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Monge assignment games

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Abstract

An assignment game is defined by a matrix A, where each row represents a buyer and each column a seller. If buyer i is matched with seller j, the market produces a_{ij} units of utility. We study Monge assignment games, that is bilateral cooperative assignment games where the assignment matrix satisfies the Monge property. These matrices can be characterized by the fact that in any submatrix of 2×2 an optimal matching is placed in its main diagonal. For square markets, we describe their cores by using only the central tridiagonal band of the elements of the matrix. We obtain a closed formula for the buyers-optimal and the sellers-optimal core allocations. Non-square markets are analyzed also by reducing them to appropriate square matrices.

Key words: assignment game, core, Monge matrix, buyers-optimal core allocation, sellers-optimal core allocation

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Resumen

Un juego de asignación se define por una matriz A, donde cada fila representa un comprador y cada columna un vendedor. Si el comprador i se empareja a un vendedor j, el mercado produce a_{ij} unidades de utilidad. Estudiamos los juegos de asignación de Monge, es decir, aquellos juegos bilaterales de asignación en los cuales la matriz satisface la propiedad de Monge. Estas matrices pueden caracterizarse por el hecho de que en cualquier submatriz 2×2 un emparejamiento óptimo está situado en la diagonal principal. Para mercados cuadrados, describimos sus núcleos utilizando sólo la parte central tridiagonal de elementos de la matriz. Obtenemos una fórmula cerrada para el reparto óptimo de los compradores dentro del núcleo y para el reparto óptimo de los vendedores dentro del núcleo. Analizamos también los mercados no cuadrados reduciéndolos a matrices cuadradas apropiadas.

Palabras clave: juego de asignación, núcleo, matriz Monge, reparto óptimo para los vendedores, reparto óptimo para los compradores

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1. Introduction

The optimal (linear sum) assignment problem is that of finding an optimal matching, given a matrix that collects the potential profit of each pair of agents. Some examples are the placement of workers to jobs, of students to colleges, of physicians to hospitals or the pairing of men and women in marriage. Once an optimal matching has been found, one question arises: how to share the output among the partners. This question, that has been mainly addressed in the field of game theory, was first considered in Shapley and Shubik (1972). They associate to each assignment problem a cooperative game, or game in coalitional form.

In the assignment game, each coalition of agents must consider the maximum profit they could attain by themselves as the worth of this coalition. The most relevant solution concept in cooperative games is the core. The core of a game consists of those allocations of the optimal profit (the worth of the grand coalition) such that no subcoalition can improve upon. Thus, if we agree to share the profit of cooperation by means of a core allocation, no coalition has incentives to depart from the grand coalition and act on its own. Shapley and Shubik prove that the core of the assignment game is a nonempty polyhedral convex set and it coincides with the set of solutions of the dual linear program related to the linear sum optimal assignment problem.

In this paper we study the assignment games, called Monge assignment games, where the matrix satisfies what is called the Monge property. Roughly speaking, the (inverse) Monge property is described by the fact that each 2×2 submatrix has an optimal matching in the main diagonal. This property can also be identified as the supermodularity of the matrix, interpreted as a function on the product of the set of indices with the usual order.

Monge matrices have been used in different fields in Operations Research, such as combinatorial optimization (see Burkard et al., 1996 or Burkard, 2007), coalitional game theory (see Okamoto, 2004), algorithmic issues (see Bein et al., 2005), or statistics (see Hou and Prékopa, 2007).

For square Monge assignment games, the central tridiagonal band of the matrix, that is the main diagonal, the sub-diagonal and the super-diagonal, is sufficient to determine the core. As a result, and differently to the general case, not all inequalities are necessary to describe the core explicitly, and in this case the buyer-seller exact representative of the matrix (Núñez and Rafels, 2002b) can be computed by a closed formula. Two important points of the core, the buyers-optimal and the sellers-optimal core allocations, are computed and related to specific 2×2 submarkets. The last part of the paper is devoted to the non-square Monge assignment games.

The paper is organized as follows. In Section 2 we describe the assignment game and the results on it we will need later. In Section 3, the Monge assignment markets are defined. We describe the core of the square Monge assignment game, and give a way to compute easily its buyer-seller exact representative matrix and in Section 4 we compute the buyers-optimal and the sellers-optimal core allocations and give a formula to obtain them. We conclude in Section 5 by analyzing non-square Monge assignment markets, their core and the buyers-optimal and sellers-optimal core allocations.

2. Preliminaries on the assignment game

A bilateral assignment market (M, M', A) is defined by a nonempty finite set of agents, usually named buyers M, a nonempty finite set of another type of agents, usually named sellers M' and a nonnegative matrix $A = (a_{ij})_{(i,j)\in M\times M'}$. Entry a_{ij} represents the profit obtained by the mixed-pair $(i,j)\in M\times M'$ if they trade. Let us assume there are $|M| = m$ buyers and $|M'| = m'$ sellers. If $m = m'$, the assignment market is said to be square. Let us denote by $\mathrm{M}^{+}_{m\times m'}$ the set of nonnegative matrices with m rows and m' columns.

A *matching* $\mu \subseteq M\times M'$ between M and M' is a bijection from $M_0\subseteq M$ to $M'_0 \subseteq M'$, such that $|M_0| = |M'_0| = \min\{|M|, |M'|\}$. We write $(i,j)\in\mu$ as well as $j=\mu(i)$ or $i=\mu^{-1}(j)$. The set of all matchings is denoted by $\mathcal{M}(M,M')$. A buyer $i\in M$ is unmatched by μ if there is no $j\in M'$ such that $(i,j)\in\mu$. Similarly, $j\in M'$ is unmatched by μ if there is no $i\in M$ such that $(i,j)\in\mu$.

A *matching* $\mu\in\mathcal{M}(M,M')$ is *optimal* for the assignment market (M,M',A) if for all $\mu'\in\mathcal{M}(M,M')$ we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and we denote the set of optimal matchings by $\mathcal{M}^{*}_{A}(M,M')$.

Shapley and Shubik (1972) associate to any assignment market a game in coalitional form (*assignment game*) with player set $N = M\cup M'$ and characteristic function w_A defined by A in the following way: for $S\subseteq M$ and $T\subseteq M'$, $w_A(S\cup T) = \max\left\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu\in\mathcal{M}(S,T)\right\}$, where $\mathcal{M}(S,T)$ is the set of matchings from S to T and $w_A(S\cup T) = 0$ if $\mathcal{M}(S,T)=\varnothing$.

The *core* of the assignment game[3],

$$Core\,(w_A) = \left\{ (x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'} \;\middle|\; \begin{array}{l} x(S) + y(T) \geq w_A\,(S \cup T)\,, \\ \text{for all } S \subseteq M \text{ and } T \subseteq M', \text{ and} \\ x(M) + y(M') = w_A\,(M \cup M') \end{array} \right\},$$

is always nonempty and, if $\mu \in \mathcal{M}_A^*(M, M')$ is an arbitrary optimal matching, the core is the set of nonnegative payoff vectors $(u, v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ such that

$$u_i + v_j \geq a_{ij} \text{ for all } (i,j) \in M \times M', \tag{1}$$

$$u_i + v_j = a_{ij} \text{ for all } (i,j) \in \mu, \tag{2}$$

and the payoff to unmatched agents by μ is null. It coincides (see Shapley and Shubik, 1972) with the set of solutions of the dual of the linear program related to the linear sum assignment problem.

Moreover, the minimum payoff that a mixed-pair $(i, j) \in M \times M'$ obtains in the core of a square assignment market (M, M', A) (see Theorem 2 in Núñez and Rafels, 2002b) is given by:

$$\min_{(x,y) \in C(w_A)} [x_i + y_j] = a_{i\mu(i)} + a_{\mu^{-1}(j)j} - w_A(N) + w_A(N \setminus \{\mu(i), \mu^{-1}(j)\}), \tag{3}$$

where $\mu \in \mathcal{M}_A^*(M, M')$ is an arbitrary optimal matching.

3. Monge assignment games

The Monge property on a matrix was named this way by Hoffman (1963) recovering the works of the 18th-century French mathematician G. Monge, who used the property in a context of a soil-transport problem. This property has been applied in many different areas such as operations research, coding theory, computational geometry, greedy algorithms, computational biology, statistics or economics. We refer to the surveys in Burkard (2007) or Burkard et al. (1996) for references, specific applications or properties. Our work addresses the following question posed in page 151 in Burkard et al. (1996): "Are there other fields where Monge matrices play a role?"

We try to contribute to this general question with a partial but interesting answer, mixing Monge matrices and assignment problems. Our main interest

[3] For any vector $z \in \mathbb{R}^N$, with $N = \{1, \ldots, n\}$ and any coalition $R \subseteq N$ we denote by $z(R) = \sum_{i \in R} z_i$. As usual, the sum over the empty set is zero.

is to describe the core and the two sectors-optimal core allocations in an easy and practical way, when dealing with Monge assignment cooperative games. Roughly speaking, our results agree with the "flavor" that adding the Monge conditions to the assignment game simplifies a lot the analysis of the aforementioned solutions. Finally, this class of assignment markets have an important economic meaning, especially when we deal with agents that can be ordered according some trait such as age, education, income, etc. (see Becker, 1973).

Definition 3.1. *An assignment market* (M, M', A) *is called a* Monge assignment market *if any* 2×2 *submarket has an optimal matching in its main diagonal, i.e.*

$$a_{ij} + a_{kl} \geq a_{il} + a_{kj} \quad \textit{for all } 1 \leq i < k \leq m, \quad \textit{and } 1 \leq j < l \leq m'.$$

This Monge property has to be checked only for consecutive 2×2 submarkets (adjacent rows and columns). That is, a matrix $A \in \mathrm{M}^+_{m\times m'}$ satisfies the Monge property[4] if and only if

$$a_{ij} + a_{i+1\, j+1} \geq a_{i\, j+1} + a_{i+1\, j} \quad \text{for all } 1 \leq i \leq m-1, \quad \text{and } 1 \leq j \leq m'-1,$$

and then it can be tested easily. Obviously any of their submarkets is also a Monge assignment market. In case of equality it is a modular matrix.

For square Monge assignment markets (see, e.g., Burkard et al., 1996) one optimal matching, maybe not unique, is placed in the main diagonal. Then the worth of the grand coalition is given by

$$w_A\left(M \cup M'\right) = \sum_{k=1}^{m} a_{kk}.$$

We want to analyze the core of a Monge assignment game. In particular, we want to characterize the buyers-optimal and the sellers-optimal core allocations in an easy way. To obtain them we must compute the marginal contribution of a player, and therefore it is crucial how to compute an optimal matching for a non-square Monge matrix.

The next proposition gives some insights on where an optimal matching is to be searched. The simple proof can be found in Aggarwal et al. (1992)

[4]This property is known in the literature as inverse Monge, anti-Monge, contra-Monge, or supermodular.

or in Lin (1992). It generalizes the fact that the main diagonal is an optimal matching if we deal with square Monge assignment markets.

Proposition 3.1. *For any Monge assignment market* (M, M', A), *with* $|M| \leq |M'|$, *at least one optimal matching* $\mu \in \mathcal{M}^*_A(M, M')$ *is monotone, i.e.*

$$\text{for all } \; i_1, i_2 \in M, \; \text{ with } i_1 < i_2, \quad \text{then } \; \mu(i_1) < \mu(i_2).$$

Monotone matchings can be seen as generalized main diagonal matchings, since they coincide with the matching given by the main diagonal entries of the square submarkets of maximal order. When the Monge assignment market is square, there is only one monotone matching, which is placed in the main diagonal. Therefore, if necessary, to distinguish the agents from the two sides of the market we will denote by $k' \in M'$ the partner of player $k \in M$ by this optimal matching.

Now we are in position to give an easy description of the core of a square Monge assignment game.

Theorem 3.1. *Let* (M, M', A) *be a square Monge assignment market. Then* $(u, v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ *belongs to the core of the market,* $C(w_A)$, *if and only if*

$$u_i + v_i = a_{ii} \qquad \text{for } \; i = 1, 2, \ldots, m, \tag{4}$$

$$u_i + v_{i+1} \geq a_{i\,i+1} \qquad \text{for } \; i = 1, 2, \ldots, m-1, \tag{5}$$

$$u_{i+1} + v_i \geq a_{i+1\,i} \qquad \text{for } \; i = 1, 2, \ldots, m-1. \tag{6}$$

Proof. The 'only if' part is obvious by the definition of the core of an assignment game, see (1) and (2), and the fact that one optimal matching is placed in the main diagonal.

Now to prove the 'if' part, consider for $i + 1 < j$, the square submarket formed by $\{i, i+1, \ldots, j-1\} \times \{i'+1, \ldots, j'\}$. One optimal matching in the square Monge submarket is placed in the main diagonal of the submatrix, that is, $\mu = \{(i, i+1), (i+1, i+2), \ldots, (j-1, j)\}$, and then:

$$\sum_{k=i}^{j-1} a_{k\,k+1} \geq a_{ij} + \sum_{k=i+1}^{j-1} a_{kk}.$$

Now, considering (4) and (5), we obtain

$$u_i + v_j = \sum_{k=i}^{j-1} u_k + \sum_{k=i+1}^{j} v_k - \sum_{k=i+1}^{j-1} a_{kk} \geq \sum_{k=i}^{j-1} a_{k\,k+1} - \sum_{k=i+1}^{j-1} a_{kk} \geq a_{ij}.$$

If $j+1<i$, just take the submarket $\{j+1,\ldots,i\}\times\{j',\ldots,(i-1)'\}$, and repeat a similar argument. $\square$

The above result makes a great reduction in terms of the number of inequalities needed to obtain or check the core of a square Monge assignment market. Apart from the positivity restrictions and the equalities in the optimally matched pairs, we only need to check $2m-2$ inequalities, that are m^2-m in the general case. Moreover, if we denote by

$$A_{i,i+1}=\left(\begin{array}{cc} a_{ii} & a_{i\,i+1} \\ a_{i+1\,i} & a_{i+1\,i+1} \end{array}\right), \text{ for } i=1,\ldots,m-1,$$

the consecutive 2×2 submarkets centered at the main diagonal, we have obtained the following characterization result:

Corollary 3.1. *Let (M,M',A) be a square Monge assignment market. The following statements are equivalent:*

1. $(u;v)\in C(w_A)$,
2. $(u_i,u_{i+1};v_i,v_{i+1})\in C(w_{A_{i,i+1}})$, *for* $i=1,\ldots,m-1$.

Also as a consequence of the above theorem we obtain that two square Monge assignment markets have the same core if and only if they have the same principal band, that is, the elements of the main diagonal and the upper and lower sub-diagonals.

Proposition 3.2. *Let (M,M',A) and (M,M',B) be two square Monge assignment markets. The following statements are equivalent:*

1. $C(w_A)=C(w_B)$,
2. $a_{ij}=b_{ij}$ *for all* $(i,j)\in M\times M'$ *such that* $|i-j|\leq 1$.

Proof. 1. $\longrightarrow$ 2. Since both matrices are square Monge assignment games, each one has one optimal matching in its main diagonal, and therefore $a_{ii}=b_{ii}$ for all $i=1,\ldots,m$. Moreover from (3), and taking into account that the main diagonal is an optimal matching, for all $(i,j)\in M\times M'$, with $j=i+1$ and $i=1,\ldots,m-1$,

$$\min_{(u,v)\in C(w_A)}[u_i+v_{i+1}]=a_{ii}+a_{i+1\,i+1}-w_A(M\cup M')+w_A(M\cup M'\setminus\{i',i+1\}).$$

We can compute $w_A(M \cup M' \setminus \{i', i+1\})$, for $i = 1, 2, ..., m-1$, since it is a square Monge assignment submarket, which implies that its main diagonal is optimal, or equivalently:

$$w_A(M \cup M' \setminus \{i', i+1\}) = \sum_{k=1}^{i-1} a_{kk} + a_{i\,i+1} + \sum_{k=i+2}^{m} a_{kk}.$$

Therefore, we obtain

$$\min_{(u,v)\in C(w_A)} [u_i + v_{i+1}] = a_{i\,i+1}. \tag{7}$$

This equality implies the existence of an allocation $(u, v) \in C(w_A)$ such that $u_i + v_{i+1} = a_{i\,i+1}$. By the hypothesis of the equality of the cores we obtain that $a_{i\,i+1} \geq b_{i\,i+1}$. A symmetric argument leads to $a_{i\,i+1} = b_{i\,i+1}$ for $i = 1, 2, ..., m-1$.

The equality between $a_{i+1\,i}$ and $b_{i+1\,i}$ is proved analogously.

2. $\longrightarrow$ 1. It is straightforward from Theorem 3.1. $\square$

We want to point out that Proposition 3.2 requires that both matrices are square and satisfy the Monge property. The consequence is that only the elements of the principal band matter to determine the core (Theorem 3.1 and Proposition 3.2). Moreover, each of these matrix entries is attainable by a core element (see (7)). From these results we see in the next section how to simplify the calculation of the buyers-optimal and the sellers-optimal core allocations for an arbitrary square Monge assignment market.

First we compute what is called the buyer-seller exact representative of the original market. The buyer-seller exact representative A^r was introduced in Núñez and Rafels (2002b) as the unique matrix which has two important properties: (1) it has the same core as the original market, i.e. $C(w_A) = C(w_{A^r})$, and (2) all its entries are attainable by a core element, i.e. for each $(i, j) \in M \times M'$ there exists $(u, v) \in C(w_{A^r})$ such that $u_i + v_j = a^r_{ij}$.

Moreover, entries in the matrix can be defined by using the core as:

$$a^r_{ij} = \min_{(u,v)\in C(w_A)} (u_i + v_j), \tag{8}$$

or by using the characteristic function (see Theorem 2 in Núñez and Rafels, 2002b) as:

$$a^r_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} - w_A(M \cup M') + w_A(M \cup M' \setminus \{\mu(i), \mu^{-1}(j)\}), \tag{9}$$

for any optimal matching $\mu \in \mathcal{M}^*_A(M, M')$, when matrix A is square.

We want to introduce an easy and practical method to determine the buyer-seller exact representative matrix, whenever we are in the presence of a square Monge assignment matrix. In Section 4, the buyer-seller exact representative matrix, A^r, is used to obtain the buyers-optimal and the sellers-optimal core allocations of the original market. Basically we concentrate in the principal band and put zeros in all other entries and finally we define a new matrix driven from the principal band elements in an additive or modular way. Let us show first this process by an example.

Example 3.1. *Let A be the following assignment matrix:*

$$A = \begin{pmatrix} 8 & 7 & 4 & 2 \\ 5 & 6 & 5 & 4 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

Matrix A is a square Monge assignment matrix, and therefore its principal band has full information to determine the core $C(w_A)$. Let us denote by

$$A^b = \begin{pmatrix} \mathbf{8} & \mathbf{7} & 0 & 0 \\ \mathbf{5} & \mathbf{6} & \mathbf{5} & 0 \\ 0 & \mathbf{2} & \mathbf{2} & \mathbf{1} \\ 0 & 0 & \mathbf{2} & \mathbf{3} \end{pmatrix}$$

the matrix where all entries outside the principal band are zero.

To compute the buyer-seller exact representative matrix A^r we change zero entries outside the band by the worths that make the Monge property as additive, with respect the principal band elements. We use the entries in matrix A^b or the ones we have constructed in previous steps, that is,

$$\begin{aligned} a^r_{13} &= a^b_{12} + a^b_{23} - a^b_{22} = 7 + 5 - 6 = 6 \\ a^r_{24} &= a^b_{23} + a^b_{34} - a^b_{33} = 5 + 1 - 2 = 4 \\ a^r_{14} &= a^r_{13} + a^r_{24} - a^b_{23} = 6 + 5 - 6 = 5. \end{aligned}$$

A similar process can be performed for the lower part of the matrix, and we obtain the buyer-seller exact representative matrix, A^r,

$$A^r = \begin{pmatrix} \mathbf{8} & \mathbf{7} & 6 & 5 \\ \mathbf{5} & \mathbf{6} & \mathbf{5} & 4 \\ 1 & \mathbf{2} & \mathbf{2} & \mathbf{1} \\ 1 & 2 & \mathbf{2} & \mathbf{3} \end{pmatrix}. \tag{10}$$

Notice that A^r turns out to be a Monge assignment matrix, and in fact, it is modular for the consecutive 2×2 submarkets when at least one player is outside the principal band. Moreover since it has the same principal band than the original matrix, we obtain that both matrices give rise to the same core (Proposition 3.2). This is what now we develop in general.

Let (M, M', A) be a square Monge assignment market. We define a new assignment matrix, that in Proposition 3.3 we will prove it is the buyer-seller exact representative matrix of A, $A^r = (a^r_{ij})_{(i,j)\in M\times M'}$ in the following way:

$$a^r_{ij} = \begin{cases} \sum_{k=i}^{j-1} a_{k\,k+1} - \sum_{k=i+1}^{j-1} a_{kk} & \text{for } \ 1 \le i < j \le m, \\ a_{ii} & \text{for } \ 1 \le i = j \le m, \\ \sum_{k=j}^{i-1} a_{k+1\,k} - \sum_{k=j+1}^{i-1} a_{kk} & \text{for } \ 1 \le j < i \le m, \end{cases} \tag{11}$$

where the summation over an empty set of indices is zero.

Notice first that entries in the principal band do not change, that is,

$$a^r_{ij} = a_{ij} \ \text{ for } \ |i-j| \le 1. \tag{12}$$

Moreover, $a^r_{ij} \ge a_{ij}$ for all $(i,j) \in M \times M'$. To see it, just consider for $i < j$ the submarket of A formed by $\{i, i+1, \ldots, j-1\} \times \{i'+1, \ldots, j'\}$. One optimal matching is placed in its main diagonal and then $\sum_{k=i}^{j-1} a_{k\,k+1} \ge a_{ij} + \sum_{k=i+1}^{j-1} a_{kk}$. The inequality follows. The case $j < i$ is similar.

Secondly, there is a recursive and practical way to compute matrix A^r, given in (11). The idea is to compute, as in the above numerical example, the parallel diagonals to the principal band, starting by the closest one. To get entries $a^r_{i\,i+2}$ for $i = 1, \ldots, m-2$, we compute them by using the formula (11):

$$a^r_{i\,i+2} = a_{i\,i+1} + a_{i+1\,i+2} - a_{i+1\,i+1} \ \text{ for } \ i = 1, \ldots, m-2.$$

Now we continue with the elements of the next parallel diagonal:

$$a^r_{i\,i+3} = a^r_{i\,i+2} + a^r_{i+1\,i+3} - a^r_{i+1\,i+2} \ \text{ for } \ i = 1, \ldots, m-3.$$

The process is repeated until we complete the entries in the upper triangle. Similarly, we can compute the entries in the lower triangle by the recursive method,

$$\begin{aligned} a^r_{i+2\,i} &= a_{i+1\,i} + a_{i+2\,i+1} - a_{i+1\,i+1} \ \text{ for } \ i = 1, \ldots, m-2, \ \text{and} \\ a^r_{i+k\,i} &= a^r_{i+k-1\,i} + a^r_{i+k\,i+1} - a^r_{i+k-1\,i+1} \ \text{ for } \ i = 1, \ldots, m-k, \end{aligned}$$

and all $k = 2, \ldots, m-1$.

Thirdly, matrix A^r satisfies the Monge property, whenever the original matrix does, just by its definition. As a consequence, square matrices A and A^r have an optimal matching in the main diagonal, and by Proposition 3.2 and (12) both matrices have the same core, $C(w_A) = C(w_{A^r})$. We are now in disposition to state the following important result.

Proposition 3.3. *For any square Monge assignment market (M, M', A), matrix A^r defined by (11) is the buyer-seller exact representative of matrix A, that is,*

1. $C(w_A) = C(w_{A^r})$,
2. *For each pair $(i, j) \in M \times M'$ there exists $(u, v) \in C(w_A)$ such that $u_i + v_j = a^r_{ij}$.*

Proof. We have already discussed that $C(w_A) = C(w_{A^r})$. Let us check that any entry of the matrix A^r is attainable by a point of the core of the market.

To this end, we introduce a new matrix $\overline{A}$, which is no more than the modular matrix generated only by the main diagonal and the super-diagonal. Formally,

$$\overline{a}_{ij} = \begin{cases} \sum_{k=i}^{j-1} a_{k\,k+1} - \sum_{k=i+1}^{j-1} a_{kk} & \text{for} \quad 1 \leq i < j \leq m, \\ a_{ii} & \text{for} \quad 1 \leq i = j \leq m, \\ \sum_{k=j}^{i} a_{kk} - \sum_{k=j}^{i-1} a_{k\,k+1} & \text{for} \quad 1 \leq j < i \leq m. \end{cases}$$

We claim that matrix $\overline{A}$ is modular, that is, $\overline{a}_{ij} + \overline{a}_{kl} = \overline{a}_{il} + \overline{a}_{kj}$ for all $1 \leq i < k \leq m$, and $1 \leq j < l \leq m'$. To see it, notice that $\overline{a}_{ij} = a^r_{ij}$ for all $1 \leq i < j \leq m$, the upper triangle, and for the lower triangle it is enough to see that equality holds for any 2×2 consecutive submarkets. It is a simple calculation to show that $\overline{a}_{ij} = \overline{a}_{i-1\,j} + \overline{a}_{i\,j+1} - \overline{a}_{i-1\,j+1}$ for $1 \leq j < i \leq m$. Joining both observations, we have that $\overline{A}$ is modular. An important consequence is that any matching is optimal in $\overline{A}$.

Moreover, notice that[5] $\overline{A} \geq A$, because the main diagonal entries have been preserved, for $1 \leq i < j \leq m$, we have $\overline{a}_{ij} = a^r_{ij} \geq a_{ij}$, and for $1 \leq j < i \leq m$, we have $\overline{a}_{ij} = \sum_{k=j}^{i} a_{kk} - \sum_{k=j}^{i-1} a_{k\,k+1} \geq a_{ij}$ since the square submarket $\{j, j+1, \ldots, i\} \times \{j', j'+1, \ldots, i'\}$ has an optimal matching in the main diagonal of the restriction of A.

[5]Let it be $A, B \in \mathrm{M}^+_{m \times m'}$. Then $A \geq B$ if $a_{ij} \geq b_{ij}$ for all $(i, j) \in M \times M'$.

Summarizing, matrices A and $\overline{A}$ have the same main diagonal and it is optimal, and from the above comments, we have $C(w_{\overline{A}}) \subseteq C(w_A)$. Moreover, since $\overline{A}$ is modular, each of its entries is in some optimal matching (all of them are optimal). This implies that for any $(i,j) \in M \times M'$ with $i \leq j$ there exists $(u,v) \in C(w_{\overline{A}})$ such that $u_i + v_j = \overline{a}_{ij} = a^r_{ij}$. This finishes the proof of statement 2 for the elements of the upper triangle. The proof for the lower triangle is similar, but defining matrix $\underline{A}$, which entries are defined as:

$$\underline{a}_{ij} = \begin{cases} \sum_{k=i}^{j} a_{kk} - \sum_{k=i}^{j-1} a_{k+1\,k} & \text{for} \quad 1 \leq i < j \leq m, \\ a_{ii} & \text{for} \quad 1 \leq i = j \leq m, \\ \sum_{k=j}^{i-1} a_{k+1\,k} - \sum_{k=j+1}^{i-1} a_{kk} & \text{for} \quad 1 \leq j < i \leq m. \end{cases}$$

It corresponds to the modular matrix associated to the main diagonal and the sub-diagonal of matrix A.

Thus, combining statements 1 and 2 we have obtained that matrix A^r defined by (11) is the buyer-seller exact representative of matrix A. $\square$

4. The buyers-optimal and the sellers-optimal core allocations

Among all core allocations of an assignment market, there exist two particular extreme core points: the *buyers-optimal core allocation* $(\overline{u}^A, \underline{v}^A)$ where each buyer attains her maximum core payoff and each seller his minimum one, and the *sellers-optimal core allocation* $(\underline{u}^A, \overline{v}^A)$ where each seller attains his maximum core payoff and each buyer her minimum one. Demange (1982) and Leonard (1983) prove that the maximum payoff of an agent is its marginal contribution, and it can be attained for all agents of the same side at the same core allocation. By Roth and Sotomayor (1990), for any assignment market (M, M', A), we have

$$\begin{aligned} \overline{u}_i^A &= w_A(M \cup M') - w_A(M \cup M' \setminus \{i\}) \text{ for all } i \in M, \text{ and} \\ \overline{v}_j^A &= w_A(M \cup M') - w_A(M \cup M' \setminus \{j\}) \text{ for all } j \in M'. \end{aligned} \tag{13}$$

Moreover, Demange et al. (1986) and Pérez-Castrillo and Sotomayor (2002) give implementations of these optimal solutions.

Notice that, if μ is an arbitrary optimal matching of (M, M', A), we obtain from the description of the core that $\underline{u}_i^A = a_{i\mu(i)} - \overline{v}_{\mu(i)}^A$ for all $i \in M$ assigned by μ and $\underline{v}_j^A = a_{\mu^{-1}(j)j} - \overline{u}_{\mu^{-1}(j)}^A$ for all $j \in M'$ assigned by μ, while agents unmatched by μ have a fixed null minimum core payoff. Therefore, the minimum core payoffs for a sector are determined by knowing an optimal matching and the maximum core payoffs of the other sector.

What can we say about these special core allocations, if we deal with square Monge assignment markets? We show two results. The first one gives an explicit formula to compute the maximum core payoffs of the agents. The formula is easy to use and allows to reach an interpretation which relates the maximum core payoff an agent can obtain with the maximum core payoff of specific 2×2 subgames.

Theorem 4.1. *For any square Monge assignment market* (M, M', A), *and any agents* $i \in M$ *and* $j \in M'$, *we have:*

$$\overline{u}_i^A = a_{ii} - \max_{t=1,\ldots,m} (a_{ti}^r - a_{tt}), \tag{14}$$

$$\overline{v}_j^A = a_{jj} - \max_{t=1,\ldots,m} (a_{jt}^r - a_{tt}), \tag{15}$$

where matrix A^r *is the buyer-seller exact representative of matrix* A, *defined by (11).*

Proof. Since matrix A is square and satisfies the Monge property, we know that $w_A(M \cup M') = \sum_{k=1}^m a_{kk}$. We need to compute, for all $i \in M$, the worth of $w_A(M \cup M' \setminus \{i\})$ to obtain $\overline{u}_i^A$, by using (13). A similar reasoning is applied to compute $\overline{v}_j^A$ for any player $j \in M'$.

Let it be $i \in M$ and denote by A_{-i} the resulting matrix from A when we remove her row. We know that matrix A_{-i} satisfies the Monge property. By Proposition 3.1 at least one optimal matching of the submarket $(M \setminus \{i\}, M', A_{-i})$ has to be monotone, and since matrix A_{-i} has $m-1$ rows and m columns, the monotone matchings can be described by $\mu_1, \ldots, \mu_m$, where,

for $1 \leq t < i$,
$$\mu_t = \{(1,1), \ldots, (t-1, t-1), (t, t+1), \ldots, (i-1, i), (i+1, i+1), \ldots, (m, m)\},$$
for $t = i$,
$$\mu_i = \{(1,1), \ldots, (i-1, i-1), (i+1, i+1), \ldots, (m, m)\}, \quad \text{and}$$
for $i < t \leq m$,
$$\mu_t = \{(1,1), \ldots, (i-1, i-1), (i+1, i), \ldots, (t, t-1), (t+1, t+1), \ldots, (m, m)\}.$$

Therefore, for any $i \in M$,

$$\begin{aligned}
\overline{u}_i^A &= w_A(M \cup M') - w_A(M \cup M' \setminus \{i\}) \\
&= \sum_{k=1}^{m} a_{kk} - \max_{t=1,\ldots,m} \left\{ \sum_{k \in M \setminus \{i\}} a_{k\mu_t(k)} \right\} \\
&= a_{ii} - \max_{t=1,\ldots,m} \left\{ \sum_{k \in M \setminus \{i\}} \left(a_{k\mu_t(k)} - a_{kk} \right) \right\}. \qquad (16)
\end{aligned}$$

Recall expression (11) of the buyer-seller exact matrix and notice now that in expression (16), $\sum_{k \in M \setminus \{i\}} \left(a_{k\mu_t(k)} - a_{kk} \right)$ becomes

$$\begin{aligned}
\sum_{k=t}^{i-1} (a_{k\,k+1} - a_{kk}) &= a^r_{ti} - a_{tt}, &\quad \text{for } 1 \leq t < i, \\
0 &= a^r_{tt} - a_{tt}, &\quad \text{for } t = i, \\
\sum_{k=i+1}^{t} (a_{k\,k-1} - a_{kk}) &= a^r_{ti} - a_{tt}, &\quad \text{for } i \leq t < m.
\end{aligned}$$

The result follows. □

Notice that any 2×2 assignment market satisfies the Monge property, up to a reordering of the agents. From Theorem 4.1 we can compute easily the buyers-optimal and the sellers-optimal core allocations.

Now it is easy to give a direct consequence. Consider for any agent, say a buyer, the 2×2 submarkets formed by herself and any other buyer, with their respective optimally matched sellers, in the buyer-seller exact matrix. There are $m - 1$ possible submarkets. Then its optimal core allocation in the original market is just the minimum of the optimal allocations for each of these 2×2 submarkets.

The formulae obtained in Theorem 4.1 to compute the maximum core payoff for each player allow to make some comments and interpretations. Firstly, once obtained the maximum core payoff, we also derive a formula to obtain the minimum core allocations, by $\underline{u}_i^A = a_{ii} - \overline{v}_i^A$ for all $i \in M$ and $\underline{v}_j^A = a_{jj} - \overline{u}_j^A$ for all $j \in M'$ since there is always an optimal matching in the main diagonal. Therefore this is an easy and practical method to compute the buyers-optimal $(\overline{u}^A, \underline{v}^A)$ and the sellers-optimal core allocation $(\underline{u}^A, \overline{v}^A)$.

As a consequence we obtain an easy formula to compute the *fair-division point* (Thompson, 1981), the midpoint between the optimal allocation for the buyers and the optimal allocation for the sellers. In Núñez and Rafels (2002a) it is proved that it coincides with the τ-value of the assignment game, a single-valued solution defined by Tijs (1981) for arbitrary coalitional games. That is, $\tau(w_A) = \frac{1}{2}(\underline{u}^A; \overline{v}^A) + \frac{1}{2}(\overline{u}^A, \underline{v}^A)$.

Corollary 4.1. *For any square Monge assignment market (M, M', A), and any players $i \in M$ and $j \in M'$, the fair-division point is given by:*

$$\tau_i(w_A) = \frac{a_{ii}}{2} + \frac{\max_{t=1,\ldots,m}(a^r_{it} - a_{tt}) - \max_{t=1,\ldots,m}(a^r_{ti} - a_{tt})}{2}$$
$$\tau_j(w_A) = \frac{a_{jj}}{2} + \frac{\max_{t=1,\ldots,m}(a^r_{tj} - a_{tt}) - \max_{t=1,\ldots,m}(a^r_{jt} - a_{tt})}{2}$$

where matrix A^r is the buyer-seller exact representative of matrix A, defined by (11).

A second remark comes directly from expressions (14) and (15). Note that for any buyer $i \in M$ and taking into account that $\max_{t=1,\ldots,m}(a^r_{ti} - a_{tt}) \geq a^r_{ii} - a_{ii} = 0$, we obtain by (14) that $\overline{u}^A_i \leq a_{ii}$, and that $\overline{u}^A_i = a_{ii}$ if and only if $\max_{t=1,\ldots,m}(a^r_{ti} - a_{tt}) = 0$, or equivalently $a_{tt} \geq a^r_{ti}$ for all $t = 1, \ldots, m$. Going now to expression (11) we obtain a characterization for the buyers-optimal core allocation to be the optimal entries for the buyers.

Corollary 4.2. *For any square Monge assignment market (M, M', A), the following statements are equivalent:*

1. *The buyers-optimal core allocation is $(\overline{u}^A, \underline{v}^A) = (a_{11}, \ldots, a_{mm}; 0, \ldots, 0)$,*
2. *The following inequalities hold:*

$$\begin{aligned} a_{tt} &\geq a_{t\,t+1} && \textit{for } 1 \leq t < m, \textit{ and} \\ a_{tt} &\geq a_{t\,t-1} && \textit{for } 1 < t \leq m. \end{aligned}$$

Proof. 1. $\longrightarrow$ 2. Since the buyers-optimal core allocation is in the core, we know that

$$\begin{aligned} \overline{u}^A_t + \underline{v}^A_{t+1} &\geq a_{t\,t+1} && \text{for } 1 \leq t < m, \text{ and} \\ \overline{u}^A_t + \underline{v}^A_{t-1} &\geq a_{t\,t-1} && \text{for } 1 < t \leq m. \end{aligned}$$

The statement follows immediately.

2. $\longrightarrow$ 1. Let it be $i \in M$. By the conditions in statement 2 we have that

$$\sum_{k=t}^{i-1} a_{kk} \geq \sum_{k=t}^{i-1} a_{k\,k+1} \quad \text{for} \;\; 1 \leq t < i,$$

which implies, using (11), that $a_{tt} \geq a^r_{ti}$ for $1 \leq t < i$. Moreover we have that

$$\sum_{k=i+1}^{t} a_{kk} \geq \sum_{k=i+1}^{t} a_{k\,k-1} \quad \text{for} \;\; i < t \leq m,$$

which implies, using (11), that $a_{tt} \geq a^r_{ti}$ for $i < t \leq m$. Now from Theorem 4.1 and taking into account that $a_{ii} = a^r_{ii}$, we have finally obtained that $\overline{u}^A_i = a_{ii}$ for all $i \in M$. □

Conditions in Corollary 4.2 (statement 2) have the advantage that they can be applied directly on the matrix entries of the original square Monge matrix A, without computing its buyer-seller exact representative A^r. In this sense, they say that the optimally matched entries a_{ii}, for $i \in M$ have to be larger than their respective predecessor (if any) $a_{ii} \geq a_{i\,i-1}$ and follower (if any) $a_{ii} \geq a_{i\,i+1}$ row entries. Therefore, by simply looking these inequalities from the square Monge matrix of Example 3.1,

$$A = \begin{pmatrix} 8 & 7 & 4 & 2 \\ 5 & 6 & 5 & 4 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

we obtain

$$(\overline{u}^A, \underline{v}^A) = (8, 6, 2, 3; 0, 0, 0, 0).$$

Similar conditions can be obtained to know when the sellers-optimal core allocation coincides with the optimally matched entries for the sellers' sector, that is, $(\underline{u}^A, \overline{v}^A) = (0, \ldots, 0; a_{11}, \ldots, a_{mm})$. Instead to compare the optimally matched pairs given by the main diagonal entries with the precedessor or follower entries in the same row, we have to compare by columns, that is,

$$\begin{aligned} a_{tt} &\geq a_{t-1\,t} \quad \text{for} \;\; 1 < t \leq m, \;\text{and} \\ a_{tt} &\geq a_{t+1\,t} \quad \text{for} \;\; 1 \leq t < m. \end{aligned}$$

We can easily see that $(\underline{u}^A, \overline{v}^A) \neq (0,0,0,0;8,6,2,3)$, since, for example, $a_{22} = 6 \not\geq a_{12} = 7$ or $a_{33} = 2 \not\geq a_{23} = 5$. By using the buyer-seller exact representative of matrix A, given in (10), and Theorem 4.1 we obtain

$$(\underline{u}^A, \overline{v}^A) = (4,3,0,0;4,3,2,3).$$

5. Non-square Monge assignment markets

Assignment markets do not need to be square and we have left to this section the analysis of the core, the buyers-optimal core allocation and the sellers-optimal core allocation of a non-square Monge assignment market. Recall that any assignment market where each 2×2 submarket has an optimal matching in its main diagonal is called a Monge assignment market (see Definition 3.1). We may assume from now on that there are less buyers than sellers, that is, $|M| \leq |M'|$.

We try to maintain the exposition of this section as practical as possible. The technique to solve the non-square markets will be the reduction to the square cases already analyzed in the previous sections.

Moreover, we will use the following 3×7 Monge assignment matrix A to illustrate some ideas, to give explanations and to introduce motivations,

$$A = \begin{pmatrix} \mathbf{12} & 11 & 2 & 8 & 30 & 1 & 9 \\ 15 & 14 & 13 & 26 & \mathbf{52} & 28 & 40 \\ 1 & 0 & 0 & 13 & 40 & 40 & \mathbf{60} \end{pmatrix}. \tag{17}$$

By Proposition 3.1 we know that matrix A has a monotone optimal matching. In this case the optimal matching (unique) is $\mu = \{(1,1),(2,5),(3,7)\}$ and its entries are marked in boldface. The worth of the grand coalition is $w_A(M \cup M') = 124$, and notice that sellers $2,3,4$ and 6 are not optimally matched. The non-optimally matched sellers receive zero payoffs in any core allocation, but they introduce significant bound payoffs for the buyers' side, since any core allocation $(u,v) \in C(w_A)$ must satisfy $u_i + v_j \geq a_{ij}$, for all $(i,j) \in M \times M'$. Since $v_2 = v_3 = v_4 = v_6 = 0$, we have,

$$u_1 \geq \max_{j=2,3,4,6}\{a_{1j}\} = \max\{11,2,8,1\} = 11,$$
$$u_2 \geq \max_{j=2,3,4,6}\{a_{2j}\} = \max\{14,13,26,28\} = 28,$$
$$u_3 \geq \max_{j=2,3,4,6}\{a_{3j}\} = \max\{0,0,13,40\} = 40.$$

In fact we have got a description of the core by simply adding to the above inequalities the ones corresponding to the core of the square assignment submarket formed by the optimally matched pairs.

Formally, let (M, M', A), be an assignment market with $|M| \leq |M'|$. For any optimal matching (monotone or not) $\mu \in \mathcal{M}_A^*(M, M')$ an allocation $(u, v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ belongs to the core, $C(w_A)$, if and only if

$$u_i + v_j \geq a_{ij} \quad \text{for all} \quad (i, j) \in M \times \mu(M), \tag{18}$$
$$u_i + v_j = a_{ij} \quad \text{for all} \quad (i, j) \in \mu, \tag{19}$$
$$u_i \geq \overline{a}_i := \max_{r \in M' \setminus \mu(M)} \{a_{ir}\}, \quad \text{for all} \quad i \in M \text{ and} \tag{20}$$
$$v_j = 0 \quad \text{for all} \quad j \in M' \setminus \mu(M). \tag{21}$$

Notice that conditions (18) and (19) are equivalent to say that the projection of the allocation (u, v) on $M \times \mu(M)$ belongs to the core of the subgame of w_A restricted to $M \cup \mu(M)$. The above description of the core says that we have to consider the core of the submarket formed by the optimally matched agents $(M, \mu(M), A_{|M \times \mu(M)})$ and inside it those allocations that satisfy certain bound payoffs for the buyers, $u_i \geq \overline{a}_i$ for $i \in M$, and no more than a zero payoff for the non-optimally matched sellers, $v_j = 0$ for $j \in M' \setminus \mu(M)$.

What is really important for our purposes is that, departing from a non-square Monge assignment market (M, M', A) with $|M| \leq |M'|$, the optimally matched pairs form a submarket $(M, \mu(M), A_{|M \times \mu(M)})$ that is both square and satisfies the Monge property. As a consequence we are able to apply all the results given in the previous sections. Let us see first that the buyers-optimal core allocation of the whole market always coincides with the buyers-optimal core allocation of the square submarket of the optimally matched pairs. This property holds even if the original assignment market do not satisfy the Monge property[6]. Therefore, the maximum core payoffs for the short side of the market only depend on the square submarket formed by the optimally matched pairs. We include a simple proof for the sake of comprehensiveness.

Proposition 5.1. *Let (M, M', A) be an arbitrary assignment market with*

[6]A classical way to analyze non-square assignment markets is to add rows or columns formed by zeroes (dummy players) until we obtain a square matrix, but this process cannot be used in our setting, since we can loose the Monge property.

$|M| \le |M'|$, *and* $\mu \in \mathcal{M}^*_A(M, M')$. *Then, we have*

$$\overline{u}_i^A = \overline{u}_i^{A_\mu} \quad \textit{for all} \;\; i \in M, \quad \textit{where} \;\; A_\mu = A_{|M \times \mu(M)}.$$

Proof. Since the projection of the buyers-optimal core allocation of the whole market $(\overline{u}^A, \underline{v}^A)$ belongs to the core of the submarket A_μ we know $\overline{u}_i^{A_\mu} \ge \overline{u}_i^A$ for all $i \in M$. To see the reverse inequality let us denote by $(x', y') \in \mathbb{R}^M \times \mathbb{R}^{M'}$ the extension of the buyers-optimal core allocation of the submarket to the whole market, that is, $x'_i = \overline{u}_i^{A_\mu}$ for $i \in M$, $y'_j = \underline{v}_j^{A_\mu}$ for $j \in \mu(M)$, and $y'_j = 0$ for $j \in M' \setminus \mu(M)$.

Note that $(x', y') \in C(w_A)$ since it satisfies (18), (19), (20) and (21), where $\overline{u}_i^{A_\mu} = x'_i \ge \overline{a}_i$ for $i \in M$ holds because for any point $(x, y) \in C(w_A)$ we have $\overline{u}_i^{A_\mu} = x'_i \ge \overline{u}_i^A \ge x_i \ge \overline{a}_i$ for all $i \in M$. □

We can apply Proposition 5.1 to the matrix A given in (17) to obtain its buyers-optimal core allocation $(\overline{u}^A, \underline{v}^A)$. To this end, we isolate the square Monge submatrix given by the optimally matched pairs, that is

$$A_\mu = \begin{pmatrix} \mathbf{12} & 30 & 9 \\ 15 & \mathbf{52} & 40 \\ 1 & 40 & \mathbf{60} \end{pmatrix}.$$

The buyer-seller representative matrix can be easily computed by the process described in (11) because it is a square Monge assignment market. This process makes modular the 2×2 submarkets outside the principal band, that is,

$$(A_\mu)^r = \begin{pmatrix} \mathbf{12} & 30 & 18 \\ 15 & \mathbf{52} & 40 \\ 3 & 40 & \mathbf{60} \end{pmatrix}.$$

From the description of $(A_\mu)^r$ and applying expressions (14) we obtain

$$\begin{aligned}
\overline{u}_1^A &= \overline{u}_1^{A_\mu} = a_{11} - \max\{0, a^r_{21} - a_{22}, a^r_{31} - a_{33}\} = 12, \\
\overline{u}_2^A &= \overline{u}_2^{A_\mu} = a_{22} - \max\{a^r_{12} - a_{11}, 0, a^r_{32} - a_{33}\} = 52 - 18 = 34, \\
\overline{u}_3^A &= \overline{u}_3^{A_\mu} = a_{33} - \max\{a^r_{13} - a_{11}, a^r_{23} - a_{22}, 0\} = 60 - 6 = 54,
\end{aligned}$$

where a_{ij} and a^r_{ij}, for $i, j \in \{1, 2, 3\}$ refers to entries of matrices A_μ and $(A_\mu)^r$.

Then, the buyers-optimal core allocation for the original 3×7 assignment market is

$$(\overline{u}^A, \underline{v}^A) = (12, 34, 54; 0, 0, 0, 0, 18, 0, 6).$$

Note that departing from a non-square Monge assignment market (M, M', A) with $|M| \leq |M'|$, and fixing a monotone optimal matching $\mu \in \mathcal{M}_A^*(M, M')$, we can derive explicit expressions for its buyers-optimal core allocation. We only have to apply Theorem 4.1 to the square market $A_\mu = A_{|M \times \mu(M)}$. Therefore, for any $i \in M$

$$\overline{u}_i^A = a_{i\mu(i)} - \max_{k=1,\ldots,m} \{a_{k\mu(i)}^r - a_{k\mu(k)}\}, \tag{22}$$

$$\underline{v}_j^A = a_{\mu^{-1}(j)j} - \overline{u}_{\mu^{-1}(j)}^A, \quad \text{for } j \in \mu(M) \quad \text{and} \tag{23}$$

$$\underline{v}_j^A = 0, \quad \text{for } j \in M' \setminus \mu(M). \tag{24}$$

The process to analyze and compute the sellers-optimal core allocation will be divided in two parts. Firstly we try to reduce as much as possible the number of columns of the original matrix. The magnitude of the reduction depends on the distribution of the non-assigned sellers. Roughly speaking, the rule is that we can merge contiguous non-optimally matched sellers into one by taking the maximum for each row. In this way we preserve the Monge property of the original matrix (see Proposition 5.2 below). Once we have reduced the matrix, we can address the calculus of the sellers-optimal core allocation by looking for some special matchings of the reduced matrix.

Next proposition shows that merging contiguous columns by the maximum operator preserves the Monge property.

Proposition 5.2. *Let matrix $D \in \mathrm{M}^+_{m\times(k+t+r)}$ be*

$$D = \left(\begin{array}{ccc|ccc|ccc} a_{11} & \ldots & a_{1k} & b_{11} & \ldots & b_{1t} & c_{11} & \ldots & c_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mk} & b_{m1} & \ldots & b_{mt} & c_{m1} & \ldots & c_{mr} \end{array} \right)$$

and denote by $\widetilde{D} \in \mathrm{M}^+_{m\times(k+1+r)}$ the associated matrix defined by merging the b_{ij} entries with the maximum operator, that is,

$$\widetilde{D} = \left(\begin{array}{ccc|c|ccc} a_{11} & \ldots & a_{1k} & \widetilde{b}_1 & c_{11} & \ldots & c_{1r} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mk} & \widetilde{b}_m & c_{m1} & \ldots & c_{mr} \end{array} \right)$$

where $\widetilde{b}_i = \max_{l=1,\ldots,t}\{b_{il}\}$ *for* $i = 1, 2, \ldots, m$. *Then, if* D *satisfies the Monge property, matrix* $\widetilde{D}$ *does also.*

Proof. We only have to prove that the 2×2 submatrices of the form

$$\begin{pmatrix} a_{ik} & \widetilde{b}_i \\ a_{i+1\,k} & \widetilde{b}_{i+1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \widetilde{b}_i & c_{i1} \\ \widetilde{b}_{i+1} & c_{i+1\,1} \end{pmatrix} \quad \text{for} \;\; i = 1, \ldots, m-1.$$

have an optimal matching in its main diagonal.

We only prove the statement for the first type of submatrix, since the other one can be proved similarly. Notice first that $\widetilde{b}_i = b_{ir^*}$ for some $r^*, 1 \leq r^* \leq t$. Then,

$$a_{ik} + \widetilde{b}_{i+1} = a_{ik} + \max_{l=1,\ldots,t}\{b_{i+1\,l}\} \geq a_{ik} + b_{i+1\,r^*} \geq a_{i+1\,k} + b_{ir^*} = a_{i+1\,k} + \widetilde{b}_i,$$

where the last inequality comes from the fact that matrix D satisfies the Monge property. $\square$

We can apply Proposition 5.2 to our 3×7 matrix A given in (17) to reduce the sellers' sector, merging sellers 2, 3 and 4 in a unique column obtaining a 3×5 Monge assignment market $(M, \widetilde{M'}, \widetilde{A})$, with

$$\widetilde{A} = \begin{pmatrix} \mathbf{12} & 11 & 30 & 1 & 9 \\ 15 & 26 & \mathbf{52} & 28 & 40 \\ 1 & 13 & 40 & 40 & \mathbf{60} \end{pmatrix}. \tag{25}$$

As the original non-optimally matched sellers get a zero payoff in any core allocation, we already know that $\overline{v}_2^A = \overline{v}_3^A = \overline{v}_4^A = \overline{v}_6^A = 0$. Let us then concentrate on the rest of the sellers, that is, $\overline{v}_1^A, \overline{v}_5^A$ and $\overline{v}_7^A$. These worths now correspond, in the reduced matrix $\widetilde{A}$ with $\overline{v}_1^{\widetilde{A}}, \overline{v}_3^{\widetilde{A}}$ and $\overline{v}_5^{\widetilde{A}}$, and $\overline{v}_1^A = \overline{v}_1^{\widetilde{A}}, \overline{v}_5^A = \overline{v}_3^{\widetilde{A}}$ and $\overline{v}_7^A = \overline{v}_5^{\widetilde{A}}$, since the core of the assignment markets given by matrix A and $\widetilde{A}$ are alike, and differ just by the dimension spaces where they are in and also in the numeration of the players. This is so because we have merged non-optimally matched sellers of the original assignment market by the maximum operator and this fact essentially does not change the core, for the original optimally matched pairs (see (18), (19) and (20)). Notice that the entries of the optimal matching will be preserved, from A to $\widetilde{A}$.

Recall (13) to obtain the maximum core payoff for a seller, which is given by his marginal contribution:

$$\overline{v}_j^{\widetilde{A}} = w_{\widetilde{A}}(M \cup \widetilde{M}') - w_{\widetilde{A}}(M \cup \widetilde{M}' \setminus \{j\}) \quad \text{for } j \in \widetilde{M}'.$$

The worth of the grand coalition is known to be $w_{\widetilde{A}}(M \cup \widetilde{M}') = 124$, and to compute $w_{\widetilde{A}}(M \cup \widetilde{M}' \setminus \{j\})$ we introduce the following notation. To any subset of m elements from $\widetilde{M}' = \{1, \dots, \widetilde{m}'\}$, that is, $S = \{j_1, j_2, \dots, j_m\}$, we assume that the elements are ordered, $1 \leq j_1 < j_2 \dots < j_m \leq \widetilde{m}'$, and we associate the sum of the main diagonal of the square submarket $(M, S, \widetilde{A}_{|M \times S})$, that is,

$$h_{\widetilde{A}}(j_1, j_2, \dots, j_m) = \sum_{k=1}^{m} \widetilde{a}_{kj_k}. \tag{26}$$

It is easy to argue that

$$w_{\widetilde{A}}(M \cup \widetilde{M}' \setminus \{j\}) = \max_{j \notin \{j_1, j_2, \dots, j_m\}} h_{\widetilde{A}}(j_1, j_2, \dots, j_m). \tag{27}$$

The reason is that when we drop out a seller $j \in \widetilde{M}'$ from a Monge assignment market, we obtain a (in general, non-square) Monge assignment market, and by Proposition 3.1 one of its optimal matchings will be monotone. The possible optimal matchings are precisely the main diagonals of its maximum-size square submarkets, that is what is described in (26) and (27).

Therefore, from the reduced matrix $\widetilde{A}$ obtained in (25), we have $\binom{5}{3} = 10$ possible submarkets and their main diagonal sums correspond to all the monotone matchings of matrix $\widetilde{A}$. Some computations yield:

$$\begin{aligned}
h_{\widetilde{A}}(1,2,3) &= \widetilde{a}_{11} + \widetilde{a}_{22} + \widetilde{a}_{33} = 78,\\
h_{\widetilde{A}}(1,2,4) &= \widetilde{a}_{11} + \widetilde{a}_{22} + \widetilde{a}_{34} = 78,\\
h_{\widetilde{A}}(1,2,5) &= \widetilde{a}_{11} + \widetilde{a}_{22} + \widetilde{a}_{35} = 98,\\
h_{\widetilde{A}}(1,3,4) &= \widetilde{a}_{11} + \widetilde{a}_{23} + \widetilde{a}_{34} = 104,\\
h_{\widetilde{A}}(1,3,5) &= \widetilde{a}_{11} + \widetilde{a}_{23} + \widetilde{a}_{35} = \mathbf{124},\\
h_{\widetilde{A}}(1,4,5) &= \widetilde{a}_{11} + \widetilde{a}_{24} + \widetilde{a}_{35} = 100,\\
h_{\widetilde{A}}(2,3,4) &= \widetilde{a}_{12} + \widetilde{a}_{23} + \widetilde{a}_{34} = 103,\\
h_{\widetilde{A}}(2,3,5) &= \widetilde{a}_{12} + \widetilde{a}_{23} + \widetilde{a}_{35} = 123,\\
h_{\widetilde{A}}(2,4,5) &= \widetilde{a}_{12} + \widetilde{a}_{24} + \widetilde{a}_{35} = 99,\\
h_{\widetilde{A}}(3,4,5) &= \widetilde{a}_{13} + \widetilde{a}_{24} + \widetilde{a}_{35} = 118.
\end{aligned}$$

From these data we obtain the sellers-optimal core allocation, by applying (27):

$$\begin{aligned}
\overline{v}_1^A &= \overline{v}_1^{\widetilde{A}} = 124 - \max\{h_{\widetilde{A}}(2,3,4), h_{\widetilde{A}}(2,3,5), h_{\widetilde{A}}(2,4,5), h_{\widetilde{A}}(3,4,5)\} = 1,\\
\overline{v}_5^A &= \overline{v}_3^{\widetilde{A}} = 124 - \max\{h_{\widetilde{A}}(1,2,4), h_{\widetilde{A}}(1,2,5), h_{\widetilde{A}}(1,4,5), h_{\widetilde{A}}(2,4,5)\} = 24,\\
\overline{v}_7^A &= \overline{v}_5^{\widetilde{A}} = 124 - \max\{h_{\widetilde{A}}(1,2,3), h_{\widetilde{A}}(1,2,4), h_{\widetilde{A}}(1,3,4), h_{\widetilde{A}}(2,3,4)\} = 20.
\end{aligned}$$

Joining all these results we obtain by the standard description of the core (see (18), (19) and (20)) that the sellers-optimal core allocation of the 3×7 assignment market given in (17) is:

$$(\underline{u}^A, \overline{v}^A) = (11, 28, 40; 1, 0, 0, 0, 24, 0, 20).$$

When we deal with non-square Monge assignment markets (M, M', A) with $|M| < |M'|$, where all the non-optimally matched agents are contiguous, an interesting recursive formula to compute the sellers-optimal core allocation can be provided under the hypothesis that the main diagonal of the original non-square Monge assignment market, $\mu = \{(1,1), \ldots, (m,m)\}$, is the optimal matching for A.

This recursive formula consists of computing first the optimal core allocation for the last assigned agent, $\overline{v}_m^A$, taking into account the maximal entry for the non-assigned sellers. The previous assigned seller has its optimal core allocation as the previous one adding the difference between the two adjacent entries in the matrix in the row of its optimally assigned buyer, $\overline{v}_{m-1}^A = \overline{v}_m^A + (a_{m-1\ m-1} - a_{m-1\ m})$. This process is repeated for all assigned sellers. Non-assigned sellers get zero at any core allocation.

Proposition 5.3. *Let (M, M', A) be a Monge assignment market with $|M| < |M'|$, with*

$$A = \left(\begin{array}{ccc|ccc} a_{11} & \ldots & a_{1m} & a_{1\,m+1} & \ldots & a_{1\,m+k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mm} & a_{m\,m+1} & \ldots & a_{m\,m+k} \end{array} \right)$$

and $\mu = \{(1,1), \ldots, (m,m)\} \in \mathcal{M}_A^*(M, M')$. Then, we have

$$\begin{aligned}
&\overline{v}_m^A = a_{mm} - \max\{a_{m\,m+1}, \ldots, a_{m\,m+k}\},\\
&\overline{v}_j^A = \overline{v}_{j+1}^A + (a_{jj} - a_{jj+1}) \quad \textit{for } j = m-1, \ldots, 1, \quad \textit{and}\\
&\overline{v}_k^A = 0 \quad \textit{for } k = m+1, \ldots, m+k.
\end{aligned}$$

Proof. By Proposition 5.2 we can reduce matrix A to matrix $\widetilde{A}$,

$$\widetilde{A} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1m} & \widetilde{a}_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mm} & \widetilde{a}_m \end{array} \right), \tag{28}$$

and recall that $\widetilde{a}_i = \max\{a_{i\,m+1}, \dots, a_{i\,m+k}\}$, for $i = 1, \dots, m$.

Notice that the matching $\mu = \{(1,1), \dots, (m,m)\}$ is also optimal for matrix $\widetilde{A}$, which satisfies the Monge property.

We know that $\overline{v}_m^A = \overline{v}_m^{\widetilde{A}} = w_{\widetilde{A}}(M \cup \widetilde{M}') - w_{\widetilde{A}}(M \cup \widetilde{M}' \setminus \{m'\})$. Notice now that

$$\begin{aligned} w_{\widetilde{A}}(M \cup \widetilde{M}') &= a_{11} + \cdots + a_{mm} \quad \text{and} \\ w_{\widetilde{A}}((M \cup \widetilde{M}') \setminus \{m'\}) &= a_{11} + \cdots + a_{m-1\,m-1} + \widetilde{a}_m, \end{aligned}$$

where the second equality holds since $\widetilde{A}_{|M \times \widetilde{M}' \setminus \{m'\}}$ is a square Monge assignment matrix. Therefore,

$$\overline{v}_m^A = \overline{v}_m^{\widetilde{A}} = a_{mm} - \widetilde{a}_m.$$

Now, continuing in the same way, we obtain the desired expressions. □

The hypothesis of Proposition 5.3 could be relaxed, because in the proof what is really needed is the fact that matrix $\widetilde{A}$ in (28) satisfies the Monge property and that $\mu = \{(1,1), \dots, (m,m)\}$ is an optimal matching of matrix $\widetilde{A}$.

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