THE DIFFERENTIABLE CHAIN FUNCTOR IS NOT HOMOTOPY EQUIVALENT TO THE CONTINUOUS CHAIN FUNCTOR

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Abstract. Let S_* and S_*^∞ be the functors of continuous and differentiable singular chains on the category of differentiable manifolds. We prove that the natural transformation $i : S_*^\infty \longrightarrow S_*$, which induces homology equivalences over each manifold, is not a natural homotopy equivalence.

Contents

1. Introduction

A basic result of Differential Topology, proved by S.Eilenberg ([E]), states that the singular homology of smooth manifolds can be calculated with differentiable singular chains: let M be a differentiable manifold, $S_*(M)$ its singular chain complex and $S_*^\infty(M)$ its singular differentiable chain complex, then Eilenberg proved that there exists a chain map

$$\theta_M : S_*(M) \longrightarrow S_*^\infty(M),$$

which is a homotopy inverse for the natural inclusion

$$i_M : S_*^\infty(M) \longrightarrow S_*(M).$$

Eilenberg's definition of θ_M depends on a triangulation on M, so it should be clear that it cannot be natural. There are other different proofs of this result (see, for example, [M], [W]), but the question remains if there is a natural homotopy inverse for i.

A classical technique in Algebraic Topology to prove that there is a homotopy equivalence between two functors is the acyclic models theorem. For example, one of the first applications of acyclic models was the proof that the functor S_* and the functor of (nondegenerated) cubical chains C_* are homotopy equivalent. M. Barr has proved a generalised acyclic models theorem, whose version for pointwise homotopy equivalences gives Eilenberg's theorem ([B1]). One may wonder whether the proof can be modified to give a natural homotopy equivalence between S_*

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and S_*^∞ (see [B2], p. ix). In this note we answer this question in negative form proving that the functors S_*, S_*^∞ are not homotopy equivalent.

2. The main result

We maintain the notation settled at the introduction.

Theorem 1. *The differentiable chain functor S_*^∞ is not homotopy equivalent to the continuous chain functor S_*. More specifically, there is no natural transformation of functors $\theta : S_* \longrightarrow S_*^\infty$ which induces isomorphisms in homology.*

Let's assume that there is a natural transformation $\theta : S \longrightarrow S_*^\infty$ inducing isomorphisms in homology. Identify the standard 1-simplex Δ^1 with the unit interval $[0,1]$ and let $\iota : \Delta^1 \longrightarrow \mathbb{R}$ be the inclusion map $\iota(t) = t$. Then ι is a singular chain of $\mathbb{R}$, $\iota \in S_1(\mathbb{R})$. Let

$$\theta_{\mathbb{R}}(\iota) = \sum_{j=0}^{n} \lambda_j \sigma_j \in S_1^\infty(\mathbb{R})$$

be its image by $\theta_{\mathbb{R}}$, where $\sigma_j : \Delta^1 \longrightarrow \mathbb{R}$ are differentiable simplexes, with $\sigma_i \neq \sigma_j$ if $i \neq j$.

Lemma. *At least one σ_j is a non-constant map.*

Proof of the lemma. Let $e : \mathbb{R} \longrightarrow \mathbb{S}^1$ denote the exponential map $e(t) = (cos(2\pi t),\ sin(2\pi t))$. By the naturality of θ we have a commutative diagram

$$\begin{array}{ccc} S_*(\mathbb{R}) & \xrightarrow{\theta_{\mathbb{R}}} & S_*^\infty(\mathbb{R}) \\ \downarrow e_* & & \downarrow e_* \\ S_*(\mathbb{S}^1) & \xrightarrow{\theta_{\mathbb{S}^1}} & S_*^\infty(\mathbb{S}^1) \end{array}$$

that is, $\theta_{\mathbb{S}^1}(e_*(\iota)) = e_*(\theta_{\mathbb{R}}(\iota))$. However, on one hand, $e_*(\iota) = e\iota$ is a generating cycle for the homology group $H_1(\mathbb{S}^1)$. On the other hand, if all σ_j were constant maps, $e_*(\theta_{\mathbb{R}}(\iota))$ would be a boundary. Therefore, $\theta_{\mathbb{S}^1} : S_*(\mathbb{S}^1) \longrightarrow S_*^\infty(\mathbb{S}^1)$, which is an isomorphism in homology, would send a generator of $H_1(\mathbb{S}^1)$ to zero.

So we may assume, for instance, that σ_0 is a non-constant map. Let $t_0 \in \Delta^1$ be such that $\sigma_0'(t_0) \neq 0$.

Now let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous bijective map satisfying the following conditions: $\alpha(u_0) = 0$, $\alpha_{|(-\infty,u_0]}$ and $\alpha_{|[u_0,\infty)}$ are $\mathcal{C}^\infty$ functions with different first derivative at u_0 and all other higher derivatives at u_0 equal to zero. To be more specific, we take

$$\alpha(x) = \begin{cases} 2(x-u_0), & \text{if } x \geq u_0, \\ x - u_0, & \text{if } x \leq u_0. \end{cases}$$

Take $\beta : \Delta^1 \longrightarrow \mathbb{R}$ to be the composition $\beta = \alpha\iota$. This is a singular simplex $\beta \in S_1(\mathbb{R})$. Put

$$\theta_{\mathbb{R}}(\beta) = \sum_{k=0}^{m} \mu_k \tau_k \in S_1^{\infty}(\mathbb{R}) \ ,$$

with $\tau_k : \Delta^1 \longrightarrow \mathbb{R}$ differentiable simplexes.

Consider a $\mathcal{C}^\infty$-function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is injective and such that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. For instance, we can take f to be

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -e^{-\frac{1}{x^2}}, & \text{if } x < 0. \end{cases}$$

The composition $f\alpha$ is a $\mathcal{C}^\infty$ function. This is clear at all points except, maybe, at $u_0 = \sigma_0(t_0)$. Let us show that this is indeed the case and also that all higher derivatives at u_0 are zero.

By induction, it suffices to prove that, for each $n > 0$, both lateral derivatives

$$(f\alpha)_+^{(n)}(u_0), \quad \text{and} \quad (f\alpha)_-^{(n)}(u_0),$$

exist and are zero. And this follows immediately from the following formula for the higher derivatives of the function $(f\alpha)_{|[u_0,\infty)}$ (respectively, $(f\alpha)_{|(-\infty,u_0]}$), a simplified version of Faà di Bruno's formula, that can easily be proved by induction:

$$(f\alpha)^{(n)}(x) = f^{(n)}(\alpha(x))\alpha'(x)^n + \sum_{i=1}^{n-1} f^{(i)}(\alpha(x)) P_{n,i}(\alpha'(x), \dots, \alpha^{(n)}(x)) \ ,$$

where $P_{n,i}$ are polynomials in the higher derivatives of α.

Hence, $f\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\mathcal{C}^\infty$ function. By the naturality of θ, we have

$$(f\alpha)_*(\theta_{\mathbb{R}}(\iota)) = \theta_{\mathbb{R}}((f\alpha)_*(\iota)) = \theta_{\mathbb{R}}(f_*(\alpha\iota)) = f_*(\theta_{\mathbb{R}}(\alpha\iota)) = f_*(\theta_{\mathbb{R}}(\beta)) \ .$$

Thus,

$$\lambda_0 f\alpha\sigma_0 + \sum_{j\neq 0} \lambda_j f\alpha\sigma_j = \sum_{k=0}^{m} \mu_k f\tau_k \ .$$

Now, $f\alpha\sigma_0 \neq f\alpha\sigma_j$, for every $j > 0$, because $f\alpha$ is an injective function, and $f\tau_i \neq f\tau_j$ if $i \neq j$, as f is also injective. So there exists some k such that $f\alpha\sigma_0 = f\tau_k$. We may assume $k = 0$. As f is injective, we may cancel it to obtain

$$\alpha\sigma_0 = \tau_0 \ .$$

But $\alpha\sigma_0$ is not a $\mathcal{C}^\infty$ function: if we compute the right and left derivatives at t_0, assuming for instance $\sigma_0'(t_0) > 0$, we obtain $2\sigma_0'(t_0)$ and $\sigma_0'(t_0)$, respectively, because $\alpha'_+(u_0) = 2$ and $\alpha'_-(u_0) = 1$. So we get a contradiction, since τ_0 is of class $\mathcal{C}^\infty$.

3. A generalization

In fact, Eilenberg's result is more general than that we have stated. What he proves is that all the inclusions

$$i_M : S_*^k(M) \longrightarrow S_*(M) \ ,$$

where $S_*^k(M)$ denotes the singular simplexes of class $\mathcal{C}^k$, $k = 1, 2, \dots, \infty$, are homotopy equivalences. We can also show that theirs (point-wise) homotopy inverses can not be natural transformations.

For $k = 0$ we take $S_*^0 = S_*$, and refer to the continuous singular chains as 0-differentiable chains.

Theorem 2. *The k-differentiable and l-differentiable chain functors, $l > k \geq 0$, are not homotopy equivalent. More specifically, there is no natural transformation of functors $\theta : S_*^k \longrightarrow S_*^l$, with $l > k$, which induces isomorphisms in homology.*

Proof. It is enough to see that there could not be such a natural transformation $\theta : S_*^k \longrightarrow S_*^l$ for the case $l = k + 1$. The proof goes in the same way as before, and all we have to do is replace our function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ with a bijective and everywhere differentiable function of class $\mathcal{C}^{k+1}$, except at $u_0 = \sigma_0(t_0)$, where it is of class $\mathcal{C}^k$, but not of class $\mathcal{C}^{k+1}$, $\alpha^{(i)}(u_0) = 0$, for all $i = 1, \dots, k$, and has different lateral derivatives $\alpha_+^{(k+1)}(u_0)$ and $\alpha_-^{(k+1)}(u_0)$. For instance, we can take α to be:

$$\alpha(x) = \begin{cases} 2(x - u_0)^{k+1}, & \text{if } x \geq u_0, \\ (-1)^k (x - u_0)^{k+1}, & \text{if } x \leq u_0. \end{cases}$$

With the same reasoning as before we come to

$$\alpha\sigma_0 = \tau_0,$$

where now $\sigma_0, \tau_0 \in S_1^{k+1}(\mathbb{R})$. Again, if $\sigma_0'(t_0) > 0$, we see that the left and right $(k+1)$-th derivatives of $\alpha\sigma_0$ at t_0 are

$$(-1)^k (k+1)! \sigma_0'(t_0)^{k+1} \qquad \text{and} \qquad 2(k+1)! \sigma_0'(t_0)^{k+1} \ ,$$

respectively. So, $\alpha\sigma_0$ has different $(k+1)$-derivatives from the right and from the left at t_0. Thus it is not of class $\mathcal{C}^{k+1}$, which contradicts the fact that it should be equal to τ_0, which is of class $\mathcal{C}^{k+1}$. □

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