Petrov type D perfect-fluid solutions in generalized Kerr–Schild form

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Generalized Kerr–Schild space-times for a perfect-fluid source are investigated. New Petrov type D perfect fluid solutions are obtained starting from conformally flat perfect-fluid metrics.

I. INTRODUCTION

This work is concerned with perfect-fluid solutions of Einstein's equations for a metric in generalized Kerr–Schild form. Since the original Kerr–Schild paper,[1] a lot of generalizations of the Kerr–Schild ansatz have appeared.[2] Bilge and Gürses[3] have shown how the Newman–Penrose spin coefficients, trace-free Ricci, Ricci scalar, and Weyl spinors transform under the most general Kerr–Schild transformation. In this paper we treat generalized Kerr–Schild metrics of the form

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + 2Hl_\alpha l_\beta, \tag{1.1}$$

where $g_{\alpha\beta}$ is the metric of any space-time, l_α is a null, geodesic vector field for the metric $g_{\alpha\beta}$, and H is a scalar field.

As far as we know, no perfect-fluid solution of the Kerr–Schild type is known. All the solutions we obtain are of Petrov type D, and most of these are new since the velocity of the fluid does not lie in the two-space defined by the principal null directions of the Weyl tensor.[4]

In Sec. II we obtain the Riemann, Ricci, and Weyl tensors of the metric $\tilde{g}$ as functions of the Riemann, Ricci, and Weyl tensors of the metric g and the spin coefficients defined by a null tetrad associated with l_α. Our notation and calculations are quite close to those of Taub (Ref. 2). Section III is devoted to writing down the equations in the case where g is a conformally flat solution of Einstein's equations for a perfect fluid. It is shown easily that the geodesic (shear-free) null vector fields in a conformally flat space-time are the geodesic (shear-free) null vector fields in flat space-time. Since the most general vector field of this kind is already known,[5] one has great freedom in choosing the vector field l_α. Two cases appear depending on whether l_α is shear-free or not. They are studied in Secs. IV and V. Finally, in Sec. VI we give some examples of how the method works and some explicit solutions.

II. THE RIEMANN, RICCI, AND WEYL TENSORS OF GENERALIZED KERR–SCHILD METRICS

It is easily shown that[6]

$$\tilde{g}^{\alpha\beta} = g^{\alpha\beta} - 2Hl^\alpha l^\beta, \tag{2.1}$$

$$\tilde{l}^\alpha = l^\alpha, \quad \tilde{l}^\alpha \tilde{l}_\alpha = 0. \tag{2.2}$$

Then, we obtain for the Christoffel symbols

$$\tilde{\Gamma}^\alpha_{\beta\lambda} = \Gamma^\alpha_{\beta\lambda} + A^{\ \alpha}_{\beta\lambda} + 2Hl^\alpha l_\beta l_\lambda l^\mu \nabla_\mu H, \tag{2.3}$$

where the $\Gamma^\alpha_{\beta\lambda}$ are the Christoffel symbols for the metric g and

$$A^{\ \alpha}_{\beta\lambda} \equiv \nabla_\beta (Hl^\alpha l_\lambda) + \nabla_\lambda (Hl^\alpha l_\beta) - \nabla^\alpha (Hl_\beta l_\lambda) \tag{2.4}$$

or

$$A^{\ \alpha}_{\beta\lambda} = Hl^\alpha S_{\beta\lambda} + l_\beta (l^\alpha \nabla_\lambda H + HA_\lambda{}^\alpha) + l_\lambda (l^\alpha \nabla_\beta H + HA_\beta{}^\alpha) - l_\beta l_\lambda \nabla^\alpha H, \tag{2.5}$$

with

$$S_{\alpha\beta} \equiv \nabla_\alpha l_\beta + \nabla_\beta l_\alpha, \quad A_{\alpha\beta} \equiv \nabla_\alpha l_\beta - \nabla_\beta l_\alpha. \tag{2.6}$$

Next, we compute the Riemann tensor from the expression (2.3) and we find the following:

$$\begin{aligned}\tilde{R}^{\ \alpha}_{\beta\lambda\mu} = {} & R^{\ \alpha}_{\beta\lambda\mu} + \nabla_\lambda A^{\ \alpha}_{\beta\mu} - \nabla_\mu A^{\ \alpha}_{\beta\lambda} \\ & + \nabla_\lambda \left[2Hl^\alpha l_\beta l_\mu l^\rho \nabla_\rho H\right] \\ & - \nabla_\mu \left[2Hl^\alpha l_\beta l_\lambda l^\rho \nabla_\rho H\right] \\ & + A^{\ \alpha}_{\rho\lambda} A^{\ \rho}_{\beta\mu} - A^{\ \alpha}_{\rho\mu} A^{\ \rho}_{\beta\lambda},\end{aligned} \tag{2.7}$$

where $R^{\ \alpha}_{\beta\lambda\mu}$ is the Riemann tensor for the metric g. Then, for the Ricci tensor we obtain

$$\tilde{R}_{\beta\mu} = R_{\beta\mu} + \nabla_\lambda A^{\ \lambda}_{\beta\mu} + 2Hl_\beta l^\rho \nabla_\lambda A^{\ \lambda}_{\rho\mu}, \tag{2.8}$$

where $R_{\beta\mu}$ is the Ricci tensor for the metric g. After a long calculation it may be shown that

$$\begin{aligned}\nabla_\rho A^{\ \rho}_{\beta\lambda} = {} & Nl_\beta l_\lambda - l^\rho \Phi_\rho (l_\beta k_\lambda + l_\lambda k_\beta) \\ & + \bar{\Sigma}(l_\beta m_\lambda + l_\lambda m_\beta) \\ & + \bar{\Omega} m_\beta m_\lambda + \Sigma(l_\beta \bar{m}_\lambda + l_\lambda \bar{m}_\beta) \\ & + \Omega \bar{m}_\beta \bar{m}_\lambda + \Gamma(m_\beta \bar{m}_\lambda + m_\lambda \bar{m}_\beta),\end{aligned} \tag{2.9}$$

where we have chosen a null tetrad $(l, k, m, \bar{m})$ for the metric g and[7]

$$\Gamma \equiv (\rho + \bar{\rho}) \nabla_\alpha (Hl^\alpha) - 2H(\rho\bar{\rho} + \sigma\bar{\sigma} + \phi_{00}), \tag{2.10}$$

$$\Omega \equiv 2\sigma \nabla_\alpha (Hl^\alpha) - 2H(\psi_0 + 2\rho\sigma), \tag{2.11}$$

$$\begin{aligned}\Sigma \equiv {} & m^\alpha \Phi_\alpha - (\beta + \bar{\alpha} + \tau) \nabla_\alpha (Hl^\alpha) \\ & - 2H\left[\psi_1 - \rho(\beta + \bar{\alpha}) - \sigma(\bar{\beta} + \alpha) - \phi_{01}\right],\end{aligned} \tag{2.12}$$

$$\begin{aligned}\Phi_\beta \equiv {} & \nabla_\beta \nabla_\alpha (Hl^\alpha) - H\nabla^\rho \nabla_\rho l_\beta \\ & - 2(\nabla^\rho l_\beta) \nabla_\rho H + Hl^\rho R_{\rho\beta},\end{aligned} \tag{2.13}$$

$$\begin{aligned}N \equiv {} & -g^{\rho\sigma} \nabla_\rho \nabla_\sigma H + 2(\gamma + \bar{\gamma}) \nabla_\alpha (Hl^\alpha) \\ & - 2H(k^\beta \nabla_\rho l_\beta)(k^\lambda \nabla^\rho l_\lambda) \\ & - 2k^\alpha \Phi_\alpha - 2H\left[\psi_2 + \bar{\psi}_2 + k^\rho l^\sigma R_{\rho\sigma} + 4\Lambda\right].\end{aligned} \tag{2.14}$$

Therefore, if the metric g is given, the Einstein equations

$$\tilde{R}_{\beta\mu} = \chi(\tilde{T}_{\beta\mu} - \tfrac{1}{2} \tilde{g}_{\beta\mu} \tilde{T}) \tag{2.15}$$

are defined by (2.8) and (2.9). In particular, from (2.8), (2.9),

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and (2.15) we obtain the interesting relation

$$\chi l^{\alpha} \widetilde{T}_{\alpha\mu} = l^{\alpha} R_{\alpha\mu} + l_{\mu}(l^{\rho}\Phi_{\rho} + (\chi/2)\widetilde{T}). \qquad (2.16)$$

We distinguish two cases.

(1) $l^{\alpha} R_{\alpha\mu} = a l_{\mu}$. In this case l_{α} is an eigenvector of $\widetilde{T}_{\alpha\beta}$ and then perfect-fluid solutions cannot exist.

(2) $l^{\alpha} R_{\alpha\mu} \neq a l_{\mu}$. In this case $\widetilde{T}_{\alpha\beta}$ can be the energy-momentum tensor of a perfect fluid. It is the purpose of this paper to study this case when both $\widetilde{T}_{\alpha\beta}$ and $T_{\alpha\beta}$ are perfect-fluid energy-momentum tensors.

Let $(l^{\alpha}, k^{\alpha}, m^{\alpha}, \overline{m}^{\alpha})$ be a null tetrad for the metric g. Then

$$\tilde{l}^{\alpha} = l^{\alpha}, \quad \tilde{k}^{\alpha} = k^{\alpha} + H l^{\alpha}, \quad \tilde{m}^{\alpha} = m^{\alpha}, \quad \widetilde{\overline{m}}^{\alpha} = \overline{m}^{\alpha}$$

is a null tetrad for the metric $\tilde{g}$. By using this null tetrad we compute the Weyl tensor and we finally obtain

$$\tilde{\psi}_0 = \psi_0, \quad \tilde{\psi}_1 = \psi_1, \qquad (2.17)$$

$$3\tilde{\psi}_2 = 3\psi_2 - 2H\phi_{00} - \tfrac{1}{2}\nabla_{\lambda}\,[l^{\lambda}\nabla_{\alpha}(Hl^{\alpha})] + 3\rho[DH - H(\rho - \bar{\rho})], \qquad (2.18)$$

$$\tilde{\psi}_3 = \psi_3 - H\tilde{\psi}_1 + \tfrac{1}{2}\overline{\Sigma} - 2H\phi_{10} - (\bar{\tau} - \bar{\beta} - \alpha)(DH - 2H\rho) - \bar{\delta}DH + \rho\bar{\delta}H - H\bar{\tau}(\rho - \bar{\rho}) + 2H\bar{\sigma}(\tau - \beta - \bar{\alpha}) + \bar{\sigma}\delta H, \qquad (2.19)$$

$$\tilde{\psi}_4 = \psi_4 + H^2\bar{\psi}_0 - 2H\Delta\bar{\sigma} - \bar{\sigma}\Delta H + 2H\bar{\sigma}\bar{\mu} + 4H\bar{\sigma}(\gamma - \bar{\gamma}) - 2H^2\bar{\sigma}(\rho - \bar{\rho}) - \bar{\delta}[\bar{\delta}H - 2H(\bar{\beta} + \alpha)] - \lambda(DH - 2H\rho) + (\bar{\beta} + 3\alpha - 2\bar{\tau})(\bar{\delta}H - 2H(\bar{\beta} + \alpha)). \qquad (2.20)$$

III. THE EINSTEIN EQUATIONS FOR A CONFORMALLY FLAT PERFECT-FLUID METRIC $g_{\alpha\beta}$

Henceforth, we choose the metric g to be conformally flat; that is,

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0 \Leftrightarrow g_{\alpha\beta} = \phi^2 \eta_{\alpha\beta}, \qquad (3.1)$$

where ϕ^2 is a positive function of the coordinates and $\eta_{\alpha\beta}$ is the metric of flat space-time. Moreover, we assume that $g_{\alpha\beta}$ is a solution of Einstein's equations for a perfect-fluid energy-momentum tensor; that is to say

$$R_{\alpha\beta} = \chi(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T), \qquad (3.2)$$

$$T_{\alpha\beta} = (q + p)u_{\alpha} u_{\beta} + p g_{\alpha\beta}, \quad g^{\alpha\beta} u_{\alpha} u_{\beta} = -1. \qquad (3.3)$$

All metrics of this kind are known: they are either generalized interior Schwarzschild solutions or generalized Friedmann solutions (Ref. 4).

It may easily be verified that if $g_{\alpha\beta}$ is a conformally flat space-time, and if l_{α} is a null geodesic (shear-free) vector field for $g_{\alpha\beta}$ then it is also a null geodesic (shear-free) vector field for flat space-time. But the general solution for vector fields of this kind in flat space-time is known and is given by[5]

$$l = du + \overline{Y} d\zeta + Y d\bar{\zeta} + Y\overline{Y} dv, \qquad (3.4)$$

where Y is a complex function of the coordinates $\{u, v, \zeta, \bar{\zeta}\}$ verifying

$$Y\frac{\partial Y}{\partial \zeta} + \overline{Y}\frac{\partial Y}{\partial \zeta} - \frac{\partial Y}{\partial v} - Y\overline{Y}\frac{\partial Y}{\partial u} = 0. \qquad (3.5)$$

When l_{α} is also shear-free, Y is defined implicitly by

$$F(Y, \bar{\zeta}Y + u, vY + \zeta) = 0, \qquad (3.6)$$

where F is an arbitrary analytic function of three complex variables. The coordinates $\{u, v, \zeta, \bar{\zeta}\}$ are related with the usual coordinates of the Minkowski space-time by

$$\sqrt{2}\,u = t - z, \quad \sqrt{2}\,v = t + z, \quad \sqrt{2}\,\zeta = x + iy, \qquad (3.7)$$

and the metric $g_{\alpha\beta}$ may be written in these coordinates as

$$g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} = 2\phi^2[-du\,dv + d\zeta\, d\bar{\zeta}]. \qquad (3.8)$$

Now, we choose the null tetrad associated with l_{α} as follows:

$$l, \quad k = \phi^2\, dv, \quad m = \phi(d\zeta + Y dv). \qquad (3.9)$$

Then, after a straightforward calculation, we obtain the spin coefficients

$$\pi = -\alpha = -\phi^{-1}\overline{m}^{\alpha}\nabla_{\alpha}\phi, \quad \gamma = -\mu = \phi^{-1}k^{\alpha}\nabla_{\alpha}\phi,$$
$$\rho = \phi^{-2}[\rho_M + \phi\, l^{\alpha}\nabla_{\alpha}\phi], \quad \rho_M = \frac{\partial Y}{\partial \zeta} - \overline{Y}\frac{\partial Y}{\partial u},$$
$$\tau = \phi^{-1}\tau_M + \bar{\alpha}, \quad \tau_M = -\frac{\partial Y}{\partial u}, \qquad (3.10)$$
$$\sigma = \phi^{-2}\sigma_M, \quad \sigma_M = \frac{\partial Y}{\partial \bar{\zeta}} - Y\frac{\partial Y}{\partial u},$$
$$\kappa = \epsilon = \lambda = \beta = \nu = 0.$$

Moreover, it is well known that the null tetrad is defined up to a transformation of the form

$$l' = l, \quad m' = e^{iC}(m + \overline{Z}l),$$
$$k' = k + Zm + \overline{Z}\overline{m} + Z\overline{Z}l. \qquad (3.11)$$

We make such a transformation choosing

$$\overline{Z} = -m^{\alpha}u_{\alpha}/l^{\alpha}u_{\alpha} \qquad (3.12)$$

so that

$$m'^{\alpha}u_{\alpha} = 0. \qquad (3.13)$$

After this change of null tetrad, the new spin coefficients are[8]

$$\pi' = \pi - DZ, \quad \kappa' = \epsilon' = 0, \quad \rho' = \rho,$$
$$\sigma' = \sigma, \quad \beta' = Z\sigma, \quad \alpha' = \alpha + Z\rho,$$
$$\mu' = \mu + \overline{Z}\pi + Z^2\sigma - \delta' Z, \quad \tau' = \tau + Z\sigma + \overline{Z}\rho,$$
$$\lambda' = Z\pi + 2Z\alpha + Z^2\rho - \bar{\delta}' Z, \qquad (3.14)$$
$$\gamma' = \gamma + Z\tau + Z^2\sigma + Z\overline{Z}\rho + \overline{Z}\alpha,$$
$$\nu' = Z\mu + Z\overline{Z}\pi + Z^2\tau + Z^3\sigma + Z^2\overline{Z}\rho + 2Z\gamma + 2Z\overline{Z}\alpha - \Delta' Z.$$

Hereafter, we shall drop the primes.

We search for solutions $\tilde{g}_{\alpha\beta}$ of Einstein's equations for a perfect-fluid energy-momentum tensor

$$\widetilde{T}_{\alpha\beta} = (\tilde{q} + \tilde{p})\tilde{u}_{\alpha}\tilde{u}_{\beta} + \tilde{p}\tilde{g}_{\alpha\beta}, \qquad (3.15)$$

$$\tilde{g}^{\alpha\beta}\tilde{u}_{\alpha}\tilde{u}_{\beta} = -1. \qquad (3.16)$$

Taking into account all previous assumptions and results, the Einstein equations (2.15), (2.8), and (2.9)—once they are projected onto the null tetrad—lead us to the following set of equations:

$$m^{\alpha}\tilde{u}_{\alpha} = 0, \qquad (3.17)$$

$$\Omega = 0, \qquad (3.18)$$

$$(\tilde{q}+\tilde{p})(l^\alpha \tilde{u}_\alpha)^2 = (q+p)(l^\alpha u_\alpha)^2, \tag{3.19}$$

$$\nabla_\lambda \,[l^\lambda \nabla_\mu (H l^\mu)] = \chi[\tfrac{1}{2}(\tilde{q}-3\tilde{p}) - \tfrac{1}{2}(q-3p) + H(q+p)(l^\alpha u_\alpha)^2], \tag{3.20}$$

$$\Gamma = (\chi/2)[\tilde{q}-\tilde{p}-q+p], \tag{3.21}$$

$$N = \chi\left[\frac{\tilde{q}+\tilde{p}}{4(l^\alpha \tilde{u}_\alpha)^2} - \frac{q+p}{4(l^\alpha u_\alpha)^2} - H^2(q+p)(l^\alpha u_\alpha)^2 - 2Hp\right], \tag{3.22}$$

$$\Sigma = 0. \tag{3.23}$$

Starting from (2.11), Eq. (3.18) becomes

$$\sigma[DH - H(\rho-\bar{\rho})] = 0. \tag{3.18$'$}$$

We can consider two cases.

(A) $\sigma \neq 0$. Then Eq. (3.18′) implies that we must have

$$\rho = \bar{\rho}, \quad DH = 0. \tag{3.24}$$

This case is studied in the following section.

(B) $\sigma = 0$. Then (3.18′) is automatically satisfied. This case is studied in Sec. V.

IV. THE CASE $\sigma \neq 0$

In this section we try to solve Eqs. (3.17)–(3.23) with the assumption $\sigma \neq 0$.

Throughout this and the next sections we shall use repeatedly (but not explicitly) the Bianchi identities and the Newman–Penrose equations for the metric $g_{\alpha\beta}$ (the Bianchi identities are given in the Appendix). Whenever we make some assumption or specialization we must restrict these equations in the appropriate fashion. The details are omitted.

First of all, from (3.20), (3.21), and (3.24) we obtain $\tilde{q}$ and $\tilde{p}$ as functions of q, p, and H,

$$\chi\tilde{p} = \chi p + 2H\phi_{00}, \tag{4.1}$$

$$\chi\tilde{q} = \chi q + 4H(\rho^2 - \sigma\bar{\sigma}) - 2H\phi_{00}. \tag{4.2}$$

Furthermore, from (3.19), (3.17), and (3.16), we get $\tilde{u}_\alpha$:

$$(l^\alpha \tilde{u}_\alpha)^2 = \chi(q+p)(l^\alpha u_\alpha)^2[\chi(q+p) + 4H(\rho^2-\sigma\bar{\sigma})]^{-1},$$
$$m^\alpha \tilde{u}_\alpha = 0. \tag{4.3}$$

Then, we only must solve Eqs. (3.22) and (3.23).

Starting from (2.12) and (2.13), making use of (3.24), and after some standard calculations, we obtain for (3.23)

$$\bar{\delta} H = 2H(\alpha + \bar{\beta}) - H\bar{\tau} - H\rho(\tau/\sigma). \tag{4.4}$$

In the same way, it follows from (2.14), (3.24), and (4.1)–(4.4) that Eq. (3.22) becomes

$$\rho\Delta H = H\{\rho(\mu+\gamma+\bar{\gamma}) + \sigma\lambda + \delta(\rho(\tau/\sigma)) + \rho(\tau/\sigma)(\bar{\alpha}-\beta) - \Delta\rho - \rho^2(\tau\bar{\tau}/\sigma\bar{\sigma}) - 4(\phi_{11}/\phi_{00})(\rho^2-\sigma\bar{\sigma}) - 2\Lambda\} - (H^2/\phi_{00})[(\rho^2-\sigma\bar{\sigma})^2 - \phi_{00}^2]. \tag{4.5}$$

From (2.18) and (2.19) with (3.24), (4.5), and (4.4) we get[9] (for the sake of brevity $\tilde{\psi}_4$ is not written here)

$$\tilde{\psi}_0 = 0, \quad 3\tilde{\psi}_2 = H(\sigma\bar{\sigma} - \rho^2 - \phi_{00}),$$
$$\tilde{\psi}_1 = 0, \quad \tilde{\psi}_3 = H(\tau/\sigma)(\sigma\bar{\sigma}-\rho^2). \tag{4.6}$$

The question now is the following: Are Eqs. (3.24), (4.4), and (4.5) compatible? Since Eqs. (3.24) and (4.4) are linear in H, their compatibility with (4.5) (which is nonlinear in H) gives us an expression for H which is not, in general, a solution of Eqs. (3.24), (4.4), and (4.5). In order to proceed we assume

$$\rho^2 - \sigma\bar{\sigma} = \phi_{00}, \tag{4.7}$$

so that Eq. (4.5) becomes linear as well.

Keeping this in mind, the compatibility condition of (3.24) and (4.4) is simply

$$\pi = 0 \tag{4.8}$$

and the compatibility condition of (4.4) with its complex conjugate is the reality condition

$$\bar{\sigma}(\delta\tau + \tau(\beta-\bar{\alpha})) = \sigma(\bar{\delta}\bar{\tau} + \bar{\tau}(\bar{\beta}-\alpha)). \tag{4.9}$$

Furthermore, Eq. (4.5) now becomes

$$\Delta H = H\left\{2(\mu+\gamma+\bar{\gamma}) - 2\rho\frac{\bar{\lambda}}{\sigma} - \frac{1}{2}\left(\frac{\tau^2}{\sigma} + \frac{\bar{\tau}^2}{\bar{\sigma}}\right) - \rho\frac{\tau\bar{\tau}}{\sigma\bar{\sigma}}\right\}, \tag{4.10}$$

which is compatible with (3.24). Finally, a new integrability condition arises from (4.4) and (4.10):

$$\Delta\tau + 3\bar{\lambda}\,\bar{\tau} + \tau(\mu+\gamma-\bar{\gamma}) + \tau(\bar{\tau}^2/\bar{\sigma}) + 2\rho\bar{\nu} = 0. \tag{4.11}$$

It is easily shown that this condition is compatible with the Newman–Penrose equations.

We can summarize our results as follows: Let us choose the conformally flat perfect-fluid metric $g_{\alpha\beta}$ and the null geodesic vector field l_α such that they verify $\rho = \bar{\rho}$, (4.7), (4.8), (4.9), and (4.11). Then, let us solve the integrable system of equations for H given by (3.24), (4.4), and (4.10). The new Kerr–Schild metric $\tilde{g}_{\alpha\beta}$ is a solution of Einstein's equations for a perfect-fluid energy-momentum tensor (3.15), where $\tilde{q}$, $\tilde{p}$, and $\tilde{u}_\alpha$ are given by (4.1), (4.2), and (4.3). The Weyl tensor of these new solutions is (τ never vanishes)

$$\tilde{\psi}_0 = \tilde{\psi}_1 = 0, \quad 3\tilde{\psi}_2 = -2H\phi_{00},$$
$$\tilde{\psi}_3 = -H(\tau/\sigma)\phi_{00}, \quad \tilde{\psi}_4 = -H(\tau^2/\sigma^2)\phi_{00},$$

so that we have $3\tilde{\psi}_2\tilde{\psi}_4 = 2\tilde{\psi}_3^2$, and therefore they are of Petrov type D. Since $\tilde{\psi}_3$ and $\tilde{\psi}_4$ do not vanish, the vector field $\tilde{k}^\alpha$ is not a multiple null eigenvector of the Weyl tensor, but l^α certainly is. From (4.4) we have

$$\tilde{u}^\alpha = a l^\alpha + b\tilde{k}^\alpha$$

and then $\tilde{u}^\alpha$ does not lie in the preferred two-space spanned by the two multiple null eigenvectors of the Weyl tensor. Excepting the Wahlquist solution, no solutions of this kind were known up to now.

V. THE CASE $\sigma = 0$

Now, we assume $\sigma = 0$ so that the function Y of (3.4) is defined by (3.6) and also we have

$$\beta = \lambda = 0. \tag{5.1}$$

We define in this section

$$U \equiv \delta H - 2H\bar{\alpha}, \quad V \equiv DH - 2H\rho. \tag{5.2}$$

From (3.20), (3.21), (3.19), (3.17), and using the same procedure of the previous section we get

$$\chi \tilde{p} = \chi p - DV - V(3\rho + \bar{\rho}) - 3H\rho^2 - 4H\rho\bar{\rho} + H\bar{\rho}^2 + 4H\phi_{00}, \quad (5.3)$$

$$\chi \tilde{q} = \chi q + V(\bar{\rho} - \rho) - DV + 3H(\rho^2 + \bar{\rho}^2), \quad (5.4)$$

$$(l^\alpha \tilde{u}_\alpha)^2 = 2\phi_{00}\{\chi(q+p) + 4H\phi_{00} - 2[DV + 2V\rho + 2H\bar{\rho}(\rho - \bar{\rho})]\}^{-1}. \quad (5.5)$$

Equations (3.23) and (3.22) become, respectively,

$$\delta V + (\rho + \bar{\rho})U + (\tau - \bar{\alpha})V + H[\bar{\rho}\bar{\alpha} + \delta\bar{\rho} + 2\tau(\bar{\rho} - \rho)] = 0, \quad (5.6)$$

$$\begin{aligned}\rho\Delta H = & -H\Delta(\rho + \bar{\rho}) - \delta\bar{U} + \bar{\alpha}\bar{U} - \bar{\tau}U - \tau\bar{U} - \mu V \\ & - 4H\phi_{11} + H(\gamma + \bar{\gamma})(\rho - \bar{\rho}) - (1/4\phi_{00}) \\ & \times [DV + 2V\rho + 2H\bar{\rho}(\rho - \bar{\rho})] \\ & \times [DV + 2V\rho + 2H\bar{\rho}(\rho - \bar{\rho}) \\ & - 4H\phi_{00} - 8\phi_{11}]. \end{aligned} \quad (5.7)$$

In order to make compatible U and V we must have

$$DU + (2\bar{\rho} - \rho)U + (\tau + \bar{\pi})V + H[\bar{\rho}\bar{\alpha} + \delta\bar{\rho} + 2\bar{\pi}(\rho - \bar{\rho})] = 0. \quad (5.8)$$

Also, U must verify the reality condition

$$\bar{\delta}U + \bar{\rho}\Delta H + \bar{\mu}\bar{V} - U\alpha + 2H\rho(\gamma + \bar{\gamma}) = \text{c.c.} \quad (5.9)$$

Now, the Weyl tensor is given by

$$\begin{aligned}&\tilde{\psi}_0 = \tilde{\psi}_1 = 0, \quad -6\tilde{\psi}_2 = [D - 2(\rho - \bar{\rho})]V, \\ &\tilde{\psi}_3 = H[\rho\alpha + \bar{\delta}\rho + \bar{\tau}(\rho - \bar{\rho})] + (2\rho - \bar{\rho})\bar{U}, \\ &\tilde{\psi}_4 = -[\bar{\delta} - (3\alpha - 2\bar{\tau})]\bar{U}.\end{aligned} \quad (5.10)$$

In this paper, we only solve these equations under the assumptions

$$\rho = \bar{\rho}, \quad V = -2H\rho, \quad (5.11)$$

and so we have

$$DH = 0. \quad (5.12)$$

Then, Equations (5.6), (5.7), (5.8), and (5.9) become, respectively,

$$\tau = 0, \quad (5.13)$$

$$\rho\Delta H = H[-2\Delta\rho + 2\mu\rho - 4\rho^2(\phi_{11}/\phi_{00})] - \delta\bar{U} + \bar{\alpha}\bar{U} - (H^2/\phi_{00})(\rho^2 + \phi_{00})(\rho^2 - \phi_{00}), \quad (5.14)$$

$$DU + \rho U = 0, \quad (5.15)$$

$$\bar{\delta}U - \alpha U = \delta\bar{U} - \bar{\alpha}\bar{U}. \quad (5.16)$$

As in the previous section, in order to avoid nonlinear terms in H we assume

$$\rho^2 = \phi_{00}, \quad (5.17)$$

so that Eq. (5.14) may be written

$$\rho\Delta H = 2H[\rho(\gamma + \bar{\gamma}) - 2(\phi_{11} + \Lambda)] - \delta\bar{U} + \bar{\alpha}\bar{U}. \quad (5.18)$$

The compatibility of this equation with (5.12) leads us to

$$\mu\rho + \phi_{11} + \Lambda = 0. \quad (5.19)$$

This condition eliminates many candidates for $g_{\alpha\beta}$ (i.e., all generalized Schwarzschild metrics).[10] Now, the integrability condition of (5.18) with U is

$$\rho\Delta U + \bar{\delta}\delta U - 2\bar{\alpha}\bar{\delta}U - 2\alpha\delta U + U[3\alpha\bar{\alpha} - \rho(5\mu + 3\bar{\gamma} + \gamma)] = 0. \quad (5.20)$$

For the Weyl tensor we have

$$3\tilde{\psi}_2 = -2H\rho^2, \quad \tilde{\psi}_3 = \rho\bar{U}, \quad \tilde{\psi}_4 = -[\bar{\delta} - 3\alpha]\bar{U}. \quad (5.21)$$

Consequently, if we want to obtain Petrov type D solutions, that is to say

$$3\tilde{\psi}_2\tilde{\psi}_4 = 2\tilde{\psi}_3^2,$$

we must have

$$\delta U = 3\bar{\alpha}U + U^2/H. \quad (5.22)$$

We put

$$f \equiv U/H \quad (5.23)$$

and then Eqs. (5.15), (5.16), and (5.22) are written as follows:

$$Df = -\rho f, \quad (5.24)$$

$$\bar{\delta}f + f(\alpha + \bar{f}) = \delta\bar{f} + \bar{f}(\bar{\alpha} + f), \quad (5.25)$$

$$\delta f = \bar{\alpha}f. \quad (5.26)$$

On the other hand, bearing Eqs. (5.23)–(5.26) in mind, Eq. (5.20) becomes

$$\rho\Delta f - \rho f(\mu + \bar{\gamma} - \gamma) + f(\bar{\alpha}\bar{f} - \alpha f - 4\alpha\bar{\alpha}) + (\bar{\alpha} + f)\bar{\delta}f = 0. \quad (5.27)$$

Equations (5.24)–(5.27) are satisfied by choosing

$$f = A\bar{\alpha},$$

where A is an arbitrary real constant, and where two supplementary conditions remain:

$$\delta\bar{\alpha} = \bar{\alpha}^2, \quad \Delta\bar{\alpha} = \bar{\alpha}[\mu + \bar{\gamma} - \gamma + (\alpha\bar{\alpha}/\rho)(5 + A)]. \quad (5.28)$$

These conditions are compatible with the Newman–Penrose equations.

Now, we summarize our results in this section: Let us choose the conformally flat perfect-fluid metric $g_{\alpha\beta}$ and the shear-free geodesic null vector field l_α verifying $\rho = \bar{\rho}$, (5.13), (5.17), (5.19), and (5.28). Then we set $U = AH\bar{\alpha}$ and we solve Eqs. (5.12), (5.18), and $\delta H = (2 + A)H\bar{\alpha}$. These equations always have solutions. The new generalized Kerr–Schild metric $\tilde{g}_{\alpha\beta}$ is a solution of the Einstein equations for a perfect-fluid energy-momentum tensor (3.15), where $\tilde{q}$, $\tilde{p}$, and $\tilde{u}_\alpha$ are given by (5.3)–(5.5) (when they are conveniently restricted to the case we have studied). The Weyl tensor of the new metrics is Petrov type D. Unless we have $A = 0$ or $\alpha = 0$, reasoning similar to that in the previous section leads us to solutions previously unknown, as $\tilde{u}_\alpha$ does not lie in the preferred two-space spanned by the two multiple null eigenvectors of the Weyl tensor. In the cases $A = 0$ or $\alpha = 0$ the solutions may belong to the family given by Wainwright.[11]

Obviously, we only have solved a very particular case in this section. Other more general cases remain for a subsequent paper.

VI. EXPLICIT EXAMPLES

In this section we give some examples of how the equations may be solved explicitly. We can assume two different forms for the metric $g_{\alpha\beta}$: the form manifestly conformally flat as given in (3.8) and other forms in which the spin coefficients of the null tetrad are adapted to the conditions obtained in Secs. IV and V, even though we do not know the conformal factor explicitly. In the first case the conditions on the spin coefficients become equations for the function Y of (3.4). Once we have obtained the function Y, we can solve the integrable equations for H. In the second case, we have the advantage that we do not need the conformal factor, which is unknown in many metrics. Next, we give some examples for both cases.

(1) In this example, we choose the conformally flat metric given by Oleson[12] in coordinates $\{x^0, x^1, x^2, x^3\} = \{u,t,x,y\}$ in the following form:

$$g_{\alpha\beta}\, dx^\alpha\, dx^\beta = t^{3/2}(dx - (2/\sqrt{t})\, G_{,x}\, du)^2 + \sqrt{t}\, (dy + 2\sqrt{t}\, G_{,y}\, du)^2 - 2G\, dt\, du + 2G^2 M\, du^2,$$

$$_{,x} \equiv \frac{\partial}{\partial x}, \quad M(t) = 2\sqrt{t}\,(a^2 + b^2 t), \quad a, b = \text{const},$$

$$G(x,y,u) = g(x,u)h(y,u), \quad g_{,xx} + a^2 g = 0,$$

$$h_{,yy} + b^2 h = 0, \quad p = (3/4t^{3/2})(a^2 - 7b^2 t),$$

$$q = p + \frac{12b^2}{\sqrt{t}}, \quad \mathbf{l} = \frac{\partial}{\partial t},$$

$$\mathbf{m} = \frac{1}{\sqrt{2t}}\left(t^{-1/4}\frac{\partial}{\partial x} + i t^{1/4}\frac{\partial}{\partial y}\right),$$

$$\mathbf{k} = G^{-1}\left(\frac{\partial}{\partial u} + GM\frac{\partial}{\partial t} + \frac{2}{\sqrt{t}}\, G_{,x}\frac{\partial}{\partial x} - 2\sqrt{t}\, G_{,y}\frac{\partial}{\partial y}\right),$$

$$\rho = \bar\rho = \frac{1}{2t}, \quad \sigma = \bar\sigma = \frac{1}{4t}, \quad (l^\alpha u_\alpha)^2 = \frac{1}{2M}.$$

This metric verifies

$$\phi_{00} = 3\sigma\bar\sigma, \quad \rho^2 = 4\sigma\bar\sigma, \quad \pi = 0,$$

so that the conditions (4.7) and (4.8) are satisfied. Then, a straightforward calculation leads us to

$$\tau = (1/\sqrt{2t}\; G)(t^{-1/4} G_{,x} + i t^{1/4} G_{,y}), \quad \alpha = 0.$$

Now, it may easily be verified that

$$\delta\tau = \bar\delta\,\bar\tau.$$

Therefore, condition (4.9) is also automatically satisfied. Next, we obtain

$$\mu = -2b^2\sqrt{t}, \quad \gamma - \bar\gamma = 2iG^{-2} G_{,x}\, G_{,y},$$

and after a little computation the condition (4.11) becomes

$$G_{,xu} - G^{-1} G_{,x}\, G_{,u} = 0, \quad G_{,yu} - G^{-1} G_{,y}\, G_{,u} = 0.$$

Consequently, we must restrict the metric $g_{\alpha\beta}$ to the case in which

$$G(x,y,u) = g(x)h(y)n(u),$$

where $n(u)$ is an arbitrary function of the variable u. Once this restriction is imposed we know that the equations for H are compatible. The integration of these equations [(3.24), (4.4), and (4.10)] is standard and we finally obtain

$$H = ch(y)/g^3(x), \quad c = \text{const}.$$

From (4.1)–(4.3) we have

$$(l^\alpha \tilde u_\alpha)^2 = [2(M+H)]^{-1}, \quad m^\alpha \tilde u_\alpha = 0,$$

$$\chi\tilde p = \chi p + 3H/8t^2, \quad \chi\tilde q = \chi(\tilde p + 12b^2/\sqrt{t}).$$

The final form for the metric $\tilde g_{\alpha\beta}$ is the following:

$$\tilde g_{\alpha\beta}\, dx^\alpha\, dx^\beta = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + 2h(y)g^{-3}(x)G^2\, du^2,$$

$$G(x,y,u) = g(x)h(y)n(u), \quad g_{,xx} + a^2 g = 0,$$

$$h_{,yy} + b^2 h = 0, \quad a, b = \text{const}.$$

(2) The most simple metric $g_{\alpha\beta}$ we can choose is the "flat" Robertson–Walker metric, that is to say

$$g_{\alpha\beta}\, dx^\alpha\, dx^\beta = 2R^2(-du\, dv + d\zeta\, d\bar\zeta),$$

$$R = R(u+v), \quad q = q(u+v), \quad p = p(u+v),$$

$$\dot q = -3(q+p)\frac{\dot R}{R}, \quad \dot R^2 = \frac{\chi}{3} qR^4, \quad \dot{} \equiv \frac{\partial}{\partial t},$$

$$t \equiv (1/\sqrt{2})(u+v), \quad l_\alpha\, dx^\alpha = du + \bar Y d\zeta + Y d\bar\zeta + Y\bar Y dv,$$

$$u_\alpha\, dx^\alpha = -(R/\sqrt{2})(du + dv).$$

Now, the function Z of (3.12) is given by

$$Z = -\bar Y R/(1 + Y\bar Y),$$

so that Eqs. (3.11) and (3.14) provide the null tetrad and the spin coefficients, respectively. The function Y is defined by (3.6).

To satisfy Eqs. (5.13) and (5.19) it is necessary that

$$Y = 0.$$

Then (5.28) is automatically verified. Finally, the condition (5.17) leads us to[13]

$$p = -\tfrac{1}{3} q,$$

and therefore we must restrict the Robertson–Walker metric such that

$$q = A^2/R^2, \quad R = Be^{\pm Ct}, \quad C \equiv \sqrt{\chi/3}\, A, \quad A, B = \text{const}.$$

Solving the integrable system of equations for H we easily obtain

$$H = \text{const}.$$

Consequently, we obtain the following solution:

$$\tilde g_{\alpha\beta}\, dx^\alpha\, dx^\beta = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + 2H\, du^2,$$

$$R = Be^{\pm Ct}, \quad \tilde q = \frac{A^2}{R^2}\left(1 + \frac{1}{3R^2}\right), \quad \tilde p = \frac{A^2}{R^2}\left(\frac{1}{3R^2} - 3\right),$$

$$(l^\alpha \tilde u_\alpha)^2 = 1/2(H + R^2), \quad m^\alpha \tilde u_\alpha = 0, \quad A, B, H = \text{const}.$$

APPENDIX: BIANCHI IDENTITIES

Next, we list the Bianchi identities for a conformally flat perfect-fluid metric. We choose the null tetrad (3.11) such that

$$\phi_{01} = \phi_{02} = \phi_{12} = \kappa = \epsilon = \rho - \bar\rho = 0,$$

$$\Lambda = (\chi/24)(q - 3p), \quad \phi_{00} = (\chi/2)(q+p)(l^\alpha u_\alpha)^2,$$

$$\phi_{11} = (\chi/8)(q+p), \quad \phi_{00}\phi_{22} = 4\phi_{11}^2,$$

and then we have

$$\phi_{00}\,\lambda = 2\phi_{11}\,\bar{\sigma}, \quad \phi_{00}\,\bar{\nu} = 2\phi_{11}(\tau + \bar{\pi}),$$
$$\delta(\phi_{11} + \Lambda) = 0, \quad \delta\,\phi_{11} = \bar{\pi}\,\phi_{11}, \quad \mu = \bar{\mu},$$
$$D(\phi_{11} + \Lambda) = \mu\,\phi_{00} - 2\rho\,\phi_{11},$$
$$\Delta(\phi_{11} + \Lambda) = 2\mu\,\phi_{11} - \rho\,\phi_{22},$$
$$\delta\,\phi_{00} = (\bar{\pi} - 2\bar{\alpha} - 2\beta)\,\phi_{00},$$
$$2\Delta\,\phi_{11} - D\,\phi_{22} + \rho\,\phi_{22} - 2\mu\,\phi_{11} = 0,$$
$$\Delta\,\phi_{00} - 2D\,\phi_{11} - 2\rho\,\phi_{11} + \phi_{00}(\mu + 2\gamma + 2\bar{\gamma}) = 0.$$

[1]R. P. Kerr and A. Schild, *Atti Del Convegno Sulla Relatività Generale: Problemi Dell'Energia E Onde Gravitazionali* (Anniversary Volume, Fourth Centenary of Galileo's Birth), edited by G. Barbéra (Firenze, 1965), p. 173.

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[3]A. H. Bilge and M. Gürses, in *Proceedings of the XI International Colloquium on Group Theoretical Methods in Physics*, edited by M. Serdaroglu and E. Inönü (Springer, Istanbul, Turkey, 1982).

[4]D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Deutscher Verlag der Wissenschaften, Berlin, 1980).

[5]D. Cox and E. J. Flaherty, Jr., Commun. Math. Phys. **47**, 75 (1976); V. P. Frolov, in *Problems in the General Theory of Relativity and Theory of Group Representations*, edited by N. G. Basov (Plenum, New York, 1979).

[6]In this paper there are three kinds of objects related to the metrics $\tilde{g}_{\alpha\beta}$, $g_{\alpha\beta}$, or $\eta_{\alpha\beta}$. We denote these, respectively, by a tilde (i.e., $\tilde{\psi}_0$, $\tilde{k}^{\alpha}$, etc.), no label (i.e., k_α, σ, ρ, etc.), and by M (i.e., σ_M, ρ_M, etc.). Consequently, we raise and lower indices of the tensors with $\tilde{g}_{\alpha\beta}$, $g_{\alpha\beta}$, or $\eta_{\alpha\beta}$, respectively.

[7]We use standard notation in Newman–Penrose formalism. Our conventions coincide with those of Kramer *et al.*[4] except for the spin coefficients where the sign is changed. The signature of the metric is $(-, +, +, +)$.

[8]B. Aronson and E. T. Newman, J. Math. Phys. **13**, 1847 (1972).

[9]It is a consequence of the expression (4.6) for the Weyl tensor that solutions of Petrov type N cannot be obtained. This may be seen as follows. Oleson[12] has shown that perfect-fluid solutions of Petrov type N with a geodesic principal null vector must satisfy $\tilde{\sigma}\neq 0$ and $\tilde{\phi}_{00} = 3\sigma\bar{\sigma}$. It is easily shown that $\tilde{\sigma} = \sigma$ and $\tilde{\phi}_{00} = \phi_{00}$, therefore we must have $\sigma\neq 0$ and $\phi_{00} = 3\sigma\bar{\sigma}$. Since $\tilde{\psi}_0 = \tilde{\psi}_1 = 0$, in order to obtain solutions of Petrov type N, it is necessary to satisfy $\tilde{\psi}_2 = 0$. But then, we have $-\rho^2 - 2\sigma\bar{\sigma} = 0$ which is not possible.

[10]Generalized Schwarzschild interior metrics verify $\phi_{11} + \Lambda = \text{const} > 0$ and $\rho = 2(l^{\alpha} u_{\alpha})^2 \mu$. Therefore $\mu\,\rho > 0$ and Eq. (5.19) is not possible.

[11]J. Wainwright, Gen. Relativ. Gravit. **8**, 797 (1977).

[12]M. Oleson, J. Math. Phys. **12**, 666 (1971). We take the metrics belonging to class I of Oleson's paper with the assumption $a = $ const so that we have a conformally flat metric. In fact, these metrics belong to the Friedmann class of perfect-fluid space-times.

[13]Although $p = -\frac{1}{3}q$ is not physically admissible we allow it since we can obtain an energy density $\tilde{q}$ and a pressure $\tilde{p}$ that are physically reasonable.