DOCUMENTS DE TREBALL

DE LA FACULTAT D'ECONOMIA I EMPRESA

Col.lecció d'Economia

E09/216

Symmetrically multilateral-bargained allocations in multi-sided assignment markets

Oriol Tejada

Carles Rafels

Adreça correspondència:
Departament of Actuarial, Financial and Economic Mathematics
Facultat d'Economia i Empresa
Universitat de Barcelona
Av. Diagonal 690
08034 Barcelona
Email: oriol.tejada@ub.edu; crafels@ub.edu

Acknowledgements: Support from grant Programa FPU of Ministerio de Educación y Ciencia of the Spanish Government and from the Barcelona Economics Program of CREA, Ministerio de Educación y Ciencia and FEDER, under grant ECO2008-02344/ECON is acknowledged.

Symmetrically multilateral-bargained allocations in multi-sided assignment markets

Abstract: We extend Rochford's (1983) notion of symmetrically pairwise-bargained equilibrium to assignment games with more than two sides. A symmetrically multilateral-bargained (SMB) allocation is a core allocation such that any agent is in equilibrium with respect to a negotiation process among all agents based on what every agent could receive -and use as a threat- in her preferred alternative matching to the optimal matching that is formed. We prove that, for balanced multi-sided assignment games, the set of SMB is always nonempty and that, unlike the two-sided case, it does not coincide in general with the kernel (Davis and Maschler, 1965). We also give an answer to an open question formulated by Rochford (1983) by introducing a kernel-based set that, together with the core, characterizes the set of SMB.

Keywords: Cooperative games, core, kernel, bargaining

JEL Classification: C71, C78

Resum: Aquest treball tracta d'extendre la noció d'equilibri simètric de negociació bilateral introduït per Rochford (1983) a jocs d'assignació multilateral. Un pagament corresponent a un equilibri simètric de negociació multilateral (SMB) és una imputació del core que garanteix que qualsevol agent es troba en equilibri respecte a un procés de negociació entre tots els agents basat en allò que cadascun d'ells podria rebre -i fer servir com a amenaça- en un 'matching' òptim diferent al que s'ha format. Es prova que, en el cas de jocs d'assignació multilaterals, el conjunt de SMB és sempre no buit i que, a diferència del cas bilateral, no sempre coincideix amb el kernel (Davis and Maschler, 1965). Finalment, responem una pregunta oberta per Rochford (1982) tot introduïnt un conjunt basat en la idea de kernel, que, conjuntament amb el core, ens permet caracteritzar el conjunt de SMB.

1 Introduction

In a m-sided assignment game (Quint, 1991) there are m different types of agents and a single worth is attached to any coalition of exactly one agent of each type -what will be called an *essential coalition*-. The worth of any arbitrary coalition is obtained by partitioning it into essential coalitions and singletons (that have zero worth), adding up their worth and, finally, keeping the maximum of such values. Multilateral assignment markets represent many situations in which multiple partnerships are formed and can be modeled by m-sided assignment games (see Tejada and Rafels, 2009, for some examples and a further justification on that link).

In their seminal paper about two-sided assignment games, Shapley and Shubik (1972) prove that two-sided sided assignment games have always a nonempty core, whereas Kaneko and Wooders (1982) -in a more general framework- and Quint (1991) show that this property does not hold for arbitrary m-sided assignment games.

Tejada and Rafels (2009) show that, for multi-sided assignment games, there is a bijection between competitive prices and core allocations. Therefore, the core represents the very first main approximation to resolve the problem of predicting which prices will emerge in a multi-sided assignment market. Nevertheless, Shapley and Shubik (1972) already noticed that one of the main drawbacks of the core as a prediction of the outcome of a cooperative game, in particular of a multi-sided assignment game, is that it is based only on what agents can do, not on what they can prevent other agents to do.

In this paper we show that, if agents do not base their behavioral strategy only on what they can do but also on what they can prevent others to do, the core of a balanced multi-sided assignment game can be non-trivially refined. This is made introducing a set of multilateral-bargained (SMB) allocations, that generalizes the set of pairwise-bargained (SPB) allocations introduced by Rochford (1983). In particular, for two-sided assignment games this refinement coincides with the intersection between the kernel and the core, as

Rochford (1983) demonstrates, which is, in fact, the kernel itself (Driessen, 1998 and Granot, 1995)[1]. We show that this coincidence fails to hold for arbitrary multi-sided assignment games. Nevertheless, we show that if the kernel is slightly modified, the set of SMB can be characterized by this new kernel-based set. For two-sided assignment games, this last set is precisely the kernel.

In fact, Rochford already noticed that, for more than two sides, the kernel might not be the most appropriate solution set. In her words (Rochford, 1983, p. 278), "In a game where the resulting coalition structure contains coalitions with more than two members, it is not clear why a solution based on pairwise equilibrium is appropriate".

The rest of the paper is organized as follows. In Section 2 we introduce the notation and background results needed. In Section 3 we define symmetrically multilateral-bargained SMB allocations. Finally, in Section 4 we present the main results of the paper.

2 Notation and Preliminaries

Consider a market in which there are m different finite sets (or types) of agents $N^1, ..., N^m$ such that $n_1 = |N^1|, ..., n_m = |N^m|$. With some abuse of notation[2], for all j, $1 \leq j \leq m$ we denote $N^j = \{1, 2, ..., n_j\}$. By convention (see Quint, 1991), we refer to the i-th agent of type j as j-i. We call any m-tuple of agents $E = (i_1, ..., i_m) \in N^1 \times ... \times N^m$ an *essential* coalition. As an abuse of notation, and when no confusion is possible, we will also use E to refer to the set $\{1\text{-}i_1, ..., m\text{-}i_m\}$.

A *m-sided assignment problem* (m-SAP), that will be denoted by $(N^1, ..., N^m; A)$, is characterized by a nonnegative m-dimensional matrix $A = (a_{i_1 \ldots i_m})_{i_1 \in N^1, \ldots, i_m \in N^m}$. An arbitrary entry of A , namely $a_{i_1 \ldots i_m}$, corresponds to the worth that is attached to any essential coalition

[1] In light of Driessen (1998) and Granot and Granot (1995), what Rochford (1983) actually did, therefore, was giving a different characterization of the kernel of a two-sided assignment game in terms of a bargaining process based on threats.

[2] Formally, for any j, $1 \leq j \leq m$, N^j should be denoted by $\{j\} \times \{1, .., n_j\}$. Then, any agent is a pair (j, i_j), j denoting the type of the agent and i_j denoting the index within the set of agents of her same type.

$E = (i_1, ..., i_m)$. Observe that A can be cast to a mapping from $N^1 \times ... \times N^m$ to $\mathbb{R}_+$ defined by $A(i_1, ..., i_m) = a_{i_1 \dots i_m}$.

If $n_1 = n_2 = ... = n_m = n$ we say that $(N^1, ..., N^m; A)$ is square. A dummy agent is such that the worth of any essential coalition 'containing' her is always zero. Dummy agents do not alter the characteristic function when we restrict to real players (see, for instance, Quint (1991) for a complete argument). Therefore, by adding dummy agents, from now on we will assume, without loss of generality, that $(N^1, ..., N^m; A)$ is square. In fact, in addition to $(N^1, ..., N^m; A)$ being square, another dummy agent will always be artificially introduced for each type. The reason to do that will be apparent below in Section 3.

A *matching* $\mu = \{E^1, ..., E^t\}$ among $N^1, ..., N^m$ is a set of essential coalitions such that $|\mu| = t = \min_{1 \le j \le m} |N^j|$ and any agent j-i belongs at most to one essential coalition $E^1, ..., E^t$. We say that agent j-i is *unassigned* by μ if she does not belong to E^k for any $1 \le k \le t$. We denote by $\mathcal{M}(N^1, ..., N^m)$ the set of all matchings among $N^1, ..., N^m$. A matching is optimal if it maximizes $\sum_{(i_1,...,i_m)\in\mu} A(i_1, ..., i_m)$ in $\mathcal{M}(N^1, ..., N^m)$. We denote by $\mathcal{M}^*_A(N^1, ..., N^m)$ the set of all optimal matchings of $(N^1, ..., N^m; A)$.

Following Shapley and Shubik (1972) and Quint (1991), for each multi-sided assignment problem $(N^1, ..., N^m; A)$ the *m-sided assignment game* (m-SAG) is the cooperative game[3] (N, ω_A) with set of players $N = \{j\text{-}i : 1 \le j \le m, 1 \le i \le n_j\}$ and characteristic function ω_A defined by

$$\omega_A(S) = \max_{\mu \in \mathcal{M}(N^1 \cap S, ..., N^m \cap S)} \left\{ \sum_{(i_1,...,i_m)\in\mu} A(i_1, ..., i_m) \right\},$$

where the summation over the empty set is zero. Notice that if $m = 2$ the setting reduces to the classic Shapley-Shubik assignment market.

The core of a game is the set of imputations that cannot be improved upon by any coalition on its own. Quint (1991) shows that, given a square multi-sided assignment game,

[3]A cooperative game is a pair (N, v), where N is the set of players and v, the *characteristic function*, associates a numerical value $v(S) \in \mathbb{R}$ to any coalition $S \subseteq N$, being $v(\varnothing) = 0$.

its core, $C(\omega_A)$, is the set of nonnegative vectors x satisfying

$$A(i_1, ..., i_m) - \sum_{j=1}^{m} x_{ji_j} \leq 0, \tag{1}$$

for any $(i_1, ..., i_m) \in N^1 \times ... \times N^m$, where (1) must be tight if $(i_1, ..., i_m)$ belongs to some optimal matching.

On the other hand, for any imputation[4] x of (N, ω_A) and any coalition $S \subseteq N$, the *excess of coalition S with respect to x* is defined by $e(S, x) = \omega_A(S) - x(S)$, where $x(S) = \sum_{i \in S} x_i$. Given (N, ω_A) an arbitrary m-SAG, we introduce the following set, that is defined for any pair of agents j-i and any k-l,

$$\Gamma_{j\text{-}i,k\text{-}l} = \{S \subseteq N : j\text{-}i \in S,\ k\text{-}l \notin S\}, \tag{2}$$

Them, for any $x \in I(\omega_A)$, the *surplus of agent j-i against agent k-l at x* is

$$s_{j\text{-}i,k\text{-}l}(x) = \max\{e(S, x) : S \in \Gamma_{j\text{-}i,k\text{-}l}\}. \tag{3}$$

The above expression can be interpreted as the maximum that agent j-i can expect to obtain (if negative, the least she can expect to lose) if she departs from x without agent k-l's collaboration and assuming that other agents are happy with what they recieve in x. Notice that $s_{j\text{-}i,k\text{-}l}(x)$ is a bilateral concept, in the sense that only two agents, j-i and k-l, are involved in the definition. Then, an idea of bilateral equilibrium is formulated in the following property:

(P) $s_{j\text{-}i,k\text{-}l}(x) = s_{k\text{-}l,j\text{-}i}(x)$ for all $1 \leq j, k \leq m$, $1 \leq i, l \leq n$ such that j-$i \neq k$-l.

Observe that (P) implies that each pair of agents are in equilibrium through the bargaining

[4] Given a cooperative game (N, v), the set of *imputations* is defined by $I(v) = \{x \in \mathbb{R}^n : x_i \geq v(\{i\}),\ \sum_{i=1}^{n} x_i = v(N)\}$.

process defined in (3). The kernel of (N, ω_A) is the set[5]

$$K(\omega_A) = \{x \in I(\omega_A) : x \text{ satisfies } (\text{P})\}.$$

3 Main definitions

Let (N, ω_A) be a balanced m-SAG and let (μ, x) be the 'current' stable outcome. That is, agents are matched under an optimal matching μ and are paid the payoff $x \in C(\omega_A)$. Then suppose that some agent wonders about what would happen if she broke the essential coalition she belongs to in μ, formed a new essential coalition, payed to her hypothetical new partners the worth they are receiving in the current payoff x, whereas she would appropriate the rest of the worth of the new formed coalition. Since x is a core allocation, such worth will never be larger than what she is currently receiving in x. In fact, the more such worth is below than what she is receiving in x, the less credible a threaten she might pose on other agents not to deviate from (μ, x) is. This is true in the sense that any change in the outcome would drive her lose part of her current payoff, whereas other agents would benefit. As any agent in the market can think in the same way, one would expect that, given a proposed allocation x, after a negotiation process among all agents, they should be paid exactly what they can threaten. However, such payoffs might not be efficient, in the sense that the market is always able to provide at least the whole resources to pay all agents their threats, but many times can pay more. In such case, the most straightforward idea to share the remaining benefits is to distribute them symmetrically.

To formalize all the above ideas we introduce the following set, that is defined for any

[5]The kernel is introduced by Davis and Maschler (1965). However, to be accurate, the definition provided in our paper is that of the pre-kernel (Maschler, Peleg and Shapley, 1972, 1979). Nevertheless, Maschler and Peleg (1967) prove that for zero-normalized and monotonic games the kernel coincides with the pre-kernel, which is always nonempty. A game (N, v) is *zero-normalized* if $v(\{i\}) = 0$ for all $i \in N$ and is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. Finally, it is straightforward to check that multi-sided assignment games are zero-normalized and monotonic.

agent j-i and any essential coalition E,

$$\Gamma^{\xi}_{j\text{-}i,E} = \left\{ \widetilde{E} \in N^1 \times ... \times N^m : j\text{-}i \in \widetilde{E},\ E \neq \widetilde{E} \right\}, \tag{4}$$

The concept of threat, that generalizes Rochford's (1983) definition to m-SAGs with more than two sides, is introduced in the following definition.

Definition 1 *Given a balanced m-SAG* (N, ω_A)*,* $\mu \in \mathcal{M}^*_A(N^1, ..., N^m)$ *and* $x \in C(\omega_A)$*, the vector of threats* $t(x) \in \mathbb{R}^{nm}$ *is defined by*

$$t_{ji}(x) = \max_{(k_1,...,k_{j-1},i,k_{j+1},...,k_m) \in \Gamma^{\xi}_{j\text{-}i,E}} \left\{ A(k_1, ..., k_{j-1}, i, k_{j+1}, ..., k_m) - \sum_{s=1, s \neq j}^{m} x_{sk_s} \right\},$$

for all $1 \leq j \leq m$ *and* $1 \leq i \leq n$*, where* j-$i \in E$ *and* $E \in \mu$*.*

Some remarks must be made on the above definition. First observe that $x_{ji} - t_{ji}(x)$ is the minimum loss that agent j-i would incur if, for some reason, the essential coalition she belongs to in some stable outcome broke. On the other hand, since $x \in C(\omega_A)$ and we have assumed that for any type there is at least one dummy agent[6], we have that $0 \leq t_{ji}(x) \leq x_{ji}$. Lastly, and most important, it can be checked that the vector of threats does not depend on the optimal matching μ chosen[7].

The idea of a symmetrically sharing of the remainders of a negotiation process based on the above threats is formally introduced in the following definition and generalizes that from Rochford (1983) to m-SAGs with more than two sides.

Definition 2 *Given a balanced m-SAG* (N, ω_A)*,* $\mu \in \mathcal{M}^*_A(N^1, ..., N^m)$ *and* $x \in C(\omega_A)$*, the vector of symmetrically bargained incomes* $b(x) \in \mathbb{R}^{nm}_{+}$ *is defined by*

$$b_{ji}(x) = t_{ji}(x) + \frac{1}{m} \left(A(i_1, ..., i_m) - t_{ji}(x) - \sum_{s=1, s \neq j}^{m} t_{si_s}(x) \right)$$

[6] I.e. there really are $n+1$ agents of each type, since for all $1 \leq j \leq m$, j_{n+1} is a dummy agent. However, we will only care about those with index lower than $n+1$. This can be made without loss of generality and it is an assumption that Rochford (1983) also makes.

[7] Let $\mu^1 \neq \mu^2$ be two optimal matchings. The non-trivial case occurs when agent j-i is matched in μ^1 and μ^2 to different partners, i.e., when j-$i \in E \in \mu^1$, j-$i \in E' \in \mu^2$ and $E \neq E'$. In such case $E' \in \Gamma^{\mu^1}_{j\text{-}i,E}$ and $E \in \Gamma^{\mu^2}_{j\text{-}i,E'}$. Lastly, since $x \in C(\omega_A)$, $t^{\mu^1}_{ji}(x) = t^{\mu^2}_{ji}(x) = x_{ji}$.

for all $1 \leq j \leq m$ *and* $1 \leq i \leq n$*, where* $(i_1, ..., i_{j-1}, i, i_{j+1}, ..., i_m) \in \mu$*.*

Again, it is important to point out that $b(x)$ does not depend on the specific optimal matching chosen. Indeed, it is not difficult to show that the set of all agents can be partitioned in two sets, both of them having the same number of agents of all types: those that belong to the same essential coalition in any optimal matching and those that do not. If agent j-i belongs to the first subset, we trivially have that $b_{ji}^{\mu_1}(x) = b_{ji}^{\mu_2}(x)$ for any $\mu_1, \mu_2 \in \mathcal{M}_A^*(N^1, ..., N^m)$. On the other hand, threats of agents of the second set coincide with their payoff in x, which, after some calculations, implies that $b_{ji}^{\mu}(x) = t_{ji}(x)$ for any agent j-i of the second set. In conclusion, $b(x)$ does not depend on the specific optimal assignment chosen. Additionally observe that, given $x \in C(\omega_A)$, the vector of symmetrically bargained incomes $b(x)$ pays to each agent j-i his threat plus a 'symmetric' part of the (negative or positive) difference between the worth of any essential coalition that contains her and belongs to some optimal matching μ and the threats of agent j-i herself and her partners in μ. In particular, given $x \in C(\omega_A)$ a new vector $b(x)$ is obtained after a process of bargaining. As usual, we focus our attention on those allocations that are invariant through the above bargaining procedure.

Definition 3 *Given a balanced m-SAG* (N, ω_A)*, the set of symmetrically multilateral-bargained (SMB) allocations is*

$$SMB(\omega_A) = \{x \in C(\omega_A) : x = b(x)\}.$$

Notice that $SMB(\omega_A)$ is only defined for balanced m-SAGs and is a subset (or a refinement) of the core. In the next section we will prove that $SMB(\omega_A)$ is always nonempty for balanced m-SAGs (see Theorem 7) and that, unlike the two-sided case, it does not coincide with the kernel (see Example 8). Furthermore, we will prove that, for more than two sides, the kernel can be slightly modified so that a characterization of $SMB(\omega_A)$ in terms of this new kernel-based set is still possible (see Theorem 6), hence extending Rochford's (1983) result and proving that her intuition about the non-appropriateness of the kernel in m-SAGs, in which the 'essential' coalitions consist on more than two agents, was right.

In addition to all the above, recall that an allocation belongs to the kernel of a m-sided assignment, $K(\omega_A)$, if any pair of agents in the market, namely j-i and k-l, are in equilibrium through a bilateral bargaining, that is $s_{j\text{-}i,k\text{-}l}(x) = s_{k\text{-}l,j\text{-}i}(x)$. However, as we have already said, in a m-SAG, it is m agents (one of each side) that are needed to make a deal. Hence, it seems natural to look for multilateral equilibria instead of bilateral equilibria. To do so we extend the surplus functions given in (3) to include surpluses of any agent either against essential coalitions containing her or singletons. Formally, given (N, ω_A) an arbitrary m-SAG, we introduce the following set, that is defined for any agent j-i and any B being either an essential coalition containing j-i or a singleton,

$$\Gamma_{j\text{-}i,B} = \{S \subseteq N : j\text{-}i \in S,\ B \nsubseteq S\}\,. \tag{5}$$

Observe that, when B is an essential coalition, let us say E, then $\Gamma_{j\text{-}i,E} \cap \left\{ \widetilde{E} : \widetilde{E} \in N^1 \times ... \times N^m \right\}$ is precisely the set introduced in (4). On the other hand, when B is a singleton, the set defined in (5) is exactly the set defined in (2). For any $x \in I(\omega_A)$, the *surplus of agent j-i against B at x* is

$$s_{j\text{-}i,B}(x) = \max\{e(S,x) : S \in \Gamma_{j\text{-}i,B}\}\,, \tag{6}$$

where, again, B is either an essential coalition containing j-i or a singleton. Notice that, for $m = 2$, (6) coincides with (3). This having been defined, we introduce a kernel-based set that requires that, after a bargaining process among all agents, none of them should be 'discriminated' against agents of her same type (P1) whereas agents of any possible essential coalition should make a 'fair' deal among them (P2).

Definition 4 *Given an arbitrary m-SAG (N, ω_A), the multi-sided kernel (in short, m-Kernel) is the set*

$$K_m(\omega_A) = \{x \in I(\omega_A) : x \ \textit{satisfies (P1) and (P2)}\}\,,$$

where

(P1) $s_{j\text{-}i,j\text{-}l}(x) = s_{j\text{-}l,j\text{-}i}(x)$ for all $1 \le j \le m, 1 \le i \ne l \le n$.

(P2) $s_{1\text{-}i_1,E}(x) = ... = s_{m\text{-}i_m,E}(x)$ for all $E = (i_1, ..., i_m) \in N^1 \times ... \times N^m$.

Notice that, in the above definition, we do not require (N, ω_A) to be balanced. Moreover, in the next section we will prove that $K_m(\omega_A)$ is always nonempty, regardless (N, ω_A) is balanced or not (see Theorem 5). Lastly, the reader can check that, given a balanced m-SAG (N, ω_A), if $x \in C(\omega_A)$ then $s_{j\text{-}i,k\text{-}l}(x) = 0$ for any agents j-i and k-l that are not assigned together under some optimal matching. This implies the following remark.

Remark 1 *Given a balanced m-SAG* (N, ω_A) *and* $x \in C(\omega_A)$*, then* $x \in K_m(\omega_A)$ *if and only if* x *satisfies* (P2).

4 Main results

This section is primarily devoted to prove that, for any balanced m-SAG (N, ω_A), the set $SMB(\omega_A)$ is nonempty and can be characterized by the multi-sided kernel. First we show that the kernel is always included in the multi-sided kernel.

Theorem 5 *For any arbitrary m-SAG* (N, ω_A)*, the kernel of* (N, ω_A) *is always included in the multi-sided kernel of* (N, ω_A)*, i.e.* $K(\omega_A) \subseteq K_m(\omega_A)$.

Proof. Take $x \in K(\omega_A)$ and assume, without loss of generality, that $E = (1, ..., 1)$ and that $s_{1\text{-}1,E}(x) \ge ... \ge s_{m\text{-}1,E}(x)$. On one hand, suppose that there is $S^* \in \Gamma_{1\text{-}1.E}$ such that m-1 $\in S^*$ and $s_{1\text{-}1,E}(x) = e(S^*, x)$. By definition of $\Gamma_{1\text{-}1.E}$, there is j, $2 \le j \le m-1$ such that j-1 $\notin S^*$. Hence, $s_{1\text{-}1,E}(x) \ge s_{m\text{-}1,E}(x) \ge s_{m\text{-}1,j\text{-}1}(x) \ge e(S^*, x) = s_{1\text{-}1,E}(x)$, which implies that all inequalities must be tight. On the other hand, suppose that, for all $S \in \Gamma_{1\text{-}1,E}$ such that m-1 $\in S$, we have that $s_{1\text{-}1,E}(x) > e(S, x)$. Then, $s_{1\text{-}1,m\text{-}1}(x) = s_{1\text{-}1,E}(x)$ and, since $x \in K(\omega_A)$, we obtain that $s_{1\text{-}1,E}(x) \ge s_{m\text{-}1,E}(x) \ge s_{m\text{-}1,1\text{-}1}(x) = s_{1\text{-}1,m\text{-}1}(x) = s_{1\text{-}1,E}(x)$.

Again, all inequalities must be tight. In conclusion, $s_{1\text{-}1,E}(x) = ... = s_{m\text{-}1,E}(x)$ and, since $E = (1, ..., 1)$ was picked without loss of generality, $K(\omega_A) \subseteq K_m(\omega_A)$. ■

Example 8 at the end of the paper shows that the inclusion of $K(\omega_A)$ into $K_m(\omega_A)$ is sometimes strict.

The main theorem of the paper is devoted to characterizing the set of symmetrically multilateral-bargained allocations of a balanced multi-sided assignment game in terms of the core and the multi-sided kernel, hence generalizing Rochford's (1983) main result.

Theorem 6 *Given a balanced m-SAG* (N, ω_A), $SMB(\omega_A) = C(\omega_A) \cap K_m(\omega_A)$.

Proof. Before proving the two inclusions we show that given $\mu \in \mathcal{M}_A^*(N^1, ..., N^m)$, $E \in \mu$ and $j\text{-}i \in E$ then, for any $x \in C(\omega_A)$, we have that

$$s_{j\text{-}i,E}(x) = -x_{ji} + t_{ji}(x). \tag{7}$$

Throughout the whole proof we will assume, without loss of generality, that $\mu = \{(i, ..., i) : 1 \leq i \leq n\}$ is an optimal matching of $(N^1, ..., N^m; A)$. In particular, given $j\text{-}i$ an arbitrary agent, $E = (i, ..., i)$ is the essential coalition of μ that contains $j\text{-}i$. By Definition 1 and (6), we have that, for any agent $j\text{-}i$,

$$\begin{aligned} s_{j\text{-}i,E}(x) &= \max\{e(S,x) : S \in \Gamma_{j\text{-}i,E}\} \\ &\geq \max\left\{e(S,x) : S \in \Gamma^{\xi}_{j\text{-}i,E}\right\} = -x_{ji} + t_{ji}(x). \end{aligned} \tag{8}$$

We will prove (7) by the counterreciprocal. Indeed, suppose that the above inequality is strict, i.e., that there is $S \in \Gamma_{j\text{-}i,E}$ such that .

$$e(S,x) > -x_{ji} + t_{ji}(x) \tag{9}$$

Since $S \in \Gamma_{j\text{-}i,E}$ and $E = (i, ..., i)$, there must be $k \in \{1, ..., n\}$ such that $k\text{-}i \notin S$. Let $\widetilde{\mu} \in \mathcal{M}_A^*(N^1 \cap S, ..., N^m \cap S)$. We distinguish two cases.

- *Case 1:* $\left|S \cap N^j\right| \leq \min_{k\in\{1,..,j-1,j+1,...,m\}} \left|S \cap N^k\right|$.

In this case, j-i must be assigned under $\widetilde{\mu}$. We denote the essential coalition that contains j-i by $\left(\widetilde{\mu}_1^{-1}(i), ..., \widetilde{\mu}_m^{-1}(i)\right)$, where $\widetilde{\mu}_j^{-1}(i) = i$ and, for $k \in \{1, ..., j-1, j+1, ..., m\}$, $\widetilde{\mu}_k^{-1}(i)$ is the index of the agent of type k that is assigned together with j-i under $\widetilde{\mu}$. Observe that, since k-$i \notin S$, we have that $\widetilde{\mu}_k^{-1}(i) \neq i$. Then,

$$\begin{aligned}
e(S, x) &= \sum_{(i_1,...,i_m)\in\widetilde{\mu}} A(i_1, ..., i_m) - \sum_{h\text{-}r\in S} x_{hr} \\
&= A(\widetilde{\mu}_1^{-1}(i), ..., \widetilde{\mu}_m^{-1}(i)) - x_{ji} - \sum_{h=1,h\neq i}^{m} x_{h\widetilde{\mu}_h^{-1}(i)} \\
&\quad + \sum_{\substack{(i_1,...,i_m)\in\widetilde{\mu} \\ i_j\neq i}} \left(A(i_1, ..., i_m) - \sum_{h=1}^{m} x_{hi_h} \right) \\
&\quad - x\left(S \backslash \left(\cup_{(i_1,...,i_m)\in\widetilde{\mu}} \{1\text{-}i_1, ..., m\text{-}i_m\}\right)\right) \\
&\leq -x_{ji} + A(\widetilde{\mu}_1^{-1}(i), ..., \widetilde{\mu}_m^{-1}(i)) - \sum_{\substack{h=1 \\ h\neq i}}^{m} x_{h\widetilde{\mu}_h^{-1}(i)} \\
&\leq -x_{ji} + t_{ji}(x),
\end{aligned}$$

where the first inequality holds since $x \in C(\omega_A)$ and the second inequality holds by Definition 1, since $\left(\widetilde{\mu}_1^{-1}(i), ..., \widetilde{\mu}_m^{-1}(i)\right) \in \Gamma^{\xi}_{j\text{-}i,E}$ because $\widetilde{\mu}_k^{-1}(i) \neq i$.

- *Case 2:* $\left|S \cap N^j\right| > \min_{k\in\{1,..,j-1,j+1,...,m\}} \left|S \cap N^k\right|$.

If j-i is assigned under $\widetilde{\mu}$, it is left to reader to prove that an analogous argument as the above demonstrates that $e(S, x) \leq -x_{ji} + t_{ji}(x)$. Thus, suppose that j-i is not assigned under $\widetilde{\mu}$ or that there is no such $\widetilde{\mu}$. In such case we have

$$\begin{aligned}
e(S, x) &= \sum_{(i_1,...,i_m)\in\widetilde{\mu}} A(i_1, ..., i_m) - \sum_{h_r\in S} x_{hr} \\
&= \sum_{(i_1,...,i_m)\in\widetilde{\mu}} \left(A(i_1, ..., i_m) - \sum_{h=1}^{m} x_{hi_h} \right) \\
&\quad - x\left(S \backslash \left(\{j\text{-}i\} \cup_{(i_1,...,i_m)\in\widetilde{\mu}} \{1\text{-}i_1, ..., m\text{-}i_m\}\right)\right) - x_{ji} \\
&\leq -x_{ji} \leq -x_{ji} + t_{ji}(x),
\end{aligned}$$

where the first inequality holds again because $x \in C(\omega_A)$ and the last inequality holds since $t_{ji}(x) \geq 0$.

In conclusion, we have proved that $e(S,x) \leq -x_{ji}+t_{ji}(x)$, which contradicts the assumption made in (9). Hence, inequality in (8) must be tight, which implies that (7) must hold.

Having (7) been proved, we are in a position to prove the theorem. First we prove that $SMB(\omega_A) \supseteq C(\omega_A) \cap K_m(\omega_A)$. Indeed, take $x \in C(\omega_A) \cap K_m(\omega_A)$. On one hand, since $x \in C(\omega_A)$, we have

$$\sum_{k=1}^{m} x_{ki} = A(i,...,i). \tag{10}$$

On the other hand, since $x \in K_m(\omega_A)$, we also have that $s_{j\text{-}i,E}(x) = s_{k\text{-}i,E}(x)$ for all $1 \leq k \neq j \leq m$, where $E = (i,...,i)$. Thus, applying (7) we obtain that

$$x_{ki} = x_{ji} - t_{ji}(x) + t_{ki}(x) \tag{11}$$

for all $1 \leq k \neq j \leq m, 1 \leq i \leq n$. Lastly, combining (10) and (11) we get

$$\begin{aligned} x_{ji} &= A(i,...,i) - \sum_{k=1,k\neq j}^{m} x_{ki} \\ &= A(i,...,i) - \sum_{k=1,k\neq j}^{m} (x_{ji} - t_{ji}(x) + t_{ki}(x)) \\ &= A(i,...,i) - (m-1)x_{ji} + mt_{ji}(x) - \sum_{k=1}^{m} t_{ki}(x), \end{aligned}$$

for $1 \leq j \leq m$ and $1 \leq i \leq n$, which is equivalent to

$$x_{ji} = t_{ji}(x) + \frac{1}{m}\left(A(i,...,i) - \sum_{k=1}^{m} t_{ki}(x)\right) = b_{ji}(x).$$

In conclusion, $x \in SMB(\omega_A)$.

Second, we prove that $SMB(\omega_A) \subseteq C(\omega_A) \cap K_m(\omega_A)$. Since, by definition, $SMB(\omega_A)$ is always included in $C(\omega_A)$, we only have to check that $SMB(\omega_A) \subseteq K_m(\omega_A)$. Indeed, let $x \in SMB(\omega_A)$. Then, for all $1 \leq j \leq m$ and $1 \leq i \leq n$, we have that

$$x_{ji} = b_{ji}(x) = t_{ji}(x) + \frac{1}{m}\left(A(i,...,i) - \sum_{s=1}^{m} t_{si}(x)\right).$$

Applying (7) to $s_{j\text{-}i,E}(x)$, where $E = (i,...,i)$, and using the above equality we obtain

$$s_{j\text{-}i,E}(x) = -x_{ji} + t_{ji}(x) = \frac{1}{m}\left(\sum_{s=1}^{m} t_{si}(x) - A(i,...,i)\right),$$

that does not depend on j. Hence, $s_{1\text{-}i,E}(x) = ... = s_{m\text{-}i,E}(x)$ for any $1 \leq i \leq n$. Since we have chosen $E = (i, ..., i) \in \mu$ without loss of generality, we have proved that (P2) holds for any essential coalition belonging to some optimal matching. It remains to show that this implies that (P2) also holds for any coalition that does not belong to any optimal matching, in particular μ. Let $\widetilde{E} = (i_1, ..., i_m)$ be one of such coalitions and take $j \in \{2, ..., m\}$. On one hand, if $i_j = i_1$, since $x \in C(\omega_A)$, by (3) we have that $s_{1\text{-}i_1,j\text{-}i_1}(x) \leq 0$. On the other hand, if $i_j \neq i_1$, we have that $s_{1\text{-}i_1,j\text{-}i_j}(x) = 0$ because 1-i_1 and j-i_j are not assigned together under μ. Hence, since $\widetilde{E} \notin \mu$ implies that there is at least one $j \in \{2, ..., m\}$ such that $i_j \neq i_1$, we obtain that

$$s_{1\text{-}i_1,\widetilde{E}}(x) = \max_{2 \leq j \leq m} \left\{ s_{1\text{-}i_1,j\text{-}i_j}(x) \right\} = 0.$$

Lastly, since above we have chosen $s_{1\text{-}i_1,\widetilde{E}}(x)$ without loss of generality, then $s_{1\text{-}i_1,\widetilde{E}}(x) = ... = s_{m\text{-}i_m,\widetilde{E}}(x) = 0$, that is, (P2) holds for any arbitrary essential coalition $\widetilde{E}$. In conclusion, together with Remark 1, we have proved that $x \in K_m(\omega_A)$. ■

A detailed revision of the above proof reveals that, in fact, given a balanced m-SAG (N, ω_A), a core allocation x belongs to $K_m(\omega_A)$ if and only if (P2) holds for all the essential coalitions of an arbitrary optimal matching.

Moreover, as a consequence of the above theorem we prove that the set of symmetrically multilateral-bargained allocations is always non-emtpy for any balanced m-SAG.

Theorem 7 *Let (N, ω_A) be any balanced m-SAG. Then, the set of symmetrically multilateral-bargained allocations, $SMB(\omega_A)$, is nonempty.*

Proof. It holds directly from Theorem 5, Theorem 6 and the fact that $C(\omega_A) \cap K(\omega_A)$ is nonempty (Schmeidler, 1969, proves it for arbitrary balanced games). ■

To illustrate the results contained in the paper, let us consider the following example.

Example 8 *Let $(N^1, N^2, N^3; A)$ be an 3-SAP with two agents of each type given by the* *following matrix, where the optimal matching is marked in bold:*

$$A = \left(\begin{array}{cc} \mathbf{3} & 0 \\ 0 & 0 \end{array} \right) \quad \left(\begin{array}{cc} 2 & 0 \\ 0 & \mathbf{0} \end{array} \right).$$

It can be checked that $K(\omega_A) = K(\omega_A) \cap C(\omega_A) = \{(\frac{5}{4}, 0; \frac{5}{4}, 0; \frac{1}{2}, 0)\}$, that $K_m(\omega_A) = K_m(\omega_A) \cap C(\omega_A) = \{(x_{11}, 0; x_{21}; 0; \frac{1}{2}, 0) : x_{11} + x_{21} = \frac{5}{2},\ x_{11} \geq \frac{1}{2},\ x_{21} \geq \frac{1}{2}\}$ and that $C(\omega_A) = \{(x_{11}, 0; x_{21}; 0; 3 - x_{11} - x_{21}, 0) : 2 \leq x_{11} + x_{21} \leq 3,\ x_{11} \geq 0,\ x_{21} \geq 0\}$. The projections of the three latter onto the space $\{(x_{11}, x_{21})\} \subset \mathbb{R}^2$ are drawn in Figure 1.

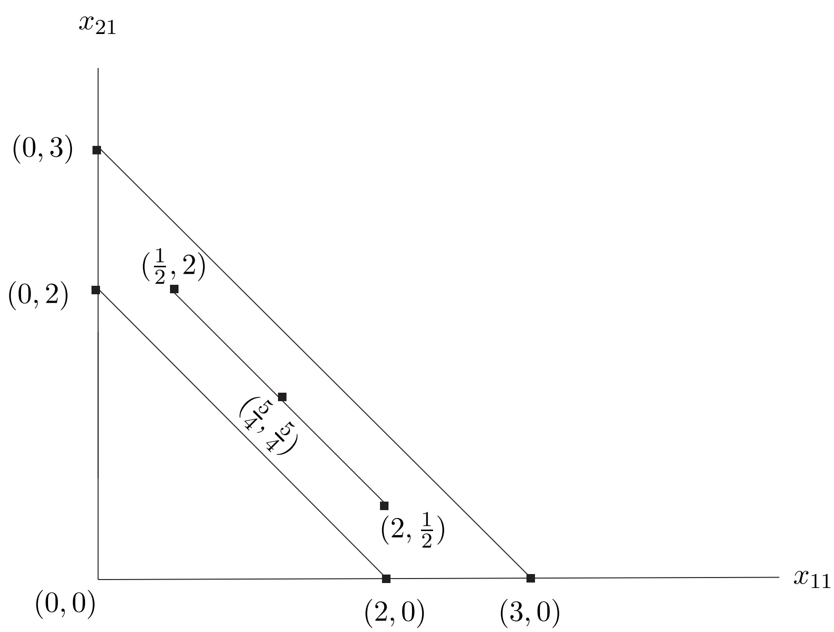

Figure 1: The core, the kernel and the m-kernel of the 3-SAG of the example.

That is, within $\{(x_{11}, x_{21})\} \subset \mathbb{R}^2_+$, $C(\omega_A)$ corresponds to the polygon delimited by $(0,2)$, $(2,0)$, $(3,0)$ and $(0,3)$, $K_m(\omega_A)$ corresponds to the segment between $(\frac{1}{2}, 2)$ and $(2, \frac{1}{2})$ and $K(\omega_A)$ corresponds to just $(\frac{5}{4}, \frac{5}{4})$. Observe that, in this example, $K(\omega_A) \subsetneq K_m(\omega_A) = SMB(\omega_A) \subsetneq C(\omega_A)$. It is still an open question to know whether, for any balanced multi-sided assignment game, the multi-sided kernel is always included in the core.

References

[1] Davis, M., Maschler, M. (1965) "The kernel of a cooperative game". Naval Research Logistics Quarterly, 12: 223-259.

[2] Driessen, T. (1998). "A note on the inclusion of the kernel in the core of the bilateral assignment game". International Journal of Game Theory 27, 301-303.

[3] Granot, D. (1995). "On a new bargaining set for cooperative games". Working Paper No. 95 MSC-005. Faculty of Commerce and Business Administration, The University of British Columbia, Vancouver, B.C., Canada.

[4] Kaneko, M. Wooders, M. (1982). "Cores of partitioning games". Mathematical Social Sciences. 3, 313-327.

[5] Maschler, M., Peleg, B. (1967). "The structure of the kernel of a cooperative game", SIAM Journal of Apllied Mathematics, 15: 569-604.

[6] Maschler, M., Peleg, B., Shapley, L. (1972). "The kernel and bragaining set for convex games". International Journal of Game Theory. 1, 73-93.

[7] Maschler, M., Peleg, B., Shapley, L. (1979). "Geometric properties of the kernel, nucleolus and related solution concepts". Math. Oer. Res. 4, 303-338.

[8] Quint, T. (1991). "The Core of an m-Sided Assignment Game". Games and Economic Behavior 3, 487-503.

[9] Rochford, S. (1983): "Symmetrically pairwise-bargained allocations in an assignment market". Journal of Economic Theory 34, 262–281.

[10] Schmeidler, D. (1969). "The nucleolus of a characteristic function game". SIAM Journal of Applied Mathematics 17,1163–1170.

[11] Shapley, L.S., Shubik, M. (1972). "The assignment game I: The Core". International Journal of Game Theory 1, 111-130.

[12] Tejada, O., Rafels, C. (2009). "Competitive prices of homogeneous goods in multilateral markets". GSE Research Network working paper n. 370.