Poincaré–Cartan integral invariant and canonical transformations for singular Lagrangians: An addendum

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The results of a previous work, concerning a method for performing the canonical formalism for constrained systems, are extended when the canonical transformation proposed in that paper is explicitly time dependent.

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In a previous paper[1] we discussed in the framework of the Poincaré–Cartan integral invariant, a method for performing the canonical formalism for constrained systems. The basic idea consists of considering a canonical transformation which brings the constraints into a subset of the canonical variables. Thus the physical variables can be easily obtained by means of a reduction of the phase space. Our method is different from the path-integral approach of Faddeev[2] (see also Ref. 3), which use in addition a set of gauge-fixing conditions, one for each first-class constraint. Two applications of our procedure concerning action-at-a-distance relativistic models have been recently studied.[4]

In this note we extend the method by considering a time-dependent general canonical transformation, such that all the constraints acquire an explicit time dependence.

Let us consider a dynamical system described in terms of $2n$ degrees of freedom in the phase space q_s, p_s $(s = 1,\dots,n)$ and constrained to the hypersurface S defined by

$$\Omega_\alpha(q,p) = 0 \quad (\alpha = 1,\dots,T - W), \tag{1}$$

$$\Omega_\beta(q,p) = 0 \quad (\beta = T - W + 1,\dots,T), \tag{2}$$

where Ω_α are $T - W$ first-class[5] and Ω_β W second-class constraints. In order to guarantee the stability of S during the evolution, the Ω_α are required to satisfy

$$\{\Omega_\alpha, H_c\} \approx 0, \tag{3}$$

where H_c is the canonical Hamiltonian. The notation "$\approx$" means equality on the hypersurface S ("weak" equality).

Now, given the set (2), according to some theorems on function groups[6] and involutory systems[7] it is possible, at least locally, to find a canonical transformation

$$\{q_s,p_s, \quad s = 1,\dots,n\} \to \{Q'_s,P'_s, \quad s = 1,\dots,n\}, \tag{4}$$

such that the equations

$$Q'_f = P'_f = 0 \quad (f = n_2 + 1,\dots,n),(n_2 = n - W/2), \tag{5}$$

define the same surface as Eqs. (2) and the following equations,

$$\{Q'_s,P'_{s'}\} = \delta_{ss'},$$
$$\{Q'_s,Q'_{s'}\} = \{P'_s,P'_{s'}\} = 0, \tag{6}$$

are identically (and not only "weakly") satisfied.

If we denote the generating function by F, defined as

$$p_s\delta q_s - H_c\delta t = P'_s\delta Q'_s - K_c\delta t - \delta F, \tag{7}$$

the Hamilton equations for the new variables are given by

$$\dot{Q}'_s \approx \{Q'_s, K(Q'_s,P'_s,t)\}, \quad \dot{P}'_s \approx \{P'_s, K(Q'_s,P'_s,t)\} \tag{8}$$

where K,

$$K = K_c + l_\alpha \tilde{\Omega}_\alpha - \tilde{\Omega}_\beta \tilde{C}_{\beta\beta'} \left[\{\tilde{\Omega}_{\beta'}, K_c\} + \frac{\partial \tilde{\Omega}_{\beta'}}{\partial t} \right], \tag{9}$$

is the extended Hamiltonian with l_α arbitrary functions. $\tilde{\Omega}_{\alpha,\beta}$ are obtained from Eqs. (1) and (2) by substitution of variables, and $\tilde{C}_{\beta\beta'}$ is defined by

$$\tilde{C}_{\beta\beta'}\{\tilde{\Omega}_{\beta'},\tilde{\Omega}_{\beta''}\} \approx \delta_{\beta\beta''}. \tag{10}$$

In I we have shown that it is possible to write the equations of motion for the reduced set of variables $R' = \{Q'_j,P'_j, j = 1,\dots,n_2\}$ which are free with respect to the second-class constraints (5) in a simple form

$$\dot{Q}'_j \approx \{Q'_j,\bar{K}\}_{R'}, \quad \dot{P}'_j \approx \{P'_j,\bar{K}\}_{R'}, \tag{11}$$

$$\bar{K} = \bar{K}(Q'_j,P'_j,t) = \bar{K}_c(Q'_j,P'_j,t) + l_\alpha \bar{\Omega}_\alpha(Q'_j,P'_j,t) \tag{12}$$

where $\{\ ,\ \}_{R'}$ denote the Poisson brackets defined on the space R' and $\bar{K}_c$ and $\bar{\Omega}_\alpha$ are obtained by setting equal to zero the variables Q'_f and P'_f, corresponding to the second-class constraints, in K_c and $\tilde{\Omega}_\alpha$ of Eq. (9). As shown in I the $\bar{\Omega}_\alpha$ so obtained are first class, i.e.,

$$\{\bar{\Omega}_\alpha,\bar{\Omega}_{\alpha'}\}_{R'} \approx 0 \tag{13}$$

and, as a consequence of $(d/dt)\Omega_\alpha(q,p) \approx 0$, satisfy the stability condition

$$\frac{d}{dt}\bar{\Omega}_\alpha = \frac{\partial \bar{\Omega}_\alpha}{\partial t} + \{\bar{\Omega}_\alpha,\bar{K}_c\}_{R'} \approx 0. \tag{14}$$

In Eq. (14) we have now supposed the $\bar{\Omega}_\alpha$ explicitly time dependent, unlike what we did for the sake of simplicity in I.

A similar procedure of reduction of the phase space can be performed also for the first-class constraints. In fact, a theorem on involutory systems[7] guarantees that it is possible, at least locally, to replace the $\bar{\Omega}_\alpha$ by an equivalent set of equations

$$P_e(Q'_j,P'_j,t) = 0 \quad (e = n_1 + 1,\dots,n_2), \tag{15}$$

$(n_1 = n - T + W/2)$, which are in involution. For instance, the set (15) can be obtained by solving the equations

$$\bar{\Omega}_\alpha(Q_j',P_j',t)=0 \quad (\alpha=1,...,n_2-n_1) \tag{16}$$

with respect to an equal number n_2-n_1 of momenta. Without loss of generality we suppose Eq. (16) be solved with respect to $P_e'(e=n_1+1,...,n_2)$, or

$$|\partial\bar{\Omega}_\alpha/\partial P_e'|\neq 0. \tag{17}$$

Let

$$P_e=P_e'-f_e(Q_e',Q_k',P_k',t) \quad (k=1,...,n_1) \tag{18}$$

be the expression of the equations in involution. The stability of the hypersurface (18) can be easily proved. In fact, from

$$\bar{\Omega}_\alpha(Q_k',Q_e',P_k',P_e'=f_e(Q_k',Q_e',P_k',t),t)=0 \tag{19}$$

we get

$$\frac{\partial\bar{\Omega}_\alpha}{\partial t}\approx-\frac{\partial\bar{\Omega}_\alpha}{\partial P_e'}\frac{\partial f_e}{\partial t}=\frac{\partial\bar{\Omega}_\alpha}{\partial P_e'}\frac{\partial P_e}{\partial t}, \tag{20}$$

$$\begin{cases}-\{P_j',\bar{\Omega}_\alpha\}_{R'}=\dfrac{\partial\bar{\Omega}_\alpha}{\partial Q_j'}\approx-\dfrac{\partial\bar{\Omega}_\alpha}{\partial P_e'}\dfrac{\partial f_e}{\partial Q_j'}=-\dfrac{\partial\bar{\Omega}_\alpha}{\partial P_e'}\dfrac{\partial P_e}{\partial Q_j'}\\ \{Q_j'\bar{\Omega}_\alpha\}_{R'}=\dfrac{\partial\bar{\Omega}_\alpha}{\partial P_j'}\approx-\dfrac{\partial\bar{\Omega}_\alpha}{\partial P_e'}\dfrac{\partial f_e}{\partial P_j'}=-\dfrac{\partial\bar{\Omega}_\alpha}{\partial P_e'}\dfrac{\partial P_e}{\partial P_j'}\end{cases} \tag{21}$$

Therefore, from Eq. (14) we get

$$\frac{\partial\bar{\Omega}_\alpha}{\partial P_e'}\left[\frac{\partial P_e}{\partial t}+\{P_e,\bar{K}_c\}_{R'}\right]\approx 0, \tag{22}$$

and using Eq. (17)

$$\frac{\partial P_e}{\partial t}+\{P_e,\bar{K}_c\}_{R'}\approx 0. \tag{23}$$

As a final step we make a transformation

$$\{Q_j',P_j',j=1,...,n_2\}\to\{Q_k,P_k,Q_e,P_e,k=1,...,n_1,\ e=n_1+1,...,n_2\} \tag{24}$$

with

$$\{Q_k,P_{k'}\}=\delta_{kk'},\quad \{Q_e,P_{e'}\}=\delta_{ee'}, \tag{25}$$

where part of the momenta are the set of functions in the involution (18) which are equivalent to the first-class constraints.

If we denote the new canonical Hamiltonian by K_c' and the new expression for the constraints by

$$\hat{\Omega}_\alpha(Q_k,P_k,Q_e,P_e,t)=\bar{\Omega}_\alpha(Q_j'(Q_k,P_k,Q_e,P_e,t),P_j'(Q_k,P_k,Q_e,P_e,t),t), \tag{26}$$

the Hamiltonian equations are given by

$$\begin{cases}\dot{Q}_k\approx\{Q_k,K_c'+l_\alpha\hat{\Omega}_\alpha\}_R\\ \dot{P}_k\approx\{P_k,K_c'+l_\alpha\hat{\Omega}_\alpha\}_R\end{cases} \tag{27}$$

$$\begin{cases}\dot{Q}_e\approx\{Q_e,K_c'+l_\alpha\hat{\Omega}_\alpha\}_R\\ \dot{P}_e\approx\{P_e,K_c'+l_\alpha\hat{\Omega}_\alpha\}_R\end{cases} \tag{28}$$

where now $\{\ ,\ \}_R$ denote the Poisson brackets with respect to the set

$$R=\{Q_k,P_k,Q_e,P_e,k=1,...,n_1,e=n_1+1,...,n_2\}.$$

With respect to the stability of the hypersurface $\hat{\Omega}_\alpha=0$, after the canonical transformations (24) we have

$$\frac{\partial}{\partial t}\hat{\Omega}_\alpha+\{\hat{\Omega}_\alpha,K_c'\}_R\approx 0. \tag{29}$$

On the other hand, due to the equivalence between $\hat{\Omega}_\alpha$ and P_e we may write

$$\hat{\Omega}_\alpha(Q_k,P_k,Q_e,P_e,t)=g_{\alpha e'}(Q_k,P_k,Q_e,P_e,t)P_{e'},\quad \det|g|\neq 0, \tag{30}$$

where we introduced the strong equality notation "$\equiv$" following Sudarshan and Mukunda.[8]

Thus from Eq. (30) we have

$$\frac{\partial\hat{\Omega}_\alpha}{\partial t}\approx 0, \tag{31}$$

and using Eqs. (30) and (29) in Eq. (28), we get

$$\dot{P}_e=\{P_e,K_c'\}=\frac{\partial K_c'}{\partial Q_e}\approx 0. \tag{32}$$

In other words the variables Q_e are ignorable variables.

Finally, the remaining equations (27) and (28) become

$$\begin{cases}\dot{Q}_k\approx\{Q_k,K_c'\}_R\\ \dot{P}_k\approx\{P_k,K_c'\}_R\end{cases} \tag{33}$$

and

$$\dot{Q}_e=\{Q_e,K_c'\}_R+\lambda_e, \tag{34}$$

where $\lambda_e=g_{e\alpha}l_\alpha$ are arbitrary functions.

We can now consider the reduced space $[Q_k,P_k,Q_e]$, where Q_k and P_k satisfy

$$\dot{Q}_k=\frac{\partial\mathscr{K}_c}{\partial P_k},\quad \dot{P}_k=-\frac{\partial\mathscr{K}_c}{\partial Q_k}\quad (k=1,...,n_1), \tag{35}$$

with

$$\mathscr{K}_c=\mathscr{K}_c(Q_k,P_k,t)=K_c'(Q_k,P_k,Q_e,P_e,t)|_{P_e=0}. \tag{36}$$

where the Q_e dependence disappears due to Eq. (32) and the Q_e's are gauge-dependent variables

$$\dot{Q}_e=\left.\frac{\partial K_c'}{\partial P_e}\right|_{P_e=0}+\lambda_e\quad (e=n_1+1,...,n_2). \tag{37}$$

In conclusion, we have isolated the set of the gauge-dependent variables Q_e from a set of physical (gauge-independent) variables Q_k,P_k.

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