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Carleson measures and Logvinenko–Sereda sets on compact manifolds

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Abstract. Given a compact Riemannian manifold M of dimension $m \geq 2$, we study the space of functions of $L^2(M)$ generated by eigenfunctions of eigenvalues less than $L \geq 1$ associated to the Laplace–Beltrami operator on M. On these spaces we give a characterization of the Carleson measures and the Logvinenko–Sereda sets.

Keywords. Logvinenko–Sereda sets, reverse Carleson inequalities, spectral analysis.

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1 Introduction and statement of the results

Let (M, g) be a smooth, connected, compact Riemannian manifold without boundary of dimension $m \geq 2$. Let dV be the volume element of M associated to the metric g_{ij}. Let Δ_M be the Laplacian on M associated to the metric g_{ij}. It is given in local coordinates by

$$\Delta_M f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x_j}\right),$$

where $|g| = \det(g_{ij})$ and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$. As M is compact, g_{ij} and all its derivatives are bounded and we assume that the metric g is non-singular at each point of M.

Since M is compact, the spectrum of the Laplacian is discrete and there is a sequence of eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

and an orthonormal basis ϕ_i of smooth real eigenfunctions of the Laplacian, i.e., $\Delta_M \phi_i = -\lambda_i \phi_i$. So $L^2(M)$ decomposes into an orthogonal direct sum of eigenfunctions of the Laplacian.

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We consider the following spaces of $L^2(M)$:

$$E_L = \left\{ f \in L^2(M) : f = \sum_{i=1}^{k_L} \beta_i \phi_i,\ \Delta_M \phi_i = -\lambda_i \phi_i,\ \lambda_{k_L} \le L \right\},$$

where $L \ge 1$ and $k_L = \dim E_L$. We see that E_L is the subspace of $L^2(M)$ generated by eigenfunctions of eigenvalues $\lambda \le L$. Thus in E_L we consider functions in $L^2(M)$ with a restriction on the support of its Fourier transform. It is, in a sense, a Paley–Wiener type space on M with bandwidth L.

The motivation of this paper is to show that the spaces E_L behave like the space defined in $\mathbb{S}^d$ $(d > 1)$ of spherical harmonics of degree less than $\sqrt{L}$. In fact, the space E_L is a generalization of the spherical harmonics and the role of them are played by the eigenfunctions. The cases $M = \mathbb{S}^1$ and $M = \mathbb{S}^d$ $(d > 1)$ have been studied in [16] and [13], respectively.

This similarity between eigenfunctions of the Laplacian and polynomials is not new; for instance, Donnelly and Fefferman showed in [3, Theorem 1] that on a compact manifold, an eigenfunction of eigenvalue λ behaves essentially like a polynomial of degree $\sqrt{\lambda}$. In this direction they proved the local L^∞-Bernstein inequality stated below.

Theorem (Donnelly–Fefferman). *Let M be as above with $m = \dim M$. If u is an eigenfunction of the Laplacian $\Delta_M u = -\lambda u$, then there exists $r_0 = r_0(M)$ such that for all $r < r_0$ we have*

$$\max_{B(x,r)} |\nabla u| \le \frac{C\lambda^{(m+2)/4}}{r} \max_{B(x,r)} |u|.$$

The proof of the above estimate is rather delicate. Donnelly and Fefferman conjectured that it is possible to replace $\lambda^{(m+2)/4}$ by $\sqrt{\lambda}$ in the inequality. If the conjecture holds, we have in particular a global Bernstein type inequality:

$$\|\nabla u\|_\infty \lesssim \sqrt{\lambda}\, \|u\|_\infty . \tag{1.1}$$

In fact, this weaker estimate holds and a proof will be given later. This fact suggests that the right metric to study the space E_L should be rescaled by a factor $1/\sqrt{L}$ because in balls of radius $1/\sqrt{\lambda}$, a bounded eigenfunction of eigenvalue λ oscillates very little.

In the present work we will study for which measures $\mu = \{\mu_L\}_L$ one has

$$\int_M |f|^2 d\mu_L \approx \int_M |f|^2\, dV, \quad \forall f \in E_L, \tag{1.2}$$

with constants independent of f and L.

We will also study the inequality

$$\int_M |f|^2 d\mu_L \lesssim \int_M |f|^2 \, dV$$

that defines the Carleson measures and we will present a geometric characterization of them. Inequality (1.2) will be studied only for the special case

$$d\mu_L = \chi_{A_L} dV,$$

where $\mathcal{A} = \{A_L\}_L$ is a sequence of sets in the manifold. In case (1.2) holds, we say that $\mathcal{A}$ is a sequence of Logvinenko–Sereda sets. Our two main results are the following:

Theorem 1.1. *The sequence of sets $\mathcal{A} = \{A_L\}_L$ is Logvinenko–Sereda if and only if there is an $r > 0$ such that*

$$\inf_L \inf_{z \in M} \frac{\operatorname{vol}(A_L \cap B(z, r/\sqrt{L}))}{\operatorname{vol}(B(z, r/\sqrt{L}))} > 0.$$

Theorem 1.2. *Suppose $\mu = \{\mu_L\}_L$ is a sequence of measures on M. Then μ is L^2-Carleson for M if and only if there exists a $C > 0$ such that for all L*

$$\sup_{\xi \in M} \frac{\mu_L(B(\xi, 1/\sqrt{L}))}{\operatorname{vol}(B(\xi, 1/\sqrt{L}))} \leq C.$$

In what follows, when we write $A \lesssim B$, $A \gtrsim B$ or $A \approx B$, we mean that there are constants depending only on the manifold such that $A \leq CB$, $A \geq CB$ or $C_1 B \leq A \leq C_2 B$, respectively. Also, the value of the constants appearing during a proof may change, but they will be still denoted by the same letter. We will denote by $B(\xi, r)$ a geodesic ball in M of center ξ and radius r and $\mathbb{B}(z, r)$ will denote an Euclidean ball in $\mathbb{R}^m$ of center z and radius r.

The structure of the paper is the following: in the second section, we will explain the asymptotics of the reproducing kernel of the space E_L. In the third section, we shall discuss one of the tools used: the harmonic extension of functions in the space E_L. Following this, we will prove Theorem 1.2 and in the last section, we will prove our main result that is Theorem 1.1.

2 The reproducing kernel of E_L

Let

$$K_L(z, w) := \sum_{i=1}^{k_L} \phi_i(z)\phi_i(w) = \sum_{\lambda_i \leq L} \phi_i(z)\phi_i(w).$$

This function is the reproducing kernel of the space E_L, i.e.,

$$f(z) = \langle f, K_L(z,\cdot)\rangle, \quad \forall f \in E_L.$$

Note that $\dim(E_L) = k_L = \#\{\lambda_i \le L\}$. The function K_L is also called the spectral function associated to the Laplacian. Hörmander proved in [7] the following estimates:

(1) $K_L(z,w) = O(L^{(m-1)/2})$, $z \neq w$.

(2) $K_L(z,z) = \frac{\sigma_m}{(2\pi)^m} L^{m/2} + O(L^{(m-1)/2})$ (uniformly in $z \in M$), where $\sigma_m = 2\pi^{m/2}/(m\Gamma(m/2))$.

(3) $k_L = \frac{\mathrm{vol}(M)\sigma_m}{(2\pi)^m} L^{m/2} + O(L^{(m-1)/2})$.

In fact, in [7] there are estimates for the spectral function associated to any elliptic operator of order $n \ge 1$ with constants depending only on the manifold.

So for L big enough we have $k_L \approx L^{m/2}$ and

$$\|K_L(z,\cdot)\|_2^2 = K_L(z,z) \approx L^{m/2} \approx k_L$$

with constants independent of L and z.

3 Harmonic extension

In what follows, given $f \in E_L$, we will denote by h the harmonic extension of f in $N = M \times \mathbb{R}$. The metric that we consider in N is the product metric, i.e., if we denote it by $\tilde{g}_{ij}$ for $i = 1,\dots,m+1$, then

$$(\tilde{g}_{ij})_{i,j=1,\dots,m+1} = \begin{pmatrix} (g_{ij})_{i,j=1}^m & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this matrix, we can calculate the gradient and the Laplacian for N. If $h(z,t)$ is a function defined on N, then

$$|\nabla_N h(z,t)|^2 = |\nabla_M h(z,t)|^2 + \left(\frac{\partial h}{\partial t}(z,t)\right)^2$$

and

$$\Delta_N h(z,t) = \Delta_M h(z,t) + \frac{\partial^2 h}{\partial t^2}(z,t).$$

Note that $|\nabla_M h(z,t)| \le |\nabla_N h(z,t)|$.

Let $f \in E_L$, we know that

$$f = \sum_{i=1}^{k_L} \beta_i \phi_i, \quad \Delta_M \phi_i = -\lambda_i \phi_i, \quad 0 \le \lambda_i \le L.$$

Define for $(z,t) \in N$

$$h(z,t) = \sum_{i=1}^{k_L} \beta_i \phi_i(z) e^{\sqrt{\lambda_i} t}.$$

Observe that $h(z,0) = f(z)$. Moreover, $|\nabla_M f(z)|^2 \le |\nabla_N h(z,0)|^2$.

The function h is harmonic in N because

$$\Delta_N h(z,t) = \sum_{i=1}^{k_L} \left[\beta_i e^{\sqrt{\lambda_i} t} \Delta_M \phi_i(z) + \beta_i \phi_i(z) \Delta_{\mathbb{R}}(e^{\sqrt{\lambda_i} t}) \right] = 0.$$

For the harmonic extension, we do not have the mean-value property because it is not true for all manifolds (only for the harmonic manifolds, see [19] for a complete characterization of them). What is always true is a "submean-value property" (with a uniform constant) for positive subharmonic functions, see for example [17, Chapter II, Section 6]).

Observe that since h is harmonic on N, $|h|^2$ is a positive subharmonic function on N. In fact, $|h|^p$ is subharmonic for all $p \ge 1$ (for a proof see [5, Proposition 1]). Therefore, we know that for all $r < R_0(M)$

$$|h(z,t)|^2 \lesssim \fint_{B(z,r/\sqrt{L}) \times I_r(t)} |h(w,s)|^2 dV(w) ds,$$

where $R_0(M) > 0$ denotes the injectivity radius of the manifold M and where $I_r(t) = (t - r/\sqrt{L}, t + r/\sqrt{L})$. In particular,

$$|f(z)|^2 \le C_r L^{(m+1)/2} \int_{B(z,r/\sqrt{L}) \times I_r} |h(w,s)|^2 dV(w) ds, \tag{3.1}$$

where $I_r = I_r(0)$. The following result relates the L^2-norm of f and h.

Proposition 3.1. *Let $r > 0$ be fixed. If $f \in E_L$, then*

$$2r e^{-2r} \|f\|_2^2 \le \sqrt{L} \, \|h\|_{L^2(M \times I_r)}^2 \le 2r e^{2r} \|f\|_2^2. \tag{3.2}$$

Therefore, for $r < R_0(M)$

$$\frac{\sqrt{L}}{2r} \|h\|_{L^2(M \times I_r)}^2 \approx \|f\|_2^2$$

with constants depending only on the manifold M.

Proof. Using the orthogonality of $\{\phi_i\}_i$, we have

$$\|h\|^2_{L^2(M\times I_r)} = \int_{I_r}\int_M \left|\sum_{i=1}^{k_L} \beta_i \phi_i(z) e^{\sqrt{\lambda_i}t}\right|^2 dV(z)dt$$
$$= \int_{I_r}\sum_{i=1}^{k_L}\int_M |\beta_i|^2 |\phi_i(z)|^2 dV(z) e^{2\sqrt{\lambda_i}t}dt \le \int_{I_r} e^{2\sqrt{L}t}dt\, \|f\|_2^2 .$$

Similarly, one can prove the left hand side inequality of (3.2). □

We recall now a result proved by Schoen and Yau that estimates the gradient of harmonic functions.

Theorem (Schoen–Yau). *Let N be a complete Riemannian manifold with Ricci curvature bounded below by $-(n-1)K$ (n is the dimension of N and K is a positive constant). Suppose B_a is a geodesic ball in N with radius a and h is an harmonic function on B_a. Then*

$$\sup_{B_{a/2}} |\nabla h| \le C_n \left(\frac{1+a\sqrt{K}}{a}\right) \sup_{B_a} |h|, \tag{3.3}$$

where C_n is a constant depending only on the dimension of N.

For a proof see [17, Corollary 3.2, page 21].

Remark 3.2. We will use Schoen and Yau's estimate in the following context. Take $N = M\times\mathbb{R}$. Observe that $\mathrm{Ricc}(N) = \mathrm{Ricc}(M)$, which is bounded from below because M is compact. Note that N is complete because it is a product of two complete manifolds. We put $a = r/\sqrt{L}$ ($r < R_0(M)$) and $B_a = B(z, r/\sqrt{L})\times I_r$ (note that this is not the ball of center $(z,0)\in N$ and radius $r/\sqrt{L}$, but it contains and it is contained in such ball of comparable radius).

Using Schoen and Yau's theorem, we deduce the global Bernstein inequality for a single eigenfunction.

Corollary 3.3 (Bernstein inequality). *If u is an eigenfunction of eigenvalue λ, then*

$$\|\nabla u\|_\infty \lesssim \sqrt{\lambda}\, \|u\|_\infty . \tag{3.4}$$

Proof. The harmonic extension of u is $h(z,t) = u(z)e^{\sqrt{\lambda}t}$. Applying inequality (3.3) to h (taking $a = R_0(M)/(2\sqrt{\lambda})$),

$$|\nabla u(z)| \lesssim \sqrt{\lambda}\, \|h\|_{L^\infty(M\times I_{R_0/2})} \approx \sqrt{\lambda}\, \|u\|_\infty . \quad \square$$

We conjecture that in inequality (3.4) one can replace u by any function $f \in E_L$, i.e.,

$$\|\nabla f\|_\infty \lesssim \sqrt{L}\,\|f\|_\infty .$$

For instance, as a direct consequence of Green's formula we have the L^2-Bernstein inequality for the space E_L:

$$\|\nabla f\|_2 \lesssim \sqrt{L}\,\|f\|_2 , \quad \forall f \in E_L .$$

For our purpose, it is sufficient to have a weaker Bernstein type inequality that compares the L^∞-norm of the gradient with the L^2-norm of the function.

Proposition 3.4. *Let $f \in E_L$. Then there exists a universal constant C such that*

$$\|\nabla f\|_\infty \le C\sqrt{k_L}\sqrt{L}\,\|f\|_2 .$$

For the proof, we need the following lemma.

Lemma 3.5. *For all $f \in E_L$ and $0 < r < R_0(M)/2$,*

$$|\nabla f(z)|^2 \le C_r L^{(m+2+1)/2} \int_{B(z,r/\sqrt{L})\times I_r} |h(w,s)|^2 dV(w)ds.$$

Proof. Using inequality (3.3) and the submean-value inequality for $|h|^2$, we have

$$\begin{aligned}
|\nabla f(z)|^2 &\le |\nabla h(z,0)|^2 \\
&\lesssim \frac{L}{r^2} \sup_{B(z,r/\sqrt{L})\times I_r} |h(w,t)|^2 \\
&\lesssim \frac{L^{(m+1+2)/2}}{\tilde{r}^{m+2+1}} \int_{B(z,\tilde{r}/\sqrt{L})\times I_{\tilde{r}}} |h(\xi,s)|^2 dV(\xi)ds,
\end{aligned}$$

where $\tilde{r} = 2r$. □

Proof of Proposition 3.4. By Lemma 3.5, given $0 < r < R_0(M)/2$, there exists a constant C_r such that

$$|\nabla f(z)|^2 \le C_r k_L L\sqrt{L} \int_{M\times I_r} |h(w,s)|^2 dV(w)ds \overset{\text{Proposition 3.1}}{\approx} C_r k_L L\,\|f\|_2^2 .$$

Taking $r = R_0(M)/4$, we get $|\nabla f(z)|^2 \le C k_L L\,\|f\|_2^2$ for all $z \in M$. □

4 Characterization of Carleson measures

Definition 4.1. Let $\mu = \{\mu_L\}_{L\geq 0}$ be a sequence of measures on M. We say that μ is an L^2-*Carleson sequence* for M if there exists a positive constant C such that for all L and $f_L \in E_L$

$$\int_M |f_L|^2 d\mu_L \leq C \int_M |f_L|^2 dV.$$

Theorem 4.2. *Let μ be a sequence of measures on M. Then μ is L^2-Carleson for M if and only if there exists a $C > 0$ such that for all L*

$$\sup_{\xi\in M} \mu_L(B(\xi, 1/\sqrt{L})) \leq \frac{C}{k_L}. \tag{4.1}$$

Remark 4.3. Condition (4.1) can be viewed as

$$\sup_{\xi\in M} \frac{\mu_L(B(\xi, 1/\sqrt{L}))}{\mathrm{vol}(B(\xi, 1/\sqrt{L}))} \lesssim 1.$$

First, we prove the following result.

Lemma 4.4. *Let μ be a sequence of measures on M. Then the following conditions are equivalent.*

(1) *There exists a constant $C = C(M) > 0$ such that for each L*

$$\sup_{\xi\in M} \mu_L(B(\xi, 1/\sqrt{L})) \leq \frac{C}{k_L}.$$

(2) *There exist $c = c(M) > 0$ ($c < 1$ small) and $C = C(M) > 0$ such that for all L*

$$\sup_{\xi\in M} \mu_L(B(\xi, c/\sqrt{L})) \leq \frac{C}{k_L}.$$

Proof. Obviously, the first condition implies the second one since

$$B(\xi, c/\sqrt{L}) \subset B(\xi, 1/\sqrt{L}).$$

Let us prove the converse. The manifold M is covered by the union of balls of center $\xi \in M$ and radius $c/\sqrt{L}$. Taking into account the 5-r covering lemma (see [15, Chapter 2, page 23] for more details), we get a family of disjoint balls, denoted by $B_i = B(\xi_i, c/\sqrt{L})$, such that M is covered by the union of $5B_i$. This union may be finite or countable. Let $\xi \in M$ and consider $B := B(\xi, 1/\sqrt{L})$.

Suppose n is the number of balls $\bar{B}_i$ such that $\bar{B} \cap 5\bar{B}_i \neq \emptyset$. Since $\bar{B}$ is compact, we have a finite number of these balls so that

$$\bar{B} \subset \bigcup_{i=1}^{n} 5\bar{B}_i .$$

We claim that n is independent of L. In this case, we get

$$\mu_L(B) \leq \sum_{i=1}^{n} \mu_L(B(\xi_i, 5c/\sqrt{L})) \lesssim \frac{n}{k_L}$$

and thus our statement is proved. Indeed, using the triangle inequality, we have for all $i = 1, \dots, n$

$$B(\xi_i, c/\sqrt{L}) \subset B(\xi, 10/\sqrt{L}).$$

Therefore,

$$\bigcup_{i=1}^{n} B(\xi_i, c/\sqrt{L}) \subset B(\xi, 10/\sqrt{L}),$$

where the union is a disjoint union of balls. Now,

$$\frac{10^m}{L^{m/2}} \approx \operatorname{vol}(B(\xi, 10/\sqrt{L})) \geq \sum_{i=1}^{n} \operatorname{vol}(B_i) \approx n \frac{c^m}{L^{m/2}}.$$

Hence $n \lesssim (10/c)^m$ and we can choose it independently of L. □

Now we can prove the characterization of the Carleson measures.

Theorem 4.2. Assume condition (4.1) holds. We need to prove the existence of a constant $C > 0$ (independent of L) such that for each $f \in E_L$

$$\int_M |f|^2 d\mu_L \leq C \int_M |f|^2 dV.$$

Let $f \in E_L$ with L and $r > 0$ (small) fixed. We have

$$\begin{aligned}\int_M |f(z)|^2 d\mu_L &\overset{(3.1)}{\leq} C_r L^{(m+1)/2} \int_M \int_{B(z,r/\sqrt{L})\times I_r} |h(w,s)|^2 dV(w) ds\, d\mu_L(z) \\ &= C_r L^{(m+1)/2} \int_{M\times I_r} |h(w,s)|^2 \mu_L(B(w,r/\sqrt{L})) dV(w) ds \\ &\leq C_r L^{(m+1)/2} \frac{1}{k_L} \int_{M\times I_r} |h(w,s)|^2 dV(w) ds \overset{\text{Proposition 3.1}}{\approx} \|f\|_2^2\end{aligned}$$

with constants independent of L. Therefore, $\mu = \{\mu_L\}_L$ is L^2-Carleson for M.

For the converse, assume that μ is L^2-Carleson for M. We have to show the existence of a constant C such that for all $L \geq 1$ and $\xi \in M$, $\mu_L(B(\xi, c/\sqrt{L})) \leq C/k_L$ (for some small constant $c > 0$). We will argue by contradiction, i.e., assume that for all $n \in \mathbb{N}$ there exists L_n and a ball B_n of radius $c/\sqrt{L_n}$ such that $\mu_{L_n}(B_n) > n/k_{L_n} \approx n/L_n^{m/2}$ (c will be chosen later). Let b_n be the center of the ball B_n. Define $F_n(w) = K_{L_n}(b_n, w)$. Note that the function $L_n^{-m/4} F_n \in E_{L_n}$ and $\|F_n\|_2^2 = K_{L_n}(b_n, b_n) \approx L_n^{m/2}$. Therefore,

$$C \approx \int_M |L_n^{-m/4} F_n|^2 dV \gtrsim \int_M |L_n^{-m/4} F_n|^2 d\mu_{L_n} \gtrsim \int_{B_n} |L_n^{-m/4} F_n|^2 d\mu_{L_n}$$
$$\geq \inf_{w \in B_n} |L_n^{-m/4} F_n(w)|^2 \mu_{L_n}(B_n) \gtrsim \inf_{w \in B_n} |F_n(w)|^2 \frac{n}{L_n^m}.$$

Now we will study this infimum. Let $w \in B_n = B(b_n, c/\sqrt{L_n})$. Then

$$|F_n(b_n)| - |F_n(w)| \leq |F_n(b_n) - F_n(w)| \leq \frac{c}{\sqrt{L_n}} \|\nabla F_n\|_\infty$$
$$\overset{\text{Proposition 3.4}}{\leq} \frac{c}{\sqrt{L_n}} C_1 \sqrt{k_{L_n}} \sqrt{L_n} \, \|F_n\|_2 \approx c C_1 k_{L_n}.$$

We pick c small enough so that

$$\inf_{B_n} |F_n(w)|^2 \geq C L_n^m.$$

Finally, we have shown that $C \gtrsim n$ for all $n \in \mathbb{N}$. This gives the contradiction. □

The following result is a Plancherel–Pólya type theorem but in the context of the Paley–Wiener spaces E_L. Before we give the statement of the result, we shall introduce the concept of a separated family of points.

Definition 4.5. Suppose $\mathcal{Z} = \{z_{Lj}\}_{j \in \{1,\dots,m_L\}, L \geq 1} \subset M$ is a triangular family of points, where $m_L \to \infty$ as $L \to \infty$. We say that $\mathcal{Z}$ is *uniformly separated* if there exists $\epsilon > 0$ such that

$$d(z_{Lj}, z_{Lk}) \geq \frac{\epsilon}{\sqrt{L}}, \quad \forall j \neq k, \ \forall L \geq 1,$$

where ϵ is called the *separation constant of* $\mathcal{Z}$.

Theorem 4.6 (Plancherel–Pólya Theorem). *Let $\mathcal{Z}$ be a triangular family of points in M, i.e., $\mathcal{Z} = \{z_{Lj}\}_{j \in \{1,\dots,m_L\}, L \geq 1} \subset M$. Then $\mathcal{Z}$ is a finite union of uniformly separated families if and only if there exists a constant $C > 0$ such that for all $L \geq 1$ and $f_L \in E_L$*

$$\frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \leq C \int_M |f_L(\xi)|^2 dV(\xi). \tag{4.2}$$

Remark 4.7. The above result is interesting because the inequality (4.2) means that the sequence of normalized reproducing kernels is a Bessel sequence for E_L, i.e.,

$$\sum_{j=1}^{m_L} |\langle f, \tilde{K}_L(\cdot, z_{Lj})\rangle|^2 \lesssim \|f\|_2^2 \quad \forall f \in E_L,$$

where $\{\tilde{K}_L(\cdot, z_{Lj})\}_j$ are the normalized reproducing kernels. Note that we have $|\tilde{K}_L(\cdot, z_{Lj})|^2 \approx |K_L(\cdot, z_{Lj})|^2 k_L^{-1}$. That is the reason why the quantity k_L appears in inequality (4.2).

Proof. This is a consequence of Theorem 4.2 applied to the measures

$$\mu_L = \frac{1}{k_L} \sum_{j=1}^{m_L} \delta_{z_{Lj}}, \quad L \geq 1. \qquad \square$$

5 Characterization of Logvinenko–Sereda sets

Before we state the characterization, we would like to recall some history of these kind of inequalities. The classical Logvinenko–Sereda (L–S) theorem describes some equivalent norms for functions in the Paley–Wiener space PW_Ω^p. The precise statements is the following:

Theorem 5.1 (L–S). *Let Ω be a bounded set and $1 \leq p < +\infty$. A set $E \subset \mathbb{R}^d$ satisfies*

$$\int_{\mathbb{R}^d} |f(x)|^p dx \leq C_p \int_E |f(x)|^p dx, \quad \forall f \in PW_\Omega^p,$$

if and only if there is a cube $K \subset \mathbb{R}^d$ such that

$$\inf_{x \in \mathbb{R}^d} |(K + x) \cap E| > 0.$$

One can find the original proof in [11] and another proof can be found in [6, p. 112–116].

Luecking studied in [12] this notion for the Bergman spaces. Following his ideas, in [14], the following result has been proved.

Theorem 5.2. *Let $1 \leq p < +\infty$, μ be a doubling measure and let $\mathcal{A} = \{A_L\}_{L \geq 0}$ be a sequence of sets in $\mathbb{S}^d$. Then $\mathcal{A}$ is $L^p(\mu)$-L–S if and only if $\mathcal{A}$ is μ-relatively dense.*

For the precise definitions and notations see [14]. Using the same ideas, we will prove the above theorem for the case of our arbitrary compact manifold M and the measure given by the volume element.

In what follows, $\mathcal{A} = \{A_L\}_L$ will be a sequence of sets in M.

Definition 5.3. We say that $\mathcal{A}$ is *L–S* if there exists a constant $C > 0$ such that for any L and $f_L \in E_L$

$$\int_M |f_L|^2 dV \leq C \int_{A_L} |f_L|^2 dV.$$

Definition 5.4. The sequence of sets $\mathcal{A} \subset M$ is *relatively dense* if there exist $r > 0$ and $\rho > 0$ such that for all L

$$\inf_{z \in M} \frac{\mathrm{vol}(A_L \cap B(z, r/\sqrt{L}))}{\mathrm{vol}(B(z, r/\sqrt{L}))} \geq \rho > 0.$$

Remark 5.5. It is equivalent to having this property for all $L \geq L_0$ for some L_0 fixed.

A natural example of relatively dense sets is the following. Consider a separated family in M, $\mathcal{Z} = \{z_{Lj}\}_{j \in \{1,\dots,m_L\}, L \geq 1}$, with separation constant s. Let us denote $A_L = M \setminus \bigcup_{j=1}^{m_L} B(z_{Lj}, \frac{s}{10\sqrt{L}})$. It is easy to check that the family $\mathcal{A} = \{A_L\}_L$ is relatively dense.

Our main statement is the following:

Theorem 5.6. *$\mathcal{A}$ is L–S if and only if $\mathcal{A}$ is relatively dense.*

We shall prove the two implications in the statement separately. First we will show that this condition is necessary. Before proceeding, we construct functions in E_L with a desired decay of its L^2-integral outside a ball.

Proposition 5.7. *Given $\xi \in M$ and $\epsilon > 0$, there exist functions $f_L = f_{L,\xi} \in E_L$ and $R_0 = R_0(\epsilon, M) > 0$ such that*

(1) $\|f_L\|_2 = 1$.

(2) *For all $L \geq 1$,*

$$\int_{M \setminus B(\xi, R_0/\sqrt{L})} |f_L|^2 dV < \epsilon.$$

(3) *For all $L \geq 1$ and any subset $A \subset M$,*

$$\int_A |f_L|^2 dV \leq C_1 \frac{\mathrm{vol}(A \cap B(\xi, R_0/\sqrt{L}))}{\mathrm{vol}(B(\xi, R_0/\sqrt{L}))} + \epsilon,$$

where C_1 is a constant independent of L, ξ and f_L.

Remark. In the above proposition, the R_0 does not depend on the point ξ.

Proof. Given $z, \xi \in M$ and $L \geq 1$, let $S_L^N(z,\xi)$ denote the Riesz kernel of index $N \in \mathbb{N}$ associated to the Laplacian, i.e.,

$$S_L^N(z,\xi) = \sum_{i=1}^{k_L} \left(1 - \frac{\lambda_i}{L}\right)^N \phi_i(z)\phi_i(\xi).$$

Note that $S_L^0(z,\xi) = K_L(z,\xi)$. The Riesz kernel satisfies the following inequality:

$$|S_L^N(z,\xi)| \leq C L^{m/2}(1 + \sqrt{L} d(z,\xi))^{-N-1}. \tag{5.1}$$

This estimate has its origins in Hörmander's article [8, Theorem 5.3]. Estimate (5.1) can be found also in [18, Lemma 2.1].

Note that on the diagonal, $S_L^N(z,z) \approx C_N L^{m/2}$. The upper bound is trivial by definition and the lower bound follows from

$$S_L^N(z,z) \geq \sum_{i=1}^{k_{L/2}} \left(1 - \frac{\lambda_i}{L}\right)^N \phi_i(z)\phi_i(z) \geq 2^{-N} K_{L/2}(z,z) \approx C_N L^{m/2}.$$

Similarly, we observe that $\|S_L^N(\cdot,\xi)\|_2^2 \approx C_N L^{m/2}$.

Given $\xi \in M$, define for all $L \geq 1$

$$f_{L,\xi}(z) := f_L(z) = \frac{S_L^N(z,\xi)}{\|S_L^N(\cdot,\xi)\|_2}.$$

We will choose the order N later. Each f_L belongs to the space E_L and has unit L^2-norm. Let us verify the second property claimed in Proposition 5.7. Fix a radius R. Using the estimate (5.1), we get

$$\int_{M\setminus B(\xi,R/\sqrt{L})} |f_L|^2 dV \leq C_N L^{m/2} \int_{M\setminus B(\xi,R/\sqrt{L})} \frac{dV}{(\sqrt{L} d(z,\xi))^{2(N+1)}} = (\star).$$

For any $t \geq 0$, consider the following set:

$$A_t := \left\{ z \in M : d(z,\xi) \geq \frac{R}{\sqrt{L}},\ d(z,\xi) < \frac{t^{-1/(2(N+1))}}{\sqrt{L}} \right\}.$$

Note that for $t > R^{-2(N+1)}$ we have $A_t = \emptyset$, and for $t < R^{-2(N+1)}$ we obtain $A_t \subset B(\xi, t^{-1/(2(N+1))}/\sqrt{L})$. Using the distribution function, we have

$$(\star) = C_N L^{m/2} \int_0^{R^{-2(N+1)}} \operatorname{vol}(A_t)dt \leq C_N \frac{1}{R^{2(N+1)-m}}$$

provided $N+1 > m/2$. Thus if we pick R_0 big enough, we get

$$\int_{M\setminus B(\xi,R_0/\sqrt{L})} |f_L|^2 dV < \epsilon. \tag{5.2}$$

Now the third property claimed in Proposition 5.7 follows from (5.2). Indeed, given any subset A in the manifold M,

$$\int_A |f_L|^2 dV \le \int_{A\cap B(\xi,R_0/\sqrt{L})} |f_L|^2 dV + \epsilon.$$

Observe that

$$\begin{aligned}\int_{A\cap B(\xi,R_0/\sqrt{L})} &|f_L|^2 dV \\ &\lesssim C_N L^{m/2} \int_{A\cap B(\xi,R_0/\sqrt{L})} \frac{dV(z)}{(1+\sqrt{L}d(z,\xi))^{2(N+1)}} \\ &\lesssim C_N R_0^m \frac{\mathrm{vol}(A\cap B(\xi,R_0/\sqrt{L}))}{\mathrm{vol}(B(\xi,R_0/\sqrt{L}))}.\end{aligned}$$

□

Now we are ready to prove one of the implications in the characterization of the L–S sets.

Proposition 5.8. *Assume $\mathcal{A}$ is L–S. Then it is relatively dense.*

Proof. Assume $\mathcal{A}$ is L–S, i.e.,

$$\int_M |f_L|^2 dV \le C \int_{A_L} |f_L|^2 dV.$$

Let $\xi \in M$ be an arbitrary point. Fix $\epsilon > 0$ and consider the R_0 and the functions $f_L \in E_L$ given by Proposition 5.7. Using the third property of Proposition 5.7 for the sets A_L, we get for all $L \ge 1$

$$1 = \|f_L\|_2^2 \le C \int_{A_L} |f_L|^2 \le CC_1 \frac{\mathrm{vol}(A_L\cap B(\xi,R_0/\sqrt{L}))}{\mathrm{vol}(B(\xi,R_0/\sqrt{L}))} + C\epsilon,$$

where C_1 is a constant independent of L, ξ and f_L. Therefore, we have proved that there exist constants c_1 and c_2 such that

$$\frac{\mathrm{vol}(A_L\cap B(\xi,R_0/\sqrt{L}))}{\mathrm{vol}(B(\xi,R_0/\sqrt{L}))} \ge c_1 - c_2\epsilon.$$

Hence $\mathcal{A}$ is relatively dense provided $\epsilon > 0$ is small enough. □

Before we continue, we will prove a result concerning the uniform limit of harmonic functions with respect to some metric.

Lemma 5.9. *Let $\{H_n\}_n$ be a family of uniformly bounded real functions defined on the ball $\mathbb{B}(0,\rho) \subset \mathbb{R}^d$ for some $\rho > 0$. Let g be a non-singular $\mathcal{C}^\infty$ metric such that g and all its derivatives are uniformly bounded and $g_{ij}(0) = \delta_{ij}$. Define $g_n(z) = g(z/L_n)$ (the rescaled metrics), where L_n is a sequence of values tending to ∞ as n increases. Assume the family $\{H_n\}_n$ converges uniformly on compact subsets of $\mathbb{B}(0,\rho)$ to a limit function $H : \mathbb{B}(0,\rho) \to \mathbb{R}$ and H_n is harmonic with respect to the metric g_n (i.e., $\Delta_{g_n} H_n = 0$). Then the limit function H is harmonic in the Euclidean sense.*

Proof. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{B}(0,\rho))$. We have

$$\int_{\mathbb{B}(0,\rho)} \Delta_g f \varphi dV = \int_{\mathbb{B}(0,\rho)} f \Delta_g \varphi dV.$$

By direct computation, we see that $\Delta_{g_n}\varphi \to \Delta\varphi$ uniformly and $\Delta_{g_n}\varphi$ is uniformly bounded on $\mathbb{B}(0,\rho)$. Then

$$0 = \int_{\mathbb{B}(0,\rho)} H_n \Delta_{g_n} \varphi dV_{g_n} \to \int_{\mathbb{B}(0,\rho)} H \Delta\varphi dm(z) = \int_{\mathbb{B}(0,\rho)} \Delta H \varphi dm(z).$$

Therefore, the limit function H is harmonic in the weak sense. Applying Weyl's lemma, we conclude that H is harmonic in the Euclidean sense. □

Remark 5.10. The above argument also holds if we have a sequence of metrics g_n converging uniformly to g whose derivatives also converge uniformly to the derivatives of g. In this case, the conclusion would be that the limit is harmonic with respect to the limit metric g.

Now, we shall prove the sufficient condition of the main result.

Proposition 5.11. *If $\mathcal{A}$ is relatively dense, then it is L–S.*

Proof. Fix $\epsilon > 0$ and $r > 0$. Let $D := D_{\epsilon,r,f_L}$ be

$$D = \left\{ z \in M : |f_L(z)|^2 = |h_L(z,0)|^2 \geq \epsilon \fint_{B(z,\frac{r}{\sqrt{L}})\times I_r} |h_L(\xi,t)|^2 dV(\xi)dt \right\},$$

where h_L is the harmonic extension of f_L defined as

$$h_L(z,t) = \sum_{i=1}^{k_L} \beta_i \phi_i(z) e^{\sqrt{\lambda_i}t}, \quad f_L(z) = \sum_{i=1}^{k_L} \beta_i \phi_i(z).$$

The norm of f_L is almost concentrated on D because

$$\begin{aligned}\int_{M\setminus D} &|f_L(z)|^2 dV(z)\\ &\lesssim \epsilon \frac{1}{l(I_r)} \int_{M\times I_r} |h_L(\xi,t)|^2 \frac{L^{m/2}}{r^m} \int_{(M\setminus D)\cap B(\xi, r/\sqrt{L})} dV(z) dV(\xi) dt\\ &\lesssim \epsilon \frac{1}{l(I_r)} \int_{M\times I_r} |h_L(\xi,t)|^2 dV(\xi) dt \overset{\text{Proposition 3.1}}{\lesssim} e^{2r}\epsilon \int_M |f_L|^2 dV.\end{aligned}$$

It is enough to prove

$$\int_D |f_L|^2 dV \lesssim \int_{A_L} |f_L|^2 dV \tag{5.3}$$

with constants independent of L, and for this it is sufficient to show that there exists a constant $C > 0$ such that for all $w \in D$

$$|f_L(w)|^2 \leq \frac{C}{\operatorname{vol}(B(w, r/\sqrt{L}))} \int_{A_L\cap B(w,r/\sqrt{L})} |f_L(\xi)|^2 dV(\xi) \tag{5.4}$$

because then (5.3) follows by integrating (5.4) over D. So we need to prove (5.4).

This is the outline of the proof: We assume that (5.4) is not true in order to construct functions that satisfy the opposite inequality. Then we will parameterize these functions and prove that their limit is harmonic with unit norm and is zero in a subset of positive measure. This will lead to a contradiction. Now we proceed with the details.

Step 1*. Parametrization and rescalement of the functions.*

If (5.4) is not true, then for all $n \in \mathbb{N}$ there exist L_n, functions $f_n \in E_{L_n}$ and $w_n \in D_n = D_{\epsilon,r,f_n}$ such that

$$|f_n(w_n)|^2 > \frac{n}{\operatorname{vol}(B(w_n, r/\sqrt{L_n}))} \int_{A_{L_n}\cap B(w_n,r/\sqrt{L_n})} |f_n|^2 dV.$$

By the compactness of the manifold M, there exists $\rho_0 = \rho_0(M) > 0$ such that for all $w \in M$ the exponential map, $\exp_w : \mathbb{B}(0,\rho_0) \to B(w,\rho_0)$, is a diffeomorphism and $(B(w,\rho_0), \exp_w^{-1})$ is a normal coordinate chart, where w is mapped to 0 and the metric g verifies $g_{ij}(0) = \delta_{ij}$.

For all $n \in \mathbb{N}$, take $\exp_n(z) := \exp_{w_n}(rz/\sqrt{L_n})$, which is defined in $\mathbb{B}(0,1)$ and acts as follows:

$$\begin{aligned}\exp_n : \mathbb{B}(0,1) &\longrightarrow \mathbb{B}(0, r/\sqrt{L_n}) \longrightarrow B(w_n, r/\sqrt{L_n})\\ z &\longmapsto \frac{rz}{\sqrt{L_n}} \longmapsto \exp_{w_n}(rz/\sqrt{L_n}) =: w.\end{aligned}$$

Consider $F_n(z) := c_n f_n(\exp_n(z)) : \mathbb{B}(0,1) \longrightarrow B(w_n, r/\sqrt{L_n}) \xrightarrow{c_n f_n} \mathbb{R}$ and the corresponding harmonic extension h_n of f_n. Set

$$H_n(z,t) := c_n h_n(\exp_n(z), rt/\sqrt{L_n}),$$

defined on $\mathbb{B}(0,1)\times J_1$ (where $J_1 = (-1,1)$), where c_n is a normalization constant such that

$$\int_{\mathbb{B}(0,1)\times J_1} |H_n(w,s)|^2 d\mu_n(w) ds = 1.$$

Step 2. *The functions H_n are uniformly bounded.*

Let μ_n be the measure such that

$$d\mu_n(z) = \sqrt{|g|(\exp_{w_n}(rz/\sqrt{L_n}))} dm(z).$$

Note that

$$\int_{B(w_n,\frac{r}{\sqrt{L_n}})} |f_n|^2 dV = \frac{r^m}{L_n^{m/2}} \frac{1}{|c_n|^2} \int_{\mathbb{B}(0,1)} |F_n(z)|^2 d\mu_n(z).$$

Therefore, we have

$$⨍_{B(w_n,r/\sqrt{L_n})} |f_n|^2 dV \approx \frac{1}{|c_n|^2} \int_{\mathbb{B}(0,1)} |F_n|^2 d\mu_n.$$

As $w_n \in D_n$, we obtain

$$\begin{aligned} |F_n(0)|^2 = |c_n|^2 |f_n(w_n)|^2 &\geq |c_n|^2 \epsilon ⨍_{B(w_n,r/\sqrt{L_n})\times I_r} |h_n(w,t)|^2 dV dt \\ &\approx \epsilon \int_{\mathbb{B}(0,1)\times J_1} |H_n(w,s)|^2 d\mu_n(w) ds = \epsilon. \end{aligned}$$

Since $|h_n|^2$ is subharmonic,

$$|F_n(0)|^2 = |c_n|^2 |h_n(w_n,0)|^2 \lesssim \int_{\mathbb{B}(0,1)\times J_1} |H_n(w,s)|^2 d\mu_n(w) ds = 1.$$

Hence we have $0 < \epsilon \lesssim |F_n(0)|^2 \lesssim 1$ for all $n \in \mathbb{N}$.

Using the assumption, we get

$$\frac{1}{n} \gtrsim \frac{|c_n|^2}{\mathrm{vol}(B(w_n, r/\sqrt{L_n}))} \int_{A_{L_n}\cap B(w_n,\frac{r}{\sqrt{L_n}})} |f_n|^2 dV \approx \int_{B_n \cap \mathbb{B}(0,1)} |F_n|^2 d\mu_n,$$

where B_n is such that $\exp_n(B_n \cap \mathbb{B}(0,1)) = A_{L_n} \cap B(w_n, r/\sqrt{L_n})$. So we have that

$$\begin{cases} \forall n & 0 < \epsilon \lesssim |F_n(0)|^2 \lesssim 1, \\ \forall n & \int_{\mathbb{B}(0,1)\cap B_n} |F_n|^2 d\mu_n \lesssim \frac{1}{n}. \end{cases}$$

In fact, $0 < \epsilon \lesssim |H_n(0,0)|^2 \lesssim 1$ (by definition) and one can prove that $|H_n|^2 \lesssim 1$. Indeed, if $(z,s) \in \mathbb{B}(0,1/2) \times J_{1/2}$, let $w = \exp_n(z) \in B(w_n, r/(2\sqrt{L_n}))$ and $t = rs/\sqrt{L_n} \in I_{r/2}$. Then

$$\begin{aligned} |H_n(z,s)|^2 &= |c_n|^2 |h_n(w,t)|^2 \\ &\lesssim |c_n|^2 \fint_{B(w,r/(2\sqrt{L_n}))\times I_{r/2}(t)} |h_n|^2 \\ &\lesssim |c_n|^2 \fint_{B(w_n,r/\sqrt{L_n})\times I_r} |h_n|^2 dV dt \approx 1. \end{aligned}$$

Therefore, working with $1/2$ instead of 1, we have $|H_n|^2 \lesssim 1$ for all n.

Step 3*. The family $\{H_n\}_n$ is equicontinuous in $\mathbb{B}(0,1) \times J_1$.*

Consider $(w,t) \in B(w_n, r/(4\sqrt{L_n})) \times I_{r/4}$ and $(\tilde{w}, \tilde{t}) \in B(w, \tilde{r}r/\sqrt{L_n}) \times I_{\tilde{r}r}(t)$. Then there exists some small $\delta > 0$ such that

$$|c_n||h_n(w,t) - h_n(\tilde{w},\tilde{t})| \le |c_n| \frac{\tilde{r}}{\sqrt{L_n}} r \sup_{B(w,\delta/\sqrt{L_n})\times I_\delta(t)} |\nabla h_n| \le (\star).$$

Taking $\tilde{r}$ small enough so that $\delta \le r/4$ and using Schoen and Yau's estimate (3.3), we have

$$\begin{aligned} (\star) &\le |c_n| \frac{\tilde{r}r}{\sqrt{L_n}} \sup_{B(w_n,r/(2\sqrt{L_n}))\times I_{r/2}} |\nabla h_n| \\ &\lesssim \frac{\tilde{r}r}{\sqrt{L_n}} \frac{1}{\frac{r}{\sqrt{L_n}}} \sup_{B(w_n,r/\sqrt{L_n})\times I_r} |c_n||h_n| \lesssim \tilde{r}. \end{aligned}$$

So we have proved that $|c_n||h_n(w,t) - h_n(\tilde{w},\tilde{t})| \le C\tilde{r}$. Take $\tilde{r}$ small enough so that $C\tilde{r} < \epsilon$. Let $(z,s) \in \mathbb{B}(0,1/4) \times J_{1/4}$ and $(\tilde{z},\tilde{s}) \in \mathbb{B}(z,\tilde{r}) \times (s-\tilde{r}, s+\tilde{r})$. Consider $w = \exp_n(z)$, $t = rs/\sqrt{L_n}$, $\tilde{w} = \exp_n(\tilde{z})$ and $\tilde{t} = r\tilde{s}/\sqrt{L_n}$. Then we have proved that for all $\epsilon > 0$ there exists $\tilde{r} > 0$ (small) such that for all $(z,s) \in \mathbb{B}(0,1/4) \times J_{1/4}$:

$$|H_n(z,s) - H_n(\tilde{z},\tilde{s})| < \epsilon \quad \text{if } |z-\tilde{z}| < \tilde{r},\ |s-\tilde{s}| < \tilde{r}, \quad \forall n \in \mathbb{N}.$$

Change $1/4$ to 1. So the sequence H_n is equicontinuous.

Step 4. *There exists a limit function of H_n that is real analytic.*

The family $\{H_n\}_n$ is equicontinuous and uniformly bounded on $\mathbb{B}(0,1) \times J_1$. Therefore, by Ascoli–Arzela's theorem, there is a partial sequence (denoted as the sequence itself) such that $H_n \to H$ uniformly on compact subsets of $\mathbb{B}(0,1) \times J_1$. Since $F_n(z) = H_n(z,0)$, we get a function $F(z) := H(z,0) : \mathbb{B}(0,1) \to \mathbb{R}$, which is the limit of F_n (uniformly on compact subsets of $\mathbb{B}(0,1)$).

Now we will prove that H is real analytic. In fact, we will show that H is harmonic. We have the following properties:

(1) Observe that the family of measure $d\mu_n$ converges uniformly to the ordinary Euclidean measure because $g_{ij}(\exp_{w_n}(rz/\sqrt{L_n})) \to g_{ij}(\exp_{w_0}(0)) = \delta_{ij}$, where w_0 is the limit point of some subsequence of w_n (recall that we are taking normal coordinate charts).

(2) If $g_n(z) := g(rz/\sqrt{L_n})$ (i.e., g_n is the rescaled metric), then we have that $\Delta_{(g_n,Id)} H_n(z,s) = 0$ for all $(z,s) \in \mathbb{B}(0,1) \times J_1$ by construction.

(3) The functions H_n are uniformly bounded and converge uniformly on compact subsets of $\mathbb{B}(0,1) \times J_1$.

We are in the hypothesis of Lemma 5.9 that guarantees the harmonicity of H in the Euclidean sense.

Step 5. *Using the hypothesis, we will construct a measure τ such that $|F| = 0$ τ-a.e. and $\tau(\overline{\mathbb{B}(a,s)}) \lesssim s^m$ for all $\overline{\mathbb{B}(a,s)} \subset \mathbb{B}(0,1)$. These two properties and the real analyticity of F will lead to a contradiction.*

By hypothesis, the sequence $\{A_L\}_L$ is relatively dense. Taking into account that $\mathrm{vol}(B(w_n, r/\sqrt{L_n})) = \frac{r^m}{L_n^{m/2}} \mu_n(\mathbb{B}(0,1))$, we get that

$$\inf_n \mu_n(B_n) \geq \rho > 0, \tag{5.5}$$

where we have denoted $B_n \cap \mathbb{B}(0,1)$ by B_n.

Let τ_n be such that $d\tau_n = \chi_{B_n} d\mu_n$. From a standard argument (τ_n are supported in a ball) we know the existence of a weak *-limit of a subsequence of τ_n, denoted by τ. This subsequence will be noted as the sequence itself. From (5.5) we know that τ is not identically 0. Now we have that

$$\int_{\mathbb{B}(0,1)} |F|^2 d\tau = 0.$$

Therefore, $F = 0$ τ-a.e. in $\mathbb{B}(0,1)$. Now for all $K \subset \mathbb{B}(0,1)$ compact,

$$\int_K |F|^2 d\tau = 0,$$

therefore $F = 0$ in $\operatorname{supp}\tau$. Let $\overline{\mathbb{B}(a,s)} \subset \mathbb{B}(0,1)$ satisfy $\overline{\mathbb{B}(a,s)} \cap \operatorname{supp}\tau \neq \emptyset$. Then using the fact $B_n \subset \mathbb{B}(0,1)$, we obtain

$$\tau_n(\overline{\mathbb{B}(a,s)}) \leq \int_{\overline{\mathbb{B}(a,s)}} d\mu_n \approx \frac{L_n^{m/2}}{r^m}\mathrm{vol}(B(\exp_n(a), sr/\sqrt{L_n})) \approx s^m.$$

Therefore, $\tau_n(\overline{\mathbb{B}(a,s)}) \lesssim s^m$ for all n. Hence in the limit case, $\tau(\overline{\mathbb{B}(a,s)}) \lesssim s^m$. In short,

(1) We have sets $B_n \subset \mathbb{B}(0,1)$ such that

$$\rho \leq \mu_n(B_n) \leq \mu_n(\mathbb{B}(0,1)) \approx 1.$$

(2) We have measures τ_n weakly-* converging to τ (not identically 0).

(3) $\tau(\overline{\mathbb{B}(a,s)}) \lesssim s^m$ for all $\overline{\mathbb{B}(a,s)} \subset \mathbb{B}(0,1)$.

(4) $|F| = 0$ τ-a.e. in $\mathbb{B}(0,1)$.

(5) $|F(0)| > 0$ and $|F| \lesssim 1$.

We know that H is real analytic, then $F(z)$ is real analytic. Federer ([4, Theorem 3.4.8]) proved that the $(m-1)$-Hausdorff measure $\mathcal{H}^{m-1}(F^{-1}(0)) < \infty$. Hence $\mathcal{H}^{m-1}(\operatorname{supp}\tau) \leq \mathcal{H}^{m-1}(F^{-1}(0)) < \infty$. This implies that the Hausdorff dimension $\dim_{\mathcal{H}}(\operatorname{supp}\tau) \leq m-1$. On the other hand, since $\tau(\overline{\mathbb{B}(a,s)}) \lesssim s^m$, we have

$$0 < \tau(\operatorname{supp}\tau) \lesssim \mathcal{H}^m(\operatorname{supp}\tau)$$

and this implies that $\dim_{\mathcal{H}}(\operatorname{supp}\tau) \geq m$ by Frostman's lemma. So we reached to a contradiction and the proof is complete. This concludes the proof of the proposition. □

Remark 5.12. A natural question is if one can replace the condition of being L–S, i.e.,

$$\int_M |f|^2 dV \leq C \int_{A_L} |f|^2 dV, \quad \forall f \in E_L, \tag{5.6}$$

by a weaker condition like

$$\int_M |f|^2 dV \leq C \int_{A_L} |f|^2 dV, \quad \forall f \in W_L, \tag{5.7}$$

where W_L is the L-eigenspace of Δ, and still obtains the fact that $\{A_L\}_L$ are relatively dense. If this was achieved, one could try to use this fact together with the recent work of Colding–Minicozzi (see [1]) in order to make some progress towards a proof of the lower bound in Yau's conjecture on the size of nodal sets. Unfortunately, a simple example shows that condition (5.6) cannot be replaced by (5.7). Indeed, take $M = \mathbb{S}^1$. Thus we are considering the space of polynomials of the form $p_n(z) = az^n + b\overline{z}^n$. Note that $|p_n(z)| = |az^{2n} + b|$ for all $z \in \mathbb{S}^1$. Now consider the sets

$$A_n = \left\{z \in \mathbb{S}^1 : \mathrm{Im}(z) < 0\right\}.$$

Trivially,

$$\int_{\mathbb{S}^1} |p_n| dV \leq 2 \int_{A_n} |p_n| dV, \quad \forall n \in \mathbb{N},$$

but the A_n are not relatively dense.

Of course, an interesting question which is left open is a geometric/metric description of the L–S sets for the L-eigenspaces.

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Bibliography

[1] T. H. Colding and W. P. Minicozzi II, Lower bounds for nodal sets of eigenfunctions, to appear in *Comm. Math. Phys.*

[2] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, *Invent. Math.* **93** (1988), no. 1, 161–183.

[3] H. Donnelly and C. Fefferman, Growth and geometry of eigenfunctions of the Laplacian, in: *Analysis and Partial Differential Equations*, pp. 635–655, Lecture Notes in Pure and Applied Mathematics 122, Marcel Dekker, New York, 1990.

[4] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften 153, Springer-Verlag, New York 1969.

[5] R. E. Greene and H. Wu, Integrals of subharmonic functions on manifolds of nonnegative curvature, *Invent. Math.* **27** (1974), 265–298.

[6] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, Springer-Verlag, Berlin, 1994.

[7] L. Hörmander, The spectral function of an elliptic operator, *Acta Math.* **121** (1968), 193–218.

[8] L. Hörmander, On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators, in: *Some Recent Advances in the Basic Sciences, Volume 2* (Proceedings Annual Science Conference, Belfer Graduate School, Yeshiva University, New York, 1965–1966), pp. 155–202, Academic Press, New York, 1966.

[9] J. Jost, *Riemannian Geometry and Geometric Analysis*, Fifth edition, Universitext, Springer-Verlag, Berlin, 2008.

[10] P. Li and R. Schoen, L^p and mean value properties of subharmonic functions on Riemannian manifolds, *Acta Math.* **153** (1984), no. 3–4, 279–301.

[11] V. N. Logvinenko and J. F. Sereda, Equivalent norms in spaces of entire functions of exponential type, *Teor. funktsii, funkt. analiz i ich prilozhenia* **20** (1974), 102–111, 175.

[12] D. H. Luecking, Equivalent norms on L^p spaces of harmonic functions, *Monatsh. Math.* **96** (1983), no. 2, 133–141.

[13] J. Marzo, Marcinkiewicz-Zygmund inequalities and interpolation by spherical harmonics, *J. Funct. Anal.* **250** (2007), no. 2, 559–587.

[14] J. Marzo and J. Ortega-Cerdà, Equivalent norms for polynomials on the sphere, *Int. Math. Res. Not. IMRN* (2008), no. 5, Art. ID rnm 154, 18p.

[15] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*, Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge, 1995.

[16] J. Ortega-Cerdà and J. Saludes, Marcinkiewicz–Zygmund inequalities, *J. Approx. Theory* **145** (2007), no. 2, 237–252.

[17] R. Schoen and S.-T. Yau, *Lectures on Differential Geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology 1, International Press, Redwood City, CA, 1994.

[18] C. D. Sogge, On the convergence of Riesz means on compact manifolds, *Ann. of Math. (2)* **126**, no. 3, 439–447.

[19] T. J. Willmore, Mean value theorems in harmonic riemannian spaces, *J. Lond. Math. Soc.* **25** (1950), 54–57.

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