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The mean-variance model from the inverse of the variance-covariance matrix

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Abstract

In this paper we obtain the main results of the Markowitz mean-variance model from the inverse of the covariance matrix, following a shorter and mathematically rigorous path. We also obtain the equilibrium expression of Sharpe's capital asset pricing model (CAPM).

JEL Classification: G11, G12

Keywords: CAPM, Beta, portfolio composition

Resumen:

En este artículo, a partir de la inversa de la matriz de varianzas y covarianzas se obtiene el modelo Esperanza-Varianza de Markowitz siguiendo un camino más corto y matemáticamente riguroso. También se obtiene la ecuación de equilibrio del CAPM de Sharpe.

1. INTRODUCTION

Consider a financial market in which there are N assets with expected returns given by the following row vector:

$$(\mu_1, \quad \mu_2, \quad \dots \quad , \mu_N) = (\mu_i)_{1xN}$$

and the following variance-covariance matrix:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2N} \\ \dots & \dots & \dots & \dots \\ \sigma_{N1} & \sigma_{N2} & \dots & \sigma_{NN} \end{pmatrix} = (\sigma_{ij})_{NxN}$$

In this paper, we obtain the main results of the Markowitz mean-variance model (1991) from the inverse of the covariance matrix, following a shorter and mathematically accurate path. We also obtain the equilibrium expression of Sharpe's CAPM (1964).

This paper consists of several sections. In Section 2, we calculate the minimum variance point. In Sections 3 and 4, we obtain the critical line and the efficient frontier. Section 5 introduces the riskless asset, which allows for the market portfolio. In Section 6, we develop the CAPM model. Finally, in Section 7 we apply the M-V model to Spanish real estate mutual funds.

2 THE MINIMUM VARIANCE POINT (MVP)

We will determine a portfolio consisting of N assets that have the minimum of all possible variances. The vector:

$$(w_1, \quad w_2, \quad \dots, \quad w_N) = (w_i)_{1xN}$$

represents the weight per unit that is held by each asset in the portfolio. The sum of the components of this vector must be one (in principle, components can be negative or greater than 1, which means that short selling can be carried out).

To obtain the minimum variance portfolio, we propose a program with one equality constraint:

$$Min\ \left(w_i\right)_{1xN}\cdot\left(\sigma_{ij}\right)_{NxN}\left(w_j\right)_{Nx1} = Min\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{ij}\cdot w_i\ w_j$$

subject to: $\sum_{i=1}^{N} w_i = 1$

($w_i < 0$ or $w_i > 1$ is possible, as borrowing is allowed).

The Lagrangian is:

$$L(w_1, w_2, \ldots.., w_N; \alpha) = \sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{ij}\cdot w_i\cdot w_j + \alpha\cdot\left(1 - \sum_{i=1}^{N} w_i\right)$$

Partially deriving with respect to the first N variables and equating to 0 gives:

$$\frac{\partial L}{\partial w_j} = 2\cdot\sum_{i=1}^{N}\sigma_{ji}\cdot w_i - \alpha = 0 \qquad (j=1,2,...,N)$$

The previous N equalities can be expressed using the following matrix:

$$\left(\sigma_{ij}\right)_{NxN}\cdot\left(w_j\right)_{Nx1} = \left(\alpha/2\right)_{Nx1}$$

By calculating the inverse matrix of $(\sigma_{ij})_{NxN}$, represented by $(d_{ij})_{NxN}$, we obtain:

$$\left(w_i\right)_{Nx1} = \left(\sigma_{ij}\right)_{NxN}\cdot\left(\alpha/2\right)_{Nx1} = \left(d_{ij}\right)_{NxN}\cdot\left(\alpha/2\right)_{Nx1} \qquad [1]$$

In addition, when these values are replaced in the constriction, the following is obtained:

$$1 = \sum_{i=1}^{N} w_i = \sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij}\cdot\frac{\alpha}{2} = \frac{\alpha}{2}\cdot\underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij}}_{A} = \frac{\alpha}{2}\cdot A \Rightarrow$$

$$\Rightarrow \alpha = \frac{2}{A} \qquad [2]$$

That is, A represents the sum of all the elements in the inverse of the variance-covariance matrix. With this notation and by substituting [2] in [1] we obtain:

$$\left(w_i\right)_{Nx1} = \left(d_{ij}\right)_{NxN} \cdot \left(1\!/\!A\right)_{Nx1} \Rightarrow w_{\mathrm{i}} = \frac{\sum_{j=1}^{N} d_{ij}}{\mathrm{A}} = \frac{\sum_{j=1}^{N} d_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij}} \qquad [3]$$

Expression [3] provides the composition, on a unit basis, of the portfolio that corresponds to the minimum variance point.

In essence, the composition of the minimum variance point is obtained by adding the rows (or columns) of the inverse of the variance-covariance matrix, divided by the sum of all the elements of this matrix.

The expected return corresponding to the minimum variance point is:

$$t^* = \sum_{i=1}^{N} \frac{\sum_{j=1}^{N} d_{ij}}{\mathrm{A}} \cdot \mu_i = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij} \cdot \mu_i}{A} = \frac{B}{A}$$

where

$$\sum_{i=1}^{N} \mu_i \cdot \sum_{j=1}^{N} d_{ij} = B$$

The minimum variance is:

$$V^* = \left(1\!/\!A\right)_{1xN} \cdot \left(d_{ij}\right)_{NxN} \left(\sigma_{ij}\right)_{NxN} \left(d_{ij}\right)_{NxN} \left(\cdot 1\!/\!A\right)_{Nx1} = \left(1\!/\!A\right)_{1xN} \left(\cdot d_{ij}\right)_{NxN} \left(\cdot 1\!/\!A\right)_{Nx1} =$$

$$= \frac{1}{A^2} \cdot \left(1\right)_{1xN} \left(d_{ij}\right)_{NxN} \left(1\right)_{Nx1} = \frac{A}{A^2} = \frac{1}{A}$$

3. THE CRITICAL LINE (CL)

The expected return of the portfolio is given by the expression:

$$E(\bar{r}_p) = \left(\mu_i\right)_{1xN} \cdot \left(w_i\right)_{Nx1} = \sum_{i=1}^{N} \mu_i \cdot w_i$$

It can easily be shown that any expected positive or negative return can be obtained by appropriately weighting the various assets ($-\infty < E(\bar{r}_p) < +\infty$).

Now we are about to obtain the optimal portfolio for each expected *t* return, i.e. the portfolio that has the least variance possible. To do this, we propose the following program with two equality constraints:

$$Min\ \left(w_i\right)_{1xN} \cdot \left(\sigma_{ij}\right)_{NxN} \left(w_j\right)_{Nx1} = Min \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}\ w_i\ w_j$$

subject to:

$$\sum_{i=1}^{N} w_i = 1$$

$$\sum_{i=1}^{N} \mu_i \cdot w_i = t$$

The Lagrangian is:

$$L(w_1, w_2, \ldots.., w_N ; \alpha, \beta) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} \cdot w_i \cdot w_j + \alpha \cdot \left(1 - \sum_{i=1}^{N} w_i \right) + \beta \cdot \left(t - \sum_{i=1}^{N} \mu_i\ w_i \right)$$

Partially deriving with respect to the first *N* variables and equating to *0* gives:

$$\frac{\partial L}{\partial w_j} = 2 \cdot \sum_{i=1}^{N} \sigma_{ji} \cdot w_i - \alpha - \beta \cdot \mu_j = 0 \qquad (j = 1, 2, \ldots, N)$$

The previous *N* equalities can be expressed using the following matrix:

$$\left(\sigma_{ij}\right)_{NxN} \left(w_j\right)_{Nx1} = \frac{1}{2} \cdot \left(\alpha + \beta \cdot \mu_j\right)_{Nx1}$$

We can obtain the following by calculating the inverse of the variance-covariance matrix and using the same notations that have been used to obtain the MVP:

$$\left(w_j\right)_{Nx1} = \frac{1}{2} \cdot \left(d_{ij}\right)_{NxN} \left(\alpha + \beta \cdot \mu_j\right)_{Nx1} \qquad [4]$$

When these are replaced in both restrictions the following is obtained:

$$\left.\begin{aligned}\frac{1}{2}\cdot\sum_{i=1}^{N}\sum_{j=1}^{N}d_{ij}\cdot(\alpha+\beta\cdot\mu_j)&=1\\ \frac{1}{2}\cdot\sum_{i=1}^{N}\mu_i\cdot\sum_{j=1}^{N}d_{ij}\cdot(\alpha+\beta\cdot\mu_j)&=t\end{aligned}\right\}\Leftrightarrow$$

$$\left\{\begin{aligned}\alpha\cdot\sum_{i=1}^{N}\sum_{j=1}^{N}d_{ij}+\beta\cdot\left(\sum_{i=1}^{N}\mu_i\cdot\sum_{j=1}^{N}d_{ij}\right)&=2\\ \alpha\cdot\left(\sum_{i=1}^{N}\mu_i\cdot\sum_{j=1}^{N}d_{ij}\right)+\beta\cdot\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\cdot\mu_j\cdot d_{ij}\right)&=2\cdot t\end{aligned}\right\} \qquad [5]$$

To simplify thel resolution of the previous system, we use the following notations:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}d_{ij}=A$$

$$\sum_{i=1}^{N}\mu_i\cdot\sum_{j=1}^{N}d_{ij}=B$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\cdot\mu_j\cdot d_{ij}=C$$

A represents the sum of the elements of the inverse variance-covariance matrix.

B is the weighted sum, determined from the expected returns of the sum of the elements of the columns (or the rows) in the inverse variance-covariance matrix.

C represents the sum of the elements of the inverse weighted variance-covariance matrix by the expected returns.

With the above notations, the system [5] remains:

$$\left.\begin{aligned}A\cdot\alpha+B\cdot\beta&=2\\ B\cdot\alpha+C\cdot\beta&=2t\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow\left\{\begin{aligned}\alpha&=\frac{\begin{vmatrix}2 & B\\ 2\cdot t & C\end{vmatrix}}{\begin{vmatrix}A & B\\ B & C\end{vmatrix}}=2\cdot\frac{C-B\cdot t}{A\cdot C-B^2}\\ \beta&=\frac{\begin{vmatrix}A & 2\\ B & 2\cdot t\end{vmatrix}}{\begin{vmatrix}A & B\\ B & C\end{vmatrix}}=2\cdot\frac{A\cdot t-B}{A\cdot C-B^2}\end{aligned}\right.$$

and by substituting in [4], we get:

$$\left(w_i\right)_{\mathrm{Nx1}} = \frac{1}{A\cdot C - B^2}\cdot\left(d_{ij}\right)_{\mathrm{NxN}} \cdot\left((C - B\,t) + (A\,t - B)\mu_j \right)_{\mathrm{Nx1}} =$$

$$= \frac{1}{A\cdot C - B^2}\cdot\left(d_{ij}\right)_{\mathrm{NxN}} \cdot\left(A\,\mu_j - B\right)_{\mathrm{Nx1}} \cdot t + \frac{1}{A\cdot C - B^2}\left(\cdot d_{ij}\right)_{\mathrm{NxN}} \left(\cdot C - \mu_j\,B\right)_{Nx1}$$

In essence, we find that the set of points of R^N corresponding to the optimal portfolios for each expected return t is a line within that space. We will call this the critical line (CL). This line has an efficient segment and an inefficient segment. The efficient segment corresponds to the values $t < t^*$. The direction vector of the critical line and the crossing point are given respectively by:

$$\vec{v} = \left(v_i\right)_{Nx1} = \frac{1}{A\cdot C - B^2}\cdot\left(d_{ij}\right)_{NxN} \left(A\,\mu_j - B\right)_{Nx1}$$

$$a = \left(a_i\right)_{Nx1} = \frac{1}{A\cdot C - B^2}\cdot\left(d_{ij}\right)_{NxN} \left(C - \mu_j\,B\right)_{Nx1}$$

The crossing point “a” has coordinates that correspond to the optimal portfolio among all those with zero expected return.

4. THE EFFICIENT FRONTIER (EF)

4.1 The efficient frontier with no constraints on the weights of assets

The MVP and the CL are in an N-dimensional space within the hyper plane:

$$\sum_{i=1}^{N} w_i = 1 .$$

Now we will relate the expected return t with the variance V for each point of the critical line. The expected return t will be the independent variable, and variance V the dependent variable. This will lead to a real function of a real variable whose graphic representation is on the plane R^2.

$$V=\frac{1}{A\cdot C-B^2}\cdot\left[t\cdot\left(A\cdot\mu_i-B\right)_{1xN}\cdot\left(d_{ij}\right)_{NxN}+\left(C-\mu_i\cdot B\right)_{1xN}\cdot\left(d_{ij}\right)_{NxN}\right]\left(\sigma_{ij}\right)_{NxN}$$

$$\left[\frac{1}{A\cdot C-B^2}\cdot\left(d_{ij}\right)_{NxN}\cdot\left(A\cdot\mu_j-B\right)_{Nx1}\cdot t+\left(d_{ij}\right)_{NxN}\cdot\left(C-\mu_j\cdot B\right)_{Nx1}\right]=$$

$$=\frac{1}{\left(A\cdot C-B^2\right)^2}\cdot\left(A\cdot\mu_i-B\right)_{1xN}\cdot\left(d_{ij}\right)_{NxN}\cdot\left(A\cdot\mu_j-B\right)_{Nx1}\cdot t^2+$$

$$+\frac{2}{\left(A\cdot C-B^2\right)^2}\cdot\left(C-\mu_i\cdot B\right)_{1xN}\cdot\left(d_{ij}\right)_{NxN}\cdot\left(A\cdot\mu_j-B\right)_{Nx1}\cdot t+$$

$$+\frac{1}{\left(A\cdot C-B^2\right)^2}\cdot\left(C-\mu_i\cdot B\right)_{1xN}\cdot\left(d_{ij}\right)_{NxN}\cdot\left(C-\mu_j\cdot B\right)_{Nx1}=$$

$$=\frac{1}{\left(A\cdot C-B^2\right)^2}\cdot\left(A\cdot\mu_i-B\right)_{1xN}\cdot\left(d_{ij}\right)_{NxN}\cdot\left(A\cdot\mu_j-B\right)_{Nx1}\cdot t^2+$$

$$+\frac{2}{\left(A\cdot C-B^2\right)^2}\cdot\left(C-\mu_i\cdot B\right)_{1xN}\cdot\left(d_{ij}\right)_{NxN}\cdot\left(A\cdot\mu_j-B\right)_{Nx1}\cdot t+$$

$$+\frac{1}{\left(A\cdot C-B^2\right)^2}\cdot\left(C-\mu_i\cdot B\right)_{1xN}\cdot\left(d_{ij}\right)_{NxN}\cdot\left(C-\mu_j\cdot B\right)_{Nx1}=$$

$$=\frac{A}{A\cdot C-B^2}\cdot t^2-\frac{2\cdot B}{A\cdot C-B^2}\cdot t+\frac{C}{A\cdot C-B^2}=\mathrm{a}\cdot t^2+\mathrm{b}\cdot t+\mathrm{c} \qquad [6]$$

where:

$$a=\frac{A}{A\cdot C-B^2}$$
$$b=\frac{-2\cdot B}{A\cdot C-B^2} \qquad [7]$$
$$c=\frac{C}{A\cdot C-B^2}$$

(The coefficient c corresponds to the minimum variance among all the portfolios with zero expected return.)

This shows that the function relating the expected return t to the variance V along the critical line is a quadratic function. We can also ensure that the second-degree coefficient “a” is positive, since the inverse of the variance-covariance matrix is defined as positive. Therefore, the graphical representation will be an upward opening parabola, with a minimum at the vertex that corresponds to the MVP.

The expected return on the MVP is given by:

$$t^* = \frac{-b}{2a} = \frac{B}{A}$$

and substituting in [6] gives:

$$V^* = \frac{A}{A \cdot C - B^2} \cdot \left(\frac{B}{A}\right)^2 - \frac{2 \cdot B}{A C - B^2} \cdot \frac{B}{A} + \frac{C}{A C - B^2} =$$

$$= \frac{B^2}{A \cdot C - B^2} \cdot \frac{1}{A} - \frac{2 \cdot B}{A C - B^2} \cdot \frac{B}{A} + \frac{C \cdot A}{\left(A \cdot C - B^2\right) \cdot A} =$$

$$= \frac{-B^2 + C \cdot A}{\left(A \cdot C - B^2\right) \cdot A} = \frac{1}{A}$$

These results agree with those obtained in Section 2.

5.2 Derivability of the efficient frontier when there are constraints on the weights of assets

When there are no constraints on the weights of assets that may make up a portfolio the efficient frontier in the space $t - V$ has the shape of a parabola and the feasible set is not bounded. However, when there are constraints on the weights of assets, the efficient frontier is formed by a continuous succession of a finite number of parabolic arcs. In addition, it can be seen that two consecutive arcs are tangents to the point of intersection.

When there are constraints on the weights of assets, the efficient set has an explicit equation $V = f(E(\tilde{r})) = f(t)$ defined by bands, in such a way that each band has a distinct second-degree polynomial. These polynomials are such that the lateral limits of the function $V = f(E(\tilde{r})) = f(t)$ coincide at the points of intersection. The lateral derivatives also coincide at these points of intersection, which indicates the derivability of the function *f*. Esteve (1995) presented a rigorous proof of this property. Nonetheless, in the demonstration of this property, there is a special case in which derivability cannot be guaranteed. This occurs when the Minimum Variance Point (*MVP*) coincides with a vertex of the feasible set in the n-dimensional space. Next, we present an example with three assets and demonstrate that the efficient frontier does not admit a derivative at the minimum point.

Counterexample Below we outline an example which proves non-derivability in a specific case. Let us suppose that three assets A, B and C have expected returns

$$(\mu_i)_{1xN} = (0.10, \quad 0.12, \quad 0.08)$$

of 0.10, 0.12 and 0.08 respectively and the following variance and covariance matrix:

$$(\sigma_{ij})_{NxN} = \begin{pmatrix} 0.0005852 & 0.0008364 & 0.0007692 \\ 0.0008364 & 0.0012348 & 0.0011244 \\ 0.0007692 & 0.0011244 & 0.0010332 \end{pmatrix}$$

We assume that the weights w_1, w_2, w_3 of assets A, B and C, respectively, are always real numbers between 0 and 1. If we create combinations of investments A, B and C with non-negative weights, the expected returns on all possible portfolios will be between 0.08 and 0.12. Under these conditions, the critical line will be formed by the infinite points (w_1, w_2, w_3), which are the solution of the program:

$$MIN\ V(w_1, w_2, w_3) = (w_1, w_2, w_3) \cdot \begin{pmatrix} 0.0005852 & 0.0008364 & 0.0007692 \\ 0.0008364 & 0.0012348 & 0.0011244 \\ 0.0007692 & 0.0011244 & 0.0010332 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \qquad [8]$$

conditional on:

$$\begin{cases} \mathrm{w}_1 + \quad \mathrm{w}_2 + \quad \mathrm{w}_3 = 1 & [9] \\ 0.10 \cdot w_1 + 0.12 \cdot w_2 + 0.08 w_3 = t & [10] \\ \mathrm{w}_1 \geq 0, \quad \mathrm{w}_2 \geq 0 \quad \mathrm{w}_3 \geq 0 & \end{cases}$$

for $t \in [0.08, 0.12]$.

Attempts to obtain the critical line by solving this program for the infinites $t \in [0.08, 0.12]$ would be highly problematic.

In this example, it can be verified that:

The critical line is constituted by two line segments $\bar{S}_1 = \overline{(0,0,1),(1,0,0)}$ and $\bar{S}_2 = \overline{(1,0,0),(0,1,0)}$ in the increasing direction of the parameter $t = E(\tilde{r})$.

The segment $\bar{S}_1 = \overline{(0,0,1),(1,0,0)}$ is constituted by points in the form $(w_1, 0, 1-w_1)$, which verifies that the expected return on the portfolio corresponding to these points is:

$$t = E(\tilde{r}) = 0.10 \cdot w_1 + 0.08 \cdot (1 - w_1) = 0.02 w_1 + 0.08 \qquad [11]$$

Specifically, for $w_1 = 0$ we have that *t = 0.08* and for $w_1 = 1$ we have that *t = 0.10*.

If we remove w_1 and w_3 from the expressions [9] and [10], which correspond to two planes, and express them as a function of w_2 and t we obtain:

$$w_1 = 2 \cdot (w_2 - 25t + 2) \qquad [12]$$
$$w_3 = w_2 - 50 \cdot t + 5 \qquad [13]$$

Substituting [12] and [13] in expression (8) gives us:

$$V(w_2, t) =$$
$$= (2 \cdot (w_2 - 25t + 2), w_2, w_2 - 50t + 5) \cdot \begin{pmatrix} 0.0005852 & 0.0008364 & 0.0007692 \\ 0.0008364 & 0.0012348 & 0.0011244 \\ 0.0007692 & 0.0011244 & 0.0010332 \end{pmatrix} \cdot \begin{pmatrix} 2 \cdot (w_2 - 25t + 2) \\ w_2 \\ w_2 - 50 \cdot t + 5 \end{pmatrix} =$$
$$= 0.0004352 \cdot w_2^2 + 0.2t^2 - 0.0184 w_2 t - 0.0584 \cdot t + 0.0027104 \cdot w_2 + 0.0044252 \qquad [14]$$

A partial derivation with respect to w_2 and t gives:

$$\frac{\partial V}{\partial w_2} = 0.0008704 \cdot w_2 - 0.0184t + 0.0027104$$
$$\frac{\partial V}{\partial t} = -0.0184 \cdot w_2 + 0.40t - 0.0584$$

For $w_2 = 0 \text{ and } t \in [0.08, 0.10]$, we can show that:

$$\frac{\partial V}{\partial w_2} > 0 \text{ and } \frac{\partial V}{\partial t} < 0$$

The positive sign of the first of these partial derivatives shows that the points of segment $\bar{S}_1$ belong to the critical line; while the negative sign of the second partial derivative indicates that this segment is contained within the non-efficient part of the critical path.

The segment $\bar{S}_2 = \overline{(1,0,0),(0,1,0)}$ is constituted by points in the form $(w_1, 0, 1 - w_1)$ which verifies that the expected return on the portfolio corresponding to these points is:

$$t = E(\tilde{r}) = 0.10 \cdot (1 - w_2) + 0.12 \cdot w_2 = 0.02w_2 + 0.10 \qquad [15]$$

Specifically, for $w_2 = 0$ we have that *t = 0.10* and for $w_2 = 1$ we have that *t = 0.12*. If we remove *w₁* and *w₂* from the expressions [9] and [10] and express them as a function of w_3 we obtain:

$$w_1 = -2 \cdot \left(w_3 - 25t + 3 \right) \qquad [16]$$

$$w_2 = \mathrm{w}_3 + 50 \cdot t - 5 \qquad [17]$$

Substituting [16] and [17] in expression [8] gives us:

$V(w_3,t) =$

$$= (-2\cdot(w_3 - 25t + 3)\ , w_3 + 50t - 5 , w_3\ )\cdot \begin{pmatrix} 0.0005852 & 0.0008364 & 0.0007692 \\ 0.0008364 & 0.0012348 & 0.0011244 \\ 0.0007692 & 0.0011244 & 0.0010332 \end{pmatrix} \cdot \begin{pmatrix} -2\cdot(w_3 - 25t + 3) \\ w_3 + 50t - 5 \\ w_3 \end{pmatrix} =$$

$$= 0.0004352\cdot w_3^2 + 0.368t^2 + 0.02512 w_3\, t - 0.04848\cdot t + 0.0016146\cdot w_2 + 0.04848044252$$

and a partial derivation with respect to w_3 and t gives:

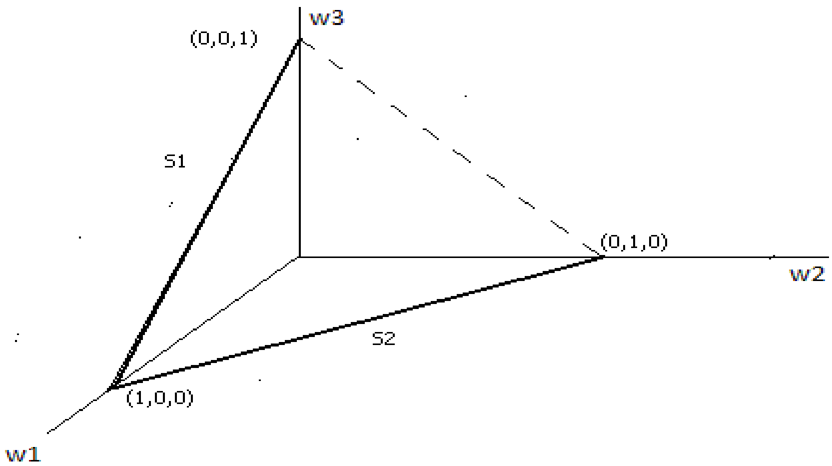
$$\frac{\partial V}{\partial w_3} = 0.0008704\cdot w_3 - 0.02512t + 0.0016416$$

$$\frac{\partial V}{\partial t} = 0.02512\cdot w_3 - 0.736t - 0.04848$$

For $w_3 = 0$ and $t \in [0.10, 0.12]$ it can be show that:

$$\frac{\partial V}{\partial w_3} > 0 \text{ and } \frac{\partial V}{\partial t} > 0$$

The positive sign of the first of these partial derivatives shows that the points of segment $\bar{S}_2$ belong to the critical line, while the positive sign of the second partial derivative also indicates that this segment is contained within the efficient part of the critical line.



Graph 1: The critical line

If we substitute the point $(w_1, 0, 1-w_1) \in \overline{S_1}$ in expression [1] we obtain:

$$V(w_1) = (w_1, 0, 1-w_1) \cdot \begin{pmatrix} 0.0005852 & 0.0008364 & 0.0007692 \\ 0.0008364 & 0.0012348 & 0.0011244 \\ 0.0007692 & 0.0011244 & 0.0010332 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ 0 \\ 1-w_1 \end{pmatrix} =$$

$$= 0.00008 \cdot w_1^2 + 0.000528 \cdot w_1 + 0.0044252 \qquad [18]$$

If for expression [11] corresponding to segment $\overline{S_1}$ we remove the expected return *t* and substitute in [18] we obtain:

$$V = f(E(\tilde{r})) = f(t) = 0.20 \cdot t^2 - 0.0584 t + 0.0044252 \quad (0.08 \leq t \leq 0.10) \qquad [19]$$

This is the expression of the efficient frontier in its decreasing section.

Similarly, if we substitute the point $(1-w_2, w_2, 0) \in \overline{S_2}$ in expression [8] we obtain:

$$V(w_2) = (w_2, 1-w_2, 0) \cdot \begin{pmatrix} 0.0005852 & 0.0008364 & 0.0007692 \\ 0.0008364 & 0.0012348 & 0.0011244 \\ 0.0007692 & 0.0011244 & 0.0010332 \end{pmatrix} \cdot \begin{pmatrix} w_2 \\ 1-w_2 \\ 0 \end{pmatrix} =$$

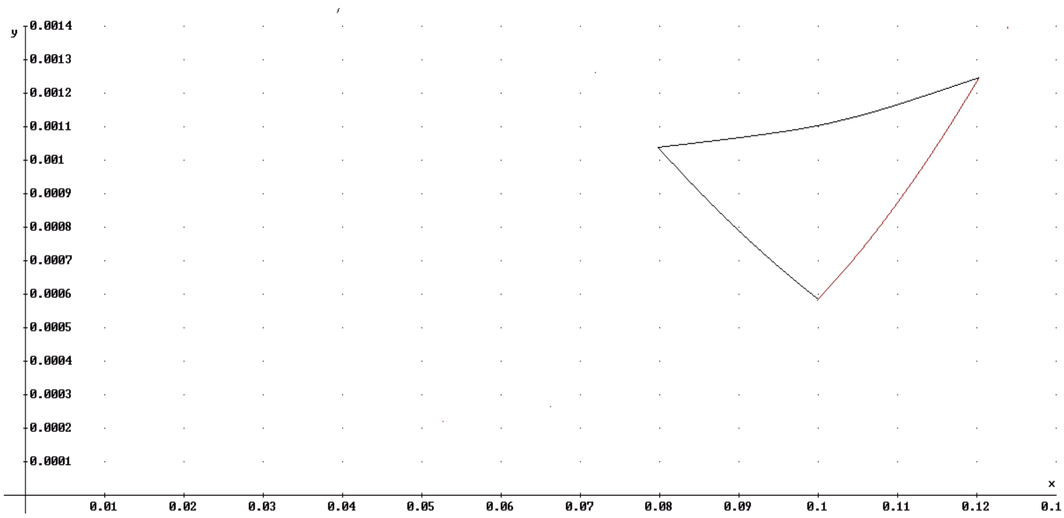
$$= 0.001472 \cdot w_2^2 + 0.0005024 \cdot w_2 + 0.0005852 \quad (0 \leq w_2 \leq 1) \qquad [20]$$

If for expression [15] corresponding to segment $\overline{S_2}$ we remove the expected return *t* and substitute in [20] we obtain:

$$V = f(E(\tilde{r})) = f(t) = 0.368 \cdot t^2 - 0.04848 t + 0.0017532 \quad (0.10 \leq t \leq 0.12) \qquad [21]$$

This is the expression of the efficient set in its increasing section. Combining [19] and [21] in a single expression gives us:

$$V = f(E(\tilde{r})) = f(t) = \begin{cases} 0.20 \cdot t^2 - 0.0584 t + 0.0044252 & (0.08 \leq t \leq 0.10) \\ 0.368 \cdot t^2 - 0.04848 t + 0.0017532 & (0.10 \leq t \leq 0.12) \end{cases}$$



Graph.2. Feasible set and efficient frontier

We can see that:

$$f(0.10) = \lim_{t \to 0.10^-} f(t) = \lim_{t \to 0.10^+} f(t) = 0.0005852$$

This proves that *f* is continuous in t = 0.10 and, consequently, it will be continuous for $t \in (0.08, 0.12)$.

Moreover, if we derive *f* taking into account the two expressions that lead to its determination, we obtain:

$$f'(t) = \begin{cases} 0.40 \cdot t - 0.05840 & (0.08 \le t < 0.10) \\ 0.736 \cdot t - 0.04848 & (0.10 < t \le 0.12) \end{cases}$$

If we calculate the lateral limits of f'(t) in t = 0.10 we obtain:

$$\lim_{t \to 0.10^-} f'(t) = -0.0184 \neq \lim_{t \to 0.10^+} f'(t) = 0.02512$$

This proves that *f(t)* cannot be derived at the point *t = 0.10*, since it has different lateral derivatives at this point. Consequently, we have to do the following.

At the point *t = 0.10*, it can be shown that the function *V = f(t)* is continuous, but not derivable.

The function $V = f(t)$ can be derived in the set $(0.08, 0.10) \cup (0.10, 0.12)$.

Bearing in mind the sign of $f'(t)$, we have that $f(t)$ is decreasing for $t \in (0.08, 0.10)$ and increasing for $t \in (0.10, 0.12)$. Given that $f(t)$ is continuous in the interval $[0.08, 0.12]$ we have that $f(t)$ shows a minimum at the point $t = 0.10$.

5. THE RISK-FREE ASSET AND THE MARKET PORTFOLIO

Now suppose that in addition to the N risk assets there is an additional asset with an expected return constant $R_0 < t^*$. We will call this a risk-free asset. Since its return is always constant, the variance of its return and the covariances with the returns on risky assets will always be zero.

Now suppose the existence of a portfolio p with a proportion $1-w$ of risk-free asset and a share w of a portfolio p' formed by a fixed basket with the N risky assets ($w \geq 0$). Suppose the expected return and the variance of portfolio $\underline{p'}$ are $t_{p'}$ and $V_{p'}$ respectively.

The expected return of portfolio p is:

$$t = (1-w) \cdot R_0 + w\,t_{p'} \qquad [22]$$

and its variance:

$$V = \left(1-w, \quad w\right) \cdot \begin{pmatrix} 0 & 0 \\ 0 & V_{p'} \end{pmatrix} \cdot \begin{pmatrix} 1-w \\ w \end{pmatrix} = w^2\, V_{p'} \qquad [23]$$

If we clear w in [22] and substitute in [23], we get:

$$V = \left(\frac{t - R_0}{t_{p'} - R_0} \right)^2 \cdot V_{p'} \qquad [24]$$

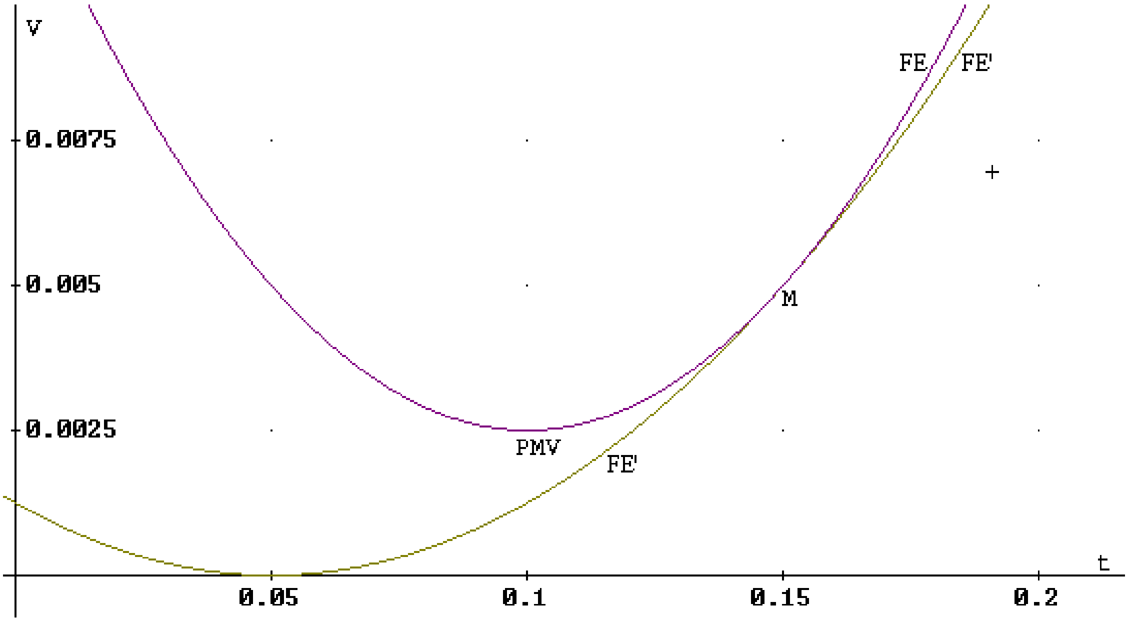
This expression, as well as the efficient frontier [6], produces a graphical representation of a parabola.

To combine risky and risk-free assets efficiently, portfolio *p' must* be on the efficient frontier obtained in Section 4. However, among the infinite portfolios, there is one optimal portfolio on the efficient frontier. This is portfolio M, which is located at the tangency point of parabola [6] with parabola [24], which is the result of equating *p`* = *M*. Since *M* is an optimal portfolio, all investors will choose it to perform combinations of risky assets with the risk-free asset. Any other composition of risky assets will be rejected for not being optimal. As all investors will choose the same portfolio *M*, this portfolio is called the market portfolio[1]. By making *p'* = *M* the expression [24] becomes:

$$V = \left(\frac{t - R_0}{t_M - R_0} \right)^2 \cdot V_M \qquad [25]$$

The parabola which corresponds to the graph of [25] is a new efficient frontier *EF '* that dominates the efficient frontier [6] obtained in Section 4, since for every expected return t provides a minor variance V, except at the point $t = t_M$ ($w = 1$), which corresponds to the *M* portfolio in which the two parabolas are tangents and hence the variances obtained by [6] and [25] are identical.

[1] The weight w_i^M of the *i*-asset within the market portfolio is the stock-exchange capitalization of this asset divided by the sum of the risky N assets. The market capitalization of an asset is derived by multiplying the number of shares (into which the capital of the company is divided) by their price.



R_0 t^* t_M

Graph 3. The parabolas [6] and [25] with the values $R_o = 0.05$, $t_M = 0.15$ and $V_M = 0.005$

We are now going to determine the point of tangency $t = t_M$ of [6] and [26]. By differentiating and equating both expressions, we obtain:

$$\frac{dV}{dt} = \frac{2(t - R_0)}{(t_M - R_0)^2} \cdot V_M = 2 \cdot a\, t + b$$

And by replacing t with t_M:

$$\frac{2(t_M - R_0)}{(t_M - R_0)^2} \cdot V_M = 2 \cdot a\, t_M + b \Leftrightarrow \frac{2}{t_M - R_0} \cdot (a\, t_M^2 + b\, t_M + c) = 2 \cdot a\, t_M + b \Leftrightarrow$$

$$\Leftrightarrow 2 \cdot (a\, t_M^2 + b\, t_M + c) = (2 \cdot a\, t_M + b)(t_M - R_0) \Leftrightarrow$$

$$\Leftrightarrow 2 \cdot a\, t_M^2 + 2 \cdot b\, t_M + 2 \cdot c = 2 \cdot a\, t_M^2 - 2 \cdot a \cdot R_0\, t_M + b\, t_M - b\, R_0 \Leftrightarrow$$

$$\Leftrightarrow 2 \cdot b\, t_M + 2 \cdot c = -2 \cdot a\, R_0\, t_M + b\, t_M - b\, R_0 \Leftrightarrow (b + 2 \cdot a \cdot R_0)\, t_M = -2 \cdot c - b\, R_0 \Leftrightarrow$$

$$\Leftrightarrow t_M = \frac{-2 \cdot c - b \cdot R_0}{b + 2 \cdot a \cdot R_0}$$

When the coefficients *a*, *b* and *c* are substituted by their values determined by [7] we get:

$$t_M = \frac{-2 \cdot c - b \cdot R_0}{b + 2 \cdot a \cdot R_0} = \frac{-2 \cdot C + 2 \cdot B\, R_0}{-2 \cdot B + 2 \cdot A\, R_0} = \frac{B\, R_0 - C}{A\, R_0 - B} \qquad [26]$$

Once the expected return t_M of the market portfolio has been determined, we will calculate its composition. However, to facilitate this calculation we need a definition and two previous results:

Definition 1. For each of the *N* risky assets, the risk premium is defined as:

$$PR_i = E(\tilde{r}_i) - R_0 = \mu_i - R_0 \qquad (1 \le \mathrm{i} \le \mathrm{N})$$

The risk premium is the excess expected return of the asset *i* over the certain return of the risk-free asset.

Similarly, the risk premium is defined as follows for a portfolio p:

$$PR_p = E(\tilde{r}_p) - R_0$$

In particular, for the market portfolio we have:

$$PR_M = E(\tilde{r}_M) - R_0$$

Lemma 1:

$$\frac{(A\cdot\mu_i - B)(C - B\cdot R_0) + (C - \mu_i\cdot B)\cdot(B - A\cdot R_0)}{B - A\cdot R_0} = \frac{A\cdot C - B^2}{B - A\cdot R_0}\cdot PR_i$$

Proof:

$$\frac{(A\cdot\mu_i - B)(C - B\cdot R_0) + (C - \mu_i\cdot B)\cdot(B - A\cdot R_0)}{B - A\cdot R_0} =$$

$$= \frac{A\cdot C\cdot\mu_i - A\cdot B\cdot\mu_i\cdot R_0 - B\cdot C + B^2\cdot R_0 + C\cdot B - C\cdot A\cdot R_0 - \mu_i\cdot B^2 + \mu_i\cdot A\cdot B\cdot R_0}{B - A\cdot R_0} =$$

$$= \frac{(A\cdot C - B^2)\cdot\mu_i - (A\cdot C - B^2)\cdot R_0}{B - A\cdot R_0} = \frac{(A\cdot C - B^2)\cdot(\mu_i - R_0)}{B - A\cdot R_0} = \frac{A\cdot C - B^2}{B - A\cdot R_0}\cdot PR_i$$

Lemma 2:

$$B - A \cdot R_0 = (1)_{1xN} \cdot \left(d_{ij}\right)_{NxN} \cdot \left(PR_j\right)_{Nx1}$$

Proof:

$$B - A \cdot R_0 = \sum_{i=1}^{N} \mu_i \cdot \sum_{j=1}^{N} d_{ij} - R_0 \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} = \sum_{i=1}^{N} \mu_i \cdot \sum_{j=1}^{N} d_{ij} - \sum_{i=1}^{N} R_0 \sum_{j=1}^{N} d_{ij} =$$

$$= \sum_{i=1}^{N} (\mu_i - R_0) \cdot \sum_{j=1}^{N} d_{ij} = \sum_{i=1}^{N} PR_i \cdot \sum_{j=1}^{N} d_{ij} = \sum_{j=1}^{N} \sum_{i=1}^{N} d_{ji} \cdot PR_i =$$

$$= (1)_{1xN} \cdot \left(d_{ij}\right)_{NxN} \cdot \left(PR_j\right)_{Nx1}$$

(In this demonstration, the variance-covariance matrix and its inverse are assumed to be symmetric.)

With the two previous results, we can now tackle the problem of calculating the composition of the market portfolio *M*. To achieve this, we replace the t_M expression obtained in [26] in the vector equation of the *CL* at the end of paragraph 2:

$$\left(w_i^M\right)_{Nx1} = \frac{1}{A \cdot C - B^2} \cdot \left(d_{ij}\right)_{NxN} \cdot \left[\left(A \cdot \mu_j - B\right)_{Nx1} \cdot t_M + \left(C - B \cdot \mu_j\right)\right] =$$

$$= \frac{1}{A \cdot C - B^2} \cdot \left(d_{ij}\right)_{NxN} \cdot \left[\left(A \cdot \mu_j - B\right)_{Nx1} \cdot \frac{B \cdot R_0 - C}{A \cdot R_0 - B} + \left(C - B \cdot \mu_j\right)\right] =$$

$$= \frac{1}{A \cdot C - B^2} \cdot \left(d_{ij}\right)_{NxN} \cdot \left(\frac{A \cdot C - B^2}{B - A \cdot R_0} \cdot PR_j\right)_{Nx1} = \frac{1}{B - A \cdot R_0} \cdot \left(d_{ij}\right)_{NxN} \cdot \left(PR_j\right)_{Nx1} =$$

$$= \frac{\left(d_{ij}\right)_{NxN} \cdot \left(PR_j\right)_{Nx1}}{(1)_{1xN} \cdot \left(d_{ij}\right)_{NxN} \cdot \left(PR_j\right)_{Nx1}}$$

In essence, the composition of the market portfolio is obtained by applying the inverse variance-covariance matrix $\left(d_{ij}\right)_{NxN}$ to the risk premium vector $\left(PR_i\right)_{Nx1}$. Finally, the components of the resulting vector are divided by the sum of the same (in this way, the sum of the resulting vector components has a value of 1).

$$\frac{\left(d_{ij}\right)_{NxN} \cdot \left(PR_j\right)_{Nx1}}{(1)_{1xN} \cdot \left(d_{ij}\right)_{NxN} \cdot \left(PR_j\right)_{Nx1}} \qquad [27]$$

6. THE CAPITAL ASSET PRICING MODEL (CAPM)

Definition 2. We call β-Sharpe of asset *i* the regression coefficient [2] of returns from this asset with respect to the return from the market portfolio

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} \qquad [28]$$

The capital asset pricing model relates β_i to the respective PR_i risk premiums and to the PR_M market portfolio risk premium (called simply the market risk premium).

To obtain this relation, we develop the definition [28] and apply the formula for the composition of the market portfolio from [27]

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{\vec{e}_i \cdot \left(\sigma_{ij}\right)_{NxN} \left(w_j^M\right)_{Nx1}}{\left(w_i^M\right)_{1xN} \cdot \left(\sigma_{ij}\right)_{NxN} \left(w_j^M\right)_{Nx1}} =$$

$$= (1)_{1xN} \cdot \left(\sigma_{ij}\right)_{NxN} \left(PR_j\right)_{Nx1} \cdot \frac{\vec{e}_i \cdot \left(\sigma_{ij}\right)_{NxN} \left(d_{ij}\right)_{NxN} \left(PR_j\right)_{Nx1}}{\left(PR_i\right)_{1xN} \cdot \left(d_{ij}\right)_{NxN} \left(\sigma_{ij}\right)_{NxN} \left(d_{ij}\right)_{NxN} \left(PR_j\right)_{Nx1}} =$$

$$= \frac{\vec{e}_i \cdot \left(PR_j\right)_{Nx1}}{\dfrac{\left(PR_i\right)_{1xN} \cdot \left(d_{ij}\right)_{NxN} \left(PR_j\right)_{Nx1}}{(1)_{1xN} \cdot \left(\sigma_{ij}\right)_{NxN} \left(PR_j\right)_{Nx1}}} = \frac{PR_i}{\left(PR_i\right)_{1xN} \cdot \left(w_i\right)_{Nx1}} = \frac{PR_i}{\sum_{j=1}^{N} PR_j \cdot w_j} = \frac{PR_i}{\sum_{j=1}^{N} (\mu_j - R_0) \cdot w_j} =$$

$$= \frac{PR_i}{\sum_{j=1}^{N} \mu_j \cdot w_j - \sum_{j=1}^{N} R_0 \cdot w_j} = \frac{PR_i}{E(\tilde{r}_M) - R_0 \sum_{j=1}^{N} w_j} = \frac{PR_i}{E(\tilde{r}_M) - R_0 \cdot 1} = \frac{PR_i}{PR_M}$$

(In the previous demonstration, the $\vec{e}_i$ vector is the *i*-th vector of the canonical basis of R^N, i.e. the vector that is null for all the components except the *i*-th component, which is equal to 1.)

---

[2] Regression line slope of the *i* asset returns over the market portfolio *M* returns.

In essence, the CAPM states that the β-Sharpe of asset *i* is equal to the quotient between the PR_i risk premium of the corresponding asset and the market risk premium PR_M.

The expression that summarizes the *CAPM* is usually presented in several ways:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{PR_i}{PR_M} = \frac{\mu_i - R_0}{E(\tilde{r}_M) - R_0} = \frac{E(\tilde{r}_i) - R_0}{E(\tilde{r}_M) - R_0}$$

By passing the denominator to multiply the Beta, the following expression is obtained:

$$E(\tilde{r}_i) - R_0 = \beta_i \cdot (E(\tilde{r}_M) - R_0)$$

This is how the final expression of *CAPM* in the literature on the subject is most frequently presented[3].

7. APPLICATION TO SPANISH REAL ESTATE MUTUAL FUNDS

During the entire period 1-2002 to 12-2008 there were only 3 Spanish real estate mutual funds[4]. Of these, we obtain the following information (Fernández, 2010):

- Expected returns: $(\mu_i)_{1xN} = (0.004652, 0.00359, 0.006217)$

  (Mean returns for 84 months from 1-2002 to 12-2008.)

- Returns variance-covariance matrix:

$$(\sigma_{ij})_{NxN} = \begin{pmatrix} 0.0000183 & 0.000008187 & 0.0000081135 \\ 0.000008187 & 0.00004505 & 0.0000002182 \\ 0.0000081135 & 0.0000002182 & 0.00004273 \end{pmatrix}$$

[3] Sharpe, W. (2000)

[4] SCH inmobiliario 1 FII, BBVA propiedad, Segurfondo

We take as the free-asset return: $R_0 = 0.002704$

(Geometric mean monthly returns of the Euribor to 1 year.)[5]

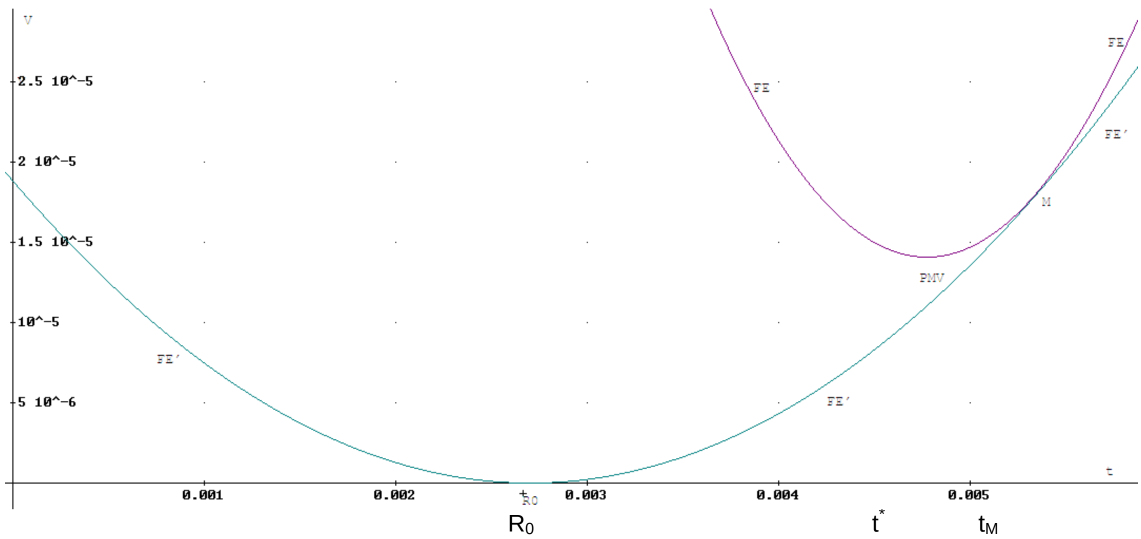
When we apply the mean-variance model, we obtain:

- Constant terms: A = 71368; B = 340.7; C = 1.709
- MVP: $(t^*, V^*) = (0.004774, 0.00001401)$
- Efficient frontier: $V = 12.14 \cdot t^2 - 0.1159t + 0.0002907$
- Market portfolio: $M = (0.4977, 0.0404, 0.4619)$

$$(t_M, V_M) = (0.005332, 0.00001779)$$

- Betas:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0.7712 \\ 0.3371 \\ 1.3368 \end{pmatrix}$$



Graph 4. The parabolas [6] and [25] for the Spanish real mutual funds in the period 2002-2008

Si all investors choose an optimal combination of risky assets with a risk-free asset, then *M* is the market portfolio and the vector (0.4977,0.0404,0.4619) would give the relative proportion of their assets.

[5]European Central Bank (2002-2008)

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