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## The proportional distribution in a cooperative model with external opportunities


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## Abstract

We study a cooperative problem where agents contribute a certain amount of money or capital in order to obtain a surplus. The proportional distribution with respect to the contributions of players is a core element of the cooperative game associated. Within this basic model, an external agent is introduced in order to evaluate the potential profit of every subcoalition of agents in the case this new agent enters. This analysis can produce that the relative bargaining power of agents may be modified. In particular, we evaluate whether the proportional distribution is still a robust proposal from the point of view of the bargaining set of a cooperative game with coalition structure (Davis and Maschler, 1963). Since, in general, the proportional distribution fails to be a bargaining set element of this game, a sufficient condition for the proportional allocation to belong to the bargaining is stated. A necessary condition is also analysed. Finally, we state a sufficient condition that guarantees the proportional distribution to be the unique element of the bargaining set of the associated game with coalition structure.

## Resum

En aquest article es considera un problema de cooperació entre agents on cada agent realitza una contribució (diners, capital, treball, esforç) per tal d'obtenir un benefici comú a repartir. El repartiment proporcional respecte a les contribucions és una distribució que pertany al nucli del joc cooperatiu associat. A partir d'aquest model bàsic s'introdueix un agent extern que pot realitzar una determinada aportació que serveix per avaluar el potencial benefici de cada subcoalició d'agents si aquest nou agent finalment entrés. Aquesta anàlisi pot produir que el poder relatiu dels agents hagi variat. en concret s'avalua si la distribució proporcional és encara robusta des del punt de vista de la seva pertinença al conjunt de negociació. Amb aquest objectiu, analitzem el problema utilitzant el model de joc cooperatius amb estructura de coalició. Donat que, en general, la distribució proporcional, no pertany al conjunt de negociació, s'estudia una condició suficient per a que així sigui. També enunciem una condició necessària, i finalment es proposa una condició suficient que garanteix que el repartiment proporcional és la única distribució existent dins del conjunt de negociació



**Keywords**: cooperative game, proportional distribution, bargaining set, coalition structure

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# 1 Introduction

Many cooperative activities involve the contribution of money, capital or labour by the agents engaged in these tasks. Some models and applications can be found in Izquierdo and Rafels (2001), Lemaire (1991) or the one-input one-output model analysed by Mas-Colell (1980). The proportional distribution with respect to agents' initial contribution arises as a natural allocation rule of the surplus generated. In this paper we approach this situation from the point of view of cooperative games. We mainly focus on analysing the effects in the proportional distribution when we allow the possibility to cooperate with an external agent. The role of this external member is merely to revise the properties of the proportional allocation considering these eventual *external opportunities* of agents.

To illustrate the above idea, let us consider the following example. Three partners in a company are contributing an amount of money $c_1 = 100$, $c_2 = 100$ and $c_3 = 300$, respectively. Let us suppose that the average return expressed in percentage is given by the following function $r$ that depends on the amount invested $x$: $r(x) = 20\%$, if $0 \leq x < 500$ and $r(x) = 30\%$, if $x \geq 500$. Notice the higher yield (30%) is only obtained when the three investors jointly contribute, obtaining $500 \times 30\% = 150$. This suggests that the proportional distribution, $p = (30, 30, 90)$, seems hard to be refuted, although the third investor contributes three times of both investor 1 or 2.

Let us consider now an external partner (investor 0) endowed with a capital $c_0 = 200$. This external agent will give new insight into the proportional distribution. In particular, investor 3 may argue that an eventual coalition with investor 0 allows him to obtain the highest yield without the contribution of partners 1 and 2, while a similar argument cannot be used by these partners. This fact indicates that the relative opportunity costs of agents is altered and so the proportional distribution might be revised.

Firstly, in this paper we analyse a cooperative model where there are increasing returns to scale with respect to some initial endowments. With this assumption, the proportional distribution always lies in the core of the cooperative game associated. This model can be embedded in the more general class of games, the average monotonic cooperative games (Izquierdo and Rafels, 2001), for which the core and the bargaining set (Davis and Maschler, 1967) do coincide. In Section 2 we introduce this basic model and we analyse some structure of the core of the associated game. Basically, we provide a necessary and sufficient condition to identify when the core of the game shrinks to the proportional distribution. Moreover, we state a technical property concerning the reduced game that result from moving out an agent. This property will be a cornerstone for proving results in Section 3.

Secondly, to capture the effects of the external agent, we consider in Section 3 the model of cooperative games with coalition structures (Davis and Maschler, 1963; Aumann and Drèze, 1974). Since the core of the game associated with coalition structure is quite often empty, we focus on the concept of bargaining set which is always non-empty (Peleg, 1963). To motivate our analysis, we begin providing an example where the proportional distribution does not belong to the bargaining set of the game with coalition structure. This example reveals important differences between the two cooperative models with and without external opportunities. Moreover, it also prompts to solve some interesting questions. Firstly, we wonder whether or not the proportional distribution belongs to the bargaining set with coalition structure (see Theorem 1 and 2). Secondly, we analyse situations where the bargaining set with coalition structure shrinks to the proportional distribution.

## 2 The basic model and the core

Let $N = \{1, 2, \ldots, n\}$ be a set of agents (players) that are engaged in a joint activity. We denote by $c_i > 0$ the contribution (capital, labor, effort) of agent $i$ and by $c = (c_1, c_2, \ldots, c_n)$ the vector of contributions. For all $S \subseteq N$, we denote by $c(S)$ the joint contribution of a subgroup $S$ of agents, that is $c(S) = \sum_{i \in S} c_i$. A cooperative game is a function $v$ that assigns to each subcoalition of agents $S \subseteq N$ the surplus generated $v(S) \in \mathbb{R}_+$, where $v(\varnothing) = 0$. We say that $v(S)$ is the worth of coalition $S$. Furthermore, and for this model, we assume increasing returns to scale; that is, for all $S, T \subseteq N$,

$$c(S) \leq c(T) \Rightarrow \frac{v(S)}{c(S)} \leq \frac{v(T)}{c(T)}. \tag{1}$$

We call a game satisfying (1) a *game with increasing returns to scale* or *IRS game* and we denote it by $(N, v, c)$. We denote the class of all *IRS games* with player set $N$ by $IRG^N$. It is easy to see that any game with increasing returns to scale is superadditive, i.e. $v(S) + v(T) \leq v(S \cup T)$, for any $S, T \subseteq N$, $S \cap T = \varnothing$.

A distribution of the worth of the grand coalition is a vector $x = (x_i)_{i\in N}$ where $x_i$ is the payoff of agent $i$ such that $\sum_{i\in N} x_i = v(N)$. In the sequel we denote, for all $S \subseteq N$, $x(S) = \sum_{i\in S} x_i$ and $x_S$ the restriction of $x$ to the members of $S$. The core of a cooperative game $v$ with player set $N$ is defined as

$$C(N, v) = \{x \in \mathbb{R}^N \mid x(S) \geq v(S), \text{ for all } S \subseteq N, \text{ and } x(N) = v(N)\}.$$

An easy way to prove the non-emptiness of the core of an IRS game is to check that the proportional distribution, $p(v, c) = (p_i)_{i\in N}$, where $p_i = c_i \cdot \frac{v(N)}{c(N)}$, is a core element of the game. The proof is straightforward taking into account (1). On the other hand, the core of a IRS game shrinks into one single point (the proportional distribution) whenever the average return of the grand coalition and all coalitions of $n-1$ members are equal.

**Proposition 1** *Let $(N, v, c) \in IRG^N$ be an increasing returns to scale game. Then*

$$C(N, v) = \{p(v, c)\} \text{ if and only if } \frac{v(N)}{c(N)} = \frac{v(N \setminus \{i\})}{c(N \setminus \{i\})}, \text{ for all } i \in N.$$

*Proof* $\Rightarrow$) Let us suppose on the contrary there exists $i^* \in N$ such that

$$\frac{v(N)}{c(N)} > \frac{v(N \setminus \{i^*\})}{c(N \setminus \{i^*\})},$$

and consider the payoff vector $x \neq p(v, c) \in \mathbb{R}^N$ defined as $x_{i^*} = v(N) - v(N \setminus \{i^*\})$ and $x_i = c_i \cdot \frac{v(N\setminus\{i^*\})}{c(N\setminus\{i^*\})}$, for all $i \in N \setminus \{i^*\}$. We check this vector is in the core of the game contradicting $C(N, v) = \{p(v, c)\}$. To this aim, first notice $x(N) = v(N)$. For any coalition $S \subseteq N \setminus \{i^*\}$, we have

$$x(S) = c(S) \cdot \frac{v(N \setminus \{i^*\})}{c(N \setminus \{i^*\})} \geq c(S) \cdot \frac{v(S)}{c(S)} \geq v(S).$$

If $S \subseteq N$ and $i^* \in S$ then

$$\begin{aligned} x(S) &= x_{i^*} + x(S \setminus \{i^*\}) = v(N) - v(N \setminus \{i^*\}) + c(S \setminus \{i^*\}) \cdot \tfrac{v(N\setminus\{i^*\})}{c(N\setminus\{i^*\})} \\ &= v(N) - c(N \setminus S) \cdot \tfrac{v(N\setminus\{i^*\})}{c(N\setminus\{i^*\})} \geq c(N) \cdot \tfrac{v(N)}{c(N)} - c(N \setminus S) \cdot \tfrac{v(N\setminus\{i^*\})}{c(N\setminus\{i^*\})} \\ &= c(S) \cdot \tfrac{v(N)}{c(N)} + c(N \setminus S) \cdot (\tfrac{v(N)}{c(N)} - \tfrac{v(N\setminus\{i^*\})}{c(N\setminus\{i^*\})}) > c(S) \cdot \tfrac{v(N)}{c(N)} \geq c(S) \cdot \tfrac{v(S)}{c(S)} = v(S). \end{aligned}$$

$\Leftarrow$) If $x \neq p(v,c)$ then, by efficiency of $x$, there exists $i \in N$ such that $x_i > c_i \cdot \frac{v(N)}{c(N)}$ and so $x(N \setminus \{i\}) < c(N \setminus \{i\}) \cdot \frac{v(N)}{c(N)} = c(N \setminus \{i\}) \cdot \frac{v(N \setminus \{i\})}{c(N \setminus \{i\})} = v(N \setminus \{i\})$ contradicting $x$ to be a core element of $v$. □

Notice the condition that guarantees the proportional distribution to be the unique core element says that no player is *essential* to obtain the higher average return, $\frac{v(N)}{c(N)}$. Otherwise, players that become essential in this sense may claim for a higher payoff within the core with respect to the proportional allocation. To be more precise, whenever there is some player $i \in N$ such that $\frac{v(N)}{c(N)} > \frac{v(N \setminus \{i\})}{c(N \setminus \{i\})}$, we have just proved in Proposition 1 that the maximum payoff to this player within the core is his *marginal contribution*, $v(N) - v(N \setminus \{i\}) > p_i$.

To end this section, let us introduce the concept of reduced games and enunciate a technical property concerning reduce IRS games that will be crucial to prove results in next section. Let $v$ be a cooperative game defined on the player set $N$, $\varnothing \neq T \subseteq N$ and $x \in \mathbb{R}^N$. We define the reduced game[1] of $v$ at $x$ on $T$, $r_x^T(v)$ as

$$r_x^T(v)(\varnothing) := 0,$$

$$r_x^T(v)(S) := \max_{Q \subseteq N \setminus T} \{v(S \cup Q) - x(Q)\}, \quad \text{for all } \varnothing \neq S \subseteq T.$$

If the game we reduce is an IRS game, and under some conditions on the allocation $x$, the reduced game is also an IRS game. The proof can be found in Izquierdo and Rafels (2001) – see Proposition 3.1 – for the more general class of average monotonic games .

**Proposition 2** *Let $(N,v,c) \in IRG^N$, $x \in \mathbb{R}^N$ and $i \in N$ such that $x_i \geq c_i \cdot \frac{v(N)}{c(N)}$. Then,*

$$(N \setminus \{i\}, r_x^{N \setminus \{i\}}(v), c_{N \setminus \{i\}}) \text{ is an IRS game.}$$

[1]This definition coincides with the classical Davis-Maschler reduced game concept (Davis and Maschler, 1965) except for the worth of the grand coalition $T$.

# 3 External opportunities

Suppose we have an IRS game $(N, v, c)$. In order to consider external opportunities, let us introduce in the problem an agent 0, an external agent that will serve as a reference to evaluate the bargaining power of the rest of agents. Let us denote by $N_0 = N \cup \{0\}$ the extended set of agents that includes agent 0. This agent might contribute an amount $c_0 > 0$. From now on, with some abuse of notation, we will denote by $c = (c_0, c_1, \ldots, c_n)$ the extended vector of contributions of agents. Let $(N_0, v_0, c)$ be the IRS game derived from this new problem, where $v_0(S) = v(S)$, for all $S \subseteq N$.

As we have mentioned in the Introduction, to analyse the new situation we use the model of cooperative games with coalition structures (Aumann and Drèze, 1974). A coalition structure is a partition of the set of agents. To our purpose we will take the specific partition (or coalition structure) of $N_0$ consisting of the set $N$ and the singleton $\{0\}$. We denote it by $\beta_0 = \{N, \{0\}\}$. An IRS cooperative game with external agent is no more than an IRS cooperative game with coalition structure $\beta_0$. Formally,

**Definition 1** *Let $N = \{1, 2, \ldots, n\}$ be a set of agents and $N_0 = N \cup \{0\}$. An IRS game with external agent is a triplet $(\beta_0, v_0, c)$ where $\beta_0 = \{N, \{0\}\}$, a vector*[2] *$c = (c_0, c_1, \ldots, c_n) \in \mathbb{R}^{N_0}_{++}$ and $(N_0, v_0, c) \in IRG^{N_0}$.*

The set of imputations (or distributions) of the game $(\beta_0, v_0, c)$ is defined as follows:

$$I(\beta_0, v_0) = \left\{ x \in \mathbb{R}^{N_0} \mid \begin{array}{l} x(N) = v_0(N),\ x_i \geq v_0(\{i\}), \text{for all } i \in N, \text{ and} \\ x_0 = v_0(\{0\}) \end{array} \right\}.$$

Notice this model is adequate to analyse the problem with an external agent since an imputation assigns an efficient distribution to the set of agents $N$, $x(N) = v_0(N)$, being the payoff to the external player equal to $v_0(\{0\})$. Hence the objective is not to distribute $v(N_0)$ but using agent $\{0\}$ to distribute $v_0(N))$ among agents in $N$.

**Definition 2** *Given an IRS game with external agent, $(\beta_0, v_0, c)$, the* extended proportional distribution $\mathbf{p}(\beta_0, v_0, c) = (p_i)_{i \in N_0}$ *is defined as*

[2] We denote by $R^{N_0}_{++} = \{x \in R^{N_0} \mid x_i > 0 \text{ for all } i \in N_0\}$.

$$p_i = c_i \cdot \frac{v(N)}{c(N)},\ \textit{for all } i \in N \textit{ and } p_0 = v_0(\{0\}).$$

Following Aumann and Drèze (1974), the core of the IRS game with external agent is then

$$C(\beta_0, v_0) = \{x \in I(\beta_0, v_0) \mid x(S) \geq v(S),\ \text{for all } S \subseteq N_0\}\,.$$

In the definition of the core, let us remark that it is not required $x(N_0) = v(N_0)$. However, all core constraints, including those involving players in $N$ cooperating with player 0, must be satisfied. Quite often, this causes the core to be empty. In fact, the set-solution concept that fits well to analyse external opportunities is the bargaining set (Davis and Maschler, 1963) which is based on objections and counter-objections and it is always a non-empty set.

Let $(\beta_0, v_0, c)$ be an IRS game with external agent and $x \in I(\beta_0, v_0)$. Following Davis and Maschler, an *objection* at $x$ of a player $i \in N$ against a player $j \in N$, with $i \neq j$, is a pair $(S, y)$ where $S \subseteq N_0$, $i \in S$, $j \notin S$ and $y_k > x_k$, for all $k \in S$, with $y(S) = v(S)$.

A *counter-objection* to the objection $(S, y)$ is a pair $(T, z)$ where $T \subseteq N_0$, $j \in T$, $i \notin T$ and $z_k \geq y_k$, for all $k \in T \cap S$, $z_k \geq x_k$, for all $k \in T \setminus S$ with $z(T) = v(T)$.

Let us remark that objections and counter-objections can only be raised by players in $N$. The external agent 0 can take part of these actions but cannot address or receive them.

The bargaining set of an IRS game with external agent is defined as:

$$\mathcal{M}_1^i(\beta_0, v_0) = \{x \in I(\beta_0, v_0) \mid \text{ every objection at } x \text{ has a counter-objection}\}.$$

As we have said, the proportional distribution is always a core element (and so it belongs to the bargaining set) when we deal with the basic model without external opportunities. Let us see that this cannot be the case in the model with external opportunities.

**Example 1** *Following the example given in the introduction we have that* $c = (c_0, c_1, c_2, c_3) =$

$(200, 100, 100, 300)$ *and the worth of the different coalitions are*

$v_0(\{0\}) = 40$, $v_0(\{1\}) = v_0(\{2\}) = 20$, $v_0(\{3\}) = 60$,

$v_0(\{0,1\}) = v_0(\{0,2\}) = 60$, $v_0(\{0,3\}) = 150$,

$v_0(\{1,2\}) = 40$, $v_0(\{1,3\}) = v_0(\{2,3\}) = 80$,

$v_0(\{0,1,2\}) = 80$, $v_0(\{0,1,3\}) = v_0(\{0,2,3\}) = 180$, $v_0(\{1,2,3\}) = 150$, $v(N_0) = 210$,

*where* $v_0(S) = c(S) \cdot r(c(S))$, *for all* $S \subseteq N_0$, *with* $r(x) = 20\%$, *if* $0 \leq x < 500$ *and* $r(x) = 30\%$, *if* $x \geq 500$.

*We claim that the proportional distribution* $\mathbf{p}(\beta_0, v_0, c) = (p_0, p_1, p_2, p_3) = (40, 30, 30, 90)$ *is not in the bargaining set* $\mathcal{M}_1^i(\beta_0, v_0)$. *To check it notice* $p_0 + p_3 = 130 < 150 = v(\{0,3\})$. *The objection of player* 3 *against player* 1 *at* $\mathbf{p}(\beta_0, v_0, c)$ *through coalition* $S = \{0,3\}$ *defined as* $y_0 = 50$ *and* $y_3 = 100$ *cannot be countered by player* 1. *To see it just notice* $\mathbf{p}(S) > v_0(S)$, *for all* $S \subseteq N_0$ *with* $1 \in S$ *and* $3 \notin S$.

*In fact, we claim the unique point in the bargaining set is* $x = (x_0, x_1, x_2, x_3) = (40, 20, 20, 110)$ *(notice all the surplus goes to player 3!!). It is an element of the bargaining set since* $x(S) \geq v_0(S)$, *for all* $S \subseteq N_0$ *with* $3 \notin S$ *(no objection can be addressed to player* 3*) and any objection addressed either to player 1 or 2 is countered by the individual worth* $v_0(\{1\})$ *or* $v_0(\{2\})$ *respectively.*

*For the uniqueness part, any other payoff vector* $x' \neq x$ *satisfies either* $x'_1 > 20$ *and* $x'_2 \geq 20$, *or* $x'_1 \geq 20$ *and* $x'_2 > 20$ *since* $x' \in I(\beta_0, v_0)$. *Let us analyse the first case* $x'_1 > 20$ *and* $x'_2 \geq 20$ *being the other case analogous. Since* $x \in I(\beta_0, v_0)$, *we have* $x'_3 < 110$. *The objection* $(S, y)$ *of player 3 against player 1 at* $x'$ *through* $\{0,3\}$ *defined as* $y_0 = 40 + \frac{110 - x'_3}{2}$ *and* $y_3 = x'_3 + \frac{110 - x'_3}{2}$, *cannot be countered. To check it simply notice* $x'(S) > v(S)$, *for all* $S \subseteq N_0$ *with* $1 \in S$ *but* $3 \notin S$.

A sufficient condition to guarantee that the proportional distribution is in the bargaining set is stated in the next proposition.

**Theorem 1** *Let $(\beta_0, v_0, c)$ be an IRS game with external agent.*

$$\text{If } \frac{v_0(N_0 \setminus \{k\})}{c(N_0 \setminus \{k\})} = \frac{v_0(N)}{c(N)}, \text{ for all } k \in N, \text{ then } \mathbf{p}(\beta_0, v_0, c) \in \mathcal{M}_1^i(\beta_0, v_0).$$

*Proof* Let $\mathbf{p}(\beta_0, v_0, c) = p = (p_i)_{i \in N_0}$. If $v_0(S) - p(S) \leq 0$ for all $S \subseteq N_0$, $S \neq N_0$, then it trivially holds $\mathbf{p}(\beta_0, v_0, c) \in \mathcal{M}_1^i(\beta_0, v_0), \beta_0)$ (there are no objections).

Let $S \subseteq N_0$, $S \neq N_0$, such that $v_0(S) - p(S) > 0$. First notice $0 \in S$, since by definition $p(R) \geq v_0(R)$, for all $R \subseteq N$. Now we prove that for any objection at $p$ there is a counter-objection. For this, consider an arbitrary objection $(S, y)$ at $p$ of a player $i \in S \setminus \{0\}$ against a player $j \in N \setminus S$, that is

$$y_k = c_k \cdot \tfrac{v_0(N)}{c(N)} + \varepsilon_k = p_k + \varepsilon_k \quad \text{for all } k \in S \setminus \{0\},$$

$$y_0 = v_0(\{0\}) + \varepsilon_0 = p_0 + \varepsilon_0,$$

where $\varepsilon_k > 0$, for all $k \in S$, and $\sum_{k \in S} \varepsilon_k = v_0(S) - p(S)$.

Let us see that player $j$ can always built a counter-objection $(N_0 \setminus \{i\}, z)$ to the objection $(S, y)$ as follows:

$$z_k := y_k \qquad \text{for all } k \in S \setminus \{i\},$$

$$z_k := p_k + \tfrac{\Delta}{|N_0 \setminus S|} \qquad \text{for all } k \in N_0 \setminus S,$$

where $\Delta = c(S) \cdot [\frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} - \frac{v_0(S)}{c(S)}] + \varepsilon_i$. Notice, since $\varepsilon_i > 0$ and $\frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} = \frac{v_0(N_0 \setminus \{j\})}{c(N_0 \setminus \{j\})} \geq \frac{v_0(S)}{c(S)}$, we have $\Delta > 0$, and so $z_k > p_k$, for all $k \in N_0 \setminus S$. Moreover, it is a valid counter-objection since

$$
\begin{aligned}
z(N_0 \setminus \{i\}) &= y(S \setminus \{i\}) + p(N_0 \setminus S) + \Delta \\
&= p(S \setminus \{i\}) + \textstyle\sum_{k \in S \setminus \{i\}} \varepsilon_k + p(N_0 \setminus S) + c(S) \cdot [\frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} - \frac{v_0(S)}{c(S)}] + \varepsilon_i \\
&= p(S \setminus \{i\}) + \textstyle\sum_{k \in S} \varepsilon_k + p(N_0 \setminus S) + c(S) \cdot \frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} - c(S) \cdot \frac{v_0(S)}{c(S)} \\
&= p(N_0 \setminus \{i\}) + v_0(S) - p(S) + c(S) \cdot \frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} - v_0(S) \\
&= p(N_0 \setminus \{0, i\}) - p(S \setminus \{0\}) + c_0 \cdot \frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} + p(S \setminus \{0\}) \\
&= p(N_0 \setminus \{0, i\}) + c_0 \cdot \frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} \\
&= c(N_0 \setminus \{0, i\}) \cdot \frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} + c_0 \cdot \frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} \\
&= c(N_0 \setminus \{i\}) \cdot \frac{v_0(N_0 \setminus \{i\})}{c(N_0 \setminus \{i\})} = v_0(N_0 \setminus \{i\}).
\end{aligned}
$$

□

The sufficient condition of the above theorem might seem a little bit *adhoc* to guarantee that the proportional distribution belongs to the bargaining set. Nevertheless, next theorem shows that, under very mild restrictions on the game, very similar relations have to hold.

**Theorem 2** *Let* $(\beta_0, v_0, c)$ *be an IRS game with external agent such that*

*(i)* $v_0(\{1\}) + v_0(\{2\}) + \ldots + v_0(\{n\}) < v_0(N)$, *and*

*(ii)* $\min_{i \in N} c_i \leq c_0$.

*If* $\mathbf{p}(\beta_0, v_0, c) \in \mathcal{M}_1^i(\beta_0, v_0)$, *then* $\dfrac{v_0(N_0 \setminus \{k\})}{c(N_0 \setminus \{k\})} = \dfrac{v_0(N_0 \setminus \{k'\})}{c(N_0 \setminus \{k'\})}$, *for all* $k$, $k' \in N$.

*Proof* Let $\mathbf{p}(\beta_0, v_0, c) = p = (p_i)_{i \in N_0}$. Without lost of generality let us suppose $c_1 \leq c_2 \leq \ldots \leq c_n$ and so $\min_{i \in N} c_i = c_1 \leq c_0$. On one hand, this implies that $c(N) \leq C(N_0 \setminus \{1\})$ and, by definition of IRS game, $\frac{v_0(N)}{c(N)} \leq \frac{v_0(N \setminus \{1\})}{c(N \setminus \{1\})}$. On the other hand, since $v_0(\{1\}) + v_0(\{2\}) + \ldots + v_0(\{n\}) < v_0(N)$, we can easily deduce $p_1 > v_0(\{1\})$.

Let us suppose now that $\mathbf{p}(\beta_0, v_0, c) \in \mathcal{M}_1^i(\beta_0, v_0)$ but there exists $k, k' \in N$ such that $\frac{v_0(N_0 \setminus \{k\})}{c(N_0 \setminus \{k\})} > \frac{v_0(N_0 \setminus \{k'\})}{c(N_0 \setminus \{k'\})}$. Hence,

$$
\frac{v_0(N_0 \setminus \{1\})}{c(N_0 \setminus \{1\})} > \frac{v_0(N_0 \setminus \{n\})}{c(N_0 \setminus \{n\})}. \tag{2}
$$

With this assumption, we can define an objection at $\mathbf{p}(\beta_0, v_0, c)$ without counter-objection which contradicts the fact that the proportional distribution is in the bargaining set $\mathcal{M}_1^i(\beta_0, v_0)$.

The objection $(N_0 \setminus \{1\}, y)$ at $\mathbf{p}(\beta_0, v_0, c)$ of player $n$ against player 1 is defined as follows:

$$y_k = c_k \cdot \tfrac{v_0(N)}{c(N)} + \tfrac{\varepsilon}{n-1} \qquad \text{for all } k \in N_0 \setminus \{0,1\},$$

$$y_0 = v_0(N_0 \setminus \{1\}) - c(N_0 \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} - \varepsilon$$

where $0 < \varepsilon < c_0 \cdot [\tfrac{v_0(N_0 \setminus \{1\})}{c(N_0 \setminus \{1\})} - \tfrac{v_0(N_0 \setminus \{n\})}{c(N_0 \setminus \{n\})}]$.

This is a valid objection since $y(N_0 \setminus \{1\}) = v_0(N_0 \setminus \{1\})$, $y_k >= c_k \cdot \tfrac{v_0(N)}{c(N)} = p_k$ for all $k \in N_0 \setminus \{0,1\}$ and

$$\begin{aligned} y_0 &= v_0(N_0 \setminus \{1\}) - c(N_0 \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} - \varepsilon \\ &> c(N_0 \setminus \{1\}) \cdot \tfrac{v_0(N_0 \setminus \{1\})}{c(N_0 \setminus \{1\})} - c(N_0 \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} - c_0 \cdot [\tfrac{v_0(N_0 \setminus \{1\})}{c(N_0 \setminus \{1\})} - \tfrac{v_0(N_0 \setminus \{n\})}{c(N_0 \setminus \{n\})}] \\ &\geq c(N \setminus \{1\}) \cdot [\tfrac{v_0(N_0 \setminus \{1\})}{c(N_0 \setminus \{1\})} - \tfrac{v_0(N)}{c(N)}] + c_0 \cdot \tfrac{v_0(N_0 \setminus \{n\})}{c(N_0 \setminus \{n\})} \\ &> c_0 \cdot \tfrac{v_0(N_0 \setminus \{n\})}{c(N_0 \setminus \{n\})} \geq c_0 \cdot \tfrac{v_0(\{0\})}{c_0} = v(\{0\}) = p_0 \end{aligned}$$

Let us see there is no counter-objection to the above objection. Consider an eventual counter-objection $(T, z)$ of player 1 against player $n$. First notice $T \neq \{1\}$ since $p_1 > v_0(\{1\})$. Furthermore we claim $0 \in T$; in other case, $T \subseteq N$, $|T| \geq 2$ and we have

$$z(T) \geq y(T \setminus \{1\}) + x_1 = p(T \setminus \{1\}) + \frac{(|T|-1) \cdot \varepsilon}{n-1} + p_1 > p(T) \geq v_0(T),$$

where the last equality follows $p(T) \geq v_0(T)$. But $z(T) > v_0(T)$ is not allowed for a counter-objection and we conclude $0 \in T$. However, in this case we have,

$$
\begin{aligned}
z(T) &\geq y(T \setminus \{1\}) + x_1 = y(T \setminus \{0,1\}) + y_0 + x_1 \\
&> c(T \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} + v_0(N_0 \setminus \{1\}) - c(N_0 \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} \\
&\quad - c_0 \cdot [\tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} - \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})}] + c_1 \cdot \tfrac{v_0(N)}{c(N)} \\
&= c(T \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} + c(N_0 \setminus \{0,1,n\}) \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_0 \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_n \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} \\
&\quad - c(N_0 \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} - c_0 \cdot [\tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} - \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})}] + c_1 \cdot \tfrac{v_0(N)}{c(N)} \\
&= c(T \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} + c(N_0 \setminus \{0,1,n\}) \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_n \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} \\
&\quad - c(N_0 \setminus \{0,1\}) \cdot \tfrac{v_0(N)}{c(N)} + c_0 \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} + c_1 \cdot \tfrac{v_0(N)}{c(N)} \\
&= -c(N_0 \setminus (T \cup \{n\})) \cdot \tfrac{v_0(N)}{c(N)} - c_n \cdot \tfrac{v_0(N)}{c(N)} + c(N_0 \setminus \{0,1,n\}) \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_n \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} \\
&\quad + c_0 \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} + c_1 \cdot \tfrac{v_0(N)}{c(N)} \\
&= -c(N_0 \setminus (T \cup \{n\})) \cdot \tfrac{v_0(N)}{c(N)} - c_n \cdot \tfrac{v_0(N)}{c(N)} + c(N_0 \setminus (T \cup \{n\})) \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} \\
&\quad + c(T \setminus \{0,1\}) \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_n \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_0 \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} + c_1 \cdot \tfrac{v_0(N)}{c(N)} \\
&= c(N_0 \setminus (T \cup \{n\})) \cdot [\tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} - \tfrac{v_0(N)}{c(N)}] + c(T \setminus \{0,1\}) \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} \\
&\quad + c_n \cdot [\tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} - \tfrac{v_0(N)}{c(N)}] + c_0 \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} + c_1 \cdot \tfrac{v_0(N)}{c(N)} \\
&\geq c(T \setminus \{0,1\}) \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_1 \cdot [\tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} - \tfrac{v_0(N)}{c(N)}] + c_0 \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} + c_1 \cdot \tfrac{v_0(N)}{c(N)} \\
&= c(T \setminus \{0,1\}) \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_1 \cdot \tfrac{v_0(N_0\setminus\{1\})}{c(N_0\setminus\{1\})} + c_0 \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} \\
&> c(T \setminus \{0,1\}) \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} + c_1 \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} + c_0 \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} = c(T) \cdot \tfrac{v_0(N_0\setminus\{n\})}{c(N_0\setminus\{n\})} \\
&\geq c(T) \cdot \tfrac{v_0(T)}{c(T)}
\end{aligned}
$$

and we reach again $z(T) > v_0(T)$ which invalidates $(T, z)$ as a counter-objection. □

In parallel with the results of the basic model, we state a sufficient condition for the proportional distribution to be the unique element of the bargaining set.

**Theorem 3** *Let $(\beta_0, v_0, c)$ be an IRS game with external agent.*

$$\textit{If } \frac{v_0(N\setminus\{k\})}{c(N\setminus\{k\})} = \frac{v_0(N_0)}{c(N_0)}, \textit{ for all } k\in N, \textit{ then } \mathcal{M}_1^i(\beta_0, v_0) = \{\mathbf{p}(\beta_0, v_0, c)\}.$$

*Proof* Let $p = \mathbf{p}(\beta_0, v_0, c)$. Since $\frac{v_0(N\setminus\{k\})}{c(N\setminus\{k\})} = \frac{v_0(N_0)}{c(N_0)}$, for all $k \in N$, and by (1) we also got $\frac{v_0(N_0\setminus\{k\})}{c(N_0\setminus\{k\})} = \frac{v_0(N_0)}{c(N_0)}$, for all $k \in N$. By Theorem 1, $p \in \mathcal{M}_1^i(\beta_0, v_0)$. To prove that $\mathcal{M}_1^i(\beta_0, v_0) \subseteq \{p\}$, let us suppose $x \neq p$ and $x \in \mathcal{M}_1^i(\beta_0, v_0)$. By Proposition 1, since $x_N \neq p_N$ and $\frac{v_0(N\setminus\{k\})}{c(N\setminus\{k\})} = \frac{v_0(N)}{c(N)}$ for all $k \in N$, we have

$$\text{there exists } S \subseteq N \text{ such that } v_0(S) - x(S) > 0. \tag{3}$$

Moreover, since $x_N \neq p_N$, there exists $k^* \in N$ such that $x_{k^*} > c_{k^*} \cdot \frac{v_0(N)}{c(N)}$ and so

$$x(N\setminus\{k^*\}) < c(N\setminus\{k^*\}) \cdot \frac{v_0(N)}{c(N)} = c(N\setminus\{k^*\}) \cdot \frac{v_0(N\setminus\{k^*\})}{c(N\setminus\{k^*\})} = v_0(N\setminus\{k^*\}).$$

Hence,

$$\begin{aligned} v_0(N_0) - x(N_0) &= c(N_0)\cdot \tfrac{v_0(N_0)}{c(N_0)} - v_0(N) - x_0 \\ &= c_0 \cdot \tfrac{v_0(N_0)}{c(N_0)} + c(N)\cdot \tfrac{v_0(N_0)}{c(N_0)} - c(N)\cdot\tfrac{v_0(N)}{c(N)} - x_0 \\ &= c_0\cdot\tfrac{v_0(N_0)}{c(N_0)} - x_0 < c_0 \cdot \tfrac{v_0(N_0)}{c(N_0)} + v_0(N\setminus\{k^*\}) - x(N\setminus\{k^*\}) - x_0 \\ &\leq v_0(N_0\setminus\{k^*\}) - x(N_0\setminus\{k^*\}). \end{aligned} \tag{4}$$

Now we define the set $\mathcal{A}$ as follows:

$$\mathcal{A} = \{\varnothing \neq T \subseteq N_0 \mid (T, r_x^T(v_0), c_T) \in IRG^T \text{ and } x(Q) \geq v(Q) \text{ for all } Q \subseteq N_0\setminus T\}.$$

Note that, by Proposition 2 and since $x_{k^*} > c_{k^*}\cdot\frac{v_0(N)}{c(N)} = c_{k^*}\cdot\frac{v_0(N_0)}{c(N_0)}$, we have $N_0\setminus\{k^*\} \in \mathcal{A}$. Let $S^* \in \mathcal{A}$ be a minimal coalition with respect to the inclusion. By the above argument $S^* \neq N_0$. Moreover, by definition, $r_x^{S^*}(v_0)(S^*) = v_0(S^*\cup Q^*) - x(Q^*)$, for some

$Q^* \subseteq N_0 \setminus S^*$. If there is more than one of such coalition $Q^*$ that fits in the definition, we take $Q^*$ maximal with respect to the inclusion, that is, if $Q^* \subsetneq Q \subseteq N \setminus S^*$ then $v_0(S^* \cup Q^*) - x(Q^*) > v_0(S^* \cup Q) - x(Q)$.

By definition of the set $\mathcal{A}$ we have:

(a) For all $i \in S^*$,

$$x_i < c_i \cdot \frac{r_x^{S^*}(v_0)(S^*)}{c(S^*)}. \tag{5}$$

In other case, there would be $i \in S^*$ such that $x_i \geq c_i \cdot \frac{r_x^{S^*}(v_0)(S^*)}{c(S^*)}$. If $S^* = \{i^*\}$, since $S^* \in \mathcal{A}$, then $x(S) \geq v_0(S)$ for all $S \subseteq N_0 \setminus \{i^*\}$ and

$$x_{i^*} \geq c_{i^*} \cdot \frac{r_x^{S^*}(v_0)(S^*)}{c(S^*)} = c_{i^*} \cdot \frac{r_x^{S^*}(v_0)(\{i^*\})}{c(\{i^*\})} = r_x^{S^*}(v_0)(\{i^*\}) \geq v_0(\{i^*\} \cup Q) - x(Q),$$

and so $x(\{i^*\} \cup Q) \geq v_0(\{i^*\} \cup Q)$, for all $Q \subseteq N_0 \setminus \{i^*\}$. Thus $x(S) \geq v_0(S)$, for all $S \subseteq N_0$, contradicting ( 3 ).

If $|S^*| > 1$ and there exist $i \in S^*$ such that $x_i \geq c_i \cdot \frac{r_x^{S^*}(v_0)(S^*)}{c(S^*)}$, by the same reasoning as before, we have $x(\{i^*\} \cup Q) \geq v_0(\{i^*\} \cup Q)$, for all $Q \subseteq N_0 \setminus S^*$. But this means that, by Proposition 2, coalition $S^* \setminus \{i^*\}$ would be in $\mathcal{A}$ contradicting[3] the minimality of $S^*$.

(b) We claim $S^* \neq \{0\}$. On the contrary, let us suppose $S^* = \{0\}$. Since $S^* \in \mathcal{A}$, we have $x(S) \geq v_0(S)$, for all $S \subseteq N$, which involves a contradiction with (3).

(c) If $S^* = \{i\}$ then $Q^* \neq \varnothing$ (or $|S^* \cup Q^*| > 1$). Otherwise, if $Q^* = \varnothing$ and by (5),

$$x_i < c_i \cdot \frac{r_x^{\{i\}}(v_0)(\{i\})}{c(\{i\})} = c_i \cdot \frac{v_0(\{i\})}{c(\{i\})} = v_0(\{i\}),$$

but this contradicts $x$ to be an imputation of the game $(\beta_0, v_0)$.

(d) $S^* \cup Q^* \neq N$ since, by (a), $x(S^*) < r_x^{S^*}(v_0)(S^*) = v_0(S^* \cup Q^*) - x(Q^*)$ and thus, if $S^* \cup Q^* = N$, we would have $x(N) < v_0(N)$ contradicting that $x \in I(\beta_0, v_0)$.

[3]The reduction process is transitive; see Lemma 3.1 in Izquierdo and Rafels (2001).

Let $i^* \in S^*$, $i^* \neq 0$ and $j^* \in N \setminus (S^* \cup Q^*)$ where the last player exists since $S^* \cup Q^* \neq N$. Then, we define the objection $(S^* \cup Q^*, y)$ at $x$ of player $i^*$ against player $j^*$ as follows:

$$y_{i^*} = c_{i^*} \cdot \frac{r_x^{S^*}(v_0)(S^*)}{c(S^*)} - \varepsilon$$

$$y_k = c_k \cdot \frac{r_x^{S^*}(v_0)(S^*)}{c(S^*)} + \frac{\varepsilon}{|S^* \cup Q^*| - 1} \qquad \text{for all } k \in S^* \setminus \{i^*\}$$

$$y_k = x_k + \frac{\varepsilon}{|S^* \cup Q^*| - 1} \qquad \text{for all } k \in Q^*,$$

where $0 < \varepsilon < c_{i^*} \cdot \frac{r_x^{S^*}(v_0)(S^*)}{c(S^*)} - x_{i^*}$.

Now consider an eventual counterobjection $(T, z)$ of player $j^*$ against player $i^*$. By definition, it is necessary $v_0(T) - x(T) \geq 0$. On the other hand, we claim $T \cap (S^* \cup Q^*) \neq \varnothing$. In other case, $T \cap (S^* \cup Q^*) = \varnothing$, and by the superadditivity of the game $(N_0, v_0)$, we have

$$v_0(S^* \cup Q^*) - x(Q^*) \leq v_0(S^* \cup Q^*) - x(Q^*) + v_0(T) - x(T) \leq v_0(S^* \cup Q^* \cup T) - x(Q^* \cup T),$$

but this contradicts the maximality of $Q^*$. Hence, if $T \cap S^* \neq \varnothing$

$$\begin{aligned}
z(T) &\geq y(T \cap (S^* \cup Q^*)) + x(T \setminus (S^* \cup Q^*)) \\
&= y(T \cap S^*) + y(T \cap Q^*) + x(T \setminus (S^* \cup Q^*)) \\
&> c(T \cap S^*) \cdot \frac{r_x^{S^*}(v_0)(S^*)}{c(S^*)} + x(T \cap Q^*) + x(T \setminus (S^* \cup Q^*)) \\
&\geq c(T \cap S^*) \cdot \frac{r_x^{S^*}(v_0)(T \cap S^*)}{c(T \cap S^*)} + x(T \cap Q^*) + x(T \setminus (S^* \cup Q^*)) \\
&= r_x^{S^*}(v_0)(T \cap S^*) + x(T \cap Q^*) + x(T \setminus (S^* \cup Q^*)) \\
&\geq v_0((T \cap S^*) \cup (T \setminus S^*)) - x(T \setminus S^*) + x(T \cap Q^*) + x(T \setminus (S^* \cup Q^*)) = v_0(T)
\end{aligned}$$

Thus, we get $z(T) > v_0(T)$ which it is not allowed for a counter-objection.

In case $T \cap S^* = \varnothing$, then we have $T \subseteq N_0 \setminus S^*$ and we obtain $z(T) > x(T)$. However, since $S^* \in \mathcal{A}$ and $T \subseteq N_0 \setminus S^*$, we also know $x(T) \geq v_0(T)$. Thus, $z(T) > v_0(T)$ and we reach the same contradiction.

□

# References


[1] Auman RJ, Drèze JH (1974). Cooperative Games with Coalition Structures. Int. J. Game Theory 3, 217–237

[2] Davis M, Maschler M (1963). Existence of Stable Payoff Configurations for Cooperative Games. Bull. Amer. Math. Soc. 69, 106–108

[3] Davis M, Maschler M (1965). The Kernel of a Cooperative Game. Naval Res. Logist. Quart. 12, 223–259

[4] Davis M, Maschler M (1967). Existence of Stable Payoff Configurations for Cooperative Games, in Essays in Mathematical Economics in Honor of Oskar Morgestern, 39–52. (M. Shubik, Ed.), Princeton, NJ: Princeton Iniv. Press.

[5] Izquierdo JM, Rafels C (2001). Average Monotonic Cooperative Games. Games Econ. Behav. 36, 174–192

[6] Lemaire J (1991). Cooperative Game Theory and its Insurance Applications. Astin Bull. 21, 17–40

[7] Mas-Colell A (1980). Remarks on the Game-theoretic Analysis of a Simple Distribution of Surplus Problem. Int. J. Game Theory 9(3), 125–140

[8] Peleg, B (1963). Existence Theorem for the Bargaining Set $\mathcal{M}_1^{(i)}$. Bull. Amer. Math. Soc. 69, 109–110