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A geometric chracterization of the nucleolus of the assignment game

Francesc Llerena, Marina Núñez and Carles Rafels[1]

Adreça correspondència:

Francesc Llerena
Departament de Gestió d'Empreses. Universitat Rovira i Virgili
Avda. Universitat 1, E-43204 Reus, email: francesc.llerena@urv.es

Marina Núñez and Carles Rafels
Departament de Matemàtica Económica, Financera i Actuarial. Universitat de Barcelona
Avd. Diagonal, 690, E-08034 Barcelona. Email: mnunez@ub.edu, crafels@ub.edu

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Abstract: Maschler et al. (1979) provide a geometrical characterization for the intersection of the kernel and the core of a coalitional game, showing that those allocations that lie in both sets are always the midpoint of certain bargaining range between each pair of players. In the case of the assignment game, this means that the kernel can be determined as those core allocations where the maximum amount, that can be transferred without getting outside the core, from one agent to his/her optimally matched partner equals the maximum amount that he/she can receive from this partner, also remaining inside the core. We now prove that the nucleolus of the assignment game can be characterized by requiring this bisection property be satisfied not only for optimally matched pairs but also for optimally matched coalitions.
Key words: cooperative games, assignment games, core, kernel, nucleolus
JEL: C71, C78

Resum: Maschler et al. (1979) caracteritzen geomètricament la intersecció del kernel i del core en els jocs cooperatius, demostrant que les distribucions que pertanyen a ambdós conjunts es troben en el punt mig d'un cert rang de negociació entre parelles de jugadors. En el cas dels jocs d'assignació, aquesta caracterització vol dir que el kernel només conté aquells elements del core on el màxim que un jugador pot transferir a una parella òptima és igual al màxim que aquesta parella li pot transferir, sense sortir-se'n del core. En aquest treball demostrem que el nucleolus d'un joc d'assignació queda caracteritzat si requerim que aquesta propietat de bisecció es compleixi no només per parelles, sinó també per coalicions entre sectors aparellades òptimament.

1 Introduction

A two-sided assignment market consists of two disjoint sets of agents, let us say buyers and sellers or firms and workers, and a non-negative real number associated with each possible partnership between two agents of different sectors, that represents the potential profit of forming that pairing. Assuming transferable utility to share the profits of these partnerships, Shapley and Shubik (1972) introduce the assignment game to model this situation in a coalitional form where only individual coalitions and mixed-pair coalitions are relevant. They show that the core of this game is non-empty and consists of those individually rational allocations that are efficient and satisfy pairwise stability, that is, no buyer-seller pair can form a partnership and produce more than the sum of their payoffs. The core of the assignment game has been widely studied in the literature and, since it very rarely reduces to only one point, it becomes necessary to make a selection inside the core.

An outstanding element of the core for arbitrary coalitional games is the nucleolus (Schmeidler, 1969), which is the unique individually rational allocation that lexicographically minimizes the vector of non-increasingly ordered excesses of coalitions. This definition can be interpreted as in Maschler et al. (1979), saying that the nucleolus is fair in the sense that it is the result of an arbitrator's desire to minimize the dissatisfaction of the most dissatisfied coalition. In the aforementioned paper it is described a finite process that iteratively reduces the set of payoffs to a singleton, called the lexicographic center, that is proved to coincide with the nucleolus. The nucleolus has also been analyzed for different classes of combinatorial optimization games, take for instance flow games (Deng et al., 2008) and cyclic permutation games (Solymosi et al., 2005).

Solymosi and Raghavan (1994) present a definition of lexicographic center specialized for assignment games, based on the fact (already pointed out by Huberman, 1980) that for assignment games, only one-player coalitions and mixed-pair coalitions play a role in the calculation of the nucleolus. Making use of that they provide an algorithm that computes the nucleolus of an assignment game. A specialization of this algorithm to the class of neighbor games is given in Hamers et al. (2003).[2]

The kernel is another solution concept for arbitrary coalitional games. It was intro-

[2]See also Raghavan and Sudhölter (2006) for examples of application of this algorithm.

duced by Davis and Maschler (1965) and it always contains the nucleolus. It is shown in Maschler et al. (1979) that for two games with the same core the intersection of the kernel and the core also coincides. In the same paper, a geometric characterization of those allocations in the intersection of the core and the kernel of a game is given. It is shown there that an outcome that lies in both the kernel and the core is always the midpoint of a certain bargaining range between each pair of players. Each endpoint of this range is in the boundary of the core, representing a maximum demand by one player, in that the other player can find a coalition to support him in resisting any greater demand. This view of the kernel gives it an intuitive interpretation as a ”fair division” scheme. However, a similar geometric characterization of the locus of the nucleolus inside the core is not possible for arbitrary games, since there are games with the same core but different nucleolus.[3] Nevertheless, it is known from Núñez (2004) that two assignment games with the same core have the same nucleolus.[4] This suggests the possibility of characterizing the locus of the nucleolus in the core of the assignment game.

The kernel of the assignment game is always included in the core (Driessen, 1998) and it is characterized in Rochford (1984) as those core elements that remain fixed after a rebargaining process. Driessen (1999) relates Rochford's bargaining procedure with the geometric interpretation of the kernel given in Maschler et al. (1979): given an allocation in the core of the assignment game and an optimally matched pair, one can consider the maximum amount that can be transferred from one member of the pair to her/his partner, the payoff to the remaining agents being unaltered, without getting outside the core. In a kernel element, and for each optimal pair, the transfers of both partners are balanced, that meaning that the kernel element is at a midpoint with respect to certain ranges of the core. The aim of the present paper is to determine which other bisection conditions in terms of transfers are necessary to individualize the nucleolus of the assignment game. Under the assumption that there are as many buyers as sellers we consider, for each core allocation and for each subset of buyers, which is the maximum equal payoff that each of them can transfer to his optimal partner without leaving the core. When this maximum

[3]See Maschler et al. (1979) page 335.

[4]An analysis of different assignment games with the same core can be found in Martínez-de-Albéniz et al. (2011a) and Martínez-de-Albéniz et al. (2011b).

transfer equals the maximum transfer of the coalition of partners we say that the initial core allocation satisfies the bisection property with respect to this coalition of buyers. Then, the nucleolus of the assignment game is characterized as the unique core allocation that has the bisection property with respect to all coalitions.

The paper is organized as follows. Section 2 includes the preliminaries about coalitional games and assignment games and Section 3 contains the geometric characterization of the nucleolus.

2 Definitions and notations

Let $N = \{1, 2, ..., n\}$ denote a finite set of players, and 2^N the set of all possible coalitions or subsets of N. The cardinality of coalition S is denoted by $|S|$. Given two coalitions S and T, $S \subseteq T$ denotes inclusion while $S \subset T$ denotes strict inclusion.

A *cooperative game in coalitional form* (a game) is a pair (N, v), where $v : 2^N \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$, is the characteristic function which assigns to each coalition S the worth $v(S)$ it can attain.

Given a game (N, v), a payoff vector is $x \in \mathbb{R}^N$, where x_i stands for the payoff to player $i \in N$. The restriction of x to a coalition S is denoted by $x_{|S}$. An *imputation* is a payoff vector x that is efficient, $\sum_{i\in N} x_i = v(N)$, and individually rational, $x_i \geq v(\{i\})$ for all $i \in N$. The set of all imputations of a game (N, v) is denoted by $I(v)$, and when $I(v) \neq \emptyset$ the game is said to be *essential*. The *excess* of a coalition S at an imputation $x \in I(v)$ is $e^v(S, x) = v(S) - \sum_{i\in S} x_i$.

A *solution concept* defined on the set of games with player set N is a rule that assigns to each such game a subset of efficient payoff vectors. The best known set-solution concept for coalitional games is the core. The *core* of a game is the set of payoff vectors that are efficient and coalitionally rational, that is, $\sum_{i\in S} x_i \geq v(S)$ for all $S \subseteq N$. A game with a non-empty core is called *balanced*. Given a balanced game (N, v), a well known single–valued core selection is *the nucleolus* (Schmeidler, 1969).

Let us define the *vector* $\theta(x) \in \mathbf{R}^{2^n-2}$ *of excesses* of all coalitions (different from the grand coalition and the empty set) at x, arranged in a nonincreasing order. That is to say, for all $k \in \{1, \ldots, 2^n - 2\}$, $\theta(x)_k = e^v(S_k, x)$, where $\{S_1, \ldots, S_k, \ldots, S_{2^n-2}\}$ is the set of all nonempty coalitions in N different from N, and $e^v(S_k,x) \geq e^v(S_{k+1},x)$. Then the *nucleolus* of the game (N,v) is the imputation $\eta(v)$ which minimizes $\theta(x)$ with respect to the lexicographic order over the set of imputations: $\theta(\eta(v)) \leq_{Lex} \theta(x)$ for all $x \in I(v)$. This means that, for all $x \in I(v)$, either $\theta(\eta(v)) = \theta(x)$ or $\theta(\eta(v))_1 < \theta(x)_1$ or there exists $k \in \{1,2,\ldots,2^n-3\}$ such that $\theta(\eta(v))_i = \theta(x)_i$ for all $1 \leq i \leq k$ and $\theta(\eta(v))_{k+1} < \theta(x)_{k+1}$.

The *kernel* (Davis and Maschler, 1965) is another set-solution concept for cooperative games. The kernel, $\mathscr{K}(v)$, of an essential cooperative game (N,v) is always nonempty and it contains the nucleolus. For *zero–monotonic games*,[5] as it is the case of assignment games, the kernel can be described by

$$\mathscr{K}(v) = \{z \in I(v) \mid s^v_{ij}(z) = s^v_{ji}(z) \text{ for all } i,\, j \in N,\, i \neq j\},$$

where the maximum surplus $s^v_{ij}(z)$ of player i over another player j with respect to the imputation z is defined by

$$s^v_{ij}(z) = \max\{\, e^v(S,z) | S \subseteq N,\, i \in S,\, j \notin S\}\,.$$

We will just write $s_{ij}(z)$ when no confusion regarding the game v can arise.

The assignment model

A two-sided assignment market (M,M',A) is defined by a finite set of buyers M, a finite set of sellers M', and a nonnegative matrix $A = \left(a_{ij}\right)_{(i,j)\in M\times M'}$. The real number a_{ij} represents the profit obtained by the mixed-pair $(i,j) \in M \times M'$ if they trade. Let us assume there are $|M| = m$ buyers and $|M'| = m'$ sellers, and $n = m + m'$ is the cardinality of $N = M \cup M'$.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M'_0 \subseteq M'$, such that $|M_0| = \left|M'_0\right| = \min\{|M|,|M'|\}$. We write $(i,j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. The set of all matchings is denoted by $\mathscr{M}(M,M')$. If $m = m'$, the assignment market is said to be square.

[5] A game (N,v) is zero-monotonic if for any pair of coalitions S,T, $S \subset T \subseteq N$ it holds $v(S) + \sum_{i\in T\setminus S} v(\{i\}) \leq v(T)$.

A *matching* $\mu \in \mathscr{M}(M,M')$ is *optimal* for the assignment market (M,M',A) if for all $\mu' \in \mathscr{M}(M,M')$ we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and we denote the set of optimal matchings by $\mathscr{M}_A^*(M,M')$.

Shapley and Shubik (1972) associate to any assignment market (M,M',A) a cooperative game in coalitional form, with player set $N = M \cup M'$ and characteristic function w_A, defined by: for $S \subseteq M$ and $T \subseteq M'$, $w_A(S \cup T) = \max\left\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathscr{M}(S,T)\right\}$, $\mathscr{M}(S,T)$ being the set of matchings between S and T. The core of the assignment game is always non-empty, and it is enough to impose coalitional rationality for one-player coalitions and mixed-pair coalitions:

$$C(w_A) = \left\{ (u,v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'} \;\middle|\; \begin{array}{l} \sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_A(N), \\ u_i + v_j \geq a_{ij}, \text{ for all } (i,j) \in M \times M' \end{array} \right\}, \tag{1}$$

where $\mathbb{R}_+$ stands for the set of non-negative real numbers. It follows from (1) that, if μ is an optimal matching, unassigned agents receive null payoff and, moreover,

$$\text{if } (i,j) \in \mu, \text{ then } u_i + v_j = a_{ij}. \tag{2}$$

Since the assignment game has a non-empty core, its nucleolus always lies in the core. Moreover, it can be deduced from Huberman (1980) that only individual coalitions and mixed-pair coalitions need to be taken into account in the computation of the nucleolus of an assignment game. Solymosi and Raghavan (1994) provide an algorithm to compute the nucleolus of the assignment game.

As for the kernel of assignment games, it turns out that it is always included in the core, $\mathscr{K}(w_A) \subseteq C(w_A)$ (Driessen, 1998). Moreover, if $(u,v) \in C(w_A)$, then *(a)* $s_{ij}(z) = 0$ whenever $i,j \in M$ or $i,j \in M'$, and *(b)* if $i \in M$ and $j \in M'$, then $s_{ij}(z)$ is always attained at the excess of some individual coalition or mixed–pair coalition:

$$s_{ij}(u,v) = \max_{k \in M' \setminus \{j\}} \{-u_i, a_{ik} - u_i - v_k\}.$$

As a consequence, given $(u,v) \in C(w_A)$, we get that $(u,v) \in \mathscr{K}(w_A)$ if and only if $s_{ij}(u,v) = s_{ji}(u,v)$ for all (i,j) belonging to all the optimal matchings, since the remaining equalities hold trivially.

By adding dummy players, that is, null rows or columns in the assignment matrix, we can assume from now on, and without loss of generality, that the number of sellers equals the number of buyers.

3 Characterization of the nucleolus

Given an arbitrary coalitional game (N,v), with any core allocation $z \in C(v)$ and any pair of agents $i, j \in N$, there is associated a non-negative real number $\delta^v_{ij}(z)$ designating the largest amount that can be transferred from player i to player j with respect to the core allocation z while remaining in the core of the game (N,v):

$$\delta^v_{ij}(z) = \max\{\varepsilon \geq 0 \mid z - \varepsilon e^i + \varepsilon e^j \in C(v)\},$$

where, for all $i \in N$, $e^i \in \mathbb{R}^N$ is the vector defined by $e^i_i = 1$ and $e^i_k = 0$ for all $k \neq i$, $k \in N$. This critical number $\delta^v_{ij}(z)$ was introduced by Maschler et al. (1979). For any core element $z \in C(v)$, this number $\delta^v_{ij}(z)$ is related to the excess $s^v_{ij}(z)$ in the definition of the kernel by $\delta^v_{ij}(z) = -s^v_{ij}(z)$. They prove in the aforementioned paper that a bisection property characterizes those elements in the intersection of the kernel and the core: $z \in C(v) \cap \mathscr{K}(v)$ if and only if z is the midpoint of the core segment with extreme points $z - \delta^v_{ij}(z)e^i + \delta^v_{ij}(z)e^j$ and $z + \delta^v_{ji}(z)e^i - \delta^v_{ji}(z)e^j$, for all $i, j \in N$. In this section we introduce a stronger bisection property that characterizes the nucleolus of the assignment game.

Let (M, M', A) be an assignment market with as many buyers as sellers, that is, $|M| = |M'| = m$. For any $R \subseteq M$ or $R \subseteq M'$, the vector $e^R \in \mathbb{R}^m$ stands for $e^R_k = 1$ if $k \in R$ and $e^R_k = 0$ if $k \notin R$. Then, for each $S \subseteq M$ and $T \subseteq M'$, $S, T \neq \emptyset$, we define *the largest amount that can be transferred from players in S to players in T with respect to the core allocation (u,v) while remaining in the core of w_A* by

$$\delta^{w_A}_{S,T}(u,v) = \max\{\varepsilon \geq 0 \mid (u - \varepsilon e^S, v + \varepsilon e^T) \in C(w_A)\}. \tag{3}$$

Similarly,

$$\delta^{w_A}_{T,S}(u,v) = \max\{\varepsilon \geq 0 \mid (u + \varepsilon e^S, v - \varepsilon e^T) \in C(w_A)\}. \tag{4}$$

We write $\delta_{S,T}(u,v)$ and $\delta_{T,S}(u,v)$, respectively, if no confusion arises regarding the assignment game $(M \cup M', w_A)$.

Notice that if there exists an optimal matching $\mu \in \mathscr{M}_A^*(M,M')$ such that S and T do not correspond each other by this optimal matching ($\mu(S) \neq T$), then $\delta_{S,T}(u,v) = \delta_{T,S}(u,v) = 0$ for all $(u,v) \in C(w_A)$. The reason is that if there exists $i \in S$ such that $\mu(i) \notin T$ (and similarly for $j \in T$ such that $\mu^{-1}(j) \notin S$) we have that the payoff vector $(u',v') = (u - \varepsilon e^S, v + \varepsilon e^T)$ will lie outside the core for all $\varepsilon > 0$, since $u'_i + v'_{\mu(i)} = u_i - \varepsilon + v_{\mu(i)} \neq a_{i\mu(i)}$. This is why we will only consider transfers between coalitions that correspond by an optimal matching.

Definition 1. *Let (M,M',A) be an assignment market, $\mu \in \mathscr{M}_A^*(M,M')$ and $S \subseteq M$, $S \neq \emptyset$. The core allocation (u,v)* has the S-bisection property *with respect to μ if and only if $\delta_{S,\mu(S)}(u,v) = \delta_{\mu(S),S}(u,v)$.*

Both for theoretical and practical purposes, it will be useful to have an explicit expression of the critical numbers $\delta_{S,T}(u,v)$ when $S \subseteq M$ and $\mu(S) = T$ by some optimal matching μ. Given $(u,v) \in C(w_A)$, if we want the allocation $(u',v') = (u - \varepsilon e^S, v + \varepsilon e^T)$ to remain in the core of the assignment game, the inequalities $u_i - \varepsilon \geq 0$ for all $i \in S$ and $u_i - \varepsilon + v_j \geq a_{ij}$ for all $i \in S$ and all $j \in M' \setminus T$ must hold. This means that, for all $(u,v) \in C(w_A)$, given a non-empty coalition $S \subseteq M$ and $\mu \in \mathscr{M}_A^*(M,M')$,

$$\delta_{S,\mu(S)}(u,v) = \min_{i \in S, j \in M' \setminus \mu(S)} \{u_i, u_i + v_j - a_{ij}\}. \tag{5}$$

and, similarly,

$$\delta_{\mu(S),S}(u,v) = \min_{i \in M \setminus S, j \in \mu(S)} \{v_j, u_i + v_j - a_{ij}\}. \tag{6}$$

At this point it is worth to remark that, by Maschler et al. (1979) and Driessen (1999), the kernel of the assignment game is the set of core allocations satisfying the $\{i\}$-bisection property for all $i \in M$. Since the nucleolus belongs to the kernel, it satisfies this property. What we state in the next theorem is that the nucleolus of the assignment game can be characterized by the S-bisection property, for all $S \subseteq M$, $S \neq \emptyset$, and with respect to any optimal matching μ.

Theorem 2. *Let (M,M',A) be a square assignment market. The nucleolus is the unique core allocation satisfying the S-bisection property, for all $S \subseteq M$, $S \neq \emptyset$. Formally, if $(u,v) \in C(w_A)$ and $\mu \in \mathscr{M}_A^*(M,M')$, then*

$$(u,v) = \eta(w_A) \text{ if and only if } \delta_{s,\mu(S)}(u,v) = \delta_{\mu(S),S}(u,v) \text{ for all } S \subseteq M, S \neq \emptyset.$$

Proof. We first prove that the nucleolus satisfies the S-bisection property for all $S \subseteq M$, $S \neq \emptyset$. Let us denote (for short) by $\eta = (\eta_{|M}, \eta_{|M'})$ the nucleolus $\eta(w_A)$, and let us fix a coalition $S \subseteq M$, $S \neq \emptyset$. We now consider the core segment $[\eta_S^-, \eta_S^+]$ that can be obtained from η by means of doing equal transfers from agents in S to agents in $\mu(S)$ (and reciprocally). By (3) and (4), the extreme points of the segment are

$$\begin{aligned} \eta_S^- &= \left(\eta_{|M} - \delta_{S,\mu(S)}(\eta)e^S, \eta_{|M'} + \delta_{S,\mu(S)}(\eta)e^{\mu(S)}\right) \text{ and} \\ \eta_S^+ &= \left(\eta_{|M} + \delta_{\mu(S),S}(\eta)e^S, \eta_{|M'} - \delta_{\mu(S),S}(\eta)e^{\mu(S)}\right). \end{aligned}$$

For simplicity of notation we will omit the subscript and write η^- and η^+.

Let $K = \delta_{S,\mu(S)}(\eta) + \delta_{\mu(S),S}(\eta)$, then the segment $[\eta^-, \eta^+]$ can be described as the set of those payoff vectors $(u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ for which there exists $\varepsilon_{(u,v)} \in [0,K]$ such that

$$\begin{array}{ll} u_i = \eta_i^+ - \varepsilon_{(u,v)} \text{ for all } i \in S, & u_i = \eta_i^+ \text{ for all } i \in M \setminus S \\ v_j = \eta_j^+ + \varepsilon_{(u,v)} \text{ for all } j \in \mu(S), & v_j = \eta_j^+ \text{ for all } j \in M' \setminus \mu(S). \end{array} \tag{7}$$

Note, from the definition of η^+ and η^-, that $\varepsilon_{\eta^+} = 0$ and $\varepsilon_{\eta^-} = K$. Moreover, the nucleolus η is described taking $\varepsilon_\eta = \delta_{\mu(S),S}$.

Since by definition the vector of ordered excesses (with respect to individual and mixed-pair coalitions) of the nucleolus, $\theta(\eta)$, satisfies $\theta(\eta) \leq_L \theta(u,v)$ for all $(u,v) \in C(w_A)$, we have, in particular, that $\theta(\eta)$ lexicographically minimizes the vector of excesses $\theta(u,v)$ over $[\eta^-, \eta^+]$. We will see that η satisfies the equation $\delta_{S,\mu(S)}(\eta) = \delta_{\mu(S),S}(\eta)$ or, equivalently, $\varepsilon_\eta = \frac{K}{2}$.

If the segment $[\eta^-, \eta^+]$ reduces to a single point, we are done. Otherwise, let us fix an arbitrary allocation $(u,v) \in [\eta^-, \eta^+]$ and analyze first the excesses of mixed-pair coalitions at (u,v):

- If $(i,j) \in S \times \mu(S)$, and taking (7) into account, there exists $\varepsilon_{(u,v)} \in [0,K]$ such that

$$\begin{aligned} e(\{i,j\},(u,v)) &= a_{ij} - u_i - v_j = a_{ij} - (\eta_i^+ - \varepsilon_{(u,v)}) - (\eta_j^+ + \varepsilon_{(u,v)}) \\ &= a_{ij} - \eta_i^+ - \eta_j^+ = e(\{i,j\}, \eta^+). \end{aligned} \tag{8}$$

- Similarly, if $(i,j) \in (M \setminus S) \times (M' \setminus \mu(S))$, then

$$e(\{i,j\},(u,v)) = a_{ij} - u_i - v_j = a_{ij} - \eta_i^+ - \eta_j^+ = e(\{i,j\}, \eta^+). \tag{9}$$

Since the excesses of the above coalitions are constant on $[\eta^-,\eta^+]$ they need not be considered to lexicographically minimize the vector of excesses $\theta(u,v)$ over the segment $[\eta^-,\eta^+]$. Thus, the relevant excesses of mixed-pair coalitions are those with either one agent in S and the other one in $M'\setminus\mu(S)$ or one agent in $M\setminus S$ and the other one in $\mu(S)$:

- If $(i,j)\in S\times(M'\setminus\mu(S))$, by (7) and the fact that $\eta_i^+=\eta_i^-+K$ and $\eta_j^+=\eta_j^-$, we have

$$\begin{aligned} e(\{i,j\},(u,v)) &= a_{ij}-u_i-v_j = a_{ij}-(\eta_i^+-\varepsilon_{(u,v)})-\eta_j^+ \\ &= a_{ij}-\eta_i^--\eta_j^--K+\varepsilon_{(u,v)} \le -K+\varepsilon_{(u,v)}. \end{aligned} \tag{10}$$

- Similarly, if $(i,j)\in(M\setminus S)\times\mu(S)$, then, by (7),

$$e(\{i,j\},(u,v)) = a_{ij}-u_i-v_j = a_{ij}-\eta_i^+-(\eta_j^++\varepsilon_{(u,v)}) \le -\varepsilon_{(u,v)}. \tag{11}$$

Let us now analyze the excesses of individual coalitions at the allocation $(u,v)\in[\eta^-,\eta^+]$. Notice that if $i\in M\setminus S$, by (7) we have $e(\{i\},(u,v))=-\eta_i^+$ and similarly, if $j\in M'\setminus\mu(S)$ it holds $e(\{j\},(u,v))=-\eta_j^+$. Again, since the excess of the above individual coalitions is constant on $[\eta^-,\eta^+]$ they need not be taken into account to compute the lexicographic minimum of the vector of ordered excess over $[\eta^-,\eta^+]$. It remains to consider the excesses of individual coalitions at (u,v) with $i\in S$ or $j\in\mu(S)$:

- If $i\in S$, then by (7) and taking into account that $\eta_i^+=\eta_i^-+K$, we have

$$e(\{i\},(u,v)) = -(\eta_i^+-\varepsilon_{(u,v)}) = -\eta_i^--K+\varepsilon_{(u,v)} \le -K+\varepsilon_{(u,v)}. \tag{12}$$

- If $j\in\mu(S)$, by (7) we get

$$e(\{j\},(u,v)) = -(\eta_j^++\varepsilon_{(u,v)}) \le -\varepsilon_{(u,v)}. \tag{13}$$

Now, by definition of η^+, there must be some core constraint that is tight at the extreme point η^+ and not at all points of the segment $[\eta^-,\eta^+]$. If this core constraint were related to a coalition $\{i\}$ with $i\in S$, then $\eta_i^+=0$ would imply, by (7), $u_i=-\varepsilon_{(u,v)}\ge 0$ or, equivalently, $\varepsilon_{(u,v)}=0$, for all $(u,v)\in[\eta^-,\eta^+]$, in contradiction with the assumption that $[\eta^-,\eta^+]$ is not a singleton. Also, if the constraint that is tight at η^+ is $\{i,j\}$ with $(i,j)\in S\times(M'\setminus\mu(S))$ we have, by the second equality in (10), that for all $(u,v)\in[\eta^-,\eta^+]$, $e(\{i,j\},(u,v))=\varepsilon_{(u,v)}$ and since excesses at core allocations are always non-positive we obtain $\varepsilon_{(u,v)}=0$ for all $(u,v)\in[\eta^-,\eta^+]$, which implies, as before, a contradiction. This means that either:

a) There exists $(i*,j*) \in (M\setminus S)\times \mu(S)$ such that $\eta^+_{i*}+\eta^+_{j*} = a_{i*j*}$, and then for all $(u,v)\in[\eta^-,\eta^+]$, and taking (11) and (13) into account, we have

$$e(\{i*,j*\},(u,v)) = a_{i*,j*} - \eta^+_{i*} - (\eta^+_{j*} + \varepsilon_{(u,v)}) = -\varepsilon_{(u,v)} \geq e(T,(u,v)), \qquad (14)$$

for all $T=\{i,j\}$ with $(i,j)\in(M\setminus S)\times\mu(S)$ and all $T=\{j\}$ with $j\in\mu(S)$.

b) Or there exists $j*\in\mu(S)$ with $\eta^+_{j*}=0$, and then for all $(u,v)\in[\eta^-,\eta^+]$, again taking (11) and (13) into account, we have

$$e(\{j*\},(u,v)) = -\varepsilon_{(u,v)} \geq e(T,(u,v)), \qquad (15)$$

for all $T=\{i,j\}$ with $(i,j)\in(M\setminus S)\times\mu(S)$ and all $T=\{j\}$ with $j\in\mu(S)$.

Similarly, there must be some core constraint that is tight at η^- and not at all other points of $[\eta^-,\eta^+]$. If this core constraint were related to a coalition $\{j\}$ with $j\in\mu(S)$, then $\eta^-_j=0$ would imply, by (7), $v_j=\eta^+_j+\varepsilon_{(u,v)}=\eta^-_j-K+\varepsilon_{(u,v)}=-K+\varepsilon_{(u,v)}$. Since $\varepsilon_{(u,v)}\in[0,K]$ and $v_j\geq 0$, we have $v_j=0$ for all $(u,v)\in[\eta^-,\eta^+]$. Also, if the constraint that is tight at η^- is $\{i,j\}$ with $(i,j)\in(M\setminus S)\times\mu(S)$ we have by (11), and the fact that $\eta^-_i=\eta^+_i$ and $\eta^-_j=\eta^+_j+K$, $e(\{i,j\},(u,v))=a_{ij}-\eta^+_i-(\eta^+_j+\varepsilon_{(u,v)})=a_{ij}-\eta^-_i-(\eta^-_j-K+\varepsilon_{(u,v)})=K-\varepsilon_{(u,v)}\leq-\varepsilon_{(u,v)}$, for all $(u,v)\in[\eta^-,\eta^+]$, which implies $K=0$ or, equivalently, the reduction of the segment $[\eta^-,\eta^+]$ to only one point, in contradiction with our assumption. This means that either:

c) There exists $(i*,j*)\in S\times(M'\setminus\mu(S))$ such that $\eta^-_{i*}+\eta^-_{j*}=a_{i*j*}$, and then for all $(u,v)\in[\eta^-,\eta^+]$, and taking (10) and (12) into account, we have

$$e(\{i*,j*\},(u,v)) = -K+\varepsilon_{(u,v)} \geq e(T,(u,v)), \qquad (16)$$

for all $T=\{i,j\}$ with $(i,j)\in S\times(M'\setminus\mu(S))$ and all $T=\{i\}$ with $i\in S$.

d) Or there exists $i*\in S$ with $\eta^-_{i*}=0$, and then for all $(u,v)\in[\eta^-,\eta^+]$, again taking (10) and (12) into account, we have

$$e(\{i*\},(u,v)) = -K+\varepsilon_{(u,v)} \geq e(T,(u,v)), \qquad (17)$$

for all $T=\{i,j\}$ with $(i,j)\in S\times(M'\setminus\mu(S))$ and all $T=\{i\}$ with $i\in S$.

To sum up, let us denote by $\mathscr{C}$ the set of coalitions that are essential for the computation of the nucleolus of the assignment game (individual and mixed-pair coalitions). Then, for all $(u,v) \in [\eta^-, \eta^+]$ we have

$$\max_{S \in \mathscr{C}} e(S,(u,v)) = \max\{-\varepsilon_{(u,v)}, -K + \varepsilon_{(u,v)}\}$$

and thus

$$\min_{(u,v)\in[\eta^-,\eta^+]} \max_{S \in \mathscr{C}} e(S,(u,v))$$

is attained at the point $(u,v) \in [\eta^-, \eta^+]$ such that $-\varepsilon_{(u,v)} = -K + \varepsilon_{(u,v)}$, that is $\varepsilon_{(u,v)} = \frac{K}{2}$. Since the nucleolus lexicographically minimizes the vector of excesses over the segment $[\eta^-, \eta^+]$ we deduce that $\varepsilon_\eta = \frac{K}{2}$ and thus, since $\varepsilon_\eta = \delta_{\mu(S),S}(\eta)$, we have $\delta_{S,\mu(S)}(\eta) = \delta_{\mu(S),S}(\eta)$, which proves the S-bisection property of the nucleolus with respect to the arbitrary coalition $S \subseteq M$.

To conclude the proof we must see that a core allocation different from the nucleolus fails to satisfy the S-bisection property for some coalition $S \subseteq M, S \neq \emptyset$. Let us consider $z \in C(w_A)$ such that $z \neq \eta$. This implies the existence of a non-empty coalition $S \subseteq M$ such that $z_i > \eta_i$ for all $i \in S$ and $z_i \leq \eta_i$ for all $i \in M \setminus S$(otherwise we interchange the roles of z and η). As a consequence, it follows from (2) that $z_j < \eta_j$ for all $j \in \mu(S)$ and $z_j \geq \eta_j$ for all $j \in M' \setminus \mu(S)$. Then, making use of expressions (5) and (6),

$$\begin{aligned}
\delta_{S,\mu(S)}(z) &= \min_{i \in S, j \in M' \setminus \mu(S)} \{z_i, z_i + z_j - a_{ij}\} > \min_{i \in S, j \in M' \setminus \mu(S)} \{\eta_i, \eta_i + \eta_j - a_{ij}\} \\
&= \min_{j \in \mu(S), i \in M \setminus S} \{\eta_j, \eta_i + \eta_j - a_{ij}\} > \min_{j \in \mu(S), i \in M \setminus S} \{z_j, z_i + z_j - a_{ij}\} = \delta_{\mu(S),S}(z),
\end{aligned}$$

where the second equality follows from the fact that $\delta_{S,\mu(S)}(\eta) = \delta_{\mu(S),S}(\eta)$. Then, $\delta_{S,\mu(S)}(z) > \delta_{\mu(S),S}(z)$ implies that z does not satisfies the S-bisection property. □

Let us stress that Theorem 2 is useful to check if a given allocation in the core of an assignment game $(M \cup M', w_A)$ is in fact its nucleolus. Moreover, whenever several optimal matchings exist, the difficulty of the problem may be reduced. Indeed, because of the remark made after expression (4), it is enough to check that the given allocation satisfies the S-bisection property for all coalitions $S \subseteq M$ that have the same image by all optimal matchings $\mu \in \mathscr{M}^*_A(M,M')$. Formally, if $\mathscr{S} = \left\{ S \subseteq M \mid \mu(S) = \mu'(S) \text{ for all } \mu, \, \mu' \in \mathscr{M}^*_A(M,M') \right\}$, then

$$\eta(w_A) = \left\{(u,v) \in C(w_A) \,|\, \delta_{S,\mu(S)}(u,v) = \delta_{\mu(S),S}(u,v) \text{ for all } S \in \mathscr{S} \right\}. \tag{18}$$

Also, after Theorem 2 one may wonder if the nucleolus of the assignment game could be characterized by imposing the bisection property for some smaller subset of coalitions of M. The next example shows that, in this sense, our characterization cannot be refined.

Example 3. *Let it be the assignment market with set of buyers* $M = \{1,2,3\}$*, set of sellers* $M' = \{4,5,6\}$ *and defined by matrix*

$$A = \begin{pmatrix} 7 & 6 & 3 \\ 5 & 4 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

This market has two optimal matchings: $\mu_1 = \{(1,4),(2,5),(3,6)\}$ and $\mu_2 = \{(1,5),(2,4),(3,6)\}$. Thus, to check that a given allocation is the nucleolus we only need to verify that it satisfies the bisection property for those $S \subseteq M$ such that $\mu_1(S) = \mu_2(S)$. In this case, $\eta(w_A) = (3.5, 1.5, 0.5; 3.5, 2.5, 0.5)$ since it satisfies the bisection property with respect to coalitions $\{3\}$, $\{1,2\}$ and $\{1,2,3\}$. Moreover, if we consider the core element $(u,v) = (3,1,0.5;4,3,0.5)$ we realize that $\delta_{\{3\},\{6\}}(u,v) = 0.5 = \delta_{\{6\},\{3\}}(u,v)$, $\delta_{M,M'}(u,v) = 0.5 = \delta_{M',M}(u,v)$, but $\delta_{\{1,2\},\{4,5\}}(u,v) = 0.5$ while $\delta_{\{4,5\},\{1,2\}}(u,v) = 1.5$. This last remark shows that the characterization of Theorem 2 cannot be improved.

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