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An axiomatization of the nucleolus of the assignment game

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An axiomatization of the nucleolus of the assignment game

Abstract: On the domain of two-sided assignment markets, the nucleolus is axiomatized as the unique solution that satisfies derived consistency (Owen, 1992) and complaint monotonicity on sectors' size. As a consequence, we obtain a geometric characterization of the nucleolus by means of a strong form of the bisection property that characterizes the intersection between the core and the kernel of a coalitional game in Maschler et al (1979).

Resum: En el domini dels jocs bilaterals d'assignació, es presenta una axiomática del nucleolus com lúnica solució que compleix les propietats de consistència respecte del joc derivat definit per Owen (1992) i monotonia de les queixes dels sectors respecte de la seva cardinalitat. Com a consequència obtenim una caracterització geomètrica del nucleolus mitjançant una propietat de bisecció més forta que la que satisfan els punts del kernel (Maschler et al, 1979).

Key words: cooperative games, assignment games, core, nucleolus

JEL: C71, C78

1 Introduction

The assignment game is a coalitional game that represents a two-sided market situation. In this market there exists a finite set of sellers, each one with an indivisible object on sell, and a finite set of buyers willing to buy at most one object each. Each agent has a reservation value that is what he or she obtains if not matched with an agent on the opposite side. Every buyer-seller pair (i, j) is attached to a real number a_{ij} that represents the value that this pair can attain if matched together. From these valuations, we obtain the assignment matrix A. The worth of each coalition is the total profit that can be obtained by optimally matching buyers and sellers in the coalition. When reservation values are null and the assignment matrix is non-negative, our game is the one introduced by Shapley and Shubik (1972).

Coalitional game theory analyzes how the agents can share the profit of an optimal pairing, taking into account the worth of all possible coalitions. The most studied solution concept in this model has been the core, the set of efficient allocations that are coalitionally rational. Shapley and Shubik prove that the core of the assignment game is non-empty and it can be described just in terms of the assignment matrix, with no need of the associated characteristic function.

Other solutions have been considered for the assignment game: Thompson's fair division point (1981), the kernel or symmetrically pairwise bargained allocations (Rochford 1984), the nucleolus (Solymosi and Raghavan 1994), the Shapley value (Hoffmann and Sudhölter 2007) and the von Neumann-Morgenstern stable sets (Núñez and Rafels 2013). However, as far as we know, axiomatic characterizations of solutions in this framework have been focused on the core. Axiomatizations of the core of assignment games are due to Sasaki (1995) and Toda (2003 and 2005).

On the general class of coalitional games, the prenucleolus (that for the assignment game coincides with the nucleolus) has been axiomatized by Sobolev (1975) by means of covariance, anonimity and the reduced game property of Davis and Maschler (1965). Potters (1991) also characterizes the nucleolus on the class of balanced games[1] by means of the above reduced game property. However, both aforementioned sets of axioms do not characterize the nucleolus on the class of assignment games since the Davis and Maschler

[1] In fact Potters characterizes the nucleolus in a more general class of games.

reduced game of an assignment game needs not remain inside this class. Moreover, it seems desirable an axiomatization of the nucleolus of the assignment game in terms of axioms that are not stated by means of the characteristic function but by means of the data of the assignment market.

In the present paper, on the domain of assignment games, the nucleolus is uniquely determined by only two axioms: derived consistency and complaint monotonicity on sectors' size. Derived consistency is based on the derived game introduced by Owen (1992). Roughly speaking, complaint monotonicity on sectors' size only requires that at each solution outcome, the most dissatisfied agent on the short side of the market is at most as well off as the most dissatisfied agent on the large side of the market, where we interpret the dissatisfaction of an agent with a given outcome as the difference between his reservation value and the amount that this outcome allocates to him.

As a by-product of the axiomatization of the nucleolus, we obtain a geometric characterization of the nucleolus. Maschler et al (1979) provide a geometrical characterization for the intersection of the kernel and the core of a coalitional game, showing that those allocations that lie in both sets are always the midpoint of certain bargaining range between each pair of players. In the case of the assignment game, this means that the kernel can be determined as those core allocations where the maximum amount that can be transferred, without getting outside the core, from one agent to his/her optimally matched partner equals the maximum amount that he/she can receive from this partner, also remaining inside the core (Rochford 1984; Driessen 1999). We now state that the nucleolus of the assignment game can be characterized by requiring this bisection property be satisfied not only for optimally matched pairs but also for optimally matched coalitions.

Preliminaries on assignment games are in Section 2. Section 3 explores the property of derived consistency. In Section 4 we prove the axiomatic characterization of the nucleolus and deduce a geometric characterization.

2 Preliminaries

Let U and U' be two countable disjoint sets, the first one formed by all potential buyers and the second one formed by all potential sellers. An *assignment market* is a quintuple $\gamma=(M,M',A,p,q)$. The sets $M\subset U$ and $M'\subset U'$ are two finite sets of buyers and sellers respectively (the two sides of the market) of cardinality $|M|=m$ and $|M'|=m'$, with $M\cup M'$ non-empty. Matrix $A=(a_{ij})_{(i,j)\in M\times M'}$ is such that for all $(i,j)\in M\times M'$, the real number a_{ij} denotes the worth obtained by the pair (i,j) if they trade. Finally, $p\in\mathbb{R}^M$ and $q\in\mathbb{R}^{M'}$ where, for all $i\in M$, p_i is the reservation value of buyer i if she remains unpaired with any seller (and similarly for q_j for all $j\in M'$). Notice that neither a_{ij} nor p_i or q_j are constrained to be non-negative. When the market has null reservation prices we will just describe it as (M,M',A).

A *matching* μ between $S\subseteq M$ and $T\subseteq M'$ is a bijection from a subset of S to a subset of T. We denote by $Dom(\mu)\subseteq S$ and $Im(\mu)\subseteq T$ the corresponding domain and image. If $i\in S$ and $j\in T$ are related by μ we indistinctly write $(i,j)\in\mu$, $j=\mu(i)$ or $i=\mu^{-1}(j)$. We denote by $\mathscr{M}(S,T)$ the set of matchings between S and T. Given an assignment market $\gamma=(M,M',A,p,q)$, for all $S\subseteq M$, $T\subseteq M'$ and $\mu\in\mathscr{M}(S,T)$ we write

$$v(S,T;\mu)=\sum_{(i,j)\in\mu}a_{ij}+\sum_{i\in S\setminus Dom(\mu)}p_i+\sum_{j\in T\setminus Im(\mu)}q_j, \tag{1}$$

with the convention that any summation over an empty set of indices is zero.

A matching $\mu\in\mathscr{M}(M,M')$ is *optimal* for the assignment market $\gamma=(M,M',A,p,q)$ if for all $\mu'\in\mathscr{M}(M,M')$ it holds $v(M,M';\mu)\geq v(M,M';\mu')$. The set of optimal matchings for the assignment market γ is denoted by $\mathscr{M}^*_\gamma(M,M')$.

With any assignment market $\gamma=(M,M',A,p,q)$, we associate a game in coalitional form $(M\cup M',w_\gamma)$ (*assignment game*) with player set $M\cup M'$ and characteristic function w_γ defined as follows: for all $S\subseteq M$ and $T\subseteq M'$,

$$w_\gamma(S\cup T)=max\{v(S,T;\mu)\mid\mu\in\mathscr{M}(S,T)\}. \tag{2}$$

Notice that by (1) and (2) we have that $w_\gamma(\{i\})=p_i$ for all $i\in M$ and $w_\gamma(\{j\})=q_j$ for all $j\in M'$. This assignment game, that allows for agents' reservation values, is a generalization of the assignment game of Shapley and Shubik (1972) (that is, an assignment game with non-negative matrix and null reservation values) and was introduced by Owen (1992) and also used by Toda (2003, 2005).

We denote by Γ_{AG} the set of all assignment markets $\gamma = (M, M', A, p, q)$, and also, for simplicity of notation, the set of their corresponding assignment games. Since we will deal with consistency properties, we allow for the emptiness of one side of the market.[2] The set of assignment games Γ_{AG} is closed by strategic equivalence. In fact, it can be shown that every assignment game in Γ_{AG} is strategically equivalent to an assignment game in the sense of Shapley and Shubik.[3] As a consequence, Shapley and Shubik's results on the core of the assignment game extend to Γ_{AG}.

Given an assignment market $\gamma = (M, M', A, p, q)$, a payoff vector is $x = (u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ where u_i stands for the payoff to buyer $i \in M$ and v_j stands for the payoff to seller $j \in M'$. We write $x_{|S}$ or $(u_{|S \cap M}, v_{|S \cap M'})$ to denote the projection of a payoff vector x to agents in coalition $S \subseteq M \cup M'$. Also, $x(S) = \sum_{i \in S} x_i$, with $x(\emptyset) = 0$. An *imputation* of γ is a payoff vector (u, v) that is efficient, $u(M) + v(M') = w_\gamma(M \cup M')$, and individually rational, $u_i \geq p_i$ for all $i \in M$ and $v_j \geq q_j$ for all $j \in M'$. We denote by $I(\gamma)$ the set of imputations of the assignment market γ.

The core of the assignment market is always non-empty and it is formed by those efficient payoff vectors $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ that satisfy coalitional rationality for mixed-pair coalitions and one-player coalitions:

$$C(\gamma) = \left\{ (u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} \sum_{i \in M} u_i + \sum_{j \in M'} v_j = w_\gamma(M \cup M'), \\ u_i + v_j \geq a_{ij} \text{ for all } (i, j) \in M \times M', \\ u_i \geq p_i \text{ for all } i \in M,\ v_j \geq q_j \text{ for all } j \in M' \end{array} \right\}.$$

[2]If γ is an assignment market with $M' = \emptyset$, then it is easy to see that the associated assignment game (M, w_γ) given by (2) is the modular game generated by the vector of reservation values $p \in \mathbb{R}^M$, that is, $w_\gamma(S) = \sum_{i \in S} p_i$, for all $S \subseteq M$. Similarly, if $\gamma = (M, M', A, p, q)$ with $M = \emptyset$, then $w_\gamma(T) = \sum_{j \in T} q_j$, for all $T \subseteq M'$.

[3]Two games (N, v) and (N, w) are strategically equivalent if and only if there exist $\alpha > 0$ and $d \in \mathbb{R}^N$ such that $w(S) = \alpha v(S) + \sum_{i \in S} d_i$. Let $\gamma = (M, M', A, p, q)$ be an assignment market where $A = (a_{ij})_{(i,j) \in M \times M'}$, $p \in \mathbb{R}^M$, $q \in \mathbb{R}^{M'}$, and let $\tilde{\gamma} = (M, M', \tilde{A})$ be an assignment market with null reservation values and matrix $\tilde{A} = (\tilde{a}_{ij})_{(i,j) \in M \times M'}$ given by $\tilde{a}_{ij} := \max\{0, a_{ij} - p_i - q_j\}$, for all $(i, j) \in M \times M'$. Then, as the reader can easily check, $w_\gamma(S \cup T) = w_{\tilde{\gamma}}(S \cup T) + \sum_{i \in S} p_i + \sum_{j \in T} q_j$, for all $S \subseteq M$ and $T \subseteq M'$.

Moreover, if μ is an optimal matching of γ, any core allocation $(u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ satisfies

$$u_i + v_j = a_{ij} \text{ for all } (i,j) \in \mu, \tag{3}$$

$$u_i = p_i \text{ for all } i \in M \setminus Dom(\mu), \tag{4}$$

$$v_j = q_j \text{ for all } j \in M' \setminus Im(\mu). \tag{5}$$

One single-valued selection in the core of the assignment market is the nucleolus. This solution, that was introduced for arbitrary coalitional games by Schmeidler (1969), only relies on the worth of individual coalitions and mixed-pair coalitions when applied to the assignment game. Given an assignent market $\gamma = (M, M', A, p, q)$, with any imputation $(u,v) \in I(\gamma)$ we associate a vector $\theta(u,v)$ whose components are $a_{ij} - u_i - v_j$, for all $(i,j) \in M \times M'$, $p_i - u_i$ for all $i \in M$ and $q_j - v_j$ for all $j \in M'$, non-increasingly ordered. Then, the *nucleolus* of the assignment market γ is the imputation $\eta(\gamma)$ that minimizes $\theta(u,v)$ with respect to the lexicographic order over the set of imputations: $\theta(\eta(\gamma)) \leq_{Lex} \theta(u,v)$ for all $(u,v) \in I(\gamma)$. This means that, for all $(u,v) \in I(\gamma)$, either $\theta(\eta(v)) = \theta(u,v)$ or $\theta(\eta(v))_1 < \theta(u,v)_1$ or there exists $k \in \{2, \ldots, mm' + m + m'\}$ such that $\theta(\eta(v))_i = \theta(u,v)_i$ for all $1 \leq i \leq k-1$ and $\theta(\eta(v))_k < \theta(u,v)_k$.

3 Derived consistency and the core of the assigment game

In this section we consider a consistency property, with respect to a certain reduction of the market, that will be satisfied not only by the core but also by the nucleolus. We begin by introducing the concept of a solution on the domain of assignment markets. The next two definitions follow Toda (2005).

Definition 1. *Let $\gamma = (M, M', A, p, q) \in \Gamma_{\text{AG}}$. A payoff vector $(u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ is* feasible *if there exists $\mu \in \mathscr{M}(M, M')$ such that*

(i) $u_i = p_i$ *for all* $i \in M \setminus Dom(\mu)$, $v_j = q_j$ *for all* $j \in M' \setminus Im(\mu)$, *and*

(ii) $u_i + v_j = a_{ij}$ *for all* $(i,j) \in \mu$.

In the above definition, μ is said to be *compatible* with (u,v). Notice that a matching that is compatible with a feasible payoff vector need not be an optimal matching.

Definition 2. *A* solution *on* Γ_{AG} *is a correspondence* σ *that associates a non-empty subset of feasible payoff vectors with each* $\gamma \in \Gamma_{\text{AG}}$.

If $\gamma = (M, M', A, p, q) \in \Gamma_{\text{AG}}$, we write $\sigma(\gamma)$ to denote the image of this assignment market by a solution σ.

Consistency is a standard property used to analyze the behavior of solutions with respect to reduction of population. Roughly speaking, a solution is consistent if whenever we reduce the game to a subset of agents and the excluded agents are paid according to a solution payoff, the projection of this payoff to the remaining agents still belongs to the solution of the reduced game. Different consistency[4] notions depend on the different definitions for the reduced game, that is, the different ways in which the remaining agents can reevaluate their coalitional capabilities. Probably, the best known notion of consistency is based on Davis and Maschler reduced game (Davis and Maschler 1965). Peleg (1986) uses the above consistency notion to characterize the core on the domain of all coalitional games. However, it turns out that the Davis and Machler reduced game of an assignment game may not be an assignment game (see Owen 1992). To overcome this drawback, Owen introduces the derived market.

Definition 3. *Let* $\gamma = (M, M', A, p, q)$ *be an assignment market,* $\emptyset \neq T \subset M \cup M'$, *and* $x = (u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$. *The* derived assignment market *relative to* T *at* x *is* $\gamma^{T,x} = \left(T \cap M, T \cap M', A^T, p^{T,x}, q^{T,x}\right)$, *where* $A^T = A_{|(T\cap M)\times(T\cap M')}$ *and*

$$\begin{aligned} p_i^{T,x} &= \max\left\{p_i, \max_{j\in M'\setminus T}\{a_{ij} - v_j\}\right\}, \text{ for all } i \in T \cap M, \\ q_j^{T,x} &= \max\left\{q_j, \max_{i\in M\setminus T}\{a_{ij} - u_i\}\right\}, \text{ for all } j \in T \cap M'. \end{aligned}$$

The derived assignment game *relative to* T *at* x *is the coalitional game associated to the derived assignment market* $\gamma^{T,x}$, *that is* $(T, w_{\gamma^{T,x}})$.

The interpretation of the derived assignment market is as follows. Once agents not in T have left the market taking their corresponding payoff in x, the agents in T interact in the submarket defined by the submatrix $A^T = A_{|(T\cap M)\times(T\cap M')}$ but must reevaluate their reservation values, since the outside option has been modified. Each agent $i \in T \cap M$ has

[4]For a comprehensive survey on the consistency principles, the reader is referred to Thomson (2003).

the choice of remaining unmatched (thus getting the original reservation value) or matching somebody, say j, outside T, which leaves an income $a_{ij} - x_j$. The best of these choices determines the new reservation value of agent $i \in T \cap M$. Similarly, agents in $T \cap M'$ reevaluate their reservation value.

Next we define consistency with respect to this derived market. A solution σ on Γ_{AG} satisfies

- **derived consistency** if for all $\gamma = (M, M', A, p, q) \in \Gamma_{\mathrm{AG}}$, all $\emptyset \neq T \subset M \cup M'$ and all $x \in \sigma(\gamma)$, then $x_{|T} \in \sigma\left(\gamma^{T,x}\right)$.

The reader will find easily that, given an assignment market $\gamma = (M, M', A, p, q)$ and any optimal matching $\mu \in \mathscr{M}^*_\gamma(M, M')$, if we consider the derived assignment market relative to a non-empty coalition $T \subseteq M \cup M'$ and at a core element $z = (u, v) \in C(\gamma)$, then the reservation value in the derived market of some agents in T can be expressed by:

$$p_i^{T,z} = a_{i\mu(i)} - v_{\mu(i)} \text{ for all } i \in M \cap T \text{ matched to } \mu(i) \in M' \setminus T, \tag{6}$$
$$p_i^{T,z} = p_i \text{ for all } i \in M \cap T \text{ unmatched by } \mu. \tag{7}$$

Similarly,

$$q_j^{T,z} = a_{\mu^{-1}(j)j} - u_{\mu^{-1}(j)} \text{ for all } j \in M' \cap T \text{ matched to } \mu^{-1}(j) \in M \setminus T, \tag{8}$$
$$q_j^{T,z} = p_j \text{ for all } j \in M' \cap T \text{ unmatched by } \mu. \tag{9}$$

Moreover, if μ is an optimal matching of the initial market γ, then its restriction to T is an optimal matching for $\gamma^{T,z}$, that is,

$$\mu_{|T} = \{(i,j) \in \mu \mid i \in T \cap M, j \in T \cap M'\} \in \mathscr{M}^*_{\gamma^{T,z}}(M \cap T, M' \cap T). \tag{10}$$

The reader may make use of the above statements (6) to (10) to prove that the core of the assignment market satisfies derived consistency.

As for the nucleolus, it is known from Potters (1991) that, in the case of balanced games, it satisfies consistency with respect to Davis and Maschler reduced game. For assignment games, Owen (1991, page 76) proves that the derived game of an assignment game relative to any coalition $T \subseteq M \cup M'$ and at a core allocation is the superadditive cover of the Davis and Maschler reduced game relative to the same coalition T and at the given core allocation. Besides that, if a game has the same efficiency level that its superadditive cover, then both games have the same nucleolus (Miquel and Núñez 2010). From the above remarks, and taking into account (10), we easily conclude that also the nucleolus of the assignment game satisfies derived consistency.

Proposition 4. *On the domain of assignment markets Γ_{AG}, the core and the nucleolus satisfy derived consistency.*

We prove in the next proposition that, on the domain Γ_{AG}, any solution σ satisfying derived consistency selects a subset of the core, that is, $\sigma(\gamma) \subseteq C(\gamma)$ for all $\gamma \in \Gamma_{AG}$. This result is needed in the axiomatization theorem (Theorem 7), but it is also of interest on its own.

Proposition 5. *On the domain of assignment markets Γ_{AG}, derived consistency implies core selection.*

Proof. Let σ be a solution on Γ_{AG} satisfying derived consistency. Let $\gamma = (M, M', A, p, q)$ be an assignment market and $z = (u, v) \in \sigma(\gamma)$. If $M \neq \emptyset$ and $M' = \emptyset$, then feasibility of the solution (Definition 1) implies $z = p$ and since $C(\gamma) = \{p\}$ we have $z \in C(\gamma)$. Similarly, if $M = \emptyset$ and $M' \neq \emptyset$, then $z = q$ and $C(\gamma) = \{z\}$.

Assume now that $M \neq \emptyset$ and $M' \neq \emptyset$. For all $i \in M$ consider the derived market relative to $T = \{i\}$ at z. By derived consistency of σ, $u_i \in \sigma(\gamma^{\{i\},z})$ and by Definitions 1 and 2, $u_i = p_i^{\{i\},z} = \max\{p_i, \max_{j \in M'}\{a_{ij} - v_j\}\}$, which implies that, for all $i \in M$, $u_i \geq p_i$ and $u_i + v_j \geq a_{ij}$ for all $j \in M'$. Similarly, for all $j \in M'$ let us consider the derived market relative to $T = \{j\}$ at z. Again by derived consistency of σ, $v_j \in \sigma(\gamma^{\{j\},z})$ and, by Definitions 1 and 2, $v_j = q_j^{\{j\},z} = \max\{q_j, \max_{i \in M}\{a_{ij} - u_i\}\}$ which implies $v_j \geq q_j$ for all $j \in M'$. Hence, $z = (u, v)$ satisfies coalitional rationality for all mixed-pair and individual coalitions. It only remains to check its efficiency, that is, $z(M \cup M') = w_\gamma(M \cup M')$.

Let $\mu \in \mathscr{M}_\gamma^*(M, M')$ be an optimal matching and $\mu' \in \mathscr{M}(M, M')$ a matching that is compatible with $z = (u, v)$. Notice that such μ' exists since $z = (u, v)$ is a feasible payoff vector by Definition 2. Then,

$$\begin{aligned}\sum_{i\in Dom(\mu)} a_{i\mu(i)} \quad + & \sum_{i\in M\setminus Dom(\mu)} p_i + \sum_{j\in M'\setminus Im(\mu)} q_j \\ \leq & \sum_{i\in Dom(\mu)} \left(u_i + v_{\mu(i)}\right) + \sum_{i\in M\setminus Dom(\mu)} u_i + \sum_{j\in M'\setminus Im(\mu)} v_j \\ = & \sum_{i\in Dom(\mu')} \left(u_i + v_{\mu'(i)}\right) + \sum_{i\in M\setminus Dom(\mu')} u_i + \sum_{j\in M'\setminus Im(\mu')} v_j \\ = & \sum_{i\in Dom(\mu')} a_{i\mu'(i)} + \sum_{i\in M\setminus Dom(\mu')} p_i + \sum_{j\in M'\setminus Im(\mu')} q_j,\end{aligned}$$

where the first equality follows by simply reordering terms. By optimality of the matching μ, the above inequality implies

$$\begin{aligned}\sum_{i\in Dom(\mu)} a_{i\mu(i)} \quad + & \sum_{i\in M\setminus Dom(\mu)} p_i + \sum_{j\in M'\setminus Im(\mu)} q_j \\ = & \sum_{i\in Dom(\mu')} a_{i\mu'(i)} + \sum_{i\in M\setminus Dom(\mu')} p_i + \sum_{j\in M'\setminus Im(\mu')} q_j\end{aligned}$$

and thus $\sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_\gamma(M\cup M')$ which concludes the proof of $z=(u,v)\in C(\gamma)$. $\square$

Toda (2005) gives two axiomatizations of the core of assignment markets by means of a consistency property (that we will refer to as Toda's consistency) and Pareto optimality, pairwise monotonicity and individual monotonicity (or population monotonicity). To this end, Toda proves that any subcorrespondence of the core that satisfies Toda's consistency selects the whole core. Now, as a consequence of Proposition 5 above, it is straightforward to characterize the core of assignment markets as the only solution that satisfies both aforementioned consistency principles: derived consistency and Toda's (2005) consistency.

4 Axiomatic and geometric characterizations of the nucleolus

In this section, we characterize axiomatically the nucleolus on the class of assignment games by means of two axioms, the first of them being derived consistency. Due to the bilateral structure of the market, we look for a second axiom that guarantees some balancedness between groups. As a by-product of the following axiomatization we will derive a geometric characterization that determines the position of the nucleolus inside the core.

Given an assignment market $\gamma = (M, M', A, p, q)$, if $j \in M'$ and (u,v) is a payoff vector, $q_j - v_j$ measures the difference between player's j reservation value q_j and the amount v_j he has been paid. Thus, the higher this difference is, the more dissatisfied the agent is with the payoff vector. Sector M' as a whole can measure its degree of dissatisfaction by $\max_{j\in M'}\{q_j - v_j\}$, and in this way we define the complaint of a sector as the maximum dissatisfaction of its agents. Analogously, the complaint of sector M is $\max_{i\in M}\{p_i - u_i\}$.

It is worth to remark that complaint monotonicity on sectors' size, although defined by means of excesses of individual coalitions, is far from the definition of the nucleolus since it does never compare these excesses across different imputations.

A solution σ on Γ_{AG} satisfies

- **complaint monotonicity on sectors' size** if for all $\gamma = (M, M', A, p, q) \in \Gamma_{\mathrm{AG}}$ with $|M| \leq |M'|$ and all $(u,v) \in \sigma(\gamma)$, then

$$\max_{i\in M}\{p_i - u_i\} \leq \max_{j\in M'}\{q_j - v_j\}. \tag{11}$$

  And similarly, if $|M| \geq |M'|$, then $\max_{i\in M}\{p_i - u_i\} \geq \max_{j\in M'}\{q_j - v_j\}$.

Thus, an interpretation of the axiom of complaint monotonicity on sectors' size is the following: the less populated sector must be at least as satisfied as the most populated one. That is, if for instance supply is shorter than demand, the sector of sellers has a better position in the market and this fact should be recognized by the solution outcomes. It follows straightforwardly that if a solution has the property above and $|M| = |M'|$, then, at any solution outcome, both sides of the market have the same complaint.

Note that, when imposed on solution concepts that are a core selection, the above property is not much demanding. Indeed, for those markets that are not square, that is, markets that have a different number of buyers than sellers, inequality (11) holds trivially at any core allocation. And for those markets with as many buyers as sellers, complaint monotonicity on sectors' size selects a hypersurface. For instance, in a square Shapley and Shubik's assignment game (M, M', A), imposing complaint monotonicity on sectors' size is equivalent to imposing $\min_{i\in M}\{u_i\} = \min_{j\in M'}\{v_j\}$ to the solution outcomes, and in general there are still infinitely many core allocations that satisfy this equality. Figure 1 depicts the core of a

2×2-assignment game and the piece-wise linear curve $A-B-C-D$ formed by the subset of core allocations at which both sectors have the same complaint.

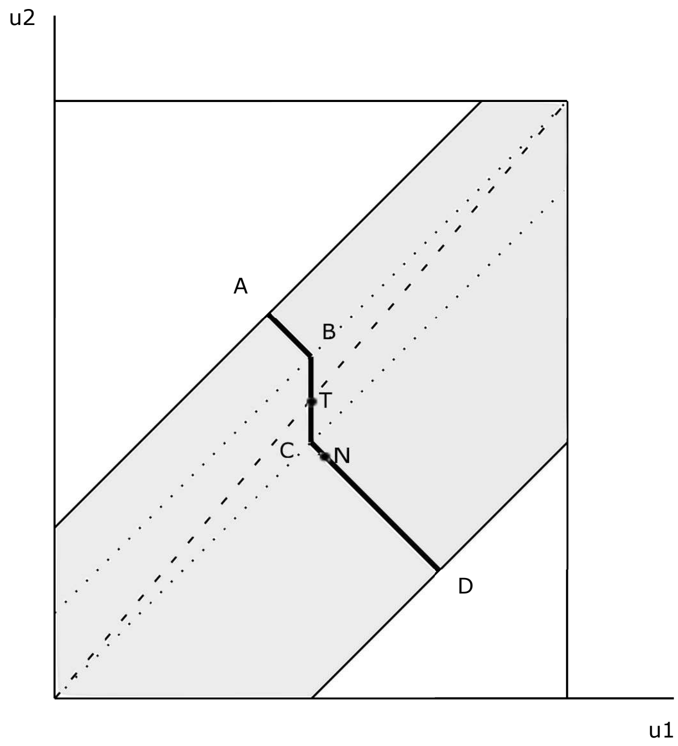

Figure 1:

A more formal geometric interpretation of complaint monotonicity on sectors' size for square markets will follow after the proof of Theorem 7. It is not difficult to realize that the midpoint between the buyers-optimal core allocation and the sellers-optimal core allocation (point T in the figure), that is known as Thompson's fair division point (Thompson 1981), satisfies complaint monotonicity on sectors' size. On the other hand, the Shapley value (Shapley 1953) does not satisfy this property. Next we prove that the nucleolus (denoted by N in Figure 1) also satisfies this complaint monotonicity property.

Proposition 6. *On the domain of assignment markets Γ_{AG}, the nucleolus satisfies complaint monotonicity on sectors' size.*

Proof. Let $\gamma = (M, M', A, p, q) \in \Gamma_{AG}$ and let $\eta = \eta(\gamma)$ be the nucleolus of the assignment market γ. If $|M| < |M'|$, for any optimal matching $\mu \in \mathscr{M}^*_\gamma(M, M')$, there exists $j^* \in M'$

that is unmatched by μ. Thus, since $\eta \in C(\gamma)$, we have $\eta_{j^*} = q_{j^*}$, that implies

$$\max_{i\in M}\{p_i - \eta_i\} \leq 0 = q_{j^*} - \eta_{j^*} = max_{j\in M'}\{q_j - \eta_j\}.$$

Analogously if $|M| > |M'|$.

If $|M| = |M'|$, let $\varepsilon_1 = -\max_{i\in M}\{p_i - \eta_i\} = \min_{i\in M}\{\eta_i - p_i\} \geq 0$, $\varepsilon_2 = -\max_{j\in M'}\{q_j - \eta_j\} = \min_{j\in M'}\{\eta_j - q_j\} \geq 0$ and assume, without loss of generality, that $\varepsilon_1 < \varepsilon_2$. Now define the payoff vectors

$$(u',v') = (\eta_{|M} - \varepsilon_1 \cdot e^M, \eta_{|M'} + \varepsilon_1 \cdot e^{M'}) \text{ and } (u'',v'') = (\eta_{|M} + \varepsilon_2 \cdot e^M, \eta_{|M'} - \varepsilon_2 \cdot e^{M'}), \quad (12)$$

where $e^M = (1,\ldots,1) \in \mathbb{R}^M$ and $e^{M'} = (1,\ldots,1) \in \mathbb{R}^{M'}$.

It can be easily checked that $(u',v'),(u'',v'') \in C(\gamma)$. Now take $z = (u,v) = \frac{1}{2}(u',v') + \frac{1}{2}(u'',v'')$. By substitution from (12), for all $(i,j) \in M \times M'$,

$$\begin{aligned} a_{ij} - u_i - v_j &= a_{ij} - \left(\eta_i + \frac{1}{2}(\varepsilon_2 - \varepsilon_1)\right) - \left(\eta_j + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\right) \\ &= a_{ij} - \eta_i - \eta_j. \end{aligned}$$

Thus, to lexicographically minimize the vector of ordered excesses over the set of core allocations, we only need to consider excesses of individual coalitions.[5]

First,

$$\begin{aligned} &\max\left\{\max_{k\in M}\{p_k - u_k\}, \max_{k\in M'}\{q_k - v_k\}\right\} \\ &= \max\left\{\max_{k\in M}\{p_k - \eta_k\} + \frac{1}{2}(\varepsilon_1 - \varepsilon_2), \max_{k\in M'}\{q_k - \eta_k\} + \frac{1}{2}(\varepsilon_2 - \varepsilon_1)\right\} \\ &= \max\left\{-\varepsilon_1 + \frac{1}{2}(\varepsilon_1 - \varepsilon_2), -\varepsilon_2 + \frac{1}{2}(\varepsilon_2 - \varepsilon_1)\right\} = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2). \end{aligned}$$

Moreover, since $\varepsilon_1 < \varepsilon_2$, we have

$$-\varepsilon_1 = \max\left\{\max_{k\in M}\{p_k - \eta_k\}, \max_{k\in M'}\{q_k - \eta_k\}\right\}.$$

But then,

$$\begin{aligned} \max\left\{\max_{k\in M}\{p_k - u_k\}, \max_{k\in M'}\{q_k - v_k\}\right\} &= -\frac{1}{2}(\varepsilon_1 + \varepsilon_2) < -\varepsilon_1 \\ &= \max\left\{\max_{k\in M}\{p_k - \eta_k\}, \max_{k\in M'}\{q_k - \eta_k\}\right\}, \end{aligned}$$

[5] The following property is well known (see Potters and Tijs 1992). For any $n \in \mathbb{N}$ we define the map $\theta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which arranges the coordinates of a point in $\mathbb{R}^n$ in non-increasing order. Take now any $z \in \mathbb{R}^p$. Then, $\theta(x) \leq_{Lex} \theta(y) \Longleftrightarrow \theta(x,z) \leq_{Lex} \theta(y,z)$.

in contradiction with $\eta = \eta(\gamma)$ being the nucleolus. Hence, $\varepsilon_1 = \varepsilon_2$ and this concludes the proof. □

We are now ready to state and prove the axiomatic characterization of the nucleolus.

Theorem 7. *On the domain of assignment markets Γ_{AG}, the only solution satisfying derived consistency and complaint monotonicity on sectors' size is the nucleolus.*

Proof. From Propositions 4 and 6 we know that the nucleolus satisfies both properties. To show uniqueness assume there exists a solution σ on Γ_{AG} satisfying derived consistency and complaint monotonicity on sectors' size.

Let $\gamma = (M, M', A, p, q) \in \Gamma_{AG}$ and $z = (u, v) \in \sigma(\gamma)$. From Proposition 5, σ satisfies core selection and thus $z \in C(\gamma)$. If $M \neq \emptyset$ and $M' = \emptyset$ (or $M = \emptyset$ and $M' \neq \emptyset$) from Definition 2 we have $z = \eta$. Assume then $M \neq \emptyset$ and $M' \neq \emptyset$.

Let $\mu \in \mathscr{M}^*_\gamma(M, M')$ be an optimal matching. If $\mu = \emptyset$, since both z and η belong to the core, we again have $z = \eta$. Assume then that $\mu \neq \emptyset$ and $z \neq \eta$. For any $\emptyset \neq S \subset M$ such that $|S| = |\mu(S)|$ let us consider the derived market relative to $T = S \cup \mu(S)$ at z. Notice that under the current assumptions, such a coalition S always exists. By derived consistency of σ, $z_{|T} \in \sigma(\gamma^{T,z})$. Since $|S| = |\mu(S)|$, by the complaint monotonicity on sectors' size of σ applied to the derived market $\gamma^{T,z}$, we have

$$\max_{i \in S}\{p_i^{T,z} - z_i\} = \max_{j \in \mu(S)}\{q_j^{T,z} - z_j\}. \tag{13}$$

From the definition of $p^{T,z}$, we obtain

$$\begin{aligned} \max_{i \in S}\left\{p_i^{T,z} - z_i\right\} &= \max_{i \in S}\left\{\max\left\{p_i, \max_{k \in M' \setminus \mu(S)}\{a_{ik} - z_k\}\right\} - z_i\right\} \\ &= \max_{\substack{i \in S \\ k \in M' \setminus \mu(S)}}\{p_i - z_i, a_{ik} - z_i - z_k\} \end{aligned} \tag{14}$$

Similarly, making use of the definition of $q^{T,z}$,

$$\max_{j \in \mu(S)}\{q_j^{T,z} - z_j\} = \max_{\substack{j \in \mu(S) \\ k \in M \setminus S}}\{q_j - z_j, a_{kj} - z_j - z_k\}, \tag{15}$$

and, as a consequence, expression (13) is equivalent to

$$\max_{\substack{i \in S \\ k \in M' \setminus \mu(S)}}\{p_i - z_i, a_{ik} - z_i - z_k\} = \max_{\substack{j \in \mu(S) \\ k \in M \setminus S}}\{q_j - z_j, a_{kj} - z_j - z_k\}, \tag{16}$$

for all non-empty coalition $S \subseteq M$ with $|S| = |\mu(S)|$ and all $z \in \sigma(\gamma)$, being σ a solution satisfying derived consistency and complaint monotonicity on sectors' size. Since the nucleolus also satisfies these two axioms, we have

$$\max_{\substack{i \in S \\ k \in M' \setminus \mu(S)}} \{p_i - \eta_i, a_{ik} - \eta_i - \eta_k\} = \max_{\substack{j \in \mu(S) \\ k \in M \setminus S}} \{q_j - \eta_j, a_{kj} - \eta_j - \eta_k\}, \tag{17}$$

for all $\emptyset \neq S \subseteq M$ with $|S| = |\mu(S)|$. Now, from $z \neq \eta$, either there exists a non-empty coalition $S^* \subseteq M$ such that $z_i > \eta_i$ for all $i \in S^*$ and $z_i \leq \eta_i$ for all $i \in M \setminus S^*$, or there exists a non-empty coalition $S^* \subseteq M$ such that $z_i < \eta_i$ for all $i \in S^*$ and $z_i \geq \eta_i$ for all $i \in M \setminus S^*$. Let us assume without loss of generality that the first case holds, since the proof in the second case is analogous. That is, assume there exists $\emptyset \neq S^* \subseteq M$ such that

$$z_i > \eta_i \text{ for all } i \in S^* \text{and } z_i \leq \eta_i \text{ for all } i \in M \setminus S^*. \tag{18}$$

Notice that all agents in S^* are matched by μ. Indeed, if there existed an agent in S^* unassigned, $i \in S^* \setminus Dom(\mu)$, from $z \in C(\gamma)$ we would have $z_i = p_i > \eta_i$, in contradiction with the nucleolus being in the core. From $z \in C(\gamma)$ follows $z_j < \eta_j$ for all $j \in \mu(S^*)$, and the reader will also check that $z_j \geq \eta_j$ for all $j \in M' \setminus \mu(S^*)$. Then,

$$\begin{aligned}
\max_{\substack{i \in S^* \\ j \in M' \setminus \mu(S^*)}} \{p_i - z_i, a_{ij} - z_i - z_j\} &< \max_{\substack{i \in S^* \\ j \in M' \setminus \mu(S^*)}} \{p_i - \eta_i, a_{ij} - \eta_i - \eta_j\} \\
&= \max_{\substack{j \in \mu(S^*) \\ i \in M \setminus S^*}} \{q_j - \eta_j, a_{ij} - \eta_i - \eta_j\} \\
&< \max_{\substack{j \in \mu(S^*) \\ i \in M \setminus S^*}} \{q_j - z_j, a_{ij} - z_i - z_j\},
\end{aligned} \tag{19}$$

where the equality follows from (17). We have then reached a contradiction with (16). Hence, $z = \eta$. □

The axioms in Theorem 7 are clearly independent because the core satisfies derived consistency on the class of assignment games but not complaint monotonicity on sectors' size and Thompson's fair division point satisfies the second axiom but fails to satisfy the first one, since it generally differs from the nucleolus.

Let us remark that, under the assumption $|M| = |M'|$, complaint monotonicity on sectors' size can also be interpreted in geometric terms when combined with core selection.

Notice that, by adding dummy players, that is, null rows or columns in the assignment matrix and null reservation values, we can assume without loss of generality that the number of buyers equals the number of sellers, since this does not modify the nucleolus payoff of the non-dummy agents. Then, saying that solution σ on Γ_{AG} satisfies complaint monotonicity on sectors' size and $|M| = |M'|$, is equivalent to saying

$$\min_{i\in M}\{u_i - p_i\} = \min_{j\in M'}\{v_j - q_j\}, \tag{20}$$

for all $(u,v) \in \sigma(\gamma)$. Suppose now that $\sigma(\gamma) \subseteq C(\gamma)$, where σ is a solution on Γ_{AG}. For all $S \subseteq M$, let the incidence vector $e^S \in \mathbb{R}^M$ be defined by $(e^S)_i = 1$ for all $i \in S$ and $(e^S)_i = 0$ for all $i \in M \setminus S$. The vector $e^T \in \mathbb{R}^{M'}$, for all $T \subseteq M'$, is defined analogously. Take $\varepsilon_1(u,v) = \min_{i\in M}\{u_i - p_i\}$ and notice that $\varepsilon_1(u,v) = \max\{\varepsilon \geq 0 \mid (u - \varepsilon \cdot e^M, v + \varepsilon \cdot e^{M'}) \in C(\gamma)\}$. The reason is that, taking into account $(u,v) \in C(\gamma)$, for all $\varepsilon \geq 0$ efficiency and coalitional rationality for mixed-pair coalitions hold trivially for the payoff vector $(u - \varepsilon \cdot e^M, v + \varepsilon \cdot e^{M'})$ and, as long as $\varepsilon \leq \varepsilon_1(u,v)$, individual rationality also holds. Similarly, if we write $\varepsilon_2(u,v) = \min_{j\in M'}\{v_j - q_j\}$, we can check that $\varepsilon_2(u,v) = \max\{\varepsilon \geq 0 \mid (u + \varepsilon \cdot e^M, v - \varepsilon \cdot e^{M'}) \in C(\gamma)\}$. As a consequence, if σ satisfies core selection, then for each chosen allocation $(u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$, the largest per capita amount that sector M can transfer to sector M' without leaving the core equals the largest per capita amount that sector M' can transfer to sector M without getting outside the core. This means that core elements satisfying complaint monotonicity on sectors' size, when $|M| = |M'|$, are at the midpoint of a certain 45°-slope range within the core.

As a consequence of the axiomatization in Theorem 7, we can show that a stronger form of the above bisection property characterizes the nucleolus.

Let $\gamma = (M, M', A, p, q)$ be an assignment market with $|M| = |M'|$. For each $\emptyset \neq S \subseteq M$, $\emptyset \neq T \subseteq M'$, $|S| = |T|$, we define *the largest equal amount that can be transferred from players in S to players in T with respect to the core allocation $(u,v) \in C(\gamma)$, while remaining in the core of γ*, by

$$\delta^{\gamma}_{S,T}(u,v) = \max\{\varepsilon \geq 0 \mid (u - \varepsilon e^S, v + \varepsilon e^T) \in C(\gamma)\}. \tag{21}$$

Similarly,

$$\delta^{\gamma}_{T,S}(u,v) = \max\{\varepsilon \geq 0 \mid (u + \varepsilon e^S, v - \varepsilon e^T) \in C(\gamma)\}. \tag{22}$$

The following geometric characterization extends the bisection property provided by Maschler et al.(1979) to characterize the intersection of the kernel[6] and the core of arbitrary coalitional games. In the case of assignment games, it turns out that the kernel is always included in the core (Driessen 1998). It is known that a core element of an assignment market γ belongs to its kernel if and only if $\delta^{\gamma}_{\{i\},\{\mu(i)\}}(u,v) = \delta^{\gamma}_{\{\mu(i)\},\{i\}}(u,v)$ for all $i \in M$ assigned by an optimal matching μ of γ.

Next theorem shows that, for assignment games, if we require this bisection property to hold not only for all optimally matched pairs but for all optimally matched coalitions we geometrically characterize the nucleolus. The main point is that if a core element (u,v) satisfies $\delta^{\gamma}_{S,\mu(S)}(u,v) = \delta^{\gamma}_{\mu(S),S}(u,v)$ for all $S \subseteq M$, then it satisfies equation (16), which has been proved to characterize the nucleolus among the set of core elements.

This result generalizes the one given in Llerena and Núñez (2011) for the classical assignment game of Shapley and Shubik (1972).

Theorem 8. *Let $\gamma = (M,M',A,p,q)$ be a square assignment market and $\mu \in \mathscr{M}^*_\gamma(M,M')$. Then, the nucleolus is the unique core allocation satisfying $\delta^{\gamma}_{S,\mu(S)}(\eta(\gamma)) = \delta^{\gamma}_{\mu(S),S}(\eta(\gamma))$, for all $\emptyset \neq S \subseteq M$.*

Figure 2 illustrates the above geometric characterization of the nucleolus.

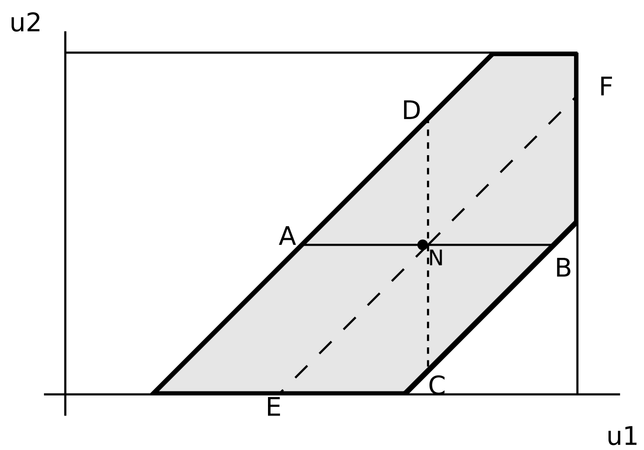

Figure 2:

[6]The kernel is a set-solution concept for coalitional games that was introduced by Davis and Maschler (1965). The kernel always contains the nucleolus.

In light grey we represent the core of a 2×2 assignment game in the plane u_1, u_2 of the buyers' payoffs. If we assume that $\mu = \{(1,1),(2,2)\}$ is an optimal matching, then the nucleolus is the unique core allocation (denoted by N in the picture) that bisects at the same time the horizontal segment $[A,B]$, the vertical segment $[C,D]$ and the 45°-slope segment $[E,F]$. The higher the dimension of the core, the more the number of bisection equalities that must be considered.

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