enseñanza referentes a la matemática educativa, ambas tienen grado de maestras en didáctica de la matemática. Las y los estudiantes que participaron en la investigación, son jóvenes que cursaron quinto semestre de bachillerato y sus edades para ese entonces eran aproximadamente de entre 16 y 17 años. Los grupos a los que se les impartió la clase eran del turno vespertino (clases por la tarde).

5.3.2. Etapas de análisis

El curso de formación constó de tres etapas: ciclo de LS, idoneidad didáctica y valoración de la clase y rediseño considerando los criterios de idoneidad didáctica. En la última etapa en la sesión de valoración de la clase con los criterios de idoneidad didáctica, se solicitó a las profesoras analizar y valorar su clase considerando los criterios de idoneidad, para esto se les compartió un archivo Excel que contenía una tabla por cada criterio de idoneidad, sus componentes y sus descriptores. En esta tabla podían escribir, teniendo en cuenta cada componente, momentos donde consideraban que la idoneidad de la clase había sido beneficiada o disminuida y algunas propuestas de mejora. Cada profesora presentó su análisis sobre su propia clase. Siendo estos momentos los puntos de partida para la reflexión y argumentación colectiva.

Para este trabajo se consideró el análisis de una sesión de la última etapa, la sesión de análisis y valoración de la clase con los criterios de idoneidad didáctica y el análisis constó de las siguientes etapas:

1) Revisión de la videograbación de la sesión del curso de formación docente, donde las profesoras evalúan su clase implementada con los criterios de idoneidad didáctica como guía, para identificar momentos donde se presentan episodios de reflexión o argumentación práctica entre las profesoras.
2) Transcripción de los episodios seleccionados. En las transcripciones las profesoras se identifican como profesora 1 y profesora 2.
3) De las transcripciones, donde se presentan episodios de reflexión o argumentación práctica, de acuerdo con la perspectiva procedimiento de la argumentación de la TAC, y considerando como se hace en Ramos (2006), se identifican algunos elementos de la trayectoria argumentativa, en la que se ubican: roles de las profesoras, consensos conseguidos o invalidados, así como razones: proponente, oponente, pretensiones de validez, invalidación de la pretensión de validez, consenso racionalmente motivado, consenso por omisión. Estos elementos permiten ver cómo de la valoración con los criterios de idoneidad emergen episodios argumentativos.
4) Se describe el tema sobre el que se reflexiona en cada episodio.

5) Después de los análisis realizados en los momentos anteriores, se hace una validación entre los autores con el objetivo de consensuar los resultados obtenidos.

5.4. Análisis, Resultados y Discusión

A continuación, se presenta el análisis de tres episodios de valoración de la idoneidad de la clase implementada, donde cada valoración involucra reflexión o argumentación. Los primeros dos episodios suceden cuando la profesora 1 analiza su clase desde el criterio ecológico con los componentes conexiones intra e interdisciplinares e innovación didáctica respectivamente y el tercer episodio se produce cuando la profesora 2 analiza su clase con el criterio mediacional.

5.4.1. Episodio A

Profesora 1: En cuanto a las conexiones intra e interdisciplinarias, yo le puse que lo que mejora el grado de idoneidad, es que se requieren algunos conceptos vistos en materia de matemática, no es en un contexto extra matemático, pero si en otras materias como en las anteriores, Matemática IV, Matemática III, como el concepto de pendiente, el concepto de recta, etcétera, pero lo que disminuye el grado de idoneidad es que en clase no se abordaron ningún contenido intra e interdisciplinario, no se abordó nada, ni nada en contexto, ni en contexto extra matemático, lo puse en mi clase porque Profesora 2 si lo mencionó, le puse que podía plantearse en el *Kahoot!* alguna pregunta que pueda llevar a la aplicación de la derivada, aunque todavía no se esté viendo ese tema, pero al menos que en la introducción se hable de que la derivada tiene una aplicación o simplemente mencionarse.

Profesora 2: Yo aquí sí quiero comentar que también puse algo muy parecido a lo tuyo, pero en mi caso yo sí traté de mencionar lo de velocidad y aceleración, pero creo que no me salió bien y entonces allí le puse que disminuye el grado de idoneidad, porque quise introducir ese concepto muy rápido y de algún modo, quiero pensar que los estudiantes no están preparados para eso, yo puse la parte de introducir el tema de velocidad y aceleración, cuando el concepto de derivada quizás aún no está interiorizado, a lo mejor mencionarlo, pero si uno va directamente puede disminuir…

Profesora 1: Hay un video de YouTube donde explican la definición de la derivada y la explican con la razón de cambio, no se meten tanto a lo que nosotros nos metimos, allí explican el uso de la derivada, más que su definición. Creo yo que hay como dos formas de hacerlo, si tú quieres irte a un nivel más básico, o no irte tan formal, son pocos los estudiantes que van a captar y comprender esa definición, entonces si te vas a una definición extra matemática, creo que es más fácil de comprender, pero al final terminas sin comprender que es la derivada, comprendes que se aplica en física y que la derivada de la aceleración es la velocidad y ya te quedas allí. No sé si en el rediseño valdría la pena hacer alguna pregunta o mejor dejarlo así, porque más adelante si se ve esta parte, pero no es el momento de verla en esta clase.

Profesora 2: En mi caso, no sé si debería evitar hacer un comentario, así como el que yo hice o sólo mantenerme con lo que corresponde, a lo mejor no debería, eso sería lo que haría yo.

Los elementos de la trayectoria argumentativa de este episodio son:

- Proponente: Profesora 1 (propone aumentar en *Kahoot!* una pregunta que considere una aplicación de la derivada).
- Razones: En la clase no hay contenido intra e interdisciplinario.
- Oponente: Profesora 2 (no está de acuerdo en considerar tareas que contemplen conexiones interdisciplinarias).
- Razones: Los estudiantes no están preparados, estás conexiones no se deben introducir con poco tiempo y el concepto de derivada no está interiorizado.
- Reflexión: Abordar la derivada una con menor nivel epistémico (enseñar la derivada como aplicación) y otra con mayor nivel epistémico (trabajar la definición formal). Cuestiona su propuesta.
- Consenso: No introducir situaciones interdisciplinares en la sesión de introducción a la derivada.

En este episodio la valoración de la profesora 1 sobre las conexiones intra e interdisciplinares, al principio fue baja por que no hizo comentarios que beneficiaran este componente, pero al escuchar la experiencia de la profesora 2, quien sí hizo comentarios de este tipo pero que no estaba satisfecha con el beneficio de esta acción a la idoneidad didáctica de la clase, cambió de opinión y más aún reflexionó sobre diferentes opciones de introducir el concepto de derivada. Al termino de esta reflexión ambas profesoras consensuaron, por omisión, que no es necesario relacionar el concepto de derivada con otras materias argumentando que se resta tiempo para trabajar con la introducción y definición de la derivada.

5.4.2. Episodio B

Profesora 1: En la innovación didáctica, yo puse que se están utilizando recursos innovadores. No sé ¿qué tan innovadores? pero en la actualidad estamos utilizando recursos como *Kahoot!*, GeoGebra, que la autoevaluación del *Kahoot!* es una forma práctica de evaluar y es un recurso tecnológico que usamos y recursos como Zoom. Le estoy poniendo que no hay contenido nuevo, realmente el currículo no se ha cambiado, este concepto es algo que se viene enseñando desde hace años y no se ha cambiado, es un contenido útil, pero no innovador. Eso disminuye el grado de idoneidad y no se aplica alguna mejora, no sé ¿si sea necesario utilizar algún otro recurso? Cuando yo estaba dando la clase, había una parte de cuando explico el documento, no se alcanza a ver bien y si los alumnos están en el celular, eso disminuye el grado cognitivo o afectivo, o no se cual grado, de que ven tan chiquita la pantalla que eso les afecta y si yo tengo mi escrito con letra muy chica, ellos no alcanzan a ver bien, en este sentido pensé que sería bueno en esta clase, que en vez de explicar los ejercicios en PDF, explicarlos en un pizarrón virtual, porque en el pizarrón virtual tienes la facilidad de tener tus puntos, los puedes compartir y hacer zoom, hacer énfasis, etcétera [hay una explicación grafica del uso del pizarrón virtual donde se muestran sus ventajas y desventajas]. Esto es algo que podemos mejorar en la clase, en vez de hacer los ejercicios en clase, tenerlos hechos, pero irlos enfocando en lo que tú quieres resaltar. Sería otro recurso que podríamos ocupar, el pizarrón virtual en vez del PDF.

Profesora 2: Allí yo hice mención, donde disminuyen el grado de idoneidad, con el *Kahoot!*, no sé si recuerdan que teníamos problemas con el que se cargaran las imágenes, lo metí como recurso tecnológico, disminuyo el grado porque no estaba bien diseñado, aparte de que no cargaban bien las imágenes, el tiempo que se les dio a los estudiantes para responder, una cosa con la otra se arruino y ya no funciono al cien por ciento, esa fue una de las cosas que yo identifique y en la mejora, sería únicamente mejorar el diseño de los recurso tecnológicos para tener mayor tiempo de respuesta y de aparición en los dispositivos que tengan los estudiantes, ya sea del celular, o la computadora o la tableta, esos fueron los comentarios que yo puse en esa parte…

Profesora 1: Lo que yo me di cuenta, ya cuando estaba revisando el video yo dividí la pantalla, la mitad tenía el video y la otra mitad la transcripción, cuando estaba haciendo eso, me di cuenta que si tú lo ves des de un monitor si se ve bien [hay una muestra grafica de como se ve el video de pantalla cuando se modifica el tamaño de la ventana de YouTube], el problema del PDF es que no es tan fácil de manipular y un pizarrón virtual se puede implementar para que los alumnos tengan mejor visualización de lo que queremos hacer énfasis y con el PDF no.

Instructor: Profesora 1, en este pizarrón, del PDF que ya tienes [el PDF que se utilizó para trabajar en clase con los estudiantes], ¿si tú copias y pegas ese recuadro de las igualdades se mantiene así y puedes trabajarlo allí? tampoco es que necesites mucho trabajo extra en editar en este pizarrón…

Los elementos de la trayectoria argumentativa de este episodio son:

- Proponente: Profesora 1.
- Acción: Utilizar el pizarrón virtual en vez del archivo PDF para explicar ejercicios resueltos.
- Razones: Los estudiantes utilizan el celular [teléfono móvil] para atender la clase o consultar materiales. En las explicaciones de la clase el PDF no siempre se ve bien. El pizarrón virtual tiene ventajas para explicar en clase.
- Proponente: Profesora 2.
- Acción: Rediseñar las actividades con recursos tecnológicos.

- Razones: Los estudiantes tenían poco tiempo para responder a las preguntas de *Kahoot!* también las imágenes se tardaban en actualizarse en el cuestionario de *Kahoot!*
- Reflexión: Se reflexiona sobre el uso de los recursos tecnológicos por un lado el pizarrón virtual para mejorar la interacción con los estudiantes, como medio e innovación. Y por otro lado consideraciones para que el uso de *Kahoot!* no afecte la idoneidad de la clase.

En el segundo episodio, la profesora 1 menciona los recursos innovadores que utilizaron para la clase y después reflexiona sobre la interacción con los alumnos, para el rediseño de la clase, utilizando una aplicación que permite leer un archivo PDF en la clase virtual por *Zoom* y que este sea legible para los alumnos. Esto permite ver la relación que tiene el criterio ecológico con otros criterios, en este caso con el mediacional e interaccional. Así que esta acción beneficiará a la idoneidad de tres criterios. Respecto a la reflexión de los recursos innovadores, la profesora 2 complementa la reflexión de la profesora 1 valorando negativamente a la aplicación *Kahoot!* argumentando que el tiempo utilizado con esta aplicación algunas veces fue limitado y haciendo una propuesta de mejora para que dicha aplicación aumente la idoneidad de la clase implementada, en particular se mejorará la interacción con los estudiantes considerando un tiempo adecuado para el uso de recursos tecnológicos, esto permite ver como la reflexión conjunta complementa a las reflexiones personales para mejorar la idoneidad didáctica.

5.4.3. Episodio C

Profesora 2: El primer componente que aparece aquí, son los recursos materiales, manipulativos, calculadoras o computadoras, aquí creo que fue fácil. Hubo bastantes cosas que mejoran en este grado de idoneidad, porque se utilizó el *JamBoard*, se utilizó el *Kahoot!* y se utilizó *Zoom* para que los estudiantes trabajaran. El *JamBoard*, como recurso manipulativo donde los estudiantes resolvieron sus ejercicios y se les pudo evaluar de manera síncrona a través de la revisión individual de sus trabajos y se usó *Kahoot!* donde los estudiantes respondieron preguntas teóricas relacionadas con el concepto de derivada. Allí hubo esa parte de materiales, creo que también usaron calculadoras en casa para resolver los ejercicios que aparecieron en el *JamBoard*. Puse que algo que disminuye el grado de idoneidad, fue él tiempo y la forma de usar el recurso referente al *Kahoot!*, porque allí hubo el diseño por los tiempos, que las imágenes propuestas y el tiempo no fueron los más adecuados e incidieron en que los alumnos se desconcentraron un poco al momento de responder o que no les aparecieran, eso les incidió a sus respuestas. La propuesta de mejora es el rediseño o mejorar esta parte del *Kahoot!*, o como mencionas [se refiere al instructor], tener el plan B para que, si llega a pasar eso, los estudiantes puedan visualizar en otro formato la pregunta que se hace. El otro componente es el número de alumnos, horario y condiciones del aula, pues aquí no puse nada en que se mejora en el grado de idoneidad porque no encontré algo positivo, pero si encontré varias que disminuyen el grado de idoneidad, pues la primera parte es que como estamos en virtualidad, las cámaras de los alumnos generalmente permanecen apagadas y no hay forma de saber si están o no están trabajando, no podemos visualizar sus rostros de sorpresa, de duda, de entender la idea, eso creo que disminuye el grado de idoneidad y otra de las cosas que yo puse por allí es, en el caso de estas clases que yo di, es el horario de la asignatura, para los alumnos creo que no causan conflicto, porque ellos fueron del turno vespertino, pero en mi caso ya era mi última sesión del día y como profesor uno se cansa, obviamente tratamos de hacerlo bien, pero también incide en esa parte.

Instructor: ¿Cuántas clases impartes?

Profesora 2: En ese semestre eran seis, no eran muchas, pero al final si es agotador. Aquí no se me ocurrió algo para mejorar, pues creo que no tengo posibilidad como para cambiar ni el horario, ni la parte del *Zoom*, tampoco la cantidad de estudiantes que hay en la clase. Y el tiempo de enseñanza colectiva tutoría de aprendizaje, aquí creo que una de las cosas que mejoraron este grado de idoneidad es que, si se tomó el tiempo para cuando se trabajó con el *Kahoot!*, explicar y dar las respuestas correctas cuando ellos se equivocaban o cuando estaban bien, explicar esa parte, se utilizó el material que se les proporciona a los estudiantes y obviamente a partir de él, se fue explicando y desarrollando. En las sesiones síncronas también había escucha por parte de los estudiantes, generalmente en las clases se preguntó si ¿había dudas?, si ¿había preguntas?, en caso de que sí hubieran se tomó el tiempo para contestarlas y ya fuera de la sesión síncrona, en mi caso, yo respondía mensajes a través de Instagram, entonces allí hubo tiempo en ese sentido para terminar de aclarar o ajustar algunas inquietudes. Algo que disminuye el grado de idoneidad en cuanto a la enseñanza colectiva, sobre todo en las sesiones síncronas, cuando estoy con ellos en *Zoom*, a mí se me hace complejo identificar a los estudiantes que requieren mayor atención, en el sentido de que si entendieron o no entendieron, porque a veces, ellos ni siquiera preguntan, entonces allí me cuesta trabajo, que si yo le puedo dedicar algún tiempo, todo depende de que ellos pregunten y en mejorar creo que no porque siempre hay disposición y tiempo para poder atender dudas y mejorar el aprendizaje de los estudiantes.

Los elementos de la trayectoria argumentativa de este episodio son:

- Proponente: Rediseño de cuestionario de *Kahoot!*
- Razones: Las imágenes propuestas y el tiempo no fueron adecuados en el cuestionario de *Kahoot!*
- Reflexión: El uso inadecuado de los recursos materiales, manipulativos, etc. incide en los estudiantes puede ser que se desconcentren o se atrasen para dar respuesta. Hay elementos que no pueden ser modificados por las docentes como el número de estudiantes y los horarios de clase. En el caso de la profesora 2 en su sexta clase ya estaba cansada. La profesora 2 comenta que, en clases virtuales, es difícil identificar a los estudiantes que requieren mayor atención cuando no preguntan.

En este último episodio, la profesora 2 reflexiona sobre tres temas relacionados con el criterio mediacional (recursos materiales y número de alumnos, aula y horario), respecto al horario la profesora reflexiona sobre su agotamiento al impartir seis cursos en un día y que esto influye en la idoneidad de la clase y comenta que algunas veces no hay ideas para mejorar la idoneidad o que hay cosas que están fuera de su alcance como horario y número de alumnos.

5.5. Conclusiones

La argumentación práctica que se identifica en este trabajo emerge de la solicitud que se hace a las profesoras de valorar su clase implementada y proponer acciones de mejora con los CID.

Los CID promueven una amplia reflexión docente, ya que cuando se les solicita a los docentes valorar su propia práctica, sin una pauta, la valoración trata pocos temas y es limitada. Mientras que, la valoración con los seis criterios de idoneidad, sus respectivos componentes e indicadores permiten considerar diferentes aspectos de la clase implementada, y son un punto de inicio para la reflexión y la propuesta de acciones de mejora para un rediseño, algunas veces las propuestas de mejora contienen argumentos para estas acciones. Los tipos de acciones que se proponen durante la reflexión son: acciones de mejora (rediseñar el cuestionario de *Kahoot!* aumentando el tiempo), acciones nuevas (el pizarrón virtual), eliminar acciones realizadas (no hablar sobre conexiones interdisciplinarias en la clase de introducción a la derivada). La valoración, algunas veces, implicó una reflexión y propuesta de mejora, otras veces, solo implicó una reflexión sin propuesta de mejora y también hubo momentos en la valoración sin reflexión. La reflexión conjunta tiene la ventaja de compartir y complementar ideas y experiencias. Algunas veces las ideas de una profesora eran nuevas para la otra profesora y esto enriqueció las posibles acciones que se pueden considerar para el diseño de futuras clases. Las etapas del curso de formación docente implementado fomentan la reflexión personal y conjunta de las profesoras en diferentes momentos.

Al finalizar el curso de formación, se observó que la interiorización de los criterios de idoneidad, por parte de las profesoras, se obtuvo parcialmente y se requiere continuidad en el uso de los criterios de idoneidad para lograr una incorporación a las prácticas docentes de una manera más natural. De este trabajo se plantean los siguientes temas respecto a la formación docente de matemática: comparar la argumentación práctica en las diferentes etapas del curso de formación en particular la argumentación que se genera en el ciclo del LS. ¿De qué manera se puede fomentar la argumentación práctica de las docentes para justificar sus acciones en un proceso de enseñanza y aprendizaje? Se pueden implementar más cursos de formación para ampliar la muestra del análisis y validar los resultados considerando más factores como: otras asignaturas de matemática, más niveles educativos y ampliar la diversidad de docentes en un curso de formación.

Capítulo 6: Tercer Artículo Científico

El artículo científico presentado en este capítulo tiene por objetivo analizar los criterios de idoneidad didáctica que emergen en la argumentación práctica de futuros profesores que participaron en un ciclo de *Lesson Study*, en el cual planificaron una clase sobre geometría plana para estudiantes de Educación Infantil, atendiendo así al cumplimiento del cuarto objetivo específico de esta investigación doctoral. Al igual que el manuscrito reportado en el Capítulo 4, aquí se reportan los análisis de episodios de *Lesson Study* donde no se tuvo participación en el diseño e implementación del ciclo, sino que sólo se tuvo acceso a los datos derivados del ciclo para su posterior análisis.

En la Tabla 9, se presentan los datos de identificación de este tercer artículo.

Tabla 9. Datos de identificación del tercer artículo científico.

Criterios	**Datos**
Título	Didactic suitability criteria that emerge in the future teachers' practical argumentation.
Autores	Telesforo Sol [1], Adriana Breda [1], Adriana Richit [2], Gemma Sala-Sebastià [1].
Afiliaciones	[1] Departamento de Educación Lingüística y Literaria y de Didáctica de las Ciencias Experimentales y de la Matemática, Facultad de Educación, Universidad de Barcelona (España). [2] Universidade Federal da Fronteira Sul (Brasil).
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Abstract and Keywords / Resumen y Palabras Clave

Abstract: The article's objective is to analyse the didactic suitability criteria that emerge in the practical argumentation of future teachers who participated in a Lesson Study cycle, in which they planned a lesson on plane geometry for pupils in Early Childhood Education. The sessions in the cycle were recorded by video and two of them were analysed, firstly, with the Pragma-dialectic and Toulmin models, and secondly, with the didactic suitability criteria. As a result, it is observed that the practical arguments of future teachers for choosing the manual materials to be used focus on the use of logical blocks and the criteria that support them are based mainly on the mediational criterion, followed by the epistemic one. It was also observed that the participants' consensus on the dynamics of the lesson is that it should begin with the exploration of the logical blocks, followed by the elaboration, recording and exhibition of artworks and that the criteria supporting this decision-making are related, respectively, to the interactional, ecological and affective criteria. This study provides a tool to show, in detail, the structure of practical arguments and the important role of the didactic suitability criteria in the didactical decision-making of a group of prospective teachers.

Keywords: Didactic suitability criteria; Initial teachers' training; Lesson study; Practical argumentation.

Resumen: El objetivo de este artículo es analizar los criterios de idoneidad didáctica que emergen en la argumentación práctica de futuros profesores que participaron en un ciclo de Estudio de Clases, en el cual planificaron una clase sobre geometría plana para estudiantes de Educación Infantil. Las sesiones en el ciclo se grabaron en video y dos de ellas se analizaron, primero, con los modelos de la Pragma-dialéctica y de Toulmin, y segundo, con los criterios de idoneidad didáctica. Como resultado, se observó que los argumentos prácticos de futuros profesores para escoger los materiales manuales para utilizarse se enfocan en el uso de bloqueos lógicos y los criterios que los sustentan se basan, principalmente, en el criterio mediacional, seguido el epistémico. También, se observó que el consenso de los participantes sobre la dinámica de la clase es que debería comenzar con la exploración de los bloques lógicos, seguida de la elaboración, grabación, y exhibición de trabajos de arte y que los criterios que sustentan esta toma de decisiones se relacionan con los criterios interaccional, ecológico y afectivo, respectivamente. Este estudio aporta una herramienta para mostrar, en detalle, la estructura de los argumentos

prácticos y el papel importante de los criterios de idoneidad didáctica en la toma de decisiones didácticas de un grupo de futuros profesores.

Palabras clave: Argumentación práctica; Criterios de idoneidad didáctica; Estudio de clases; Formación inicial de profesores.

6.1. Introduction

Initial teacher training should provide future teachers with situations in which they can step away from their role as university students to assume the role of teachers (Correa Molina, 2013, 2015). Thus, teacher degree courses need to work on the professional development of teachers and provide future teachers with the acquisition of knowledge related to the profession, including reflections made as teachers in decision-making processes (Correa Molina, 2014).

On the one hand, the development of teacher reflection can be provided through the Lesson Study (LS) approach (Murata, 2011), which provides conditions for teachers to collaboratively plan and implement a lesson or learning sequence on a given topic and then reflect on it, making decisions about the teaching to be carried out and experimenting with new practices (Fernández & Yoshida, 2004; Nguyen & Tran, 2023; Richit, 2020; Richit, Agranionih et al., 2024; Richit, da Ponte, & Tomasi, 2021; Richit, da Ponte, & Tomkelski, 2024; Richit & Tomkelski, 2020). On the other hand, there are the Didactic Suitability Criteria (DSC) (Breda et al., 2018; Font et al., 2010; Godino et al., 2007), which are proposed as a tool for organizing teacher reflection aimed at analysing and improving processes of teaching and learning mathematics. Different authors have attempted to compile criteria based on the results of research in math education to guide teachers' practice so that it is of high quality, suitable, competent, etc. (Praetorius & Charalambous, 2018). The DSC are one of these widely accepted compilations in Mathematics Education.

Concerning this theoretical construct, one phenomenon has been regularly observed: some of the components and indicators of DSC can be used to reinterpret the practical orientations that appear in the discourse of teachers or future teachers when they evaluate an episode or justify the improvement of a learning sequence, without having been taught how to use this tool (among others, Breda, 2020; Breda & Lima, 2016; Breda, Seckel et al., 2021; Frade & Tinti, 2023; Horta & Silva, 2023; Morales-López & Font, 2017;

Moreira et al., 2018). In other words, their comments can be considered as implicit evidence of an orientation towards action (agreed upon by the participants and derived from their knowledge, beliefs and values), that is, as a norm that should guide teaching practice so that it is a quality practice. The evidence of these criteria also emerges as consensus in the spaces for collective reflection through the LS cycles (Erbilgin & Arikan, 2021; Hummes, Breda, & Font, 2022; Hummes et al., 2019).

To analyse in depth the discourse of teachers or future teachers about actions related to teaching practice, we can resort to the study of argumentation about actions they agree to carry on, known as practical argumentation. Practical argumentation is defined as a mode of reasoning situated, in a particular social context, whose objective is to select an action for solving a practical problem (Gómez, 2017). It is also a promising analytical agenda for studies focused on issues related to teacher education and the professional development of teachers. This agenda makes it possible to analyse, treat, and understand the different professional, theoretical, social, cultural, and practical aspects intrinsic to teacher training processes and their role in decision-making, solving problems in the teaching field and proposing educational changes.

Previous studies have shown that teachers present arguments about the actions they agree to conduct and that these arguments use criteria about what they consider best, which can be reinterpreted as components and indicators of the Didactic Suitability Criteria. These analyses have been carried out by serving math teachers (Hummes et al., 2021), professors of future teachers (Sol, Breda et al., 2023; Sol, Sánchez et al., 2023) and teachers of compulsory secondary education in Spain (Ledezma, Sol et al., 2022). Research focusing on practical argumentation in initial teacher training is recent. It includes the work of Correa Molina (2015) and Ramos (2021), however, we have not found studies that reveal the emergence of criteria from future teachers, that is, with future professionals who will work in Early Childhood Education (ECE) and initial years of Primary Education.

The Didactic Suitability Criteria that emerge from the argumentative paths that guide didactic decision-making in initial teacher training analysis from the perspective of practical argumentation, can contribute to making the different aspects that underpin decisions and guide classroom practice explicit. Thus, modes of argumentation, conceived as forms of discursive interaction developed by subjects (Ramos, 2021), are an aspect of teaching that converges with the basic principles of Lesson Study, reflection, and collaboration.

With this in mind, this work responds to the following question: What are the Didactic Suitability Criteria that emerge in the practical argumentation of future teachers, when they are involved in a Lesson Study cycle in which they make decisions related to planning a lesson on flat geometric shapes for Early Childhood Education? In this sense, this paper aims to analyse the Didactic Suitability Criteria that emerge in the episodes of argumentation undertaken by future teachers when planning an investigative lesson on flat geometric shapes for Early Childhood Education students.

6.2. Theoretical Approach

6.2.1. Didactic suitability criteria

Godino et al. (2019) suggest that Mathematics Education is a scientific and technological discipline since it must address descriptive, explanatory and predictive issues, which are characteristic of scientific knowledge, as well as prescriptive and evaluative issues, which are characteristic of technological knowledge. From the perspective of these authors, the prescriptive and evaluative view of Mathematics Education leads to the generation of theoretical constructs such as Didactic Suitability Criteria (DSC).

The didactic suitability of a teaching and learning process is defined as the degree to which the characteristics of these processes can be considered optimal or adequate for attaining overlap between the personal meanings achieved by students (learning) and the institutional meanings intended or implemented by teachers (teaching), considering the circumstances and available resources (environment) (Godino et al., 2007). A teaching and learning process will achieve a high degree of didactic suitability if it can coherently and systematically articulate the six following partial didactic suitability criteria (DSC) (Font et al., 2010): a) Epistemic criteria. Assessing whether the mathematics taught is *good maths*; b) Cognitive criteria. Assessing, before beginning the instructional process, whether what is to be taught is within a reasonable distance of what the students know; c) Interactional criteria. Assessing whether the interaction resolves the students' doubts and difficulties; d) Mediational criteria. Evaluating the suitability of the resources and time used in the instructional process; e) Affective criteria. Evaluating student involvement (interest, motivation) in the instructional process; f) Ecological criteria.

Evaluating the suitability of the instructional process to the school's pedagogical project, curriculum guidelines, social and professional environment conditions, etc.

In turn, each DSC has its respective components, the usefulness of which involves defining a set of observable indicators that allow evaluating the degree of suitability of each facet of the teaching and learning process. Table 5 shows the components of each DSC.

Table 5. Criteria and components of didactic suitability.

Criteria	Components
Epistemic	Errors; Ambiguities; Richness of processes; Representativeness of the complexity of the mathematical object.
Cognitive	Prior knowledge; Curricular adaptation to individual differences. Learning; High cognitive demand.
Interactional	Teacher-student interaction; Interaction among students; Autonomy; Educational evaluation.
Mediational	Material resources; Number of students, class schedule, and classroom conditions; Time.
Affective	Interests and needs; Attitudes; Emotions.
Ecological	Curriculum adaptation; Intra and interdisciplinary connections; Social and professional utility; Didactic innovation.

Source: Adapted from Breda et al. (2017).

The DSC and its components are based on the principles and standards of the National Council of Teachers of Mathematics (NCTM), current trends in Mathematics Education and research in this field (Breda et al., 2018). Therefore, they constitute a consensual tool that is used to structure the reflections of teachers in teacher training programs in different Ibero-American countries (Burgos & Castillo, 2021; Calle et al., 2022; Esqué & Breda, 2021; Giacomone et al., 2018; Hummes, Breda et al., 2023; Morales-Maure et al., 2019; Seckel, 2016; Sol, Lujambio et al., 2023; etc.).

6.2.2. Practical argumentation

Practical argumentation is defined as a way of reasoning situated in a particular social context that aims to choose an action to solve a practical problem (Gómez, 2017). Supported by Vignaux (1976), Gómez affirms that practical argumentation is a process that deconstructs, constructs, and reconstructs propositions and discourses, transforming them. In this understanding, practical argumentation is a social device that regulates individual and collective interactions. This study considers practical argumentation to be “argumentation that aims to decide on a direction for action” (Lewiński, 2018, p. 219).

We analysed the argumentation in this study from the perspective of Pragma-dialectics (van Eemeren & Grootendorst, 2004). Pragma-dialectics proposes an ideal model for critical discussion, and four phases can occur in this technique:

1) The confrontation phase establishes the difference of opinion. In a mixed difference of opinion, this means that one party's point of view is not immediately accepted by the other but is met with doubt or criticism. In a mixed difference of opinion, the other party presents their opposing point of view.
2) The opening phase refers to the starting points of the discussion and assigns the roles of protagonist and antagonist (in a mixed difference of opinion, there are protagonists and antagonists). In addition, the rules of the debate and the starting points are agreed upon.
3) The argumentation phase is when the protagonist defends their point of view against the antagonist's persistent criticism, presenting arguments to meet the antagonist's objections or to resolve their doubts.
4) The conclusion phase is when the parties assess how far the resolution of the difference of opinion has gone and in whose favour. If the protagonist withdraws their point of view, the difference of opinion is resolved in favour of the antagonist; if the antagonist abandons their questioning, it is resolved in favour of the protagonist.

About analysing teachers' pedagogical arguments (Fenstermacher & Richardson, 1993; Metaxas et al., 2016), Fenstermacher and Richardson (1993) describe the structure of a practical argument. The authors consider that a practical argument can be described by four types of premises (empirical, axiological, normative, and circumstantial) and an action or intention to act. Metaxas et al. (2016) examine changes in a teacher's pedagogical argumentation, using Toulmin's model and argumentation schemes. Thus, research into collective practical argumentation – arguments whose claim is an action that is carried out or argumentations about actions that are considered in the discourse but are not carried out (Gómez, 2017) – can provide tools for analysing how consensus is reached in dialogues between participants, where they offer reasons for and against carrying out certain actions. In particular, agreement on the action to be taken is based on criteria of what is considered or could be considered best, as well as other elements. For example, in Kinach (2002), some future teachers in the later years of elementary school have to decide on different ways to explain negative integers and agreement is reached because one teacher argued that one explanation was better than the other because one favoured meaningful understanding while the other only favoured memorisation of the concept and all the participants except one agreed that meaningful understanding was better than simple memorization.

The argumentative model proposed by Toulmin (1958/2003) works as follows: based on some evidence (data), an affirmation is formulated (claim). A warrant connects the data with the claim, which is based on a theoretical, practical, or experimental foundation: the backing. Modal qualifiers (certainly, definitely, etc.) indicate how the statement is interpreted; as true, possible, or probable. Finally, possible rebuttals or objections are considered.

6.3. Methodology

In this qualitative research (Creswell, 2009), we focused on analysing practical argumentation about collaboratively negotiated decisions in a group of future teachers planning a lesson on flat geometric figures for primary school students participating in a Lesson Study cycle. To do this, we first identified episodes of practical argumentation (Lewiński, 2018) and then analysed them using the following theoretical models: a) the ideal model, proposed by Pragma-dialectics (van Eemeren & Grootendorst, 2004); and b) the Toulmin model (1958/2003).

6.3.1. Context and participants

The research was carried out in a Lesson Study cycle, composed of ten sessions of approximately three hours each, which involved nine future teachers (T1, T2, T3, T4, T5, T6, T7, T8, T9) who are undergraduates in education at a public university in southern Brazil who were studying mathematical literacy in the tenth semester of the course, and the teacher trainer, a specialized coordinator in LS cycle (TT). The students, organized in groups and in a collaborative way, prepared a research lesson for childhood education pupils, focusing on flat geometric shapes.

The LS cycle carried out considered four different stages: defining objectives and goals, planning the research lesson, carrying out the research lesson and reflecting on the lesson (Richit, 2020; Richit et al., 2019). In total, the cycle involved eight three-hour sessions, two of which were dedicated to defining the school level of the students and the mathematical topic to be developed, as well as formulating the research question; five sessions were dedicated to planning the research lesson, studying the curriculum guidelines for ECE, discussing the construction of mathematical notions in ECE and preparing materials for the lesson; and a reflection session after the end of the cycle. It also involved two 80-minute sessions in which the research lesson was developed. The cycle thus totalled ten meetings. The research lesson was taught by the class teacher (T1) in a class of fourteen children and was organized in two stages over two consecutive days. The main of the lesson was to identify and analyse what criteria the children used to classify geometric shapes when creating a work of art with logic blocks.

6.3.2. Data collection and analysis

The research data was based on the Lesson Study sessions recorded and transcribed, and the textual interactions of a WhatsApp group created specifically for the LS were incorporated into the empirical material. In particular, the dialogues relating to two sessions of the LS cycle were considered (session two, which focused on the objectives stage definition, and session six, which addressed the planning stage of the research lesson). These sessions were chosen because they show the most evidence of decision-making arguments.

The analysis of argumentation considered the Pragma-dialectic model, Toulmin's model and the characterization of practical argumentation, which are previously explained in the theoretical approach section. The steps are:

1) Description of the dialogue.
2) Review and scrutiny of the empirical material to identify episodes of practical argumentation.
3) Identification of the different arguments that the future teachers express when they take part in the discussion.
4) Identification of words that account for the existence of arguments, such as: then, therefore, for example, if, that is, I understand that every discourse, it is supposed, it is said, etc.
5) Application of the Pragma-dialectical model.

6) In the argumentation phase detailed in 5), presentation of the arguments identified, considering Toulmin's model.
7) Relation of the arguments identified with the Didactic Suitability Criteria. The initial analysis of the data was carried out by two researchers and later triangulated by the others to clarify interpretations.

6.3.3. Ethical aspects and limitations

The investigation meets the ethical requirements of research and was approved by Research Ethics Committee of the Universidade Federal da Fronteira Sul (Opinion no. 4.764.981, approved on 10 June 2021), prioritizing maintaining the anonymity and integrity of the participants. The limitations of the study refer to the fact that the Lesson Study participants were unfamiliar with this approach and, therefore, with the perspective of collaborative planning. As a result, the interactions between the participants were more discrete at the beginning, becoming more developed in the second half of the Lesson Study cycle.

6.4. Results

The analysis of the LS sessions, which were recorded and transcribed, revealed different episodes of argumentation related to teaching decisions when planning a lesson to address flat geometric shapes in Childhood Education. Of these, two episodes are analysed from the perspective of practical argumentation, the first related to the first stage of the cycle (study of the curriculum and goals) and the other related to the second stage of the LS cycle (planning the research lesson). In particular, the episodes relate, respectively, to the choice of mathematical topic to be worked on, followed by the choice of material to be used and, finally, the dynamics (sequence of activities) to be carried out in the research lesson.

6.4.1. First argumentative episode

In the first Lesson Study sessions, participants made decisions about the school level at which the lesson would be taught, the mathematical topic to be covered in the research lesson, the objective of the lesson, the thematic context for addressing the topic, and also shared classroom experiences from their supervised practice and professional experience.

The discussion presented below took place in the second meeting of the cycle (the stage of defining the objectives and goals) after the group had decided that the lesson would be held at the kindergarten level. The socialization of classroom experiences, confronted with the guidelines for primary school education, delineated the choice of curriculum topic (flat geometric shapes). For reasons of space, we present the dialogue between the future teachers in partial form.

T2: At the time [of the internship] I worked with Tangram, but I did not work in the sense of teaching something […]. I worked more spontaneously with the children, like a game […]. It was more like construction. Some things they recognized, for example: What does a rectangle look like? They said it is more like this, longer. What does a square look like? It is like this, all the same. The activity was very good. They assembled lots of good things.

T3: The Tangram is very good.

T4: Tangram is great for children to assemble things, and figures like boats and animals. There are even some ready-made examples to create with it. But you can also work on lots of things with this game [Tangram], such as sorting the pieces by colour or shape and ordering them by size.

T5: Tangram is a very interesting material, and children like to play with it, but one piece is a parallelogram. So, I would have to explain to them what a parallelogram is, I do not know if it would not be a bit complicated.

TT: Perfect! So, you see, you have worked with classifying geometric shapes!

TT: That is the idea. Let us think of an activity whose only objective is to work with math. But, yes, it is a game, it is telling a story, it is a context where, through it, we work with another theme together. So, you worked on classifying or recognizing geometric shapes, that was very interesting.

T6: And there are the logic blocks [too].

T4: The logic blocks are very good even for primary school.

T1: With Tangram or logic blocks you can work on geometric shapes, which is one of the mathematical concepts that we need to develop in primary school.

T3: We did something every day that really interested them, which was the calendar. We would see what day it was, then each one would look for the day, sometimes they would want to count down to that day, and then they would make connections, you know. Another thing I mentioned was the flags, because they have the experience of colour, the sequence of three different things with the same quantity, they were doing these things, you know? Like hide and seek, and then they started one, two, three, four, ten, twenty.

T3: Now I also remembered the clock, “What time is lunch? Is lunch coming?”, they were always worried. They said: “Snack time is coming”. They always asked every day, every hour. I said: “Ah, so it's like this, look, when the hand is here and here, it will be …”. So, they never asked me again because they knew.

TT: Yes, and recognizing the time on the clock is very difficult.

T3: Very difficult. […]

TT: We are talking, everyone is telling us about an experience of something in primary school. Or what you did at work, or what you did on your internship. P2, P3 and P1 have already spoken.

T3: But the calendar thing was very significant because, by the end of the year, they knew that on Saturday and Sunday, they did not come [to school]; they had two days at home, five days at school. So, they already had that understanding. It was great.

T6: Oh yes, the playlist of songs, in the workshops, always the rhythm. So, I also count, one, one, two, three, one, one, two, three…

T6: That is all […] but there are other things. This week we did this, exploring broccoli and so on. So, we brought the microscope, and there [the students] saw the differences, I think it is very good, he said he saw a little animal under the microscope, and it had lots of legs, another one had hair, then he started counting the hairs, the legs. […]

T3: I have just remembered something. The recipes we work with, are homemade dough recipes. And they love it.

T1: I did the dough recipe and then they wrote down the recipe in their way. And then they drew, right? In the recipe it was two tablespoons of salt, they took out two tablespoons, etc.

TT: There are various things here, there is the calendar, there is… so in the case of broccoli, we would have to make a very close approximation, but it could be something else.

TT: Yes, and we can create something else, we can use another game, another material, in short…

TT: The hopscotch idea that came up the other day… P2 worked with Tangram. Tangram is also very cool. You can make coloured soup, a soup of shapes, for example, a soup just of triangles, and then they must identify and classify, identify, and classify. I can play around, if the subject is, for example, cooking, making a recipe and doing that, it would be possible to explore.

T3: If it is Tangram, great. They love it…

TT: And they did this, it seems to me that it was a work of art, they had to assemble it, but at the same time they were handling this material to make a design, an artwork, they were exploring these geometric elements. So, it could also be an artistic context and then work with geometric shapes.

T6: Logic blocks […]

TT: The logic blocks are great, we have here as many as you need.

T7: There are lots of activities you can do with logic blocks. There are those strong attributes, which Ursula Simons developed, she has a marvellous book, if I am not mistaken it is "150 activities with logic blocks", it is a book just with logic blocks. So, it explains "n" ways of working with logic blocks, from thickness, colour, and size, it is very interesting.

TT: Yes, I presented it, I do not know if it was in your class, or after some activities. One of them was, working with classification. First with size, thicker, less thick, medium; then by colour, various possibilities. There was soup, block soup, and then they created criteria, everyone made their soup and then explained to their classmates what the ingredients were, and the ingredients were the pieces. So, you could do that, I think it is a good idea. I think that at this stage we can think about the logic blocks: I am going to help you define the objective, and then you will have about twenty minutes to think about at least the beginning of this activity. What will it be? Will it be a song, will it be a joke? Today we will get to this point, which is the start of the lesson. And then, in the next lesson, we will organize the stages for, more or less sixty, eighty minutes at most, this activity with them.

T6: So, let us get to the recipe, shall we?

TT: Yes, I think it could be with logic blocks, and then we can create the recipe, the artwork, and a lot of things inside. Objective: What would we like to learn about children's learning from this? Remembering that the blocks allow us to work with geometric shapes, to work with the properties of the elements, which P7 talked about, the thicker ones, the thinner ones, which he called "thins". The thinner, the thicker, etc., the greater the notion we have of the properties. What is yellow, what is bigger, what is smaller, what is medium? So, there is also something in the BNCC [Common National Curriculum Base], more or less, that you can do with just one activity. So, you can help them get started. What do you want to learn about learning for the youngest ones?

TT: Perfect. Criteria and classification of shapes, so I have put it here: creating an artwork, or I don t know…

T3: They love playing with wooden blocks and building blocks. They love making castles, they even stack things up high just to watch them fall. [And they play kitchen too.] They pick up the wooden game pieces to make food. They use the pieces as if they were the ingredients.

T1: So that is it. We can work on geometric shapes with wooden pieces, and logic blocks, and see how they choose the pieces, and what criteria they use to choose the pieces they use to play, build things, castles, or play with food. Teacher, could you put this objective in the plan?

The dialogue continues […]

6.4.2. Analysis of the argument of the first episode from the perspective of pragma-dialectics and Toulmin's model

Opening: A round of ideas is held, looking for activities that have some aspect of mathematics to define the session with primary school students.

Confrontation: A discussion between future teachers and the rules of argumentation are not explicit, but to discard a suggestion, instead of arguing against it, only another suggestion is proposed, that is, counterarguments are rarely made.

Argumentation: The argumentation phase, analysed according to Toulmin's model, is presented in a sequence of ten arguments, each with its respective backing, described in chronological order. Arguments 1 and 6 were refuted. In addition, data, a warrant and a claim were identified. Each argument identified presents the claim, implicit or explicit, that the material mentioned is a good option for working on a mathematical aspect at the primary school level.

- *Data*: To think of activities, considering the experiences of future teachers, whose objective is to work with math; considering creation and motor skills.
- *Argument 1*: Use of the Tangram.
- *Backing*: Tangram allows you to work on the process of exploration and construction; you can recognize geometric figures (rectangles and squares); you can encourage students' autonomy; you can put things together and there are ready-made examples to create them; you can work on classifying different characteristics (colour, shape, size); children like it; you can work on representing numbers; you could work in an artistic context.
- *Rebuttal*: But we must explain some geometric figures (parallelograms), so it might get a bit complicated.
- *Argument 2*: Use of logic blocks.
- *Backing*: With these, you can work with geometric shapes (a mathematical concept that must be developed in primary education); this material is available; you can work on classification considering different characteristics and the students' explanation can be encouraged.
- *Argument 3*: Use of the calendar.
- *Backing*: To see what day it was and to count the days (this interested them a lot); it allowed them to make connections; to work with a unit of time (this is significant); to work on the idea of quantity (a lot, a little); to count; various mathematical notions considered in the Common National Curriculum Base (BNCC) were worked on; it is significant because they knew they would not be coming on Saturday and Sunday.
- *Argument 4*: Use of flags.
- *Backing*: They have the experience of colour; a sequence of three different things with the same quantity. Working on classification.
- *Argument 5*: Use of scripted games like hide-and-seek.
- *Backing*: They (the students) can count.
- *Argument 6*: Use of a clock.

- *Backing*: Working on the students' interests (lunchtime), recognizing the time on the clock (it is very difficult); they have assimilated the school routine (starting time, snack and dismissal); organization of time.
- *Rebuttal*: Recognizing the time on the clock is very difficult.
- *Argument 7*: Using the music playlist.
- *Backing*: Rhythms is considered; harmony is related to mathematics; it is also intrinsic in music; the relationship between math and music.
- *Argument 8*: Exploring broccoli.
- *Backing*: Observation is worked on; students are interested; use of a microscope; you can work on shapes and symmetries; you can count; it can be related to fractals and beauty in geometry; fractals can be related to art.
- *Argument 9*: To work with a recipe and the use of dough.
- *Backing*: You work with three-dimensional construction; you work with different units of measurement.
- *Argument 10*: Work with logic blocks.
- *Backing*: The material (logic blocks are available) and there is a book by Ursula Simon with 150 activities for working with logic blocks.
- *Warrant*: Many activities can be conducted with logic blocks (this is a strong point).
- *Claim*: An activity will be planned with logic blocks.

Below is the diagram that organises the argumentation phase according to Toulmin's model (Figure 25).

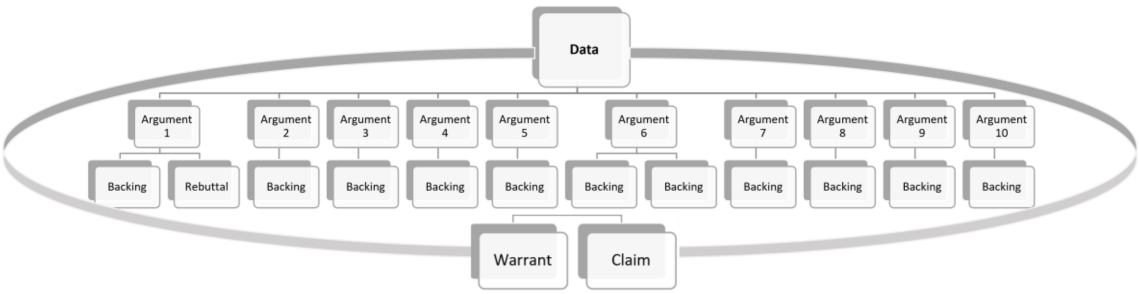

Figure 25. Organisation of the first episode of argumentation following the Toulmin's model.
Source: The authors.

In the argumentation phase, it can be seen that of all the experiences shared by the future teachers, the three that were of most interest were: working with Tangram, making recipes with dough, and logic blocks. In conclusion, the material selected to plan the research lesson for Early Childhood Education students was logic blocks. The warrant of working with logic blocks is based mainly on the existence of a bibliography with various activities designed and that the TT makes the material available. Another observation is that the teacher who first mentioned logic blocks comes back to them when defining the activity, thus reaffirming her interest in using this material. Tangram, despite having several comments in its favor, was not taken up by the teachers as material to be used in the lesson to be implemented.

6.4.3. Analysis of the argumentation of the first episode based on the didactic suitability criteria

When the prospective teachers' arguments are analysed, it can be seen that they focused on mentioning the materials and commenting on some of their characteristics and functionalities for student learning. In the discussion, the prospective teachers' values and beliefs can be identified when they comment on which material should be used in class and why. The discussion focused on the mediational criterion, mainly the manipulative material resources component (Tangram, logic blocks, pasta, clock, calendar, broccoli, etc.), intending to try to create an activity in which the "creation process" is developed. This process is considered in the epistemic criterion, in its richness of process components.

The backings identified for considering the use of Tangram are related to different Didactic Suitability Criteria. Regarding the epistemic criterion, the following components emerged: richness of processes, when considering the processes of exploration, construction and classification; representativeness of complexity, since the theme of geometric figures makes it possible to work on representativeness of figures, such as the rectangle and the square, and also makes it possible to work on numerical representativeness (counting, ordering, etc.). In terms of the interactional criteria, the autonomy component emerges when we consider that students can explore the material autonomously. Regarding the affective criteria, in particular, there is evidence of the student's interests and needs, when we consider that the students enjoy working with Tangram. Finally, concerning the ecological criteria, there are intra- and interdisciplinary connections since the material used can be related to the arts. On the other hand, the rebuttal of the use of Tangram, by mentioning that the lesson could be complicated if the geometric figures were explained, is related to the cognitive criteria, considering students' prior knowledge, since having to explain the parallelogram could be outside the students' zone of proximal development.

In terms of the other materials mentioned, the criteria considered for using the calendar were: affective, student interests; ecological, socio-labour usefulness; and epistemic, when working on concepts of counting and quantity. The criterion considered for the use of the flags was epistemic, considering the process of classification and sequencing. For the use of the clock, we considered the ecological criteria, the component of socio-professional usefulness, and the affective criteria, considering students' interests and needs. For the use of the musical playlist, ecological criteria were considered, in particular, the component of interdisciplinary connections when linking music and mathematics. Finally, when exploring broccoli, the following criteria were considered: interactional, in particular, the autonomy component when working on the autonomous exploration of materials; ecological in its interdisciplinary component, when relating fractals to broccoli; and affective, in its emotion component, when highlighting the aesthetic qualities of mathematics.

As for the arguments for the use of logic blocks (the material chosen by the participants), their backing is related to the following criteria: ecological, in particular to the curricular adaptation component, considering that geometric shapes must be worked with in primary education; epistemic, because working on "classification" generates a wealth of relevant mathematical processes; interactional, in the component of interaction between students, by mentioning that students can explain their work to each other, which connects to the wealth-of-processes component, of epistemic suitability, emphasizing the process of explanation and justification. In addition, about the mediational, it is argued that the material (logic blocks) is available for use and that there is a book by Ursula Simon with 150 activities for working with logic blocks. The warrant mentioned for the selection of logic blocks is related to the mediational criteria, in their material resources component; and to the epistemic one, given that the material (logic blocks) is a versatile resource that favours the introduction of good situations, languages, procedures and arguments.

6.4.4. Second argumentative episode

The second argumentative episode refers to the analysis of the dialogues from the sixth Lesson Study meeting, in particular the discussion about the dynamics of the research lesson, for which the group planned to carry out two central activities using logic blocks (the material chosen in session two, the first argumentative episode). For reasons of space, we present a partial transcript of the dialogue between the participants and then show the analysis of the argumentation, first from the perspective of Pragma-dialectics and Toulmin's model, and then from the Didactic Suitability Criteria perspective.

TT: We will now analyse the class with logic blocks for the lesson of P1 [...].

T1: What if we gave the children the logic blocks and left them free to explore the material, as well as the images of the works of art? And then ask them: what do you think we can create with these pieces?

T4: It is interesting, but I think only after they have explored [the material] well. You must let them explore the material well. Then you can do this, leading, at a later point in the lesson, to this activity you suggested [creating things with the pieces of material].

T3: […] Then, if they have explored everything, if they have exhausted their desire to play and do what they wanted, then they will be more open, more willing to do what we are going to propose, which is to create the work of art.

T7: Yes. I even realize that […] they will be more focused because then the euphoria of the first contact will have passed. Then they will be more focused on what we are going to propose.

T1: That is true.

T7: But that does not mean we cannot project the [art] work without saying anything. Just do the projection [while they explore the space and the material] and leave the projection without saying anything while they carry out the activity. Set up a space with the projection and the logic blocks for them to explore.

T1: Right. A slide presentation without teacher intervention.

T3: So, on the first day it was like this: taking the children to the planned space to familiarize and interact with each other, with free exploration of the blocks. How are we going to relate this exploration activity to the presentation of the works of art on the slides?

T4: Then the next day you start with the material, you start with the slide show, asking what is similar to these figures that we explored, that we played with [in the previous class]

T7: Okay. So, [this was defined], for the first day there will be the "Planned space for exploring logic blocks". [For the second day, the round table discussion], and to start this round table discussion, before the presentation of the slides, as on the first day, we will probably take photos of them or take photos of them playing, or we will make videos of them. Then we will play it back to them and say: Ah, remember that day you [worked with the logic block pieces?] Because they will see each other, and then they will remember [the activity we did]. Then we move on to the creation of the works, to the round table discussion and we do it all. […]

6.4.5. Analysis of the argument of the second episode from the perspective of pragma-dialectics and Toulmin's model

Opening: FP comments that the lesson will be planned with logic blocks, considering what was defined in the previous episode.

Confrontation: [none].

Argument: The argumentation phase, analysed with Toulmin's model, is presented in a sequence of six arguments, each with its respective backing, warrant and claim, described in chronological order. No argument was rebuttal. In addition, common data was identified for all the arguments identified.

- *Data*: Work on flat geometric shapes by exploring works of art and logic block pieces.
- *Argument 1*: Begin a class with free exploration of the logic blocks.
- *Backing*: If the blocks are handed out beforehand, [the students] will already know that they will have to use squares, rectangles, circles and triangles to do the activity; it does not make much sense to start with the artwork without first talking about the blocks.
- *Warrant*: Students should be given a specific time to familiarize themselves with the logic blocks.
- *Claim*: Begin the class with the free exploration of the logic blocks.

- *Argument 2*: Start the lesson by handing out the logic blocks for exploration and projecting the works of art at the same time.
- *Backing*: You have to let them explore the material well [logic blocks] and then ask questions; if they explore the logic blocks first, they will be more willing to create works of art later; if they explore first, then they will be more focused on the lesson proposals; if you start with works of art, the students might not be very interested.
- *Warrant*: Artistic images can be projected without saying anything while students explore the logic blocks.
- *Claim*: The first activity will be the exploration of logic blocks with the projection of works of art [if the teacher intervenes].
- *Argument 3*: Do the class in two sessions.
- Support: 30 to 40 minutes is not enough to do the first part of the session; the time will depend on the interaction between the students; the ideal is always to do the lesson over two days; one day is not enough to work on everything; the time of the lessons will depend on the interaction between the students.
- *Warrant*: The first part will be on the first day, it is not possible to do more.
- *Claim*: The class will be conducted over two days.
- *Argument 4*: Take photos on the first day.
- *Backing*: A summary of the first day can be made by showing photos and videos of the first day; with photos it would be more interesting; photographs are used to review and comment.
- *Warrant*: The photos help you to remember what the students did on the first day.
- *Claim*: Take photos on the first day.
- *Argument 5*: Begin the second class with photos.
- *Backing*: Photos of the work will only be shown in class [for privacy]; students should not appear in the photos [for privacy].
- *Warrant*: A presentation with photos of what was done on the first day allows the students to revise and comment on the second day of the class.
- *Claim*: Begin lesson two by showing photos of what students did in lesson one.
- *Argument 6*: Create an exhibit of the students' work.

- *Backing*: Making an exhibition will please the students, as one teacher put up clotheslines with the works hanging from them, with great music; the exhibition can be made like a museum exhibition; questions can be asked so that the students can present their work at the exhibition.
- *Warrant*: The students will be happy to see the exhibition of their work.
- *Claim*: The second lesson will have the following structure: it will return to the previous activity; it will show works by famous artists; and it will end with an exhibition of the artworks assembled by the children.

Below is the diagram that organises the argumentation phase according to Toulmin's model (Figure 26).

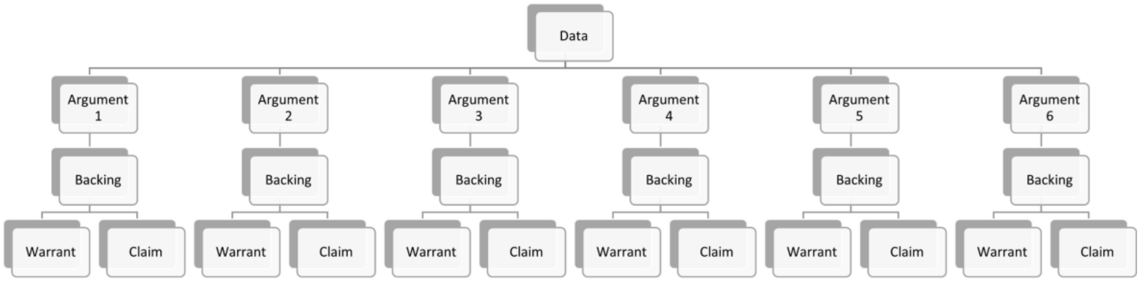

Figure 26. Organisation of the second episode of argumentation based on Toulmin's model.
Source: The authors.

The arguments from the critical discussion for starting the session with the logic blocks considered the option of the students having first contact with the blocks so that they could play, and then present artworks to the students with the idea of being able to influence them to create art with this material. The arguments for taking photos on the first day of class were mostly related to students remembering the artwork they made in the previous lesson. The arguments for the students to show their work relate to the data that they would be happy to see their work on display.

6.4.6. Analysis of the argumentation of the second episode based on the didactic suitability criteria

The first argument may be related to the interactional criteria, in the sense that the students explore the logic blocks autonomously, reinforcing the autonomy component, while the richness of the process component (the process of experimentation and exploration) of the epistemic criteria also emerges subtly. The second argument aims to complement the exploration of the logic blocks by proposing that while students explore the blocks works of art by different artists be projected without the intervention of the teacher, which increases the degree of suitability of the interactional criteria, in particular the autonomy component. The first proposal for creating art with logic blocks is related to the ecological criteria, in its component called interdisciplinary connections, by linking mathematics and art.

The third argument, which proposes doing the lesson in two sessions, is related to the time component of the mediational criteria. Arguments four and five are about taking photographs of classmates' work in the first session so that the students remember and comment on what they have done in the second session, which would improve the degree of the interactional criteria, in particular the interaction between the students and the cognitive criteria, by recalling the knowledge learned in the previous class. Finally, argument six deals with the creation of an exhibition of the student's work. This proposal would benefit the degree of suitability of the affective criteria, in particular the emotions component, by seeking to promote students' self-esteem and highlight the aesthetic qualities of mathematics.

6.5. Discussion and Final Considerations

This work aimed to investigate the Didactic Suitability Criteria that emerge in the argumentation episodes developed by future teachers, and participants in a Lesson Study cycle, when planning a research lesson on flat geometric shapes aimed at ECE students. To meet this objective, we first identified episodes of practical argumentation (Lewiński, 2018) and then analysed them using the theoretical models of Pragma-dialectics (van Eemeren & Grootendorst, 2004) and Toulmin's model (1958/2003).

Based on the analysis, our first conclusion is that we can identify several moments of practical argumentation in which DSC emerged. In fact, in both the first and second episodes, all six DSC appear. This result corroborates the origin of the construction of the DSC theoretical tool (Breda et al., 2018; Godino et al., 2023). Since DSC are the product of an existing consensus in the educational community, particularly in Mathematics Education, the probability of them appearing in the discourse of teachers or future teachers is quite high. However, depending on the context, the weight given to the use of one or the other set of criteria is different.

In the first episode, for example, the decision not to use the Tangram is strongly related to the cognitive criteria, since the material features flat geometric figures (such as parallelograms), which may be difficult to understand for some students of ECE. The same applies to the decision not to use a clock since learning how to tell time is difficult for ECE pupils. This result corroborates the studies by Hummes, Sol et al. (2022) since the weight of the decision by a group of teachers participating in a Lesson Study not to take irrational numbers into account when working on the Pythagorean Theorem was related to cognitive criteria. On the other hand, the decision to use logic blocks had an argumentative weight related to mediational criteria, when it was argued that the material is viable because it is available for use and because there is a book by Ursula Simon with 150 activities for working with logic blocks. In addition, important weight was given to epistemic criteria, since the logic blocks are a versatile resource that favours the introduction of good problem situations, languages, procedures, and arguments.

In the second episode, for example, the weight given to the argument related to the epistemic criteria for defining the dynamics of the lesson (playing with the blocks, assembling works of art with the blocks, photographing the work and exhibiting the works of art built by the students) was low (only the process of exploring the manipulative material appears). On the other hand, more weight was given to interactional criteria, particularly the autonomy component since this sequence of activities would encourage student autonomy. In addition, it is mentioned that the dynamics could foster positive emotions in students (affective criteria) since they could relate mathematics to art (ecological criteria).

We could say that prospective teachers who oppose the implementation of a particular action mostly use certain Didactic Suitability Criteria, while teachers who are in favour of its implementation generally base their positions on other DSC (Ramos, 2006). Furthermore, this conflict between principles that should guide practice is resolved through a consensus that depends on the context, because the participants consider the different DSC that are in conflict to be more or less relevant based on the context.

We understand that this type of analysis only happens because the Lesson Study facilitates collective argumentation, since practical argumentation in initial training when influenced by the student teachers' previous school experiences and restricted to supervised internship experiences, can be enhanced in the context of the LS (Richit, 2020). Therefore, the analysis of practical argumentation in initial teacher training, considering the context of our study, characterized a mode of reasoning situated in a particular context, influenced by the student teachers' previous experiences, which underpinned the decisions and actions relating to the planned lesson (Gómez, 2017).

The analysis showed that the practical argumentation of the future teachers when making didactic decisions during the planning of a research lesson for ECE students favoured the development of aspects of teaching at this school level, such as the importance of adapting the materials used and classroom management to the objective of the proposed activity. This result corroborates research carried out by Amâncio and Zaidan (2023), which identified that a training process carried out with future teachers, in a particular context, can establish links with the fundamental principles of Lesson Study, such as the participants' choice of topic and teaching material, consideration of the active participation of students and their learning, as well as spaces for collective reflection.

One limitation of this study is that we only analysed two episodes of argumentation. However, we believe that the conclusions can go beyond these specific cases. The results of this study are a relevant contribution to research on the role of Didactic Suitability Criteria in the reflection of teachers (among other studies, Breda, 2020) and future teachers (Breda, Seckel et al., 2021) since this study provides a tool that allows us to show, in detail, the structure of practical arguments and the important role of DSC in these arguments.

A possible line of development for this research is to investigate teachers' knowledge and beliefs that are inferred from practical argumentation. It could be interesting to study the possible relationship between the data, backing, warrants, and some aspects of the didactic dimension of teacher knowledge (Breda et al., 2017; Pino-Fan & Godino, 2015).

Additional statements:

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Chapter 7: Conclusions

This chapter presents the conclusions and final reflections of the present doctoral thesis. The first section describes the conclusions regarding the specific objectives proposed; the second section explains the main contributions of this doctoral research; the third section details the limitations of the study and future lines of research; and the fourth section presents the instances of dissemination of the results of this doctoral thesis.

6.6. Conclusions Regarding the Objectives

In this doctoral thesis, research focused on reflection on teaching practice was presented, which was developed in different training contexts and countries, with prospective and practising mathematics teacher and teacher educators. The analyses were focused on practical argumentation and its relationship with the Didactic Suitability Criteria construct, addressing the relationships that can be established between both references in teaching reflection. The different publications presented in this doctoral thesis have allowed to show these relationships between Didactic Suitability Criteria and practical argumentation, which varied depending on the contexts and implementation of the analyses, the knowledge and use of these Criteria by the participating teachers, and the focus of the training instance for mathematics teachers that was reported.

In general terms, the objective proposed for this doctoral research was as follows:

> To investigate the practical argumentation of mathematics teachers (prospective and practising teachers) and teacher educators during their reflection on the design, implementation, and redesign of didactic sequences and their relationship with the Didactic Suitability Criteria

To achieve this general objective, four specific objectives were proposed, the conclusions of which are described in the following subsections, highlighting the relationship between the proposed objectives and the most relevant approaches of each publication that has addressed them.

6.6.1. On the fulfilment of the first specific objective

The first specific objective focused on investigating the practical argumentation of teacher educators, experts in the use of the Didactic Suitability Criteria, during the reflection on the design of a didactic unit in a Lesson Study cycle. This objective was fundamental to analyse the practical argumentation when the teachers participating in a Lesson Study cycle have a bread knowledge and make explicit use of the Didactic Suitability Criteria in the reflections. The fulfilment of this first specific objective was achieved through the publication of the article presented in Chapter 2.

A first aspect to highlight in this article is its methodology for analysing the practical argumentation and its relationship with the Didactic Suitability Criteria. The assessment of the design of the didactic unit on functions reported, guided by the complete guidelines of criteria, components, and indicators, made it possible to identify the practical arguments that emerged from the reflection of the participating teacher educators. Once these practical arguments were identified, Toulmin's model (1958/2003) played a relevant role, first, in structuring these arguments and, second, in relating their elements to the Didactic Suitability Criteria.

A second aspect to highlight in this article is that its results revealed that the components of this construct allowed the identification of elements of the didactic unit that functioned as *data* in the practical arguments, white at the same acting as *backings*, since the participating teacher educators had a broad knowledge of the complete guidelines of Didactic Suitability Criteria. Finally, this construct facilitated the creation of specific *warrants* to propose an action in the *claim*.

In conclusion, practical argumentation was related to Didactic Suitability Criteria, specifically, during the assessment guided by the components of this construct, since it favoured the emergence of practical argumentation. Furthermore, by representing the practical arguments with Toulmin's model, it was observed that the components of the Didactic Suitability Criteria were directly related to the elements of this model.

6.6.2. On the fulfilment of the second specific objective

The second specific objective focused on investigating the practical argumentation of prospective mathematics teachers, familiar with the Didactic Suitability Criteria, during the systematised reflection on a didactic unit implemented in their educational internship experiences. This objective was fundamental to analyse the practical argumentation of prospective teachers when they use the Didactic Suitability Criteria, having an initial knowledge of this construct, but without being experts in its use, to reflect on their practice and justify their actions in a teaching and learning process (educational internship experiences). The fulfilment of this second specific objective was achieved through the publication of the article presented in Chapter 3.

A first aspect to highlight in this article is its methodology for analysing the practical argumentation and its relationship with the Didactic Suitability Criteria. Because in this research data were collected from two sources, namely, video recording of group reflection sessions of the study subject with his MFP tutor professors and the MFP itself, two additional tools were used to structure his practical argumentation. On one hand, the ideal model of Pragma-dialectics to analyse the group reflection sessions and, on the other hand, the diagramming technique to analyse the study subject's MFP.

A second aspect to highlight in this article is that its results revealed that practical argumentation was fostered because, by considering the complete guidelines of Didactic Suitability Criteria, the prospective teacher had several aspects on which to reflect and formulate proposals for actions to improve his mathematical teaching and learning process. The change in beliefs about modelling that could be seen in the study subject was in line with the approach of Didactic Suitability Criteria and of practical argumentation from a normative perspective, since with both references, improvements are sought in the teaching and learning process, promoting changes that benefit said processes.

In conclusion, practical argumentation was related to Didactic Suitability Criteria through, on one hand, the ideal model of Pragma-dialectics and the structuring of his written reflection with the diagramming technique. This double analysis of the practical arguments of a prospective teacher allowed not only to delve deeper into the reasons underlying his actions, but also into his knowledge and beliefs about the modelling process when he reflected on its implementation in the classroom.

6.6.3. On the fulfilment of the third specific objective

The third specific objective focused on investigating the practical argumentation of practising mathematics teachers during their reflection in training courses on Lesson Study and Didactic Suitability Criteria. This objective was fundamental to analyse how the use of the Didactic Suitability Criteria, learned during one of the stages of a Lesson Study cycle, affects the practical argumentation of the teachers participating in these training courses, comparing their reflection before and learning about this construct. The fulfilment of this third specific objective was achieved through the publication of the book chapter presented in Chapter 5 and was complemented with the results of the manuscript presented in Chapter 4.

A first aspect to highlight in this book chapter is its methodology for relating the practical argumentation with the Didactic Suitability Criteria. First, the participating teachers were asked to use this construct to assess the implemented lessons, considering the moments in which they estimated that the suitability of the lesson had been benefitted or diminished, and making proposals for improvement. Second, these assessments were analysed by adapted the methodology proposed by Ramos (2006), based on argumentative trajectories, in order to identify and describe the episodes of argumentation in the assessments of the lessons. Finally, these trajectories were described using the Didactic Suitability Criteria to explain what happened.

A second aspect to highlight in this book chapter is that its results revealed a broader reflection on practice after teaching the Didactic Suitability Criteria to the participating teachers, where their practical argumentation included different types of actions, such as modifying, eliminating, and proposing a new action.

In conclusion, practical argumentation was related to Didactic Suitability Criteria when, in the assessment of the implemented lesson, guided by the criteria and components of this construct, more practical arguments arose when elements that reduced the degree of suitability of the lesson were identified (seeking to avoid them) than when elements that increased it were identified.

Regarding the methodology for analysing the practical argumentation and its relationship with the Didactic Suitability Criteria proposed in the manuscript, an adaptation of Toulmin's model was made in which these Criteria nuanced an element of the model to define a specific action in the *claim*; that is, they acted as principles to select an action within the practical argumentation of the participating teachers. However, this manuscript is in the process of evaluation, so no definitive conclusions are raised about this proposal.

6.6.4. On the fulfilment of the fourth specific objective

The fourth specific objective focused on investigating the emerging Didactic Suitability Criteria in the practical argumentation of prospective teachers during their reflection in a

Lesson Study cycle. This objective was fundamental to understand how the Didactic Suitability Criteria emerge implicitly in the practical argumentation of prospective teachers who, unlike the participants in the other cases analysed, do not have knowledge of this construct. The fulfilment of this fourth specific objective was achieved through the publication of the article presented in Chapter 6.

A first aspect to highlight in this article is its methodology for analysing the practical argumentation and its relationship with the Didactic Suitability Criteria. In this context, the ideal model of Pragma-dialectics and Toulmin's model were used to structure and represent the practical arguments of the participating teachers and, subsequently, to relate them to the Didactic Suitability Criteria.

A second aspect to highlight in this article is that its results revealed that, in the group reflections, it was possible to identify instances of practical argumentation in which the Didactic Suitability Criteria emerged implicitly. In this sense, the three criteria with the highest presence were the cognitive (to rule out some proposals for the use of didactic materials), mediational (to select the appropriate didactic material for the lesson), and interactional criteria (to promote student autonomy).

In conclusion, practical argumentation was related to Didactic Suitability Criteria in the following way: since the Lesson Study cycle fostered reflection on practice in the prospective teachers, practical arguments emerged in their discussions, where the Didactic Suitability Criteria appeared implicitly; by representing these practical arguments with Toulmin's model, some elements of this model could be associated with these Criteria.

6.7. Contributions of This Doctoral Research to Mathematics Education

The research reported in this doctoral thesis proposes three contributions to Mathematics Education, specifically, in the context of teacher education and reflection on teaching practice, which have been reflected both in the publications presented and in the instances of dissemination of this research (detailed in the last section of this chapter).

The first contribution of this research is the deepening of the analysis of reflection on teaching practice by establishing a relationship between the Didactic Suitability Criteria construct and practical argumentation. The relationships described in the previous section allow to highlight and reformulate a phenomenon previously described in the literature on this construct, in which the components of the Didactic Suitability Criteria act as regularities in the teachers' discourse when they assess a classroom episode or justify that a didactic proposal can be improved, even without formally knowing this tool. In terms of practical argumentation, this phenomenon can be interpreted as the components of the Didactic Suitability Criteria emerging – implicitly or explicitly – in the practical arguments that arise during teachers' reflections when evaluating a classroom episode or the choice of a particular didactic material, without them consciously using this construct.

The second contribution of this research is the application of different approaches and tools to study practical argumentation and its relationship with the Didactic Suitability Criteria, such as Toulmin's model (1958/2003), the diagramming technique (Guevara, 2011), the ideal model of Pragma-dialectics (van Eemeren & Grootendorst, 2004), and the argumentative trajectory (Ramos, 2006). Although the procedures for conducting these analyses varied depending on the context in which they were developed, they all followed three fundamental steps:

1) Identifying a practical argument (using the ideal model of Pragma-dialectics or argumentative trajectory), which implies recognising a proposal for action for a teaching and learning process.
2) Structuring and representing practical arguments (using Toulmin's model or the diagramming technique).
3) Relating the practical arguments to the Didactic Suitability Criteria.

The third contribution of this research consists of the adaptation and implementation of a course based on the Lesson Study methodology and the Didactic Suitability Criteria with the aim of promoting reflection on teaching practice by the participating teachers and studying practical argumentation by the researchers. It is worth noticing that, in the contexts where a lesson was designed and its implementation was assessed, the reflection guided by the Didactic Suitability Criteria fostered reflection on the teaching practice of the participating teachers and made their argumentation richer and broader.

6.8. Limitations of the Study and Future Research Lines

Based on the research reported in this doctoral thesis, some limitations were identified. The first limitation identified is that, in the contexts analysed, only a few episodes of argumentation were considered. This means that, although the most relevant episodes were selected to make evident the relationship between Didactic Suitability Criteria and practical argumentation, a broader analysis, including a larger number of episodes or some less relevant ones, could provide new nuances or deepen the relationships already identified between both references.

The second limitation identified point to the diversity of mathematics teacher education contexts in which the analyses were conducted, which included early childhood education in Brazil, secondary education in Spain, and baccalaureate education in Mexico. Although reflection on teaching practice and the search for relationships between Didactic Suitability Criteria and practical argumentation were relevant at all these educational levels, a more detailed study at a more specific educational level could enrich the understanding of this phenomenon.

The third limitation, derived from the previous one, is that there was no figure who actively and consciously promoted practical argumentation in the teacher training contexts in which the analyses were conducted, that is, someone who questioned and delved deeper into the proposals for action that emerged during the reflections of the participating teachers. This lack resulted in some of the proposed actions not being justified (by those who proposed them) or discussed in depth (by the rest of the participants), which limited the possibility of structuring some practical arguments in a more complete manner or of analysing certain episodes in greater depth.

A final limitation is that, as stated at the beginning of the publications presented in Chapters 4 and 6, there was no participation in the design and implementation of the Lesson Study cycles conducted in Brazil (unlike those conducted in Spain and Mexico). This lack of involvement implied certain restrictions when analysing the practical arguments and identifying the implicit emergence of the Didactic Suitability Criteria in the reflections of the participating teachers.

In addition to the limitations described above, some future research lines. The first research line proposed is based on the fact that the study of practical argumentation offers a fertile field for developing a research agenda for reflection on teaching practice in mathematics teacher education. In this sense, one of the avenues to explore would be the inference of teachers' knowledge and beliefs through the analysis of the practical arguments generated during their reflection on teaching practice, taking as a possible starting point the teacher's Didactic-Mathematical Knowledge and Competences model (Breda et al., 2017; Godino et al., 2017; Pino-Fan et al., 2015, 2017).

The second research line proposed aims to develop teacher training instances that use, in a balanced way, all the components of the Didactic Suitability Criteria and where practical argumentation is fostered. This would lead to refining the proposals for courses that integrate the use of the Lesson Study methodology with the teaching and application of the Didactic Suitability Criteria, so that this construct promotes a comprehensive reflection and a detailed promotion and analysis of the practical argumentation of the participating teachers. Regarding the latter, it is important that whoever moderates these sessions is able to actively foster practical argumentation, encouraging justifications and discussions of the actions proposed by the participating teachers.

A final research line proposed would be the analysis of the levels of practical argumentation in reflections on teaching practice. This could be achieved by exploring the structure of the arguments using Toulmin's model or the diagramming technique, seeking coherence between the proposed actions and the reasons given, and the type of actions suggested (incorporation of new actions, improvements to previous actions, or elimination of actions taken). In this way, these levels could act as indicators to promote deeper, more detailed reflection that covers more aspects of teaching practice.

6.9. Difusión de Resultados

Del presente trabajo de investigación se han derivado las siguientes publicaciones en revistas, participación en congresos y actas de congresos.

6.9.1. Publicaciones en revistas

- Ledezma, C., Sol, T., Sala-Sebastià, G., y Font, V. (2022). Knowledge and beliefs on mathematical modelling inferred in the argumentation of a prospective teacher when reflecting on the incorporation of this process in his lessons. *Mathematics, 10*(18), Artículo 3339. https://doi.org/10.3390/math10183339
- Sol, T., Breda, A., Richit, A., y Sala-Sebastià, G. (2024). Didactic suitability criteria that emerge in the future teachers' practical argumentation. *Zetetiké, 32*, Artículo e024004. https://doi.org/https://doi.org/10.20396/zet.v32i00.8676234

- Sol, T., Ledezma, C., Breda, A., y Font, V. (en prensa). Criterios de idoneidad didáctica en la argumentación práctica de formadores de profesores de matemática. *Uniciencia*.

6.9.2. Publicaciones en actas de congresos

- Hummes, V., Sol, T., y Breda, A. (2021). Argumentación práctica sobre la enseñanza del Teorema de Pitágoras realizada por profesores de matemáticas en un curso de formación. En P. D. Diago, D. F. Yáñez, M. T. González-Astudillo, y D. Carrillo (Eds.), *Investigación en Educación Matemática XXIV* (pp. 343–350). SEIEM.
- Hummes, V., Sol, T., Sánchez, A., Breda, A., y Font, V. (2022). Irrational numbers in the teaching of the Pythagorean theorem: Practical argumentation in a Lesson Study and Didactic Suitability training course. En J. Hodgen, E. Geraniou, G. Bolondi, y F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of European Research in Mathematics Education (CERME12)* (pp. 3402–3409). Free University of Bozen-Bolzano; ERME.
- Sol, T., Ledezma, C., Sala-Sebastià, G., y Font, V. (2022). Conocimientos y creencias sobre modelización matemática en la argumentación reflexiva de un futuro profesor. En A. Rodrigues, A. Domingos, H. Martins, L. Serrazina, y P. Teixeira (Eds.), *Investigação em Educação Matemática 2022: Desenvolvimento Curricular* (pp. 308–320). SPIEM.
- Sol, T., Sánchez, A., Breda, A., Font, V., y Hummes, V. (2022). Análisis de la reflexión de futuros profesores de matemáticas sobre los errores que cometen en su práctica docente. En T. F. Blanco, C. Núñez-García, M. C. Cañadas, y J. A. González-Calero (Eds.), *Investigación en Educación Matemática XXV* (pp. 549–557). SEIEM.
- Sala-Sebastià, G., Breda, A., Sol, T., Sánchez, A., Inglada, N., y Cortés, A. (2023). ¿Tareas abiertas o guiadas? Emergencia de los criterios de idoneidad didáctica en la argumentación práctica de formadores de profesores en un ciclo de Estudio de Clases. En A. Richit, D. Fiorentini, y R. d. S. P. Neves (Eds.), *Atas do II Seminário Internacional de Lesson Study No Ensino de Matemática* (pp. 259–269). Editora da Universidade Federal da Fronteira Sul.

- Sol, T., Breda, A., Sánchez, A., Font, V., Díez-Palomar, J., y Richit, A. (2023). Critérios de Adequação Didática evidenciados na argumentação prática de professores de matemática: Uma experiência de Estudo de Aula sobre o ensino de funções. En A. Richit, D. Fiorentini, y R. d. S. P. Neves (Eds.), *Atas do II Seminário Internacional de Lesson Study No Ensino de Matemática* (pp. 148–158). Editora da Universidade Federal da Fronteira Sul.
- Sol, T., Breda, A., Sánchez, A., Font, V., y Sala-Sebastià, G. (2023). Criterios de idoneidad didáctica en la argumentación práctica docente al seleccionar problemas para introducir el tema de función. En C. Jiménez-Gestal, Á. Magreñán, E. Badillo, y P. Ivars (Eds.), *Investigación en Educación Matemática XXVI* (pp. 515–522). SEIEM.
- Sol, T., Sánchez, A., Breda, A., y Font, V. (2023). Didactic suitability criteria in teachers' practical argumentation in the phase of design of a lesson study cycle about functions. En M. Ayalon, B. Koichu, R. Leikin, L. Rubel, y M. Tabach (Eds.), *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 227–234). PME.
- Sol, T., Lujambio, M., Lara, N., y Font, V. (2023). Reflexión docente sobre la enseñanza de la introducción a la derivada en un proceso de instrucción. En S. Caviedes, J. G. Lugo-Armenta, L. R. Pino-Fan, y A. Sánchez (Eds.), *Actas del Primer Congreso Internacional de Didáctica de la Matemática* (pp. 197–205). Universidad de Los Lagos.
- Sol, T., Breda, A., Font, V., y Sala-Sebastià, G. (2024). Argumentación práctica sobre ambigüedades en la valoración del diseño de una unidad didáctica sobre funciones. En N. Adamuz-Povedano, E. Fernández-Ahumada, N. Climent, y C. Jiménez (Eds.), *Investigación en Educación Matemática XXVII* (pp. 497–504). SEIEM.

6.9.3. Presentaciones en eventos académicos

- *II Seminário Internacional de Educação em Ciências, Educação Matemática e tecnologias Educativas (SIECEMTE)*. Título: Criterios de idoneidad didáctica en la argumentación práctica de formadores de profesores de matemática. Fecha: 18 al 20 de septiembre 2024.

- *II Seminário Internacional de Educação em Ciências, Educação Matemática e tecnologias Educativas (SIECEMTE)*. Título: Critérios de adequação didática emergentes na argumentação prática de futuros pedagogos em um ciclo de Estudos de Aula. Fecha: 18 al 20 de septiembre 2024.
- *Investigación en Educación Matemática XXVII – SEIEM, 2024.* Título: Argumentación práctica sobre ambigüedades en la valoración del diseño de una unidad didáctica sobre funciones. Fecha: del 4 al 6 de septiembre 2024.
- *CIMIE24 – XII Congreso Internacional Multidisciplinar de Investigación Educativa.* Título: Análisis de la argumentación práctica docente con los criterios de idoneidad en un Lesson Study. Fecha: 4 al 5 de julio de 2024.
- *Investigación en Educación Matemática XXVI – SEIEM, 2023*. Título: Criterios de idoneidad didáctica en la argumentación práctica docente al seleccionar problemas para introducir el tema de función. Fecha: 6 al 8 de septiembre de 2023.
- *Simposio en Educación Matemática Primaria y Secundaria*. Título: Argumentación práctica en la valoración de una unidad didáctica sobre funciones diseñada en un ciclo de *Lesson Study*. Fecha: 26 y 27 de octubre de 2023.
- *Reunión Latinoamericana de Matemática Educativa – RELME 36*. Título: Criterios de idoneidad didáctica en la argumentación práctica de docentes sobre el diseño de actividades para de clase sobre función en un ciclo de *Lesson Study*. Fecha: 24 al 28 de julio 2023.
- *46th Conference of the International Group for the Psychology of Mathematics Education (PME)*. Título: Didactic suitability criteria in teachers' practical argumentation in the phase of design of a lesson study cycle about functions. Fecha: 16 al 21 de julio de 2023.
- *II Seminario Internacional de Lesson Study en Enseñanza de Matemáticas (II SILSEM)*. Título: ¿Tareas abiertas o guiadas? Emergencia de los criterios de idoneidad didáctica en la argumentación práctica de formadores de profesores en un ciclo de Estudio de Clases. Fecha: 17 al 19 de mayo de 2023.
- *II Seminario Internacional de Lesson Study en Enseñanza de Matemáticas (II SILSEM)*. Título: Critérios de Adequação Didática evidenciados na argumentação prática de professores de matemática: uma experiência de Estudo de Aula sobre o ensino de funções. Fecha: 17 al 19 de mayo de 2023.

- *Encontro de Investigação em Educação Matemática. (EIEM 2022).* Título: Conocimientos y creencias sobre modelización matemática en la argumentación reflexiva de un futuro profesor. Fecha: 19 y 20 de noviembre de 2022.
- *Investigación en Educación Matemática XXV – SEIEM, 2022*. Título: Análisis de la reflexión de futuros profesores de matemáticas sobre los errores que cometen en su práctica docente. Fecha: 1 al 3 de septiembre de 2022.
- *Reunión Latinoamericana de Matemática Educativa – RELME 35*. Título: Argumentación práctica sobre la enseñanza de introducción a la derivada en un curso de formación docente. Fecha: 3 al 8 de julio 2022.
- *Reunión Latinoamericana de Matemática Educativa – RELME 35*. Título: Experiencia sobre el diseño e implementación de una clase de introducción a la derivada en un curso de formación docente que combina el *lesson study* y la idoneidad didáctica. Fecha: 3 al 8 de julio de 2022.
- *Primer Congreso Internacional de Didáctica de la Matemática – CIDIDMAT*. Título: Reflexión docente sobre la enseñanza de la introducción a la derivada en un proceso de instrucción. Fecha: 26 de febrero al 26 de marzo de 2022.
- *Congress of the European Society for Research in Mathematics Education (CERME12)*. Título: Irrational numbers in the teaching of the Pythagorean theorem: practical argumentation in a Lesson Study and Didactic Suitability training course. Fecha: 2 al 6 de febrero de 2022.
- *Investigación en Educación Matemática XXIV – SEIEM, 2021*. Título: Argumentación práctica sobre la enseñanza del teorema de Pitágoras realizada por profesores de matemáticas en un curso de formación. Fecha: 8 al 10 de septiembre de 2021.

6.9.4. Presentación de conferencias, talleres, y jornadas

- Cátedra Doctoral en Educación Matemática 2024-II, realizada por la Universidad Pedagógica Nacional, Facultad de Ciencia y Tecnología, Departamento de Matemáticas. Colombia. Título: Argumentación del profesor de matemáticas: algunas perspectivas y dimensiones. 17 de septiembre de 2024.
- Seminario Multidisciplinario de divulgación científica de vanguardia. Facultad de Ciencias, Universidad Autónoma del Estado de México, México. Conferencia

magistral: Aplicación de la argumentación práctica y la idoneidad didáctica en la formación docente. 23 de agosto de 2024.

- Seminario de Doctorado del grupo de investigación en la Universidad de Barcelona, España. Título de la conferencia: Argumentación práctica sobre ambigüedades en la valoración del diseño de una unidad didáctica. 19 de abril de 2024.
- Seminario Latinoamericano de Colaboración sobre el Enfoque Onto-Semiótico. Universidad de los Lagos, Chile. Título de la conferencia: Argumentación práctica en cursos de formación docente que combinan los Criterios de Idoneidad Didáctica y el *Lesson Study.* 12 de abril de 2023.
- Actividades formativas transversales del Doctorado en Filología Española del Departamento de Filología Española de la Universidad Autónoma de Barcelona, España. Seminario: Introducción a la argumentación. 10 y 12 de mayo de 2022.
- Grupo de Estudio sobre Argumentación en Educación Matemática – GEAEM de Colombia. Conferencia: Argumentación práctica en un curso de formación de profesores de matemáticas que combina *Lesson Study* y Criterios de Idoneidad Didáctica. 29 de julio de 2021.
- Clase en Máster, Departamento de Matemáticas 'Federico Enriques', Universidad de Milán, Italia Título: Didactic Suitability Criteria. 11 de octubre de 2024.
- Grupo de investigación de Didáctica de Matemáticas. Departamento de Matemáticas 'Federico Enriques', Universidad de Milán, Italia. Título de la conferencia: Practical Argumentation and the Didactical Suitability Criteria in Mathematics Teacher Education. 29 de octubre de 2024.

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Anexo: Carta de aceptación del artículo

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CONSTANCIA DE ACEPTACIÓN
A QUIEN INTERESE

Por medio de la presente, en calidad de director de la Revista Uniciencia certifico que el artículo: Criterios de idoneidad didáctica en la argumentación práctica de formadores de profesores de matemática de los autores Telesforo Sol, Carlos Ledezma, Adriana Breda y Vicenç Font fue valorado favorablemente por nuestro comité evaluador (dos pares en revisión simple ciega) y fue aceptado por el comité editor de nuestra revista para publicación.

El artículo sería publicado en nuestro sitio web en los próximos meses, según se atiendan las correcciones que surgirán en los siguientes procesos editoriales.

Atentamente,

YURI MORALES LOPEZ (FIRMA)
PERSONA FISICA, CPF-01-1025-0819.
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