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Vertical Syndication-Proof Competitive Prices in Multilateral Markets

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Abstract

A multi-sided Böhm-Bawerk assignment game (Tejada, to appear) is a model for a multilateral market with a finite number of perfectly complementary indivisible commodities owned by different sellers, and inflexible demand and support functions. We show that for each such market game there is a unique vector of competitive prices for the commodities that is *vertical syndication-proof*, in the sense that, at those prices, syndication of sellers each owning a different commodity is neither beneficial nor detrimental for the buyers. Since, moreover, the benefits obtained by the agents at those prices correspond to the nucleolus of the market game, we provide a syndication-based foundation for the nucleolus as an appropriate solution concept for market games. For different solution concepts a syndicate can be disadvantageous and there is no escape to Aumman's paradox (Aumann, 1973). We further show that vertical syndication-proofness and horizontal syndication-proofness – in which sellers of the same commodity collude – are incompatible requirements under some mild assumptions.

Our results build on a self-interesting link between multi-sided Böhm-Bawerk assignment games and bankruptcy games (O'Neill, 1982). We identify a particular subset of Böhm-Bawerk assignment games and we show that it is isomorphic to the whole class of bankruptcy games. This isomorphism enables us to show the uniqueness of the vector of vertical syndication-proof prices for the whole class of Böhm-Bawerk assignment market using well-known results of bankruptcy problems.

Resum

Un joc d'assignació *multi-sided Böhm-Bawerk* (Tejada, per aparèixer) és un model per a un mercat multilateral en el que hi ha un nombre finit de béns indivisibles i perfectament complementaris (cadascun en mans de diferents venedors) i en el que les funcions d'oferta i demanda són inflexibles. En aquest treball demostrem que per a cada mercat d'aquest tipus existeix un únic vector de preus competitius (per als diferents béns) que és *vertical syndication-proof*, en el sentit que, a aquests preus, el fet que venedors de diferents béns s'agrupin en sindicats no afecta els compradors. Curiosament, els beneficis distribuits a tots els agents a aquests preus corresponen al nucleolus del joc, de manera que, com a subproducte del nostre resultat, obtenim una justificació per a l'ús del nucleolus. Per a d'altres solucions un sindicat pot ser perjudicial i, per tant, no hi ha sortida a la paradoxa trobada per Aumann (1973). Addicionalment, provem que *vertical syndication-proof* i *horizontal syndication-proof* són, sota unes certes condicions, restriccions incompatibles entre sí. Les proves dels nostres resultats es fonamenten en un relació, interessant per sí mateixa, entre els *multi-sided Böhm-Bawerk assignment games* i els *bankruptcy games* (O'Neill, 1982). En efecte, identifiquem un subconjunt d'aquells i provem que són isomorfs a la clase sencera d'aquests. Aquest isomorfisme ens permet provar la unicitat anteriorment citada.

JEL codes: C71; D43

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1 Introduction

Standard tools of cooperative game theory fail to reflect the intuitive idea that syndication in markets is unequivocally profitable for those agents undertaking it. Whether this fact should lead to a renounce of cooperative game theory as a framework to analyze syndication in markets is, however, controversial. Aumann (1973) and Postlewaite and Rosenthal (1974) present examples of markets in which the core does not exhibit the property that syndication is advantageous. Aumann (1973) considers a pure exchange economy with a non-atomic continuum of agents and provides examples in which the syndication of several agents makes them worse off. Postlewaite and Rosenthal (1974) consider a bilateral market with finitely many agents and show that, by colluding and acting as a single entity, all agents of one side of the market can never do better at any core allocation of the syndicated market than the worst they can do in the core of the market when they are unsyndicated.

Whereas Aumann (1973) concludes that the core might not be suitable for explaining the advantage of the syndicate, Postlewaite and Rosenthal (1974) argue that the disincentives for syndication might in fact be economically reasonable in certain situations. Legros (1987) extends the analysis of Postlewaite and Rosenthal (1974) to finite bilateral markets and show that, when considering the nucleolus (Schmeidler, 1969) of the cooperative game associated to the bilateral market, the disadvantageousness of a syndicate depends upon the relative rarity of the commodity that its members own. Besides, some empirical papers (see e.g. Asch and Seneca (1976)) also identify circumstances under which there is a negative relationship between syndication and profitability of firms.

In this paper we deal with a certain class of market games (Shapley and Shubik, 1969): *multi-sided Böhm-Bawerk assignment games* (Tejada, to appear; Tejada and Núñez, 2012). These games are special instances of multi-sided assignment games (Quint, 1991) and they constitute a model for a multilateral market in which *(i)* there are a finite number of complementary indivisible commodities each owned by a different seller, *(ii)* the demand and support functions are inflexible, meaning that sellers are willing to sell exactly one unit of a commodity and buyers are willing to buy a lot consisting of exactly one unit of each commodity, and *(iii)* different units of the same commodity are homogeneous from each buyers' point of view, i.e., buyers' willingness-to-pay does not depend on who they buy the goods from. Any multi-sided Böhm-Bawerk assignment game is obtained from a market situation, called *multi-sided Böhm-Bawerk assignment market*, that specifies the valuations of sellers and buyers. As shown in Tejada (2010), for arbitrary multi-sided assignment games the core

A syndicate is a coalition of agents who agree on operating as a single entity.

Other cooperative solution concepts like the Shapley value (see Gardner (1977) and Guesnerie (1977)) as well as other non-cooperative solution concepts (see Okuno *et al.* (1980) and Salant *et al.* (1983)) also contradict the intuitive idea that syndication is never disadvantageous.

More generally, a *multi-sided assignment problem* (Quint, 1991) corresponds to any situation in which there are different types of agents and a positive benefit can be reached only through the formation of tuples consisting exactly of one individual of each type. A multi-sided assignment market is just a particular case of a multi-sided assignment market. To any assignment problem Quint (1991) assigns a cooperative game, which he calls *multi-sided assignment game*. The case where there are only two types of agents (Shapley and Shubik, 1972) has usually captured the most attention because many markets are typically bilateral. Unlike for arbitrary multi-sided assignment games, the core of multi-sided Böhm-Bawerk assignment games is in one-to-one correspondence with the set of competitive prices.

If the bilaterality of the market is to be preserved, for the two-sided market analyzed in Legros (1987) the most natural syndicate to consider is *horizontal*, in the sense that only sellers of the same commodity are allowed to collude into a single entity. Nevertheless, in multilateral markets like those analyzed in the present paper, it is also worth investigating *vertical syndication*, where sellers of different commodities might collude and form a syndicate.

In the case of horizontal syndication, the creation of a syndicate does not change the number and nature of the different types of agents in a m-sided market, but it just alters the bargaining position of the syndicated players. In the case of vertical syndication, however, when k different agents of a m-sided market, with $k \in \{1, \ldots, m-1\}$, each owning one different commodity collude they create a new entity, the syndicate, which does not correspond to any type of agents in the unsyndicated market. We assume that the expected private benefit of syndicating for each of the k syndicated sellers is taken given that the structure of the market will switch from m-sidedness to $(m-k+1)$-sidedness. This assumption enables us to accommodate vertical syndication into the framework of multi-sided assignment markets.

We show that for each multi-sided Böhm-Bawerk assignment market there is a unique vector of competitive prices for the commodities that is *vertical syndication-proof*, in the sense that it offers no incentives to *(a)* sellers of different commodities to collude and act as a single seller, and *(b)* to a single seller to split into more sellers of different (possibly virtual) commodities. The vector of benefits obtained by all agents at this unique vector of prices turn out to correspond with the nucleolus of the cooperative game associated with the market. This finding provides a syndication-based foundation for the consideration of the nucleolus in market games. Such a justification was, to the best of our knowledge, lacking in the literature.

We note that, like most of the aforementioned papers, we study syndication in markets by means of tools borrowed from cooperative games. Cooperative game theory is widely used to study situations where coalitions play an important role in the creation of worth, which then has to be shared among the individuals. A variety of classes of cooperative games arise as further structure is imposed on the worths attached to coalitions. Each of the different models, e.g. multi-sided assignment problems, is typically aimed at describing a situation in which individuals interact in a particular way. Given a game, different solution concepts correspond to different normative criteria or behavior of the players.

Our approach to study syndication builds on a self-interesting link between multi-sided Böhm-Bawerk assignment games and another class of cooperative games that has long been studied: bankruptcy games (O'Neill, 1982). We identify a subclass of Böhm-Bawerk assign-

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is always nonempty.

We point out that the class of bilateral Böhm-Bawerk assignment markets does not contain nor it is contained in the class of markets considered by Legros (1987).

Notice that for vertical syndication in a market to make real sense, the market has to be at least three-sided.

See Section 4 for a further discussion on how to treat syndication within the framework of cooperative games.

ment games and we show that it is isomorphic to the whole class of bankruptcy games. This isomorphism enables us to use previous results on bankruptcy games to obtain results for the whole class of multi-sided Böhm-Bawerk assignment markets.

Bankruptcy problems refer to situations in which a group of individuals have claims over an estate, but the estate is not enough to satisfy all claims. The cooperative approach to this problem was first addressed in O'Neill (1982), and since a large literature has grown following the same approach. Bankruptcy problems are relevant in economic theory for their simplicity and, very especially, because they embody situations other than bankruptcy, e.g. taxation, cost sharing or surplus sharing. In this paper we find a relationship between bankruptcy games and a particular class of market games which emphasize the usefulness of bankruptcy problems.

Our analysis provides further insights on syndication in multilateral markets. First, we show that considering any solution concept different than the nucleolus, vertical syndication can be strictly harmful, as long as the considered solution concept satisfies some mild properties. These properties, which are discussed in detail in the corresponding section, refer to the prices selected in bilateral markets. Second, we prove that vertical syndication-proofness and horizontal syndication-proofness are incompatible requirements under the aforementioned mild assumptions.

The rest of the paper is organized as follows. In Section 2 we present the main definitions and we illustrate them by means of examples. In Section 3 we prove the fundamental link between Böhm-Bawerk assignment games and bankruptcy games. In Section 4 we exploit the previous link to find the unique vertical syndication-proof vector of competitive prices for Böhm-Bawerk assignment markets. In Section 5 we analyze other type of syndication. Section 6 concludes.

2 Notation and Preliminaries

A *cooperative game* is a pair (N, v), where N is a finite set of players and v, the *characteristic function*, is a real valued function on $2^N := \{S : S \subseteq N\}$ with $v(\emptyset) = 0$. We denote by $\mathcal{G}$ the set of all cooperative games. For every finite set N, (N, v_0) stands for the *null game*, where for every $S \subseteq N$, $v_0(S) = 0$. The set of non-negative real numbers is denoted by $\mathbb{R}_+$, and for every $x \in \mathbb{R}$ we use the notation $(x)_+ := \max\{0, x\}$. For an arbitrary finite set N, we denote by $\mathbb{R}^N$ the $|N|$-dimensional Euclidean space with elements $x \in \mathbb{R}^N$ having components x_i, $i \in N$. For every $S \subseteq N$ and $x \in R^N$, let as usual $x(S) := \sum_{i \in S} x_i$. The *core* of a game is the set of efficient allocations that cannot be improved upon by any coalition on its own, i.e.,

$$C(N, v) = \left\{ x \in \mathbb{R}^N \ : x(N) = v(N) \text{ and, for every } S \subsetneq N, x(S) \geq v(S) \right\}.$$

2.1 Böhm-Bawerk Assignment Games

Definition 2.1. *Given an integer $m > 1$, an m-*sided Böhm-Bawerk assignment market *is a tuple $(N^1, \ldots, N^m, \mathbf{c}, w)$, where $N^1, \ldots, N^{m-1}$ are the finite sets of sellers, N^m is the*

See Thomson (2003) for an extensive survey on the subject.

finite set of buyers, $\mathbf{c} = (c^1, \ldots, c^{m-1}) \in \mathbb{R}_+^{N^1} \times \cdots \times \mathbb{R}_+^{N^{m-1}}$*, and* $w \in \mathbb{R}_+^{N_m}$.

The m sets of agents $N^1, \ldots, N^m$ are called *sectors*. Each seller owns exactly a unit of an indivisible commodity, whereas each buyer is willing to buy a bundle composed of exactly one unit from each commodity. The goods owned by two sellers belong to the same commodity if and only if both sellers belong to the same sector, so in a m-sided Böhm-Bawerk assignment market there are $m-1$ different commodities. For each $k \in \{1, \ldots, m-1\}$, c_i^k stands for the valuation of seller $i \in N^k$ of her own good (we may also call it simply the cost), whereas w_i stands for the valuation of buyer $i \in N^m$ of a lot consisting of one good of each of the commodities. W.l.o.g., we will assume that $|N^1| = \cdots = |N^m| = n$ by introducing dummy sellers with very large reservation prices and dummy buyers with zero willingness-to-pay. We will also assume that sellers' valuations of sellers are arranged within each sector in a nondecreasing way whereas buyers' valuations are arranged in nonincreasing way, i.e., for every $k \in \{1, \ldots, m-1\}$, $c_1^k \leq \cdots \leq c_n^k$ and $w^1 \geq \cdots \geq w^n$. We denote by $\mathcal{BBM}$ the set of all multi-sided Böhm-Bawerk assignment markets. When no confusion regarding the set of agents may arise we will denote a (multi-sided) Böhm-Bawerk assignment market simply by $(\mathbf{c}, w)$.

We call any m-tuple of agents $E \in \prod_{k=1}^m N^k$ an *essential coalition*. A *matching* among $N^1, \ldots, N^m$ is a maximal set of disjoint essential coalitions such that every player belongs to at most one essential coalition. We denote by $\mathcal{M}(N^1, \ldots, N^m)$ the set of all matchings among $N^1, \ldots, N^m$. Given an m-sided Böhm-Bawerk assignment market $(\mathbf{c}, w)$, we denote by $A(\mathbf{c}, w)$ the m-dimensional matrix that indicates the potential benefit obtained by all essential coalitions, i.e., the matrix defined for every $E = (i_1, \ldots, i_m) \in \prod_{k=1}^m N^k$ by

$$a_E = \left(w_{i_m} - \sum_{k=1}^{m-1} c_{i_k}^k \right)_+ . \tag{1}$$

If money exists in the market and side payments between agents are allowed, an m-sided Böhm-Bawerk assignment market $(\mathbf{c}, w)$ generates an *assignment game* $(N, \omega^{\mathbf{c},w}) \in \mathcal{G}$, where $N = \cup_{k=1}^m N^k$ and, for every $S \subseteq N$,

$$\omega^{\mathbf{c},w}(S) = \max_{\mu \in \mathcal{M}(N^1 \cap S, \ldots, N^m \cap S)} \left\{ \sum_{E \in \mu} a_E \right\}, \tag{2}$$

where $a_E \in A(\mathbf{c}, w)$ and the summation over the empty set is zero. That is, the assignment game describes the best that players of each coalition can do organizing themselves in essential coalitions and is a tool for the analysis of the division of the net profit $\omega^{\mathbf{c},w}(N)$ between buyers and sellers. We denote by $\mathcal{BBG}$ the set of all assignment games associated to markets in $\mathcal{BBM}$.

Tejada (to appear) studies the core of an arbitrary game in $\mathcal{BBG}$ by defining the so-called *sectors game* in which the (fictitious) players are the sectors of the market. We briefly recall the main results in Tejada (to appear), the comprehension of which requires introducing

This assumption enables us to properly speak about the i^{th} agent of any sector $k \in \{1, \ldots, m\}$.

We abuse notation and use essential coalition to denote the m-tuple as well as the coalition composed of all agents in the tuple.

additional concepts. For every $i \in \mathbb{N}$, let $E^i := (i, \ldots, i) \in \mathbb{R}^m$. The highest number of Pareto-improving trades that can take place simultaneously in $(\mathbf{c}, w)$ is

$$r = \max_{1 \leq i \leq n} \{i : a_{E^i} > 0\}, \tag{3}$$

with the convention that $r = 0$ if all entries of $A(\mathbf{c}, w)$ are zero. Let $M := \{1, \ldots, m\}$. For every $S \subseteq M$ we introduce the notation $E^S := r\mathbb{1}_S + (r+1)\,\mathbb{1}_{M \setminus S} \in \mathbb{R}^m$, where, for every $T \subseteq M$, $\mathbb{1}_T \in \mathbb{R}^m$ is the vector such that $\mathbb{1}_T(k) = 1$ if $k \in T$ and $\mathbb{1}_T(k) = 0$ if $k \notin T$. Notice that, by introducing dummy players, we can always ensure that $r < n$, so there are at least $r+1$ agents in all sectors. Also note that $\{E^S : S \subseteq M\}$ are precisely all essential coalitions that can be obtained by combining the r^{th} and $r+1^{th}$ agents of each sector.

Definition 2.2. *Given an m-sided Böhm-Bawerk assignment problem $(\mathbf{c}, w)$, the associated* sectors game $(M, v^{\mathbf{c},w}) \in \mathcal{G}$ *is the cooperative game with set of players M and characteristic function defined, for every $S \subseteq M$, by*

$$v^{\mathbf{c},w}(S) = \begin{cases} a_{E^S} & \text{if } r > 0, \\ 0 & \text{if } r = 0. \end{cases}$$

We denote by $\mathcal{SG}$ the set of all sectors games associated to markets in $\mathcal{BBM}$. For every $(\mathbf{c}, w) \in \mathcal{BBM}$ we introduce the *replica operator*, $\mathcal{R}_{\mathbf{c},w}$, which is an injective linear function defined by

$$\begin{array}{cccccccc} \mathcal{R}_{\mathbf{c},w}: & \mathbb{R}^M & \longrightarrow & \mathbb{R}^{N^1} & \times & \ldots & \times & \mathbb{R}^{N^m} \\ & (x_1, \ldots, x_m) & \longrightarrow & (\overbrace{x_1, \ldots, x_1}^{r}, \overbrace{0, \ldots, 0}^{n-r}; & & \ldots & & ;\overbrace{x_m, \ldots, x_m}^{r}, \overbrace{0, \ldots, 0}^{n-r}) \end{array}$$

We also introduce for every $(\mathbf{c}, w) \in \mathcal{BBM}$ the *translation vector* $t_{\mathbf{c},w} = (t_1^1, \ldots, t_n^1; \ldots; t_1^m, \ldots, t_n^m) \in \mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^m}$, which is defined for every $i \in \{1, \ldots, n\}$ by

$$t_i^k = \begin{cases} \left(c_r^k - c_i^k\right)_+ & \text{if } k \in \{1 \ldots, m-1\}, \\ (w_i - w_r)_+ & \text{if } k = m. \end{cases}$$

Lastly, for every $t \in \mathbb{R}^l$ and $B \subseteq \mathbb{R}^l$, let $t + B$ denote the translated set B by the vector t, i.e., $t + B = \{x = t + x' : x' \in B\}$. We are now in the position to state the main result in Tejada (to appear).

Theorem 2.1. (Tejada, to appear) *Let $(\mathbf{c}, w)$ be an m-sided Böhm-Bawerk assignment problem. Then,*

$$C(N, \omega^{\mathbf{c},w}) = t_{\mathbf{c},w} + \mathcal{R}_{\mathbf{c},w}(C\,(M, v^{\mathbf{c},w})).$$

In order to illustrate the above definitions we introduce the following example.

Example 2.1. *Consider the 24-person 3-sided Böhm-Bawerk assignment market given by*

$$\begin{aligned}(c_1, c_2, w) = (&30, 33, 40, 48, \mathbf{50}, 55, 70, 78; \\ &25, 35, 58, 59, \mathbf{60}, 67, 70, 82; \\ &200, 180, 160, 140, \mathbf{130}, 120, 100, 90).\end{aligned}$$

It is an easy exercise to check that the associated sectors game $(M, v^{\mathbf{c},w})$ is given by $M = \{1, 2, 3\}$ and

$$
\begin{array}{lll}
v^{\mathbf{c},w}(\{1\}) = 3, & v^{\mathbf{c},w}(\{1,2\}) = 10, & \\
v^{\mathbf{c},w}(\{2\}) = 5, & v^{\mathbf{c},w}(\{1,3\}) = 13, & v^{\mathbf{c},w}(\{1,2,3\}) = 20. \\
v^{\mathbf{c},w}(\{3\}) = 8, & v^{\mathbf{c},w}(\{2,3\}) = 15, &
\end{array}
$$

According to Theorem 2.1, the core of the assignment game associated to the 24-person market in Example 2.1 is determined (up to a bijection) by the core of the 3-player game $(M, v^{\mathbf{c},w})$ described above. Moreover, in the definition of the sectors game the only relevant information of the assignment problem is contained in rows r and $r+1$ of $(\mathbf{c}, w)$. Thus we can associate to every Böhm-Bawerk assignment market another Böhm-Bawerk assignment market with only two agents in each of the sectors as follows. Given an arbitrary Böhm-Bawerk assignment problem $(\mathbf{c}, w) = (N^1, \ldots, N^m, \mathbf{c}, w)$ with $r > 0$, let $(\widetilde{\mathbf{c}}, \widetilde{w}) = (\widetilde{N}^1, \ldots, \widetilde{N}^m, \widetilde{\mathbf{c}}, \widetilde{w})$ be the Böhm-Bawerk assignment problem with $|\widetilde{N}^1| = \ldots = |\widetilde{N}^m| = 2$ defined by

$$
\widetilde{\mathbf{c}} = (c_r^1, c_{r+1}^1; \ldots; c_r^{m-1}, c_{r+1}^{m-1}) \quad \text{and} \quad \widetilde{w} = (w_r, w_{r+1}). \tag{4}
$$

It is straightforward to check that $(M, v^{\mathbf{c},w}) = (M, v^{\widetilde{\mathbf{c}},\widetilde{w}})$. For markets $(\mathbf{c}, w) \in \mathcal{BBM}$ with $r = 0$ we take $(\widetilde{\mathbf{c}}, \widetilde{w}) = ((c_1^1, c_2^1; \ldots; c_1^m, c_2^m), (w_1, w_2))$. In this later case $(M, v^{\mathbf{c},w}) = (M, v^{\widetilde{\mathbf{c}},\widetilde{w}}) = (M, v_0)$. Therefore, to study the core allocations of an arbitrary Böhm-Bawerk assignment market we may restrict the attention to the following type of market.

Definition 2.3. *Given $m > 1$, a* 2-regular m-sided Böhm-Bawerk assignment market *(just 2-regular market henceforth) is an m-sided Böhm-Bawerk assignment market* $(N^1, \ldots, N^m, \mathbf{c}, w)$ *such that $|N^1| = \cdots = |N^m| = 2$ and*

$$
w_2 - \sum_{k=1}^{m-1} c_2^k \leq 0. \tag{5}
$$

As before, when no confusion may arise, we will simply denote a 2-regular market by $(\mathbf{c}, w)$. We will also denote by *2-$\mathcal{BBM}$* the set of all 2-regular markets and by *2-$\mathcal{BBG}$* the set of the assignment games associated to 2-regular markets.

Remark 2.1. *By Definitions 2.2 and 2.3, the set of sectors games associated to 2-regular markets is precisely $\mathcal{SG}$.*

We point out that by considering 2-regular markets we, however, deliberately focus on markets with a very specific structure that does not encompass the more general market situations covered by the whole class of Böhm-Bawerk assignment markets. A further justification for considering the set of 2-regular markets is given in the following result.

Proposition 2.1. *The set of games associated to 2-regular markets, 2-$\mathcal{BBG}$, and the set of sectors games, $\mathcal{SG}$, are isomorphic.*

Since the proof of Proposition 2.1 is rather technical, it has been relegated to the Appendix.

2.2 Bankruptcy Games

Definition 2.4. *A* bankruptcy problem *is a tuple (N, E, d), where N is the finite set of agents, $E \geq 0$ is the estate to be divided, and $d \in \mathbb{R}_+^N$ is the vector of claims satisfying $\sum_{i \in N} d_i \geq E$.*

We denote by $\mathcal{BP}$ the set of all bankruptcy problems. Given a bankruptcy problem $(N, E, d) \in \mathcal{BP}$, the associated *bankruptcy game* $(N, v^{E,d}) \in \mathcal{G}$ is defined for every $S \subseteq N$ by

$$v^{E,d}(S) = (E - d(N \backslash S))_+ . \tag{6}$$

The worth of a coalition according to the bankruptcy game is the amount that is left after all other claims are satisfied, whenever there is something left. We denote by $\mathcal{BG}$ the set of all bankruptcy games obtained from bankruptcy problems. Let us consider an example of a bankruptcy game.

Example 2.2. *Consider the bankruptcy problem where $N = \{1, 2, 3\}$, $E = 20$ and $d = (5, 7, 10)$. Then, the associated bankruptcy game $(N, v^{E,d})$ is given by:*

$$\begin{array}{lll} v^{E,d}(\{1\}) = 3, & v^{E,d}(\{1,2\}) = 10, & \\ v^{E,d}(\{2\}) = 5, & v^{E,d}(\{1,3\}) = 13, & v^{E,d}(\{1,2,3\}) = 20. \\ v^{E,d}(\{3\}) = 8, & v^{E,d}(\{2,3\}) = 15, & \end{array}$$

3 Relation between Böhm-Bawerk Assignment Games and Bankruptcy Games

In this section we find a deep connection between bankruptcy problems and Böhm-Bawerk assignment markets.

Theorem 3.1. *The set of bankruptcy games is precisely the set of sectors games associated to (2-regular) Böhm-Bawerk assignment markets, i.e.,*

$$\mathcal{BG} = \mathcal{SG}.$$

Proof. On the one hand, for every $(M, E, d) \in \mathcal{BP}$ we define $\mathbf{c}(E, d) \in \mathbb{R}_+^{2(m-1)}$ and $w(E, d) \in \mathbb{R}_+^2$ as follows:

$$\begin{aligned} \mathbf{c}(E,d) &= \left(\frac{\delta}{m-1}, d_1 + \frac{\delta}{m-1}; \ldots; \frac{\delta}{m-1}, d_{m-1} + \frac{\delta}{m-1} \right), \\ w(E,d) &= (E + \delta, E - d_m + \delta), \end{aligned} \tag{7}$$

where $\delta := \max\{0, d_m - E\}$. By definition of the bankruptcy problem, it is immediate to check that $(\mathbf{c}(E, d), w(E, d)) \in$ *2-$\mathcal{BBM}$*, i.e., the above defined m-sided Böhm-Bawerk assignment market satisfies Eq. (5). The sectors game associated to $(\mathbf{c}(E, d), w(E, d))$ is defined, for every $S \subseteq M \setminus \{m\}$, by

$$v^{\mathbf{c}(E,d),w(E,d)}(S) = \left(E - d_m + \delta - \sum_{k \in M \backslash (S \cup \{m\})} \left[d_k + \frac{\delta}{m-1} \right] \right)_+ = (E - d(S))_+ = v^{E,d}(S),$$

and

$$\begin{aligned} v^{\mathbf{c}(E,d),w(E,d)}(S \cup \{m\}) = \left(E + \delta - \sum_{k \in M \backslash (S \cup \{m\})} \left[d_k + \frac{\delta}{m-1} \right] \right)_+ &= (E - d(S \cup \{m\}))_+ \\ &= v^{E,d}(S \cup \{m\}). \end{aligned}$$

On the other hand, for every $(\mathbf{c}, w) \in \mathit{2}\text{-}\mathcal{BBM}$ we define $E(\mathbf{c}, w) \in \mathbb{R}_+$ and $d(\mathbf{c}, w) \in \mathbb{R}_+^m$ as follows:

$$\begin{aligned} E(\mathbf{c}, w) &= \left(w_1 - \sum_{k \in M \setminus \{m\}} c_1^k \right)_+, \\ d(\mathbf{c}, w) &= \left(c_2^1 - c_1^1, \ldots, c_2^{m-1} - c_1^{m-1}, w_1 - w_2 \right). \end{aligned} \tag{8}$$

From Definition 2.3 it immediately follows that $(M, E(\mathbf{c}, w), d(\mathbf{c}, w)) \in \mathcal{BP}$, i.e., $d(M) \geq E$. The associated bankruptcy game is then defined, for every $S \subseteq M \setminus \{m\}$, by

$$\begin{aligned} v^{E(\mathbf{c},w),d(\mathbf{c},w)}(S) &= \left(\left(w_1 - \sum_{k \in M \setminus m} c_1^k \right)_+ - (w_1 - w_2) - \sum_{k \in M \setminus (S \cup \{m\})} (c_2^k - c_1^k) \right)_+ \\ &= \left(w_2 - \sum_{k \in S} c_1^k - \sum_{k \in M \setminus (S \cup \{m\})} c_2^k \right)_+, = v^{\mathbf{c},d}(S) \end{aligned}$$

and

$$\begin{aligned} v^{E(\mathbf{c},w),d(\mathbf{c},w)}(S \cup \{m\}) &= \left(\left(w_1 - \sum_{k \in M \setminus m} c_1^k \right)_+ - \sum_{k \in M \setminus (S \cup \{m\})} \left(c_2^k - c_1^k \right) \right)_+ \\ &= \left(w_1 - \sum_{k \in S} c_1^k - \sum_{k \in M \setminus (S \cup \{m\})} c_2^k \right)_+ = v^{\mathbf{c},d}(S \cup \{m\}). \end{aligned}$$

□

The above theorem shows that for every game that can be obtained from a bankruptcy problem there exists (a not unique) Böhm-Bawerk market whose associated sectors game is precisely the former game, and vice versa. In particular, the coincidence of the games in Examples 2.2 and 2.1 is not just a happy coincidence, but rather it illustrates the above link between two classes of games. Furthermore, from Proposition 2.1 it immediately follows that the class of 2-regular markets and the class of bankruptcy games are isomorphic.

4 Exploiting the Link between Games

In this section, we explore the role of vertical syndication in Böhm-Bawerk assignment markets by exploiting Theorem 3.1. More specifically, we focus on point-valued solution concepts, i.e., rules that propose for every Böhm-Bawerk assignment market a payoff to each agent that participates in the market. Formally, a *rule on* $\mathcal{BBM}$ is a map, f, that associates to every m-sided Böhm-Bawerk assignment market $(\mathbf{c}, w) \in \mathcal{BBM}$ a payoff vector $\mathsf{f}(\mathbf{c}, w) = (\mathsf{f}^1(\mathbf{c}, w), \ldots, \mathsf{f}^m(\mathbf{c}, w))$ where, for every $k \in M$, $\mathsf{f}^k(\mathbf{c}, w) \in \mathbb{R}^{N^k}$. We restrict the attention to rules that propose payoff vectors in the core of the assignment game.

CS A rule on $\mathcal{BBM}$, f, satisfies *core selection* if, for every $(\mathbf{c}, w) \in \mathcal{BBM}$,

$$\mathsf{f}(\mathbf{c}, w) \in C(N, \omega^{\mathbf{c},w}).$$

The above property requires a rule to completely share the spoils generated by the market so that no coalition of agents has incentives to search for an agreement on their own. Moreover, CS can be spelled out into two different conditions.

Remark 4.1. *A rule on $\mathcal{BBM}$, f, satisfies* CS *if and only if*

1) Let $(\mathbf{c}, w) \in \mathcal{BBM}$ and r be the parameter defined in Eq. (3). Then, for every $k \in M$,

1.a) if $i \in \{r+1, \ldots, n\}$,

$$\mathsf{f}_i^k(\mathbf{c}, w) = 0,$$

1.b) if $i, j \in \{1, \ldots, r\}$,

$$\mathsf{f}_i^k(\mathbf{c}, w) - \mathsf{f}_j^k(\mathbf{c}, w) = \begin{cases} c_j^k - c_i^k & \text{if } k \in M \setminus \{m\}, \\ w_i - w_j & \text{if } k = m. \end{cases}$$

2) f gives core allocations of 2-regular markets, i.e., for every $(\mathbf{c}, w) \in$ 2-$\mathcal{BBM}$,

$$\mathsf{f}(\mathbf{c}, w) \in C(N, \omega^{\mathbf{c}, w}).$$

Proof. Let f be a rule on $\mathcal{BBM}$ satisfying CS, then it trivially satisfies Condition 2). Condition 1) can also be easily checked taking into account Theorem 2.1. In particular, 1.a) holds by the definition of the replica operator and 1.b) from the definition of the translation vector. The reverse implication follows from Theorem 2.1 and the properties of the Böhm-Bawerk assignment market defined in Eq. (4). □

In what follows we propose and characterize a certain rule on $\mathcal{BBM}$. To do so we take advantage of the large literature related to bankruptcy problems. Indeed, note that Theorems 2.1 and 3.1 suggest a natural procedure to find rules on $\mathcal{BBM}$ based on bankruptcy rules. A *rule on $\mathcal{BP}$* is a map, fb, that associates to every bankruptcy problem $(N, E, d) \in \mathcal{BP}$ a payoff vector $\mathsf{fb}(E, d) \in \mathbb{R}^N$ that shares the whole estate and allocates to every player no more than her own claim, i.e., for every $(N, E, d) \in \mathcal{BP}$, $\sum_{i \in N} \mathsf{fb}_i(E, d) = E$ and, for every $i \in N$, $\mathsf{fb}_i(E, d) \leq d_i$.

Whereas payoffs in a bankruptcy problem have a straightforward interpretation, payoffs in a Böhm-Bawerk assignment market require some subtleties. Indeed, in such a market the natural way in which agents interact is by exchanging money for goods, i.e., by means of prices that fix the amount of money paid by a buyer to a seller in exchange for a good. That is, if a seller $i \in N^k$ obtains a utility of x_i^k when bargaining with a buyer she must charge a price

$$p_i^k = x_i^k + c_i^k, \tag{9}$$

to be paid by the buyer. Tejada (2010) shows that, for arbitrary assignment markets – in particular for Böhm-Bawerk assignment markets – there is a one-to-one correspondence between competitive prices and prices obtained from core allocations by means of (9). Roughly speaking, a vector of prices (one price for each seller) is *competitive* if it covers costs for the sellers, there exists a matching of sellers and buyers such that each buyer is assigned by the matching a bundle of goods from her demand set (given the prices), and no seller can benefit from lowering the price she charges. For every multi-sided Böhm-Bawerk assignment market the parameter r defined in Eq. (3) describes the number of actual transactions that take place in the market at any optimal matching. Due to Theorem 2.1, it holds that, for every vector of competitive prices of a given market, the price charged by any two sellers of the same commodity that participate in an actual transaction must coincide. Formally, for every $(\mathbf{c}, w) \in \mathcal{BBM}$ let $p \in \mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^{m-1}}$ be a vector of competitive prices. Then, for every $k \in M \setminus \{m\}$ and $i, j \in \{1, \ldots, r\}$, by Theorem 2.1 it holds that $p_i^k = p_j^k$. This fact enables us to denote by $p(x) = \big(p^1(x), \ldots, p^{m-1}(x)\big) \in \mathbb{R}_+^{m-1}$ the (unique) *vector of competitive prices associated to a core allocation* x of the assignment game, where for every $k \in M \setminus \{m\}$, $p^k(x)$ is the *competitive price of the commodity* k.

Next, we present a property for rules on $\mathcal{BBM}$ that will be of key importance in the characterization result. Loosely speaking, in a Böhm-Bawerk assignment market, like in any market, a strategy for a firm (a seller) could be more complex than just setting a price: it could decide to syndicate with more firms to increase its market power. The immediate consequence of such a move for the buyers is that they would only have to bargain with the resulting merged seller instead of all initial sellers. We are interested in finding rules that are immune to this kind of syndication.

The usual approach – see e.g. Postlewaite and Rosenthal (1974) or Legros (1987) – to treat the impact of a syndicate $\emptyset \neq S \subsetneq N$ in a market game (N, v) is to consider a representative $k \notin N$ for the syndicate and the restricted game (N^S, v^S), where $N^S = (N \cup \{k\}) \setminus S$ is the set of players and the characteristic function give for every $T \subseteq N^S$ by

$$v^S(T) = \begin{cases} v(T \cup S \setminus \{k\}) & \text{if } k \in T, \\ v(T) & \text{if } k \notin T. \end{cases}$$

Given a rule, f, on the set of market games, the syndicate $S \subseteq N$ is *advantageous* (resp. *disadvantageous*) if $\sum_{i \in S} \mathsf{f}_i(N, v) < \mathsf{f}_k(N^S, v^S)$ (resp. $\sum_{i \in S} \mathsf{f}_i(N, v) > \mathsf{f}_k(N^S, v^S)$). A rule is *syndication-proof* if there are no advantageous nor disadvantageous syndicates. Intuitively it may seem reasonable to think that disadvantageous syndicates will not exist. However, Aumann (1973) has shown that this idea is not always correct. The Aumann paradox applies to a rule and a market when there exists at least one disadvantageous syndicate.

The above approach is, however, not without its flaws. Indeed, there are classes of market games, e.g. Böhm-Bawerk assignment games, that are not closed by the restricted game operation. We recall that a defining feature of a Böhm-Bawerk assignment market is that the demand and supply functions are inflexible, that is, buyers are only interested on buying one unit of each commodity and sellers are only endowed which one unit of a good. It follows from the aforementioned characteristic that the restricted game of a Böhm-Bawerk assignment game might not be a Böhm-Bawerk assignment game itself. Our approach in this paper is, however, to assume that the above feature – together with the fact that each

See Tejada (2010) for a formal definition of competitive prices. Also notice that, unlike core allocations, a vector of competitive prices specifies a number (the price) to all sellers but nothing to buyers.

We note that (N^S, v^S) is a market game if (N, v) is a market game too.

There are obviously other standard criteria to compare payoffs to all agents in S and the payoff to the representative of the syndicate S, k.

buyer values each lot equally – is inherent to the market, which leads us to consider the property below – which is initially stated only for 2-regular markets.

2-VSP A rule on $\mathcal{BBM}$, f, satisfies *vertical syndication-proof for 2-regular markets* if, for every $(N^1, \ldots, N^m, \mathbf{c}, w) \in$ *2-*$\mathcal{BBM}$ and every $k \in M \setminus \{m\}$,

$$\mathsf{f}^m(N^1, \ldots, N^m, \mathbf{c}, w) = \mathsf{f}^m\left(N^k, N^m, \bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right),$$

where the *reduced two-sided market*

$$\left(N^k, N^m, \bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right)$$

is composed of the same sector of buyers, with valuations given by w, and a unique sector of sellers with costs, $\bar{c}^k(\mathsf{f}(\mathbf{c}, w))$, defined for every $i \in N^k = \{1, 2\}$ by

$$\bar{c}_i^k(\mathsf{f}(\mathbf{c}, w)) = c_i^k + \sum_{l \in M \setminus \{k, m\}} p^l(\mathsf{f}(\mathbf{c}, w)). \tag{10}$$

When no confusion may arise we omit the specification of the sets N^k, N^m and we write $\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right)$ instead of $\left(N^k, N^m, \bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right)$. We also point out the following fact.

Remark 4.2. *Let* $(\mathbf{c}, w)) \in$ *2-*$\mathcal{BBM}$*. If* f *satisfies* CS *then* $\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right) \in$ *2-*$\mathcal{BBM}$*.*

Proof. Indeed, $|N^k| = |N^m| = 2$ since $(\mathbf{c}, w) \in$ *2-*$\mathcal{BBM}$ and

$$\begin{aligned}
&w_2 - \bar{c}_2^k(\mathbf{c}, w) \\
=&w_2 - c_2^k - \sum_{l \in M \setminus \{k, m\}} \left[\mathsf{f}_1^l(\mathbf{c}, w) + c_1^l\right] = w_2 - c_2^k - \sum_{l \in M \setminus \{k, m\}} c_1^l - \sum_{l \in M \setminus \{k, m\}} \mathsf{f}_i^l(\mathbf{c}, w) \\
\leq& \left(w_2 - c_2^k - \sum_{l \in M \setminus \{k, m\}} c_1^l\right)_+ - \left[\mathsf{f}_2^k(\mathbf{c}, w) + \mathsf{f}_2^m(\mathbf{c}, w) + \sum_{l \in M \setminus \{k, m\}} \mathsf{f}_1^l(\mathbf{c}, w)\right] \leq 0,
\end{aligned}$$

where the penultimate inequality holds because $\mathsf{f}_2^k(\mathbf{c}, w) = \mathsf{f}_2^m(\mathbf{c}, w) = 0$ due to Theorem 2.1 and the fact that $(\mathbf{c}, w)$ is 2-regular market and f satisfies CS, and the last inequality holds since $\mathsf{f}(\mathbf{c}, w)$ is a core allocation of $(N, v^{\mathbf{c}, w})$. □

Vertical syndication-proof for 2-regular markets states that the benefits assigned by f to each buyer in the original 2-regular market $(\mathbf{c}, w)$ have to coincide with the benefits assigned by f to each buyer in the reduced two-sided 2-regular market $\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right)$. This latter market consists, on the one hand, of the same buyers of $(\mathbf{c}, w)$ with the same willingnesses-to-pay and, on the other hand, of a unique sector of sellers: sector k of $(\mathbf{c}, w)$, which contains the *representatives* of the different syndicates. The costs for the syndicated sellers in the latter bilateral market are obtained from those in $(\mathbf{c}, w)$ by adding the (competitive) price of all goods in sectors $l \in M \setminus \{k, m\}$ prescribed by f in $(\mathbf{c}, w)$. Thus, if prices are determined by a rule that satisfies 2-VSP, buyers are indifferent to the above simultaneous syndication of sellers. The above property, 2-VSP, is particularly interesting since it relates the payoffs in a multi-sided market to payoffs in a two-sided market.

There are, of course, other ways in which agents might syndicate among them, some of which are described later in Section 5.

Whereas the literature on bilateral assignment games has grown in the last years very rapidly, even after the more general multilateral model was introduced by Quint (1991), there are much fewer papers addressing

We note two other important features of the above property. On the one hand, instead of focusing on the supply side of the market, 2-VSP deals with the demand side of the market. We are interested in rules that set prices that cannot be manipulated by sellers in a way that the buyer' surplus, i.e. the aggregate buyers' benefit, is reduced regardless of how the aggregate sellers' surplus is shared among sellers. On the other hand, 2-VSP abstracts from the exact bargaining process within a syndicate composed of $m-1$ sellers, each one owning a different commodity: it just requires that a representative of a certain sector $k \in M$, the same for all syndicates, is chosen to bargain with buyers.

We point out that if we demand CS, the payoffs to the reduced sector of sellers are fixed in the same way as payoffs to buyers are.

Remark 4.3. *If a rule on $\mathcal{BBM}$, f, satisfies* CS *and* 2-VSP *then, for every $(\mathbf{c}, w) \in 2\text{-}\mathcal{BBM}$ and every $k \in M \setminus \{m\}$,*

$$\mathsf{f}^k(\mathbf{c}, w) = \mathsf{f}^k\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right),$$

where $\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right) \in 2\text{-}\mathcal{BBM}$ is defined in Eq. (10).

Proof. By Remark 4.2 the reduced market is a 2-regular market, so the parameter r – defined in Eq. (3) – is at most 1. Then it suffices to check that $\mathsf{f}_2^k(\mathbf{c}, w) = \mathsf{f}_2^k\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right) = 0$ and

$$\begin{aligned}
\mathsf{f}_1^k(\mathbf{c}, w) &= \omega^{\mathbf{c},w}(N) - \sum_{l \in M\setminus\{k,m\}} \mathsf{f}_1^l(\mathbf{c}, w) - \mathsf{f}^m(\mathbf{c}, w) \\
&= \omega^{\mathbf{c},w}(N) - \sum_{l \in M\setminus\{k,m\}} \mathsf{f}_1^l(\mathbf{c}, w) - \mathsf{f}_1^m\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right) \\
&= \omega^{\mathbf{c},w}(N) - \sum_{l \in M\setminus\{k,m\}} \mathsf{f}_1^l(\mathbf{c}, w) - \omega^{\bar{c}^k(\mathsf{f}(\mathbf{c},w)),w}\left(N^k \cup N^m\right) + \mathsf{f}_1^k\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right) \\
&= \mathsf{f}_1^k\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right),
\end{aligned}$$

where the first and the third equalities hold since f yields efficient allocations, the second equality holds since f satisfies 2-MP, and the last equality holds since, by definition of the reduced market,

$$\omega^{\bar{c}^k(\mathsf{f}(\mathbf{c},w)),w}\left(N^k \cup N^m\right) = \omega^{\mathbf{c},w}(N) - \sum_{l \in M\setminus\{k,m\}} \mathsf{f}^l(\mathbf{c}, w).$$

□

Hence, 2-VSP (together with CS) implies that no seller obtains an advantage by syndicating with other sellers, compensating the disappeared sellers with the competitive price prescribed by the rule, and then bargaining directly with the buyers.

For bilateral Böhm-Bawerk assignment markets the core is a segment determined by the buyers' optimal allocation and the sellers' optimal allocation. There seems to be widespread consensus that the most "fair" solution to such a market game is to select the midpoint of

this latter model. To the best of our knowledge, there is no other property that connects the multilateral case to the bilateral case.

the core (see Núñez and Rafels (2005) for instance). This assertion is supported by the fact that such an allocation coincides with the nucleolus of the game. The following property requires that for every bilateral 2-regular market the nucleolus (or midpoint) of the core is selected.

2-CBS A rule on $\mathcal{BBM}$, f, coincides with the *canonical bilateral solution for 2-regular markets* if, for every bilateral 2-regular market $(\mathbf{c}, w) \in \mathit{2}\text{-}\mathcal{BBG}$,

$$\begin{aligned} \mathsf{f}^1(\mathbf{c}, w) &= \left((w_2 - c_1)_+ + \frac{(w_1 - c_1)_+ - (w_1 - c_2)_+ - (w_2 - c_1)_+}{2}, 0 \right) \in \mathbb{R}^2_+, \\ \mathsf{f}^2(\mathbf{c}, w) &= \left((w_1 - c_2)_+ + \frac{(w_1 - c_1)_+ - (w_1 - c_2)_+ - (w_2 - c_1)}{2}, 0 \right) \in \mathbb{R}^2_+. \end{aligned} \tag{11}$$

We show the promised characterization result in two steps. First we show that if a rule on $\mathcal{BBM}$ exists satisfying CS, 2-VSP, and 2-CBS then it must be unique.

Proposition 4.1. *There is at most one rule on $\mathcal{BBM}$ that satisfies* CS, 2-VSP*, and* 2-CBS.

Proof. Let $(\mathbf{c}, w) \in \mathcal{BBM}$ and suppose that the payoffs in the 2-regular market $(\widetilde{\mathbf{c}}, \widetilde{w}) \in \mathit{2}\text{-}\mathcal{BBM}$ are uniquely determined. Then, from 1) in Remark 4.1 the uniqueness for the payoffs of the remaining agents follows. Hence, it suffices to show that if a rule on $\mathcal{BBM}$ satisfies the properties then the payoffs in every 2-regular market are uniquely determined.

We argue by the counter reciprocal. Let f and g be two rules on $\mathcal{BBM}$ that satisfy CS, 2-VSP, and 2-CBS and such that there is a 2-regular market for which $\mathsf{f}(\mathbf{c}, w) \neq \mathsf{g}(\mathbf{c}, w)$.

We define two rules on $\mathcal{BP}$, fb and gb, as follows. For every $(M, E, d) \in \mathcal{BP}$ and $k \in M$, let

$$\begin{aligned} \mathsf{fb}_k(M, E, d) &= \mathsf{f}^k_1(\mathbf{c}(E, d), w(E, d)) \quad \text{and} \\ \mathsf{gb}_k(M, E, d) &= \mathsf{g}^k_1(\mathbf{c}(E, d), w(E, d)), \end{aligned}$$

where $(\mathbf{c}(E, d), w(E, d))$ is the 2-regular market defined in Eq. (7). Then, fb is a well-defined bankruptcy rule. Indeed, since f satisfies CS,

$$\sum_{k \in M} \mathsf{fb}_k(N, E, d) = \sum_{k \in M} \mathsf{f}^k_1(\mathbf{c}(E, d), w(E, d)) = \omega^{\mathbf{c}(E,d), w(E,d)}(N) = E.$$

Moreover, since a player's payoff in the core is bounded from above by her marginal contribution to the grand coalition, we obtain that, for every $k \in M \setminus \{m\}$,

$$\begin{aligned} \mathsf{fb}_k(M, E, d) &= \mathsf{f}^k_1(\mathbf{c}(E, d), w(E, d)) \leq \omega^{\mathbf{c}(E,d), w(E,d)}(N) - \omega^{\mathbf{c}(E,d), w(E,d)}(N \setminus \{1^k\}) \\ &= \left(w_1(E, d) - \sum_{l \in M \setminus \{m\}} c^l_1(E, d) \right)_+ - \left(w_1(E, d) - \sum_{l \in M \setminus \{k,m\}} (c^l_1(E, d) - c^k_2(E, d)) \right)_+ \\ &\leq c^k_2(E, d) - c^k_1(E, d) = d_k, \end{aligned}$$

The *nucleolus* of a cooperative game is the allocation that minimizes the vector of non-increasingly ordered excesses according to the lexicographic order.

In the rest of the proof we will only need 2) in Remark 4.1, but we may refer to it simply by CS.

and

$$\begin{aligned}\mathsf{fb}_m(M,E,d) &= \mathsf{f}_1^k(\mathbf{c}(E,d),w(E,d)) \le \omega^{\mathbf{c}(E,d),w(E,d)}(N) - \omega^{\mathbf{c}(E,d),w(E,d)}(N\setminus\{1^m\})\\ &= \left(w_1(E,d) - \sum_{l\in M\setminus\{m\}} c_1^l(E,d)\right)_+ - \left(w_2(E,d) - \sum_{l\in M\setminus\{m\}} c_1^l(E,d) - (c_2^k(E,d)-c_1^k(E,d))\right)_+\\ &\le w_1(E,d) - w_2(E,d) = d_m,\end{aligned}$$

where 1^k denotes the first agent of sector $k \in M$. To check that gb is well-defined the same argument is valid.

Second, fb coincides with the Contested Garment Rule for 2-player bakruptcy problems. Indeed, let $(M,E,d)\in\mathcal{BP}$ with $m=2$. Then, since f satisfies 2-CBS,

$$\begin{aligned}\mathsf{fb}_1(M,E,d) &= \mathsf{f}_1^1(\mathbf{c}(E,d),w(E,d)) = (w_2(E,d)-c_1(E,d))_+\\ &\quad + \frac{1}{2}\Big((w_1(E,d)-c_1(E,d))_+ - (w_1(E,d)-c_2(E,d))_+ + (w_2(E,d)-c_1(E,d))_+\Big)\\ &= (E-d_2)_+ + \frac{1}{2}\Big(E-(E-d_1)_+ - (E-d_2)_+\Big) = \mathsf{CG}_1(M,E,d).\end{aligned}$$

and since fb is efficient, $\mathsf{fb}_2(M,E,d) = \mathsf{CG}_2(M,E,d)$. Following the same lines it can be checked that gb also coincides with the Contested Garment Rule.

Third, we show that fb is bilaterally consistent.

Suppose that $S=\{k,m\}$ for some $k\in M\setminus\{m\}$. On the one hand, by definition,

$$\begin{aligned}&\mathsf{fb}_m(\{k,m\},\mathsf{fb}_k(M,E,d)+\mathsf{fb}_m(M,E,d),(d_k,d_m))\\ &= \mathsf{f}_1^m\left(\begin{pmatrix}0\\ d_k\end{pmatrix},\begin{pmatrix}\mathsf{f}_1^k(\mathbf{c}(E,d),w(E,d))+\mathsf{f}_1^m(\mathbf{c}(E,d),w(E,d))\\ \mathsf{f}_1^k(\mathbf{c}(E,d),w(E,d))+\mathsf{f}_1^m(\mathbf{c}(E,d),w(E,d))-d_m\end{pmatrix}\right).\end{aligned} \tag{12}$$

On the other hand, by definition and the fact that f satisfies 2-VSP,

$$\mathsf{fb}_m(M,E,d) = \mathsf{f}_1^m(\mathbf{c}(E,d),w(E,d)) = \mathsf{f}_1^m(\bar{\mathbf{c}}^k(\mathsf{f}(\mathbf{c}(E,d),w(E,d)),w(E,d)). \tag{13}$$

Moreover since f satisfies CS (in particular it shares the whole surplus of the market)

$$\bar{\mathbf{c}}^k(\mathsf{f}(\mathbf{c}(E,d))) = \begin{pmatrix}0+\sum_{l\in M\setminus\{k,m\}}\mathsf{f}_1^l(\mathbf{c}(E,d),w(E,d))\\ d_k+\sum_{l\in M\setminus\{k,m\}}\mathsf{f}_1^l(\mathbf{c}(E,d),w(E,d))\end{pmatrix} = \begin{pmatrix}E-\mathsf{fb}_k(M,E,d)-\mathsf{fb}_m(M,E,d)\\ E+d_k-\mathsf{fb}_k(M,E,d)-\mathsf{fb}_m(M,E,d)\end{pmatrix}.$$

It follows easily that the assignment games associated to the 2-regular markets of the last terms of Eqs. (12) and (13) are indeed equal. Again, since f satisfies CS it allocates the same payoff in two markets whose associated assignment game coincides and hence,

$$\mathsf{fb}_m(M,E,d) = \mathsf{fb}_m(\{k,m\},\mathsf{fb}_k(M,E,d)+\mathsf{fb}_m(M,E,d),(d_k,d_m)).$$

The Contested Garment Rule (also known as Concede and Divide), CG, introduced in Aumann and Maschler (1985) is the bankruptcy rule defined for two agents problems, $(M,E,d)\in\mathcal{BP}$ with $m=2$ by

$$\mathsf{CG}(M,E,d) = \Big((E-d_2)_+ + \frac{1}{2}\left(E-(E-d_1)_+ - (E-d_2)_+\right), (E-d_1)_+ + \frac{1}{2}\left(E-(E-d_1)_+ - (E-d_2)_+\right)\Big)$$

A bankruptcy rule, fb, is *consistent* if for every $(M,E,d)\in\mathcal{BP}$ and $S\subseteq M$, if we set $x=\mathsf{fb}(M,E,d)$, then $x_S = \mathsf{fb}(S,x(S),d_S)$. A bankruptcy rule is *bilaterally consistent* if it is consistent for every $S\subseteq M$ with $|S|=2$.

By notational convenience we may write vectors in columns.

Finally since fb is a well-defined rule on $\mathcal{BP}$ it satisfies bilateral consistency when $S = \{k, m\}$.

Next, suppose that $S = \{k, l\}$ with $k, l \in M \setminus \{m\}$. On the one hand, by notational convenience let $A := \mathsf{f}_1^k(\mathbf{c}(E,d), w(E,d))$ and $B := \mathsf{f}_1^m(\mathbf{c}(E,d), w(E,d))$. Then,

$$\begin{aligned}
\mathsf{fb}_k(M,E,d) &= \mathsf{f}_1^k(\mathbf{c}(E,d), w(E,d)) = \mathsf{f}_1^k(\bar{\mathbf{c}}^k(E,d), w(E,d)) \\
&= \frac{1}{2}\left[E - \sum_{j\neq k,m} \mathsf{f}_1^j(\mathbf{c}(E,d), w(E,d)) - \left(E - \sum_{j\neq k,m} \mathsf{f}_1^j(\mathbf{c}(E,d), w(E,d)) - d_k\right)_+ \right. \\
&\quad \left. + \left(E - \sum_{j\neq k,m} \mathsf{f}_1^j(\mathbf{c}(E,d), w(E,d)) - d_m\right)_+ \right] \\
&= \frac{1}{2}\left[A + B - (A+B-d_k)_+ + (A+B-d_m)_+\right],
\end{aligned} \tag{14}$$

where the second equality holds by 2-VSP (Remark 4.3), the third equality holds by CBS, and the last equality holds by CS (only by efficiency). Next, following the same lines

$$\mathsf{fb}_m(M,E,d) = \frac{1}{2}\left[A + B + (A+B-d_k)_+ - (A+B-d_m)_+\right]. \tag{15}$$

Hence, Eqs. (14) and (15) together define the following system of equations:

$$\begin{cases} 2A = A + B - (A+B-d_k)_+ + (A+B-d_m)_+ \\ 2B = A + B + (A+B-d_k)_+ - (A+B-d_m)_+ \end{cases} \tag{16}$$

On the other hand, let $\pi : M \longrightarrow M$ be the permutation defined by $\pi(m) = k$, $\pi(k) = m$ and $\pi(j) = j$ for every $j \in M \setminus \{k, m\}$. Let $(\pi(M), E, d^\pi) \in \mathcal{BP}$, where $d_j^\pi = d_{\pi(j)}$ for every $j \in M$. by notational convenience let $A^\pi = \mathsf{f}_1^k(\mathbf{c}(E,d^\pi), w(E,d^\pi))$ and $B^\pi = \mathsf{f}_1^m(\mathbf{c}(E,d^\pi), w(E,d^\pi))$, then following the same lines as in Eq. (14)

$$\begin{aligned}
\mathsf{fb}_m(\pi(M),E,d^\pi) &= \mathsf{f}_1^m(\mathbf{c}(E,d^\pi), w(E,d^\pi)) = \mathsf{f}_1^m(\bar{\mathbf{c}}^k(E,d^\pi), w(E,d^\pi)) \\
&= \frac{1}{2}\left[E - \sum_{j\neq k,m} \mathsf{f}_1^j(\mathbf{c}(E,d^\pi), w(E,d^\pi)) + \left(E - \sum_{j\neq k,m} \mathsf{f}_1^j(\mathbf{c}(E,d^\pi), w(E,d^\pi)) - d_m\right)_+ \right] \\
&\quad - \left(E - \sum_{j\neq k,m} \mathsf{f}_1^j(\mathbf{c}(E,d^\pi), w(E,d^\pi)) - d_k\right)_+ \\
&= \frac{1}{2}\left[A^\pi + B^\pi + (A^\pi+B^\pi-d_m)_+ - (A^\pi+B^\pi-d_k)_+\right].
\end{aligned} \tag{17}$$

Following the same lines as in Eq. (15),

$$\mathsf{f}_1^k(\mathbf{c}(E,d^\pi), w(E,d^\pi)) = \frac{1}{2}\left[A + B - (A+B-d_m)_+ + (A+B-d_k)_+\right]. \tag{18}$$

Hence, Eqs. (17) and (18) together define the following system of equations:

$$\begin{cases} 2B^\pi = A^\pi + B^\pi + (A^\pi+B^\pi-d_m)_+ - (A^\pi+B^\pi-d_k)_+ \\ 2A^\pi = A^\pi + B^\pi - (A^\pi+B^\pi-d_m)_+ + (A^\pi+B^\pi-d_k)_+ \end{cases} \tag{19}$$

Note that the two systems of equations are equal and solvable, so we obtain that

$$\begin{aligned}
\mathsf{fb}_k(M,E,d) = A = B^\pi = \mathsf{fb}_m(\pi(M),E,d^\pi) &= \mathsf{fb}_m(\{l,m\}, \mathsf{fb}_l(E,d) + \mathsf{fb}_m(E,d), (d_l, d_m)) \\
&= \mathsf{fb}_k(\{l,k\}, \mathsf{fb}_l(E,d) + \mathsf{fb}_k(E,d), (d_l, d_k)),
\end{aligned}$$

where the fourth equality holds from the proof of the case $m \in S$ above and the last equality is obtained following the same reasoning done with the systems of equations above. Since fb is a well-defined rule on $\mathcal{BP}$ (efficiency) we conclude that it satisfies bilateral consistency also when $S = \{k, l\}$. The same argument is also valid for gb and shows that it also satisfies bilateral consistency.

Lastly, from the characterization result in Aumann and Maschler (1985), which states that there is a unique rule on $\mathcal{BP}$ that satisfies bilateral consistency and coincides with the Contested Garment Rule, we contradict the inequality in the beginning of the proof.

□

To present the characterization result we need to show that there is a rule on $\mathcal{BBM}$ that satisfies the properties. To do so, we first propose the so-called *Talmud Assignment Rule*.

Definition 4.1. *The* Talmud Assignment Rule, T, *is the rule on* $\mathcal{BBM}$ *that satisfies Condition 1) in Remark 4.1 and that associates to every 2-regular market,* $(\mathbf{c}, w) \in 2\text{-}\mathcal{BBG}$, *the following payoffs:*

- *If* $\sum_{k=1}^{m-1}(c_1^k + c_2^k) \geq w_1 + w_2$, *then*

$$T^k(\mathbf{c}, w) = \begin{cases} \left(\min\left\{\frac{c_2^k - c_1^k}{2}, \lambda\right\}, 0\right) & \text{if } k \in M \setminus \{m\}, \\ \left(\min\left\{\frac{w_1 - w_2}{2}, \lambda\right\}, 0\right) & \text{if } k = m, \end{cases}$$

- *If* $\sum_{k=1}^{m-1}(c_1^k + c_2^k) \leq w_1 + w_2$, *then*

$$T^k(\mathbf{c}, w) = \begin{cases} \left(c_2^k - c_1^k - \min\left\{\frac{c_2^k - c_1^k}{2}, \lambda\right\}, 0\right) & \text{if } k \in M \setminus \{m\}, \\ \left(w_1 - w_2 - \min\left\{\frac{w_1 - w_2}{2}, \lambda\right\}, 0\right) & \text{if } k = m, \end{cases}$$

where λ *is chosen so that* $\sum_{k \in M} \sum_{i \in N^k} T_i^k(\mathbf{c}, w) = \omega^{\mathbf{c}, w}(N)$.

Recall that, by definition of 2-regular markets, whenever $r > 0$ it holds that $\sum_{k=1}^{m-1} c_1^k < w_1$ and $\sum_{k=1}^{m-1} c_2^k \geq w_2$. Also notice that $c_2^k - c_1^k$, with $k \in M \setminus \{m\}$, can be interpreted as a measure of the bargaining power of the firm in sector k with the lowest cost with respect to the firm in the same sector with the largest cost. A similar argument applies to $w_1 - w_2$. The interpretation of the Talmud Assignment Rule is the following: *(i)* if aggregate costs are larger than aggregate buyers valuations, i.e. $\sum_{k=1}^{m-1}(c_1^k + c_2^k) \geq w_1 + w_2$, sellers with the smallest cost in every sector and the buyer with largest valuation are assigned equal payoffs subject to no one receiving more than half of its bargaining power, and *(ii)* if aggregate costs are smaller than aggregate buyers valuations, i.e. $\sum_{k=1}^{m-1}(c_1^k + c_2^k) \leq w_1 + w_2$, sellers with the smallest cost in every sector and the buyer with largest valuation are subtracted the same utility from their bargaining power subject to no one receiving less than half of its bargaining power.

We next prove that CS, 2-MP, and 2-CBS are compatible properties for rules on $\mathcal{BBM}$.

Proposition 4.2. *The Talmud assignment rule,* T, *satisfies* CS, 2-VSP, *and* 2-CBS.

In Aumann and Maschler (1985) bilateral consistency is denoted consistency

Proof. Note that T can be written, for every $(\mathbf{c}, w) \in 2\text{-}\mathcal{BBM}$ and $k \in M$, as

$$\mathsf{T}^k(\mathbf{c}, w) = (\mathsf{Tb}_k(M, E(\mathbf{c}, w), d(\mathbf{c}, w)), 0), \tag{20}$$

where Tb is the Talmud Bankruptcy Rule (Aumann and Maschler, 1985). Then, since Tb selects exactly the nucleolus of the bankruptcy game and the nucleolus of a game always belongs to the core whenever the core is nonempty. Then, from Theorem 3.1, it follows that $\mathsf{T}(\mathbf{c}, w) \in C(N, \omega^{\mathbf{c},w})$. That is, T satisfies CS.

Next, we show that T satisfies 2-VSP. Note that we only need to check it for $i = 1$. On the one hand, from Eq. (8) and the efficiency of the Talmud Bankruptcy Rule,

$$\begin{aligned} E\left(\bar{c}^k(\mathsf{T}(\mathbf{c}, w)), w\right) &= \left(w_1 - \sum_{l \in M\setminus\{m\}} c_1^l - \sum_{l \in M\setminus\{k,m\}} \mathsf{T}_1^k(\mathbf{c}, w)\right)_+ \\ &= E(\mathbf{c}, w) - \sum_{l \in M\setminus\{k,m\}} \mathsf{Tb}_l(M, E(\mathbf{c}, w), d(\mathbf{c}, w)) \\ &= \mathsf{Tb}_k(M, E(\mathbf{c}, w), d(\mathbf{c}, w)) + \mathsf{Tb}_m(M, E(\mathbf{c}, w), d(\mathbf{c}, w)). \end{aligned}$$

On the other hand, it can be easily checked that

$$d_k(\bar{c}^k(\mathsf{T}(\mathbf{c}, w)), w) = d_k(\mathbf{c}, w) \quad \text{and} \quad d_m(\bar{c}^k(\mathsf{T}(\mathbf{c}, w)), w) = d_m(\mathbf{c}, w)$$

Thus, from the equations above, Eq. (20), and the fact that Tb is bilaterally consistent

$$\begin{aligned} &\mathsf{T}_1^m(\mathbf{c}, w) = \mathsf{Tb}_m(M, E(\mathbf{c}, w), d(\mathbf{c}, w)) \\ &\quad = \mathsf{Tb}_m(\{k, m\}, \mathsf{Tb}_k(M, E(\mathbf{c}, w), d(\mathbf{c}, w)) + \mathsf{Tb}_m(M, E(\mathbf{c}, w), d(\mathbf{c}, w)), (d_k(\mathbf{c}, w), d_m(\mathbf{c}, w))) \\ &\quad = \mathsf{Tb}_m(\{k, m\}, E\left(\bar{c}^k(\mathsf{T}(\mathbf{c}, w)), w\right), w), (d(\bar{c}^k(\mathsf{T}(\mathbf{c}, w)), w), d_m(\bar{c}^k(\mathsf{T}(\mathbf{c}, w)), w))) \\ &\quad = \mathsf{T}_1^m(\bar{c}^k(\mathsf{T}(\mathbf{c}, w)), w) \end{aligned}$$

Finally, T trivially satisfies 2-CBS by Eq. (20) and the fact that Tb coincides with the Contested Garment Rule. □

The characterization result follows as a direct consequence of Propositions 4.1 and 4.2.

Theorem 4.1. *The Talmud Assignment Rule is the unique rule on $\mathcal{BBM}$ that satisfies* CS, 2-VSP*, and* 2-CBS.

We next show that no property can be left out from the characterization result in Theorem 4.1, so the set of properties used to characterize the Talmud Assignment Rule is minimal with respect to inclusion.

Lemma 4.1. *The properties that characterize the Talmud Bankruptcy Rule,* T*, are logically independent.*

The *Talmud Bankruptcy Rule* is defined for every $(M, E, d) \in \mathcal{BP}$ and each $i \in N$

- If $\sum_{j\in N} d_j/2 \geq E$, $\mathsf{Tb}_i(M, E, d) = \min\{d_i/2, \lambda\}$,
- If $\sum_{j\in N} d_j/2 \leq E$, $\mathsf{Tb}_i(M, E, d) = d_i - \min\{d_i/2, \lambda\}$,

where λ is chosen so that $\sum_{j\in M} \mathsf{Tb}_j(M, E, d) = E$.

Proof. Let us consider the following three rules on $\mathcal{BBM}$:

(i) Let the rule on $\mathcal{BBM}$, $\hat{\mathsf{T}}$, be defined, for every $(\mathbf{c}, w) \in \mathcal{BBM}$, as

$$\hat{\mathsf{T}}(\mathbf{c}, w) = \begin{cases} \mathsf{T}(\mathbf{c}, w) & \text{if } (\mathbf{c}, w) \in \mathit{2}\text{-}\mathcal{BBM}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then, $\hat{\mathsf{T}}$ satisfies 2-VSP and 2-CBS but not CS.

(ii) Let the rule on $\mathcal{BBM}$, $\bar{\mathsf{T}}$, assign, for every $(\mathbf{c}, w) \in \mathcal{BBM}$ the core-center (see González-Díaz and Sánchez-Rodríguez (2007)) of $(N, \omega^{\mathbf{c},w})$. By Núñez and Rafels (2005) and Tejada and Núñez (2012) we know that this allocation is different from the nucleolus (of the game) for the whole class of market games in $\mathcal{BBG}$, but that they coincide for bilateral markets. Then, $\bar{\mathsf{T}}$ satisfies CS and 2-CBS, but not 2-VSP.

(iii) On the one hand, consider the following subset of *2*-$\mathcal{BBM}$:

$$\mathcal{V} := \{(\mathbf{c}, w) \in \mathit{2}\text{-}\mathcal{BBM} : w_2 = 0\}.$$

Note that for every $k \in M \setminus \{m\}$ and every rule on $\mathcal{BBM}$, f, $(\mathbf{c}, w) \in \mathcal{V}$ if and only if $(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w) \in \mathcal{V}$. Let $\tilde{\mathsf{T}}$ be the rule on $\mathcal{BBM}$ that selects, for every $(\mathbf{c}, w) \in \mathcal{BBM}$, the translation by the isomorphism defined in Theorem 2.1 of the marginal worth vector of $(M, v^{\mathbf{c},w})$ associated to the order $(1, 2, \ldots, m)$, i.e., sector $1 \in M$ comes first, sector $2 \in M$ comes second, and so on and so forth. By Part 2 of Theorem 1 in Tejada (to appear) we know that the proposed allocation is an extreme core allocation of $(N, v^{\mathbf{c},w})$. Moreover, from the definitions of a marginal worth vector and the sectors game, it holds that

$$\tilde{T}_i^k(\mathbf{c}, w) = 0 \text{ for any } (\mathbf{c}, w) \in \mathcal{V}, k \in M \setminus \{m\}, i \in N^k. \tag{21}$$

Lastly, let the rule on $\mathcal{BBM}$, $\tilde{\tilde{\mathsf{T}}}$, be defined, for every $(\mathbf{c}, w) \in \mathcal{BBM}$ by

$$\tilde{\tilde{\mathsf{T}}}(\mathbf{c}, w) = \begin{cases} \tilde{\mathsf{T}}(\mathbf{c}, w) & \text{if } (\mathbf{c}, w) \in \mathcal{V}, \\ \mathsf{T}(\mathbf{c}, w) & \text{otherwise.} \end{cases}$$

By construction, it immediately follows that $\tilde{\tilde{\mathsf{T}}}$ satisfies CS. Given $(\mathbf{c}, w) \in \mathcal{V}$, we obtain

$$\begin{aligned}\tilde{\tilde{\mathsf{T}}}_1^m(\mathbf{c}, w) =& v^{\mathbf{c},w}(M) - v^{\mathbf{c},w}(M \setminus \{m\}) = \left(w_1 - \sum_{l \in M \setminus \{m\}} c_1^l\right)_+ - \left(w_2 - \sum_{l \in M \setminus \{m\}} c_1^l\right)_+ \\ =& \left(w_1 - \sum_{l \in M \setminus \{m\}} c_1^l\right)_+ = \left(w_1 - \bar{c}_1^k(\mathsf{f}(\mathbf{c}, w)) + \sum_{l \in M \setminus \{k,m\}} f_1^l(\mathbf{c}, w)\right)_+ \\ =& \left(w_1 - \bar{c}_1^k(\mathsf{f}(\mathbf{c}, w))\right)_+ = \left(w_1 - \bar{c}_1^k(\mathsf{f}(\mathbf{c}, w))\right)_+ - \left(w_2 - \bar{c}_1^k(\mathsf{f}(\mathbf{c}, w))\right)_+ \\ =& v^{\bar{c}^k(\mathsf{f}(\mathbf{c},w)),w}(k, m) - v^{\bar{c}^k(\mathsf{f}(\mathbf{c},w)),w}(\{k\}) = \tilde{\tilde{\mathsf{T}}}_1^m(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w),\end{aligned}$$

where the fifth equality holds by Eq. (21). Thus, $\tilde{\tilde{\mathsf{T}}}$ satisfies 2-VSP. We note that it is as an easy exercise to check that $\mathcal{V}$ contains bilateral 2-regular markets with a core being a non-degenerate segment, which implies that $\tilde{\tilde{\mathsf{T}}}$ is really a different solution than the Talmud Assignment Rule, T. Therefore, according to Theorem 4.1, $\tilde{\tilde{\mathsf{T}}}$ does not satisfy 2-CBS.

□

It is worth mentioning that, as a consequence of Theorem 3 in Tejada and Núñez (2012), we know that the Talmud Assignment Rule, T, selects for each arbitrary Böhm-Bawerk assignment market precisely the nucleolus of the corresponding assignment game. Thus, Theorem 4.1 provides a syndication-based foundation for the nucleolus as the appropriate solution concept for price stability in multilateral markets.

Theorem 4.1 also implies that for any rule on $\mathcal{BBM}$, f, different than T that satisfies CS and 2-VSP there is a 2-regular market for which 2-VSP does not hold. There are two possibilities: either the formation of the syndicate is advantageous for the sellers or it is disadvantageous. Following the techniques in the proof of Proposition 4.1 it can be proved that there is always a Böhm-Bawerk market in which vertical syndicating is disadvantageous, so there is no escape to Aumman's paradox (Aumann, 1973) outside from the nucleolus.

We illustrate the result in Theorem 4.1 by means of the following graph, in which the syndication-proof prices for both commodities are shown for the market in Example 2.1, with the modification that the valuation of the sixth buyer ranges, w_6, from 110 to 125.

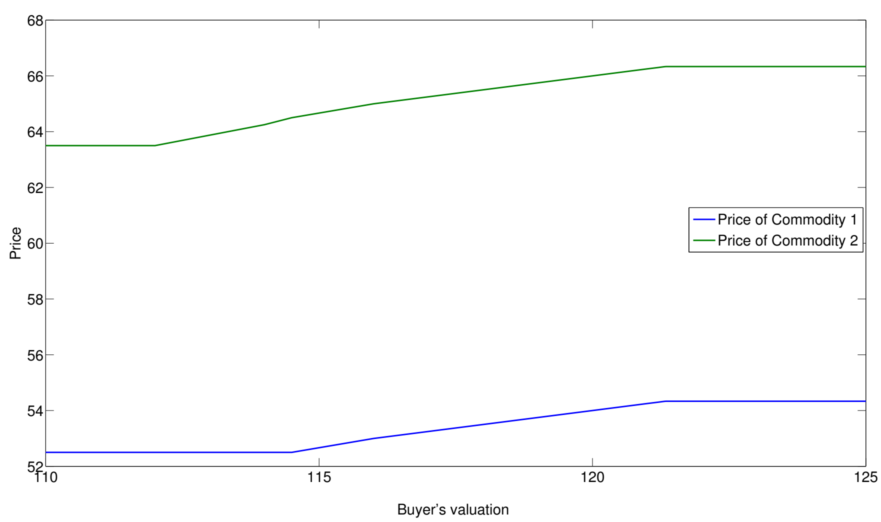

Figure 1: Behavior of vertical syndication-proof prices in the market of Example 2.1 for varying buyer's valuation.

Loosely speaking the argument goes as follows. First, take a Böhm-Bawerk market $(\mathbf{c}, w)$ such that $\mathsf{f}^m(\mathbf{c}, w) \lneq \mathsf{f}^m\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right)$. Second, define another Böhm-Bawerk market $(\mathbf{c}', w')$ with the property that $\mathsf{f}^i(\mathbf{c}, w) = \mathsf{f}^j(\mathbf{c}', w')$ and $\mathsf{f}^i\left(\bar{c}^k(\mathsf{f}(\mathbf{c}, w)), w\right) = \mathsf{f}^j\left(\bar{c'}^k(\mathsf{f}(\mathbf{c}', w')), w'\right)$ for $\{i, j\} = \{k, m\}$, with $k \neq m$.

We note that when $w_6 > 120$ the buyer is, according to our ordering assumptions, the fifth buyer.

We point out that the prices for the particular case that corresponds exactly to Example 2.1 are:

$$p^1 = 54 + \frac{1}{3} \quad \text{and} \quad p^2 = 66 + \frac{1}{3}.$$

5 Other Ways of Syndication

In the present section we investigate an extension and a variation of the syndication-proofness property used so far, i.e., 2-VSP. First, we note that, as it is stated, 2-VSP only applies to 2-regular markets and, moreover, it does not allow for syndication of a number of sellers lower than the number of commodities in the market. Second, as discussed in the Introduction, a syndicate may be created among sellers of different commodities – vertical syndication –, but also among sellers of the same commodity – horizontal syndication.

We present one positive result and one impossibility result. First, we show that the Talmud Assignment Rule satisfies a stronger version of 2-VSP, which applies to general Böhm-Bawerk markets and permits the emergence of vertical syndicates with an arbitrary number of members, possibly lower than the number of commodities in the market. Second, we show that among the solution concepts that satisfy 2-CBS and CS there is none that simultaneously satisfies vertical and horizontal syndication-proofness.

Let us formally introduce the properties above described. We start by considering one possible extension of 2-VSP, the interpretation of which resembles that of the former property. Given $S \subseteq M \setminus \{m\}$, let $N^{S \cup \{m\}} := (N^l)_{l \in M \setminus (S \cup \{m\})}$ and $\mathbf{c}^{M \setminus (S \cup \{m\})} := (c^l)_{l \in M \setminus (S \cup \{m\})}$.

SVSP A rule on $\mathcal{BBM}$, f, satisfies *strong vertical syndication-proof* if, for every $(\mathbf{c}, w) \in \mathcal{BBM}$, every $\emptyset \neq S \subseteq M \setminus \{m\}$ and some $k \in S$,

$$\mathsf{f}^m(N^1, \ldots, N^m, \mathbf{c}, w) = \mathsf{f}^m\left(N^k, N^{S \cup \{m\}}, N^m, \bar{c}^k(\mathsf{f}(\mathbf{c}, w)), \mathbf{c}^{M \setminus (S \cup \{m\})}, w\right),$$

where the *reduced* $(m - |S| + 1)$*-sided market*

$$\left(N^k, (N^l)_{l \in M \setminus (S \cup \{m\})}, N^m, \bar{c}^k(\mathsf{f}(\mathbf{c}, w)), \mathbf{c}^{M \setminus (S \cup \{m\})}, w\right)$$

is composed of the same sector of buyers, with valuations given by w, of the same sectors of sellers N^l, $l \in M \setminus (S \cup \{m\})$, with costs given by $(c^l)_{l \in M \setminus (S \cup \{m\})}$, and a new sector of sellers, N^k, with costs, $\bar{c}^k(\mathsf{f}(\mathbf{c}, w))$, defined for every $i \in N^k$ by

$$\bar{c}_i^k(\mathsf{f}(\mathbf{c}, w)) = c_i^k + \sum_{l \in S \setminus \{k\}} p^l(\mathsf{f}(\mathbf{c}, w)).$$

We note that, following the lines of Remark 4.2, it can be checked that the above reduced market is a 2-regular market whenever f satisfies CS and $(\mathbf{c}, w) \in$ *2-*$\mathcal{BBM}$.

As mentioned in the Introduction, our approach in this paper is to consider that the defining features of a Böhm-Bawerk market, with special emphasis on the one outlined in Section 4, are to be preserved after any syndicate is created. Particularly this implies that the creation of a syndicate by some sellers of the same commodity – what we have called horizontal syndication – cannot have any consequence on the market as long as the whole pool of goods owned by the syndicate remains available for all buyers.

In order to make horizontal syndication more interesting we consider that a syndicate of sellers of the same commodity can decide not to dump into the market the whole pool of goods. By doing so the syndicate can affect the market − and hence the associated cooperative game – and possibly obtain an additional benefit. In particular, we assume that members of horizontal syndicate can always decide to behave as if they were not syndicated, so, by definition, an horizontal syndicate can never be disadvantageous for its members. For the sake of simplicity, we state the property only for 2-regular markets.

2-HSP A rule on $\mathcal{BBM}$, f, satisfies *horizontal syndication-proofness for 2-regular markets* if, for every $(\mathbf{c}, w) \in$ *2-*$\mathcal{BBM}$, every $k \in M \setminus \{m\}$ and every $\emptyset \neq S \subseteq N^k$,

$$\mathsf{f}^m(N^1, \ldots, N^m, \mathbf{c}, w) = \mathsf{f}^m(S, (N^l)_{l \in M \setminus \{k,m\}}, N^m, \tilde{c}^{k,S}, \mathbf{c}^{M \setminus \{k,m\}}, w), \quad (22)$$

and

$$\sum_{i \in N^k} \mathsf{f}_i^k(\mathbf{c}, w) \leq \sum_{i \in S} \mathsf{f}_i^k(\tilde{c}^{k,S}, \mathbf{c}^{M \setminus \{k,m\}}, w), \quad (23)$$

where $\mathbf{c}^{M \setminus \{k,m\}} = (c^l)_{l \in M \setminus \{k,m\}}$ and $\tilde{c}^{k,S} = (c_i^k)_{i \in S}$.

The above property, 2-HSP, requires the benefits assigned by a rule to each of the buyers not be affected by the horizontal syndicate composed of all sellers in N^k when they decide to dump into the markets only the goods that belong to sellers in S – see Eq. (22) – as long as the syndicate behaves in a rational way, i.e., the aggregate payoff to all agents in N^k does not decrease – see Eq. (23).

We show now the following results concerning the above two properties.

Proposition 5.1. *The Talmud Assignment Rule,* T*, satisfies* SVSP.

Proof. First, we show that T satisfies SVSP for 2-regular markets. Indeed, for such a market it can be easily checked, following the lines in the proof of Proposition 4.2 and using consistency instead of bilateral consistency..

Second, consider an arbitrary multisided Böhm-Bawerk assignment market. Since T satisfies CS, it immediately follows from Remark 4.1 and the above part that T satisfies SVSP. □

Proposition 5.1 reveals that the Talmud Assignment Rule is vertical syndication-proof for all Böhm-Bawerk assignment markets, but Theorem 4.1 specifies that this stronger property is not necessary to characterize the rule.

Proposition 5.2. *Let* f *be a rule on* $\mathcal{BBM}$ *that satisfies* 2-CBS *and* CS*. Then* f *does not satisfy simultaneously* 2-VSP *and* 2-HSP .

Notice that for single sellers or vertical syndicates such manipulation can never be profitable since it suppose a renounce to make any profit.

This assumption is in sharp contrast with the usual assumptions in the literature. Nevertheless, in a Böhm-Bawerk assignment market all buyers have a strong market power – inherited from the fact that they are indifferent between who they buy the goods from – which permits us to make such an assumption.

To be consistent with the assumptions along the paper, dummy sellers might be added to sector k so that there is the same number of agents in all sectors.

It is well-known that Tb is consistent (Thomson, 2003). A comprehensive argument for this part can be provided by the authors upon request.

Proof. Consider the bilateral 2-regular market $(\mathbf{c}, w)$ defined by $\mathbf{c} = c^1 = (1, 2)$ and $w = (3, 0)$, and let $k = 1$ and $S = \{1\} \subseteq N^1$. By Theorem 4.1 we know that if f exists, it has to coincide with the Talmud Assignment Rule, T, which can be easily computed from the nucleolus of the corresponding sectors game. It follows that

$$\mathsf{T}_1^1(c^1, w) + \mathsf{T}_2^1(c^1, w) = \frac{1}{2} \leq \frac{3}{2} = \mathsf{T}_1^1(\tilde{c}^{1,S}, w) + \mathsf{T}_2^1(\tilde{c}^{1,S}, w),$$

and

$$\mathsf{T}^2(c^1, w) = \left(\frac{3}{2}, 0\right) \neq (1, 0) = \mathsf{T}^2(\tilde{c}^{1,S}, w),$$

which concludes the proof. □

We note that, when $m = 3$, 2-VSP and SVSP are equivalent properties when applied to 2-regular markets. The main conclusion we can extract from Proposition 5.2 is the non-existence, for the market games considered in this paper and also for any superclass containing them, of a rule that specifies prices so that buyers are immune to any arbitrary syndication of sellers.

6 Conclusion

This paper connects three different strands of the literature. First, it deals with syndication in a certain class of market games. As discussed in the Introduction, the role of syndication in markets was challenged by the disturbing paper by Aumann (1973). Syndicates have since been extensively tackled from both a theoretical and empirical points of view, the main goal of these efforts being the understanding of the formation process and stability of syndicates. We also note that to the best of our knowledge, there was a lack of a foundation for the use of the nucleolus in markets.

Second, the class of market games considered in this paper lies within a specific class of multi-sided assignment games that possess some nice properties and have a clear economic interpretation. The literature on multi-sided assignment games is relatively short, and has mainly focused on finding sufficient conditions that guarantee the non-emptiness of the core but has not been able to discover, among many other issues, a connection between assignment games of different number of sides.

Third, a key result of the paper reveals a (surprising) connection between multi-sided assignment games and bankruptcy games. Whereas few papers are concerned with the latter since they were established by Quint (1991), not least because of the higher complexity that these latter problems involve, for bankruptcy games there are available a large number of rules and characterization results. Exploiting this link appears to be an adequate direction to explore in the future.

All in all, besides shedding some light on the three open problems pointed out above, the main contributions of this paper are *(i)* to bring together the three aforementioned strands of the literature and *(ii)* to find positive and negative results concerning the role of (vertical

A worthy exception is Tejada and Rafels (2010), that proves the existence of certain proper subset of the core for arbitrary balanced multi-sided assignment games.

and horizontal) syndication in multilateral markets, that can serve as baseline results for future research on the role of syndicates.

Appendix

Proof of Proposition 2.1 .

First of all note that, due to Remark 2.1,

$$\mathcal{SG} = \left\{ (M, v^{\mathbf{c},w}) : (N^1, \ldots, N^m, \mathbf{c}, w) \in \mathit{2}\text{-}\mathcal{BBM}, M = \{1, \ldots, m\} \right\},$$

and, by Definition 2.3,

$$\mathit{2}\text{-}\mathcal{BBG} = \left\{ (N, \omega^{\mathbf{c},w}) : (N^1, \ldots, N^m, \mathbf{c}, w) \in \mathit{2}\text{-}\mathcal{BBM}, N = \cup_{k=1}^m N^k \right\}.$$

Second, consider the following mapping:

$$\begin{array}{rrcl} \Psi : & \mathcal{SG} & \to & \mathit{2}\text{-}\mathcal{BBG} \\ & (M, v^{\mathbf{c},w}) & \mapsto & (N, \omega^{\mathbf{c},w}) \end{array}$$

It is straightforward to check that Ψ is surjective. To check that it is also injective, let $(M, v^{\mathbf{c},w}), (M', v^{\mathbf{c}',w'}) \in \mathcal{SG}$ such that

$$\Psi(M, v^{\mathbf{c},w}) = \Psi(M', v^{\mathbf{c}',w'}). \tag{24}$$

From Eq. (24) we immediately deduce that $M = M'$. On the one hand, suppose that the parameter r – see Eq. (3) – associated to $(M, v^{\mathbf{c},w})$ is 0. Then both the sectors game and the assignment game associated to $(M, v^{\mathbf{c},w})$ are null games. Due to Eq. (24), the assignment game associated to $(M, v^{\mathbf{c}',w'})$ is the null game. This implies that the sectors game associated to $(M, v^{\mathbf{c}',w'})$ is also the null game, so $(M, v^{\mathbf{c},w}) = (M, v^{\mathbf{c}',w'})$.

On the other hand, assume that the parameter r is 1 for both markets. We prove by contradiction that $(M, v^{\mathbf{c},w}) = (M, v^{\mathbf{c}',w'})$. Suppose on the contrary that

$$(M, v^{\mathbf{c},w}) \neq (M, v^{\mathbf{c}',w'}). \tag{25}$$

Then there is $S \subseteq M$ such that $v^{\mathbf{c},w}(S) \neq v^{\mathbf{c}',w'}(S)$. Let $A(\mathbf{c}, w) = \{a_E\}_{E \in \prod_{k=1}^m N^k}$ and $A(\mathbf{c}', w') = \{a'_E\}_{E \in \prod_{k=1}^m N^k}$ be the corresponding assignment matrices. From Eq. (25) we deduce that $a_{E^S} \neq a'_{E^S}$. Next, let $T \subseteq N$ be the coalition of N that contains all the first agents of the sectors in S and the second agents of the sectors not in S, i.e., $T = \{1^k \in N^k : k \in S\} \cup \{2^k \in N^k : k \in M \setminus S\}$. Then, it is holds that

$$\omega^{\mathbf{c},w}(T) = a_{E^S} \neq a'_{E^S} = \omega^{\mathbf{c}',w'}(T),$$

which contradicts Eq. (24) and concludes the proof.

$\square$

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