Classical predictive electrodynamics of two charges with radiation: General framework. I

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Outgoing radiation is introduced in the framework of the classical predictive electrodynamics using Lorentz–Dirac's equation as a subsidiary condition. In a perturbative scheme in the charges the first radiative "self-terms" of the accelerations, momentum and angular momentum of a two charge system without external field are calculated.

INTRODUCTION

This is the first of a series of two papers dealing with the classical dynamics of a radiating system consisting of two structureless interacting charges. We assume that each charge is moving in the retarded field of the other according to Lorentz–Dirac's equation.[1]

We take "the absorber" point of view of Wheeler–Feynman[2,3] and use the framework of predictive electrodynamics.[4,5] This theory is seen to be consistent with the phenomena of classical radiation and more precisely with the Lorentz–Dirac equation.

In Sec. 2 we show within a perturbative scheme in the charges how to construct the dynamical predictive system (the accelerations) of two classical interacting charges when radiation is present and there is no external field. Then in Sec. 3 we give explicitly the first radiative "self-terms" of the accelerations.

To fourth order in the charges ($n + m \leqslant 4$, e_1^n, e_2^m) the other terms in the accelerations, i.e., terms in e_1e_2 and $e_1^2e_2^2$, are shown to be those of Refs. 4 and 5.

Section 4 contains a review of the definitions of Hamilton–Jacobi coordinates, momentum and angular momentum in predictive relativistic mechanics together with some techniques to calculate them in our perturbative scheme. Proofs and explanations are omitted and the reader is referred to the work of Bel and Martin.[6]

Next, in Sec. 5, we calculate the first radiative "self-term" of Hamilton–Jacobi's coordinates, momentum and angular momentum in the perturbative scheme. For all those magnitudes the terms in e_1e_2 can be found in Ref. 6, while terms in $e_1^2e_2^2$ for the Hamilton–Jacobi's momenta will be given in paper II (this issue). Our calculations show that our radiative system is not conservative in the sense of Ref. 6: Angular momentum does not recover its free particle expression after the two particles have undergone mutual interaction. This fact allows us to compute the lower "self-term" of the total intrinsic angular momentum radiated by the system. This is done by calculating the limit for the "future infinity of the first radiative "self-term" in the intrinsic angular momentum.

Finally we calculate the 3-accelerations of the two charges to third order in $1/c$. This gives us the first correction to the accelerations which are derived from Darwin's Lagrangian,[7] when outgoing radiation is accounted for.

More detailed calculations, including scattering cross sections and the 4-momentum balance of a scattering process, will be given in paper II.

1. LORENTZ–DIRAC EQUATION FROM THE POINT OF VIEW OF PREDICTIVE RELATIVISTIC MECHANICS (PRM)

Let us consider a system of n point structureless classical particles. In PRM the dynamics of such a system is governed by a differential system of the form

$$\frac{dx_a^\alpha}{ds_a} = u_a^\alpha,$$

$$\frac{du_a^\alpha}{ds_a} = \xi_a^\alpha(x_b^\beta, u_c^\gamma)$$

$$(\alpha,\beta,\gamma,\cdots = 0,1,2,3, a,b,c,\cdots = 1,2,\ldots,n), \qquad (1.1)$$

where x_a^α, u_a^α, and s_a^α stand for the 4-position, 4-velocity, and proper time of the particle a. The functions ξ_a^α (the accelerations) are Poincaré invariant 4-vectors which are the solution of the system

$$u_{a'}^\rho \frac{\partial \xi_a^\alpha}{\partial x^{a'\rho}} + \xi_{a'}^\rho \frac{\partial \xi_a^\alpha}{\partial u^{a'\rho}} = 0, \qquad (1.2)$$

where we have raised index a' to invalidate the summation convention, which will only work in the case where equal indices stand in covariant and contravariant positions, respectively. According to this, the index ρ is summed in (1.2). Let us note that we will raise and lower latin indices without change of sign. Finally a' means "different from" a.

The functions ξ_a^α also satisfy the constraints

$$\xi_a^\alpha u_{a\alpha} = 0. \qquad (1.3)$$

This guarantees that solutions of (1.1) initially satisfying $u_a^2 = -1$ (We choose signature $+2$) will maintain this relation forever.

We summarize here the main results of PRM. Further details can be found in the original papers.[6,8]

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Using perturbation methods, and imposing the compatibility of PRM with classical electrodynamics Bel and Martin and co-workers have singled out the unique accelerations ξ^{α}_{a} describing the classical particle-particle electromagnetic interaction.[4,5,9] Classical electrodynamics, through the Lorentz force law and the formula of retarded (alternatively advanced or time-reversal) potentials, specifies the values of functions ξ^{α}_{a} for arguments x^{β}_{b} standing on null configurations,

$$(x^{\alpha}_{\alpha} - x^{\alpha}_{a'})(x_{\alpha a} - x_{a'\alpha}) = 0. \tag{1.4}$$

Because of (1.1) PRM is a dynamical theory of "Newtonian type" in the sense that a finite number of initial conditions (more precisely initial positions and velocities) are enough to determine the trajectories. Hence the word predictive in the name (predictive relativistic mechanics) of the theory. Following this terminology we will speak about *predictive electrodynamics* as it refers to the electrodynamics built into the framework of PRM.

PRM, as it has been described here, concerns itself with isolated systems of particles. In this sense it seems at first sight that predictive electrodynamics is unable to account for the fundamental phenomena of electromagnetic radiation. However, that this is not the case can be clearly seen if we take the point of view of Wheeler–Feynman[2,3] and others. According to these authors the theory of classical electromagnetic radiation is only a way to account for the interaction of a given system of charges with all other charges of the entire universe (theory of the absorber). In particular, the Lorentz–Dirac equation for an accelerated radiating charge is given by

$$\xi^{\alpha} = \frac{e}{m} F^{\alpha}{}_{\beta} u^{\beta} + \frac{2e^2}{3m}(\dot{\xi}^{\alpha} - \xi^2 u^{\alpha}), \tag{1.5}$$

where u^{α}, ξ^{α}, e, m are the 4-velocity, 4-acceleration, charge, and mass of the electric charge, respectively. $F^{\alpha\beta}$ is the given external *retarded* electromagnetic field acting on the charge. Finally $\dot{\xi}^{\alpha}$ stands for $(d/ds)\xi^{\alpha}$ and s is the proper time of the charge.

If we take Eq. (1.5) as the differential equation describing the charges' motion we would have to abandon predictive relativistic mechanics, since Eq. (1.5) is a third-order differential equation. Thus if we want to keep predictive electrodynamics, Eq. (1.5) has to be taken as a differential equation for the acceleration. In fact, we always have differential equations for the accelerations in PRM; e.g., Eqs. (1.2), which are by no means the differential equations of the particle motion. In each physical situation we must supply (1.2) with the good asymptotic conditions in order to get a unique acceleration ξ^{α}_{a}. Then if we plug into (1.1) the acceleration obtained in this way, we will be able to write down the equation of motion. Analogously, Eq. (1.5) has many solutions. (For example those accelerations which allow for the run away trajectories.) The problem consists in finding the right asymptotic conditions in order to select the physical ones. We assume now that the physical solutions, ξ^{α}, of (1.5) can be expanded in powers of e,

$$\xi^{\alpha} = e\ {}^{1}\xi^{\alpha} + e^2\ {}^{2}\xi^{\alpha} + \cdots. \tag{1.6}$$

It is obvious that (1.5) has a unique solution of the form (1.6). Furthermore, accelerations such as (1.6) exclude automatically the pathological solutions of (1.5) called run away solutions,[10] since these solutions are not analytical functions in e (see for instance Ref. 11). For instance we get for the first terms of (1.6)

$${}^{1}\xi^{\alpha} = \frac{1}{m} F^{\alpha}{}_{\beta} u^{\beta}, \quad {}^{2}\xi^{\alpha} = 0, \quad {}^{3}\xi^{a} = \frac{2}{3m^2} \dot{F}^{\alpha}{}_{\beta} u^{\beta}. \tag{1.7}$$

2. THE CASE OF TWO PARTICLES

We will consider, within the framework of predictive electrodynamics, the case of two charged particles mutually interacting, without external fields, but taking account of their electromagnetic radiation. We write an equation similar to (1.5) for each particle. (Remember that we raise and lower latin indices $a,b,\cdots$ without any change: $\xi^{a\alpha} \equiv \xi^{\alpha}_{a}$. Here again we have raised the index a to invalidate the summation convention.)

$$\xi^{\alpha}_{a} = \frac{e_a}{m_a} F^{\alpha}_{a'\beta} u^{\beta}_{a} + \frac{2e_a^2}{3m_a}(\dot{\xi}^{\alpha}_{a} - \xi^2_{a} u^{\alpha}_{a})$$
$$\left(\dot{\xi}^{\alpha}_{a} \equiv \frac{d\xi^{\alpha}_{a}}{ds_a}\right), \tag{2.1}$$

where a relates to the particle we are dealing with and $F^{\alpha}_{a'\beta}$ is the *retarded* electromagnetic field created by particle $a'(a' \neq a)$ on the particle a. The problem with equations (2.1) is that they are not differential equations—since the term $(e_a/m_a)F^{\alpha}_{a'\beta}u^{\beta}$ is not defined for any x^{α}_{1} and x^{α}_{2}, but only for null configurations

$$(x^{\alpha}_{2} - x^{\alpha}_{1})(x_{2\alpha} - x_{1\alpha}) = 0. \tag{2.2}$$

A similar problem must be faced when one considers the Lorentz equations of two interacting charges

$$\hat{\xi}^{\alpha}_{a} = (e_a/m_a) F^{\alpha}_{a'\beta} u^{\beta}, \tag{2.3}$$

where we have written $\hat{\xi}^{\alpha}_{a}$ instead of ξ^{α}_{a} to denote the accelerations. Using (2.3) as asymptotic conditions and making the assumption that accelerations can be expanded in powers of e_1e_2, a unique acceleration to be used in (1.1) can be obtained,[4,5] as explained before. This acceleration $\hat{\xi}^{\alpha}_{a}$, could, in principle, be determined by using a perturbative scheme in the coupling constant $g = e_1e_2$. The first two terms in the series, which we write in evident notation ${}^{(1,1)}\hat{\xi}^{\alpha}_{a}$, ${}^{(2,2)}\hat{\xi}^{\alpha}_{a}$, are given explicitly in Refs. 4 and 5. [Attention must be paid to the fact that the acceleration notation is slightly ambiguous. The acceleration $\hat{\xi}^{\alpha}_{a}$ about which we are speaking here is not the same as in (2.3). In (2.3), it is only defined for null configurations (2.2) and here it is defined for any pair of four positions x^{α}_{1} and x^{α}_{2} and coincides with $\hat{\xi}^{\alpha}_{a}$ in (2.3) for null configurations. Something similar can be said about functions ξ^{α}_{a} in (2.4) and (2.1), respectively.] Now we substitute Eq. (2.1) for the new equations,

$$\xi^{\alpha}_{a} = \hat{\xi}^{\alpha}_{a} + \frac{2e_a^2}{3m_a}(\dot{\xi}^{\alpha}_{a} - \xi^2_{a} u^{\alpha}_{a}). \tag{2.4}$$

That is to say

$$\xi_a^{\alpha} = \frac{2e_a^2}{3m_a}(\dot{\xi}_a^{\alpha} - \xi_a^2 u_a^{\alpha}) + {}^{(1,1)}\hat{\xi}_a^{\alpha} + {}^{(2,2)}\hat{\xi}_a^{\alpha} + \cdots. \quad (2.5)$$

Assuming that ξ_a^{α} can be expanded in powers of e_1, e_2 we get for the first terms

$$^{(1,1)}\xi_a^{\alpha} = {}^{(1,1)}\hat{\xi}_a^{\alpha}, \quad {}^{(2,2)}\xi_a^{\alpha} = {}^{(2,2)}\hat{\xi}_a^{\alpha}, \quad (2.6)$$

$$^{(3,1)}\xi_a^{\alpha} = \frac{2}{3m_a}\, {}^{(1,1)}\dot{\xi}_a^{\alpha}, \quad (2.7)$$

where ${}^{(3,1)}\xi_a^{\alpha}$ is the coefficient of $e_a^3 e_{a'}$ in the expansion of ξ_a^{α}.

Then we have a crucial point to clarify: Does ξ_a^{α}, as determined by (2.4), satisfy Eqs. (1.2)? It can be easily seen that this is the case for all orders. Furthermore, ξ_a^{α} in (2.4) satisfies (1.3), so it is also easy to see that (1.3) is satisfied by ξ_a^{α} given by (2.4).

Summing up, we have two accelerations, ξ_a^{α}, which describe a predictive relativistic dynamical interaction because Eqs. (1.2) and (1.3) are satisfied. Also ξ_a^{α} satisfies Eqs. (2.4) with Eq. (2.1) as asymptotic conditions; hence this predictive interaction describes two mutually interacting charged particles with outgoing radiation and without external fields. [For a more rigorous approach to Eq. (2.4) see Sanz,[12] who first and independently of us has adopted most of the points of view that we have developed in Secs. 1 and 2.]

3. THE FIRST RADIATIVE "SELF-TERM" IN THE ACCELERATIONS

As it has been pointed out in the last section, the accelerations ξ_a^{α} corresponding to the predictive electrodynamics of two isolated particles (no Dirac term, no external field) are in principle calculable within a perturbative scheme on the coupling constant. $g \equiv e_1 e_2$ and ${}^{(1,1)}\hat{\xi}_a^{\alpha}, {}^{(2,2)}\hat{\xi}_a^{\alpha}$ are explictly given in Refs. 4 and 5. Here we calculate the first radiative "self-term" in the accelerations; this is ${}^{(3,1)}\xi_a^{\alpha}$. According to (2.6), [(2.7)] we only need ${}^{(1,1)}\xi_a^{\alpha}\ (= {}^{(1,1)}\hat{\xi}_a^{\alpha})$ to calculate ${}^{(3,1)}\xi_a^{\alpha}$. For ${}^{(1,1)}\xi_a^{\alpha}$ we have the expression[4]

$$^{(1,1)}\xi_a^{\alpha} = \frac{\eta_a}{m_a r_a^3}[(\chi u_a)u_{a'}^{\alpha} - (u_1u_2)x^{\alpha}] \quad (\eta_1 \equiv 1,\ \eta_2 \equiv -1), \quad (3.1)$$

where $x_1^{\alpha} - x_2^{\alpha} \equiv x^{\alpha}$ and $(xu_a) \equiv x^{\alpha}u_{a\alpha}$, $(u_1u_2) \equiv u_1^{\alpha}u_{2\alpha}$, and

$$r_a \equiv [x^2 + (xu_{a'})^2]^{1/2}. \quad (3.2)$$

Now we have to compute $\dot{\xi}_a^{\alpha}$. In order to make the calculations easier, let us introduce a system of new variables which will replace $x^{\alpha}, u_1^{\alpha}, u_2^{\alpha}$. We define the three linearly independent 4-vectors

$$h^{\alpha} \equiv x^{\alpha} - z_1 u_1^{\alpha} + z_2 u_2^{\alpha}, \quad t_a^{\alpha} \equiv u_a^{\alpha} + (u_1u_2)u_{a'}^{\alpha} \quad (3.3)$$

where

$$z_a \equiv \frac{\eta_a[(xu_a) + (u_1u_2)(xu_{a'})]}{[(u_1u_2)^2 - 1]}. \quad (3.4)$$

These two 4-scalars, z_a, with $h^{\alpha}h_{\alpha}$ and $(u_1u_2)^2 - 1$ constitute a set of four independent variables:

$$h^2 \equiv h^{\alpha}h_{\alpha}, \quad \Lambda^2 \equiv (u_1u_2)^2 - 1, \quad z_a, \quad (3.5)$$

and they replace the 4-scalars x^2, (xua), (u_1u_2). With these new variables, using the definition of (3.2), r_a can be written

$$r_a = [h^2 + \Lambda^2 z_a^2]^{1/2}. \quad (3.6)$$

From (3.1) we get

$$^{(1,1)}\xi_a^{\alpha} = \frac{\eta_a k}{m_a r_a^3}h^{\alpha} - \frac{z_a}{m_a r_a^3}t_{a'}^{\alpha}, \quad (3.7)$$

where

$$k \equiv -(u_1u_2). \quad (3.8)$$

Now since we want the first order in $\dot{\xi}_a^{\alpha}$, ${}^{(1,1)}\dot{\xi}_a^{\alpha}$ in our notation, we can write

$$^{(1,1)}\dot{\xi}_a^{\alpha} = u_a^{\rho}\frac{\partial\, {}^{(1,1)}\xi_a^{\alpha}}{\partial x^{a\rho}}. \quad (3.9)$$

On the other hand, it can be seen that

$$u_a^{\rho}\frac{\partial h^{\alpha}}{\partial x^{a\rho}} = 0, \quad u_a^{\rho}\frac{\partial t_b^{\rho}}{\partial x^{a\rho}} = 0, \quad (3.10)$$

$$u_a^{\rho}\frac{\partial \Lambda^2}{\partial x^{a\rho}} = 0, \quad u_a^{\rho}\frac{\partial z_b}{\partial x^{a\rho}} = \delta_{ab}, \quad (3.11)$$

thus

$$u_a^{\rho}\frac{\partial}{\partial x^{a\rho}} = \frac{\partial}{\partial z^a} \quad (3.12)$$

in the system of variables (3.5). Taking into account (3.10), (3.11), and (3.7), the calculation of (3.9) is straightforward. We get

$$^{(1,1)}\dot{\xi}_a^{\alpha} = -\frac{3\eta_a k\Lambda^2 z_a}{m_a r_a^5}h^{\alpha} + \left(\frac{3\Lambda^2 z_a^2}{m_a^2 r_a^5} - \frac{1}{m_a^2 r_a^3}\right)t_{a'}^{\alpha} \quad (3.13)$$

and using (2.7)

$$^{(3,1)}\xi_a^{\alpha} = -\frac{2\eta_a k\Lambda^2 z_a}{m_a^2 r_a^5}h^{\alpha} + \left(\frac{2\Lambda^2 z_a^2}{m_a^2 r_a^5} - \frac{2}{3m_a^2 r_a^3}\right)t_{a'}^{\alpha}. \quad (3.14)$$

4. MOMENTUM AND ANGULAR MOMENTUM FOR TWO PARTICLES

In this section we review the definitions of momentum and angular momentum in PRM, the asymptotic conditions and calculational techniques. Proofs and detailed explanations are omitted. We refer the reader to Ref. 6 for them.

The momentum P^{α} is a 4-vector *invariant by M_4 translations* such that

$$\frac{dP^{\alpha}}{ds_a} = 0 \quad \left(a = 1,2; \frac{d}{ds_a} = u_a^{\alpha}\frac{\partial}{\partial x^{a\alpha}} + \xi_a^{\alpha}\frac{\partial}{\partial u^{a\alpha}}\right). \quad (4.1)$$

The angular momentum is an antisymmetric 4-tensor, $J^{\alpha\beta}$, such that

$$\frac{dJ^{\alpha\beta}}{ds_a} = 0 \quad (4.2)$$

and its behavior under M_4 translations is given by

$$\frac{\partial J^{\alpha\beta}}{\partial x_1^{\gamma}} + \frac{\partial J^{\alpha\beta}}{\partial x_2^{\gamma}} = \delta_{\gamma}^{\beta}P^{\alpha} - \delta_{\gamma}^{\alpha}P^{\beta}. \quad (4.3)$$

Let us consider a canonical coordinate system q_a^α, p_a^α in PRM. If $q_a^\alpha - x_a^\alpha$, p_a^α are invariant vectors under the Poincaré group they are said to form an *adapted* canonical coordinate system. For such a coordinate system we have

$$P^\alpha = p_1^\alpha + p_2^\alpha, \tag{4.4}$$

$$J^{\alpha\beta} = q_a^\alpha p^{a\beta} - q_a^\beta p^{a\alpha}, \tag{4.5}$$

where P^α, $J^{\alpha\beta}$ given by these expressions are a solution of (4.1) and (4.2), (4.3), respectively. A particular kind of adapted canonical coordinates are the so-called Hamilton–Jacobi coordinates, which are characterized by the supplementary relations

$$\frac{dp_a^\alpha}{ds_b} = 0, \quad \frac{dq_a^\alpha}{ds_b} = \delta_{ab} p_a^\alpha. \tag{4.6}$$

We will first calculate a system of Hamilton–Jacobi coordinates and then (4.4) and (4.5) will give us P^α and $J^{\alpha\beta}$. In this way when determining momentum and angular momentum we get a coordinate canonical system and this is interesting if one wants to quantize the system. Calculations will be made in the perturbative scheme that we have mentioned in paragraph two. So, we make the assumption that p_a^α, q_a^α can be expanded in a power series of e_1, e_2. Accelerations ξ_a^α are known explicitly in this perturbative scheme up to fourth order: ${}^{(1,1)}\xi_a^\alpha$ is given in (3.1), ${}^{(3,1)}\xi_a^\alpha$ (term relative to $e_a^3 e_{a'}$) in (3.6) and ${}^{(2,2)}\xi_a^\alpha$ is not given here because we will not use it (it can be found in Ref. 4).

To the same fourth order we have to calculate ${}^{(1,1)}p_a^\alpha$, ${}^{(3,1)}p_a^\alpha$, and ${}^{(2,2)}p_a^\alpha$ and the same for q_a^α. The first terms, ${}^{(1,1)}p_a^\alpha$, and ${}^{(1,1)}q_a^\alpha$, can be found in Ref. 6: Obviously they are the same as in the more conventional case where the Dirac term is absent. We will limit ourselves to the calculation of the terms (3.1) which represent the first radiative "self-terms." We are able to do this because in the differential equations (4.6) they are not coupled to terms ${}^{(2,2)}p_a^\alpha$, ${}^{(2,2)}q_a^\alpha$. Neither are they coupled to ${}^{(2,2)}\xi_a^\alpha$.

In order to get a family of Hamilton–Jacobi coordinates giving us to this order one unique momentum and one unique intrinsic angular momentum, we need to define appropriate asymptotic conditions for Eqs. (4.6). To do so we work with the system of new variables (3.3).

The asymptotic conditions that we are going to define are of two different kinds, corresponding, roughly speaking, to the assumption that we have a free particle system when (a) $x^2 \to +\infty$ or when (b) $e_a e_{a'} \to 0$. As far as case (a) is concerned it can be seen that in the system of variables h^2, z_a, Λ^2, the limit $x^2 \to +\infty$ corresponds to one or both of these two different situations,

$$x^2 \to +\infty \Rightarrow \begin{cases} \text{(I)} \quad h^2 \to \infty, \quad \forall\, z_a, \\ \text{(II)} \quad z_a \to \gamma\infty, \quad \forall\, h^2, \end{cases} \tag{4.7}$$

where γ takes the values $+1$ or -1. Situation (I) means that we consider successive pairs of trajectories more and more further away. Situation (II) means that we go to future infinity ($\gamma = 1$) or past infinity ($\gamma = -1$) along the straight lines defined by the two initial 4-velocities. To distinguish both cases—future or past infinity—we will put $x^2 \to \infty_f$ or $x^2 \to \infty_p$, respectively. With these notations the first group of asymptotic conditions to be attached to (4.6) is

$$\lim_{x^2 \to \infty_p} p_a^\alpha = m_a u_a^\alpha, \quad \lim_{x^2 \to \infty_p} \frac{1}{x}(q_a^\alpha - x_a^\alpha) = 0, \tag{4.8}$$

or

$$\lim_{x^2 \to \infty_f} p_a^\alpha = m_a u_a^\alpha, \quad \lim_{x^2 \to \infty_f} \frac{1}{x}(q_a^\alpha - x_a^\alpha), \tag{4.9}$$

while the second group leads to

$${}^{(0,0)}p_a^\alpha = m_a u_a^\alpha, \quad {}^{(0,0)}q_a^\alpha = x_a^\alpha. \tag{4.10}$$

Because of (4.8) and (4.9), p_a^α, q_a^α are called "regular" in past infinity or future infinity, respectively.

It can be proved that if for a PRM system there exist invariant Poincaré vectors $p_a^\alpha, q_a^\alpha - x_a^\alpha$, such that (4.6), (4.8) [or alternatively (4.9)] are satisfied, then p_a^α, q_a^α are a set of canonical coordinates and so a set of Hamilton–Jacobi coordinates. Finally it can also be shown in a perturbative framework that Hamilton–Jacobi coordinates regular at infinity really do exist.[13]

Under supplementary assumptions which roughly speaking reduce again to the general assumption that we get a free particle system when $x^2 \to +\infty$, it can be proved that

$$p_a^\alpha p_{a\alpha} = -m_a^2. \tag{4.11}$$

These identities show that, in the language of Dirac,[14] we are in the presence of a dynamical system with primary constraints. These constraints have their origin in the identities $u_a^2 = -1$. Primary constraints introduce difficulties when one tries to quantize classical dynamical systems. One way of getting round this problem is to substitute the dynamical system (1.1) by a new auxiliary one, the so-called "auxiliary dynamical system":

$$\frac{dx_a^\alpha}{d\lambda} = \pi_a^\alpha, \quad \frac{d\pi_a^\alpha}{d\lambda} = \theta_a^\alpha(x_b^\beta, \pi_c^\gamma), \tag{4.12}$$

where λ is a 4-scalar parameter and ($\pi_a^2 \equiv -\pi_a^\alpha \pi_{a\alpha}$)

$$\theta_a^\alpha(x_b^\beta, \pi_c^\gamma) = \pi_a^2 \xi_a^\alpha(x_b^\beta, \pi_c^{-1}\pi_c^\gamma, m_d \to \pi_d). \tag{4.13}$$

In this way we go round the constraints $u_a^2 = -1$ since the π_a^2 are now to be considered as two new independent variables to add to the other four: x^2, $(x\pi a)$, $(\pi_1\pi_2)$. Definitions (3.3) and (3.4) now become

$$\tilde{h}^\alpha \equiv x^\alpha - \tilde{z}_1 \pi_1^\alpha + \tilde{z}_2 \pi_2^\alpha, \quad \tilde{t}_a^\alpha \equiv \pi_{a'}^2 \pi_a^\alpha + (\pi_1\pi_2)\pi_{a'}^\alpha, \tag{4.14}$$

and

$$\tilde{z}_a \equiv \frac{\eta_a[\pi_{a'}^2(x\pi_a) + (\pi_1\pi_2)(x\pi_{a'})]}{\tilde{\Lambda}_2}, \tag{4.15}$$

with

$$\tilde{\Lambda}^2 \equiv (\pi_1\pi_2)^2 - \pi_a^2 \pi_{a'}^2. \tag{4.16}$$

Now we can determine Hamilton–Jacobi coordinates, $\tilde{p}_a^\alpha, \tilde{q}_a^\alpha$, regular at past (or future) infinity for the dynamical system (4.12), taking 4-vectors $\tilde{p}_a^\alpha, \tilde{q}_a^\alpha$ such that $\tilde{p}_a^\alpha$, $q_a^\alpha - x_a^\alpha$ are Poincaré invariant and such that they are solutions of (4.6) satisfying the asymptotic conditions

$$\lim_{x^i\to\infty_{\prime\prime}} \tilde p^\alpha_a = \pi^\alpha_a, \quad \lim_{x^i\to\infty_{\prime\prime}} \frac{1}{x}(\tilde q^\alpha_a - x^\alpha_a), \tag{4.17}$$

or the equivalent ones for the infinite future, all of which are the subsidiary conditions corresponding to (4.8) or (4.9), respectively. On the other hand, conditions (4.10) become

$$^{(0,0)}\tilde p^\alpha_a = \pi^\alpha_a, \quad ^{(0,0)}\tilde q^\alpha_a = x^\alpha_a. \tag{4.18}$$

The differential operator d/ds_a is now

$$\frac{d}{ds_a} = \pi^\alpha_a \frac{\partial}{\partial x^{a\alpha}} + \theta^\alpha_a \frac{\partial}{\partial \pi^{a\alpha}}$$

instead of the similar expression given in (4.1). Finally the identity (4.11) becomes

$$\tilde p^\alpha_a \tilde p_{a\alpha} = -\pi^\alpha_a \pi_{a\alpha} \equiv \pi^2_a. \tag{4.19}$$

From $\tilde p^\alpha_a$, $\tilde q^\alpha_a$ we get the momentum, $\widetilde P^\alpha$, and angular momentum, $\widetilde J^{\alpha\beta}$ of the auxiliary dynamical system (4.11) through expressions like (4.4), (4.5). Then it can be proved that we obtain P^α and $J^{\alpha\beta}$, the momentum and angular momentum, respectively, of the original dynamical system (1.1) by "mass geometrization" of $\widetilde P^\alpha$ and $\widetilde J^{\alpha\beta}$, that is, making the substitution $\pi^\alpha_a \to m_a u^\alpha_a$.

5. CALCULATION OF THE LOWEST RADIATIVE "SELF-TERM" IN MOMENTUM AND ANGULAR MOMENTUM

Taking (3.14) into account we get from (4.13)

$$^{(3,1)}\theta^\alpha_a = -\frac{2\eta_a \pi^2_{a'} \widetilde k \widetilde\Lambda^2 \tilde z_a}{\pi^2_a \tilde r^5_a} \widetilde h^\alpha + \frac{2\pi^2_{a'}}{\pi^2_a \tilde r^3_a}\left(\frac{\widetilde\Lambda^2 \tilde z^2_a}{\tilde r^2_a} - \frac{1}{3}\right)\widetilde t^\alpha_{a'}, \tag{5.1}$$

where

$$\tilde r_a \equiv (\pi^2_{a'} \widetilde h^2 + \widetilde\Lambda^2 \tilde z^2_a)^{1/2}, \quad \widetilde k \equiv -(\pi_1 \pi_2). \tag{5.2}$$

Let us write $\tilde p^\alpha_a$ in the general form

$$\tilde p^\alpha_a = \eta_a \widetilde\alpha_a \widetilde h^\alpha + \tilde\mu_{aa} \widetilde t^\alpha_a + \tilde\mu_{aa'} \widetilde t^\alpha_{a'}. \tag{5.3}$$

In Ref. 6 the reader will find the first order expressions for $\tilde p^\alpha_a$ and $\tilde q^\alpha_a$, which in our notations are $^{(1,1)}\tilde p^\alpha_a, {}^{(1,1)}\tilde q^\alpha_a$. Here we are interested in $^{(3,1)}\tilde p^\alpha_a$ (and later on in $^{(3,1)}\tilde q^\alpha_a$). From the first group of Eqs. (4.6) we get to this order

$$\pi^\rho_b \frac{\partial^{(3,1)}\tilde p^\alpha_a}{\partial x^{b\rho}} + {}^{(3,1)}\theta^\rho_b \frac{\partial^{(0,0)}\tilde p^\alpha_a}{\partial \pi^{b\rho}} = 0, \tag{5.4}$$

or taking (4.19) into account

$$D_b\, {}^{(3,1)}\tilde p^\alpha_a = -\delta_{ab}\, {}^{(3,1)}\theta^\alpha_b, \tag{5.5}$$

where δ_{ab} is the Kronecker delta and D_b is the differential operator

$$D_b \equiv \pi^\rho_b \frac{\partial}{\partial x^{b\rho}}. \tag{5.6}$$

Now it can be seen that

$$D_b \widetilde h^\alpha = D_b \widetilde t^\alpha_a = 0. \tag{5.7}$$

From (4.19) and the first equation of (4.18) we get

$$^{(3,1)}\tilde\mu_{aa} = 0. \tag{5.8}$$

Taking this and (5.7) into account, Eqs. (5.4) become

$$\frac{\partial^{(3,1)}\widetilde\alpha_a}{\partial \tilde z_b} = \frac{2\pi^2_{a'} \widetilde k \widetilde\Lambda^2 \tilde z_a}{\pi^2_a \tilde r^5_a}\delta_{ab}, \tag{5.9}$$

$$\frac{\partial^{(3,1)}\tilde\mu_{aa'}}{\partial \tilde z_b} = \left(\frac{2\pi^2_{a'}}{3\pi^2_a \tilde r^3_a} - \frac{2\Lambda^2 \tilde z^2_a \pi^2_{a'}}{\pi^2_a \tilde r^5_a}\right)\delta_{ab}, \tag{5.10}$$

where use has been made of the following results:

$$D_b \pi^2_a = D_b \widetilde h^2 = D_b \widetilde\Lambda^2 = 0, \quad D_b \tilde z_a = \delta_{ab}. \tag{5.11}$$

[We can get (3.10) and (3.11) from (5.7) and (5.11), respectively, by making the substitutions: $\pi^\alpha_a \to m_a u^\alpha_a$.] On the other hand, we have the asymptotic conditions (4.20) (or the equivalent ones for the future). The only solution to (5.8), (5.9) "regular" at infinity is

$$^{(3,1)}\widetilde\alpha_a = \frac{2\pi^2_{a'} \widetilde k \widetilde\Lambda^2}{\pi^2_a}\int_{\gamma\infty}^{\tilde z_a} \frac{\tilde z_a d\tilde z_a}{\tilde r^5_a} = -\frac{2\widetilde k \pi^2_{a'}}{3\pi^2_a \tilde r^3_a}, \tag{5.12}$$

$$^{(3,1)}\tilde\mu_{aa'} = \frac{2\pi^2_{a'}}{3\pi^2_a}\int_{\gamma\infty}^{\tilde z_a} \frac{d\tilde z_a}{\tilde r^3_a} - \frac{2\widetilde\Lambda^2 \pi^2_{a'}}{\pi^2_a} \times \int_{\gamma\infty}^{\tilde z_a} \frac{\tilde z^2_a d\tilde z_a}{\tilde r^5_a} = \frac{2\pi^2_{a'} \tilde z_a}{3\pi^2_a \tilde r^3_a}. \tag{5.13}$$

So we have

$$^{(3,1)}\tilde p^\alpha_a = -\frac{2\eta_a \pi^2_{a'} \widetilde k}{3\pi^2_a \tilde r^3_a}\widetilde h^\alpha + \frac{2\pi^2_{a'} \tilde z_a}{3\pi^2_a \tilde r^3_a}\widetilde t^\alpha_{a'}. \tag{5.14}$$

Let us calculate $^{(3,1)}\tilde q^\alpha_a$. From the second equation (4.6) we get

$$D_b\, {}^{(3,1)}\tilde q^\alpha_a = {}^{(3,1)}\tilde p^\alpha_a \delta_{ab} \tag{5.15}$$

and if we write

$$^{(3,1)}\tilde q^\alpha_a = \eta_a\, {}^{(3,1)}\widetilde\gamma_a \widetilde h^\alpha + {}^{(3,1)}\tilde\nu_{aa} \widetilde t^\alpha_a + {}^{(3,1)}\nu_{aa'} \widetilde t^\alpha_{a'}, \tag{5.16}$$

Eqs. (5.15) become

$$\frac{\partial^{(3,1)}\widetilde\gamma_a}{\partial \tilde z_b} = \frac{-2\pi^2_{a'} \widetilde k}{3\pi^2_a \tilde r^3_a}\delta_{ab}, \tag{5.17}$$

$$\frac{\partial^{(3,1)}\tilde\nu_{aa'}}{\partial \tilde z_b} = \frac{2\pi^2_{a'} \tilde z_a}{3\pi^2_a \tilde r^3_a}\delta_{ab}, \tag{5.18}$$

$$\frac{\partial^{(3,1)}\tilde\nu_{aa'}}{\partial \tilde z_b} = 0. \tag{5.19}$$

The general solution of these equations satisfying the asymptotic condition given by the second of the equations (4.17) (or the equivalent ones for the future) gives the following expression for $^{(3,1)}\tilde q^\alpha_a$,

$$^{(3,1)}\tilde q^\alpha_a = \frac{2\eta_a \widetilde k}{3\pi^2_a \widetilde h^2}\left(\frac{\gamma}{\widetilde\Lambda} - \frac{\tilde z_a}{\tilde r_a}\right) h^\alpha - \frac{2\pi^2_{a'}}{3\pi^2_a \widetilde\Lambda^2 \tilde r_a} t^\alpha_{a'} + {}^{(3,1)}_{\ *}\nu_{aa} \widetilde t^\alpha_a + {}^{(3,1)}_{\ *}\nu_{aa'} \widetilde t^\alpha_{a'}, \tag{5.20}$$

where ${}^{(3,1)}_{\ \bullet}\nu_{ab}$ are arbitrary functions of $\widetilde{h}^2, \widetilde{\Lambda}^2, \pi_b$ except for the condition that $\lim \widetilde{h}^{-1}\ {}_{\bullet}\nu_{ab} = 0$. We can see that at variance with ${}^{(3,1)}\tilde{p}^{\alpha}_a$, we do not have uniqueness for the coordinates ${}^{(3,1)}\tilde{q}^{\alpha}_a$.

Using expressions (5.14), the formula (4.4), and making the substitution $\pi^{\alpha}_a \to m_a u^{\alpha}_a$ we get one unique radiative "self-term" in linear momentum,

$$ {}^rP^{\alpha} \equiv e_1^3 e_2\, {}^{(3,1)}p_1^{\alpha} + e_2^3 e_1\, {}^{(3,1)}p_2^{\alpha} = \tfrac{2}{3}\, e_1 e_2 k \left(\frac{e_2^2}{m_2 r_2^3} - \frac{e_1^2}{m_1 r_1^3} \right) h^{\alpha} + \tfrac{2}{3} e_1 e_2 \left(\frac{e_1^3 z_1}{m_1 r_1^3} t_2^{\alpha} + \frac{e_2^3 z_2}{m_2 r_2^3} t_1^{\alpha} \right). \tag{5.21} $$

Since the ${}^{(3,1)}\tilde{q}^{\alpha}_a$ given by (5.20) are not unique, the radiative "self-term" in angular momentum ${}^rJ^{\alpha\beta}$, given by (4.5), (5.14), (5.20) is not unique. Nevertheless it can be seen that the radiative "self-term," ${}^rW^{\alpha}$, of the intrinsic angular momentum W^{α}, defined by

$$ W^{\alpha} \equiv \frac{1}{2P}\, \delta^{\alpha\beta\lambda\mu} P_{\beta} J_{\lambda\mu} [P \equiv (-P^{\alpha}P_{\alpha})^{1/2},\ \delta^{0123} = 1] \tag{5.22} $$

is unique. We first get

$$ e_1^3 e_2\, {}^{(3,1)}\widetilde{W}^{\alpha} + e_2^3 e_1\, {}^{(1,3)}\widetilde{W}^{\alpha} = \frac{1}{[-({}^{(0,0)}\widetilde{P}^{\alpha}\, {}^{(0,0)}\widetilde{P}_{\alpha})]^{1/2}} \Big[e_1^3 e_2 ({}^{(3,1)}\gamma_1 - \tilde{z}_1\, {}^{(3,1)}\widetilde{\alpha}_1) + e_2^3 e_1 ({}^{(3,1)}\gamma_2 - \tilde{z}_2\, {}^{(3,1)}\widetilde{\alpha}_2) $$

$$ - \left(\frac{\widetilde{\Lambda}^2}{{}^{(0,0)}\widetilde{P}^{\alpha}\, {}^{(0,0)}\widetilde{P}_{\alpha}} + k \right) (e_1^3 e_2\, {}^{(3,1)}\widetilde{\mu}_{12} + e_2^3 e_1\, {}^{(3,1)}\widetilde{\mu}_{21}) \Big] \delta^{\alpha\beta\gamma\delta} x_{\beta} \pi_{1\gamma} \pi_{2\delta} \tag{5.23} $$

and then

$$ {}^rW^{\alpha} \equiv e_1^3 e_2\, {}^{(3,1)}W^{\alpha} + e_2^3 e_1\, {}^{(1,3)}W^{\alpha} = \frac{1}{(m_1^2 + m_2^2 + 2m_1 m_2 k)^{1/2}} \Big\{ e_1^3 e_2 \Big[\frac{2m_2 k}{3m_1 h^2} \Big(\frac{\gamma}{\Lambda} - \frac{z_1}{r_1} \Big) $$

$$ + \frac{2m_2^2 \Lambda^2 z_1}{3(m_1^2 + m_2^2 + 2m_1 m_2 k) r_1^3} \Big] + e_2^3 e_1 \Big[\frac{2m_1 k}{3m_2 h^2} \Big(\frac{\gamma}{\Lambda} - \frac{z_2}{r_2} \Big) $$

$$ + \frac{2m_1^2 \Lambda^2 z_2}{3(m_1^2 + m_2^2 + 2m_1 m_2 k) r_2^3} \Big] \Big\} n^{\alpha}, \tag{5.24} $$

where we have put

$$ \delta^{\alpha\beta\lambda\mu} x_{\beta} u_{1\lambda} u_{2\mu} \equiv n^{\alpha}. \tag{5.25} $$

One could ask why we have not considered the terms ${}^{(1,3)}\tilde{p}^{\alpha}_a, {}^{(1,3)}\tilde{q}^{\alpha}_a$. The answer is that if one considers them, then a similar calculation such as the one that has been used to calculate ${}^{(3,1)}\tilde{p}^{\alpha}_a, {}^{(3,1)}\tilde{q}^{\alpha}_a$, gives ${}^{(1,3)}\tilde{p}^{\alpha}_a = 0$, ${}^{(1,3)}\tilde{q}^{\alpha}_a = {}^{(1,3)}_{\ \bullet}\nu_{aa'}\tilde{t}^{\alpha}_{a'} + {}^{(1,3)}_{\ \bullet}\nu_{aa'}\tilde{t}^{\alpha}_{a'}$ and these expressions do not change either (5.21) or (5.24). ${}^rW^{\alpha}$ depends on γ and so it has different values at past or future infinity (it goes to zero at one of these two infinities depending on γ). In the language of Ref. 6, when radiation is present, W^{α} is not "conservative." Of course, W^{α} maintains its numerical values along a given pair of trajectories, but it does not keep the form of the expressions corresponding to free particles, which W^{α} takes at the past (or future) infinity. On the other hand, P^{α} does not depend on γ, so it is "conservative."

When the Dirac term is not considered, P^{α}, W^{α} are conservative to first order[6]: That is, ${}^{(1,1)}P^{\alpha}$, ${}^{(1,1)}W^{\alpha}$ do not depend on γ. [It can be easily recognized that terms like ${}^{(n,n)}P^{\alpha}$, ${}^{(n,n)}W^{\alpha}$ are the same if we consider the Dirac term (Lorentz–Dirac equation) as if we do not (Lorentz equation with retarded potentials).].

So, to fourth order in $e_1^n e_2^m$ $(n + m \leqslant 4)$, we have that the momentum P^{α}

$$ P^{\alpha} = m_1 u_1^{\alpha} + m_2 u_2^{\alpha} + e_1 e_2\, {}^{(1,1)}P^{\alpha} + e_1^2 e_2^2\, {}^{(2,2)}P^{\alpha} + {}^rP^{\alpha}, \tag{5.26} $$

with ${}^rP^{\alpha}$ given by (5.21), is numerically conserved along any given pair of trajectories and the same can be said about W^{α}. In the spirit of field theory and to this order the first part of (5.26)— ${}^pP^{\alpha} \equiv m_1 u_1^{\alpha} + m_2 u_2^{\alpha} + e_1 e_2\, {}^{(1,1)}P^{\alpha} + e_1^2 e_2^2\, {}^{(2,2)}P^{\alpha}$— could be interpreted as the total momentum of the system, consisting of the two charges plus their electromagnetic field, excluding self-interactions, while ${}^rP^{\alpha}$ would represent the momentum of the radiated electromagnetic field that takes the self-interaction into account. Neither ${}^pP^{\alpha}$, nor ${}^rP^{\alpha}$ are conserved numerically by themselves; but their sum to this order is conserved. In a similar way we could define ${}^pW^{\alpha}$ and then decompose W^{α} as ${}^pW^{\alpha} + {}^rW^{\alpha}$.

Now we make the choice $\gamma = -1$ in (5.24), so we take the intrinsic angular momentum W^{α} to be "regular" at past infinity.

Let us introduce ${}^rW$ such that ${}^rW^{\alpha} = {}^rWn^{\alpha}$. Then obviously we have

$$ \lim_{x^2 \to \infty_p} {}^rW = 0 $$

and for future infinity

$$ {}^{r}W_{+\infty} \equiv \lim_{x^{0}\to\infty, i} {}^{r}W = -\frac{4m_1m_2e_1e_2(e_1^2/m_1^2 + e_2^2/m_2^2)k}{3(m_1^2 + m_2^2 + 2m_1m_2k)^{1/2}\Lambda h^2}. \quad (5.27) $$

Since

$$ \lim_{x^{0}\to\infty, i} {}^{(1,1)}W^{\alpha} = 0, $$

we have the evident notation

$$ \lim_{x^{0}\to\infty, i} W^{\alpha} = {}^{(0,0)}W^{\alpha} + e_1^2e_2^2\,{}^{(2,2)}W_{+\infty}n^{\alpha} + {}^{r}W_{+\infty}n^{\alpha}, \quad (5.28) $$

where it is clear that having $W^{\alpha} = {}^{(0,0)}W^{\alpha}$ at past infinity we do not recover the form of this expression in the infinite future. [From the definition (5.25) we get

$$ {}^{(0,0)}W^{\alpha} = m_1m_2n^{\alpha}/(m_1^2 + m_2^2 + 2m_1m_2k)^{1/2}.] $$

On the other hand, if we calculate

$$ {}^{r}P^{\alpha}_{+\infty} \equiv \lim_{x^{0}\to\infty, i} {}^{r}P^{\alpha} $$

we will get zero since ${}^{r}P^{\alpha}$ does not depend on γ.

From the point of view of the "theory of the absorber" (see Sec. 1) $e_1^2e_2^2\,{}^{(2,2)}W_{+\infty}n^{\alpha} + {}^{r}W_{+\infty}n^{\alpha}$ could be interpreted as all the intrinsic angular momentum that our two particles, have delivered during all their history to the charges of the entire universe. The term ${}^{r}W_{+\infty}n^{\alpha}$ accounts for the intrinsic angular momentum which corresponds, in the language of field theory to the "self-terms" of the radiated electromagnetic field, while the other term $e_1^2e_2^2\,{}^{(2,2)}W_{+\infty}n^{\alpha}$ belongs to the system consisting of the two charges plus their electromagnetic field (self-interaction excluded). This term would be absent if the two charges were to interact through time-reversal invariant potentials instead of through retarded ones, as it is actually the case in the Lorentz–Dirac equation (2.1).

Similar considerations can be made about the momentum P^{α} but now the term ${}^{r}P^{\alpha}_{+\infty}$ cancels, that is to say, in our approximation, the all radiated "self-term" in the momentum by the two interacting particles, is zero.

Probably we would have to go on with our expansion in $e_1^ne_2^m$ up to order $n + m = 6$ to get ${}^{r}P^{\alpha}_{+\infty} \neq 0$. In fact dipolar radiation in conventional electrodynamics begins at sixth order in the charges involved.

We end this paper by making some physical considerations. First of all, we could recover the formal expansions in the charges of this article by making the reasonable assumption that the magnitudes we have calculated can be expanded in powers of the dimensionless quantities $e_ae_b/c^2m_c x$ ($x \equiv 3$ – distance between the two charges). These are not Lorentz scalar quantities. So, we cannot attach an invariant meaning to the fact that these quantities were small in a given inertial system. In spite of this, expansions in $e_ae_b/c^2m_c x$ are meaningful since, because of the Poincaré invariance, accelerations, momentum, etc., are always the same functions of positions and velocities, no matter which inertial system we are talking about. Then each inertial observer must only make sure that quantities $e_ae_b/c^2m_c x$ are small enough for him in order to get a fast convergence of the expansions.

For electrons we have $e_ae_b/c^2m_c x \sim 1$ when $x \sim 3$ Fermi. So we can see that $e_ae_b/c^2m_c x$ will be very small in all the physical situations where the classical theory developed here can be used.

In the case of the accelerations, where the terms ${}^{(2,2)}\xi^{\alpha}_{a}$ are known,[4] we can easily get the 3-accelerations $\mu^i_a = d^2x^i_a/dt^2$ up to third order in $1/c$. (see, for instance, Ref. 8 about the three-dimensional formalism of the PRM.) From ${}^{(1,1)}\xi^{\alpha}_{a}$, ${}^{(3,1)}\xi^{\alpha}_{a}$, given by (3.1), (3.14), respectively, from ${}^{(2,2)}\xi^{\alpha}_{a}$,[4] and making use of the relation[7]

$$ \mu^i_a = \left(1 - \frac{v_a^2}{c^2}\right)\left[(\xi^{\,i}_{a})_{t_a = t} - \frac{1}{c}(\xi^{\,0}_{a})_{t_a = t}V^i_a\right] \quad (5.29) $$

(where V^i_a is the 3-velocity of the particle a) we get to third order in $1/c$

$$ \mu^i_a = \frac{\eta_a e_1e_2}{m_ar^3}r^i + \frac{\eta_a e_1e_2}{c^2m_ar^3}\left[\left(\frac{1}{2}V^2_{a'} - \mathbf{V}_1\cdot\mathbf{V}_2 - \frac{3}{2}\frac{(\mathbf{r}\cdot\mathbf{V}_{a'})^2}{r^2}\right)r^i - \eta_a(\mathbf{r}\cdot\mathbf{V}_a)V^i\right] + \frac{\eta_a e_1^2e_2^2}{c^2m_1m_2r^4}r^i + \frac{2\eta_a e_1^2e_2^2\mathbf{r}\cdot\mathbf{V}}{c^3m_1m_2r^5}r^i - \frac{2\eta_a e_a^3e_{a'}\mathbf{r}\cdot\mathbf{V}}{c^3m_a^2r^5}r^i - \frac{2\eta_a e_1^2e_2^2}{3c^3m_1m_2r^3}V^i + \frac{2\eta_a e_a^3e_{a'}}{3c^3m_a^2r^3}V^i, \quad (5.30) $$

where we have put

$V^i \equiv V^i_1 - V^i_2$, $\quad r^i \equiv x^i$, $\quad V \equiv (\mathbf{V}\cdot\mathbf{V})^{1/2}$, $\quad r \equiv (\mathbf{r}\cdot\mathbf{r})^{1/2}$

and $\mathbf{V}\cdot\mathbf{V}$, $\mathbf{r}\cdot\mathbf{r}$ are R^3 scalar products.

In (5.30), to zeroth order, we recognize, Coulomb's law. The second order is the correction of Darwin's Lagrangian to Coulomb's law.[7] In the third we have two kind of terms: Only those in $e_a^3e_{a'}$, come from the Dirac term in the Lorentz–Dirac equation. We can think about it as a radiative term. The other two terms in $1/c^3$ have nothing to do with the Dirac term. They would be absent if we had taken time-reversal potentials instead of the retarded ones that must be used in the case of the Lorentz–Dirac equation.

Notice the fact that the term of μ^i_{α} in $1/c^3$ vanishes when $e_1/m_1 = e_2/m_2$.

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[1]There is extensive literature on the Lorentz–Dirac equation, see for instance, the review article of G.N. Plass. [Rev. Mod. Phys. **33**, 37 (1961)]. References to later works can be found in E.G.P. Rowe, Phys. Rev. D **12**, 1576 (1975); R.G. Beil, *ibid.* **12**, 2266 (1975); and J.C. Herrera, *ibid.* **15**, 453 (1977), and in the papers to which these authors refer.

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