A family of inhomogeneous cosmological Einstein–Rosen metrics

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Some generalized soliton solutions of the cosmological Einstein–Rosen type defined in the space-time region $t^2 \geqslant z^2$ in terms of canonical coordinates are considered. Vacuum solutions are studied and interpreted as cosmological models. Fluid solutions are also considered and are seen to represent inhomogeneous cosmological models that become homogeneous at $t \to \infty$. A subset of these evolve toward isotropic Friedmann–Robertson–Walker metrics.

I. INTRODUCTION

Many known cosmological solutions of the Einstein–Rosen form can be deduced as generalized soliton solutions.[1,2] Soliton solutions are obtained by application of the Belinskii and Zakharov soliton transformation[3] and can be easily generalized by taking advantage of the linearity of Einstein's equations for the Einstein–Rosen metric.

The main ingredients in the construction of a soliton solution are the seed metric, which is the starting solution to be transformed, and the so-called pole trajectories, a set of well defined functions that may be real or complex. Although solutions with real pole trajectories may be obtained as the limit of solutions with complex poles when some of the parameters are null, they also form a class on their own.

In terms of canonical coordinates (t,z)[3] (for instance, Einstein–Rosen coordinates), the generalized soliton solutions with real poles are defined either in the space-time region $z^2 \geqslant t^2$ or in the region $t^2 \geqslant z^2$. These solutions may be completed by matching them to the seed solutions in the region $t^2 \geqslant z^2$ or in $z^2 \geqslant t^2$, respectively.[4,5] The completed solutions, however, have discontinuous first derivatives along the matching light cones $z^2 = t^2$. These light cone discontinuities disappear if we take complex poles: the metrics are then defined in the whole (t,z) coordinate range and the two regions are smoothly matched.

All soliton solutions can be understood in terms of one-pole and two-pole solutions.[6] Real one-pole generalized soliton solutions valid in $z^2 \geqslant t^2$ have been seen[1,2] to include well-known solutions such as the spatially homogeneous Ellis and MacCallum[7] metrics and their Wainwright, Ince, and Marshman[8] inhomogeneous generalizations; the first have the cosmological singularity only, at $t = 0$, but the second are also singular at $|z| \to \infty$ and $z^2 = t^2$. The Wainwright *et al.* solutions completed with the seed solutions in $t^2 \geqslant z^2$ are better interpreted as the limits of complex pole solutions[9] or as composite universes.[4,10] Real two-pole generalized soliton solutions, on the other hand, have been seen to include the Carmeli and Charach[11,12] pulse wave solutions, which are not singular at $|z| \to \infty$. In all of these solutions the spatially homogeneous Kasner metric has been taken as the seed metric.

In this paper we consider the generalized soliton solutions with real poles defined in $t^2 \geqslant z^2$. To our knowledge these solutions have not been studied previously. In some sense they are complementary to the solutions just mentioned and their possible relevance as cosmological models should be emphasized.

We consider vacuum one-pole and two-pole solutions. For the one-pole solutions we see that, unlike the metrics in $z^2 \geqslant t^2$, they do not include homogeneous metrics. They have only the cosmological singularity at $t = 0$ and the light cone singularity at $z^2 = t^2$. This second singularity, however, disappears when one takes complex poles. Therefore, all of these solutions are potentially interesting as limits of perfectly regular inhomogeneous cosmological models, but are much simpler and easier to study. On the other hand, two-pole solutions give, as in the previous case, pulse wave type metrics. All of the generalized soliton solutions evolve in time to the seed solution (the Kasner metric in our case) and are classified as Petrov type I metrics.

We also consider solutions representing the coupling with a massless scalar field. Such solutions are easily obtained from the vacuum metrics. The new solutions admit a fluid interpretation[8,13,14] according to the space-time properties of the scalar field; in some regions it is a perfect fluid with a stiff equation of state whereas it is an anisotropic fluid in others. The scalar field also admits a generalized soliton solution and we show that the final metric can be seen as a generalization of a Tabensky and Taub fluid plane symmetric metric.[13] The most interesting aspect of this solution is that it approaches spatially homogeneous metrics for $t \to \infty$ and for some values of the parameters it approaches the isotropic Friedmann–Robertson–Walker (FRW) metric. Thus this is an example of cosmological isotropization of initially inhomogeneous metrics.

The vacuum solutions with one and two poles are studied in Sec. II by means of the curvature tensor. The fluid solutions, which are easier to interpret because of the existence of a coordinate system attached to the fluid, are considered in Sec. III.

II. VACUUM SOLUTIONS

In this section we consider the generalized soliton solutions with real poles that are defined in the space-time region $t^2 \geqslant z^2$. We also consider briefly the solutions defined in $z^2 \geqslant t^2$.

The Einstein–Rosen metrics can be written as

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$$ds^2 = f(dz^2 - dt^2) + t(e^{\Phi}\, dx^2 + e^{-\Phi}\, dy^2), \tag{1}$$

where f and Φ are functions of t and z only.

According to Einstein's equations the potential function $\Phi(z,t)$ verifies a linear wave equation. Using the linearity of this equation, soliton solutions of Φ may be generalized easily.[1,2,5] In fact, if we take the spatially homogeneous Kasner metric as the seed metric,

$$\Phi_0 = d \ln t, \quad f_0 = (d^2 - 1)/2 \ln t, \tag{2}$$

where d is an arbitrary real parameter, the soliton solutions with n simple poles may be written as[6]

$$\Phi \equiv \Phi_0 + \Phi_s = d \ln t + \sum_{i=1}^{n} \ln\left(\frac{\mu_i}{t}\right), \tag{3}$$

$$f = f_0 t^{n(4-n)/2}\left[\prod_{k=1}^{n}\left(\frac{\mu_k}{t}\right)\right]^{2+d-n} \times \prod_{\substack{k,l=1\\k>l}}^{n} (\mu_k - \mu_l)^2 \prod_{k=1}^{n} (\mu_k^2 - t^2)^{-1}, \tag{4}$$

where

$$\mu_i^{\pm} = z_i \pm (z_i^2 - t^2)^{1/2}, \quad z_i \equiv z_i^0 - z, \tag{5}$$

are the pole trajectories that are real if the parameters z_i^0 are real or complex otherwise; they verify that $\mu_i^+/t = t/\mu_i^-$.

The analysis of these solutions is usually performed by considering one and two poles only.[6]

By taking z_i^0 real we see that Φ_s, for μ_i^+, is a linear superposition of terms of type $\cosh^{-1}(z_1/t)$. Therefore, the one-pole solution may be generalized as

$$\Phi_s = h \cosh^{-1}(z_1/t), \quad |z_1| \geqslant t, \tag{6a}$$

where h is a real parameter. The corresponding f coefficient is easily found from (4) by taking appropriate limits:

$$f = t^{(d^2+h^2-1)/2}(z_1^2 - t^2)^{-h^2/2} \exp[hd \cosh^{-1}(z_1/t)]. \tag{6b}$$

Solution (6), which is defined in the space-time region $|z_1| \geqslant t$, is the Wainwright *et al.* solution.[8] Generally it has singularities at $t=0$, $|z| \to \infty$, and $t^2 = z_1^2$. When $h^2 = d^2 + 3$ it has the cosmological singularity only ($t=0$) and is the Ellis and MacCallum[7] spatially homogeneous anisotropic solution: Bianchi V if $d=0$, Bianchi III if $d^2=1$, and Bianchi VI$_h$ otherwise. In this case it is better to use coordinates (T,Z) adapted to spatial homogeneity[12] $t = \exp(-2aZ)\sinh(2aT)$, $z = \exp(-2aZ)\cosh(2aT)$, where a is a positive constant.

When this solution is completed by matching it to the Kasner solution in the space–time region $t^2 \geqslant z_1^2$, it may be seen as the limit of the one complex pole solution with no light cone singularities[9] and it may be interpreted as a composite universe.[4,10]

To complete this short review of soliton solutions defined in $z_1^2 \geqslant t^2$ we now consider the two-pole solutions that are necessary in order to give an overview of all the soliton solutions. These are the solutions obtained with μ_1^+ and μ_2^-,

$$\Phi_s = (h/2)[\cosh^{-1}(z_1/t) - \cosh^{-1}(z_2/t)], \quad \min(|z_1|,|z_2|) \geqslant t, \tag{7a}$$

and the f coefficient is

$$f = t^{(d^2-1)/2}(\mu_2 - \mu_1)^{h^2/2}(\mu_1/\mu_2)^{h(2d-h)/4} \times ((z_1^2 - t^2)(z_2^2 - t^2))^{-h^2/8}. \tag{7b}$$

Metrics (7), which are not singular at $|z| \to \infty$, are the Carmeli and Charach[11,12] pulse wave solutions. They may be seen as the limit of the corresponding complex pole solutions that describe gravisolitons propagating on a Kasner background.[16]

The solutions we wish to consider here are the family of real pole generalized soliton solutions defined in the space-time region $t \geqslant |z_1|$. In some sense they may be considered as complementary to the solutions just mentioned. For one pole such a family may be obtained in a way similar to (6a) but changing h to ih (Refs. 5 and 17):

$$\Phi_s = h \cos^{-1}(z_1/t), \quad |z_1| \leqslant t. \tag{8a}$$

The f coefficient may be obtained also from (4) by taking appropriate limits:

$$f = t^{(d^2-h^2-1)/2}(t^2 - z_1^2)^{h^2/2} \exp[dh \cos^{-1}(z_1/t)]. \tag{8b}$$

This solution may be matched to the Kasner metric in the region $|z_1| \geqslant t$.

We may now study the intrinsic properties of solution (8). By taking the null tetrad

$$n = (2f)^{-1/2}(\partial_t + \partial_z), \quad l = (2f)^{-1/2}(\partial_t - \partial_z),$$
$$m = (2g_{xx})^{-1/2}\,\partial_x + i(2g_{yy})^{-1/2}\,\partial_y,$$

and the complex conjugate of m, m^*, the Riemann tensor has three non-null components only.[15] For the metric (8) these are

$$\begin{aligned}
\Psi_2 &= -(8f)^{-1}[(1+h^2-d^2)t^{-2} - 2hz_1\, dt^{-2}(t^2 - z_1^2)^{-1/2}],\\
\Psi_0 &= -(2f)^{-1}X^+, \quad \Psi_4 = -2(f)^{-1}X^-,\\
X^{\pm} &= (t^2 - z^2)^{-1/2}\{hzt^{-2}(3d^2 - h^2 - 1) \pm ht^{-1}(3d^2 - h^2 - 3)\}/4\\
&\quad + (t^2 - z^2)^{-1}h^2 d(2 + z^2 t^{-2} \pm 3zt^{-1})/2\\
&\quad + (t^2 - z^2)^{-3/2}h\{z + z^3 \pm t(2 + h^2 + h^2 z^2 t^{-2})/2\} + dt^{-2}(d^2 - h^2 - 1)/4.
\end{aligned} \tag{9}$$

The algebraic classification of this metric is easily done by following the d'Inverno and Russell-Clark algorithm.[15,18] For $h \neq 0$ the metrics are of Petrov type I. Of course for $h=0$ the metric reduces to the Kasner seed, which is of Petrov type D for $d=0$, flat space for $d^2=1$, and Petrov type I otherwise.

The metric has only the cosmological singularity at $t=0$ and the light cone singularity at $z_1^2 = t^2$; but this last one may be avoided with complex poles. Unlike the solution (6), there are no values of the parameters for which the metric is spatially homogeneous. Therefore for a cosmological interpretation we match metric (8) with the Kasner metric in $|z_1| \geqslant t$ and the new solution may be seen as the limit of inhomogeneous complex pole solutions that have the cosmological singularity only. Such complex pole solutions are acceptable cosmological models, and may be considered as composite universes,[4,10]; however, they are not so easily deduced and analyzed.

Metric (8) evolves to the spatially homogeneous Kasner metric when $t\to\infty$.

We may now consider the two-pole solutions. They are obtained, similarly to (8), by changing h to ih in (7a):

$$\Phi_s = (h/2)[\cos^{-1}(z_1/t) - \cos^{-1}(z_2/t)],$$
$$t \geqslant \max(|z_1|,|z_2|), \tag{10a}$$

and the metric coefficient f is found to be

$$f = t^{(d^2-h^2-1)/2}[(t^2-z_1^2)(t^2-z_2^2)]^{h^2/8}$$
$$\times (z_2\sqrt{t^2-z_1^2} - z_1\sqrt{t^2-z_2^2})^{h^2/4}$$
$$\times (\sqrt{t^2-z_2^2} + \sqrt{t^2-z_1^2})^{-h^2/4}$$
$$\times \exp[(dh/2)(\cos^{-1}(z_1/t) - \cos^{-1}(z_2/t))]. \tag{10b}$$

This metric is complementary to the Carmeli and Charach[11] pulse wave solutions (7). It may be understood as the (destructive) superposition of the two solutions (8). In (7) this superposition was essential in order to avoid the singularity at $|z|\to\infty$ of the one-pole solutions (6). Here this is not necessary because the solution (8) is not singular at $|z|\to\infty$ (it is not defined there). The interpretation of the solution, however, is similar. The space-time may be divided by the light cones $|z_1| = t$ and $|z_2| = t$. In the intersection region we have solution (10); it is matched to the one-pole solution at the "inner" light cones and to the seed solution at the "outer" cones. At $t\to\infty$ the completed metric becomes the Kasner metric and thus it may be interpreted as pulse waves propagating on a Kasner background. It is the limit of the corresponding complex pole solution; in such a solution an observer sitting at some fixed z will start in an homogeneous model and will end up in the same model after having gone through inhomogeneous regions of type (8).

III. SOLUTIONS WITH FLUIDS

We now consider the coupling of Einstein's equations with a massless scalar field σ. These equations read[13]

$$R_{\mu\nu} = \sigma_{,\mu}\sigma_{,\nu}, \tag{11a}$$
$$\sigma_{;\mu}{}^{;\mu} = 0. \tag{11b}$$

It is well known that a solution of this system may have a fluid interpretation.[8,13,14] Given σ this is done in the following way. If $\sigma_{,\mu}$ is a timelike vector, $\sigma_{,\mu}\sigma^{,\mu} < 0$, σ may be considered as the potential of a perfect fluid with a stiff equation of state $p = \rho$ (p = pressure, ρ = energy density). This is achieved by defining the density, pressure, and four-velocity of the fluid as

$$\rho = p = -\tfrac{1}{2}\sigma_{,\mu}\sigma^{,\mu}, \quad u_\mu = (-\sigma_{,\mu}\sigma^{,\mu})^{-1/2}\sigma_{,\mu}. \tag{12}$$

The energy-momentum tensor of the fluid is identified from the rhs of (11a) as $T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T$, where

$$T_{\mu\nu} = 2\rho u_\mu u_\nu + \rho g_{\mu\nu}, \tag{13}$$

that is, a perfect fluid with a stiff equation of state.

If $\sigma_{,\mu}$ is a spacelike vector, $\sigma_{,\mu}\sigma^{,\mu} > 0$, the above identification is still formally valid but now u_μ is a spacelike vector and the perfect fluid interpretation does not hold. Following Tabensky and Taub[13] we can see that the rhs of (11a) may be identified with an anisotropic fluid. For this we define an orthonormal tetrad $(\hat\tau_\mu, \hat\sigma_{,\mu}, \hat x_\mu, \hat y_\mu)$, where $\hat\tau_\mu$ is a timelike vector, $\hat\sigma_\mu \equiv u_\mu$, and $\hat x_\mu$, $\hat y_\mu$ are spacelike vectors. Now $g_{\mu\nu} = -\hat\tau_\mu\hat\tau_\nu + \hat\sigma_{,\mu}\hat\sigma_{,\nu} + \hat x_\mu \hat x_\nu + \hat y_\mu \hat y_\nu$ and the rhs of (11a) can be written as $T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T$ with

$$T_{\mu\nu} = \tfrac{1}{2}\sigma_{,\lambda}\sigma^{\lambda}(\tau_\mu\tau_\nu + \sigma_{,\mu}\sigma_{,\nu} - x_\mu x_\nu - y_\mu y_\nu), \tag{14}$$

which corresponds to the energy-momentum tensor of an anisotropic fluid with energy density $\rho = \tfrac{1}{2}\sigma_{,\lambda}\sigma^{\lambda}$ and vanishing heat-flow vector. The weak and strong energy conditions[19] are satisfied and the fluid interpretation is reasonable.[13]

For the Einstein–Rosen metrics (1) the solutions of (11) are easily found. In such a case the scalar field $\sigma(t,z)$ verifies the same linear wave equation, (11b), that the potential field $\Phi(t,z)$ verifies. Only the metric coefficient $f(t,z)$ is modified by the presence of such a scalar field and it is simply found as the product of two functions, each one of them determined, respectively, from $\Phi(t,z)$ and $\sigma(t,z)$ by similar equations.[13,20]

Therefore we may take generalized soliton solutions for σ. For the real one-pole case we take, as in (8a),

$$\sigma = a\ln t + b\cos^{-1}(z_1/t), \quad t \geqslant |z_1|, \tag{15}$$

where a and b are arbitrary parameters. This is the solution used by Tabensky and Taub[13] in their study of plane symmetric metrics that evolve to FRW models.

Now the f coefficient is easily obtained by making use of (4) and (8b) as

$$f = t^{(d^2-h^2+2a^2-2b^2-1)/2}(t^2 - z_1)^{(h^2+2b^2)/2}$$
$$\times \exp[(dh + 2ab)\cos^{-1}(z_1/t)]. \tag{16a}$$

The potential Φ has not been modified, i.e., $\Phi = \Phi_0 + \Phi_s$ with Φ_s being (8a),

$$\Phi = d\ln t + h\cos^{-1}(z_1/t). \tag{16b}$$

Metric (16) gives a solution to Einstein's equations (11a) with the coupling of the massless scalar field (15). It reduces to the Tabensky and Taub plane symmetric metric when $d = h = 0$. For t this metric approaches a spatially homogeneous metric.

The space-time regions where $\sigma_{,\mu}$ is, respectively, timelike and spacelike are divided by the straight line,

$$t = -(a^2+b^2)(a^2-b^2)^{-1}z_1. \tag{17}$$

According to the previous discussion we have a perfect fluid in the space-time region between the straight line (17) and $t = z_1 > 0$ and an anisotropic fluid in the complementary region.

The presence of a fluid makes this metric easier to study and interpret because we may adapt the coordinate system to the fluid. Following Refs. 13 and 21 we shall introduce comoving coordinates. In the region where $\sigma_{,\mu}$ is timelike we may use $\sigma(t,z)$ as the time coordinate and define a space coordinate $Z(t,z)$ by

$$dZ = a^{-1}t(\sigma_{,z}\,dt + \sigma_{,t}\,dz); \tag{18}$$

this ensures that $Z_{,\mu}\sigma^{,\mu} = 0$ and that $Z_{,\mu}$ is spacelike.

Equation (18) is easily integrated as

$$Z = z_1 - a^{-1}b(t^2 - z_1^2)^{1/2}. \tag{19a}$$

The fluid lines are the hyperbolas defined by $Z=$ const; they approach straight lines for $t\to\infty$. The time coordinate may be defined as

$$T=\exp[a^{-1}\sigma(t,z)-a^{-1}b\cos^{-1}(b(a^2+b^2)^{-1/2})], \tag{19b}$$

where the constant parameters have been introduced for convenience and $\sigma(t,z)$ is given in (15).

In the region where $\sigma_{,\mu}$ is spacelike, T and Z are space and time coordinates, respectively, and the fluid lines are defined by $\sigma(t,z)=$ const.

The coordinate change defined by (19) is not explicitly invertible. However, for large t, which is the region we are interested in, it is

$$t\simeq T+b(a^2+b^2)^{-1/2}Z,\quad z\simeq Z+b(a^2+b^2)^{-1/2}T,$$

and the metric (16) can be written in comoving coordinates as

$$\begin{aligned}ds^2=T^{(d^2+h^2+2a^2+2b^2-1)}&\{1+(Z/T)(a^2+b^2)^{-1/2}\\&\times[(b/2)(d^2-h^2-2a^2-2b^2-1)-ahd]\}\\&\times\{[1-2(Z/T)b(a^2+b^2)^{-1/2}]dZ^2-dT^2\}\\&+T[1+(Z/T)b(a^2+b^2)^{-1/2}\\&\times(T^dA\,dx^2+T^{-d}A^{-1}\,dy^2)],\end{aligned} \tag{20}$$

where

$$\begin{aligned}A=1+(Z/T)(a^2+b^2)^{-1/2}(db-ah)\\ \times\exp\{h\cos^{-1}[b(a^2+b^2)^{-1/2}]\}.\end{aligned}$$

For $d=h=0$, i.e., the Tabensky and Taub plane symmetric solution, and $2(a^2+b^2)=3$ the metric approaches at $T\to\infty$ the flat FRW metric with a stiff perfect fluid

$$ds^2=T(dZ^2+dx^2+dy^2-dT^2). \tag{21}$$

Metric (20) also approaches the isotropic flat FRW model, (21), when $d=0$ and $2(a^2+b^2)=3-h^2$. For all other values of the parameters the metric approaches a spatially homogeneous but anisotropic model.

To finite values of time the metric is spatially inhomogeneous and may be interpreted as representing inhomogeneous finite perturbations on homogeneous metrics. Therefore the solutions (20) may be considered as an example of inhomogeneous cosmologies that become spatially homogeneous and, for some values of the parameters, isotropic, as a result of cosmological evolution.

Solution (16) may be matched to the space-time region $|z_1|\geqslant t$ with the seed soution obtained by setting $h=b=0$. There we again have discontinuities in the first derivatives for the metric and the fluid potential. Rather than interpreting it as a solution with shock waves, it is better interpreted as the limit of the corresponding complex one-pole solutions that have no fluid discontinuities in the pressure or the density. The asymptotic behavior at $t\to\infty$ of such a solution is that of the real pole solution described above, i.e., it evolves to spatially homogeneous metrics.

We could also describe fluid solutions with two or more poles; the interpretations, however, are now rather obvious.

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