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Stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$

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We consider the Cauchy problem for a stochastic delay differential equation driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. We prove an existence and uniqueness result for this problem, when the coefficients are sufficiently regular. Furthermore, if the diffusion coefficient is bounded away from zero and the coefficients are smooth functions with bounded derivatives of all orders, we prove that the law of the solution admits a smooth density with respect to Lebesgue measure on $\mathbb{R}$.

Keywords: fractional Brownian motion; stochastic delay differential equation

1. Introduction

A general theory for stochastic differential equations (SDEs) driven by a fractional Brownian motion (fBm) has not yet been established and in fact only a few results have been proved (see for example, Nualart and Rascanu 2002; Nualart and Ouknine 2003; Coutin and Qian 2000) for various approaches to the problem. Moreover, even the definition of stochastic integration with respect to fBm is not yet completely established. Several approaches have been proposed in recent years (see, for example, Alòs and Nualart 2003; Carmona and Coutin 2000; Coutin and Qian 2002). Given the rudimentary status of the general theory, an attempt to study the class of stochastic delay differential equations (SDDEs) might appear somewhat rash. Indeed, these equations are typically an infinite-dimensional problem, and so are much more difficult to solve than the usual SDEs. Nevertheless, they also include some problems that are more easily solved than SDEs, but which are still of great interest in applications, for example, to finance (see Arriojas *et al.* 2003; and, more generally, Hobson and Rogers 1998).

In the present paper we shall consider the Cauchy problem for an SDDE

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t-r))\mathrm{d}B(t), \qquad t \in [0, T],$$

$X(s) = \phi(s), s \in [-r, 0]$, where $\phi \in C([-r, 0])$ and the noise process $\{B(t), t \geqslant 0\}$ represents an fBm with Hurst parameter $H > \frac{1}{2}$. As a solution to this problem, we shall define a process $\{X(t), t \in [-r, T]\}$ satisfying

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$$X(t) = \phi(0) + \int_0^t b(X(s))\mathrm{d}s + \int_0^t \sigma(X(s-r))\mathrm{d}B(s), \qquad t \in [0, T],$$
$$X(s) = \phi(s), \qquad s \in [-r, 0],$$

where $\phi \in C([-r, 0])$. The stochastic integral that appears in this equation is the Stratonovich integral with respect to the fBm, introduced by Russo and Vallois (1993) (see Definition 1 below). Nevertheless, all the results presented in this paper can also be obtained via Skorohod integrals. To this class of problems belongs, for example, the Langevin equation with fBm, whose solution is the fractional Ornstein–Uhlenbeck process thoroughly studied in Cheridito *et al.* (2003) – since the diffusion coefficient is constant, the equations with and without delay coincide.

To obtain the solution, we shall first solve the equation within the interval $[0, r]$; then we use this solution process as the initial data to solve the equation within the interval $[r, 2r]$, and so on. This procedure allows us to construct a solution step by step, demonstrating at each stage its uniqueness and its regularity. With the same approach, we can prove, under the customary assumption of non-degeneracy of the diffusion coefficient, that the law of the solution at any time t admits a density with respect to Lebesgue measure on $\mathbb{R}$. Note that we use the classical techniques of stochastic calculus combined with some special properties of fBm. The delay allows us to use classical methods – such as Picard's iterations – and to avoid some of the usual problems connected with fBm.

The paper is organized as follows. In the next section we introduce the basic notation for fBm and recall some results from Nualart (2003) and Alòs and Nualart (2003). Section 3 is devoted to studying the existence of a unique solution to our SDDE driven by fBm. Finally, in Section 4, we obtain the smoothness of the density of the solution.

2. Fractional Brownian motion

Let us start with some basic facts about fractional Brownian motion and the stochastic calculus that can be developed with respect to this process.

Fix a parameter $\frac{1}{2} < H < 1$. The fBm of Hurst parameter H is a centred Gaussian process $B = \{B(t), t \in [0, T]\}$ with the covariance function

$$R(t, s) = \tfrac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

Let us assume that B is defined on a complete probability space $(\Omega, \mathcal{F}, P)$. One can show (see, for example, Alòs and Nualart 2003) that

$$R(t, s) = \int_0^{t\wedge s} K(t, r)K(s, r)\mathrm{d}r, \tag{1}$$

where $K(t, s)$ is the kernel defined by

$$K(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (r-s)^{H-\frac{3}{2}} r^{H-\frac{1}{2}} \mathrm{d}r,$$

for $s < t$, where

$$c_H = \left[\frac{H(2H-1)}{B(2-2H,\, H-\frac{1}{2})}\right]^{1/2}$$

and $B(\alpha, \beta)$ is the beta function. We assume that $K(t, s) = 0$ if $s > t$. It is worth noting that equation (1) implies that R is non-negative definite and, therefore, there exists a Gaussian process with this covariance.

Let us denote by $\mathcal{E}$ the set of step functions on $[0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}} = R(t, s).$$

One can show that

$$R(t, s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2}\, \mathrm{d}u\, \mathrm{d}r,$$

where $\alpha_H = H(2H-1)$. This implies that

$$\langle \varphi, \psi\rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi(r)\psi(u)\mathrm{d}u\, \mathrm{d}r,$$

for all φ and ψ in $\mathcal{E}$. The mapping $\mathbf{1}_{[0,t]} \to B(t)$ can be extended to an isometry between $\mathcal{H}$ and the first chaos H_1 associated with B. We denote this isometry by $\varphi \to B(\varphi)$. The elements of $\mathcal{H}$ may not be functions, but just distributions of negative order. Hence it is convenient to introduce the Banach space $|\mathcal{H}|$ of measurable functions φ on $[0, T]$ satisfying

$$\|\varphi\|^2_{|\mathcal{H}|} := \alpha \int_0^T \int_0^T |\varphi(r)|\, |\varphi(u)|\, |r-u|^{2H-2}\, \mathrm{d}r\, \mathrm{d}u < \infty.$$

One can prove (Pipiras and Taqqu 2000) that the space $|\mathcal{H}|$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{H}}$ is not complete. It is isometric to a subspace of $\mathcal{H}$, which we will therefore identify with $|\mathcal{H}|$. The continuous embedding $L^{1/H}([0, T]) \subset |\mathcal{H}|$ has been proved in Mémin *et al.* (2001).

2.1. Malliavin calculus and stochastic integrals for fractional Brownian motion

To construct a stochastic calculus of variations with respect to the Gaussian process B, we shall follow the general approach introduced by Nualart (1995), among others.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)), \tag{2}$$

where $n \geqslant 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and its partial derivatives of all orders are bounded), and $\phi_i \in \mathcal{H}$. The derivative operator D of a smooth and cylindrical random variable F of the form (2) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(B(\phi_1), \ldots, B(\phi_n))\phi_j.$$

The derivative operator D is then a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geqslant 1$. For any $k \geqslant 1$, let D^k be the kth iteration of the derivative operator. For any $p \geqslant 1$, the Sobolev space $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = \mathrm{E}|F|^p + \mathrm{E}\left(\sum_{i=1}^{k} \|D^i F\|_{\mathcal{H}^{\otimes i}}^p\right).$$

Proceeding as before, given a Hilbert space V, we denote by $\mathbb{D}^{1,p}(V)$ the corresponding Sobolev space of V-valued random variables.

The divergence operator δ is the adjoint of the derivative operator, defined by means of the duality relationship

$$\mathrm{E}(F\delta(u)) = \mathrm{E}\langle DF, u\rangle_{\mathcal{H}},$$

where u is a random variable in $L^2(\Omega; \mathcal{H})$. We say that u belongs to the domain of the operator δ, denoted by Dom δ, if the mapping $F \mapsto \mathrm{E}\langle DF, u\rangle_{\mathcal{H}}$ is continuous in $L^2(\Omega)$. A basic result says that the space $\mathbb{D}^{1,2}(\mathcal{H})$ is included in Dom δ.

Two basic properties of the divergence operator are the following:

(i) For any $u \in \mathbb{D}^{1,2}(\mathcal{H})$,

$$\mathrm{E}\delta(u)^2 = \mathrm{E}\|u\|_{\mathcal{H}}^2 + \mathrm{E}\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}},$$

where $(Du)^*$ is the adjoint of (Du) in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

(ii) For any $u \in \mathbb{D}^{2,2}(\mathcal{H})$, $\delta(u)$ belongs to $\mathbb{D}^{1,2}$, and for any h in $\mathcal{H}$,

$$\langle D\delta(u), h\rangle_{\mathcal{H}} = \delta(\langle Du, h\rangle_{\mathcal{H}}) + \langle u, h\rangle_{\mathcal{H}}.$$

Let us now consider the space $|\mathcal{H}| \otimes |\mathcal{H}| \subset \mathcal{H} \otimes \mathcal{H}$ of measurable functions φ on $[0, T]^2$ such that

$$\|\varphi\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 := \alpha_H^2 \int_{[0,T]^4} |\varphi(r, s)|\,|\varphi(r', s')|\,|r - r'|^{2H-2}\,|s - s'|^{2H-2}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}r'\,\mathrm{d}s' < \infty.$$

Let us denote by $\mathbb{D}^{1,2}(|\mathcal{H}|)$ the space of processes u such that

$$\mathrm{E}\|u\|_{|\mathcal{H}|}^2 + \mathrm{E}\|Du\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 < \infty.$$

Then $\mathbb{D}^{1,2}(|\mathcal{H}|)$ is included in $\mathbb{D}^{1,2}(\mathcal{H})$, and for a process u in $\mathbb{D}^{1,2}(|\mathcal{H}|)$ we can write

$$\mathrm{E}\|u\|_{\mathcal{H}}^2 = \alpha_H \int_{[0,T]^2} u(s)u(r)|r - s|^{2H-2}\,\mathrm{d}r\,\mathrm{d}s$$

and

$$\mathrm{E}\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}} = \alpha_H^2 \int_{[0,T]^4} D_r u(s) D_{r'} u(s')|r - s'|^{2H-2}|r' - s|^{2H-2}\,\mathrm{d}r\,\mathrm{d}r'\mathrm{d}s\,\mathrm{d}s'.$$

The elements of $\mathbb{D}^{1,2}(|\mathcal{H}|)$ are stochastic processes; we will make use of the integral notation

$\delta(u) = \int_0^T u(t)\delta B(t)$ and we will call this integral the Skorohod integral with respect to the fBm. Moreover, if $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ one can also define an indefinite integral process given by $X_t = \int_0^t u(s)\delta B(s)$.

Let us now define a Stratonovich type integral with respect to B (we put $B(t) = 0$ if $t \notin [0, T]$). Following the approach by Russo and Vallois (1993), we have:

Definition 1. *Let $u = \{u(t),\ t \in [0, T]\}$ be a stochastic process with integrable trajectories. The Stratonovich integral of u with respect to B is defined as the limit in probability as ε tends to zero of*

$$(2\varepsilon)^{-1}\int_0^T u(s)(B(s+\varepsilon) - B(s-\varepsilon))\mathrm{d}s,$$

provided this limit exists. When the limit exists, it is denoted by $\int_0^T u(t)\mathrm{d}B(t)$.

It has been shown in Alòs and Nualart (2003) that a process $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, for which it holds almost surely that

$$\int_0^T\int_0^T |D_s u(t)|\,|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t < \infty,$$

is Stratonovich integrable and that

$$\int_0^T u(s)\mathrm{d}B(s) = \int_0^T u(s)\delta B(s) + \alpha_H\int_0^T\int_0^T D_s u(t)|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t. \tag{3}$$

On the other hand, if the process u has λ-Hölder continuous trajectories almost surely, with $\lambda > 1 - H$, then its Stratonovich integral $\int_0^T u(s)\mathrm{d}B(s)$ exists and coincides with the pathwise Riemann–Stieltjes integral.

When $pH > 1$, we define the space $\mathbb{L}_H^{1,p}$ of processes $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ such that

$$\|u\|_{p,1} = \left[\int_0^T \mathrm{E}(|u(s)|^p)\mathrm{d}s + \mathrm{E}\left(\int_0^T\left(\int_0^T |D_r u(s)|^{1/H}\,\mathrm{d}r\right)^{pH}\mathrm{d}s\right)\right]^{1/p} < \infty.$$

It is known (see Nualart, 2003) that

$$\mathrm{E}\left(\sup_{t\in[0,T]}\left|\int_0^t u(s)\delta B(s)\right|^p\right) \leqslant C\|u\|_{p,1}^p,$$

where the constant $C > 0$ depends on p, H and T.

2.2. Some useful results

In the next section we shall need some additional results which provide sufficient conditions for processes $Z = \{Z(t),\ t \in [0, T]\}$ to belong to the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$. Let us recall once more that these results hold in the case $H > \frac{1}{2}$.

Lemma 1. *Let* $Z = \{Z(t),\ t \in [0,\ T]\}$ *be a stochastic process such that, for any* $t \in [0,\ T]$, $Z(t) \in \mathbb{D}^{1,2}$ *and*

$$\sup_s \mathrm{E}(|Z(s)|^2) \leqslant c_1 \quad \text{and} \quad \sup_{r,s} \mathrm{E}(|D_r Z(s)|^2) \leqslant c_2.$$

Then the stochastic process Z *belongs to* $\mathbb{D}^{1,2}(|\mathcal{H}|)$ *and*

$$\mathrm{E}\|Z\|^2_{|\mathcal{H}|} + \mathrm{E}\|DZ\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} < c_{H,T}(c_1 + c_2).$$

Proof. The proof follows the computations given in Nualart (2003). For instance, we can compute

$$\begin{aligned}
\mathrm{E}\|DZ\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} &= \mathrm{E}\left(\alpha_H^2 \int_{[0,T]^4} |D_r Z(s)|\,|D_{r'} Z(s')|\,|r - s'|^{2H-2}\,|s' - r|^{2H-2}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}r'\,\mathrm{d}s'\right)\\
&\leqslant \mathrm{E}\left(\alpha_H^2 \int_{[0,T]^4} |D_r Z(s)|^2\,|r - s|^{2H-2}\,|s' - r|^{2H-2}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}r'\,\mathrm{d}s'\right)\\
&\leqslant \alpha_H^2 \left(\frac{T^{2H-1}}{H - \frac{1}{2}}\right)^2 \int_{[0,T]^2} \mathrm{E}(|D_r Z(s)|^2)\mathrm{d}r\,\mathrm{d}s \leqslant c_{H,T}\,c_2.
\end{aligned}$$

□

Lemma 2. *Let* $Z = \{Z(t),\ t \in [0,\ T]\}$ *be a stochastic process such that for any* $t \in [0,\ T]$, $Z(t) \in \mathbb{D}^{1,2}$ *and*

$$\sup_s \mathrm{E}(|Z(s)|^2) \leqslant c_1 \quad \text{and} \quad \sup_{r,s} \mathrm{E}(|D_r Z(s)|^2) \leqslant c_2.$$

Then given $r > 0$ *and* f *a deterministic continuous function, the stochastic process* $V = \{V(t),\ t \in [0,\ T]\}$ *defined as*

$$V(t) = \begin{cases} Z(t - r), & \text{if } t > r,\\ f(t), & \text{if } t < r, \end{cases}$$

belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$.

Proof. It is clear that V belongs to $\mathbb{D}^{1,2}(\mathcal{H})$ and that

$$D_s V(t) = \begin{cases} D_s Z(t - r)\quad, & \text{if } t > r,\\ 0, & \text{if } t < r. \end{cases}$$

Then it is enough to apply Lemma 1. □

Given $s = (s_1, \ldots, s_k) \in [0,\ T]^k$, we denote by s the length of the finite sequence s, that is, k. For a random variable $Y \in \mathbb{D}^{k,p}$ and $s \in [0,\ T]^k$, we denote by $D_s^k Y$ the iterated derivative $D_{s_k} D_{s_{k-1}} \cdots D_{s_1} Y$. If $f \in C_b^{0,\infty}(\mathbb{R})$, the space of infinitely differentiable functions defined on $\mathbb{R}$ with bounded derivatives, we set

$$\Gamma_s(f;\,Y) = \sum_{m=1}^{|s|} \sum f^{(m)}(Y) \prod_{i=1}^{m} D_{p_i}^{|p_i|} Y,$$

where the second summation extends to all partitions $p_1, \ldots, p_m$ of length m of s.

Finally, we need the following lemma, an extension of a result proved in Rovira and Sanz-Solé (1996).

Lemma 3. *Let $\{F_n,\, n \geqslant 1\}$ be a sequence of random variables in $\mathbb{D}^{k,p}$, $k \geqslant 1$, $p \geqslant 2$. Assume that there exists $F \in \mathbb{D}^{k-1,p}$ such that $\{D^{k-1}F_n,\, n \geqslant 1\}$ converges to $D^{k-1}F$ in $L^p(\Omega;\, [0,\, T]^{\otimes(k-1)})$ as n goes to infinity and that, moreover, the sequence $\{D^k F_n,\, n \geqslant 1\}$ is bounded in $L^p(\Omega;\, [0,\, T]^{\otimes(k-1)})$. Then $F \in \mathbb{D}^{k,p}$.*

3. Existence and uniqueness

Let $B = \{B(t),\, t \in [0,\, T]\}$ be a one-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

Let us define the following stochastic delay differential equation driven by an fBm:

$$X(t) = \phi(0) + \int_0^t b(X(s))\mathrm{d}s + \int_0^t \sigma(X(s-r))\mathrm{d}B(s), \qquad t \in [0,\, T], \tag{4}$$

where $r > 0$ and $\phi \in C([-r,\, 0])$. For simplicity, let us assume $T = \overline{M}r$. The stochastic integral in (4) must be understood as the Stratonovich type integral defined above (see Definition 1).

We shall assume that b and σ are real valued functions that satisfy the following conditions:

(H) b and σ are $\overline{M}$-times differentiable functions with bounded derivatives up to order $\overline{M}$. Moreover, σ is bounded and $|b(0)| \leqslant c_1$ for some constant c_1.

Theorem 4. *Under hypotheses* (H)*, the SDDE* (4) *admits a unique solution X on* $[0,\, T]$.

The proof of this theorem is based on the following lemmas and propositions.

Lemma 5. *Let $M = \{M(t),\, t \in [0,\, T]\}$ be a quadratic integrable stochastic process. Assume that b is a Lipschitz function defined on $\mathbb{R}$, such that $|b(0)| \leqslant c_1$ for some constant c_1. Then, for any $T_1 \leqslant T$, the stochastic integral equation*

$$X(t) = x + \int_0^t b(X(s))\mathrm{d}s + M(t), \qquad t \in [0,\, T_1], \tag{5}$$

with $X(t) = 0$ *if $t > T_1$, admits a unique solution X on* $[0,\, T]$.

Proof. In order to prove the existence and uniqueness, we can prove that the classical Picard–Lindelöf iterations converge to a solution of (5). Consider

$$\begin{cases} X^{(n+1)}(t) = x + \int_0^t b(X^{(n)}(s))\mathrm{d}s + M(t), \\ X^{(0)}(t) = x + M(t), \end{cases} \tag{6}$$

for $t \in [0, T_1]$, and $X^{(n)}(t) = 0$ for any $t \geqslant T_1$ and all n. Notice that we only need to deal with $t \in [0, T_1]$. We have

$$\mathrm{E}(|X^{(1)}(t) - X^{(0)}(t)|^2) \leqslant K_2 \mathrm{E}\left(\int_0^t (x^2 + |M(s)|^2)\mathrm{d}s\right) + K_2 \leqslant K_2$$

uniformly in t. For a generic n, we thus have

$$\begin{aligned} \mathrm{E}(|X^{(n+1)}(t) - X^{(n)}(t)|^2) &\leqslant K_2 \int_0^t \mathrm{E}(|X^{(n)}(s) - X^{(n-1)}(s)|^2)\mathrm{d}s \\ &\leqslant K_2^{n-1} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \mathrm{E}(|X^{(1)}(s) - X^{(0)}(s)|^2)\mathrm{d}s_n \ldots \mathrm{d}s_1 \leqslant \frac{K K_2^{n-1}}{n!}. \end{aligned}$$

From this we can easily prove that the Picard–Lindelöf iterations converge in $L^2(\Omega)$ to a solution for equation (5) on $[0, T_1]$. A similar argument gives the uniqueness. □

Let us introduce some new notation. For fixed $m \geqslant 1$ and $p \geqslant 2$, we will say that a stochastic process $Z = \{Z(t), t \in [0, T]\}$ satisfies condition $(D_{(*,m,p)})$ if $Z(t)$ belongs to $\mathbb{D}^{m,p}$ for any $t \in [0, T]$, and

$$\mathrm{E}\left(\sup_t |Z(t)|^p\right) \leqslant c_{1,p} \quad \text{and} \quad \mathrm{E}\left(\sup_t \sup_{u,|u|=k} |D_u^k Z(t)|^p\right) \leqslant c_{2,k,p},$$

for any $k \leqslant m$ and for some constants $c_{1,p}$, $c_{2,k,p}$. Notice that if Z satisfies condition $(D_{(*,m,p)})$, then Lemma 1 yields that Z belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$.

Proposition 6. *Let $M = \{M(t), t \in [0, T]\}$ be a stochastic process satisfying condition $(D_{(*,m,p)})$. Assume that b has bounded derivatives up to order m and that $|b(0)| \leqslant c_1$, for some constant c_1.*

Then, given $T_1 \leqslant T$, the stochastic integral equation

$$X(t) = x + \int_0^t b(X(s))\mathrm{d}s + M(t), \qquad t \in [0, T_1], \tag{7}$$

with $X(t) = 0$ if $t > T_1$, admits a unique solution X on $[0, T]$. Moreover, the stochastic process $X = \{X(t), t \in [0, T]\}$ satisfies condition $(D_{(*,m,p)})$.

Proof. The existence of a unique solution was proved in Lemma 5, where we proved that the Picard–Lindelöf iterations converge to a solution of (7). A simple computation gives

$$\mathrm{E}\left(\sup_{r\leqslant t} |X(r)|^p\right) \leqslant K_p\left(\mathrm{E}\left(\sup_{r\leqslant t}\left|\int_0^r b(X(s))\mathrm{d}s\right|^p\right) + \mathrm{E}\left(\sup_{r\leqslant t} |M(r)|^p\right) + x^p\right),$$

so

$$\mathrm{E}\left(\sup_{r\leqslant t}|X(r)|^p\right)\leqslant K_p\left(1+\int_0^t\mathrm{E}\left(\sup_{r\leqslant s}|X(r)|^p\right)\mathrm{d}s\right)$$

and, finally, using a Gronwall's lemma argument, one finds that

$$\mathrm{E}\left(\sup_t |(X(t)|^p\right) < c_{1,p} < \infty.$$

In order to check that, for all $k\leqslant m$, $X(t)$ belongs to $\mathbb{D}^{k,p}$ for every $t\in[0, T]$ and that

$$\mathrm{E}\left(\sup_t \sup_{u,|u|=k} |D_u^k X(t)|^p\right)\leqslant c_{2,k,p},$$

we will use the Picard–Lindelöf iterations defined in (6). We consider the induction hypothesis $(\hat{H}_k)$, for $k\leqslant m$:

(a) For all $n\geqslant 0$, $X^{(n)}(t)\in\mathbb{D}^{k,p}$ for all t.
(b) $D^{k-1}X^{(n)}(t)$ converges to $D^{k-1}X(t)$ in $L^p(\Omega, [0, T]^{k-1})$ when n tends to ∞.
(c) $\sup_n \sup_t \sup_{u,u=k} \mathrm{E}(|D_u^k X^{(n)}(t)|^p)\leqslant K_{p,k}<\infty$.

Notice that hypothesis $(\hat{H}_k)$ implies that $X(t)\in\mathbb{D}^{k,p}$ and that we need only study the case $t\in[0, T_1]$, since for $t>T_1$ all the results are obvious.

Step 1. We prove $(\hat{H}_1)$, that is, the case $k=1$. Since $X^{(n)}(t)$ converges to $X(t)$ in $L^p(\Omega)$, when n tends to ∞, we know that *(b)* is true. In order to prove *(a)* and *(c)*, we will proceed with a second induction to prove, for all $n\geqslant 0$, the hypothesis $(\tilde{H}_n)$:

(i) $X^{(n)}(t)\in\mathbb{D}^{1,p}$ for all $t\in[0, T]$.
(ii) $\sup_t \sup_u \mathrm{E}(|D_u X^{(n)}(t)|^p)\leqslant K_{n,p,1}<\infty$.

By its very definition, it is clear that $X^{(0)}(t)\in\mathbb{D}^{1,p}$ for all t and that, for $t\in[0, T_1]$,

$$D_u X^{(0)}(t) = D_u M(t).$$

So $(\tilde{H}_0)$ is proved. Assume now that the hypothesis of induction $(\tilde{H}_n)$ is true. Then, from the definition of $X^{(n+1)}$, it follows that $X^{(n+1)}(t)\in\mathbb{D}^{1,p}$ and, for any $t\in[0, T_1]$,

$$D_u X^{(n+1)}(t) = \int_0^t b'(X^{(n)}(s))D_u X^{(n)}(s)\mathrm{d}s + D_u M(t).$$

Moreover, since M satisfies condition $(D_{*,m,p})$, we have

$$\begin{aligned}\sup_u \mathrm{E}(|D_u X^{(n+1)}(t)|^p) &\leqslant K_p\left(\int_0^t \sup_u \mathrm{E}(|D_u X^{(n)}(s)|^p)\mathrm{d}s + \sup_u \mathrm{E}(|D_u M(t)|^p)\right)\\ &\leqslant K_{n+1,p}<\infty.\end{aligned}$$

From this, $(\tilde{H}_{n+1})$ can be easily proved. Furthermore, we have the inequality

$$\sup_u \mathrm{E}(|D_u X^{(n+1)}(t)|^p)\leqslant K_p\int_0^t \sup_u \mathrm{E}(|D_u X^{(n)}(s)|^p)\mathrm{d}s + K_p.$$

Iterating this inequality n times, we obtain

$$\sup_u \mathrm{E}(|D_u X^{(n+1)}(t)|^p) \leqslant (K_p)^2 \int_0^t \int_0^s \sup_u \mathrm{E}(|D_u X^{(n-1)}(v)|^p) \mathrm{d}v\, \mathrm{d}s + K_p^2 t + K_p$$

$$\leqslant \sum_{k=0}^n (K_p)^{k+1} \frac{t^k}{k!} \leqslant K_p \exp(K_p t).$$

Thus we have that

$$\sup_n \sup_t \sup_u \mathrm{E}(|D_u X^{(n)}(t)|^p) \leqslant K_p \exp(K_p T) < \infty$$

and $(\hat{H}_1)$ is proved.

Notice now that by applying Lemma 3 we obtain that, for all $t \in [0, T]$, $X(t) \in \mathbb{D}^{1,p}$. Moreover, we can also conclude that

$$D_u X(t) = \int_0^t b'(X(s)) D_u X(s) \mathrm{d}s + D_u M(t),$$

and we easily see that

$$\mathrm{E}\left(\sup_t \sup_u |D_u X(t)|^p \right) < \infty.$$

Step 2. Let us assume that $(\hat{H}_i)$ holds for $i \leqslant k \leqslant m-1$. We wish to check $(\hat{H}_{k+1})$. We will prove (a) again by induction on n. Let us consider, for all $n \geqslant 0$, the hypothesis $(\tilde{H}_n)$:

(i) $X^{(n)}(t) \in \mathbb{D}^{k+1,p}$ for all $t \in [0, T]$.
(ii) $\sup_t \sup_{u,|u|=k+1} \mathrm{E}(|D_u^{k+1} X^{(n)}(t)|^p) \leqslant K_{n,p,1} < \infty$.

Since $M(t) \in \mathbb{D}^{k+1,p}$ for all t, from the definition of $X^{(0)}$ it is clear that $X^{(0)}(t) \in \mathbb{D}^{k+1,p}$ for all t and that, for $|u|$, with $u = k+1$,

$$D_u^{k+1} X^{(0)}(t) = D_u^{k+1} M(t),$$

for any $t \in [0, T_1]$. Thus $(\tilde{H}_0)$ is true. Assuming now that $(\tilde{H}_i)$ is true for $i \leqslant n$, from the definition of $X^{(n+1)}$ it follows that, for all t, $X^{(n+1)}(t) \in \mathbb{D}^{k+1,p}$ and for u, $|u| = k+1$,

$$D_u^{k+1} X^{(n+1)}(t) = \int_0^t \Gamma_u(b;\, X^{(n)}(s)) \mathrm{d}s + D_u^{k+1} M(t),$$

for any $t \in [0, T_1]$. The proof of (b) can now be obtained easily from the expressions for $D_u^k X^{(n)}(t)$ and $D_u^k X(t)$. Finally, to prove (c), set

$$\Delta_u(b;\, X^{(n)}(s)) = \Gamma_u(b;\, X^{(n)}(s)) - b^{(k+1)}(X^{(n)}(s)) D_u^{k+1} X^{(n)}(s).$$

Using the inductive hypothesis, we obtain

$$\sup_{u,|u|=k+1} \sup_s \mathrm{E}(|\Delta_u(b;\, X^{(n)}(s))|^p) \leqslant K_p.$$

Then

$$D_u^{k+1}X^{(n+1)}(t) = \int_0^t \Delta_u(b;\ X^{(n)}(s))\mathrm{d}s$$

$$+ D_u^{k+1}M(t) + \int_0^t b^{(k+1)}(X^{(n)}(s))D_u^{k+1}X^{(n)}(s)\mathrm{d}s.$$

Reproducing the same calculations as in the proof of $(\hat{H}_1)$, we can complete the proof of $(\hat{H}_{k+1})$.

Notice now that by applying Lemma 3 once again, we have that $X(t) \in \mathbb{D}^{k+1,p}$ for every t and we easily obtain that

$$\mathrm{E}\left(\sup_t \sup_{u,|u|=k+1} |D_u^{k+1}X(t)|^p\right) \leqslant K_p.$$

□

The following proposition studies the behaviour of the stochastic integral.

Proposition 7. *Let* $Y = \{Y(t),\ t \in [0,\ T]\}$ *be a stochastic process satisfying condition* $(D_{*,m+1,p})$. *Then the stochastic Stratonovich integral*

$$M(t) := \int_0^t Y(s)\mathrm{d}B(s), \qquad t \in [0,\ T],$$

is well defined and the stochastic process $M = \{M(t),\ t \in [0,\ T]\}$ *satisfies condition* $(D_{*,m,p})$.

Proof. Clearly, Y is Stratonovich integrable, but the Stratonovich integral and the divergence operator do not coincide and we have

$$\int_0^t Y(s)\mathrm{d}B(s) = \delta(Y1_{[0,t]}) + \alpha_H \int_0^t \left(\int_0^T D_v Y(s)|s-v|^{2H-2}\mathrm{d}v\right)\mathrm{d}s.$$

We have

$$\begin{aligned}
\mathrm{E}\left(\sup_t |M(t)|^p\right) &= \mathrm{E}\left(\sup_t \left|\int_0^t Y(s)\mathrm{d}B(s)\right|^p\right) \qquad (8)\\
&\leqslant K_p\mathrm{E}\left(\sup_t |\delta(Y1_{[0,t]})|^p\right) + K_p\mathrm{E}\left(\sup_t \left|\alpha_H \int_0^t \left(\int_0^T D_v Y(s)|s-v|^{2H-2}\mathrm{d}v\right)\mathrm{d}s\right|^p\right)\\
&\leqslant K_p\|Y\|_{p,1}^p + K_p c_H \mathrm{E}\left(\sup_{s,v} |D_v Y(s)|^p\right)\left|\int_0^T \left(\int_0^T |s-v|^{2H-2}\,\mathrm{d}v\right)\mathrm{d}s\right|^p\\
&\leqslant K_p < \infty.
\end{aligned}$$

As a consequence of this, we find that

$$\mathrm{E}\left(\sup_t |M(t)|^p\right) \leqslant K_p.$$

Applying an induction argument again, it is easy to check that $M(t) \in \mathbb{D}^{k,p}$ for any t and $k \leqslant m$, and that

$$D^k_{(u_1,\ldots,u_k)} M(t) = \sum_{i=1}^{k} D^{k-1}_{(u_1,\ldots,\hat{u}_i,\ldots,u_k)} Y(u_i) \mathbf{1}_{[0,t]}(u_i) + \delta(D^k_u Y \mathbf{1}_{[0,t]}),$$

where $(u_1, \ldots, \hat{u}_i, \ldots, u_k)$ denotes the vector u without the component u_i.

By the same arguments used in (8), we obtain that for all $k \leqslant m$,

$$\mathrm{E}\left(\sup_t \sup_{u,|u|=k} |D^k_u M(t)|^p \right) \leqslant K_p.$$

Proof of Theorem 4. To prove that equation (4) admits a unique solution on $[0, T]$, we shall first prove the result for $t \in [0, r]$. Then, by induction, we shall prove that if equation (4) admits a unique solution on $[0, Nr]$, we can further extend this solution to the interval $[0, (N+1)r]$ and that this extension is unique.

Actually our induction hypothesis, for $N \leqslant \overline{M}$, is the following:

(H_N) *The equation*

$$X(t) = \phi(0) + \int_0^t b(X(s))\mathrm{d}s + \int_0^t \sigma(X(s-r))\mathrm{d}B(s), \qquad t \in [0, Nr],$$

with $X(t) = 0$ *if* $t > Nr$, *has a unique solution. Moreover, for all* $p \geqslant 2$, $X(t)$ *satisfies condition* $(D_{(*,\overline{M}-N,p)})$.

For simplicity, we omit the dependence on N of the solution X. Notice that at each step we loose one degree of regularity.

Check (H_1). Let $t \in [0, r]$. Equation (4) can be written in the following easy form:

$$X(t) = \phi(0) + \int_0^t b(X(s))\mathrm{d}s + \int_0^t \sigma(\phi(s-r))\mathrm{d}B(s). \tag{9}$$

Let us define the process

$$M(t) := \int_0^t \sigma(\phi(s-r)) \mathbf{1}_{\{t<r\}}\, \mathrm{d}B(s),$$

for $t \in [0, T]$. Since ϕ is deterministic, it is immediately evident that $\sigma(\phi(\cdot - r)) \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ and that $D\sigma(\phi(s-r)) = 0$. For this reason the stochastic integral in (9) is well defined and, in view of (3), coincides with the divergence operator. Again, since ϕ is a deterministic continuous function, we have that $M(t) \in \mathbb{D}^{k,p}$ for all $k \geqslant 1$, $p \geqslant 2$. Moreover,

$$D_u M(t) = \sigma(\phi(u-r)) \mathbf{1}_{\{u<t<r\}},$$

and $D^k M(t) = 0$, when $k \geqslant 2$. Then, for all $k \geqslant 1$, $p \geqslant 2$, we have that

$$\mathrm{E}\left(\sup_t |M(t)|^p\right) \leqslant \|\sigma(\phi(.-r))\|_{p,1}^p \leqslant c_{3,1,p}$$

and

$$\mathrm{E}\left(\sup_t \sup_{u,|u|=k} |D_u^k M(t)|^p\right) \leqslant c_{4,k,p},$$

for some constants $c_{3,p}$, $c_{4,k,p}$. We have thus proved that M satisfies condition $(D_{(*,k,p)})$ for any $k \geqslant 1$.

From Proposition 6 we have that there exists a unique solution X and that this solution satisfies condition $(D_{(*,\overline{M}-1,p)})$.

Induction. Assume that (H_i) is true for $i \leqslant N$, with $N < \overline{M}$. We wish to check (H_{N+1}).

Consider the stochastic process $\{Z(t),\ t \in [0,\ T]\}$ defined as

$$Z(t) = \begin{cases} \varphi(t-r), & \text{if } t \leqslant r, \\ X(t-r), & \text{if } r < t \leqslant (N+1)r, \\ 0, & \text{if } t > (N+1)r, \end{cases}$$

where X is the solution obtained in (H_N). Set $Y(t) = \sigma(Z(t))$.

Thus, for $t \in [0,\ (N+1)r]$, our problem has become the equation

$$X(t) = \phi(0) + \int_0^t b(X(s))\mathrm{d}s + \int_0^t Y(s)\mathrm{d}B(s). \tag{10}$$

Let us define the process

$$M(t) := \int_0^t Y(s)\mathbf{1}_{\{t<(N+1)r\}}\,\mathrm{d}B(s), \qquad t \in [0,\ T].$$

and prove that the Stratonovich integral is well defined. To this end, we must prove that (as pointed out in the previous section):

1. $Y \in \mathbb{D}^{1,2}(|\mathcal{H}|)$;
2. $\int_0^T \int_0^T |D_u Y(s)|\,|s-u|^{2H-2}\,\mathrm{d}s\,\mathrm{d}u < \infty$

These two conditions can be obtained from Lemma 2 and the following facts: σ is a bounded function with bounded derivatives, Z is in $\mathbb{D}^{1,2}(|\mathcal{H}|)$, $\sup_t \sup_u \mathrm{E}(|D_u Z(t)|^p) \leqslant c_{2,p}$, and

$$D_u Y(t) = \sigma'(Z(t)) D_u Z(t).$$

Since the stochastic process Z satisfies condition $(D_{(*,\overline{M}-N,p)})$ and σ has derivatives up to order $\overline{M}$, it is clear that $Y(t) \in \mathbb{D}^{k,p}$ for any $t \in [0,\ T]$ and $k \leqslant \overline{M} - N$, with

$$D_u Y(t) = \sigma'(Z(t)) D_u Z(t), \qquad D_u^k Y(t) = \Gamma_u(\sigma,\ Z(t)).$$

Furthermore, Y will also satisfy condition $(D_{(*,\overline{M}-N,p)})$.

By Proposition 7 we obtain that M satisfies condition $(D_{(*,\overline{M}-N-1,p)})$. Finally, from Proposition 6 and using the same arguments as in step 1, we complete the proof of this theorem. □

Remark 1. In order to define the stochastic integrals appearing in the previous Picard iterations, we need only study the Malliavin derivatives up to order $\overline{M}$. For this reason, we have to assume that the coefficients σ and b have bounded derivatives up to order $\overline{M}$, which is unnecessary in the case of SDEs driven by a standard Brownian motion. As a by-product of our method, we can easily prove that if the coefficients σ and b have bounded derivatives of any order, then the solution $X(t)$ belongs to $\mathbb{D}^{\infty}$. Moreover, assuming the non-degeneracy of σ, we can also prove the smoothness of the density.

4. Regularity of the density

Let us consider now a different set of hypotheses:

($\hat{H}$) b and σ are real functions with bounded derivatives of all orders. Moreover, σ is bounded and $|b(0)| \leqslant c_1$ for some constant c_1.

Theorem 8. *Assume Hypotheses* ($\hat{H}$). *If there exists a positive constant c_0 such that $|\sigma(x)| > c_0$ for all x, then, for any $t \in (0, T]$ the solution $X(t)$ of the SDDE* (4) *has an infinitely differentiable density with respect to Lebesgue measure on $\mathbb{R}$.*

Proof. Fix $t \in (0, T]$; to apply the Malliavin criterion for the existence of a smooth density, we have to check:

1. $X(t) \in \mathbb{D}^{\infty}$;
2. $(\int_0^T |D_u X(t)|^2 \,\mathrm{d}u)^{-1} \in \bigcap_{p \geqslant 1} L^p(\Omega)$.

Following the same lines as in the proof of Theorem 4, we obtain that $X(t) \in \mathbb{D}^{\infty}$ for any $t \in [0, T]$. In order to prove the second condition it is enough to check that for any $p \geqslant 1$ there exists $\varepsilon_0 > 0$ such that

$$P\left(\int_0^T |D_u X(t)|^2 \,\mathrm{d}u \leqslant \varepsilon\right) \leqslant \varepsilon^p,$$

for all $\varepsilon \leqslant \varepsilon_0$.

From equation (4) we can write

$$X(t) = \phi(0) + \int_0^t b(X(s))\mathrm{d}s + \int_0^t \sigma(X(s-r))\delta B(s) + \alpha_H \int_0^t \left(\int_0^T D_v \sigma(X(s-r))|s-v|^{2H-2}\mathrm{d}v\right)\mathrm{d}s.$$

So, for any $u \leqslant t - r$, we obtain

$$
\begin{aligned}
D_u X(t) = {} & \int_u^t b'(X(s)) D_u X(s) \mathrm{d}s + \sigma(X(u-r)) \\
& + \int_{u+r}^t \sigma'(X(s-r)) D_u X(s-r) \delta B(s) \\
& + \alpha_H \int_{u+r}^t \left(\int_0^{s-r} D_u D_v \sigma(X(s-r)) |s-v|^{2H-2} \mathrm{d}v \right) \mathrm{d}s,
\end{aligned}
$$

and, when $u \in (t-r, t)$, we have

$$
D_u X(t) = \int_u^t b'(X(s)) D_u X(s) \mathrm{d}s + \sigma(X(u-r)).
$$

Then, again using a Gronwall's inequality argument, we have that

$$
\mathrm{E}\left(\sup_{u \in (t-r,t), s \in (t-r,t)} |D_u X(s)|^q \right) \leqslant K_q.
$$

On the other hand, we can write

$$
P\left(\int_0^T |D_u X(t)|^2 \, \mathrm{d}u \right) \leqslant p_{1,\varepsilon} + p_{2,\varepsilon},
$$

with

$$
\begin{aligned}
p_{1,\varepsilon} = P\Bigg(& \int_{t-\varepsilon^\alpha}^t \left| \int_u^t b'(X(s)) D_u X(s) \mathrm{d}s + \sigma(X(u-r)) \right|^2 \mathrm{d}u \leqslant \varepsilon, \\
& \sup_{u \in (t-\varepsilon^\alpha, t)} \int_{t-\varepsilon^\alpha}^t |b'(X(s)) D_u X(s)| \mathrm{d}s \leqslant \varepsilon^\beta \Bigg) \\
p_{2,\varepsilon} = P\Bigg(& \sup_{u \in (t-\varepsilon^\alpha, t)} \int_{t-\varepsilon^\alpha}^t |b'(X(s)) D_u X(s)| \mathrm{d}s > \varepsilon^\beta \Bigg).
\end{aligned}
$$

Since $|\sigma(x)| > c_0$ for all x, when $\alpha < 1$ we clearly have that $p_{1,\varepsilon} = 0$. However, using Chebyshev's inequality, for any $q > 1$,

$$
\begin{aligned}
p_{2,\varepsilon} & \leqslant \frac{1}{\varepsilon^{\beta q}} \mathrm{E}\left(\sup_{u \in (t-\varepsilon^\alpha, t)} \left| \int_{t-\varepsilon^\alpha}^t |b'(X(s)) D_u X(s)| \mathrm{d}s \right|^q \right) \\
& \leqslant \varepsilon^{(\alpha-\beta)q} K E\left(\sup_{u \in (t-\varepsilon^\alpha, t), s \in (t-\varepsilon^\alpha, t)} |D_u X(s)|^q \right).
\end{aligned}
$$

So, choosing $\beta < \alpha < 1$, the proof is complete. $\square$

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References

Alòs, E. and Nualart, D. (2003) Stochastic integration with respect to the fractional Brownian motion. *Stochastics Stochastics Rep.*, **75**, 129–152.

Arriojas, M., Hu, Y., Mohammed, S.A.D. and Pap, Y. (2003) A delayed Black and Scholes formula. Preprint.

Carmona, P. and Coutin, L. (2000) Stochastic integration with respect to fractional Brownian motion. *C. R. Acad. Sci. Paris Sér. I. Math.*, **330**, 231–236.

Cheridito, P., Kawaguchi, H. and Maejima, M. (2003) Fractional Ornstein–Uhlenbeck processes. *Electron. J. Probab.*, **8**(3).

Coutin, L. and Qian, Z. (2000) Stochastic differential equations for fractional Brownian motions. *C. R. Acad. Sci. Paris Sér. I Math.*, **331**, 75–80.

Coutin, L. and Qian, Z. (2002) Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields*, **122**, 108–140.

Hobson, D.G. and Rogers, L.C.G. (1998) Complete models with stochastic volatility. *Math. Finance*, **8**, 27–48.

Mémin, J., Mishura, Y. and Valkeila, E. (2001) Inequalities for the moments of Wiener integrals with respect to fractional Brownian motions. *Statist. Probab. Lett.*, **51**, 197–206.

Nualart, D. (1995) *The Malliavin Calculus and Related Topics*. Berlin: Springer-Verlag.

Nualart, D. (2003) Stochastic integration with respect to fractional Brownian motion and applications. In J.M. González-Barrios, J.A. León and A. Meda (eds), *Stochastic Models, Contemp. Math.* 336, pp. 3–39. Providence, RI: American Mathematical Society.

Nualart, D. and Ouknine, Y. (2003) Stochastic differential equations with additive fractional noise and locally unbounded drift. In E. Giné, C. Houdré and D. Nualart (eds), *Stochastic Inequalities and Applications*, Progr. Probab. 56, pp. 353–365. Basel: Birkhäuser.

Nualart, D. and Rascanu, A. (2002) Differential equations driven by fractional Brownian motion. *Collect. Math.*, **53**, 55–81.

Rovira, C. and Sanz-Solé, M. (1996) The law of the solution to a nonlinear hyperbolic SPDE. *J. Theoret. Probab.*, **9**, 863–901.

Pipiras, V. and Taqqu, M.S. (2000) Integration questions related to fractional Brownian motion. *Probab. Theory Related Fields*, **118**, 251–291.

Russo, F. and Vallois, P. (1993) Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, **97**, 403–421.

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