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INTRODUCTION TO LORENTZ SPACES

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Introduction

In this work we study Lorentz spaces $L^{p,q}$. That is, let $(\mathcal{R},\mu)$ be a σ-finite complete measure space, and suppose $p \geq 1$ and $1 \leq q \leq \infty$. The Lorentz space $L^{p,q}(\mathcal{R},\mu)$ consist of all f in $\mathcal{M}_0(\mathcal{R},\mu)$ for which the quantity

$$\|f\|_{L^{p,q}} = \begin{cases} \left\{ \int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \right\}^{\frac{1}{q}}, & (0 < q < \infty), \\ \sup_{0<t<\infty} \{t^{\frac{1}{p}} f^*(t)\}, & (q = \infty), \end{cases}$$

is finite.

The motivation comes from the Marcinkiewicz interpolation theorem, which says:

Let T be a bounded linear operator from L^{p_0} to L^{p_0} and from L^{p_1} to L^{p_1} with $p_0 \neq p_1$ and $0 < p_0, p_1 \leq \infty$. Then

$$T : L^{p,q} \longrightarrow L^{p,q},$$

is a bounded linear operator for all $0 < q \leq \infty$ and p such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1.$$

This theorem is very important, because it studies the boundedness of operators, which is fundamental to solve many mathematical problems of real life arising for example in physic.

I have specially enjoyed the mathematical Analysis courses and hence I did Real Analysis, Fourier Analysis and the Functions Theory optional subjects. Then I decided to relate my final work with something related with this matter. María Jesús Carro expose me some issues and I chose the topic of Lorenz spaces. One of the reasons was because in Fourier analysis we saw a weaker version of the Marcinkiewicz theorem. Moreover, I was also interested in initiating myself into research.

In order to study the Lorentz spaces, we introduce all the necessary theory to define them well, we see their most basic properties and we study their normability. Our main reference has been [1].

The work is structured in six chapters, for which I'll give a little summary:

- In the first chapter we introduce all the necessary notation so there's no confusion, we study the most important results about convergences and we also see some classic inequalities about L^p spaces. These last two things will be fundamental because we'll be using them very often through all the work.
- In the second chapter we study three fundamental theorems from functional analysis, such as the analytic version of the Hanh-Banach Theorem 3.29, the Baire Theorem 2.14 and the open map Theorem 2.24. These three theorems are the key ingredients for the separation theorem proof, which we'll see in chapter three.
- In the third chapter we introduce the topological vectorial spaces, their most basic properties and we will prove the separation theorem. This theorem will be used in the Lorentz-Luxemburg theorem 4.18 proof, in chapter four.
- In chapter four, we introduce the Banach function spaces, their most important properties and their associated space. All these will be the basic tools to construct the Lorentz spaces.
- In the next chapter we will study the involved tools in their definition, such as the decreasing-rearrangement f^* (Definition 5.5) and the maximal function f^{**} (Definition 5.16). We will study their main properties which will be essential to study the Lorentz spaces.
- Finally, in the last chapter we introduce the Lorentz spaces and study their most important properties such as the normability.

When I was studying the Lorentz-Luxemburg theorem, I found a big problem that I didn't expected and it was the Hanh-Banach theorem. This is one of the most important theorems in Functional Analysis. It was a problem for me because I hadn't coursed the functional analysis subject, where this theorem was explained, that's why María Jesús and I decided that we had to work deeply with it as it is so important. I also had to study other important theorems such as the Baire theorem and the open map. But, at the end, this has really been a solution more than a problem since it has helped me to review some forgotten concepts. It's been perfect to introduce myself so much in the functional analysis too.

Then, my conclusion is really positive for many reasons: To know what a research work is, to refresh forgotten concepts, to help me with topics related to my future, to learn about Lorentz spaces and to introduce myself in Functional Analysis, because without this work, I would probably hadn't done it.

Thanks

I thank my tutor, María Jesús Carro, how she's carried on this work and her support in those moments in which it was difficult for me to go forward. Apart from these, I also thank her for being honest in all the doubts I had, related to the work and personal ones, which have pooped out during these four months.

Chapter 1

Notation and previous results

In this chapter, we shall fix the notation used in the work. We shall also present the three main convergence theorem: monotone convergence theorem, Fatou's lemma and dominated convergence theorem and we see some of the most classical inequalities of L^p spaces: Hölder's inequality, Minkovsky's inequality.

1.1 Notation

- Let $(\mathcal{R}, \mu)$ be a σ-finite complete measure space.
- $\mathcal{M}$ is the set of μ-measurable functions on $\mathcal{R}$.
- $\mathcal{M}^+$ is the set of non-negative μ-measurable functions on $\mathcal{R}$.
- The characteristic function of a μ-measurable subset E of $\mathcal{R}$ will be denoted by χ_E. The function is defined by

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

- $\mathcal{M}_0 \subset \mathcal{M}$ is the set of finite functions μ-a.e., that is, if $f \in \mathcal{M}_0$, the set

$$F = \{x \in \mathcal{R} : f(x) = \infty\}$$

  has measure 0.
- If $1 \leq p \leq \infty$, we denote by p' the conjugate exponent; that is,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

- If $z \in \mathbb{C}$, then its real part will be denoted by $\Re z$ and its complex part will be denoted by $\Im z$.

Definition 1.1. *A sequence $(f_n)_n \subset \mathcal{M}$ is said to converge to f in measure if given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$\mu\{x \in \mathcal{R} : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon.$$

Definition 1.2. *A sequence $(f_n)_n \subset \mathcal{M}$ is said to converge pointwise to f if for all $x \in \mathcal{R}$,*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Definition 1.3. *A sequence $(f_n)_n \subset \mathcal{M}$ is said to converges μ-a.e. to f if there is $E \subset \mathcal{R}$ a set with $\mu(E) = 0$ such that*

$$\lim_{n\to\infty} f_n(x) = f(x),$$

for each $x \in E^c$.

Definition 1.4. *A sequence $(f_n)_n \subset \mathcal{M}$ is said to converge in $\mathcal{M}_0$, if for every $\mu(E) < \infty$,*

$$\mu\{x \in E : |f_n(x) - f(x)| > \varepsilon\} \to 0.$$

Definition 1.5. *S is the space of simple functions, that is, $f \in S$ if only if*

$$f(x) = \sum_{i=1}^{k} \alpha_i \chi_{E_i}(x),$$

where $\alpha_i \in \mathbb{R}$ and $0 \leq \mu(E_i) < \infty$ for each i.

1.2 Main Convergence Theorems

Let $(f_n)_n$ be a sequence of $\mathcal{M}$. We say that $f_n \uparrow f$ if

$$f_1(x) \leq f_2(x) \leq ...f_n(x) \leq ... \leq \lim_{n\to\infty} f_n(x) = f(x),$$

for each $x \in \mathcal{R}$. This symbol $f_n \uparrow f$ can be used for μ-a.e., that is, $f_n \uparrow f$ μ-a.e., if

$$f_1(x) \leq f_2(x) \leq ...f_n(x) \leq ... \leq \lim_{n\to\infty} f_n(x) = f(x),$$

for each $x \in \mathcal{R} \setminus F$, where $\mu(F) = 0$.

Theorem 1.6. *[Monotone convergence theorem] Let $(f_n)_n$ be a sequence of $\mathcal{M}^+$ such that $f_n \uparrow f$. Then $f \in \mathcal{M}^+$ and*

$$\lim_{n\to\infty} \int_{\mathcal{R}} f_n(x)d\mu(x) = \int_{\mathcal{R}} f(x)d\mu(x).$$

Proof. Since $0 \leq \lim_{n\to\infty} f_n(x) = \sup_n f_n(x) = f(x)$ and the supreme of measurable functions is also a measurable function, we have that f is a measurable function. For all n,

$$\int_{\mathcal{R}} f_n(x)d\mu(x) \leq \int_{\mathcal{R}} f(x)d\mu(x),$$

and hence,

$$\lim_{n\to\infty} \int_{\mathcal{R}} f_n(x)d\mu(x) \leq \int_{\mathcal{R}} f(x)d\mu(x).$$

Let $s \in S$ such that $0 \leq s(x) \leq f(x)$ for all $x \in \mathcal{R}$. We define, for all $n \in \mathbb{N}$,

$$A_n = \{x \in \mathcal{R} : f_n(x) \geq \alpha s(x)\},$$

with $0 < \alpha < 1$.

We have that A_n is a measurable set and

$$\int_{\mathcal{R}} f_n(x) d\mu(x) \geq \int_{A_n} f_n(x) d\mu(x) \geq \alpha \int_{A_n} s(x) d\mu(x).$$

If $s(x) = \sum_{j=1}^{k} \beta_j \chi_{B_j}(x)$,

$$\int_{A_n} s(x) d\mu(x) = \sum_{j=1}^{k} \int_{A_n} \beta_j \chi_{B_j}(x) d\mu(x) = \sum_{j=1}^{k} \beta_j \mu(B_j \cap A_n).$$

Thus,

$$\int_{\mathcal{R}} f_n(x) d\mu(x) \geq \alpha \sum_{j=1}^{k} \beta_j \mu(B_j \cap A_n).$$

We observe that the sequence $(A_n)_n$ is non-decreasing, that is, $A_n \subset A_{n+1}$ for all n.

$$x \in A_n \implies f_n(x) \geq \alpha s(x).$$

Since $(f_n)_n$ is a non-decreasing sequence,

$$f_{n+1}(x) \geq f_n(x) \geq \alpha s(x) \implies x \in A_{n+1}.$$

We also observe that $\mathcal{R} = \bigcup_{n=1}^{\infty} A_n$. If $x \notin \bigcup_{n=1}^{\infty} A_n$ then $x \notin A_n$ for all n. Therefore $f_n(x) \leq \alpha s(x) \leq \alpha f(x)$ for all n, then

$$\lim_{n\to\infty} f_n(x) \leq \alpha f(x) < f(x),$$

which is a contradiction.

If we fix $1 \leq j \leq k$, $(B_j \cap A_n)_n$ is a non-decreasing sequence.

$$\bigcup_{n=1}^{\infty} (B_j \cap A_n) = B_j \cap \left(\bigcup_{n=1}^{\infty} A_n \right) = B_j \cap \mathcal{R} = B_j.$$

Thus, $\lim_{n\to\infty} \mu(A_n \cap B_j) = \mu(B_j)$. Therefore,

$$\lim_{n\to\infty} \int_{\mathcal{R}} f_n(x) d\mu(x) \geq \lim_{n\to\infty} \alpha \sum_{j=1}^{k} \beta_j \mu(B_j \cap A_n) = \alpha \sum_{j=1}^{k} \beta_j \mu(B_j) =$$

$$= \alpha \sum_{j=1}^{k} \beta_j \int_{B_j} d\mu(x) = \alpha \sum_{j=1}^{k} \beta_j \int_{\mathcal{R}} \chi_{B_j}(x) d\mu(x) = \int_{\mathcal{R}} \alpha s(x) d\mu(x).$$

Using now that $\int_{\mathcal{R}} f(x) d\mu(x) = \sup_{\{s \in S: 0 \leq s \leq f\}} \int_{\mathcal{R}} s(x) d\mu(x)$ and letting $\alpha \to 1$,

$$\lim_{n\to\infty} \int_{\mathcal{R}} f_n(x) d\mu(x) \geq \int_{\mathcal{R}} f(x) d\mu(x).$$

$\square$

Lemma 1.7. *[Fatou's Lemma] Let $(f_n)_n$ be a sequence of functions of $\mathcal{M}^+$. Then*

$$\int_{\mathcal{R}} \liminf_{n\to\infty} f_n(x)d\mu(x) \leq \liminf_{n\to\infty} \int_{\mathcal{R}} f_n(x)d\mu(x).$$

Proof. We have that

$$\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} \big(\inf_{n\geq k} f_n(x)\big) = \sup_k \inf_{n\geq k} f_n(x).$$

We define $F_k(x) = \inf_{n\geq k} f_n(x)$ and then clearly $F_k(x) \leq F_{k+1}(x)$ for every $x \in \mathcal{R}$. Therefore $F_k \uparrow \liminf_{n\to\infty} f_n$ and $F_k \leq f_k$. Consequently,

$$\int_{\mathcal{R}} F_k(x)d\mu(x) \leq \int_{\mathcal{R}} f_k(x)d\mu(x).$$

By Theorem 1.6, we obtain that

$$\int_{\mathcal{R}} \lim_{k\to\infty} F_k(x)d\mu(x) = \int_{\mathcal{R}} \liminf_{n\to\infty} f_n(x)d\mu(x)$$

$$= \lim_{k\to\infty} \int_{\mathcal{R}} F_k(x)d\mu(x) \leq \liminf_{k\to\infty} \int_{\mathcal{R}} f_k(x)d\mu(x).$$

□

Theorem 1.8. *[Dominated Convergence Theorem] Let $(f_n)_n$ be a sequence of $\mathcal{M}$ and let $f \in \mathcal{M}$ such that $f_m \to f$ μ-a.e. Let $g \in \mathcal{M}^+$ such that $\int_{\mathcal{R}} g(x)d\mu(x) < \infty$. Suppose that for every n and for every $x \in \mathcal{R}$ we have that $|f_n(x)| \leq g(x)$ μ-a.e. then*

$$\lim_{n\to\infty} \int_{\mathcal{R}} f_n(x)d\mu(x) = \int_{\mathcal{R}} f(x)d\mu(x).$$

Proof. Since $|f_n(x)| \leq g(x)$ μ-a.e. then

$$\int_{\mathcal{R}} f_n(x)d\mu(x) < \infty \quad \forall n, \text{ and } \int_{\mathcal{R}} f(x)d\mu(x) < \infty.$$

For the same reason, clearly, $g + f_m \geq 0$ μ-a.e. and $g - f_m \geq 0$ μ-a.e.

By Lemma 1.7 and since $\liminf_{n\to\infty} -f_n(x) = -\limsup_{n\to\infty} f_n(x)$, we have that

$$\int_{\mathcal{R}} g(x)d\mu(x) + \int_{\mathcal{R}} f(x)d\mu(x) \leq \liminf_{n\to\infty} \int_{\mathcal{R}} (g + f_n)(x)d\mu(x)$$

$$= \int_{\mathcal{R}} g(x)d\mu(x) + \liminf_{n\to\infty} \int_{\mathcal{R}} f_n(x)d\mu(x),$$

and

$$\int_{\mathcal{R}} g(x)d\mu(x) - \int_{\mathcal{R}} f(x)d\mu(x) \leq \liminf_{n\to\infty} \int_{\mathcal{R}} (g - f_n)(x)d\mu(x)$$

$$= \int_{\mathcal{R}} g(x)d\mu(x) - \limsup_{n\to\infty} \int_{E} f_n(x)d\mu(x).$$

Therefore,

$$\limsup_{n\to\infty} \int_E f_n(x)d\mu(x) \leq \int_{\mathcal{R}} f(x)d\mu(x) \leq \liminf_{n\to\infty} \int_{\mathcal{R}} f_n(x)d\mu(x)$$

Hence,

$$\begin{aligned}\limsup_{n\to\infty} \int_E f_n(x)d\mu(x) &= \liminf_{n\to\infty} \int_{\mathcal{R}} f_n(x)d\mu(x) \\ &= \lim_{n\to\infty} \int_{\mathcal{R}} f_n(x)d\mu(x) = \int_{\mathcal{R}} f(x)d\mu(x).\end{aligned}$$

□

1.3 Inequalities of L^p Spaces

Now, we define the L^p spaces that will be very important for us.

Definition 1.9. *The $L^p = L^p(\mathcal{R}, \mu)$ space consists of all f in $\mathcal{M}$ for which the quantity*

$$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathcal{R}} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ ess \sup_{x\in\mathcal{R}} |f(x)|, & \text{if } p = \infty. \end{cases}$$

is finite.

Now, we see some important inequalities for L^p spaces. But before, we see a necessary technical lemma.

Lemma 1.10. *Let $\alpha, \beta > 0$ and $0 < \lambda < 1$. Then $\alpha^\lambda \beta^{1-\lambda} \leq \alpha\lambda + (1-\lambda)\beta$.*

Proof. We define $\phi(t) = (1-\lambda) + \lambda t - t^\lambda$ for $t \geq 0$. Then $\phi'(t) = \lambda - \lambda t^{\lambda-1} = 0$ when $t = 1$. It is easy to prove that 1 is a minimum. Therefore $\phi(t) \geq \phi(1) = 0$ then $(1-\lambda) + \lambda t \geq t^\lambda$. Doing $t = \frac{\alpha}{\beta}$, the result follows. □

Theorem 1.11. *[Holder's inequality] For every $1 \leq p \leq \infty$,*

$$\int_{\mathcal{R}} |f(x)g(x)|d\mu(x) \leq \|f\|_{L^p}\|g\|_{L^{p'}}.$$

Proof. Let $A = \|f\|_{L^p}\|$ and $B = \|g\|_{L^{p'}}$. We can assume the A, B are finite. If $A = 0$ o $B = 0$ the inequality is clearly true. So, let us assume that $A, B > 0$.

Let $\alpha = \frac{|f(x)|^p}{A^p}$, $\beta = \frac{|g(x)|^{p'}}{B^{p'}}$ and $\lambda = \frac{1}{p}$. Then, by Lemma 1.10,

$$\frac{|f(x)g(x)|}{AB} \leq \frac{|f(x)|^p}{A^p p} + \frac{|g(x)|^{p'}}{B^{p'} p'}.$$

Therefore,

$$\int_{\mathcal{R}} \frac{|f(x)g(x)|}{AB} d\mu(x) \leq \int_{\mathcal{R}} \frac{|f(x)|^p}{A^p p} d\mu(x) + \int_{\mathcal{R}} \frac{|g(x)|^{p'}}{B^{p'} p'} d\mu(x),$$

and since,

$$\int_{\mathcal{R}} \frac{|f(x)|^p}{A^p p} d\mu(x) + \int_{\mathcal{R}} \frac{|g(x)|^{p'}}{B^{p'} p'} d\mu(x) \leq \frac{1}{A^p p} A^p + \frac{1}{B^{p'} p'} B^{p'} = 1,$$

the result follows. □

Theorem 1.12. *[Minkovsky's Inequality] If $f, g \in L^p$, then $f+g \in L^p$ and*

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof. The case $p=1$ follows trivially since $|f+g| \le |f| + |g|$. The case $p=\infty$ follows from $\operatorname{ess\,sup}|A+B| \le \operatorname{ess\,sup}|A| + \operatorname{ess\,sup}|B|$. For the case $1<p<\infty$, we use that x^p is a convex function therefore

$$\big(tf(x)+(1-t)g(x)\big)^p \le tf(x)^p + (1-t)g(x)^p.$$

In particular,

$$\left(\frac{1}{2}f(x)+\frac{1}{2}g(x)\right)^p \le \frac{1}{2}f(x)^p + \frac{1}{2}g(x)^p,$$

and hence,

$$(f(x)+g(x))^p \le 2^{p-1}\left(\frac{1}{2}f(x)^p + \frac{1}{2}g(x)^p\right) = 2^{p-1}\big(f(x)^p + g(x)^p\big).$$

Since $f, g \in L^p$, we obtain that $f+g \in L^p$.

Let us now prove the desired inequality.

$$\begin{aligned}
\|f+g\|_{L^p}^p = \int_{\mathcal{R}} |f(x)+g(x)|^p d\mu(x) &= \int_{\mathcal{R}} |f(x)+g(x)||f(x)+g(x)|^{p-1} d\mu(x) \\
&\le \int_{\mathcal{R}} (|f(x)|+|g(x)|)|f(x)+g(x)|^{p-1} d\mu(x) \\
= \int_{\mathcal{R}} |f(x)||f(x)+g(x)|^{p-1} d\mu(x) &+ \int_{\mathcal{R}} |f(x)+g(x)|^{p-1}|g(x)| d\mu(x).
\end{aligned}$$

We now prove that $|f+g|^{p-1} \in L^{p'}$. Since $p' = \frac{p}{p-1}$ and $f+g \in L^p$,

$$\left(\int_{\mathcal{R}} |f(x)+g(x)|^{p-1})^{p'} d\mu(x)\right)^{\frac{1}{p'}} = \left(\int_{\mathcal{R}} |f(x)+g(x)|^p d\mu(x)\right)^{\frac{1}{p'}} < \infty.$$

By Theorem 1.11,

$$\|f+g\|_{L^p}^p \le (\|f\|_{L^p} + \|g\|_{L^p})\|(f+g)^{p-1}\|_{L^{p'}},$$

then since $\|(f+g)^{p-1}\|_{L^{p'}} = \|f+g\|_{L^p}^{p-1}$,

$$\|f+g\|_{L^p}^p \le (\|f\|_{L^p} + \|g\|_{L^p})\|(f+g)\|_{L^p}^{p-1},$$

and the result follows. $\square$

Chapter 2

Main Functional Analysis Theorems

In this chapter, we prove the main functional analysis theorems used in the work. In Section 1 and 2, we prove the analytic version of the Hahn-Banach theorem. In Section 3, we prove the Baire theorem, and in the section 4, we prove the open map theorem.

Each one of them will also take an important role in the next chapter.

2.1 Hahn-Banach Theorem

First, we introduce the Zorn's Lemma.

Definition 2.1. *A partial order is a binary relation R in a set X such that for all $a, b, c \in X$:*

- *aRa. (reflexive)*
- *aRb and bRa then $a = b$. (antisymmetric)*
- *aRb and bRc then aRc. (transitive)*

Definition 2.2. *A set with a partial order is called partially ordered set.*

Definition 2.3. *A total order $(\leq)$ in X is a partial order with the totality property: for all $a, b \in X$, $a \leq b$ or $b \leq a$.*

Definition 2.4. *Let Φ be a subset of X. Φ is called a chain if it has a total order.*

Definition 2.5. *Let $(X, \leq)$ be a partially ordered set. $M \in X$ is a maximal element if it is an element of X that it is not smaller than any other element in X.*

Lemma 2.6. *[Zorn's lemma] Suppose a partially ordered set X has the property that every chain has an upper bound in X. Then the set X contains at least one maximal element.*

Definition 2.7. *A linear functional u is a linear map from a vector space E to its field of scalars K, that is, for every $x, y \in E$ and $\alpha, \beta \in K$.*

$$u(\alpha x + \beta y) = \alpha u(x) + \beta u(y).$$

Definition 2.8. *Let E be a vector space, p is a convex functional if for all $x, y \in E$ and $\alpha \geq 0$:*

- $p(x + y) \leq p(x) + p(y)$.
- $p(\alpha x) = \alpha p(x)$.

Theorem 2.9. *[First Hahn-Banach theorem] Let E be a real vector space, and p a convex functional in E. Let u be a linear functional of a vector subspace F of E. If $u(z) \leq p(z)$ for all $z \in F$, then there exists an extension of u to a linear functional v in E such that $v(x) \leq p(x)$ for all $x \in E$.*

Proof. The proof has two parts:

(a) If $z, z' \in F$, by Definition 2.8, we have that:

$$u(z) - u(z') = u(z - z') \leq p(z - z') \leq p(z + y) + p(-z' - y).$$

So $-p(-z' - y) - u(z') \leq p(z + y) - u(z)$. Since $z, z' \in F$ are arbitrary, there exists $s \in \mathbb{R}$ such that:

$$\sup_{z' \in F}[-p(-z' - y) - u(z')] \leq s \leq \inf_{z \in F}[p(z + y) - u(z)] \tag{2.1}$$

Let $y \in E \setminus F$ and s as in (2.1). Over $F + [y] = F \oplus \{\, ty| \quad t \in \mathbb{R}\}$, we define $v(z + ty) := u(z) + ts$ with $z \in F$ and $t \in \mathbb{R}$. Then v is a extension of u and it is well defined, since $F + [y]$ is a direct sum. Let us prove that $v(x) \leq p(x)$ for all $x \in F + [y]$.

If $x \in F + [y]$, then $x = z + ty$. We consider two cases, $t > 0$ and $t < 0$.

- If $t > 0$, by (2.1), $s \leq p(z/t + y) - u(z/t)$, and hence,

  $$s + \frac{1}{t}u(z) \leq \frac{1}{t}p(z + ty) \implies v(z + ty) = ts + u(z) \leq p(z + ty) \implies v(x) \leq p(x),$$

  for all $x \in F + [y]$.

- If $t < 0$, we consider $\alpha = -t$. By (2.1), $-p(z/\alpha - y) - u(-z/\alpha) \leq s$, and hence

  $$-\frac{1}{\alpha}p(z - \alpha y) - \frac{1}{\alpha}u(-z) \leq s \implies -p(z + ty) - u(-z) \leq -ts \implies$$

  $$v(z + ty) = ts + u(z) \leq p(z + ty) \implies v(x) \leq p(x),$$

  for all $x \in F + [y]$.

(b) We say that (H, h) is an extension of (F, u) if $F \subset H \subset E$, and h is a linear functional on H, which is an extension of u, and $h(x) \leq p(x)$ for all $x \in H$.

We define Φ as the set of all the extensions of (F, u). We define an order $(\leq)$ in Φ such that $(H, h) \leq (K, k)$, if $H \subset K$ and k is an extension of h.

By Definition 2.1, we have to prove three properties about $(\leq)$. We suppose that $(H, h), (L, h), (K, k)$ are extensions of (F, u).

- The reflexive property is obvious.
- The antisymmetric property follows easily, since $H \subset L$ and $L \subset H$ implies $L = H$, and clearly $h(x) = l(x)$.
- The transitive property follows from $H \subset L$ and $L \subset K$ implies $H \subset K$. Since $h(x) = k(x)$ in H, then $(H, h) \leq (K, k)$.

Let $\Psi \subset \Phi$ be a chain. We define $K := \bigcup_{H \in \Psi} H$ and we also define $k(x) := h(x)$ if $x \in H$ and $(H, h) \in \Psi$.

K is a vector subspace: If $x_1, x_2 \in K$, then there exist $(H_1, h_1), (H_2, h_2) \in \Psi$ such that $x_1 \in H_1$ and $x_2 \in H_2$. If $H_1 \subset H_2$, then $x_1 + x_2 \in H_2 \subset K$. In the same way, if $x \in K$ and $\lambda \in \mathbb{R}$, then $\lambda x \in K$.

k is well defined: If $x \in H_1$, $x \in H_2$ and $(H_1, h_1) \leq (H_2, h_2)$, then h_2 is a extension of h_1, therefore $h_1(x) = h_2(x)$.

k is linear: If $z, y \in K$, then there exist $(H, h), (H', h') \in \Psi$ such that $z \in H$ and $y \in H'$. Suppose that $(H, h) \leq (H', h')$, then $z \in H'$. Since h' is linear, we obtain that

$$k(z + y) = h'(z + y) = h'(z) + h'(y) = k(z) + k(y).$$

(K, k) is an upper bound of Ψ. For each $(H, h) \in \Psi$, we have that $H \subset K$ and $k(x) := h(x) \leq p(x)$, therefore $(H, h) \leq (K, k)$.

By Zorn's Lemma 2.6, there exists a maximal element (V, v) of Φ. Let us prove that $V = E$, and v is an extension of u.

If there exists y such that $y \in E \setminus V$, by (a) we could find an extension of (V, v). But since (V, v) is a maximal element, we have a contradiction. $\square$

2.2 Analytic Version of the Hahn-Banach Theorem

Definition 2.10. *Let v be a map such that $v : E \to \mathbb{C}$. We say that v is real linear functional, if $v(\alpha x) = \alpha v(x)$ for every $x \in E$ and $\alpha \in \mathbb{R}$. We say that v is complex linear functional , if $v(\alpha x) = \alpha v(x)$ for every $x \in E$ and $\alpha \in \mathbb{C}$.*

Definition 2.11. *Let E be a vector space. p is a seminorm if for all $x, y \in E$ and $\alpha \in \mathbb{R}$:*

- $p(x + y) \leq p(x) + p(y)$.
- $p(\alpha x) = |\alpha| p(x)$.

Theorem 2.12. *[Analytic version of the Hahn-Banach theorem] Let E be a real or complex vector space, and p a seminorm in E. Let u be a linear functional in a vector subspace F of E. If $|u(z)| \leq p(z)$ for all $z \in F$, then there exists an extension of u to a linear functional v in E such that $|v(x)| \leq p(x)$ for all $x \in E$.*

Proof. We consider the real case and the complex case.

(a) Real case. We observe that a seminorm is a convex functional. By Theorem 2.9, we obtain that v is an extension of u such that $v(x) \leq p(x)$ for all $x \in E$. Since v is linear,

$$-v(x) = v(-x) \leq p(-x) = p(x),$$

therefore $|v(x)| \leq p(x)$.

(b) Complex case. We consider E as a real vector space. Since $|\Re u(z)| \leq |u(z)| \leq p(z)$, by (a), there is an extension f of $\Re u$ such that $|f(x)| \leq p(x)$.

We define $v(x) := f(x) - if(ix)$. Then $v : E \longrightarrow \mathbb{C}$ is a real linear functional. Let us see thay v is also a linear complex functional, that is, $v(ix) = iv(x)$,

$$v(ix) = f(ix) - if(-x) = f(ix) + if(x) = iv(x).$$

Let us prove that v is an extension of u. If $z \in F$,

- $\Re v(z) = f(z) = \Re u(z)$.
- Since u is complex linear and $iu(z) = i\Re u(z) - \Im u(z)$, we obtain that

$$\Im v(z) = -f(iz) = -\Re u(iz) = -\Re iu(z) = \Im u(z).$$

Finally, we have to prove that $|v(x)| \leq p(x)$. We write $|v(x)| = \mu v(x)$ with $|\mu| = 1$ (polar form of $v(x)$).

Since $f(x) = v(x) + if(ix)$, then $|v(x)| \leq |f(x)|$. Therefore,

$$|v(x)| = v(\mu x) \leq |f(\mu x)| \leq p(\mu x) = p(x).$$

□

2.3 Baire's Theorem

This theorem is the key to prove the open map theorem.

Definition 2.13. *Let (X,d) be a metric space. We say that X is complete if every Cauchy sequence in X is convergent in X.*

Theorem 2.14. *[Baire's Theorem] If X is a complete metric space, then the intersection of every countable collection of open dense subsets of X is dense in X.*

Proof. Suppose $\{G_n; n \in \mathbb{N}\}$ is a sequence of dense open sets. Let $A = \bigcap_n G_n$. We have to prove that every open set G of X intersects A.

Let $a_1 \in G_1 \cap G$. We consider the closed ball $\overline{B}(a_1, r_1) \subset G_1 \cap G$. Let $B(a_1, r_1)$ be its open ball.

Similarly, let $a_2 \in B(a_1, r_1) \cap G_2$. We consider $\overline{B}(a_2, r_2) \subset B(a_1, r_1) \cap G_2$ with $r_2 < \frac{r_1}{2}$.

By recurrence, we build $(a_n)_n \subset X$ and $(r_n)_n \subset (0, \infty)$ such that

$$r_n < \frac{r_{n-1}}{2} < \frac{r_1}{2^n},$$

and

$$\overline{B}(a_n, r_n) \subset B(a_{n-1}, r_{n-1}) \cap G_n \subset \overline{B}(a_{n-1}, r_{n-1}) \cap G_n.$$

The sequence $(a_n)_n$ is a Cauchy sequence. If $p, q \geq m$, then

$$d(a_p, a_q) \leq d(a_p, a_m) + d(a_q, a_m),$$

but since $a_p, a_q \in \overline{B}(a_m, r_m)$, then

$$d(a_p, a_q) \leq d(a_p, a_m) + d(a_q, a_m) < 2r_m < \frac{r_1}{2^{m-1}}.$$

Therefore $(a_n)_n$ is convergent since X is complete. Let $a = \lim_{n\to\infty} a_n$ and let us see that $a \in A \cap G$.

By recurrence, we know that $a_n \in \overline{B}(a_m, r_m)$ for every $m \leq n$. Therefore

$$a \in \bigcap_m \overline{B}(a_m, r_m) \subset \bigcap_n G_n = A.$$

Moreover, $a \in G$ because $\overline{B}(a_1, r_1) \subset G$, and hence $G \cap A \neq \emptyset$, as we wanted to see. □

Corollary 2.15. *Let X be a complete metric space. If $X = \bigcup_n F_n$ is the union of a sequence of closed sets $(F_n)_n$, then at least there is $n \in \mathbb{N}$ such that F_n is a closed set which has interior points.*

Proof. If F is a closed set such that it has no interior points, then $G = X \setminus F$ is open and dense. It is clearly open. Let us see that G is dense, that is, for every open set of X its intersection with G is not empty.

Since F has not interior points, then there does not exist an open set U such that $U \subset F$. Therefore $U \cap G \neq \emptyset$.

For all n, we suppose that F_n has no interior points. Let $G_n = X \setminus F_n$, then

$$\bigcap_n G_n = \bigcap_n X \setminus F_n = X \setminus \bigcup_n F_n = \emptyset,$$

which is a contradiction, since by Theorem 2.14, $\bigcap_n G_n = \emptyset$ is dense in X. □

2.4 Open Map Theorem

Definition 2.16. *X is called a Banach space if X is a complete normed vector space.*

Definition 2.17. *A series $\sum_{n=1}^{\infty} x_n$ in X is absolutely convergent if*

$$\sum_{n=1}^{\infty} \|x_n\|_X < \infty.$$

Proposition 2.18. *X is a Banach space if only if all absolutely convergent series are convergent.*

Proof. Let X be a Banach space and let be a sequence $(x_n)_n$ such that $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. We define the partial sums

$$S_N = \sum_{n=1}^{N} x_n, \quad r_N = \sum_{n=1}^{N} \|x_n\|_X.$$

$(r_N)_N$ is a convergent sequence, therefore it is a Cauchy sequence. Let us prove that $(S_N)_N$ is also a Cauchy sequence:

$$\|S_N - S_M\|_X = \left\| \sum_{n=N+1}^{M} x_n \right\|_X \leq \sum_{n=N+1}^{M} \|x_n\|_X = |r_M - r_N|.$$

Since X is complete, then $(s_N)_N$ is a convergent sequence, that is, there exists x such that $x = \sum_{n=1}^{\infty} x_n$.

Now, we suppose that all absolutely convergent series is convergent. Let $(x_n)_n$ be a sequence of X. We choose $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

$$\|x_n - x_{n_1}\|_X < \frac{1}{2}.$$

Now, we choose $n_2 > n_1$ such that for all $n \geq n_2$,

$$\|x_n - x_{n_2}\|_X < \frac{1}{2^2}.$$

By recurrence, we obtain a subsequence such that

$$\|x_{n_{k+1}} - x_{n_k}\|_X \leq \frac{1}{2^k}.$$

Therefore

$$\sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\|_X \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

Then $\sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k})$ is absolutely convergent, and hence, it is a convergent sequence. If

$$x = \lim_{N\to\infty} \sum_{k=1}^{N} (x_{n_{k+1}} - x_{n_k}) = \lim_{N\to\infty} (x_{n_{N+1}} - x_{n_1}) = \lim_{N\to\infty} (x_{n_N}) - x_{n_1},$$

then (x_{n_k}) is a convergent subsequence. Since $(x_n)_n$ is a Cauchy sequence and it contains a convergent subsequence, then $(x_n)_n$ is a convergent sequence. $\square$

Definition 2.19. *T is a linear operator if T is a linear map between two vector spaces X and Y.*

Definition 2.20. *Let T be a linear operator. A bounded linear operator T is a linear map between normed vector spaces X and Y such that for all $x \in X$,*

$$\|Tx\|_Y \leq M\|x\|_X.$$

Proposition 2.21. *Let T be a linear operator. T is continuous if only if T is a bounded linear operator.*

Proof. Let $x, x' \in X$ such that $\|x - x'\|_X < \delta$. Since T is bounded and linear,

$$\|Tx - Tx'\|_Y \leq \|T(x - x')\|_Y \leq M\|x - x'\|_X < M\delta.$$

Now, we have that T is continuous, in particular, is continuous in 0. Taking $\varepsilon = 1$, there exists $\delta > 0$ such that $\|Tx\|_Y \leq 1$ when $\|x\|_X \leq \delta$. For all $x \neq 0 \in X$, we define

$$\overline{x} = \delta \frac{x}{\|x\|_X}.$$

Then $\|\overline{x}\|_X \leq \delta$ and therefore $\|T\overline{x}\|_Y \leq 1$. Since T is linear, we obtain that

$$\|Tx\|_Y = \frac{\|x\|_X}{\delta}\|T\overline{x}\|_Y \leq M\|x\|_X.$$

Taking $M = \frac{1}{\delta}$, we obtain that T is a bounded linear operator. □

Remark 2.22. *Let X be a complete metric space. Let E, F be two subsets of X. Then $\overline{E} + \overline{F} \subset \overline{E + F}$.*

Proof. If $x \in \overline{E}$ and $y \in \overline{F}$, there exist $(x_n)_n \subset E$ and $(y_n)_n \subset F$ such that $x_n \to x$ and $y_n \to y$. For all n, $x_n + y_n \in E + F$, therefore

$$x_n + y_n \to x + y \in \overline{E + F}.$$

□

Definition 2.23. *A map T is said to be an open map if for every open set U, $T(U)$ is open.*

Theorem 2.24. *[Open Map Theorem] If $T : E \longrightarrow F$ is an exhaustive bounded linear operator between two Banach Spaces, then T is an open map.*

Proof. We prove it in three steps:

(1) We suppose that $T(B(0, r))$ are neighbourhoods of 0 in F, then let us see that T is open.

We consider G an open set of E and $z \in T(G)$, then there exists a such that $Ta = z$ and $a \in G$. Let $B(a, r) = a + B(0, r) \subset G$. Let us prove that

$$Ta \in T(B(a, r)) = Ta + T(B(0, r)) \subset T(G). \tag{2.2}$$

- First, let us see that $T(B(a, r)) = Ta + T(B(0, r))$.

  - If $z \in T(B(a,r))$, there is $x \in B(a,r)$ such that $T(x) = z$. Since $B(a,r) = a + B(0,r)$, then $x = a + x'$ with $x' \in B(0,r)$. Since T is a linear operator, we obtain that
$$T(x) = Ta + T(x') \in Ta + T(B(0,r)).$$
  - If $z \in Ta + T(B(0,r))$, then $z = Ta + Tx$ with $x \in B(0,r)$. Since $a + x \in B(a,r)$ and T is linear, we obtain that
$$z = Ta + Tx = T(a+x) \in T(B(a,r)).$$

- Let us see that $T(B(a,r)) \subset T(G)$. If $z \in T(B(a,r))$, there is $x \in B(a,r)$ such that $z = T(x)$. Since $x \in B(a,r) \subset G$, then $z = Tx \in T(G)$.

Thus $T(G)$ is open. For every point of $T(G)$, there is a neighbourhood such that it is contained in $T(G)$.

(2) Let us prove that for every $r > 0$, $\overline{T(B(0,r))}$ is a neighbourhood of 0 in F. That is, there is an open set $B(0,\sigma)$ such that $B(0,\sigma) \subset \overline{T(B(0,r))}$. Now, let us see some observations.

- Since $\|y\|_E = d(y,0)$, we observe that
$$nB(0,r) = B(0,nr).$$
  - If $x \in nB(0,r)$, then $x = ny$ with $y \in B(0,r)$. Therefore
$$d(x,0) = d(ny,0) = nd(y,0) < nr.$$
    Thus $x \in B(0,nr)$.
  - If $x \in B(0,nr)$, then
$$d(x,0) < nr \implies d(\frac{x}{n},0) < r.$$
    Therefore $\frac{x}{n} \in B(0,r)$ and hence $x \in nB(0,r)$.

- Let us see that
$$E = \bigcup_{n=1}^{\infty} nB(0,r).$$
  If $x \in E$, there is $n \in \mathbb{N}$ such that $d(x,0) < nr$. Therefore $x \in B(0,nr)$.

- Finally, let us see that
$$F = \bigcup_{n=1}^{\infty} \overline{T(B(0,nr))}.$$
  To prove it, we need to note three facts.
  - (a) Since T is exhaustive, then $T(E) = F$, that is,
$$F = T\big(\bigcup_{n=1}^{\infty} B(0,nr)\big).$$

– (b) Let us see that

$$T\big(\bigcup_{n=1}^{\infty} B(0,nr)\big) = \bigcup_{n=1}^{\infty} T(B(0,nr)).$$

* If $y \in T\big(\bigcup_{n=1}^{\infty} B(0,nr)\big)$, there exists $z \in \bigcup_{n=1}^{\infty} B(0,nr)$ such that $Tz = y$. Hence, there exists $n \in \mathbb{N}$ such that $z \in B(0,nr)$. Therefore,

$$y = Tz \in T(B(0,nr)) \subset \bigcup_{n=1}^{\infty} T(B(0,nr)).$$

* If $y \in \bigcup_{n=1}^{\infty} T(B(0,nr))$, there is $n \in \mathbb{N}$ such that $y \in T(B(0,nr))$. Hence, there exists $z \in B(0,nr)$ such that $Tz = y$. Therefore, $z \in \bigcup_{n=1}^{\infty} B(0,nr)$, and hence,

$$y = Tz \in T\big(\bigcup_{n=1}^{\infty} B(0,nr)\big).$$

– (c) Since $F = \bigcup_{n=1}^{\infty} T(B(0,nr)) \subset \bigcup_{n=1}^{\infty} \overline{T(B(0,nr))}$, we obtain that

$$F = \bigcup_{n=1}^{\infty} \overline{T(B(0,nr))}.$$

By Corollary 2.15, there is a closed set $\overline{T(B(0,nr))} = n\overline{T(B(0,r))}$ which has interior points. For all n, $\overline{T(B(0,nr))}$ are homeomorphic each other, therefore we obtain that $\overline{T(B(0,r))}$ has interior points.

Let $B_1 = B(0,r/2)$. Since $\overline{T(B_1)}$ has interior points, there is $y \in \overline{T(B_1)}$ such that $B(y,\sigma) \subset \overline{T(B_1)}$. Writing $B(y,\sigma) = y + B(0,\sigma)$, we obtain that $y + B(0,\sigma) \subset \overline{T(B_1)}$.

Consequently, we have that

$$B_1 = -B_1 \implies T(B_1) = -T(B_1) \implies \overline{T(B_1)} = -\overline{T(B_1)},$$

and $B_1 + B_1 \subset B(0,r)$. By Remark 2.22, we obtain that

$$B(0,\sigma) \subset -y + \overline{T(B_1)} \subset \overline{T(B_1)} + \overline{T(B_1)} \subset \overline{T(B_1 + B_1)} \subset \overline{T(B(0,r)}.$$

(3) In the finally part of the proof, let us prove that $T(B(0,s))$ is a neighbourhood of 0. We will see this, using (2), then by (1), T will be open.

Let $r < s$. We consider $y \in B(0,\sigma)$, therefore $\|y\|_F < \sigma$. We write $s = \sum_{n\geq 1} r_n$ with $r_1 = r$. We take a sequence $(\sigma_n)_n$ such that $\sigma_n \downarrow 0$ and $\sigma_1 = \sigma$. Furthermore, this sequence must hold $B(0,\sigma_n) \subset \overline{T(B(0,r_n))}$, as in (2).

- Since $y \in B(0,\sigma)$, we choose $z_1 \in E$ such that $\|z_1\|_E < r$ and $\|y - Tz_1\|_F < \sigma_2$. Therefore, $y - Tz_1 \in B(0,\sigma_2)$.

- We choose $z_2 \in E$ such that $\|z_2\|_E < r_2$ and $\|y - Tz_1 - Tz_2\|_F < \sigma_3$. Therefore, $y - Tz_1 - Tz_2 \in B(0,\sigma_3)$.

- By recurrence, $y - Tz_1 - ... - Tz_{n-1} \in B(0, \sigma_n)$. We choose $z_n \in E$ such that $\|z_n\|_E < r_n$ and $\|y - Tz_1 - .... - Tz_n\|_F < \sigma_{n+1}$.

Since $\|z_n\|_E < r_n$, then $\sum_{n\geq 1} z_n$ is convergent, by Theorem 2.18. Since E is complete, then there exists

$$x = \lim_{n\to\infty} (z_1 + ... + z_n).$$

Moreover, since $\sum_n \|z_n\|_E < \sum_n r_n = s$, then $x \in B(0, s)$.

By Proposition 2.21,

$$Tx = \lim_{n\to\infty} T(z_1 + ... + z_n) = \lim_{n\to\infty} (Tz_1 + ... + Tz_n) = y \in T(B(0, s)).$$

$\square$

Chapter 3

Topological Vector Spaces

In this chapter, we introduce the topological vector space concept and its most important properties. The motivation of this is to prove a separation theorem which will be fundamental to prove the Lorentz-Luxemburg theorem in Chapter 5.

3.1 Introduction to Topological Vector Spaces

Definition 3.1. *Let τ be a topology on a vector space X with Φ its field of scalars such that:*

- *Every point of X is a closed set.*
- *The vector space operations are continuous respect to the topology τ.*

In these conditions, we say that τ is a vector topology on X, and X is a topological vector space.

Proposition 3.2. *The sum application*

$$+ : X \times X \longrightarrow X$$

$$(x, y) \longrightarrow x + y$$

is continuous: if $x, y \in X$ and V is a neighbourhood of $x + y$, then there exist V_x and V_y neighbourhoods of x and y respectively such that:

$$V_x + V_y \subset V.$$

The scalar multiplication

$$\cdot : \Phi \times X \longrightarrow X$$

$$(\alpha, x) \longrightarrow \alpha x$$

is continuous: if $x \in X$, $\alpha \in \Phi$ and V is a neighbourhood of αx, then for some $r > 0$ and some neighbourhood W of x it holds than $\beta W \subset V$ as $|\beta - \alpha| < r$.

Proof. A function $f : X \longrightarrow Y$ between two topological spaces X and Y is continuous in $x \in X$ if for every neighbourhood $U \subset Y$ of $f(x)$, then there exists V a neighbourhood of x such that $f(V) \subset U$.

- The open sets of $X \times X$ are arbitrary unions of $U \times V$ where U and V are open sets of X. Let $(x, y) \in X \times X$ and U a neighbourhood of $x + y$. Then there exists $V = V_x \times V_y$ a neighbourhood of (x, y) such that $+(V) \subset U$, where V_x, V_y are neighbourhoods of x, y respectively. Therefore

$$+(V) = +(V_x \times V_y) = V_x + V_y \subset U.$$

- Let $(\alpha, x) \in \Phi \times X$ and let U be a neighbourhood of αx. Then there exists $V = V_\alpha \times V_x$ a neighbourhood of (α, x) such that $\cdot(V) \subset U$, where V_α, V_x are neighbourhood of α and x respectively. Let $r > 0$ such that

$$V_\alpha = \{\beta \in \Phi : |\beta - \alpha| < r\},$$

taking $\beta \in V_\alpha$, we obtain that $\cdot(\beta \times V_x) = \beta V_x \subset U$.

□

Remark 3.3. *$E \subset X$ is bounded if for every neighbourhood V of 0 there exists $s > 0$ such that $E \subset tV$ for all $t > s$.*

Definition 3.4. *Let X be a topological vector space, $a \in X$ and $\lambda \neq 0$ then:*

- *$T_a(x) = a + x$ is a translation operator.*
- *$M_\lambda(x) = \lambda x$ is a multiplication operator.*

Proposition 3.5. *Let $\lambda \neq 0$. T_a and M_λ are homeomorphisms of X onto X.*

Proof. The applications T_a and M_λ are bijective from X to X with T_{-a} and $M_{\frac{1}{\lambda}}$ their inverse applications, respectively. By Proposition 3.2, T_{-a}, $M_{\frac{1}{\lambda}}$ T_a and M_λ are continuous, therefore T_a and M_λ are homeomorphisms from X to X. □

Corollary 3.6. *τ is invariant under translation. E is an open set if only if for all $a \in X$, $a + E$ is a open set.*

Corollary 3.7. *τ is completely determined by any local base β of 0. The open sets of X are precisely those that are unions of translates of members of β.*

Proof. Taking $\beta = \{U \text{ open} : 0 \in U\}$, we have a local base of 0.

Let V be an open set. We consider $V_x = V - x$ for every $x \in U$. Then V_x is a neighbourhood of 0, by Corollary 3.6.

Since β is a local base of 0, there exists $\beta_x \in \beta$ such that $\beta_x \subset V_x = V - x$. Thus $\beta_x + x \subset V$. Therefore $\bigcup_{x \in V}\{x + \beta_x\} \subset V$.

Consequently, we obtain that

$$\bigcup_{x \in V}\{x + \beta_x\} = V,$$

since trivially, $V \subset \bigcup_{x \in V}\{x + \beta_x\}$. □

3.2 Basic Results of Topological Vector Spaces

Definition 3.8. *Let X be a topological vector space and let U be a subset of X. We say that U is symmetric if $U = -U$.*

Lemma 3.9. *If W is a neighbourhood of 0, then there is a neighbourhood U of 0 which is symmetric and satisfies $U + U \subset W$.*

Proof. To see this, note that $0+0 = 0$, that addition map is continuous by Definition 3.2, so that 0 has V_1 and V_2 as neighbourhoods, and moreover, $V_1 + V_2 \subset W$. Taking

$$U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2),$$

we see that U has the required properties. □

Remark 3.10. *Applying Lemma 3.9 to U, there is neighbourhood U' of 0 which is symmetric and satisfies $U' + U' \subset U$. Since $U + U \subset W$, then $U' + U' + U' + U' \subset W$. We also note that $U \subset U + U$.*

Remark 3.11. *Let $A, B, C \subset X$. If B is symmetric,*

$$A + B \subset C \implies A \subset C + B.$$

Proof. If $x \in A + B$, then $x = a + b \in C$, where $a \in A$ and $b \in B$. Therefore $a = x - b \in C - B = C + B$. □

Proposition 3.12. *Suppose K and C are two subsets of a topological vector space X such that K is compact, C is closed and $K \cap C = \emptyset$. Then there is a neighbourhood V of 0 such that*

$$(K + V) \cap (C + V) = \emptyset.$$

Proof. If $K = \emptyset$, then $K + V = \emptyset$ since $\emptyset + V = \emptyset$, so the proposition is true. Therefore we assume that $K \neq \emptyset$ and consider a point $x \in K$. Since C is closed, then C^c is an open and $x \in C^c$.

We define $V_x = C^c - x$. Then V_x is a neighbourhood of 0. By Lemma 3.9 and Remark 3.10, there is a symmetric neighbourhood U_x of 0 such that

$$U_x + U_x + U_x \subset U_x + U_x + U_x + U_x \subset V_x = C^c - x,$$

therefore

$$x + U_x + U_x + U_x \subset C^c.$$

Since U_x is symmetric,

$$(x + U_x + U_x) \cap (C + U_x) = \emptyset. \tag{3.1}$$

Let us consider the covering $K \subset \bigcup_{x \in K} x + V_x$. Since K is a compact set, there exists a finite subcover of K. Therefore, there are finitely many points $x_1, x_2, ...x_n$ in K such that

$$K \subset (x_1 + V_{x_1}) \cup (x_2 + V_{x_2}) \cup ..... \cup (x_n + V_{x_n}),$$

where V_{x_i} is a neighbourhood of 0 for each i.

Taking $V = V_{x_1} \cap V_{x_2} \cap .... \cap V_{x_n}$ we obtain that:

$$K + V = \bigcup_{i=1}^{n} (x_i + V_{x_i} + V) \subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + V_{x_i}),$$

and no terms in this last union intersects $C + V$, by (3.1). This completes the proof. □

Definition 3.13. *Let X be a topological vector space. $C \subseteq X$ is a balanced set if for every $|\alpha| < 1 \in \Phi$,*

$$\alpha C \subset C.$$

Proposition 3.14. *Let X be a topological vector space. Then every neighbourhood of 0 contains a balanced neighbourhood of 0.*

Proof. Let U be a neighbourhood of 0. By Proposition 3.2, there exist $\delta > 0$ and a neighbourhood V of 0 such that $\alpha V \subset U$ for $|\alpha| < \delta$. We define $W = \bigcup_{|\alpha|<\delta} \alpha V$. If $x \in W$ and $|\lambda| < 1$, then

$$\lambda x \in \lambda W = \bigcup_{|\alpha|<\delta} \lambda \alpha V \subset W$$

because $|\alpha\lambda| < \delta$. □

Proposition 3.15. *Let V be a neighbourhood of 0 in a topological vector space X. If $r_1 < r_2 < .... < r_n \to \infty$ as $n \to \infty$, then*

$$X = \bigcup_{n=1}^{\infty} r_n V.$$

Proof. Fix $x \in X$. Since $\alpha \to \alpha x$ is a continuous map of the scalar field into X, let us see that the set $W = \{\alpha : \alpha x \in V\}$ is an open set. By Proposition 3.2, for every $\alpha \in W$, there is a neighbourhood V_α of α such that for all $\beta \in V_\alpha$, $\beta x \in V$.

W contains 0, hence it contains $\frac{1}{r_n}$ for all large n. Thus $\frac{1}{r_n} x \in V$ for large n, and hence, $x \in r_n V$, for large n. □

Proposition 3.16. *Let X and Y be two topological vector spaces. If $\Lambda : X \longrightarrow Y$ is linear and continuous at 0, then Λ is continuous. In fact, Λ is uniformly continuous, in the following sense: To each neighbourhood W of 0 in Y corresponds a neighbourhood V of 0 in X such that*

$$y - x \in V \quad \Longrightarrow \quad \Lambda(y) - \Lambda(x) \in W.$$

Proof. Once W is chosen, the continuity of Λ at 0 shows that $\Lambda V \subset W$ for some neighbourhood V of 0. If $y - x \in V$, the linearity of Λ shows that $\Lambda(x) - \Lambda(y) = \Lambda(y - x) \in W$.

Then Λ maps the neighbourhood $x + V$ of x into the preassigned $\Lambda x + \Lambda V \subset \Lambda x + W$, which says that Λ is continuous at x. □

Definition 3.17. *Let X and Y be two topological vector spaces. If $\Lambda : X \longrightarrow Y$ is linear, then we define:*

$$N(\Lambda) = \Lambda^{-1}(0) = \{x \in X : \Lambda(x) = 0\}.$$

Proposition 3.18. *Let Λ be a linear functional on a topological vector space X. Assume $\Lambda(x) \neq 0$, for some $x \in X$. Then each one of the four properties implies the other three:*
(a) Λ is continuous.
(b) $N(\Lambda)$ is a closed set.
(c) $N(\Lambda)$ is not dense in X.
(d) Λ is bounded in some neighbourhood V of 0.

Proof.

- (a) $\implies$ (b): Since Λ is continuous, $N(\Lambda) = \Lambda^{-1}(0)$ is a closed set since 0 is a closed set.

- (b) $\implies$ (c): Since $N(\Lambda)$ is closed and for some x, $\Lambda(x) \neq 0$, then

$$X \neq N(\Lambda) = \overline{N(\Lambda)}.$$

Therefore $N(\Lambda)$ is not dense in X.

- (c) $\implies$ (d). Since $N(\Lambda)$ is not dense in X, there exits U an open set such that $U \cap N(\Lambda) = \emptyset$. Then $N(\Lambda)^c$ has not empty interior.

Let $x \in N(\Lambda)^c$. Let U be a neighbourhood of x such that $U \subset N(\Lambda)^c$. Then $W = U - x$ is a neighbourhood of 0. By Theorem 3.14, there is a balanced neighbourhood V of 0 such that $V \subset W = U - x$, and hence, $V + x \subset U$, therefore

$$(x + V) \cap N(\Lambda) = \emptyset, \tag{3.2}$$

Let us see that ΛV is a balanced subset of the field Φ. If $\alpha \in \Lambda V$, then exists $z \in V$ such that $\alpha = \Lambda z$. Let $|\lambda| < 1$. Since V is a balanced set then

$$\lambda\alpha = \lambda\Lambda(z) = \Lambda(\lambda z) \in \Lambda V.$$

Thus either ΛV is bounded, in which case (d) holds, or ΛV is not bounded.

Let us see that if ΛV is not bounded then $\Lambda V = \Phi$. Since ΛV is a neighbourhood of 0, for every $\alpha \in \Phi$, then there is a interval I of 0 such that $0 \in I \subset \Lambda V$. Therefore, there is $|\beta| \leq 1$ such that $\beta\alpha \in \Lambda V$. Since ΛV is balanced, $\alpha \in \Lambda V$.

But, if $\Lambda V = \Phi$, then there exists $y \in V$ such that $\Lambda y = -\Lambda x$. Therefore $x + y \in N(\Lambda)$ which is a contradiction (3.2).

- (d) $\implies$ (a), if (d) holds, then $|\Lambda x| < M$ for every $x \in V$ and some $M < \infty$. If $r > 0$ and $W = (r/M)V$, then $|\Lambda x| < r$ for all x in W. Therefore, Λ is continuous at 0. By Proposition 3.16, we have (a).

$\square$

3.3 Convexity

In this section, we introduce the concept of convexity, and we study the Minkovsky functional and its properties which will also be very important for us.

Definition 3.19. *Let A be a subset of a topological vector space X. We say that A is a convex set if only if for all $a, b \in A$*

$$at + (1-t)b \in A,$$

for all $t \in [0,1]$.

Definition 3.20. *Let A be a convex subset of a topological vector space X. We say that A is an absorbing set if only if for all $x \in X$, there exists $t_x \in \Phi$ such that $x \in t_x A$. Consequently, $0 \in A$.*

Definition 3.21. *Let A be an absorbing convex subset of a topological vector space X. We define the Minkovsky functional μ_A of A as:*

$$\mu_A(x) = \inf\{t > 0 : t^{-1}x \in A\},$$

for all $x \in X$. Since A is an absorbing set, then $\mu_A(x) < \infty$ for all $x \in X$.

Proposition 3.22. *Let A be an absorbing convex subset of a topological vector space X. Then*
(a) $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$.
(b) $\mu_A(tx) = t\mu_A(x)$ if $t \geq 0$.
(c) If $B = \{x : \mu_A(x) < 1\}$, then $B \subset A$.

Proof. For every $x \in X$, we define

$$H_A(x) = \{t > 0 : t^{-1}x \in A\}.$$

Suppose $t \in H_A(x)$ and $s > t$. Since $0 \in A$ and A is convex, then

$$\frac{1}{s}x = \left(\frac{t}{s}\right)\frac{x}{t} + \left(1 - \frac{t}{s}\right)0 \in A,$$

therefore $s \in H_A(x)$. Therefore, $H_A(x)$ is a half line whose left end point is $\mu_A(x)$.

Let us prove (a). Suppose $\mu_A(x) < s$, $\mu_A(y) < t$ and $u = t + s$. Then $s^{-1}x \in A$, $t^{-1}y \in A$. Since A is convex,

$$u^{-1}(x+y) = \left(\frac{s}{u}\right)(s^{-1}x) + \left(\frac{t}{u}\right)(t^{-1}y) \in A.$$

Therefore $\mu_A(x+y) \leq u$. Let $\varepsilon > 0$. Let $s = \mu_A(x) + \varepsilon$ and $t = \mu_A(y) + \varepsilon$ then

$$\mu_A(x+y) \leq \mu_A(x) + \mu_A(y) + 2\varepsilon.$$

Letting $\varepsilon \to 0$, this gives us (a).

It is clear that (b) holds.

(c) If $x \in B$, then $\mu_A(x) < 1$, therefore $1 \in H_A(x)$, and hence, $x \in A$. □

Remark 3.23. *Let C be an absorbing convex subset of X. If $\mu_C(x) > 1$, then $x \notin C$.*

Proof. If $\mu_C(x) > 1$, then $1 \notin H_A(x) = \{t > 0 : t^{-1}x \in C\}$. Hence, $x \notin C$. □

Remark 3.24. *If C is an absorbing convex open subset of X, then for each $x \in C$, $\mu_C(x) < 1$.*

Proof. If C is open, for every $x \in C$ there is a neighbourhood V of x such that $x \in V \subset C$. Therefore, there exists $\alpha > 1$ such that $\alpha x \in V \subset C$.

If we suppose that $\mu_C(x) = 1$, then $\mu_C(\alpha x) > 1$. Therefore, by Remark 3.23, $\alpha x \notin C$ which is a contradiction. □

Remark 3.25. *If A, B are two convex subsets of X, then $A + B$ is convex.*

Proof. We have to prove that

$$tx + (1-t)y \in A + B, \quad \forall t \in [0,1].$$

Let $x, y \in A + B$. Then $x = a_x + b_x$ and $y = a_y + b_y$ with $a_x, a_y \in A$ and $b_x, b_y \in B$. Since A, B are convex, we obtain that:

$$(a_x + b_x)t + (1-t)(a_y + b_y) = \big[ta_x + (1-t)a_y\big] + \big[tb_x + (1-t)b_y\big] \in A + B.$$

□

3.4 Third Hahn-Banach Theorem

Definition 3.26. *The dual space of a topological vector space X is the vector space X^* whose elements are the continuous linear functional on X.*

Definition 3.27. *X is locally convex if there is a local base β of 0 whose members are convex.*

Remark 3.28. *If A is a convex subset of X and f is a linear continuous functional, then $f(A)$ is a convex set.*

Proof. If $\alpha, \beta \in f(A)$ then $\alpha = f(x)$ and $\beta = f(y)$ for some $x, y \in A$. If $t \in [0,1]$ then

$$t\alpha + (1-t)\beta = tf(x) + (1-t)f(y) = f\big(tx + (1-t)y\big) \in f(A).$$

□

It will be necessary to use the fact that every complex vector space is also a real vector space. If u is the real part of a complex linear functional f on X, then u is a real linear functional and

$$f(x) = u(x) - iu(ix) \quad (x \in X).$$

Since $z = \Re z - i\Re(iz)$ for every $z \in \mathbb{C}$. Conversely, if $u : X \to \mathbb{R}$ is a real linear functional on a complex vector space X, and if f is defined by $f(x) = u(x) - iu(ix)$, then f is a complex linear functional. This was seen in the proof of Theorem 2.12.

The above facts imply that complex linear functional f on X is in X^* if only if its real part is continuous. This observation shows that in the next theorem we can suppose that $\mathbb{R}$ is the scalar field of X.

Theorem 3.29. *[Third version of Hanh-Banach theorem] Suppose A and B are disjoints, nonempty, convex sets in a Banach space X.*

(a) If A is open, then there exist $\Lambda \in X^*$ and $\gamma \in \mathbb{R}$ such that

$$\Re\Lambda x < \gamma \leq \Re\Lambda y,$$

for each $x \in A$ and $y \in B$.

(b) If A is compact, B is closed, then there exist $\Lambda \in X^*$, $\gamma_1 \in \mathbb{R}$ and $\gamma_2 \in \mathbb{R}$ such that

$$\Re\Lambda x < \gamma_1 < \gamma_2 < \Re\Lambda y,$$

for each $x \in A$ and for each $y \in B$.

Proof. (a) Fix $a_0 \in A$, $b_0 \in B$. Put $x_0 = b_0 - a_0$, and $C = A - B + x_0$. Then C is a convex neighbourhood of 0, by Remark 3.25. Let p be the Minkovsky functional of C. By Proposition 3.22 (a) and (b), p is a convex functional.

Since $A \cap B = \emptyset$, then $x_0 \notin C$. If $x_0 \in C$, this means that there exists $x \in A$ and $x \in B$ such that $x_0 = x - x + x_0$, therefore $A \cap B \neq \emptyset$ which is a contradiction.

Therefore, by Proposition 3.22 (c), $p(x_0) \geq 1$.

Now, we define $f(tx_0) = t$ on the subspace M of X generated by x_0. If $t \geq 0$,

$$f(tx_0) = t \leq tp(x_0) = p(tx_0).$$

If $t < 0$, then $f(tx_0) < 0 \leq p(tx_0)$. Thus $f \leq p$ in M. By Theorem 2.9, f extends to a linear functional Λ on X that also satisfies $\Lambda \leq p$.

In particular, $\Lambda \leq 1$ on C, hence $\Lambda \geq -1$ on $-C$: If $y \in -C$,

$$-\Lambda(y) = \Lambda(-y) \leq 1,$$

therefore $\Lambda(y) \geq -1$.

We obtain that $|\Lambda| \leq 1$ on the neighbourhood $C \cap (-C)$ of 0. Thus $\Lambda \in X^*$, by Theorem 3.18.

Let $a \in A$ and $b \in B$. Noting that $\Lambda(x_0) = f(1x_0) = 1$ and $a - b + x_0 \in C$, we have that:

$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + x_0) \leq p(a - b + x_0).$$

Since C is open, by Remark 3.24, we obtain that $p(a - b + x_0) < 1$. Thus $\Lambda a < \Lambda b$.

$\Lambda(A)$ and $\Lambda(B)$ are disjoint subsets of $\mathbb{R}$. By Remark 3.28, they also are convex subset, with $\Lambda(A)$ to the left of $\Lambda(B)$. Also, $\Lambda(A)$ is an open set since Λ is an open map, by Theorem 2.24. Since $\Lambda(A)$ is an open convex set, then $\Lambda(A) = (c, d)$. Taking $\gamma = d$, we obtain the conclusion of (a).

(b) By Theorem 3.12, there is a convex neighbourhood V of 0 in X such that $(A + V) \cap (B + V) = \emptyset$. Since $B \subset (B + V)$, we obtain that $(A + V) \cap B = \emptyset$.

Using (a) with $A + V$ in place of A, we obtain that there exists $\Lambda \in X^*$ such that $\Lambda(A + V)$ and $\Lambda(B)$ are disjoints convex subsets of $\mathbb{R}$, with $\Lambda(A + V)$ to the left of $\Lambda(B)$. That is, for each $a \in A + V$ and $b \in B$

$$\Lambda a < \gamma' \leq \Lambda b.$$

Since A is compact, then $\Lambda(A) = [c, d]$ is also compact. Therefore,

$$\Lambda(A) = [c, d] \subset \Lambda(A + V) = (e, f).$$

Taking $d < \gamma_1 < f$, we have that for all $a \in A$ and $b \in B$,

$$\Lambda a < \gamma_1 < \gamma' \leq \Lambda b.$$

Choosing γ_2 so that $\gamma_1 < \gamma_2 < \gamma'$, we have the conclusion of (b). $\square$

Chapter 4

Banach Function Spaces

In this chapter, we introduce the definition of Banach Function Spaces and their most important properties, such as the fact of being complete.

4.1 Definition of Banach Function Spaces

Definition 4.1. *A map $\rho : \mathcal{M}^+ \longrightarrow [0, \infty]$ is called Banach function norm or simply function norm if, for all $f, g, f_n, n = (1, 2, 3...)$ in $\mathcal{M}^+$, for all constants $a \geq 0$, and for all μ-measurable subsets E of $\mathcal{R}$, the following properties hold:*

- ***P1*** $\rho(f) = 0 \Leftrightarrow f = 0$ *μ-a.e. ;* $\quad \rho(af) = a\rho(f);$ $\quad \rho(f + g) \leq \rho(f) + \rho(g)$.
- ***P2*** $0 \leq g \leq f$ *μ-a.e.* $\implies \rho(g) \leq \rho(f)$.
- ***P3*** $0 \leq f_n \uparrow f$ *μ-a.e.* $\implies \rho(f_n) \uparrow \rho(f)$.
- ***P4*** $\mu(E) < \infty \implies \rho(\chi_E) < \infty$.
- ***P5*** $\mu(E) < \infty \implies \int_E f(x)d\mu(x) \leq C_E\rho(f)$ *for some constant* $0 < C_E < \infty$, *depending on E and ρ, but independent of f.*

The classic examples of Banach function norms are those associated with the Lebesgue spaces L^p, $1 \leq p \leq \infty$.

Theorem 4.2. *If $f \in \mathcal{M}^+$, then $\|f\|_{L^p} = \rho_p(f)$ are function norms for every $1 \leq p \leq \infty$.*

Proof. Case $1 \leq p < \infty$.

- **P1**:

$$\rho_p(f) = 0 \implies \int_{\mathcal{R}} f^p(x)d\mu(x) = 0 \implies f^p = 0 \ \mu\text{-a.e.} \implies f = 0 \ \mu\text{-a.e.},$$

$$\rho(af) = (a^p \int_{\mathcal{R}} f^p(x)d\mu(x))^{1/p} \implies a\rho_p(f) = \rho_p(af).$$

By Theorem 1.12, we also have the triangle inequality.

- **P2** If $0 \leq g \leq f$ μ-a.e. then

$$g^p \leq f^p \ \mu\text{-a.e.} \implies \int_{\mathcal{R}} g^p(x)d\mu(x) \leq \int_{\mathcal{R}} f^p(x)d\mu(x) \implies \rho_p(g) \leq \rho_p(f).$$

- **P3** If

$$f_n \uparrow f \implies f_n^p \uparrow f^p,$$

then by Theorem 1.6, we obtain that $\rho_p(f_n) \uparrow \rho_p(f)$.

- **P4** If $0 < \mu(E) < \infty$,

$$\int_{\mathcal{R}} \chi_E^p(x)d\mu(x) = \int_{\mathcal{R}} \chi_E(x)d\mu(x) = \mu(E) < \infty.$$

- **P5** If $\mu(E) < \infty$, then by P4, we obtain that $\chi_E \in L^{p'}$ and hence,

$$\int_{\mathcal{R}} f(x)\chi_E(x)d\mu(x) \leq \left[\int_{\mathcal{R}} f^p(x)d\mu(x)\right]^{\frac{1}{p}} \left[\int_{\mathcal{R}} \chi_E^p(x)d\mu(x)\right]^{\frac{1}{p'}} = \rho_p(f)C_E.$$

Case p=∞,

- **P1** If $\rho_\infty(f) = \operatorname{ess\,sup}_{x\in\mathcal{R}} f(x) = 0$ μ-a.e. and $f \in \mathcal{M}^+$, then $f = 0$ μ-a.e.
Since $\operatorname{ess\,sup}_{x\in\mathcal{R}} af(x) = a \operatorname{ess\,sup}_{x\in\mathcal{R}} f(x)$, then $\rho_\infty(af) = a\rho_\infty(f)$.
Since $\operatorname{ess\,sup}_{x\in\mathcal{R}}(f(x)+g(x)) \leq \operatorname{ess\,sup}_{x\in\mathcal{R}} f(x) + \operatorname{ess\,sup}_{x\in\mathcal{R}} g(x)$, then

$$\rho_\infty(f+g) \leq \rho_\infty(f) + \rho_\infty(g).$$

- **P2** If $f \leq g$ μ-a.e., then $\operatorname{ess\,sup}_{x\in\mathcal{R}} f(x) \leq \operatorname{ess\,sup}_{x\in\mathcal{R}} g(x)$, and therefore $\rho_\infty(f) \leq \rho_\infty(g)$.

- **P3** If $f_n \uparrow f$ μ-a.e., then $(\rho_\infty(f_n))_n$ is a increasing sequence, so that

$$K := \lim_{n\to\infty} \rho_\infty(f_n) = \sup_{n\in\mathbb{N}} \rho_\infty(f_n) \leq \rho_\infty(f).$$

Therefore

$$K = \sup_{n\in\mathbb{N}} \inf\{c : f_n \leq c \ \mu\text{-a.e.}\}.$$

Then, for all n, $\rho_\infty(f_n) \leq K$ μ-a.e. Therefore, $\sup_{n\in\mathbb{N}} \rho_\infty(f_n) \leq K$ μ-a.e. Since $\rho_\infty(f) = \sup_{n\in\mathbb{N}} \rho_\infty(f_n)$, we obtain that $\rho_\infty(f) \leq K$ μ-a.e. Therefore $\rho_\infty(f) \leq K$. We conclude that $\rho_\infty(f_n) \uparrow \rho_\infty(f)$.

- **P4** If $\mu(E) < \infty$, then $\rho_\infty(\chi_E) = \operatorname{ess\,sup}_{x\in\mathcal{R}} \chi_E(x) = 1$.

- **P5** If $\mu(E) < \infty$,

$$\int_E f(x)d\mu(x) \leq \int_E \operatorname*{ess\,sup}_{x\in\mathcal{R}} f(x)d\mu(x)$$

$$= \int_E \rho_\infty(f)d\mu(x) = \rho_\infty(f)\int_E d\mu(x) = \rho_\infty(f)\mu(E).$$

□

Definition 4.3. *Let ρ be a function norm. The collection $X = X(\rho)$ of all functions f in $\mathcal{M}$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, we define*

$$\|f\|_X = \rho(|f|).$$

4.2 Basic properties of Banach Function Spaces

To prove the next theorem, we need a previous result:

Proposition 4.4. *Let $(f_n)_n$ be a sequence of measurable functions such that converges in measure to f. Then there is a subsequence $(f_{n_k})_k$ which converges to f a.e.*

Proof. By Definition 1.2, we have that given ν there is a $n_\nu \in \mathbb{N}$ such that for all $n \geq n_\nu$:

$$\mu(\{x : |f(x) - f_n(x)| \geq 2^{-\nu}\}) < 2^{-\nu}.$$

We consider $E_\nu := \{x : |f_{n_\nu}(x) - f(x)| \geq 2^{-\nu}\}$, and let

$$x \notin A = \bigcap_{k=1}^{\infty} \bigcup_{\nu=k}^{\infty} E_\nu.$$

Therefore, there exists k' such that for all $\nu > k'$, $x \notin E_\nu$, and hence

$$x \in \{z : |f_{n_\nu}(z) - f(z)| \leq 2^{-\nu}\}.$$

Thus $|f_{n_\nu}(x) - f(x)| \to 0$. Therefore $f_{n_\nu}(x) \to f(x)$. Let now us see that $\mu(A) = 0$.

$$\mu(A) \leq \mu\left(\bigcup_{\nu=k}^{\infty} E_\nu\right) \leq \sum_{\nu=k}^{\infty} \mu(E_\nu) = \sum_{\nu=k}^{\infty} 2^{-\nu} = \frac{2^{-k}}{1 - \frac{1}{2}} = 2^{-k+1}.$$

Letting $k \to \infty$, we obtain that $\mu(A) = 0$. By Definition 1.2, $(f_{n_\nu})_\nu$ converges to f a.e. □

Theorem 4.5. *Let ρ be a function norm and let $X = X(\rho)$ and $\|\cdot\|_X$ be as in Definition 4.3. Then, under the natural vector space operations, $(X, \|\cdot\|_X)$ is a normed space for which the inclusions*

$$S \subset X \hookrightarrow \mathcal{M}_0 \tag{4.1}$$

hold. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges μ-a.e. to f.

Proof. Let us prove that $X \hookrightarrow \mathcal{M}_0$. Let E be a measurable such that $\mu(E) < \infty$. If $f \in X$, by P5,

$$\int_E f(x) d\mu(x) \leq C_E \|f\|_X < \infty.$$

Suppose now that $f \in X$ and is not in $\mathcal{M}_0$. Then the set $F = \{x : f(x) = \infty\}$ has positive measure. Let us consider two cases:

- Case 1: If $\mu(F) = \beta$ with $0 < \beta < \infty$ then $\int_F f(x)d\mu(x) = \infty$. Using P5,

$$\int_F f(x)d\mu(x) \leq C_F \|f\|_X < \infty,$$

which is a contradiction.

- Case 2: If $\mu(F) = \infty$ and since μ is a σ-finite measure then $F = \bigcup_{k=1}^{\infty} F_k$ with $\mu(F_k) < \infty$ for all k. Therefore if we fix F_k, we have that $\int_{F_k} f(x)d\mu(x) = \infty$. Using P5 as in case 1, we have a contradiction.

Let us see that $S \subset X$. Let $f(x) = \sum_{i=0}^{n} \alpha_i \chi_{E_i}(x) \in S$. By properties P1 and P4,

$$\|f\|_X = \rho(|f|) = \rho\bigg(|\sum_{i=0}^{n} \alpha_i \chi_{E_i}|\bigg) \leq \rho\bigg(\sum_{i=0}^{n} |\alpha_i \chi_{E_i}|\bigg) \leq$$

$$\leq \sum_{i=0}^{n} \rho(|\alpha_i \chi_{E_i}|) = \sum_{i=0}^{n} |\alpha_i| \rho(|\chi_{E_i}|) < \infty.$$

Let us now see that X is a normed space.

- Let $f \in X$. If $\|f\|_X = \rho(|f|) = 0$, then $|f| = 0$ μ-a.e., and hence $f = 0$ μ-a.e.
- Let $f, g \in X$, then using P1:

$$\|\alpha f + \beta g\|_X \leq \rho(|\alpha f + \beta g|) \leq |\alpha|\rho(|f|) + |\beta|\rho(|g|) \leq |\alpha|\|f\|_X + |\beta|\|g\|_X.$$

Let us now prove that every convergent sequence in X is also convergent in $\mathcal{M}_0$. If $f_n \to f$ in X, then $\rho(|f - f_n|) \to 0$. Let $\varepsilon > 0$ and let E be a set with finite measure. We define $A_n = \{x \in E : |f(x) - f_n(x)| \geq \varepsilon\}$. By P5,

$$\mu(A_n) = \int_{A_n} 1 d\mu \leq \int_{A_n} \frac{1}{\varepsilon}|f(x) - f_n(x)| d\mu(x) \leq \int_E \frac{1}{\varepsilon}|f(x) - f_n(x)| d\mu(x)$$

$$\leq C_E \rho\bigg(\frac{1}{\varepsilon}|f - f_n|\bigg) = C_E \frac{1}{\varepsilon} \rho(|f - f_n|). \tag{4.2}$$

Thus $\mu(A_n) \to 0$ as $n \to \infty$. That shows that $f_n \to f$ in measure on every set of finite measure. By Definition 1.4, $f_n \to f$ in $\mathcal{M}_0$.

Since μ is a σ-finite measure, similarly to Proposition 4.4, there is a subsequence $(f_{n_k})_k$ which converges to f μ-a.e. □

The next lemma will be very important for us.

Lemma 4.6. *[Fatou's lemma] Let $X = X(\rho)$ be a Banach function space and suppose $f_n \in X$ for all n.*

- *(i) If $0 \leq f_n \uparrow f$ μ-a.e., then either $f \notin X$ and $\|f_n\|_X \uparrow \infty$, or $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.*

- *(ii) If $f_n \to f$ μ-a.e. and if $\liminf_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$ and*
$$\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X.$$

Proof. i) If $f \notin X$ then $\|f\|_X = \infty$. By P3, $\|f_n\|_X \uparrow \infty$. If $f \in X$, then $\|f_n\|_X \uparrow \|f\|_X$.

ii) We define $h_n(x) = \inf_{m\geq n} |f_m(x)|$. Since $f_n \to f$ μ-a.e., then $|f_n| \to |f|$ μ-a.e. Thus $0 \leq h_n \uparrow |f|$ μ-a.e. By P3 and P2:

$$\|f\|_X = \rho(|f|) = \lim_{n\to\infty} \rho(|h_n|) \leq \lim_{n\to\infty} (\inf_{m\geq n} \rho(|f_m|) = \liminf_{n\to\infty} \|f_n\|_X < \infty.$$

Since μ is a complete measure and f is limit of a sequence of measurable functions, then f is a measurable function. Since $\|f\|_X < \infty$, then $f \in X$. $\square$

Theorem 4.7. *[Riesz-Fischer property] Let $(f_n)_n$ be a sequence of X. If*

$$\sum_{n=1}^{\infty} \|f_n\|_X < \infty,$$

then $\sum_{n=1}^{\infty} f_n$ converges in X to a function f in X and

$$\|f\|_X \leq \sum_{n=1}^{\infty} \|f_n\|_X.$$

Proof. Let $t = \sum_{n=1}^{\infty} |f_n|$ and let $t_N = \sum_{n=1}^{N} |f_n|$ therefore $0 \leq t_N \uparrow t$. Since

$$\|t_N\|_X \leq \sum_{n=1}^{N} \|f_n\|_X \leq \sum_{n=1}^{\infty} \|f_n\|_X < \infty, \quad (N = 1, 2, ...).$$

Since $t_N \uparrow t$ μ-a.e. and $\|t\|_X < \infty$, by Lemma 4.6 (i), t belongs X.

Thus $t \in \mathcal{M}_0$, by Theorem 4.5. Hence $t = \sum_{n=1}^{\infty} |f_n|$ is finite μ-a.e. Therefore $\sum_{n=1}^{\infty} |f_n|$ converges μ-a.e. and hence so does $\sum_{n=1}^{\infty} f_n(x)$. If

$$f = \sum_{n=1}^{\infty} f_n, \quad s_N = \sum_{n=1}^{N} f_n, \quad (N = 1, 2...),$$

then $s_N \to f$ μ-a.e. Hence, for any $\mathcal{M}$, we have that $s_N - s_M \to f - s_M$ μ-a.e. as $N \to \infty$. Furthermore, if $N > M$:

$$\liminf_{N\to\infty} \|s_N - s_M\|_X \leq \liminf_{N\to\infty} \sum_{n=M+1}^{N} \|f_n\|_X = \sum_{n=M+1}^{\infty} \|f_n\|_X < \infty.$$

Using the Fatou's lemma to $s_N - s_M$, we have that $f - s_M$ belongs to X and

$$\|f - s_M\|_X \leq \sum_{n=M+1}^{\infty} \|f_n\|_X.$$

Consequently, $\|f - s_M\|_X \to 0$ as $M \to \infty$.

$$\|f\|_X \leq \|f - s_M\|_X + \|s_M\|_X \leq \|f - s_M\|_X + \sum_{n=1}^{M} \|f_n\|_X.$$

Letting $M \to \infty$, we obtain the inequality. $\square$

Corollary 4.8. *If X is a Banach function space, then X is complete.*

Proof. It follows from Proposition 2.18. □

Theorem 4.9. *Let X and Y be Banach function spaces over the same measure space. If $X \subset Y$, then in fact $X \hookrightarrow Y$.*

Proof. Suppose $X \subset Y$ but $X \hookrightarrow Y$ fails. Then there exist functions f_n in X for which

$$\|f_n\|_X \leq 1, \quad \|f_n\|_Y > n^3, \quad (n = 1, 2, ...).$$

Replacing each f_n with its absolute value, we may assume $f_n \geq 0$ for all n. Since

$$\left\| \sum_{n=1}^{\infty} n^{-2} f_n \right\|_X \leq \sum_{n=1}^{\infty} n^{-2} \|f_n\|_X < \infty,$$

we obtain that $\sum_{n=1}^{\infty} n^{-2} f_n$ converges in X to some function f in X, by Theorem 4.7. Since $X \subset Y$, then f belongs to Y and hence, $\|f\|_Y < \infty$.

But this is impossible, because $0 \leq n^{-2} f_n \leq f$, and therefore

$$\|f\|_Y \geq n^{-2} \|f_n\|_Y > n,$$

for all n. We conclude that $f \notin Y$, which is a contradiction. □

The next theorem summarizes for future references the basic properties of Banach function spaces.

Theorem 4.10. *Suppose ρ is a function norm and let*

$$X = \{f \in \mathcal{M} : \rho(|f|) < \infty\}.$$

For each $f \in X$, let $\|f\|_X = \rho(|f|)$. Then $(X, \|\cdot\|_X)$ is a Banach space and the following properties hold for all f, g_n, f_n, $(n = 1, 2, 3...)$ in $\mathcal{M}$ and all measurable subsets E of $\mathcal{R}$:

- *(i) If $|g| \leq |f|$ μ-a.e. and $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$; in particular, a measurable function f belongs to X if and only if $|f|$ belongs to X, and in that case f and $|f|$ have the same norm in X.*
- *(ii) If $0 \leq f_n \uparrow f$ μ-a.e., then either $f \notin X$ and $\|f_n\|_X \uparrow \infty$, or $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.*
- *(iii) If $f_n \to f$ μ-a.e. and if $\liminf_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$ and*

$$\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X.$$

- *(iv) Every simple function belongs to X.*
- *(v) If $\mu(E) < \infty$ then $\int_E f(x) d\mu(x) \leq C_E \|f\|_X$ for some constant $0 < C_E < \infty$, depending on E and ρ but independent of f.*
- *(vi) If $f_n \to f$ in X, then $f_n \to f$ on every set of finite measure; in particular, some subsequence of $(f_n)_n$ converges to f μ-a.e.*

4.3 The associate space

In this section, we introduce the associate space X' of the Banach function space X and its properties.

Definition 4.11. *If ρ is a function norm, its associate norm ρ' is defined on $\mathcal{M}^+$ by*

$$\rho'(g) = \sup\left\{ \int_{\mathcal{R}} f(x)g(x)d\mu(x) : f \in \mathcal{M}^+, \rho(f) \leq 1 \right\}.$$

Theorem 4.12. *Let ρ be a function norm. Then the associate norm ρ' is a function norm.*

Proof. We have to prove all properties of a function norm.

- P1 and P2: If $\rho(f) \leq 1$, then f is finite μ-a.e., by Theorem 4.5. If $g = 0$ μ-a.e., we obtain that $\int_{\mathcal{R}} f(x)g(x)d\mu(x) = 0$ for $f \in \mathcal{M}^+$. Therefore $\rho'(g) = 0$.

  We now suppose that $\rho'(g) = 0$. Let E be a measurable set of $\mathcal{R}$ such that $0 < \mu(E) < \infty$, then $\rho(\chi_E) < \infty$, by P4. Moreover, $\rho(\chi_E) > 0$. If $\rho(\chi_E) = 0$, then $\chi_E = 0$ μ-a.e. Therefore, we have that $\mu(E) = 0$, which is a contradiction.

  Taking $f(x) = \chi_E(x)/\rho(\chi_E)$, we have that $\rho(f) = 1$. Then

  $$0 = \int_{\mathcal{R}} f(x)g(x)d\mu(x) = \int_{\mathcal{R}} \frac{\chi_E(x)}{\rho(\chi_E)} g(x)d\mu(x) = \rho(\chi_E)^{-1} \int_E g(x)d\mu(x),$$

  and hence $g = 0$ μ-a.e. on E. Since E is an arbitrary measurable set of finite measure and μ is σ-finite measure, then we obtain that $g = 0$ μ-a.e.

  The other properties of P1 and P2 follow from the linearity of integral.

- P3: We suppose that $g_n, g \in \mathcal{M}^+$ for all $n \in \mathbb{N}$ and $0 \leq g_n \uparrow g$ μ-a.e. By P2, the sequence $\rho'(g_n)$ increases with n and $\rho'(g_n) \leq \rho'(g)$ for all n. Then, if $\rho'(g_n) = \infty$ for some n, that implies $\rho'(g) = \infty$. Therefore, we may suppose $\rho'(g_n) < \infty$ for all n.

  Let ξ be any number satisfying $\xi < \rho'(g)$. By Definition 4.11, there is a function f in $\mathcal{M}^+$ with $\rho(f) \leq 1$ such that $\int_{\mathcal{R}} f(x)g(x)d\mu(x) > \xi$.

  Since $0 \leq g_n \uparrow g \quad \mu$-a.e., then $0 \leq fg_n \uparrow fg$ μ-a.e. Theorem 1.6 shows that

  $$\int_{\mathcal{R}} f(x)g_n(x)d\mu(x) \uparrow \int_{\mathcal{R}} f(x)g(x)d\mu(x),$$

  and hence, there exists $N \in \mathbb{N}$ such that $\int_{\mathcal{R}} f(x)g_n(x) > \xi$ for all $n \geq N$. Therefore

  $$\rho'(g) \geq \rho'(g_n) \geq \int_{\mathcal{R}} f(x)g_n(x)d\mu(x) > \xi,$$

  for all $n > N$. This shows that $\rho'(g_n) \uparrow \rho'(g)$, because if, $\rho'(g_n) \uparrow z$ where $z < \rho'(g)$, then taking $z = \xi$ we have a contradiction.

- P4: Let E be a measurable set with $\mu(E) < \infty$ and let $f \in \mathcal{M}^+$ such that $\rho(f) \leq 1$. By P5 of ρ, we have that

$$\int_{\mathcal{R}} \chi_E(x) f(x) d\mu(x) = \int_E f(x) d\mu(x) \leq C_E \rho(f) \leq C_E,$$

  with $0 < C_E < \infty$. By Definition 4.11, we obtain that $\rho'(\chi_E) \leq C_E$.

- P5: We fix E a measurable set such that $\mu(E) < \infty$. If $\mu(E) = 0$ there is nothing to prove so we may assume $\mu(E) > 0$. We know that $0 < \rho(\chi_E) < \infty$ and $f(x) = \frac{\chi_E(x)}{\rho(\chi_E)}$ satisfies that $\rho(f) = 1$. For any $g \in \mathcal{M}^+$,

$$\int_E g(x) d\mu(x) = \int_{\mathcal{R}} \chi_E(x) g(x) d\mu(x) = \rho(\chi_E) \int_{\mathcal{R}} f(x) g(x) d\mu(x).$$

  By Definition 4.11,

$$\int_E g(x) d\mu(x) = \rho(\chi_E) \int_{\mathcal{R}} f(x) g(x) d\mu(x) \leq \rho(\chi_E) \rho'(g),$$

  which shows that P5 holds for ρ'.

□

Definition 4.13. *Let ρ be a function norm and let $X = X(\rho)$ be the Banach function space determined by ρ as in the Definition 4.1. Let ρ' be the associate norm of ρ. The Banach function space $X(\rho')$ determined by ρ' is called the associate space of X and is denoted by X'.*

By Definition 4.3, if $g \in \mathcal{M}$ then

$$\begin{aligned}
\|g\|_{X'} = \rho'(|g|) &= \sup \left\{ \int_{\mathcal{R}} f(x)|g(x)| d\mu(x) : f \in \mathcal{M}^+, \rho(f) \leq 1 \right\} \\
&= \sup \left\{ \int_{\mathcal{R}} |f(x)||g(x)| d\mu(x) : |f| \in \mathcal{M}^+, \rho(|f|) \leq 1 \right\} \\
&= \sup \left\{ \int_{\mathcal{R}} |f(x) g(x)| d\mu(x) : f \in X, \|f\|_X \leq 1 \right\}.
\end{aligned}$$

Theorem 4.14. *[Holder's inequality] Let X be a Banach function space with associate space X'. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\mathcal{R}} |f(x) g(x)| d\mu(x) \leq \|f\|_X \|g\|_{X'}.$$

Proof. If $\|f\|_X = 0$, then $f = 0$ μ-a.e., so both sides of the inequality are zero. We suppose that $\|f\|_X > 0$. Then $f(x)/\|f\|_X$ has a norm 1. By Definition 4.13, we have that:

$$\int_{\mathcal{R}} \left| \frac{f(x)}{\|f\|_X} g(x) \right| d\mu(x) \leq \|g\|_{X'},$$

from which the result follows. □

Theorem 4.15. *$L^{p'}$ is the associate space of L^p.*

Proof. We have to prove that

$$\|g\|_{L^{p'}} = \sup\left\{ \int_{\mathcal{R}} |f(x)g(x)|d\mu(x),\ f \in L^p : \|f\|_{L^p} \leq 1 \right\}.$$

By Theorem 1.11, we obtain that

$$\|g\|_{L^{p'}} \geq \sup\left\{ \int_{\mathcal{R}} |f(x)g(x)|d\mu(x),\ f \in L^p : \|f\|_{L^p} \leq 1 \right\}.$$

We define $h(x) = \frac{|g(x)|^{p'-1}}{\|g\|_{L^{p'}}^{p'-1}}$. Then we observe that

$$\int_{\mathcal{R}} |h(x)g(x)|d\mu(x) = \|g\|_{L^{p'}}.$$

Let us see that $\|h\|_{L^p} = 1$,

$$\|h\|_{L^p} = \left[\int_{\mathcal{R}} \frac{|g(x)|^{p(p'-1)}}{\|g\|_{L^{p'}}^{p(p'-1)}} d\mu(x) \right]^{\frac{1}{p}}.$$

Since $p' = p(p'-1)$,

$$\|h\|_{L^p} = \frac{1}{\|g\|_{L^{p'}}^{\frac{p'}{p}}} \left[\int_{\mathcal{R}} |g(x)|^{p'} d\mu(x) \right]^{\frac{1}{p}} = 1.$$

Thus

$$\|g\|_{L^{p'}} \leq \sup\left\{ \int_{\mathcal{R}} |f(x)g(x)|d\mu(x),\ f \in L^p : \|f\|_{L^p} \leq 1 \right\}.$$

□

Lemma 4.16. *[Landau's resonance theorem] In order that a measurable function g belongs to the associate space X', it is necessary and sufficient that fg be integrable for every f in X.*

Proof. The necessity follows from Theorem 4.14. Because if $f \in X$ and fg is not integrable then

$$\infty = \int_{\mathcal{R}} |f(x)g(x)|d\mu(x) \leq \|f\|_X \|g\|_{X'}.$$

Therefore $g \notin X'$.

In the other direction, we suppose that $\|g\|_{X'} = \rho(|g|) = \infty$ but that fg is integrable for every f in X. By Definition 4.11, there exist non-negative functions f_n satisfying

$$\|f_n\|_X \leq 1, \quad \int_{\mathcal{R}} |f_n(x)g(x)|d\mu(x) > n^3, \quad \forall n \in \mathbb{N}.$$

By Theorem 4.7, we obtain that $f(x) = \sum_{n=1}^{\infty} n^{-2} f_n(x)$ belongs to X because $\|f\|_X \le \sum_{n=1}^{\infty} n^{-2} < \infty$.

However, the product fg cannot be integrable because

$$\int_{\mathcal{R}} |f(x)g(x)|d\mu(x) \ge n^{-2} \int_{\mathcal{R}} |f_n(x)g(x)|d\mu(x) > n, \quad (n = 1, 2...),$$

This contradiction establishes the sufficiency. □

Proposition 4.17. *The norm of a function g in the associate space X' is given by*

$$\|g\|_{X'} = \sup\left\{\left|\int_{\mathcal{R}} f(x)g(x)d\mu(x)\right| : f \in X, \|f\|_X \le 1\right\}.$$

Proof. By Definition 4.13 and $\left|\int_{\mathcal{R}} f(x)g(x)d\mu(x)\right| \le \int_{\mathcal{R}} |f(x)g(x)|d\mu(x)$, it follows that

$$\|g\|_{X'} \ge \sup\left\{\left|\int_{\mathcal{R}} f(x)g(x)d\mu(x)\right| : f \in X, \|f\|_X \le 1\right\}.$$

Hence, we need only to establish the reverse inequality, which, by Definition 4.13, may be written

$$\sup_{f\in S} \int_{\mathcal{R}} |f(x)g(x)|d\mu(x) \le \sup_{f\in S} \left|\int_{\mathcal{R}} f(x)g(x)d\mu(x)\right|,$$

where both supreme extended over all f in the unit ball S of X, that is, $\|f\|_X \le 1$.

We define

$$E = \{x \in \mathcal{R} : g(x) \neq 0\}.$$

We may write $g(x)$ in polar form $g(x) = |g(x)|\phi(x)$, where $|\phi| = 1$. Hence, $|g(x)| = g(x)\overline{\phi}(x)$ on E, because $|\phi(x)| = \phi(x)\overline{\phi(x)} = 1$. For any $f \in S$, we thus have

$$\int_{\mathcal{R}} |f(x)g(x)|d\mu(x) = \int_E |f(x)g(x)|d\mu(x) = \int_E |f(x)|\overline{\phi(x)}g(x)d\mu(x).$$

We define $h(x) = |f(x)|\overline{\phi(x)}$ on E and $h = 0$ off E. Then $|h| \le |f|$ on $\mathcal{R}$ and so h belongs to S. Hence,

$$\int_{\mathcal{R}} |f(x)g(x)|(x)d\mu(x) = \int_E |f(x)|\overline{\phi(x)}g(x)d\mu(x) = \int_E h(x)g(x)d\mu(x)$$

$$\le \left|\int_{\mathcal{R}} h(x)g(x)d\mu(x)\right| \le \sup_{f\in S}\left|\int_{\mathcal{R}} f(x)g(x)d\mu(x)\right|.$$

Taking the supreme on the left over all f in S, we obtain that

$$\|g\|_{X'} \le \sup\left\{\left|\int_{\mathcal{R}} f(x)g(x)d\mu(x)\right| : f \in X, \|f\|_X \le 1\right\}.$$

□

Theorem 4.18. *[Lorentz-Luxemburg Theorem] Every Banach function space X coincides with its second associate space X''. In other words, a function f belongs to X if ond only if it belongs to X'', and in that case*

$$\|f\|_X = \|f\|_{X''}.$$

Proof. By Theorem 4.14, if $f \in X$, then fg is integrable for every $g \in X'$. If we apply Lemma 4.16 to X' instead of X, we obtain that $f \in X''$. Hence $X \subset X''$. We also obtain from Definition 4.13 that

$$\|f\|_{X''} = \sup\left\{ \int_{\mathcal{R}} |f(x)g(x)|d\mu(x) : \|g\|_{X'} \leq 1 \right\}.$$

By Theorem 4.14

$$\int_{\mathcal{R}} |f(x)g(x)|d\mu(x) \leq \|f\|_X \|g\|_{X'} \leq \|f\|_X.$$

Therefore $\|f\|_{X''} \leq \|f\|_X$.

Hence, in order to complete the proof we need only show $X'' \subset X$ and

$$\|f\|_X \leq \|f\|_{X''}.$$

- First, we choose an increasing sequence of sets R_N such that $\mu(R_N) < \infty$ for all N and $\bigcup_{N\geq 1} R_N = \mathcal{R}$. This is possible because μ is a totally $\sigma-$finite measure. For each N and each $f \in X''$ we define

  $$f_N(x) = \min(|f(x)|, N)\chi_{R_N}(x).$$

  By Theorem 4.10 (iv) and since $f_N(x) \leq N\chi_{R_N}(x)$, we obtain that $\|f_N\|_X \leq \|N\chi_{R_N}\|_X < \infty$. Therefore f_N belongs to X and to X''.

- Let us see that $f_N \uparrow f$. Let $N \in \mathbb{N}$ such that $x \in R_N$ and $N > |f(x)|$. Thus $f_N(x) = |f(x)|$. For all $N' > N$ we have that $f_{N'}(x) = |f(x)|$. Since $R_N \subset R_{N+1}$, then $0 \leq f_N \uparrow |f|$. Therefore

  $$\|f_N\|_X \uparrow \|f\|_X, \quad \|f_N\|_{X''} \uparrow \|f\|_{X''},$$

  by Fatou's Lemma.

- Now, it will suffice to show that

  $$\|f_N\|_X \leq \|f_N\|_{X''}, \qquad (N = 1, 2, ...),$$

  to obtain that $\|f\|_X \leq \|f\|_{X''}$.

For the remainder of the proof, we suppose therefore that f and N are fixed. Clearly, we may assume $\|f_N\|_X > 0$ since otherwise there is nothing to prove.

- Let L^1_N be the space of $\mu-$integrable functions on $\mathcal{R}$ having a supports in R_N. With norm $g \to \int_{R_N} |g|d\mu$, it is clear that L^1_N is a Banach space.

- Let $S = \{f \in X : \|f\|_X \leq 1\}$, then the set $U = S \cap L^1_N$ is a convex subset, because if, $f, g \in U$ then $\|tf+(1-t)g\|_X \leq t+(1-t) \leq 1$ so $tf+(1-t)g \in S$. It is obvious that $tf + (1-t)g \in L^1_N$ by linearity of the integral.

- Let us see that U is a closed subset of L^N_1. If $h_n \in U$ for all n and $h_n \to h$ in L^1_N, there is a subsequence $h_{n(k)}$ converges to h $\mu-$a.e. on $\mathcal{R}$,(view in(4.2)).

  Since every $h_{n(k)}$ belongs to S, Fatou's Lemma then shows that

$$\|h\|_X \leq \liminf_{n\to\infty} \|h_{n(k)}\|_X \leq 1,$$

  therefore $h \in S$. Since $h \in L^1_N$ and $h \in S$, we have that $h \in U$. It follows therefore that U is a closed convex set of L^1_N. Hence, U is a convex closed set of L^1.

For every $\lambda > 1$, the function $g(x) = \lambda \frac{f_N(x)}{\|f_N\|_X}$ belongs to L^1_N but not to U. We remember that the dual space of L^1 is L^∞. Since U is closed and $\{g\}$ is compact, by Theorem 3.29, we obtain that there exists a nonzero $\phi \in L^\infty(\mathcal{R}, \mu)$, which may be chosen with support in R_N because g and all $h \in U$ have support in R_N, such that

$$\Re\Bigg(\int_{R_N} \phi(x)h(x)d\mu(x)\Bigg) < \gamma < \Re\Bigg(\int_{R_N} \phi(x)g(x)d\mu(x)\Bigg), \tag{4.3}$$

for some real number γ and all h in U.

Let us now see some observations to prove that

$$\|\phi\|_{X'} < \frac{\lambda}{\|f_N\|_X} \int_{R_N} |\phi(x)f_N(x)|d\mu.$$

- Writing $\phi(x) = |\phi|\psi(x)$ in polar form and observing that $\tilde{h} = \psi|h|$ belongs to U if only if h does (using Theorem 4.10 (i) and $|\psi| = 1$). We observe that

$$\int_{R_N} |\phi(x)h(x)|d\mu(x) = \int_{R_N} \phi(x)\overline{\psi(x)}|h(x)|d\mu(x) = \Re\Bigg(\int_{R_N} \phi(x)\tilde{h}(x)d\mu(x)\Bigg).$$

- Taking the supremum over $h \in U$ in (4.3) we obtain that

$$\sup_{h\in U} \int_{R_N} |\phi(x)h(x)|d\mu(x) = \sup_{\tilde{h}\in U} \Re\Bigg(\int_{R_N} \phi(x)\tilde{h}(x)d\mu(x)\Bigg) \leq \gamma$$

$$< \Re\Bigg(\int_{R_N} \phi(x)g(x)d\mu(x)\Bigg) \leq \int_{R_N} |\phi(x)g(x)|d\mu(x) \tag{4.4}$$

- Let h be a function in S, when restricted to R_N, is the pointwise limit of the increasing sequence of truncations

$$h_n(x) = \min(h(x), n)\chi_{R_N}(x),$$

that is, $h_n \uparrow h$. We have that $h_n \in X$ by the argument used when we define f_N. Let us see that each h_n is in L^1_N. It follows from

$$\int_{R_N} h_n(x) d\mu(x) \leq C_{R_N} \|h_n\|_X < \infty.$$

Since $\|h_n\|_X \leq \|h\|_X \leq 1$, we obtain that $h_n \in S$ and hence $h_n \in U$.

- Now, by the monotone convergence theorem we obtain that

$$\int_{R_N} |\phi(x)h(x)| d\mu(x) = \lim_{n\to\infty} \int_{R_N} |\phi(x)h_n(x)| d\mu(x)$$

$$\leq \sup_{h\in U} \int_{R_N} |\phi(x)f(x)| d\mu(x).$$

Therefore,

$$\|\phi\|_{X'} = \sup_{h\in S} \int_{R_N} |\phi(x)h(x)| d\mu(x) \leq \sup_{h\in U} \int_{R_N} |\phi(x)f(x)| d\mu(x).$$

Finally, by (4.4) and $g(x) = \lambda \frac{f_N(x)}{\|f_N\|_X}$ we obtain that

$$\|\phi\|_{X'} = \sup_{h\in S} \int_{R_N} |\phi(x)h(x)| d\mu(x) \leq \gamma < \frac{\lambda}{\|f_N\|_X} \int_{R_N} |\phi(x)f_N(x)| d\mu(x).$$

Equivalently,

$$\|f_N\|_X < \frac{\lambda}{\|\phi\|_{X'}} \int_{R_N} |\phi(x)f_N(x)| d\mu(x) \leq \frac{\lambda}{\|\phi\|_{X'}} \int_{\mathcal{R}} |\phi(x)f_N(x)| d\mu(x)$$

$$= \lambda \int_{\mathcal{R}} \left| \frac{\phi(x)}{\|\phi\|_{X'}} f_N(x) \right| d\mu(x).$$

By Theorem 4.14, we obtain that

$$\|f_N\|_X < \lambda \int_{\mathcal{R}} \left| \frac{\phi(x)}{\|\phi\|_{X'}} f_N(x) \right| d\mu(x) \leq \lambda \|f_N\|_{X''}.$$

Letting $\lambda \to 1$, we obtain that

$$\|f_N\|_X \leq \|f_N\|_{X''}.$$

$\square$

Chapter 5

Rearrangement-Invariant Banach Function Spaces

The objective of this chapter is to introduce all the necessary tools to define the Lorentz spaces.

5.1 Distribution Function

Definition 5.1. *The distribution function μ_f of a function f in $\mathcal{M}_0$ is given by*

$$\mu_f(\lambda) = \mu\{x \in \mathcal{R} : |f(x)| > \lambda\}, \quad (\lambda \geq 0).$$

Observe that μ_f depends only on the absolute value $|f|$ of the function f, and μ_f may assume the value $+\infty$.

Definition 5.2. *Let (S, ν) be a complete σ-finite measure space. Two functions $f \in \mathcal{M}_0(\mathcal{R}, \mu)$ and $g \in \mathcal{M}_0(S, \nu)$ are said to be equimeasurable if they have the same distribution function, that is, if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geq 0$.*

Proposition 5.3. *Suppose $f, g, f_n, (n = 1, 2, ...)$ belong to $\mathcal{M}_0(\mathcal{R}, \mu)$ and let a be any non-zero scalar. The distribution function μ_f is a non-negative, decreasing, and right-continuous function on $[0, \infty)$. Furthermore,*

$$|g| \leq |f| \; \mu\text{-a.e.} \implies \mu_g \leq \mu_f; \quad (1)$$

$$\mu_{af}(\lambda) = \mu_f(\lambda/|a|), \quad (\lambda \geq 0); \quad (2)$$

$$\mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2), \quad (\lambda_1, \lambda_2 \geq 0); \quad (3)$$

$$|f| \leq \liminf_{n\to\infty} |f_n| \; \mu\text{-a.e.} \implies \mu_f \leq \liminf_{n\to\infty} \mu_{f_n}; \quad (4.1)$$

in particular,

$$|f_n| \uparrow |f| \; \mu\text{-a.e.} \implies \mu_{f_n} \uparrow \mu_f. \quad (4.2)$$

Proof. First, let us prove that μ_f is a non-negative, decreasing, and right-continuous function on $[0, \infty)$.

- It is clear that μ_f is non-negative. Let us see that μ_f is decreasing, let

$$E_1 = \{x \in \mathcal{R} : |f(x)| > \lambda_1\}, \quad E_2 = \{x \in \mathcal{R} : |f(x)| > \lambda_2\},$$

  then if $\lambda_1 > \lambda_2$, $E_1 \subset E_2$. Thus $\mu_f(\lambda_1) \leq \mu_f(\lambda_2)$.

- Let us prove that μ_f is right-continuous, let

$$E(\lambda) = \{x \in \mathcal{R} : |f(x)| > \lambda\}, \quad (\lambda \geq 0),$$

  and fix $\lambda_0 \geq 0$. The sets $E(\lambda)$ increase as λ decreases, and we have that

$$E(\lambda_0) = \bigcup_{\lambda > \lambda_0} E(\lambda) = \bigcup_{n=1}^{\infty} E\left(\lambda_0 + \frac{1}{n}\right).$$

  Let us prove it:

  – $E(\lambda_0) = \bigcup_{\lambda>\lambda_0} E(\lambda)$. If $x \in E(\lambda_0)$, then $|f(x)| > \lambda_0$. Let $\lambda \geq 0$ such that $|f(x)| > \lambda > \lambda_0$. Then $x \in E(\lambda)$. Thus

$$E(\lambda_0) \subset \bigcup_{\lambda>\lambda_0} E(\lambda).$$

    Clearly, $\bigcup_{\lambda>\lambda_0} E(\lambda) \subset E(\lambda_0)$.

  – Now, we prove that $\bigcup_{\lambda>\lambda_0} E(\lambda) = \bigcup_{n=1}^{\infty} E\big(\lambda_0 + \frac{1}{n}\big)$. If $x \in \bigcup_{\lambda>\lambda_0} E(\lambda)$, then there exists a $\lambda > \lambda_0$ such that $|f(x)| > \lambda$. Since $\lambda > \lambda_0$ and $1/n \to 0$, then there exists $n \in \mathbb{N}$ such that $\lambda > \lambda_0 + \frac{1}{n}$. Hence,

$$\bigcup_{\lambda>\lambda_0} E(\lambda) \subset \bigcup_{n=1}^{\infty} E\left(\lambda_0 + \frac{1}{n}\right).$$

    Clearly, $\bigcup_{n=1}^{\infty} E\big(\lambda_0 + \frac{1}{n}\big) \subset \bigcup_{\lambda>\lambda_0} E(\lambda)$.

  Hence, since the sets $E(\lambda)$ increase as λ decreases, by monotone convergence theorem,

$$\mu_f\left(\lambda_0 + \frac{1}{n}\right) = \mu\left(E\left(\lambda_0 + \frac{1}{n}\right)\right) \uparrow \mu(E(\lambda_0)) = \mu_f(\lambda_0),$$

  and this establishes the right-continuity.

- Let us prove (1). For any $\lambda \geq 0$ such that $|g(x)| > \lambda$ we have that $|f(x)| \geq |g(x)| > \lambda$. Therefore $\mu_g \leq \mu_f$. The property (2) is clear.

- Let us prove (3). If $|f(x) + g(x)| > \lambda_1 + \lambda_2$, then either $|f(x)| > \lambda_1$ or $|g(x)| > \lambda_2$. Therefore

$$\{x \in \mathcal{R} : |f(x) + g(x)| > \lambda_1 + \lambda_2\}$$
$$\subset \{x \in \mathcal{R} : |f(x)| > \lambda_1\} \cup \{x \in \mathcal{R} : |g(x)| > \lambda_2\},$$

  and thus $\mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2)$.

- Let us see (4.1). Fix $\lambda \geq 0$ and let

  $$E = \{x \in \mathcal{R} : |f(x)| > \lambda\}, \quad E_n = \{x \in \mathcal{R} : |f_n(x)| > \lambda\}, \quad (n = 1, 2, ...).$$

  Since $|f| \leq \liminf_{n\to\infty} |f_n| = \sup_{n>0} \left(\inf_{k>n} |f_n|\right)$, we obtain that

  $$E \subset \liminf_{n\to\infty} E_n = \bigcup_{m=1}^{\infty} \bigcap_{n>m} E_n. \tag{5.1}$$

  Hence, for each $m = 1, 2, ...$:

  $$\mu\left(\bigcap_{n>m} E_n\right) \leq \inf_{n>m} \mu(E_n) \leq \sup_m \inf_{n>m} \mu(E_n) = \liminf_{n\to\infty} \mu(E_n) \tag{5.2}$$

  By (5.1), we obtain that

  $$\mu_f(\lambda) = \mu(E) \leq \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n>m} E_n\right).$$

  We observe that $\bigcap_{n>m} E_n$ increases with m. Let $m, m' \in \mathbb{N}$ such that $m > m'$. If $x \in \bigcap_{n>m'} E_n$ then $x \in E_n$ for all $n > m'$. Consequently, $x \in E_n$ for all $n > m$., therefore $x \in \bigcap_{n>m} E_n$. This observation shows that we can apply the monotone convergence theorem, so that

  $$\mu_f(\lambda) = \mu(E) \leq \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n>m} E_n\right) = \lim_{n\to\infty} \mu\left(\bigcap_{n>m} E_n\right),$$

  and by (5.2)

  $$\mu_f(\lambda) \leq \lim_{n\to\infty} \mu\left(\bigcap_{n>m} E_n\right) \leq \liminf_{n\to\infty} \mu(E_n) = \liminf_{n\to\infty} \mu_{f_n}(\lambda).$$

- Let us prove (4.2):

  - Since $|f_n| \uparrow |f|$, then $\lim_{n\to\infty} |f_n| = |f|$ μ-a.e, and

    $$|f| \leq \lim_{n\to\infty} |f_n| = \liminf_{n\to\infty} |f_n| \ \mu\text{-a.e.}$$

    By (4.1), we obtain that $\mu_f \leq \liminf_{n\to\infty} \mu_{f_n}$.

  - But, on the other hand, since $(f_n)_n$ is increasing, by Property (1),

    $$\lim_{n\to\infty} \mu_{f_n} \leq \mu_f.$$

    Finally, we obtain that

    $$\lim_{n\to\infty} \mu_{f_n} \leq \mu_f \leq \liminf_{n\to\infty} \mu_{f_n},$$

    so that $\mu_{f_n} \uparrow \mu_f$.

□

Remark 5.4. *It will be worthwhile to formally compute the distribution function μ_f of a nonnegative simple function f.*

Suppose

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x), \tag{5.3}$$

where the sets E_j are pairwise disjoint subsets of $\mathcal{R}$ with finite μ-measure and $a_1 > a_2 > ... > a_n > 0$.

If $\lambda \geq a_1$, then $|f(x)| \leq \lambda$, therefore $\mu_f(\lambda) = 0$. However, if $a_2 \leq \lambda < a_1$, then $f(x)$ exceeds λ precisely on the set E_1, and so $\mu_f(\lambda) = \mu(E_1)$. Similarly, if $a_3 \leq \lambda < a_2$, then $f(x)$ exceeds λ precisely on $E_1 \cup E_2$, an so $\mu_f(\lambda) = \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$. In general, we have

$$\mu_f(\lambda) = \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}(\lambda), \quad (\lambda \geq 0), \quad (a_{n+1} = 0). \tag{5.4}$$

where

$$m_j = \sum_{i=1}^{j} \mu(E_i), \quad (j = 1, 2, ..., n).$$

See Appendix Figure 6.1 and Figure 6.2.

5.2 Decreasing Rearrangement

Definition 5.5. *Suppose f belongs to $\mathcal{M}_0$. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by*

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad (t \geq 0).$$

Observe that f^* depends only on the absolute value $|f|$ of the function f. Let us observe five consequences of the definition of f^*.

Remark 5.6. *If f and g belongs to $\mathcal{M}_0$ and they are equimeasurable, then $f^* = g^*$.*

Remark 5.7. *$f^*(t) = \sup\{\lambda \geq 0 : \mu_f(\lambda) > t\} = m_{\mu_f}(t)$, where m is Lebesgue measure.*

Proof. Since μ_f is right-continuous and decreasing, then

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\} = \sup\{\lambda \geq 0 : \mu_f(\lambda) > t\}.$$

The set $\{\lambda \geq 0 : \mu_f(\lambda) > t\} = \big[0, \sup\{\lambda \geq 0 : \mu_f(\lambda) > t\}\big)$. Then

$$m_{\mu_f}(t) = m\{\lambda \geq 0 : \mu_f(\lambda) > t\} = \sup\{\lambda \geq 0 : \mu_f(\lambda) > t\}.$$

□

Remark 5.8. *We compute the decreasing rearrangement of the simple function f as (5.3).*

Referring to Definition 5.5 and Figure 1, we see that $f^*(t) = 0$ if $t \geq m_3$. Also, if $m_3 > t \geq m_2$, $f^*(t) = a_3$, and if $m_2 > t \geq m_1$, $f^*(t) = a_2$. If $0 \leq t < m_1$, then $f^*(t) = a_1$. Hence,

$$f^*(t) = \sum_{j=1}^{n} a_j \chi_{[m_{j-1}, m_j)}(t), \quad (t \geq 0),$$

where $m_0 = 0$.

See Appendix Figure 6.1 and Figure 6.3.

Since $m([m_{j-1}, m_j)) = \mu(E_j)$, f and f^* are equimeasurable, that is, $m_{f^*} = \mu_f$.

Remark 5.9. *If f as (5.3), then it may be represented also as follows:*

$$f(x) = \sum_{k=1}^{n} b_k \chi_{F_k}(x), \tag{5.5}$$

where the coefficients b_k are positive and for every k the sets F_k has finite measure. Moreover, $F_1 \subset F_2 \subset ... \subset F_n$.

This follows from

$$b_k = a_k - a_{k+1}, \quad F_k = \bigcup_{j=1}^{k} E_j, \quad (k = 1, 2.., n).$$

$$f(x) = \sum_{k=1}^{n} b_k \chi_{F_k}(x) = \sum_{k=1}^{n} (a_k - a_{k+1}) \chi_{\left[\bigcup_{j=1}^{k} E_j\right]}(x) = \sum_{k=1}^{n} (a_k - a_{k+1}) \big(\sum_{j=1}^{k} \chi_{E_j}(x) \big)$$

$$= \chi_{E_1}(x) \sum_{k=1}^{n} (a_k - a_{k+1}) + ... + \chi_{E_k}(x) \sum_{k=1}^{n} (a_k - a_{k+1})$$

$$= \chi_{E_1}(x) a_1 + ... + \chi_{E_n}(x) a_n = \sum_{j=1}^{n} a_j \chi_{E_j}(x).$$

Thus,

$$f^*(t) = \sum_{k=1}^{n} b_k \chi_{[0, \mu(F_k))}(t).$$

See Appendix Figure 6.1 and Figure 6.4.

Remark 5.10. *If f is right-continuous and decreasing, then $f = f^*$.*

Proposition 5.11. *Suppose $f, g, f_n, (n = 1, 2, ...)$ belong to $\mathcal{M}_0(\mathcal{R}, \mu)$ and let a be any non-zero scalar. The decreasing rearrangement f^* is a non-negative, decreasing, and right-continuous function on $[0, \infty)$. Furthermore,*

$$|g| \leq |f| \; \mu\text{-}a.e. \implies g^* \leq f^*; \quad (1)$$

$$(af)^* = |a|f^* \quad (2)$$
$$(f+g)^*(t_1+t_2) \le f^*(t_1) + g^*(t_2), \quad (t_1, t_2 \ge 0); \quad (3)$$
$$|f| \le \liminf_{n\to\infty} |f_n| \; \mu\text{-}a.e. \implies f^* \le \liminf_{n\to\infty} f_n^*; \quad (4)$$

in particular,

$$|f_n| \uparrow |f| \; \mu\text{-}a.e. \implies f_n^* \uparrow f^*.$$
$$f^*(\mu_f(\lambda)) \le \lambda, \quad (\mu_f(\lambda) < \infty) \quad (5.1); \quad \mu_f(f^*(t)) \le t, \quad (f^*(t) < \infty) \quad (5.2);$$
$$f \quad \textit{and} \quad f^* \textit{ are equimeasurable} \quad (6);$$
$$(|f|^p)^* = (f^*)^p, \quad (0 < p < \infty), \quad (7).$$

Proof. Since f^* is a distribution function, by Remark 5.7, then f^* is a non-negative, decreasing, right-continuous function on $[0, \infty)$, and (1), and (4) hold.

- Let us establish (2). We observe that, by Proposition 5.3 (2),
$$(af)^*(t) = \inf\{\lambda \ge 0 : \mu_{af}(\lambda) \le t\} = \inf\left\{\lambda \ge 0 : \mu_f\left(\frac{\lambda}{|a|}\right) \le t\right\}$$
$$= |a| \inf\{z \ge 0 : \mu_f(z) \le t\} = |a| f^*(t),$$

- Let us establish (5.1). Fix $\lambda \ge 0$ and suppose $t = \mu_f(\lambda)$ is finite. By Definition 5.5, we obtain that
$$f^*(\mu_f(\lambda)) = f^*(t) = \inf\{\lambda' \ge 0 : \mu_f(\lambda') \le t = \mu_f(\lambda)\} \le \lambda.$$

- Let us establish (5.2), fix $t \ge 0$ and suppose $\lambda = f^*(t)$ is finite. By Definition 5.5 and μ_f is decreasing, there is a sequence $\lambda_n \downarrow \lambda$ such that $\mu_f(\lambda_n) \le \mu_f(\lambda) \le t$, so the right continuity of μ_f gives
$$\mu_f(f^*(t)) = \mu_f(\lambda) = \lim_{n\to\infty} \mu_f(\lambda_n) \le t.$$

- Let us prove (3). We may assume that $\lambda = f^*(t_1) + g^*(t_2)$ is finite since otherwise there is nothing to prove. Let $t = \mu_{f+g}(\lambda)$. Since $|f(x) + g(x)| > f^*(t_1) + g^*(t_2)$, then $|f(x)| > f^*(t_1)$ or $|g(x)| > g^*(t_2)$. Therefore,
$$t = \mu\{x \in \mathcal{R} : |f(x) + g(x)| > f^*(t_1) + g^*(t_2)\}$$
$$\le \mu\{x \in \mathcal{R} : |f(x)| > f^*(t_1)\} + \mu\{x \in \mathcal{R} : |g(x)| > g^*(t_2)\},$$
then by (5.2),
$$t \le \mu_f(f^*(t_1)) + \mu_g(g^*(t_2)) \le t_1 + t_2,$$
This shows in particular that t is finite. Hence, using that $(f+g)^*$ is decreasing, we obtain that
$$(f+g)^*(t_1+t_2) \le (f+g)^*(t) = (f+g)^*(\mu_{f+g}(\lambda)),$$
by (5.1), we have that
$$(f+g)^*(t_1+t_2) \le (f+g)^*(\mu_{f+g}(\lambda)) \le \lambda = f^*(t_1) + g^*(t_2).$$

- Let us establish (6). For an arbitrary function f in $\mathcal{M}_0$ we can find a sequence of non-negative simple functions $(f_n)_n$, such that $f_n \uparrow |f|$. By Remark 5.8, for each n the functions f_n and f_n^* are equimeasurable, that is,

$$\mu_{f_n}(\lambda) = m_{f_n^*}(\lambda), \quad (\lambda \geq 0).$$

Since $f_n \uparrow |f|$, $f_n^* \uparrow |f^*|$ by (4). By Proposition 5.3 (4), we have that

$$\mu_f(\lambda) = m_{f^*}(\lambda), \quad (\lambda \geq 0).$$

- Let us see (7). We have that

$$\mu_{|f|^p}(\lambda) = \mu_f(\lambda^{\frac{1}{p}}) = m_{f^*}(\lambda^{\frac{1}{p}}) = m_{(f^*)^p}(\lambda), \quad (\lambda \geq 0).$$

By Definition 5.8, we obtain that

$$(|f|^p)^* = ((f^*)^p)^*.$$

Since $(f^*)^p$ is decreasing and right-continuous, by Remark 5.10, we have that

$$(|f|^p)^* = (f^*)^p.$$

$\square$

The next result gives alternative descriptions of the L^p-norm in terms of the distribution function and the decreasing rearrangement.

Proposition 5.12. *Let $f \in \mathcal{M}_0$. If $0 < p < \infty$, then*

$$\int_{\mathcal{R}} |f(x)|^p d\mu(x) = p \int_0^\infty \lambda^{p-1} \mu_f(\lambda) d\lambda = \int_0^\infty f^*(t)^p dt.$$

Furthermore, in the case $p = \infty$,

$$ess \sup_{x \in \mathcal{R}} |f(x)| = \inf\{\lambda \geq 0 : \mu_f(\lambda) = 0\} = f^*(0).$$

Proof. First, we prove it to arbitrary non-negative simple function f. Let f as (5.3). Since $\mu(E_j) = m([m_{j-1}, m_j))$, we obtain that

$$\int_{\mathcal{R}} |f(x)|^p d\mu(x) = \sum_{j=1}^n a_j^p \mu(E_j) = \sum_{j=1}^n a_j^p m([m_{j-1}, m_j)) = \int_0^\infty f^*(t)^p dt.$$

By (5.4), we have that

$$p \int_0^\infty \lambda^{p-1} \mu_f(\lambda) d\lambda = p \sum_{j=1}^n m_j \int_{a_{j+1}}^{a_j} \lambda^{p-1} d\lambda$$

$$= \sum_{j=1}^n (a_j^p - a_{j+1}^p) m_j = \sum_{j=1}^n a_j^p \mu(E_j) = \int_{\mathcal{R}} |f(x)|^p d\mu(x).$$

The third equality follows from:

$$\sum_{j=1}^{n}(a_j^p - a_{j+1}^p)m_j = \sum_{j=1}^{n}(aj^p - a_{j+1}^p)\Big(\sum_{i=1}^{n}\mu(E_j)\Big)$$

$$= \mu(E_1)\sum_{j=1}^{n}(a_j^p - a_{j+1}^p) + ... + \mu(E_n)\sum_{j=1}^{n}(a_j^p - a_{j+1}^p)$$

$$= \mu(E_1)a_1^p + ... + \mu(E_n)a_n^p = \sum_{j=1}^{n} a_j^p \mu(E_j).$$

If $f \in \mathcal{M}_0$, we can find a sequence of non-negative simple function $(f_n)_n$ such that $f_n \uparrow |f|$. Then, since f_n and f_n^* are equimeasurable we obtain that $f_n^* \uparrow f^*$. By the monotone convergence theorem,

$$\int_{\mathcal{R}} |f(x)|^p d\mu(x) = \lim_{n\to\infty}\int_{\mathcal{R}} f_n^p(x)d\mu(x) = \lim_{n\to\infty}\lim_{n\to\infty}\int_0^\infty (f_n^*)^p(t)dt = \int_0^\infty (f^*)^p(t)d\mu.$$

If $f \in \mathcal{M}_0$,

$$\int_{\mathcal{R}} |f(x)|^p d\mu(x) = p\int_0^\infty \lambda^{p-1}\mu_f(\lambda)d\lambda,$$

follows from Proposition 5.3 (4) and the monotone convergence theorem.

Now, we prove the case $p = \infty$. Remember that

$$\operatorname*{ess\,sup}_{x\in\mathcal{R}} |f(x)| = \inf\{M \geq 0 : |f(x)| \leq M\}.$$

Since μ_f in non-negative, then

$$\inf\{\lambda \geq 0 : \mu_f(\lambda) = 0\} = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq 0\} = f^*(0).$$

Let $\lambda \geq 0$ such that $\mu_f(\lambda) = 0$. Then, the set $\{x \in \mathcal{R} : |f(x)| > \lambda\}$ has measure equal 0, so that $|f(x)| \leq \lambda$ μ-a.e. Therefore,

$$\inf\{\lambda \geq 0 : \mu_f(\lambda) = 0\} = \inf\{\lambda \geq 0 : |f(x)| \leq \lambda\} = \operatorname*{ess\,sup}_{x\in\mathcal{R}} |f(x)|.$$

□

5.3 Hardy-Littlewood Inequality

Lemma 5.13. *Let g be a nonnegative simple function on $(\mathcal{R}, \mu)$ and let E be an arbitrary measurable subset of $\mathcal{R}$. Then*

$$\int_E g(x)d\mu(x) \leq \int_0^{\mu(E)} g^*(s)ds.$$

Proof. Let g be a simple function as (5.5), therefore $g^*(s) = \sum_{k=1}^n b_k \chi_{[0,\mu(F_k))}(s)$, by Remark 5.9. Consequently,

$$\int_E g(x)d\mu(x) = \sum_{j=1}^n b_j \mu(E \cap F_j) \leq \sum_{j=1}^n b_j \min(\mu(E), \mu(F_j))$$

$$= \sum_{j=1}^n b_j \int_0^{\mu(E)} \chi_{(0,\mu(F_j))}(s)ds = \int_0^{\mu(E)} g^*(s)ds.$$

Observe that if $\mu(E) \leq \mu(F_j)$ then

$$\int_0^{\mu(E)} \chi_{(0,\mu(F_j))}(s)ds = \mu(E) = \min(\mu(E), \mu(F_j)).$$

The case $\mu(E) \geq \mu(F_j)$ is analogue. □

Theorem 5.14. *[Hardy-Littlewood Inequality] If f and g belong to $\mathcal{M}_0 = \mathcal{M}_0(\mathcal{R}, \mu)$, then*

$$\int_{\mathcal{R}} |f(x)g(x)|d\mu(x) \leq \int_0^\infty f^*(s)g^*(s)ds.$$

Proof. Since f^* and g^* depend only on the absolute values of f and g, it is enough to establish the inequality for non-negative functions f and g. Therefore, we can prove the theorem for non-negative simple functions f and g (view the proof of Proposition 5.12).

Let f a simple function as (5.3). By Remark 5.9,

$$f^*(s) = \sum_{j=1}^n a_j \chi_{[m_{j-1},m_j)}(s).$$

Hence, by Lemma 5.13,

$$\int_{\mathcal{R}} |f(x)g(x)|d\mu(x) = \sum_{j=1}^m a_j \int_{E_j} g(x)d\mu(x) \leq \sum_{j=1}^m a_j \int_0^{\mu(E_j)} g^*(s)ds$$

$$= \int_0^\infty \sum_{j=1}^n a_j \chi_{[0,\mu(E_j))}(s)g^*(s)ds = \int_0^\infty f^*(s)g^*(s)ds.$$

□

Definition 5.15. *A totally σ-finite measure space $(\mathcal{R}, \mu)$, is said to be resonant, if, for each f and g in $\mathcal{M}_0$, the identity*

$$\int_0^\infty f^*(t)g^*(t)dt = \sup \int_{\mathcal{R}} |f(x)\overline{g}(x)|d\mu(x),$$

holds, where the supremum is taken over all functions $\overline{g}$ on $\mathcal{R}$ equimeasurable with g.

5.4 Maximal Function

Definition 5.16. *Let f belong to $\mathcal{M}_0(\mathcal{R},\mu)$. Then f^{**} will denote the maximal function of f^* defined by*

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \quad (t > 0).$$

Proposition 5.17. *Suppose $f, g, f_n, (n = 1, 2, ...)$ belong to $\mathcal{M}_0(\mathcal{R},\mu)$ and let a be any non-zero scalar. Then f^{**} is a non-negative, decreasing, and continuous function on $(0,\infty)$. Furthermore,*

$$f^{**} \equiv 0 \Leftrightarrow f = 0\, \mu\text{-}a.e.; \quad (1)$$

$$f^* \leq f^{**}; \quad (2)$$

$$|g| \leq |f|\, \mu\text{-}a.e. \implies g^{**} \leq f^{**}; \quad (3)$$

$$(af)^{**} = |a| f^{**}; \quad (4)$$

$$|f_n| \uparrow |f|\, \mu\text{-}a.e. \implies f_n^{**} \uparrow f^{**}; \quad (5).$$

Proof. First, we observe that f^{**} is finite for any one value of t if and only if it is finite for every $t > 0$. In other words, the function f^{**} is either everywhere finite or everywhere infinite. Since $\chi_{(0,t)}$ is continuous, f^{**} is continuous. It is obvious that f^{**} is non-negative.

- Let us see (1). If $f^{**} \equiv 0$, then $f^* = 0$ μ-a.e. Note that, if $f^*(t) = 0$ then $\mu_f(0) = \mu\{x \in \mathcal{R} : |f(x)| > 0\} \leq t$. Since $f^* = 0$ μ-a.e.,

  $$\mu_f(0) = \mu\{x \in \mathcal{R} : |f(x)| > 0\} \leq 0,$$

  then $f = 0$ μ-a.e.

- Let us see (2) follows from the fact f^* is decreasing,

  $$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds \geq f^*(t)\frac{1}{t}\int_0^t ds = f^*(t).$$

  Consequently, f^* is decreasing so $f^*(v) \leq f^*(tv/s)$ if $0 < t \leq s$. Hence,

  $$f^{**}(s) = \frac{1}{s}\int_0^s f^*(v)dv \leq \frac{1}{s}\int_0^s f^*\left(\frac{tv}{s}\right)dv.$$

  Doing a change of variables, $u = \frac{tv}{s}$, we obtain that

  $$f^{**}(s) = \frac{1}{s}\int_0^s f^*(v)dv \leq \int_0^s f^*\left(\frac{tv}{s}\right)dv = \frac{1}{t}\int_0^t f^*(u)du = f^{**}(t).$$

  Therefore, f^{**} is decreasing.

- Let us establish (3). If $|g| \leq |f|$ μ-a.e., then $g^* \leq f^*$, by Proposition 5.11 (1). Hence, $g^{**} \leq f^{**}$.

- The property (4) follows from by Proposition 5.11 (2).
- Let us see (5). If $|f_n| \uparrow |f|$ μ-a.e., then $f_n^* \uparrow f^*$, by Proposition 5.11 (4). By monotone convergence theorem, $f_n^{**} \uparrow f^{**}$.

□

Proposition 5.18. *Let $(\mathcal{R}, \mu)$ be a resonant space. If $f \in \mathcal{M}_0(\mathcal{R}, \mu)$ and t is in the range of μ, then*

$$f^{**}(t) = \frac{1}{t} \sup \left\{ \int_E |f(x)| d\mu(x) : \mu(E) = t \right\}.$$

Proof. Since t is in the range of μ, there exist a measurable set F such that $\mu(F) = t$. Let $g(x) = X_F(x)$, so $g^*(s) = \chi_{[0,t)}(s)$, by Remark 5.8.

We must observe that $\tilde{g}$ is equimeasurable with g if only if $|\tilde{g}|$ is μ-a.e. equal to the characteristic function of some set E with measure $\mu(E) = \mu(F) = t$.

- If $\tilde{g}$ is equimeasurable with $g = X_F$, then $\mu_g = \mu_{\tilde{g}}$. Then

$$\mu_{\tilde{g}}(\lambda) = \begin{cases} 0, & \text{if } \lambda \geq 1, \\ t, & \text{if } 0 \leq \lambda < 1. \end{cases}$$

  If $\lambda \geq 1$, then $\mu_{\tilde{g}}(\lambda) = \mu\{x \in \mathcal{R} : |\tilde{g}(x)| > \lambda\} = 0$, therefore, $|\tilde{g}(x)| \leq 1$ μ-a.e.

  If $0 \leq \lambda < 1$, then $\mu_{\tilde{g}}(\lambda) = \mu\{x \in \mathcal{R} : |\tilde{g}(x)| > \lambda\} = t$. Therefore, there exists a set E such that $\mu(E) = t$ and $|\tilde{g}(x)| > \lambda$.

  Therefore on E,

$$\lambda < |\tilde{g}| \leq 1$$

  for all $0 \leq \lambda < 1$. Thus $|\tilde{g}| = 1$ on E and 0 off.

- Suppose that $|\tilde{g}| = \chi_E$ μ-a.e. , where $\mu(E) = t$. Since a distribution function depends only on the absolute value of the function,

$$\mu_g(\lambda) = \mu_{|\tilde{g}|}(\lambda) = \mu_{\tilde{g}}(\lambda) = \begin{cases} 0, & \text{if } \lambda \geq 1, \\ t, & \text{if } 0 \leq \lambda < 1. \end{cases}$$

  Thus g and $\tilde{g}$ are equimeasurable.

Since $(\mathcal{R}, \mu)$ is resonant,

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds = \frac{1}{t} \int_0^\infty f^*(s) \chi_{[0,t)}(s) ds = \frac{1}{t} \int_0^\infty f^*(s) g^*(s) ds.$$

By previous observation, the supremum is taken over all functions $\tilde{g}$ on $\mathcal{R}$ equimeasurable with g.

$$f^{**}(t) = \frac{1}{t} \sup \left\{ \int_{\mathcal{R}} |f(x)\tilde{g}(x)| d\mu(x) \right\} = \frac{1}{t} \sup \left\{ \int_E |f(x)| d\mu(x) : \mu(E) = t \right\}.$$

□

Remark 5.19. *If $(\mathcal{R}, \mu)$ is resonant, then*

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t). \tag{5.6}$$

In fact, it can be proved that (5.6) holds without the resonant condition on $(\mathcal{R}, \mu)$.

Proof. We have that

$$\frac{1}{t}\int_E |f(x)+g(x)|d\mu(x) \leq \frac{1}{t}\int_E |f(x)|d\mu(x) + \frac{1}{t}\int_E |g(x)|d\mu(x).$$

Since $(\mathcal{R}, \mu)$ is resonant, taking the supremum over all measurable sets E such that $\mu(E) = t$, then the result follows. □

Consequently, we have the next result.

Remark 5.20. *Let $(\mathcal{R}, \mu)$ be a complete σ-finite measure space. Then for all $t > 0$,*

$$\int_0^t (f+g)^*(s)ds \leq \int_0^t f^*(s)ds + \int_0^t g^*(s)ds.$$

5.5 Rearrangement-Invariant Spaces

Definition 5.21. *Let ρ be a function norm over a totally σ-finite measure space $(\mathcal{R}, \mu)$. Then ρ is said to be rearrangement-invariant (r.i.) if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions f and g in $\mathcal{M}_0^+(\mathcal{R}, \mu)$. In that case, the Banach function space $X = X(\rho)$ generated by ρ is said to be a rearrangement-invariant space.*

Remark 5.22. *The Lebesgue space $L^p(\mathcal{R}, \mu)$ are rearrangement-invariant.*

Proof. Let $f, g \in \mathcal{M}_0^+(\mathcal{R}, \mu)$ which are equimeasurable. Thus $f^* = g^*$, by Remark 5.6. By Proposition 5.12,

$$\int_{\mathcal{R}} |f(x)|^p d\mu(x) = \int_0^\infty f^*(t)^p dt = \int_0^\infty g^*(t)^p dt = \int_{\mathcal{R}} |g(x)|^p d\mu(x).$$

Hence, $\|f\|_{L^p} = \|g\|_{L^p}$. □

Proposition 5.23. *Let ρ be a rearrangement-invariant function norm over a resonant measure space $(\mathcal{R}, \mu)$. Then the associate norm ρ' is also r.i. Furthemore, the norms ρ and ρ' are given by*

$$\rho'(g) = \sup\left\{\int_0^\infty f^*(s)g^*(s)ds : f \in \mathcal{M}_0^+, \rho(f) \leq 1\right\}, \quad (g \in \mathcal{M}_0^+),$$

and

$$\rho(f) = \sup\left\{\int_0^\infty f^*(s)g^*(s)ds : g \in \mathcal{M}_0^+, \rho'(g) \leq 1\right\}, \quad (f \in \mathcal{M}_0^+).$$

Proof. Let $g \in \mathcal{M}_0^+$. We have that if $\rho(f) \leq 1$, then f is finite and therefore

$$\rho'(g) = \sup\left\{ \int_{\mathcal{R}} f(x)g(x)d\mu(x) : f \in \mathcal{M}^+, \rho(f) \leq 1 \right\}$$
$$= \sup\left\{ \int_{\mathcal{R}} f(x)g(x)d\mu(x) : f \in \mathcal{M}_0^+, \rho(f) \leq 1 \right\}.$$

Since $(\mathcal{R}, \mu)$ is resonant, we have that by Definition 5.15,

$$\rho'(g) = \sup\left\{ \int_0^\infty f^*(s)g^*(s)ds : f \in \mathcal{M}_0^+, \rho(f) \leq 1 \right\}.$$

Since any two equimeasurable g and $\tilde{g}$ function have the same decreasing rearrangement, then

$$\rho'(g) = \sup\left\{ \int_0^\infty f^*(s)g^*(s)ds : f \in \mathcal{M}_0^+, \rho(f) \leq 1 \right\} = \rho'(\tilde{g}).$$

Thus ρ' is rearrangement-invariant. Similarly,

$$\rho''(f) = \sup\left\{ \int_0^\infty f^*(s)g^*(s)ds : g \in \mathcal{M}_0^+, \rho'(g) \leq 1 \right\}.$$

By Theorem 4.18, we have that $\rho'' = \rho$. Thus

$$\rho(f) = \sup\left\{ \int_0^\infty f^*(s)g^*(s)ds : g \in \mathcal{M}_0^+, \rho'(g) \leq 1 \right\}.$$

□

Corollary 5.24. *Let X be a Banach function space over a resonant measure space. Then X is rearrangement-invariant if and only if the associate space X' is also r.i., and in that case the norms are given by*

$$\|g\|_{X'} = \sup\left\{ \int_0^\infty f^*(s)g^*(s)ds : \|f\|_X \leq 1 \right\}, \quad (f \in X),$$

and

$$\|f\|_X = \sup\left\{ \int_0^\infty f^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\}, \quad (g \in X').$$

Proof. If X is rearrangement-invariant, then ρ is a r.i. Hence ρ' is also r.i., by Proposition 5.23. Therefore, X' is r.i. The other direction is analogous.

We observe that since μ_f only depends $|f|$, then $\mu_f = \mu_{|f|}$. Therefore $f^* = |f|^*$. Since $\|f\|_X = \rho(|f|)$ and $f^* = |f|^*$, by the Proposition 5.23, we obtain that

$$\|f\|_X = \rho(|f|) = \sup\left\{ \int_0^\infty f^*(s)g^*(s)ds : \rho'(g) \leq 1 \right\}.$$

Since $g \in \mathcal{M}_0^+$, then $\|g\|_{X'} = \rho'(|g|) = \rho'(g)$ and the result follows.

The other result is analogous. □

Chapter 6

The Lorentz Spaces

6.1 Hardy's Lemmas

Lemma 6.1. *Let ξ_1 and ξ_2 be non-negative measurable functions on $(0, \infty)$ and suppose*

$$\int_0^t \xi_1(s)ds \leq \int_0^t \xi_2(s)ds, \tag{6.1}$$

for all t. Let η be any non-negative decreasing function on $(0, \infty)$. Then

$$\int_0^\infty \xi_1(s)\eta(s)ds \leq \int_0^\infty \xi_2(s)\eta(s)ds.$$

Proof. First, we takeη a non-negative decreasing simple function. In that case, η may be expressed in the form

$$\eta(s) = \sum_{j=1}^n a_j \chi_{(0,t_j)}(s), \tag{6.2}$$

where the coefficients a_j are positive and $0 < t_1 < ... < t_n$. Clearly, η is non-negative and decreasing. Using (6.1), we obtain that

$$\int_0^\infty \xi_1(s)\eta(s)ds = \sum_{j=1}^n a_j \int_0^{t_j} \xi_1(s)ds \leq \sum_{j=1}^n a_j \int_0^{t_j} \xi_2(s)ds = \int_0^\infty \xi_2(s)\eta(s)ds.$$

Let $(\eta_n)_n$ be a sequence of non-negative decreasing simple functions as (6.2) such that $\eta_n \uparrow \eta$. Using the Monotone convergence theorem as in the proof of Proposition 5.12, the result follows. $\square$

Lemma 6.2. *Let ψ be a non-negative measurable function on $(0, \infty)$ and suppose $-\infty < \lambda < 1$ and $1 \leq q < \infty$. Then*

$$\left\{ \int_0^\infty \left(t^\lambda \frac{1}{t} \int_0^t \psi(s)ds \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \leq \frac{1}{1-\lambda} \left\{ \int_0^\infty (t^\lambda \psi(t))^q \frac{dt}{t} \right\}^{\frac{1}{q}}.$$

Proof. Writing $\psi(s) = s^{\frac{-\lambda}{q'}} s^{\frac{\lambda}{q'}} \psi(s)$ and applying Hölder's inequality, we obtain that

$$\frac{1}{t}\int_0^t \psi(s)ds \leq \left(\frac{1}{t}\int_0^t s^{-\lambda}ds\right)^{\frac{1}{q'}} \left(\frac{1}{t}\int_0^t s^{\frac{\lambda q}{q'}}\psi(s)^q ds\right)^{\frac{1}{q}},$$

Since $q' = \frac{q}{q-1}$,

$$\frac{1}{t}\int_0^t \psi(s)ds \leq (1-\lambda)^{1-q} t^{\frac{-\lambda}{q'}-\frac{1}{q}} \left(\int_0^t s^{\lambda(q-1)}\psi(s)^q ds\right)^{\frac{1}{q}}.$$

Hence, by an interchange in the order of integration,

$$\int_0^\infty \left(t^\lambda \frac{1}{t}\int_0^t \psi(s)ds\right)^q \frac{dt}{t} \leq (1-\lambda)^{1-q}\int_0^\infty t^{\lambda-2}\int_0^t s^{\lambda(q-1)}\psi(s)^q ds\,dt$$

$$= (1-\lambda)^{1-q}\int_0^\infty s^{\lambda(q-1)}\psi(s)^q \int_s^\infty t^{\lambda-2}dt\,ds = (1-\lambda)^{-q}\int_0^\infty s^{\lambda q}\psi(s)^q \frac{ds}{s}.$$

The result follows. □

6.2 Definition of Lorentz Spaces

Definition 6.3. *Let $(\mathcal{R}, \mu)$ be a complete σ-finite measure space, and suppose $p \geq 1$ and $1 \leq q \leq \infty$. The Lorentz space $L^{p,q}(\mathcal{R}, \mu)$ consist of all f in $\mathcal{M}_0(\mathcal{R}, \mu)$ for the which quantity*

$$\|f\|_{L^{p,q}} = \begin{cases} \left\{ \int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \right\}^{\frac{1}{q}}, & (0 < q < \infty), \\ \sup_{0<t<\infty}\{t^{\frac{1}{p}} f^*(t)\}, & (q = \infty), \end{cases}$$

is finite .

Remark 6.4. *If $p \geq 1$ then $L^{p,p} = L^p$, that is,*

$$\|f\|_{L^{p,p}} = \|f\|_{L^p}.$$

Proof. Let $1 \leq p < \infty$. By Proposition 5.12,

$$\|f\|_{L^{p,p}} = \left\{\int_0^\infty (f^*(t))^p dt\right\}^{\frac{1}{p}} = \left\{\int_{\mathcal{R}} |f(x)|^p d\mu(x)\right\}^{\frac{1}{p}} = \|f\|_{L^p}.$$

Let $p = \infty$. By Proposition 5.12 and f^* decreasing,

$$\|f\|_{L^{\infty,\infty}} = \sup_{0<t<\infty} f^*(t) = f^*(0) = \operatorname{ess\,sup}_{x\in\mathcal{R}} |f(x)| = \|f\|_{L^\infty}.$$

□

Remark 6.5. *The Lorentz space $L^{\infty,q}$, for finite q, contains only the zero-function.*

Proof. Let $f \in \mathcal{M}_0(\mathcal{R}, \mu)$ different of 0. Let $s > 0$. Since f^* is decreasing,

$$\int_0^\infty [f^*(t)]^q \frac{dt}{t} \geq \int_0^s [f^*(t)]^q \frac{dt}{t} \geq [f^*(s)]^q \int_0^s \frac{dt}{t} = \infty.$$

Therefore $\|f\|_{L^{\infty,q}} = \infty$. If $f = 0$, therefore $f^* = 0$. So that, $\|f\|_{L^{\infty,q}} = 0$ □

The next result shows that, for any fixed p, the Lorentz spaces $L^{p,q}$ increase as the second exponent q increases.

Proposition 6.6. *Suppose* $1 \leq p \leq \infty$ *and* $1 \leq q \leq r \leq \infty$. *Then*

$$\|f\|_{L^{p,r}} \leq c\|f\|_{L^{p,q}},$$

for all f *in* $\mathcal{M}_0(\mathcal{R}, \mu)$, *where* c *is a constant depending only* p, q *and* r.

Proof. We divide the proof in four cases:

- The case $p = \infty$, there is nothing to prove, by Remark 6.5.
- The case $q = r$, it is obvious.

We may assume $p < \infty$ and $q < r$.

- The case $r = \infty$. Using the fact that f^* is decreasing, we have that

$$t^{\frac{1}{p}} f^*(t) = \left\{ \frac{p}{q} \int_0^t [s^{\frac{1}{p}} f^*(t)]^q \frac{ds}{s} \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \frac{p}{q} \int_0^t [s^{\frac{1}{p}} f^*(s)]^q \frac{ds}{s} \right\}^{\frac{1}{q}} \leq \left(\frac{p}{q} \right)^{\frac{1}{q}} \|f\|_{L^{p,q}}.$$

  Hence, taking the supremum over all $t > 0$, we obtain that

$$\|f\|_{L^{p,\infty}} \leq \left(\frac{p}{q} \right)^{\frac{1}{q}} \|f\|_{L^{p,q}}. \tag{6.3}$$

- The case $r < \infty$. We have that

$$\|f\|_{L^{p,r}} = \left\{ \int_0^\infty [t^{\frac{1}{p}} f^*(t)]^{r-q+q} \frac{dt}{t} \right\}^{\frac{1}{r}} \leq \left\{ \int_0^\infty \|f\|_{L^{p,\infty}}^{r-q} [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \right\}^{\frac{1}{r}}$$

$$= \|f\|_{L^{p,\infty}}^{1-\frac{q}{r}} \left\{ \int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \right\}^{\frac{1}{q} \cdot \frac{q}{r}} = \|f\|_{L^{p,\infty}}^{1-\frac{q}{r}} \|f\|_{L^{p,q}}^{\frac{q}{r}}.$$

  By (6.3),

$$\|f\|_{L^{p,r}} \leq \left(\frac{p}{q} \right)^{\frac{r-q}{rq}} \|f\|_{L^{p,q}}.$$

□

6.3 Normability of Lorentz Spaces

Theorem 6.7. *Suppose* $1 \leq q \leq p < \infty$ *or* $p = q = \infty$. *Then* $(L^{p,q}, \|\cdot\|_{L^{p,q}})$ *is a rearrangement-invariant Banach function space.*

Proof. The result is clear when $p = q = 1$ or $p = q = \infty$ because by Remark 6.4, $L^{p,q}$ reduces to the Lebesgue spaces L^1 and L^∞, respectively. By Remark 5.22, $(L^{1,1}, \|\cdot\|_{L^{1,1}})$ and $(L^{\infty,\infty}, \|\cdot\|_{L^{\infty,\infty}})$ are rearrangement-invariant Banach function space. Hence, we may assume that $1 < p < \infty$ and $1 \leq q \leq p$.

Let us now prove the triangle inequality. In that case, with $q' = \frac{q}{q-1}$. We have from Definition 6.3

$$\|f+g\|_{L^{p,q}} = \left\{ \int_0^\infty [t^{\frac{1}{p}-\frac{1}{q}}(f+g)^*(t)]^q dt \right\}^{\frac{1}{q}} = \|t^{\frac{1}{p}-\frac{1}{q}}(f+g)^*\|_{L^q}.$$

Since $t^{1/p-1/q}(f+g)^*$ is a decreasing function and L^q is a rearrangement-invariant Banach function space, we obtain that

$$\|f+g\|_{L^{p,q}} = \|t^{\frac{1}{p}-\frac{1}{q}}(f+g)^*\|_{L^q}$$

$$= \sup\left\{ \int_0^\infty t^{\frac{1}{p}-\frac{1}{q}}(f+g)^*(t)h^*(t)dt : \|h\|_{L^{q'}(0,\infty)} \leq 1 \right\}. \tag{6.4}$$

By Remark 5.20,

$$\int_0^\infty t^{\frac{1}{p}-\frac{1}{q}}(f+g)^*(t)dt \leq \int_0^\infty t^{\frac{1}{p}-\frac{1}{q}}f^*(t)dt + \int_0^\infty t^{\frac{1}{p}-\frac{1}{q}}g^*(t)dt.$$

But applying Lemma 6.1 and since h^* is decreasing , we obtain that

$$\int_0^\infty t^{\frac{1}{p}-\frac{1}{q}}(f+g)^*(t)h^*(t)dt \leq \int_0^\infty t^{\frac{1}{p}-\frac{1}{q}}f^*(t)h^*(t)dt + \int_0^\infty t^{\frac{1}{p}-\frac{1}{q}}g^*(t)h^*(t)dt.$$

By Hölder's inequality,

$$\int_0^\infty t^{\frac{1}{p}-\frac{1}{q}}(f+g)^*(t)h^*(t)dt \leq \left\{ \int_0^\infty [t^{\frac{1}{p}-\frac{1}{q}}(f)^*(t)]^q dt \right\}^{\frac{1}{q}} \|h\|_{L^{q'}}$$

$$+\left\{ \int_0^\infty [t^{\frac{1}{p}-\frac{1}{q}}g^*(t)]^q dt \right\}^{\frac{1}{q}} \|h\|_{L^{q'}} \leq (\|f\|_{L^{p,q}} + \|g\|_{L^{p,q}})\|h\|_{L^{q'}} \leq \|f\|_{L^{p,q}} + \|g\|_{L^{p,q}}.$$

This, together with (6.4) establishes the triangle inequality for $\|\cdot\|_{L^{p,q}}$.

Now, let us see that $\|\cdot\|_{L^{p,q}}$ is a Banach function norm (Definition 4.1).

- P1: If $\|f\|_{L^{p,q}} = 0$, then $f^* = 0$ μ-a.e. Therefore $f = 0$ μ-a.e. (view in the proof property (1) of f^{**}). If $f = 0$ μ-a.e., then $f^* = 0$ so that$\|f\|_{L^{p,q}} = 0$. Then $\|af\|_{L^{p,q}} = |a|\|f\|_{L^{p,q}}$, follows from Proposition 5.11 (2).

- P2: It follows from Proposition 5.11 (1).

- P3: It follows from Proposition 5.11 (4) and monotone convergence theorem.
- P4: We have to see that if $\mu(E) = s$ then $\|\chi_E\|_{L^{p,q}} < \infty$. If $\mu(E) = s$, then $\chi_E^* = \chi_{(0,s)}$. Since $p \geq q$, we obtain that

$$\|\chi_E\|_{L^{p,q}} = \left\{ \int_0^s t^{\frac{q}{p}-1} dt \right\} < \infty.$$

- P5: If $\mu(E) < \infty$, then by Lemma 5.13,

$$\int_E f(x) d\mu(x) \leq \int_0^{\mu(E)} f^*(s) ds = \int_0^{\mu(E)} f^*(s) s^{\frac{1}{p}} s^{\frac{-1}{p}+1} \frac{ds}{s},$$

by Hölder's inequality,

$$\int_E f(x) d\mu(x) \leq \left(\int_0^{\mu(E)} (f^*(s))^q s^{\frac{q}{p}} \frac{ds}{s} \right)^{\frac{1}{q}} \left(\int_0^{\mu(E)} s^{-\frac{q'}{p}+q'} \frac{ds}{s} \right)^{\frac{1}{q'}}.$$

Since $p \geq q$, then $-\frac{q'}{p} + q' \geq 1$. Hence,

$$\int_E f(x) d\mu(x) \leq \|f\|_{L^{p,q}} C_E.$$

Finally, $\|\cdot\|_{L^{p,q}}$ is a rearrangement function norm, because if g and $\tilde{g}$ are equimeasurable then $g^* = \tilde{g}^*$. □

Although the restriction $q \leq p$ in the previous result is necessary, it can be missed in the case $p > 1$ by replacing $\|\cdot\|_{L^{p,q}}$ with an equivalent functional which is norm for all $q \geq 1$. The trick simply to replace f^* with f^{**} in the Definition 6.3 of $\|\cdot\|_{L^{p,q}}$.

Definition 6.8. *Suppose $1 < p \leq \infty$ and $1 \leq q \leq \infty$. If f in $\mathcal{M}_0(\mathcal{R}, \mu)$, let*

$$\|f\|_{L^{(p,q)}} = \begin{cases} \left\{ \int_0^\infty [t^{\frac{1}{p}} f^{**}(t)]^q \frac{dt}{t} \right\}^{\frac{1}{q}}, & (0 < q < \infty), \\ \sup_{0<t<\infty} \{t^{\frac{1}{p}} f^{**}(t)\}, & (q = \infty), \end{cases}$$

Lemma 6.9. *If $1 < p \leq \infty$ and $1 \leq q \leq \infty$, then*

$$\|f\|_{L^{p,q}} \leq \|f\|_{L^{(p,q)}} \leq p' \|f\|_{L^{p,q}},$$

for all f in $\mathcal{M}_0(\mathcal{R}, \mu)$, where $p' = \frac{p}{p-1}$. In particular, $L^{p,q}$ consists of all f for which $\|f\|_{L^{(p,q)}}$ is finite.

Proof. The first inequality is an immediate consequence of the fact that $f^* \leq f^{**}$, by Proposition 5.17 (2). The second follows directly from Lemma 6.2 and Definition 5.16, taking $\lambda = \frac{1}{p}$. □

Theorem 6.10. *If $1 < p < \infty$, $1 \leq q \leq \infty$ or if $p = q = \infty$, then $(L^{p,q}, \|\cdot\|_{L^{(p,q)}})$ is rearrangement-invariant Banach function space.*

Proof. We have the case $p < \infty$ and $q < \infty$. First, we prove that $\|\cdot\|_{L^{(p,q)}}$ is a Banach function norm.

- P1: If $\|f\|_{L^{(p,q)}} = 0$, then $\|f\|_{L^{p,q}} = 0$ by Lemma 6.9, then $f = 0$ μ-a.e. by Theorem 6.7. If $f = 0$ μ-a.e, then $\|f\|_{L^{(p,q)}} = 0$, by Proposition 5.17 (1).

  Since $f \to f^{**}$ is subadditive (Remark 5.19), the triangle inequality for $\|\cdot\|_{L^{(p,q)}}$ follows from Minkovsky's inequality, that is,

$$\|f+g\|_{L^{(p,q)}} = \left\{ \int_0^\infty [t^{\frac{1}{p}}(f+g)^{**}(t)]^q \frac{dt}{t} \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_0^\infty [t^{\frac{q}{p}-1} f^{**}(t) + t^{\frac{q}{p}-1} g^{**}(t)]^q dt \right\}^{\frac{1}{q}}$$

$$= \|t^{\frac{q}{p}-1} f^{**} + t^{\frac{q}{p}-1} g^{**}\|_{L^p} \leq \|t^{\frac{q}{p}-1} f^{**}\|_{L^p} + \|t^{\frac{q}{p}-1} g^{**}\|_{L^p} = \|f\|_{L^{(p,q)}} + \|g\|_{L^{(p,q)}}$$

  The other follows directly from Proposition 5.17 (4).

- P2: It follows from Proposition 5.17 (3).

- P3: It follows from Proposition 5.11 (4) and monotone convergence theorem.

- P4: If $\mu(E) < \infty$, then by Theorem 6.7 and by Lemma 6.9,

$$\|\chi_E\|_{L^{(p,q)}} \leq p' \|\chi_E\|_{L^{p,q}} < \infty.$$

- P5: If $\mu(E) < \infty$, then by Theorem 6.7 and by Lemma 6.9,

$$\int_E f(x) d\mu(x) \leq C_E \|f\|_{L^{p,q}} \leq C_E p' \|f\|_{L^{(p,q)}}.$$

Clearly, $\|\cdot\|_{L^{(p,q)}}$ is a rearrangement-invariant, because if f and g are equimeasurable, $f^* = g^*$, then $f^{**} = g^{**}$. Therefore $\|f\|_{L^{(p,q)}} = \|g\|_{L^{(p,q)}}$.

Now we prove the case $q = \infty$ and $1 < p \leq \infty$, the case $q = \infty$ and $p = \infty$ is analogous.

- P1,P2: They follow from the properties of the supremum.

- P3: If $f_n \uparrow f$ μ-a.e., then by Theorem 5.17,

$$t^{\frac{1}{p}} f_n^{**} \uparrow t^{\frac{1}{p}} f^{**} \implies \sup_{0<t<\infty} t^{\frac{1}{p}} f_n^{**} \uparrow \sup_{0<t<\infty} t^{\frac{1}{p}} f^{**}.$$

- P4, P5: They are analogous to above P4 and P5.

For the same above argument, $\|\cdot\|_{L^{(p,\infty)}}$ is a rearrangement-invariant. □

Remark 6.11. *The Lorentz spaces are not always rearrangement-invariant spaces. For example, the Lorentz space $L^{1,\infty}$ is not a rearrangement-invariant space.*

Appendix

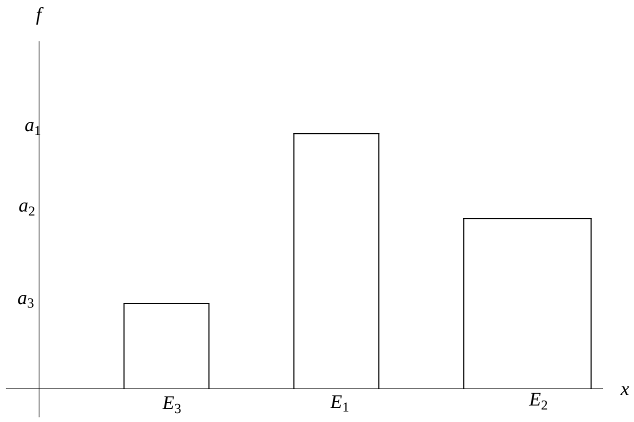

Figure 6.1: Graphs of f

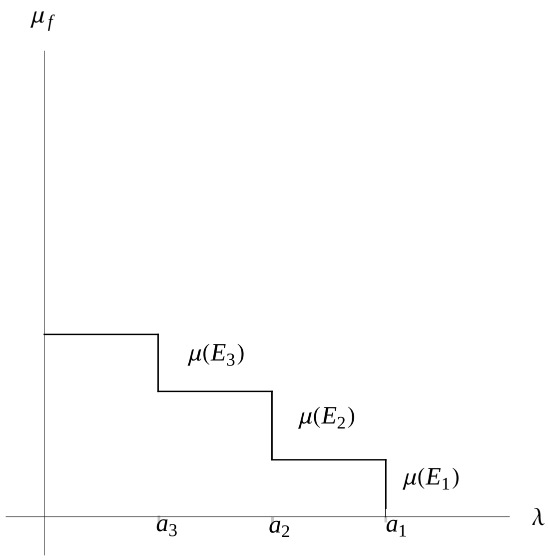

Figure 6.2: Graph of μ_f

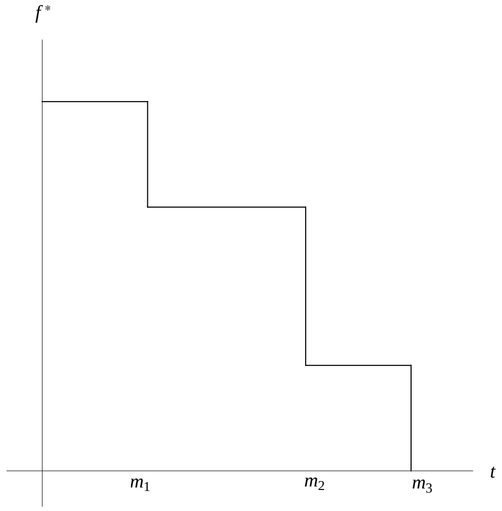

Figure 6.3: Graph of f^*

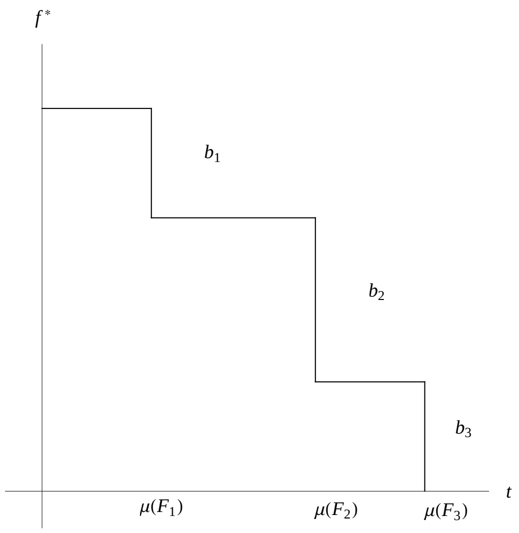

Figure 6.4: Graph of f^*

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