A CARTAN-EILENBERG APPROACH TO HOMOTOPICAL ALGEBRA

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ABSTRACT. In this paper we propose an approach to homotopical algebra where the basic ingredient is a category with two classes of distinguished morphisms: strong and weak equivalences. These data determine the cofibrant objects by an extension property analogous to the classical lifting property of projective modules. We define a Cartan-Eilenberg category as a category with strong and weak equivalences such that there is an equivalence of categories between its localisation with respect to weak equivalences and the relative localisation of the subcategory of cofibrant objets with respect to strong equivalences. This equivalence of categories allows us to extend the classical theory of derived additive functors to this non additive setting. The main examples include Quillen model categories and categories of functors defined on a category endowed with a cotriple (comonad) and taking values on a category of complexes of an abelian category. In the latter case there are examples in which the class of strong equivalences is not determined by a homotopy relation. Among other applications of our theory, we establish a very general acyclic models theorem.

CONTENTS

In their pioneering work [CE], H. Cartan and S. Eilenberg defined the notion of derived functors of additive functors between categories of modules. Their approach is based on the characterisation of projective modules over a ring A in terms of the notions of homotopy between morphisms of complexes of A-modules and quasi-isomorphisms of complexes. Projective modules can be characterised from them: an A-module P is projective if for every solid diagram

$$
\begin{array}{ccc}
 & & Y \\
 & {\scriptstyle g}\nearrow & \downarrow {\scriptstyle w} \\
P & \xrightarrow{\ f\ } & X
\end{array}
$$

where w is a quasi-isomorphism of complexes, and f a chain map, there is a lifting g such that the resulting diagram is homotopy commutative, and the lifting g is unique up to homotopy.

A. Grothendieck, in his Tohoku paper [Gr], introduced abelian categories and extended Cartan-Eilenberg methods to derive additive functors between them. Later on, Grothendieck stressed the importance of complexes, rather than modules, and promoted the introduction of derived categories by J.L. Verdier.

In modern language the homotopy properties of projective complexes can be summarised in the following manner. If $\mathcal{A}$ is an abelian category with enough projective objects, then there is an equivalence of categories

$$\mathbf{K}_{+}(Proj(\mathcal{A})) \xrightarrow{\sim} \mathbf{D}_{+}(\mathcal{A}), \tag{0.1}$$

where $\mathbf{K}_{+}(Proj(\mathcal{A}))$ is the category of bounded below chain complexes of projective objects modulo homotopy, and $\mathbf{D}_{+}(\mathcal{A})$ is the corresponding derived category. Additive functors can therefore be derived as follows. If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor, it induces a functor $F' : \mathbf{K}_{+}(Proj(\mathcal{A})) \longrightarrow \mathbf{K}_{+}(\mathcal{B})$ and by the equivalence (0.1), we obtain the derived functor $\mathbb{L}F : \mathbf{D}_{+}(\mathcal{A}) \longrightarrow \mathbf{D}_{+}(\mathcal{B})$.

In order to derive non additive functors, D. Quillen, inspired by topological methods, introduced model categories in his notes on Homotopical Algebra [Q]. Since then, Homotopical Algebra has grown considerably as can be seen, for example, in [DHKS], [Ho], [Hi]. Quillen's approach applies to classical homotopy theory as well as to rational homotopy, Bousfield localisation, or more recently to simplicial sheaves or motivic homotopy theory.

In a Quillen model category $\mathcal{C}$, a homotopy relation for morphisms is defined from the axioms and one of the main results of [Q] is the equivalence

$$\pi\mathcal{C}_{cf} \xrightarrow{\sim} \mathcal{C}[\mathcal{W}^{-1}] \ , \tag{0.2}$$

where $\pi\mathcal{C}_{cf}$ is the homotopy category of the full subcategory $\mathcal{C}_{cf}$ of fibrant-cofibrant objects, and $\mathcal{C}[\mathcal{W}^{-1}]$ is the localised category with respect to weak equivalences. The equivalence (0.2) extends the one for projective complexes (0.1) and allows derivation of functors in this setting.

The set of axioms of model categories is, in some sense, somewhat strong because there are interesting categories in which to do homotopy theory that do not satisfy all of them. Several authors (see [Br], [Ba] and others) have developed simpler alternatives, all of them focused on laterality, asking only for a left- (or right-) handed version of Quillen's set of axioms. All these alternatives are very close to Quillen's formulation.

Here we propose another approach which is closer to the original development by Cartan-Eilenberg. The initial data are two classes of morphisms $\mathcal{S}$ and $\mathcal{W}$ in a category $\mathcal{C}$, with $\mathcal{S} \subset \mathcal{W}$, which we call strong and weak equivalences, respectively. We define an object M of $\mathcal{C}$ to be cofibrant if for every solid diagram

$$
\begin{array}{ccc}
 & & Y \\
 & {}^{g}\nearrow & \downarrow w \\
M & \xrightarrow{\ f\ } & X
\end{array} \ ,
$$

where w is a weak equivalence and $f : M \longrightarrow X$ is a morphism in $\mathcal{C}$, there is a unique lifting g in $\mathcal{C}[\mathcal{S}^{-1}]$ such that the diagram is commutative in $\mathcal{C}[\mathcal{S}^{-1}]$. We say that $\mathcal{C}$ is a Cartan-Eilenberg category if it has enough cofibrant objects, that is, if each object X in $\mathcal{C}$ is isomorphic in $\mathcal{C}[\mathcal{W}^{-1}]$ to a cofibrant object. In that case the functor

$$\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \xrightarrow{\sim} \mathcal{C}[\mathcal{W}^{-1}] \tag{0.3}$$

is an equivalence of categories, where $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ is the full subcategory of $\mathcal{C}[\mathcal{S}^{-1}]$ whose objects are the cofibrant objects of $\mathcal{C}$.

In a Cartan-Eilenberg category we can derive functors exactly in the same way as Cartan and Eilenberg. If $\mathcal{C}$ is a Cartan-Eilenberg category and $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor which sends strong equivalences to isomorphisms, F induces a functor $F' : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{D}$ and by the equivalence (0.3), we obtain the derived functor $\mathbb{L}F : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}$.

Each Quillen model category produces a Cartan-Eilenberg category: the category of its fibrant objects, with $\mathcal{S}$ the class of left homotopy equivalences and $\mathcal{W}$ the class of weak equivalences. Nevertheless, note the following differences with Quillen's theory. First, in the Quillen context

the class $\mathcal{S}$ appears as a consequence of the axioms while fibrant/cofibrant objects are part of them. Second, cofibrant objects in our setting are homotopy invariant, in contrast with cofibrant objects in Quillen model categories. Actually, in a Quillen category of fibrant objects, an object is Cartan-Eilenberg cofibrant if and only if it is homotopy equivalent to a Quillen cofibrant one.

Another example covered by our presentation is that of Sullivan's minimal models. We define minimal objects in a Cartan-Eilenberg category, and call it a Sullivan category, if any object has a minimal model. As an example, we interpret some results of [GNPR1] as saying that the category of modular operads over a field of characteristic zero is a Sullivan category.

In closing this introduction, we want to highlight the definition of Cartan-Eilenberg structures coming from a cotriple. If $\mathcal{X}$ is a category with a cotriple G, $\mathcal{A}$ is an abelian category and $\mathbf{C}_{\geq 0}(\mathcal{A})$ denotes the category of non-negative chain complexes of $\mathcal{A}$, we define a structure of Cartan-Eilenberg category on the functor category $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ (see Theorem 5.1.4). We apply this result to obtain theorems of the acyclic models kind, extending results in [B] and [GNPR2]. We stress that in these examples the class of strong equivalences $\mathcal{S}$ does not come from a homotopy relation.

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1. Localisation of Categories

In this section we collect for further reference some mostly well-known facts about localisation of categories, and we introduce the notion of relative localisation of a subcategory, which plays an important role in the sequel.

1.1. Categories with weak equivalences.

1.1.1. By a *category with weak equivalences* we understand a pair $(\mathcal{C}, \mathcal{W})$ where $\mathcal{C}$ is a category and $\mathcal{W}$ is a class of morphisms of $\mathcal{C}$. Morphisms in $\mathcal{W}$ will be called *weak equivalences*.

We always assume that $\mathcal{W}$ is stable by composition and contains all the isomorphisms of $\mathcal{C}$, so that we can identify $\mathcal{W}$ with a subcategory of $\mathcal{C}$.

1.1.2. Recall that the *category of fractions*, or *localisation, of $\mathcal{C}$ with respect to $\mathcal{W}$* is a category $\mathcal{C}[\mathcal{W}^{-1}]$ together with a functor $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ such that:

(i) For all $w \in \mathcal{W}$, $\gamma(w)$ is an isomorphism.
(ii) For any category $\mathcal{D}$ and any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ that transforms morphisms $w \in \mathcal{W}$ into isomorphisms, there exists a unique functor $F' : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}$ such that $F' \circ \gamma = F$.

The uniqueness condition on F' implies immediately that, when it exists, the localisation is uniquely defined up to isomorphism. The localisation exists if $\mathcal{W}$ is small and, in general, the localisation always exists in a higher universe.

1.1.3. We say that the class of weak equivalences $\mathcal{W}$ is *saturated* if a morphism f of $\mathcal{C}$ is in $\mathcal{W}$ when γf is an isomorphism. The *saturation* $\overline{\mathcal{W}}$ of $\mathcal{W}$ is the pre-image by γ of the isomorphisms of $\mathcal{C}[\mathcal{W}^{-1}]$. It is the smallest saturated class of morphisms of $\mathcal{C}$ which contains $\mathcal{W}$. Maybe it is worth pointing out that we do *not* assume that $\mathcal{W}$ verifies the usual *2 out of 3 property*. In any case, the saturation $\overline{\mathcal{W}}$ always does.

1.2. **Hammocks.** We describe the localisation of categories by using Dwyer-Kan hammocks ([DK]). Given a category with weak equivalences $(\mathcal{C}, \mathcal{W})$ and two objects X and Y in $\mathcal{C}$, a $\mathcal{W}$-*zigzag* f from X to Y is a finite sequence of morphisms of $\mathcal{C}$, going in either direction, between X and Y,

$$f: \ X \relbar\joinrel\relbar \bullet \relbar\joinrel\relbar \bullet \cdots \bullet \relbar\joinrel\relbar \bullet \relbar\joinrel\relbar Y \ ,$$

where the morphisms going from right to left are in $\mathcal{W}$. We call the number of morphisms in the sequence the *length* of the $\mathcal{W}$-zigzag. Because each $\mathcal{W}$-zigzag is a diagram, it has a *type*, its index category. A morphism from a $\mathcal{W}$-zigzag f to a $\mathcal{W}$-zigzag g of the same type is a commutative diagram in $\mathcal{C}$,
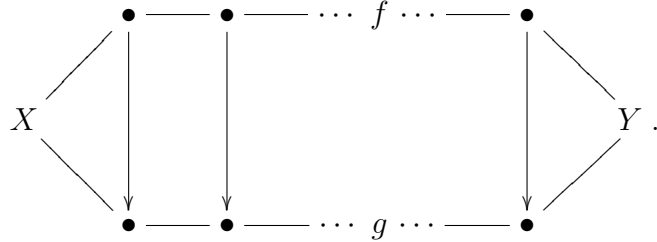
A hammock between two $\mathcal{W}$-zigzags f and g from X to Y of the same type is a finite sequence of morphisms of zigzags going in either direction. More precisely, it is a commutative diagram H in $\mathcal{C}$
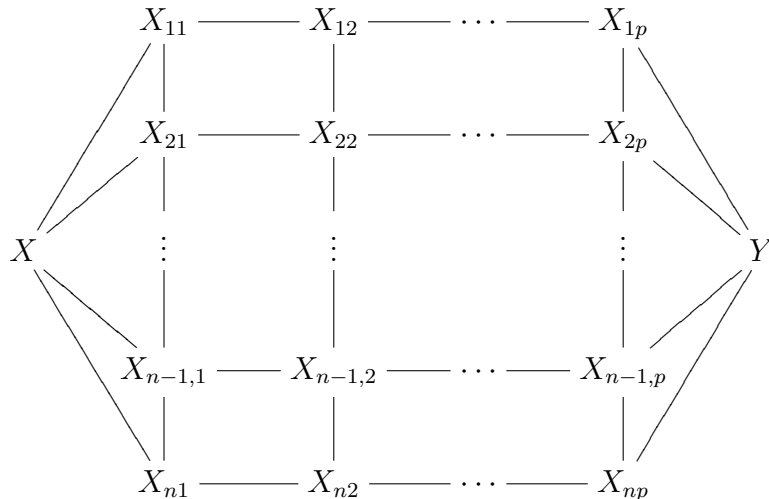
such that

(i) in each column of arrows, all (horizontal) maps go in the same direction, and if they go to the left they are in $\mathcal{W}$ (in particular, any row is a $\mathcal{W}$-zigzag),

(ii) in each row of arrows, all (vertical) maps go in the same direction, and they are arbitrary maps in $\mathcal{C}$,

(iii) the top $\mathcal{W}$-zigzag is f and the bottom is g.

If there is a hammock H between f and g, and f' is a $\mathcal{W}$-zigzag obtained from f adding identities, then adding the same identities in the hammock H and in the $\mathcal{W}$-zigzag g we obtain a new $\mathcal{W}$-zigzag g' and a hammock H' between f' and g'.

We say that two $\mathcal{W}$-zigzags f, g between X and Y are *related* if there exist $\mathcal{W}$-zigzags f' and g' of the same type, obtained from f and g by adding identities, and a hammock H between f' and g'. This is an equivalence relation between $\mathcal{W}$-zigzags. For instance, if in a $\mathcal{W}$-zigzag f there exist two consecutive arrows in the same direction, then f is equivalent to the $\mathcal{W}$-zigzag obtained from f composing these two arrows, as follows from the following diagram

$$
\begin{array}{ccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \mathrm{id}} \\
X_1 & \xrightarrow{f_2 f_1} & X_3 & \xrightarrow{\mathrm{id}} & X_3 \, .
\end{array}
$$

Furthermore, since $\mathcal{W}$ is closed by composition and contains the isomorphisms, we can add identities, if necessary, and compose two consecutive arrows in the same direction in such a way that each $\mathcal{W}$-zigzag f is related to a $\mathcal{W}$-zigzag of the form

$$
X \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \cdots \bullet \longleftarrow \bullet \longrightarrow Y \, ,
$$

that is, two consecutive morphisms always go in opposite directions and the first and the last morphisms go to the right. One such $\mathcal{W}$-zigzag will be called an *alternating $\mathcal{W}$-zigzag*.

Let $\mathcal{C}_{\mathcal{W}}$ be the category whose objects are the objects of $\mathcal{C}$ where, for any two objects X, Y, the morphisms from X to Y are the equivalence classes of $\mathcal{W}$-zigzags from X to Y, with composition being the juxtaposition of $\mathcal{W}$-zigzags.

Theorem 1.2.1. ([DHKS], 33.10). *The category $\mathcal{C}_{\mathcal{W}}$, together with the obvious functor $\mathcal{C} \longrightarrow \mathcal{C}_{\mathcal{W}}$ is a solution to the universal problem of the category of fractions $\mathcal{C}\left[\mathcal{W}^{-1}\right]$.*

In the cited reference there is a general hypothesis which concerns the class $\mathcal{W}$, which is not necessary for this result.

1.2.2. The localisation functor $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ induces a bijective map on the class of objects. In order to simplify the notation, if X is an object of $\mathcal{C}$, sometimes we will use the same letter X to denote its image $\gamma(X)$ in the localised category $\mathcal{C}[\mathcal{W}^{-1}]$.

We denote by $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$ that send morphisms in $\mathcal{W}$ to isomorphisms. The definition of the category of fractions means that for any category $\mathcal{D}$, the functor

$$
\gamma^* : \mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \longrightarrow \mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D}), \quad G \mapsto G \circ \gamma
$$

induces a bijection on the class of objects. From the previous description of the localised category we deduce that γ^* is an isomorphism of categories. In particular, the functor

$$
\gamma^* : \mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \longrightarrow \mathbf{Cat}(\mathcal{C}, \mathcal{D})
$$

is fully faithful.

1.3. **Categories with a congruence.** There are some situations where it is possible to give an easier presentation of morphisms of the category $\mathcal{C}[\mathcal{W}^{-1}]$, for example, when there is a calculus of fractions (see [GZ]). In this section we present an even simpler situation which will occur later, namely the localisation provided by some quotient categories.

1.3.1. Let $\mathcal{C}$ be a category and $\sim$ a *congruence* on $\mathcal{C}$, that is, an equivalence relation between morphisms of $\mathcal{C}$ which is compatible with composition ([ML], page 51). We denote by $\mathcal{C}/\sim$ the *quotient category*, and by $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\sim$ the universal canonical functor. We denote by $\mathcal{S}$ the class of morphisms $f : X \longrightarrow Y$ for which there exists a morphism $g : Y \longrightarrow X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$. We will call $\mathcal{S}$ the class of equivalences associated to $\sim$.

1.3.2. If $\sim$ is a congruence, in addition to the quotient category $\mathcal{C}/\sim$, one can also consider the localised category $\delta : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ of $\mathcal{C}$ with respect to the class $\mathcal{S}$ of equivalences defined by this congruence. We study when they are equivalent.

Proposition 1.3.3. *Let $\sim$ be a congruence and $\mathcal{S}$ the associated class of equivalences. If $\mathcal{S}$ and $\sim$ are compatible, that is, if $f \sim g$ implies $\delta f = \delta g$, then the categories $\mathcal{C}/\sim$ and $\mathcal{C}[\mathcal{S}^{-1}]$ are canonically isomorphic.*

Proof. If $\mathcal{S}$ and $\sim$ are compatible, the canonical functor $\delta : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ induces a functor $\phi : \mathcal{C}/\sim \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ such that $\phi \circ \pi = \delta$. Therefore, any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which sends morphisms in $\mathcal{S}$ to isomorphisms factors in a unique way through π, hence $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\sim$ has the universal property of localisation. $\square$

Example 1.3.4. The congruence $\sim$ is compatible with its class $\mathcal{S}$ of equivalences when it may be expressed by a cylinder object, or dually by a path object.

Given $X \in \mathrm{Ob}\, \mathcal{C}$, a *cylinder object over* X is an object $\mathrm{Cyl}(X)$ in $\mathcal{C}$ together with morphisms $i_0, i_1 : X \longrightarrow \mathrm{Cyl}(X)$ and $p : \mathrm{Cyl}(X) \longrightarrow X$ such that $p \in \mathcal{S}$ and $p \circ i_0 = \mathrm{id}_X = p \circ i_1$.

Now, suppose that the congruence is determined by cylinder objects in the following way:

"Given $f_0, f_1 : X \longrightarrow Y$, $f_0 \sim f_1$ if and only if there exists a morphism $H : \mathrm{Cyl}(X) \to Y$ such that $Hi_0 = f_0$ and $Hi_1 = f_1$".

Then $\sim$ and $\mathcal{S}$ are compatible. In fact, if $f_0 \sim f_1$, then we have the $\mathcal{S}$-hammock

between f_0 and f_1, which shows that $\delta(f_0) = \delta(f_1)$ in $\mathcal{C}[\mathcal{S}^{-1}]$.

More generally, $\sim$ and $\mathcal{S}$ are compatible if $\sim$ is the equivalence relation transitively generated by a cylinder object.

1.4. **Relative localisation of a subcategory.** Let $\sim$ be a congruence on a category $\mathcal{C}$. If $i : \mathcal{M} \longrightarrow \mathcal{C}$ is a full subcategory, there is an induced congruence on $\mathcal{M}$ and the quotient category $\mathcal{M}/\sim$ is a full subcategory of $\mathcal{C}/\sim$. Nevertheless, if $\mathcal{S}$ denotes the class of equivalences associated to $\sim$, and $\mathcal{S}_\mathcal{M}$ the morphisms in $\mathcal{M}$ which are in $\mathcal{S}$, the functor $\bar{i} : \mathcal{M}[\mathcal{S}_\mathcal{M}^{-1}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ is not faithful, in general. More generally, if $\mathcal{E}$ is an arbitrary class of morphisms in $\mathcal{C}$, the functor $\bar{i} : \mathcal{M}[\mathcal{E}_\mathcal{M}^{-1}] \longrightarrow \mathcal{C}[\mathcal{E}^{-1}]$ is neither faithful nor full.

To simplify the notation, in the situation above we write $\mathcal{M}[\mathcal{E}^{-1}]$ for $\mathcal{M}[\mathcal{E}_\mathcal{M}^{-1}]$.

Definition 1.4.1. Let $(\mathcal{C}, \mathcal{E})$ be a category with weak equivalences and $\mathcal{M}$ a full subcategory. The *relative localisation of the subcategory* $\mathcal{M}$ of $\mathcal{C}$ with respect to $\mathcal{E}$, denoted by $\mathcal{M}[\mathcal{E}^{-1}, \mathcal{C}]$, is the full subcategory of $\mathcal{C}[\mathcal{E}^{-1}]$ whose objects are those of $\mathcal{M}$.

This relative localisation is necessary in order to express the main results of this paper (e.g. Theorem 2.3.2). In Remark 4.2.4 we will see an interesting example where the relative localisation $\mathcal{M}[\mathcal{E}^{-1}, \mathcal{C}]$ is not equivalent to the localisation $\mathcal{M}[\mathcal{E}^{-1}]$. However, in some common situations there is no distinction between them, as for example in the proposition below, which is an abstract generalised version of Theorem III.2.10 in [GMa].

Proposition 1.4.2. *Let $(\mathcal{C}, \mathcal{E})$ be a category with weak equivalences and $\mathcal{M}$ a full subcategory. Suppose that $\mathcal{E}$ has a right calculus of fractions and that for every morphism $w : X \longrightarrow M$ in $\overline{\mathcal{E}}$, with $M \in Ob\,\mathcal{M}$, there exists a morphism $N \longrightarrow X$ in $\overline{\mathcal{E}}$, where $N \in Ob\,\mathcal{M}$. Then $\bar{i} : \mathcal{M}[\mathcal{E}^{-1}] \longrightarrow \mathcal{M}[\mathcal{E}^{-1}, \mathcal{C}]$ is an equivalence of categories.*

Proof. Let's prove that $\bar{i}$ is full: if $f = g\sigma^{-1} : M_1 \longleftarrow X \longrightarrow M_2$ is a morphism in $\mathcal{C}[\mathcal{E}^{-1}]$ between objects of $\mathcal{M}$, where $\sigma \in \mathcal{E}$, take a weak equivalence $\rho : N \longrightarrow X$ with $N \in Ob\,\mathcal{M}$, whose existence is guaranteed by hypothesis. Then $f = g\rho(\sigma\rho)^{-1}$ is a morphism of $\mathcal{M}[\mathcal{E}^{-1}]$. The faithfulness is proved in a similar way. $\qquad\square$

2. Cartan-Eilenberg categories

In this section we define cofibrant objects in a relative setting given by two classes of morphisms, as a generalisation of projective complexes in an abelian category. Then we introduce Cartan-Eilenberg categories and give some criteria to prove that a given category is Cartan-Eilenberg. We also relate these notions with Adams' study of localisation in homotopy theory, [A].

2.1. **Models in a category with strong and weak equivalences.** Let $\mathcal{C}$ be a category and $\mathcal{S}, \mathcal{W}$ two classes of morphisms of $\mathcal{C}$. Recall that our classes of morphisms are closed under composition and contain all isomorphisms, but, generally speaking, they are not saturated.

Definition 2.1.1. We say that $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a *category with strong and weak equivalences* if $\mathcal{S} \subset \overline{\mathcal{W}}$. Morphisms in $\mathcal{S}$ are called *strong equivalences* and those in $\mathcal{W}$ are called *weak equivalences*.

The basic example of category with strong and weak equivalences is the category of bounded below chain complexes of A-modules $\mathbf{C}_+(A)$, for a commutative ring A, with $\mathcal{S}$ the class of homotopy equivalences and $\mathcal{W}$ the class of quasi-isomorphisms.

Notation 2.1.2. It is convenient to fix some notation for the rest of the paper. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. We denote by $\delta : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ and $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ the canonical functors. Since $\mathcal{S} \subset \overline{\mathcal{W}}$, the functor γ factors through δ in the form

$$\mathcal{C} \xrightarrow{\quad\gamma\quad} \mathcal{C}[\mathcal{W}^{-1}] \cong \mathcal{C}[\mathcal{S}^{-1}][\delta(\mathcal{W})^{-1}].$$

with lower maps δ down to $\mathcal{C}[\mathcal{S}^{-1}]$ and γ' up to $\mathcal{C}[\mathcal{W}^{-1}]$.

Definition 2.1.3. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences, $\mathcal{M}$ a full subcategory of $\mathcal{C}$ and X an object of $\mathcal{C}$. A *left $(\mathcal{S}, \mathcal{W})$-model* of X, or simply a *left model*, in $\mathcal{M}$ is an object M in $\mathcal{M}$ together with a morphism $\varepsilon : M \longrightarrow X$ in $\mathcal{C}[\mathcal{S}^{-1}]$ which is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$.

We say that there are *enough left models* in $\mathcal{M}$, or that $\mathcal{M}$ is a *subcategory of left models* of $\mathcal{C}$, if each object of $\mathcal{C}$ has a left model in $\mathcal{M}$.

2.2. Cofibrant objects.

Definition 2.2.1. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. An object M of $\mathcal{C}$ is called *$(\mathcal{S}, \mathcal{W})$-cofibrant*, or simply *cofibrant*, if for each morphism $w : Y \longrightarrow X$ of $\mathcal{C}$ which is in $\mathcal{W}$ the map

$$w_* : \mathcal{C}[\mathcal{S}^{-1}](M, Y) \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](M, X), \ g \mapsto w \circ g$$

is bijective.

That is to say, cofibrant objects are defined by a lifting property, in $\mathcal{C}[\mathcal{S}^{-1}]$, with respect to weak equivalences: for any solid-arrow diagram such as

$$\begin{array}{ccc} & & Y \\ & \overset{g}{\nearrow} & \big\downarrow w \\ M & \underset{f}{\longrightarrow} & X \end{array}$$

with $w \in \mathcal{W}$ and $f \in \mathcal{C}[\mathcal{S}^{-1}](M, X)$, there exists a *unique* morphism $g \in \mathcal{C}[\mathcal{S}^{-1}](M, Y)$ making the triangle commutative in $\mathcal{C}[\mathcal{S}^{-1}]$.

Proposition 2.2.2. *Every retract of a cofibrant object is cofibrant.*

Proof. If N is a retract of a cofibrant object M and $w : Y \longrightarrow X$ is a weak equivalence, the map $w_*^N : \mathcal{C}[\mathcal{S}^{-1}](N, Y) \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](N, X)$ is a retract of the bijective map $w_*^M : \mathcal{C}[\mathcal{S}^{-1}](M, Y) \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](M, X)$, hence it is also bijective. Therefore N is cofibrant. $\square$

Cofibrant objects are characterised as follows (cf. [Sp], Proposition 1.4).

Theorem 2.2.3. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences, and M an object of $\mathcal{C}$. The following conditions are equivalent.*

 (i) *M is cofibrant.*
 (ii) *For each $X \in Ob\,\mathcal{C}$, the map $\gamma'_X : \mathcal{C}[\mathcal{S}^{-1}](M, X) \longrightarrow \mathcal{C}[\mathcal{W}^{-1}](M, X)$ is bijective.*

Proof. Firstly, let us see that (i) implies (ii). First of all, if M is cofibrant, the functor

$$F : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathbf{Sets}, \ X \mapsto \mathcal{C}[\mathcal{S}^{-1}](M, X)$$

sends morphisms in $\delta(\mathcal{W})$ to isomorphisms in $\mathbf{Sets}$. Therefore this functor induces a functor on the localisation

$$F' : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathbf{Sets}$$

such that $F'(\gamma'(f)) = F(f)$ for each $f \in \mathcal{C}[\mathcal{S}^{-1}](X, Y)$. In addition, γ' induces a natural transformation

$$\gamma' : F' \longrightarrow \mathcal{C}[\mathcal{W}^{-1}](M, -)\,.$$

Let X be an object of $\mathcal{C}$. To see that

$$\gamma'_X : F'(X) = \mathcal{C}[\mathcal{S}^{-1}](M, X) \longrightarrow \mathcal{C}[\mathcal{W}^{-1}](M, X)$$

is bijective we define a map

$$\Phi : \mathcal{C}[\mathcal{W}^{-1}](M, X) \longrightarrow F'(X)$$

which is inverse of γ'_X. Let $f \in \mathcal{C}[\mathcal{W}^{-1}](M, X)$, then, since F' is a functor, we have a map

$$F'(f) : F'(M) \longrightarrow F'(X).$$

We define $\Phi(f) := F'(f)(\mathrm{id}_M)$.

By the commutativity of the diagram

$$
\begin{array}{ccc}
F'(M) & \xrightarrow{\ \ \ F'(f)\ \ \ } & F'(X) \\
\big\downarrow{\scriptstyle \gamma'_M} & & \big\downarrow{\scriptstyle \gamma'_X} \\
\mathcal{C}[\mathcal{W}^{-1}](M, M) & \xrightarrow{\ \ \ f_*\ \ \ } & \mathcal{C}[\mathcal{W}^{-1}](M, X)
\end{array}
$$

we obtain

$$\gamma'_X(\Phi(f)) = \gamma'_X(F'(f)(\mathrm{id}_M)) = f_*(\gamma'_M(\mathrm{id}_M)) = f.$$

Also, given a morphism $g \in \mathcal{C}[\mathcal{S}^{-1}](M, X)$, we have

$$\Phi(\gamma'_X(g)) = F'(\gamma'_X(g))(\mathrm{id}_M) = F(g)(\mathrm{id}_M) = g,$$

so Φ is the inverse of γ'_X, thus we obtain (ii).

Next, (i) follows from (ii), since, if (ii) is satisfied, for each $w \in \mathcal{C}(Y, X)$ which is in $\mathcal{W}$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}[\mathcal{S}^{-1}](M, Y) & \xrightarrow[\cong]{\gamma'_Y} & \mathcal{C}[\mathcal{W}^{-1}](M, Y) \\
\downarrow{\scriptstyle w_*} & & \downarrow{\scriptstyle \cong} \; {\scriptstyle w_*} \\
\mathcal{C}[\mathcal{S}^{-1}](M, X) & \xrightarrow[\cong]{\gamma'_X} & \mathcal{C}[\mathcal{W}^{-1}](M, X)
\end{array}
$$

where three of the arrows are bijective; thus, so is the fourth. $\qquad\square$

2.2.4. We denote by $\mathcal{C}_{cof}$ the full subcategory of $\mathcal{C}$ whose objects are the cofibrant objects of $\mathcal{C}$, by

$$i : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$$

the inclusion functor, and by

$$j : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$$

the composition $j := \gamma' \circ i$.

From Definition 2.2.1, it follows that an object isomorphic in $\mathcal{C}[\mathcal{S}^{-1}]$ to a cofibrant object is also a cofibrant object, therefore $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ is a replete subcategory of $\mathcal{C}[\mathcal{S}^{-1}]$. (We recall that a full subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is said to be *replete* when every object of $\mathcal{B}$ isomorphic to an object of $\mathcal{A}$ is in $\mathcal{A}$.)

Now we can establish a basic fact of our theory which includes a formal version of the Whitehead theorem in the homotopy theory of topological spaces, and which is an easy corollary of Theorem 2.2.3. This theorem is no longer true with $\mathcal{M}[\mathcal{S}^{-1}]$ in the place of $\mathcal{M}[\mathcal{S}^{-1}, \mathcal{C}]$ (see Remark 4.2.4).

Theorem 2.2.5. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences and $\mathcal{M}$ be a full subcategory of $\mathcal{C}_{cof}$. The functor j induces a full and faithful functor*

$$\mathcal{M}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}].$$

In particular this induced functor reflects isomorphisms, that is to say, if $w \in \mathcal{C}[\mathcal{S}^{-1}](M, N)$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$, where M and N are in $\mathcal{M}$, then w is an isomorphism in $\mathcal{C}[\mathcal{S}^{-1}]$.
$\square$

2.3. **Cartan-Eilenberg categories.** For a category $\mathcal{C}$ with strong and weak equivalences the general problem is to know if there are enough cofibrant objects. This problem is equivalent to the orthogonal category problem for $(\mathcal{C}[\mathcal{S}^{-1}], \delta(\mathcal{W}))$ (see [Bo](I.5.4)), which has been studied by Casacuberta and Chorny in the context of homotopy theory (see [CCh]).

Definition 2.3.1. A category with strong and weak equivalences $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is called a *left Cartan-Eilenberg category* if each object of $\mathcal{C}$ has a cofibrant left model (see Definitions 2.2.1 and 2.1.3).

A category with weak equivalences $(\mathcal{C}, \mathcal{W})$ is called a left Cartan-Eilenberg category when the triple $(\mathcal{C}, \mathcal{S}, \mathcal{W})$, with $\mathcal{S}$ the class of isomorphisms of $\mathcal{C}$, is a left Cartan-Eilenberg category.

Theorem 2.3.2. *A category with strong and weak equivalences $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category if and only if*

$$j : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$$

is an equivalence of categories.

Proof. By Theorem 2.2.5, j is fully faithful. If $\mathcal{C}$ is a left Cartan-Eilenberg category, for each object X there exists a cofibrant left model $\varepsilon : M \longrightarrow X$ of X, hence $\gamma'(\varepsilon) : M \longrightarrow X$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$, so j is essentially surjective.

Conversely, if j is an essentially surjective functor, for each object X, there exists a cofibrant object M and an isomorphism $\rho : M \longrightarrow X$ in $\mathcal{C}[\mathcal{W}^{-1}]$. By Theorem 2.2.3, there exists a morphism $\sigma : M \longrightarrow X$ in $\mathcal{C}[\mathcal{S}^{-1}]$ such that $\gamma'(\sigma) = \rho$, therefore $\sigma : M \longrightarrow X$ is a cofibrant left model of X, hence $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category. $\square$

From now on, we will use also the notation $*$ for the Godement product between natural transformations and functors (see [G], Appendice), and apply its properties freely.

In a left Cartan-Eilenberg category the cofibrant left model is functorial in the localised category $\mathcal{C}[\mathcal{S}^{-1}]$. More precisely we have the following result.

Corollary 2.3.3. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a left Cartan-Eilenberg category. There exists a functor*

$$r : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$$

and a natural transformation

$$\varepsilon' : ir \Rightarrow \mathrm{id}_{\mathcal{C}[\mathcal{S}^{-1}]}$$

such that:

(1) For each object X, $\varepsilon'_X : ir(X) \longrightarrow X$ is a cofibrant left model of X.
(2) r sends morphisms in $\delta(\mathcal{W})$ into isomorphisms, and induces an equivalence of categories

$$\overline{r} : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$$

   quasi-inverse of j, such that $\overline{r}\gamma' = r$.
(3) There exists a natural isomorphism $\overline{\varepsilon} : j\overline{r} \Rightarrow \mathrm{id}_{\mathcal{C}[\mathcal{W}^{-1}]}$ such that $\gamma' * \varepsilon' = \overline{\varepsilon} * \gamma'$.
(4) The natural transformations

$$\gamma' * \varepsilon' : \gamma'ir \Rightarrow \gamma', \quad \varepsilon' * i : iri \Rightarrow i, \quad r * \varepsilon' : rir \Rightarrow r$$

   are isomorphisms.

Proof. By the previous theorem, there exists a functor

$$\overline{r} : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$$

that is the quasi-inverse of j, together with an isomorphism $\overline{\varepsilon} : j\overline{r} \Rightarrow \mathrm{id}$. Let

$$r := \overline{r}\gamma' : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}].$$

For each object X in $\mathcal{C}[\mathcal{S}^{-1}]$, $ir(X)$ is a cofibrant object, and $\bar{\varepsilon}_{\gamma'X} : \gamma'irX \longrightarrow \gamma'X$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$, hence, by Theorem 2.2.3, there exists a unique morphism $\varepsilon'_X : ir(X) \longrightarrow X$ in $\mathcal{C}[\mathcal{S}^{-1}]$ such that $\gamma'(\varepsilon'_X) = \bar{\varepsilon}_{\gamma'X}$. If $f : X \longrightarrow Y$ is a morphism in $\mathcal{C}[\mathcal{S}^{-1}]$, since $\bar{\varepsilon}$ is a natural transformation, we have

$$\gamma'(f \circ \varepsilon'_X) = \gamma'(f) \circ \bar{\varepsilon}_{\gamma'X} = \bar{\varepsilon}_{\gamma'Y} \circ \gamma'ir(f) = \gamma'(\varepsilon'_Y \circ (ir)(f)),$$

hence $f \circ \varepsilon'_X = \varepsilon'_Y \circ (ir)(f)$, because $ir(X)$ is cofibrant. As a consequence $\varepsilon' : ir \Rightarrow$ id is a natural transformation. Therefore $\varepsilon'_X : ir(X) \longrightarrow X$ is a functorial cofibrant left model of X.

On the other hand, $\gamma' * \varepsilon' = \bar{\varepsilon} * \gamma'$ and $r * \varepsilon' = \bar{r}\gamma' * \varepsilon' = \bar{r} * \bar{\varepsilon} * \gamma'$ are isomorphisms, since $\bar{\varepsilon}$ is an isomorphism. By Theorem 2.2.5, $\varepsilon' * i$ is also an isomorphism. $\qquad\square$

When proving that a category with strong and weak equivalences is a Cartan-Eilenberg category, recognising cofibrant objects may prove difficult, as the definition is given in terms of a lifting property in $\mathcal{C}[\mathcal{S}^{-1}]$. The sufficient conditions we state in the next result are basic properties of the category of bounded below chain complexes of modules over a commutative ring in the Cartan-Eilenberg approach to homological algebra ([CE]).

These conditions are also the basic properties of the category of **k**-cdg algebras in Sullivan's theory of minimal models (see [GM]). We followed the same approach to study the homotopy theory of modular operads in [GNPR1]: see Theorem 4.2.9 in this paper.

Theorem 2.3.4. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences and $\mathcal{M}$ a full subcategory of $\mathcal{C}$. Suppose that*

(i) for any $w : Y \longrightarrow X \in \mathcal{W}$ and any $f \in \mathcal{C}(M, X)$, where $M \in Ob\,\mathcal{M}$, there exists a morphism $g \in \mathcal{C}[\mathcal{S}^{-1}](M, Y)$ such that $w \circ g = f$ in $\mathcal{C}[\mathcal{S}^{-1}]$;
(ii) for any $w : Y \longrightarrow X \in \mathcal{W}$ and any $M \in Ob\,\mathcal{M}$, the map

$$w_* : \mathcal{C}[\mathcal{S}^{-1}](M, Y) \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](M, X)$$

is injective; and
(iii) for each object X of $\mathcal{C}$ there exists a morphism $\varepsilon : M \longrightarrow X$ in $\mathcal{C}$ such that $\varepsilon \in \mathcal{W}$ and $M \in Ob\,\mathcal{M}$;

Then,

(1) every object in $\mathcal{M}$ is cofibrant;
(2) $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category; and
(3) the functor $\mathcal{M}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is an equivalence of categories.

Proof. Property (2) follows immediately from (1) and (iii). Property (3) follows from (iii), (1) and Theorem 2.2.5. So it is enough to prove (1), that is: given $w : Y \longrightarrow X \in \mathcal{W}$, M in $\mathcal{M}$ and $f \in \mathcal{C}[\mathcal{S}^{-1}](M, X)$, there exists a unique $g \in \mathcal{C}[\mathcal{S}^{-1}](M, Y)$ such that $wg = f$ in $\mathcal{C}[\mathcal{S}^{-1}]$. By (ii) it is enough to prove the existence of g.

Suppose that $f \in \mathcal{C}[\mathcal{S}^{-1}](M, X)$ can be represented as an alternating $\mathcal{S}$-zigzag of $\mathcal{C}$ of length m, from M to X. We proceed by induction on m. The case $m = 1$ follows from hypothesis (i).

Let $m > 1$. Then $f = f_2 s^{-1} f_1$, where $f_1 \in \mathcal{C}(M, X_1)$, $s : X_2 \longrightarrow X_1 \in \mathcal{S}$ and $f_2 : X_2 \longrightarrow X$ is an alternating $\mathcal{S}$-zigzag of $\mathcal{C}$ of length $m - 2$. By (iii), there exists a morphism $\varepsilon : M_2 \longrightarrow X_2$ in $\mathcal{W}$ such that $M_2 \in Ob\,\mathcal{M}$, hence, by (i), there exists $g_1 \in \mathcal{C}[\mathcal{S}^{-1}](M, M_2)$ such that $f_1 = s\varepsilon g_1$. In addition, by the induction hypothesis, since $f_2\varepsilon$ can be represented as an alternating $\mathcal{S}$-zigzag of $\mathcal{C}$ of length $m - 2$, there exists $g_2 \in \mathcal{C}[\mathcal{S}^{-1}](M_2, Y)$ such that $f_2\varepsilon = wg_2$. Then $g := g_2 g_1 \in \mathcal{C}[\mathcal{S}^{-1}](M, Y)$ satisfies $wg = f$.

$$
\begin{array}{ccccc}
& & M_2 & \overset{g_2}{\dashrightarrow} & Y \\
& \overset{g_1}{\nearrow} & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle w} \\
M \overset{f_1}{\longrightarrow} X_1 & \overset{s}{\longleftarrow} & X_2 & \overset{f_2}{\longrightarrow} & X
\end{array}
$$

$\square$

Example 2.3.5. Let $\mathcal{A}$ be an abelian category with enough projective objects and let $\mathbf{C}_+(\mathcal{A})$ be the category of bounded below chain complexes of $\mathcal{A}$. Let $\mathcal{S}$ be the class of homotopy equivalences, and $\mathcal{W}$ the class of quasi-isomorphisms. Let $\mathcal{M}$ be the full subcategory of projective degree-wise complexes. Because the localisation $\mathbf{C}_+(\mathcal{A})[\mathcal{S}^{-1}]$ is the homotopy category $\mathbf{K}_+(\mathcal{A})$, by Proposition 1.3.3 and Example 1.3.4, the hypothesis of the previous theorem are well known facts (see [CE] and [GMa]), hence $(\mathbf{C}_+(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category and $\mathcal{M}$ is a subcategory of cofibrant left models of $\mathbf{C}_+(\mathcal{A})$.

2.4. **Idempotent functors and reflective subcategories.** In some cases, localisation of categories may be realised through reflective subcategories or, equivalently, by Adams idempotent functors (see [Bo](3.5.2) and [A], section 2). These notions are also related with the Bousfield localisation (see [N] for this notion in the context of triangulated categories). The following Theorem 2.4.2 relates left Cartan-Eilenberg categories with the dual notions of coreflective subcategories and coidempotent functors. Some of the parts of the theorem are a reinterpretation of well known results when $\mathcal{S}$ is the trivial class of the isomorphisms, which is in fact the key to the problem. For triangulated categories, the fourth condition in Theorem 2.4.2 corresponds to the notion of Bousfield colocalization (see [N]).

We recall that a replete subcategory (see 2.2.4) $\mathcal{A}$ of a category $\mathcal{B}$ is called *coreflective* if the inclusion functor $i : \mathcal{A} \longrightarrow \mathcal{B}$ admits a right adjoint $r : \mathcal{B} \longrightarrow \mathcal{A}$, called a *coreflector*. We recall also that a *coidempotent* functor on a category $\mathcal{B}$ is a pair (R, ε), where $R : \mathcal{B} \longrightarrow \mathcal{B}$ is an endofunctor of $\mathcal{B}$ and ε is a morphism $\varepsilon : R \Rightarrow \mathrm{id}_{\mathcal{B}}$, called *counit*, such that

$$
R * \varepsilon, \varepsilon * R : R^2 \Rightarrow R
$$

are isomorphisms, and $R * \varepsilon = \varepsilon * R$ (see [A]). In fact, the equality $R * \varepsilon = \varepsilon * R$ is a consequence of the first condition, as proved in the following lemma.

Lemma 2.4.1. *Let $\mathcal{B}$ be a category together with an endofunctor $R : \mathcal{B} \longrightarrow \mathcal{B}$ and a morphism $\varepsilon : R \Rightarrow \mathrm{id}_{\mathcal{B}}$ such that the morphisms*

$$
\varepsilon * R, R * \varepsilon : R^2 \Rightarrow R
$$

are isomorphisms. Then (R, ε) is a coidempotent functor on $\mathcal{B}$.

Proof. In the semi-simplicial object associated to (R, ε) (see [G], App.),

$$\cdots R^3 \rrrarrow R^2 \rightrightarrows R,$$

with face morphisms

$$\delta_i^n = R^i * \varepsilon * R^{n+1-i} : R^{n+1} \longrightarrow R^n, \quad 0 \leq i \leq n, \ 1 \leq n,$$

the arrows $\delta_0^1 = \varepsilon * R$, $\delta_1^1 = R * \varepsilon$ are isomorphisms. From the simplicial relations

$$\delta_0^1 \delta_0^2 = \delta_0^1 \delta_1^2, \ \delta_1^1 \delta_1^2 = \delta_1^1 \delta_2^2$$

we deduce $\delta_0^2 = \delta_1^2 = \delta_2^2$. Since $\delta_0^1 \delta_2^2 = \delta_1^1 \delta_0^2$, and $\delta_2^2 = \delta_0^2 = \varepsilon * R^2$ is also an isomorphism, we conclude that $\delta_0^1 = \delta_1^1$. $\qquad\square$

Theorem 2.4.2. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. Then the following conditions are equivalent.*

   (i) *$(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category.*
   (ii) *There exists a coidempotent functor (R', ε') on $\mathcal{C}[\mathcal{S}^{-1}]$ such that $\overline{\mathcal{W}}$ is the pre-image by $R'\delta$ of the class of isomorphisms in $\mathcal{C}[\mathcal{S}^{-1}]$, and $\gamma'\varepsilon'$ is an isomorphism.*
   (iii) *The inclusion functor $i : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ admits a right adjoint*

   $$r : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}],$$

   with a counit $\varepsilon' : ir \Rightarrow \mathrm{id}$, such that $\overline{\delta(\mathcal{W})}$ is the pre-image by r of the class of iso-morphisms in $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$, and $r\varepsilon'$ is an isomorphism. In particular $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ is a coreflective subcategory of $\mathcal{C}[\mathcal{S}^{-1}]$.
   (iv) *The localisation functor $\gamma' : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ admits a left adjoint*

   $$\lambda : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}].$$

Assuming that these conditions are satisfied, $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ is the essential image of R' (and λ).

Proof. We prove the theorem in several steps. Firstly we recall, from Corollary 2.3.3, that if $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category there exists a functor

$$r : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}],$$

together with a morphism $\varepsilon' : ir \Rightarrow \mathrm{id}$ such that $\varepsilon' * i$, $r * \varepsilon'$ and $\gamma' * \varepsilon'$ are isomorphisms.

Step 1: (i) implies (ii). Let $R' : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ be the functor $R' = ir$. Then $\varepsilon' : R' \Rightarrow \mathrm{id}$ is a natural transformation, and $\varepsilon' * R' = \varepsilon' * (ir) = (\varepsilon' * i) * r$ and $R' * \varepsilon' = (ir) * \varepsilon' = i * (r * \varepsilon')$ are isomorphisms, because so are $\varepsilon' * i$ and $r * \varepsilon'$. Therefore, by Lemma 2.4.1, (R', ε') is a coidempotent functor.

Let us see that $\overline{\mathcal{W}}$ is the pre-image by $R'\delta$ of the class of isomorphisms in $\mathcal{C}[\mathcal{S}^{-1}]$. It is enough to see that, given a morphism $f : X \longrightarrow Y$ in $\mathcal{C}[\mathcal{S}^{-1}]$, $R'(f)$ is an isomorphism if and only if $\gamma'(f)$ is an isomorphism. From the naturality of ε' we have

$$\varepsilon'_Y \circ R'(f) = f \circ \varepsilon'_X ,$$

therefore, by Theorem 2.2.5, $\gamma'(f)$ is an isomorphism if and only if $R'(f)$ is an isomorphism.

Step 2: (i) implies (iii). For each category $\mathcal{X}$, the functor

$$i_* : \mathbf{Cat}(\mathcal{X}, \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]) \longrightarrow \mathbf{Cat}(\mathcal{X}, \mathcal{C}[\mathcal{S}^{-1}])$$

is fully faithful; hence, to define a natural transformation $\eta : \mathrm{id} \Rightarrow ri$, it is enough to define a natural transformation $i * \eta : i \Rightarrow iri$. Since $\varepsilon' * i : iri \Rightarrow i$ is an isomorphism, we define η to be such that $i * \eta = (\varepsilon' * i)^{-1}$. Let us check that η and ε' are the unit and the counit, respectively, of an adjunction $i \vdash r$, that is to say (see for example [ML]),

$$(r * \varepsilon') \circ (\eta * r) = 1_r, \quad (\varepsilon' * i) \circ (i * \eta) = 1_i.$$

By step 1, $(ir) * \varepsilon' = \varepsilon' * (ir)$, and by the definition of η we obtain

$$i * ((r * \varepsilon') \circ (\eta * r)) = ((ir) * \varepsilon') \circ (i * \eta * r) = (\varepsilon' * (ir)) \circ ((\varepsilon' * i)^{-1} * r)$$

$$= ((\varepsilon' * i) * r) \circ ((\varepsilon' * i)^{-1} * r) = ((\varepsilon' * i) \circ (\varepsilon' * i)^{-1}) * r = 1_i * r = i * 1_r.$$

Since i_* is fully faithful, we obtain $(r * \varepsilon') \circ (\eta * r) = 1_r$. The other identity being trivial, we conclude that r is a right adjoint for i.

The other assertions are consequence of step 1.

Step 3: (i) implies (iv). By Corollary 2.3.3 there is a functor $\bar{r} : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ such that $\bar{r}\gamma' = r$. Let $\lambda = i\bar{r}$. Since

$$\gamma'^* : \mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{C}[\mathcal{W}^{-1}]) \longrightarrow \mathbf{Cat}(\mathcal{C}[\mathcal{S}^{-1}], \mathcal{C}[\mathcal{W}^{-1}])$$

is fully faithful, and $\gamma' * \varepsilon' : \gamma'\lambda\gamma' \Rightarrow \gamma'$ is an isomorphism, there exists a unique morphism $\eta : \mathrm{id} \Rightarrow \gamma'\lambda$ such that

$$\eta * \gamma' = (\gamma' * \varepsilon')^{-1}.$$

Then, (η, ε') are the unit and the counit of an adjunction $\lambda \dashv \gamma'$, that is to say,

$$(\gamma' * \varepsilon') \circ (\eta * \gamma') = 1_{\gamma'}, \quad (\varepsilon' * \lambda) \circ (\lambda * \eta) = 1_\lambda.$$

Indeed, the first identity follows trivially from the definition of η. For the second one, we have $\lambda\gamma' = i\bar{r}\gamma' = ir$ by the definitions, and $(ir) * \varepsilon' = \varepsilon' * (ir)$ by step 1, so we have

$$((\varepsilon' * \lambda) \circ (\lambda * \eta)) * \gamma' = (\varepsilon' * (\lambda\gamma')) \circ (\lambda * \eta * \gamma') = (\varepsilon' * (ir)) \circ ((i\bar{r}) * (\gamma' * \varepsilon')^{-1})$$

$$= ((ir) * \varepsilon') \circ ((i\bar{r}) * (\gamma' * \varepsilon')^{-1}) = ((i\bar{r}) * (\gamma' * \varepsilon')) \circ ((i\bar{r}) * (\gamma' * \varepsilon')^{-1})$$

$$= (i\bar{r}) * ((\gamma' * \varepsilon') \circ (\gamma' * \varepsilon')^{-1}) = (i\bar{r}) * 1_{\gamma'} = \lambda * 1_{\gamma'} = 1_\lambda * \gamma',$$

therefore, since γ'^* is fully faithful, the second identity of the adjunction is also satisfied.

Step 4: (ii) implies (i). Firstly, for each object X, let us check that $R'X$ is cofibrant. Let $w : A \longrightarrow B$ be a morphism in $\delta(\mathcal{W})$. By hypothesis $R'(w)$ is an isomorphism, therefore we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}[\mathcal{S}^{-1}](R'X, R'A) & \xrightarrow{\;\varepsilon'_{A*}\;} & \mathcal{C}[\mathcal{S}^{-1}](R'X, A) \\
\Big\downarrow{\scriptstyle R'w_*} & & \Big\downarrow{\scriptstyle w_*} \\
\mathcal{C}[\mathcal{S}^{-1}](R'X, R'B) & \xrightarrow{\;\varepsilon'_{B*}\;} & \mathcal{C}[\mathcal{S}^{-1}](R'X, B)
\end{array}
$$

where $R'w_*$ is bijective. The maps ε'_{A*}, and ε'_{B*} are also bijective. Indeed, we prove it for ε'_A and we omit the superscript $'$ in the proof. Since $R\varepsilon_X = \varepsilon_{RX}$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}[\mathcal{S}^{-1}](RX, RA) & \xrightarrow{\;\varepsilon_{A*}\;} & \mathcal{C}[\mathcal{S}^{-1}](RX, A) \\
\downarrow{\scriptstyle R\varepsilon_X^*} & \;\;{\scriptstyle R} & \downarrow{\scriptstyle R\varepsilon_X^*} \\
\mathcal{C}[\mathcal{S}^{-1}](R^2X, RA) & \xrightarrow{\;\varepsilon_{A*}\;} & \mathcal{C}[\mathcal{S}^{-1}](R^2X, A)
\end{array}
$$

where the vertical arrows are bijective. We deduce that the diagonal arrow is bijective, hence ε_{A*} is also bijective. Therefore $w_* : \mathcal{C}[\mathcal{S}^{-1}](R'X, A) \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](R'X, B)$ is bijective, thus $R'X$ is cofibrant.

Since $\varepsilon'_X : R'(X) \longrightarrow X \in \overline{\delta(\mathcal{W})}$, each object has a cofibrant left model, hence $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category.

Step 5: (iii) implies (i). For each object X, $\varepsilon'_X : ir(X) \longrightarrow X$ is a cofibrant left model of X, therefore $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category.

Step 6: (iv) implies (i). This is an easy consequence of the dual of Proposition I.1.3 of [GZ]. In fact, let $\eta : \mathrm{id} \Rightarrow \gamma'\lambda$ and $\varepsilon' : \lambda\gamma' \Rightarrow \mathrm{id}$ be the unit and the counit of the adjunction, respectively. The functor $\mathcal{C}[\mathcal{S}^{-1}][\delta(\mathcal{W})^{-1}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ induced by γ' is an isomorphism, thus, by *loc. cit.*, η is an isomorphism. Therefore the identity of the adjunction

$$(\eta * \gamma') \circ (\gamma' * \varepsilon') = 1_{\gamma'}$$

proves that $\gamma' * \varepsilon'$ is an isomorphism. So, for each object X, $\varepsilon'_X : \lambda\gamma'(X) \longrightarrow X$ is a left model. On the other hand, for each pair of objets X and Y, the composition

$$
\mathcal{C}[\mathcal{S}^{-1}](\lambda\gamma'(X), Y) \xrightarrow{\;\gamma'_Y\;} \mathcal{C}[\mathcal{W}^{-1}](\gamma'\lambda\gamma'(X), \gamma'(Y)) \xrightarrow{\;\eta^*_{\gamma'(X)}\;} \mathcal{C}[\mathcal{W}^{-1}](\gamma'(X), \gamma'(Y))
$$

is the adjunction map, and as $\eta^*_{\gamma'(X)}$ is bijective, so is γ'_Y. Therefore, by Proposition 2.2.3, $\lambda\gamma'(X)$ is cofibrant. Hence, $\varepsilon'_X : \lambda\gamma'(X) \longrightarrow X$ is a cofibrant left model of X, which proves (i).

Finally, in step 5 (resp. step 6) we have just proved that $R'X$ (resp. $\lambda\gamma'(X)$) is cofibrant, for each object X. Conversely, if M is cofibrant, $\varepsilon'_M : R'M \longrightarrow M$ (resp. $\varepsilon'_M : \lambda\gamma'M \longrightarrow M$) is a morphism in $\overline{\delta(\mathcal{W})}$ between cofibrant objects, therefore, by Theorem 2.2.5, it is an isomorphism in $\mathcal{C}[\mathcal{S}^{-1}]$. So $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ is the essential image of R' (resp. λ). $\qquad\square$

2.4.3. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a left Cartan-Eilenberg category. We summarise the different functors we have encountered between the categories associated to $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ in the following diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \ \delta\ \ } & \mathcal{C}[\mathcal{S}^{-1}] \\
\downarrow{\scriptstyle\gamma} & {\scriptstyle\lambda}\ \ {\scriptstyle\gamma'} & \uparrow{\scriptstyle i}\ \uparrow{\scriptstyle r} \\
\mathcal{C}[\mathcal{W}^{-1}] & \overset{j}{\underset{\overline{r}}{\rightleftarrows}} & \mathcal{C}_{cof}[\mathcal{S}^{-1},\mathcal{C}],
\end{array}$$

where:

(a) The functors γ, δ and γ' are the localisation functors (see 2.1.2).
(b) The functor i is the inclusion functor (see 2.2.4) and r is the functorial cofibrant left model (see Theorem 2.3.3).
(c) The functor r is the right adjoint of i (see Theorem 2.4.2 (iii)).
(d) The functor $\overline{r}$ is the unique functor such that $r = \overline{r}\gamma'$.
(e) The functor j is defined by $j := \gamma'i$ (see 2.2.4).
(f) The functors j and $\overline{r}$ are quasi-inverse equivalences (see Corollary 2.3.3).
(g) The functor λ is defined by $\lambda := i\overline{r}$. It is left adjoint to γ' (see Theorem 2.4.2, (iv)).

Remark 2.4.4. If $\mathcal{S}$ is just the class of isomorphisms, then $\mathcal{C}_{cof}$ is the class of objects which are left orthogonal (see [Bo](5.4)) to $\mathcal{W}$, therefore $(\mathcal{C},\mathcal{W})$ is a left Cartan-Eilenberg category if and only if $\mathcal{C}_{cof}$ is a coreflective subcategory of $\mathcal{C}$.

2.5. **Resolvent functors.** Sometimes the coidempotent functor $R' : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ in Theorem 2.4.2 comes from an endofunctor of $\mathcal{C}$ itself. We formalise this situation in the following definition.

Definition 2.5.1. Let $(\mathcal{C},\mathcal{S},\mathcal{W})$ be a category with strong and weak equivalences. A *left resolvent functor* on $\mathcal{C}$ is a pair (R,ε) where

(i) $R : \mathcal{C} \longrightarrow \mathcal{C}$ is a functor such that $R(X)$ is a cofibrant object, for each $X \in Ob\,\mathcal{C}$; and
(ii) $\varepsilon : R \Rightarrow \mathrm{id}_{\mathcal{C}}$ is morphism such that $\varepsilon_X : R(X) \longrightarrow X$ is in $\overline{\mathcal{W}}$, for each $X \in Ob\,\mathcal{C}$.

A left resolvent functor is also called a functorial cofibrant replacement.

Lemma 2.5.2. *Let $(\mathcal{C},\mathcal{S},\mathcal{W})$ be a category with strong and weak equivalences, and let (R,ε) be a left resolvent functor on $\mathcal{C}$. Then,*

(1) *we have $\overline{\mathcal{W}} = R^{-1}(\overline{\mathcal{S}})$, in particular $R(\mathcal{S}) \subset \overline{\mathcal{S}}$;*
(2) *we have $R(\varepsilon_X), \varepsilon_{R(X)} \in \overline{\mathcal{S}}$, for each $X \in Ob\,\mathcal{C}$; and*
(3) *(R,ε) induces a coidempotent functor (R',ε') on $\mathcal{C}[\mathcal{S}^{-1}]$.*

Proof. Since $R^{-1}(\overline{\mathcal{S}})$ is a saturated class of morphisms, in order to prove that $\overline{\mathcal{W}} \subset R^{-1}(\overline{\mathcal{S}})$ it is enough to check that $\mathcal{W} \subset R^{-1}(\overline{\mathcal{S}})$. In fact, if $w : X \longrightarrow Y$ is a morphism in $\mathcal{W}$, we have a

commutative diagram

$$
\begin{array}{ccc}
R(X) & \xrightarrow{\ R(w)\ } & R(Y) \\
\downarrow{\scriptstyle \varepsilon_X} & & \downarrow{\scriptstyle \varepsilon_Y} \\
X & \xrightarrow{\ w\ } & Y
\end{array},
$$

where w, ε_X and ε_Y are morphisms in $\overline{\mathcal{W}}$, hence $R(w)$ is also in $\overline{\mathcal{W}}$, since $\overline{\mathcal{W}}$ has the *2 out of 3* property. By Theorem 2.2.5, $R(w)$ is in $\overline{\mathcal{S}}$, therefore $\mathcal{W} \subset R^{-1}(\overline{\mathcal{S}})$. Conversely, if $w \in R^{-1}(\overline{\mathcal{S}})$, then $R(w) \in \overline{\mathcal{S}}$, and, from the previous diagram, we obtain $w \in \overline{\mathcal{W}}$.

From the hypothesis and part (1) we obtain $R\varepsilon_X \in \overline{\mathcal{S}}$. Next, from $\varepsilon_{RX} \in \overline{\mathcal{W}}$ and Theorem 2.2.5, we obtain $\varepsilon_{RX} \in \overline{\mathcal{S}}$. Finally (3) follows from (2) and Lemma 2.4.1. $\qquad\square$

A category with a left resolvent functor is a particular type of left Cartan-Eilenberg category where both localisations $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ and $\mathcal{C}_{cof}[\mathcal{S}^{-1}]$ agree.

Proposition 2.5.3. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences, and let (R, ε) be a left resolvent functor on $\mathcal{C}$. Then,*

(1) $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category;
(2) the canonical functor $\alpha : \mathcal{C}_{cof}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is an equivalence of categories; and
(3) an object X of $\mathcal{C}$ is cofibrant if and only if $\varepsilon_X : RX \longrightarrow X$ is an isomorphism in $\mathcal{C}[\mathcal{S}^{-1}]$.

Proof. First of all, for each object X of $\mathcal{C}$, we have $\varepsilon_X : RX \longrightarrow X \in \overline{\mathcal{W}}$, where RX is cofibrant. In particular, $\varepsilon_X : RX \longrightarrow X$ is a cofibrant left model of X, therefore $\mathcal{C}$ is a left Cartan-Eilenberg category, which proves (1).

Next, let us see (2). Since $R(X)$ is cofibrant and $R(\mathcal{W}) \subset \overline{\mathcal{S}}$, by Lemma 2.5.2, the functor R induces a functor

$$\beta : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}]$$

such that $\delta R = \beta\gamma$. Let us see that β is a quasi-inverse of α. Indeed, for each object X of $\mathcal{C}$, the counit $\varepsilon_X : R(X) \longrightarrow X$ induces a morphism in $\mathcal{C}[\mathcal{W}^{-1}]$

$$\gamma(\varepsilon_X) : \alpha\beta(\gamma(X)) = \gamma(R(X)) \longrightarrow \gamma(X)$$

which is an isomorphism. On the other hand, for each cofibrant object M, the morphism

$$\delta(\varepsilon_M) : \beta\alpha(\delta(M)) = \delta(R(M)) \longrightarrow \delta(M)$$

satisfies $\alpha\delta(\varepsilon_M) = \gamma(\varepsilon_M)$, which is an isomorphism. Therefore, by Theorem 2.2.5, $\varepsilon_M \in \overline{\mathcal{S}}$. So $\delta(\varepsilon_M)$ is an isomorphism, which proves (2).

Finally, since R is a left resolvent functor, $R(X)$ is a cofibrant object for each object X, hence, if ε_X is an isomorphism in $\mathcal{C}[\mathcal{S}^{-1}]$, X is also cofibrant. Conversely, if X is cofibrant, then $\varepsilon_X : RX \longrightarrow X$ is a morphism in $\overline{\mathcal{W}}$ between cofibrant objects, hence, by Theorem 2.2.5, it is an isomorphism in $\mathcal{C}[\mathcal{S}^{-1}]$. $\qquad\square$

The following result gives a useful criterion in order to obtain left resolvent functors, as we will see in section 6.

Theorem 2.5.4. *Let $\mathcal{C}$ be a category, $\mathcal{S}$ a class of morphisms in $\mathcal{C}$, $R : \mathcal{C} \longrightarrow \mathcal{C}$ a functor and $\varepsilon : R \Rightarrow \mathrm{id}$ a morphism such that*

$$R(\mathcal{S}) \subset \mathcal{S}, \quad R(\varepsilon_X) \in \mathcal{S}, \quad \varepsilon_{R(X)} \in \mathcal{S},$$

for each $X \in Ob\,\mathcal{C}$. If we take $\mathcal{W} = R^{-1}(\overline{\mathcal{S}})$, then $\mathcal{S} \subset \mathcal{W}$ and (R, ε) is a left resolvent functor for $(\mathcal{C}, \mathcal{S}, \mathcal{W})$, which is therefore a left Cartan-Eilenberg category satisfying conditions (1), (2) and (3) of Proposition 2.5.3.

Proof. The pair (R, ε) induces a coidempotent functor (R', ε') on $\mathcal{C}[\mathcal{S}^{-1}]$ which satisfies the hypothesis (ii) of Theorem 2.4.2, therefore $\varepsilon_X : R(X) \longrightarrow X$ provides a cofibrant left model of X, for each X. Hence (R, ε) is a left resolvent functor for $(\mathcal{C}, \mathcal{S}, \mathcal{W})$. $\qquad\square$

Example 2.5.5. Let $\mathbf{C}_+(A)$ be the category of bounded below chain complexes of A-modules, where A is a commutative ring, and let $\mathcal{S}$ be the class of homotopy equivalences. Let R be the endofunctor on $\mathbf{C}_+(A)$ defined by the free functorial resolution induced by the functor on the category of A-modules, $X \mapsto A^{(X)}$, where $A^{(X)}$ denotes the free A-module with base X, and $\varepsilon : R \Rightarrow \mathrm{id}$ is the augmentation morphism. Since the objects of $\mathbf{C}_+(A)$ are bounded below chain complexes, a quasi-isomorphism between two such complexes which are free component-wise is a homotopical equivalence. Hence the hypothesis of the previous theorem are verified and, therefore, (R, ε) is a left resolvent functor on $\mathbf{C}_+(A)$. Moreover, the class $\mathcal{W}$ is the class of quasi-isomorphisms (as in Example 2.3.5), and the cofibrant objects are the complexes which are homotopically equivalent to a free component-wise complex.

In the next sections 3 and 6 we will see other examples of resolvent functors.

Remark 2.5.6. The dual notions of cofibrant object and left Cartan-Eilenberg category, are the notions of *fibrant object* and *right Cartan-Eilenberg category*. All the preceding results have their corresponding dual. For example, dual of Theorem 2.3.2 says that a category with strong and weak equivalences $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a right Cartan-Eilenberg category if and only if the functor $\mathcal{C}_{fib}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is an equivalence of categories.

3. Models of functors and derived functors

In this section we study functors defined on a Cartan-Eilenberg category $\mathcal{C}$ and taking values in a category $\mathcal{D}$ with a class of weak equivalences. We prove that, subject to some hypotheses, certain categories of functors are also Cartan-Eilenberg categories. In this context we can realise derived functors, when they exist, as cofibrant models in the functor category. The classic example is the category of additive functors defined on a category of complexes of an abelian category with enough projective objects.

3.1. **Derived functors.** To begin with, we recall the definition of derived functor as set up by Quillen ([Q]).

Let $(\mathcal{C}, \mathcal{W})$ be a category with weak equivalences, and $\mathcal{D}$ an arbitrary category. Recall that the category $\mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D})$ is identified, by means of the functor

$$\gamma^* : \mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \longrightarrow \mathbf{Cat}(\mathcal{C}, \mathcal{D}),$$

with the full subcategory $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ of $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ whose objects are the functors which send morphisms in $\mathcal{W}$ to isomorphisms in $\mathcal{D}$.

If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, a right Kan extension (see [ML], Chap. X) of F along $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is a functor

$$\mathrm{Ran}_{\gamma} F : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D},$$

together with a natural transformation $\theta_F = \theta_{\gamma, F} : (\mathrm{Ran}_{\gamma} F)\gamma \Rightarrow F$, satisfying the usual universal property.

Definition 3.1.1. Let $(\mathcal{C}, \mathcal{W})$ be a category with weak equivalences, and $\mathcal{D}$ an arbitrary category. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called *left derivable* if the right Kan extension of F along γ exists. The functor

$$\mathbb{L}_{\mathcal{W}} F := (\mathrm{Ran}_{\gamma} F)\gamma$$

is called a *left derived functor of F* with respect to $\mathcal{W}$.

We will denote by $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$ the full subcategory of $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ of left derivable functors with respect to $\mathcal{W}$.

3.1.2. The left derived functor $\mathbb{L}_{\mathcal{W}} F$ is endowed with a natural transformation $\theta_F : \mathbb{L}_{\mathcal{W}} F \Rightarrow F$ such that, for each functor $G \in Ob\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ the map

$$\mathbf{Nat}(G, \mathbb{L}_{\mathcal{W}} F) \longrightarrow \mathbf{Nat}(G, F), \quad \phi \mapsto \theta_F \circ \phi$$

is bijective.

If $\mathcal{W}$ has a right calculus of fractions, the definition of left derived functor agrees with the definition given by Deligne in [D].

Functors in $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ are tautologically derivable functors as ensues from the following easy lemma.

Lemma 3.1.3. *Let $(\mathcal{C}, \mathcal{W})$ be a category with weak equivalences, and $\mathcal{D}$ an arbitrary category. Then,*

(1) *any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which takes $\mathcal{W}$ into isomorphisms induces a unique functor*

$$F' : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}$$

such that $F'\gamma = F$. This functor F' satisfies $F' = Ran_{\gamma} F$, with $\theta_F = Id$. In particular, F is left derivable and $\mathbb{L}_{\mathcal{W}} F = F$; and

(2) *$\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ is a full subcategory of $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$.* $\square$

3.1.4. For each $F \in Ob\,\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$, we have $\mathbb{L}_{\mathcal{W}} F \in Ob\,\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$, so, by the previous lemma, part (1), it results that $\mathbb{L}_{\mathcal{W}} F \in Ob\,\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$. Therefore, taking the left derived functor $\mathbb{L}_{\mathcal{W}}$ defines a functor

$$\mathbb{L}_{\mathcal{W}} : \mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D}) \longrightarrow \mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D}),$$

and the canonical morphism $\theta_F : \mathbb{L}_{\mathcal{W}} F \longrightarrow F$ gives a natural transformation $\theta : \mathbb{L}_{\mathcal{W}} \Rightarrow \mathrm{id}$.

Theorem 3.1.5. *With the notation above we have*

(1) the pair $(\mathbb{L}_{\mathcal{W}}, \theta)$ is a coidempotent functor on $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$;

(2) the category with weak equivalences $(\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D}), \widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is the class of morphisms whose image by $\mathbb{L}_{\mathcal{W}}$ is an isomorphism, is a left Cartan-Eilenberg category; and

(3) the category $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ is the subcategory of its cofibrant objects.

In particular, if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a left derivable functor, a left derived functor of F is the same as a cofibrant left model of F.

Proof. In the sequel we shorten $\mathbb{L}_{\mathcal{W}}$ to $\mathbb{L}$. First of all, by Lemma 3.1.3, for each left derivable functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, $\mathbb{L}\mathbb{L}F = \mathbb{L}F$ and $\theta_{\mathbb{L}F}$ is the identity, hence $\theta_{\mathbb{L}F}$ is an isomorphism. On the other hand, the naturality of θ implies that the following diagram is commutative

$$
\begin{array}{ccc}
\mathbb{L}^2 F & \xrightarrow{\;\theta_{\mathbb{L}F}\;} & \mathbb{L}F \\
{\scriptstyle \mathbb{L}\theta_F}\big\downarrow & & \big\downarrow {\scriptstyle \theta_F} \\
\mathbb{L}F & \xrightarrow{\;\theta_F\;} & F \, ,
\end{array}
$$

hence, by the universal property of Definition 3.1.1, we obtain $\mathbb{L}(\theta_F) = \theta_{\mathbb{L}F}$, so $\mathbb{L}(\theta_F)$ is also an isomorphism. Therefore $(\mathbb{L}, \theta)$ is a coidempotent functor on $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$. So, by Theorem 2.4.2, $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$, is a left Cartan-Eilenberg category, taking the isomorphisms as strong equivalences, and the class of morphisms of $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$ whose image by $\mathbb{L}$ is an isomorphism as weak equivalences. Finally, the cofibrant objects are the functors isomorphic to functors $\mathbb{L}F$, that is to say, the functors in $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$. $\qquad\square$

3.2. **A derivability criterion for functors.** In this section we give a derivability criterion for functors defined on a left Cartan-Eilenberg category, which is a non additive extension of the standard derivability criterion for additive functors, and we obtain a Cartan-Eilenberg category structure for functors satisfying such derivability criterion.

In the following results we use the notation settled in 2.4.3.

Theorem 3.2.1. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a left Cartan-Eilenberg category. Let λ denote the left adjoint to γ', and let $\varepsilon' : \lambda\gamma' \Rightarrow \mathrm{id}$ denote the counit of the adjunction. For any category $\mathcal{D}$,*

(1) $\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ is a full subcategory of $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$;

(2) if $F \in Ob\ \mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$, then

$$
\mathbb{L}_{\mathcal{W}}F = F'\lambda\gamma,
$$

where $F' : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{D}$ denotes the functor induced by F; and the canonical morphism $\theta_F : \mathbb{L}_{\mathcal{W}}F \longrightarrow F$ is defined by $\theta_F = F' * \varepsilon' * \delta$, that is to say,

$$
(\theta_F)_X = F'(\varepsilon'_{\delta X}),
$$

for each object X of $\mathcal{C}$.

Proof. The functors λ and γ' induce a pair of functors

$$\mathbf{Cat}(\mathcal{C}[\mathcal{S}^{-1}], \mathcal{D}) \underset{\lambda^*}{\overset{\gamma'^*}{\longleftarrow\!\!\!\longrightarrow}} \mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}),$$

which are also adjoint, where λ^* is right adjoint to γ'^*, and $\varepsilon'^* : \gamma'^*\lambda^* \Rightarrow \mathrm{id}$ is the counit of the adjunction, as is easily seen. Hence, for each functor $G \in \mathbf{Cat}(\mathcal{C}[\mathcal{S}^{-1}], \mathcal{D})$, $\lambda^*(G) = G \circ \lambda$ is a right Kan extension of G along γ' (see [ML](X.3)), so G is left derivable with respect to γ'. Moreover, the canonical morphism

$$\theta_{\gamma',G} : (Ran_{\gamma'}G)\gamma' = G\lambda\gamma' \longrightarrow G$$

is defined by $G(\varepsilon'_X)$, for each object X of $\mathcal{C}[\mathcal{S}^{-1}]$.

By Lemma 3.1.3, $F' = Ran_\delta F$ and $\theta_{\delta,F} = \mathrm{id}$. Since $Ran_{\gamma'}F' = F'\lambda$ we have, by Lemma 3.2.2 below,

$$Ran_\gamma F = Ran_{\gamma'}(Ran_\delta F) = F'\lambda$$

so $\mathbb{L}_\mathcal{W}F = (Ran_\gamma F)\gamma = F'\lambda\gamma$. In addition, for each object X, the canonical morphism $(\theta_{\gamma,F})_X$ is defined by

$$(\theta_{\gamma,F})_X = (\theta_{\gamma',F'})_{\delta X} \circ (\theta_{\delta,F})_X = F'(\varepsilon'_{\delta X}).$$

$\square$

Lemma 3.2.2. *Let $\gamma_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ and $\gamma_2 : \mathcal{C}_2 \longrightarrow \mathcal{C}_3$ be two composable functors, and $\gamma = \gamma_2\gamma_1$. If $F : \mathcal{C}_1 \longrightarrow \mathcal{D}$ is a functor such that $Ran_{\gamma_2}(Ran_{\gamma_1}(F))$ exists, then*

(1) $Ran_\gamma F$ exists, $Ran_\gamma F = Ran_{\gamma_2}(Ran_{\gamma_1}(F))$; and
(2) $\theta_{\gamma,F} = \theta_2\gamma_1 \circ \theta_1$, where $\theta_2 = \theta_{\gamma_2,Ran_{\gamma_1}(F)}$ and $\theta_1 = \theta_{\gamma_1,F}$. $\square$

Proof. It is enough to check that $(\theta_2 * \gamma_1) \circ \theta_1 : Ran_{\gamma_2}(Ran_{\gamma_1}(F))\gamma \Rightarrow F$ satisfies the corresponding universal property. $\square$

Example 3.2.3. The previous theorem is an extension to a non-necessarily additive setting of the standard derivability criterion for additive functors (see [GM], III.6, th. 8). In fact, let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Suppose that $\mathcal{A}$ has enough projective objects, hence, by Example 2.3.5, $(\mathbf{C}_+(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category. Let $F : \mathbf{C}_+(\mathcal{A}) \longrightarrow \mathbf{K}_+(\mathcal{B})$ be a functor induced by an additive functor $\mathcal{A} \longrightarrow \mathcal{B}$. Then, since F is additive, it sends homotopy equivalences to isomorphisms, hence, by Theorem 3.2.4, F is left derivable and $\mathbb{L}_\mathcal{W}F = F' \circ \lambda \circ \gamma$.

Next we study the Cartan-Eilenberg structure on the category $\mathbf{Cat}_\mathcal{S}(\mathcal{C}, \mathcal{D})$.

Theorem 3.2.4. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a left Cartan-Eilenberg category and $\mathcal{D}$ any category. Consider the category with weak equivalences $(\mathbf{Cat}_\mathcal{S}(\mathcal{C}, \mathcal{D}), \widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is the class of morphisms of functors $\phi : F \Rightarrow G : \mathcal{C} \longrightarrow \mathcal{D}$ such that ϕ_M is an isomorphism for all cofibrant objects M of $\mathcal{C}$. The functor*

$$\mathbb{L}_\mathcal{W} : \mathbf{Cat}_\mathcal{S}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathbf{Cat}_\mathcal{S}(\mathcal{C}, \mathcal{D}), \quad \mathbb{L}_\mathcal{W}F := F'\lambda\gamma,$$

together with the natural transformation $\theta : \mathbb{L}_\mathcal{W}F \Rightarrow F$ defined by $(\theta_F)_X = F'(\varepsilon'_{\delta(X)})$, for each object X of $\mathcal{C}$, satisfy

(1) $(\mathbb{L}_\mathcal{W}, \theta)$ is a left resolvent functor on $(\mathbf{Cat}_\mathcal{S}(\mathcal{C}, \mathcal{D}), \widetilde{\mathcal{W}})$;

(2) $(\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}), \widetilde{\mathcal{W}})$ *is a left Cartan-Eilenberg category; and*
(3) $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ *is the subcategory of its cofibrant objects.*

Proof. Since $\mathcal{S} \subset \mathcal{W}$, the category $\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ contains $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ as a full subcategory. On the other hand, by Theorem 3.2.1, $\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ is a full subcategory of $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$. Therefore, by Theorem 3.1.5, $(\mathbb{L}, \theta)$ induces a coidempotent functor on $\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$, whose essential image is $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$. In addition, by Theorem 2.5.4, $(\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}), \widetilde{\mathcal{W}})$ is a left Cartan-Eilenberg category whose cofibrant objects are functors in $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$, and $(\mathbb{L}, \theta)$ is a resolvent functor, where $(\theta_F)_X = F'(\varepsilon'_{\delta(X)})$, by Theorem 3.1.5.

Next, by Theorem 3.2.1, $\mathbb{L}F = F'\lambda\gamma$ and, by Theorem 2.5.4, the class of weak equivalences is the class of morphisms $\phi : F \Rightarrow G$ such that $\mathbb{L}(\phi)$ is an isomorphism, that is to say, $\phi_{\lambda(\gamma(X))}$ is an isomorphism, for each X. Since the objects $\lambda(\gamma(X))$ are the cofibrant objects up to strong equivalences, a morphism ϕ is a weak equivalence if and only if ϕ_M is an isomorphism for each cofibrant object M, that is to say, $\widetilde{\mathcal{W}}$ is the class of weak equivalences. $\square$

3.3. **Models of functors.** When the target category $\mathcal{D}$ of functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ is endowed with a class of weak equivalences $\mathcal{E}$, the previous results can be applied to the functor $\gamma_{\mathcal{D}}F : \mathcal{C} \longrightarrow \mathcal{D}[\mathcal{E}^{-1}]$ to obtain a model of this functor. However, in some situations, it is desirable to have cofibrant models for the functor F itself. We prove that this is possible if $\mathcal{C}$ is a left Cartan-Eilenberg category with a left resolvent functor and F sends strong equivalences to weak equivalences.

3.3.1. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a Cartan-Eilenberg category with a left resolvent functor (R, ε) and $\mathcal{D}$ a category with a saturated class of weak equivalences $\mathcal{E}$. Denote by $\mathbf{Cat}_{\mathcal{S}, \mathcal{E}}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ whose objects are the functors which send $\mathcal{S}$ to $\mathcal{E}$.

Definition 3.3.2. Let F, G be objects of $\mathbf{Cat}_{\mathcal{S}, \mathcal{E}}(\mathcal{C}, \mathcal{D})$ and $\phi : F \Rightarrow G$ a morphism.

   (i) ϕ is called a *weak equivalence* if ϕ_M is in $\mathcal{E}$, for all $M \in Ob\, \mathcal{C}_{cof}$.
   (ii) ϕ is called a *strong equivalence* if ϕ_X is in $\mathcal{E}$, for all $X \in Ob\, \mathcal{C}$.

We denote by $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{S}}$ the classes of weak and strong equivalences of $\mathbf{Cat}_{\mathcal{S}, \mathcal{E}}(\mathcal{C}, \mathcal{D})$, respectively.

If $F(\mathcal{S}) \subset \mathcal{E}$, then $R^*(F)(\mathcal{S}) = F(R(\mathcal{S})) \subset F(\overline{\mathcal{S}}) \subset \mathcal{E}$, thus the resolvent functor R induces the functor

$$R^* : \mathbf{Cat}_{\mathcal{S}, \mathcal{E}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathbf{Cat}_{\mathcal{S}, \mathcal{E}}(\mathcal{C}, \mathcal{D})$$

given by $R^*(F) := FR$, and the counit $\varepsilon : F \Rightarrow \mathrm{id}$ induces a counit $\varepsilon^* : R^* \Rightarrow \mathrm{id}$ by

$$\varepsilon^*_F := F\varepsilon : FR \longrightarrow F.$$

Theorem 3.3.3. *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with a left resolvent functor (R, ε), and $\mathcal{D}$ a category with a saturated class of weak equivalences $\mathcal{E}$. With the previous notation we have*

(1) (R^*, ε^*) is a left resolvent functor for $\left(\mathbf{Cat}_{\mathcal{S}, \mathcal{E}}(\mathcal{C}, \mathcal{D}), \widetilde{\mathcal{S}}, \widetilde{\mathcal{W}} \right)$;

(2) $\left(\mathbf{Cat}_{\mathcal{S}, \mathcal{E}}(\mathcal{C}, \mathcal{D}), \widetilde{\mathcal{S}}, \widetilde{\mathcal{W}} \right)$ is a left Cartan-Eilenberg category; and
(3) a functor $F \in Ob\, \mathbf{Cat}_{\mathcal{S}, \mathcal{E}}(\mathcal{C}, \mathcal{D})$ is cofibrant if and only if $F(\mathcal{W}) \subset \mathcal{E}$.

Proof. We first observe that, by (2) of Lemma 2.5.2, for each object X of $\mathcal{C}$, ε_{RX} and $R(\varepsilon_X)$ are in $\overline{\mathcal{S}}$, therefore, for each functor F in $\mathbf{Cat}_{\mathcal{S},\mathcal{E}}(\mathcal{C},\mathcal{D})$, the morphisms $F(\varepsilon_{RX})$ and $F(R(\varepsilon_X))$ are in $\mathcal{E}$, hence $R^*\varepsilon_F$ and $\varepsilon_{R^*(F)}$ are in $\widetilde{\mathcal{S}}$.

Moreover, by (3) of Proposition 2.5.3, it is easy to check that $\widetilde{\mathcal{W}} = (R^*)^{-1}(\widetilde{\mathcal{S}})$. In particular $R^*(\widetilde{\mathcal{S}}) \subset \widetilde{\mathcal{S}}$. Hence we can apply Theorem 2.5.4 to obtain (1) and (2).

By part (1) and Proposition 2.5.3, F is cofibrant if and only if $\varepsilon_F^* : R^*(F) \longrightarrow F$ is a strong equivalence, that is to say, $F(\varepsilon_X) : F(RX) \longrightarrow F(X) \in \mathcal{E}$, for each X.

If $F(\mathcal{W}) \subset \mathcal{E}$, since $\varepsilon_X \in \mathcal{W}$, we obtain $F(\varepsilon_X) \in \mathcal{E}$, that is to say $\varepsilon_F^* : R^*F \longrightarrow F$ is a strong equivalence, whence F is cofibrant.

To prove the converse, observe that if F is a functor such that $F(\mathcal{S}) \subset \mathcal{E}$, then we have also $F(\overline{\mathcal{S}}) \subset \mathcal{E}$ since $\mathcal{E}$ is saturated. By Lemma 2.5.2, for each $w \in \mathcal{W}$, we have $R(w) \in \overline{\mathcal{S}}$, so $F(R(w)) \in \mathcal{E}$. Hence $F(R(\mathcal{W})) \subset \mathcal{E}$.

Now, suppose that F is cofibrant, and let $w : X \longrightarrow Y \in \mathcal{W}$. We have a commutative diagram

$$
\begin{array}{ccc}
FRX & \xrightarrow{\ F\varepsilon_X\ } & F(X) \\
{\scriptstyle FRw}\Big\downarrow & & \Big\downarrow{\scriptstyle Fw} \\
FRY & \xrightarrow{\ F\varepsilon_Y\ } & F(Y)
\end{array}
$$

Since $F\varepsilon_X$, $F\varepsilon_Y$ and FRw are in $\mathcal{E}$, we obtain $Fw \in \mathcal{E}$, since $\mathcal{E}$ is saturated, that is to say $F(\mathcal{W}) \subset \mathcal{E}$. $\square$

Finally, by Theorems 3.3.3 and 3.2.1, we obtain:

Corollary 3.3.4. *With the previous notation, for each $F \in \mathbf{Cat}_{\mathcal{S},\mathcal{E}}(\mathcal{C},\mathcal{D})$, $F\varepsilon : FR \longrightarrow F$ is a cofibrant left model of F, the left derived functor $\mathbb{L}_{\mathcal{W}}(\gamma_{\mathcal{E}}F)$ of $\gamma_{\mathcal{E}}F$ is $\gamma_{\mathcal{E}}FR$, and the total left derived functor $\mathbb{L}F$ of F (see [Q], Definition 2, §I.4) is the functor induced by $\mathbb{L}_{\mathcal{W}}(\gamma_{\mathcal{E}}F)$, so we have a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F \circ R\ } & \mathcal{D} \\
{\scriptstyle \gamma_{\mathcal{W}}}\Big\downarrow & \searrow^{\mathbb{L}_{\mathcal{W}}(\gamma_{\mathcal{E}}F)} & \Big\downarrow{\scriptstyle \gamma_{\mathcal{E}}} \\
\mathcal{C}[\mathcal{W}^{-1}] & \xrightarrow{\ \mathbb{L}F\ } & \mathcal{D}[\mathcal{E}^{-1}]
\end{array}
$$

Example 3.3.5. Let $\mathbf{C}_+(A)$ be the Cartan-Eilenberg category of bounded below chain complexes of A-modules, where A is a commutative ring, and $\varepsilon : R \Rightarrow \mathrm{id}$ the resolvent functor defined by the free functorial resolution (see Example 2.5.5). Let $\mathcal{B}$ be an abelian category and $F : \mathbf{C}_+(A) \longrightarrow \mathbf{C}_+(\mathcal{B})$ a functor induced by an additive functor $A - \mathrm{mod} \longrightarrow \mathcal{B}$. Then F sends homotopy equivalences to quasi-isomorphisms, therefore $F\varepsilon : FR \Rightarrow F$ is a cofibrant

left model of F in $\mathbf{Cat}_{\mathcal{S},\mathcal{E}}(\mathbf{C}_+(A), \mathbf{C}_+(B))$, where $\mathcal{S}$ are the homotopy equivalences and $\mathcal{E}$ the quasi-isomorphisms.

4. Quillen model categories and Sullivan minimal models

In this section we describe how Cartan-Eilenberg categories relate to some other axiomatisations for homotopy theory.

4.1. **Quillen model categories.** Let $\mathcal{C}$ be a Quillen model category, that is, a category equipped with three classes of morphisms: weak equivalences $\mathcal{W}$, cofibrations $cofib$, and fibrations fib, satisfying Quillen's axioms for a model category ([Q], see also [DS]).

In a Quillen model category there are the notions of cofibrant, fibrant and cylinder objects. To distinguish between these objects and the cofibrant/fibrant/cylinder objects as introduced in this paper, the former ones will be called Quillen cofibrant/fibrant/cylinder objects. Denote by $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ the full subcategories of Quillen fibrant and cofibrant-fibrant objects of $\mathcal{C}$, respectively.

In a Quillen model category there are the notions of left and right homotopy. For instance, if $f, g : X \longrightarrow Y$ are two morphisms, a left homotopy from f to g is a morphism $h : X' \longrightarrow Y$, where X' is a Quillen cylinder object for X (that is, $\partial_0 \vee \partial_1 : X \vee X \longrightarrow X'$ is a cofibration, $p : X' \longrightarrow X$ is a weak equivalence, and $p\partial_0 = Id = p\partial_1$, see Definition I.4 of [Q]), such that $h\partial_0 = f$ and $h\partial_1 = g$. Let $\sim_l$ be the equivalence relation transitively generated by the left homotopy, and let $\mathcal{S}_l$ be the class of homotopy equivalences coming from $\sim_l$. We denote by $\pi^l(X, Y)$ the set of equivalence classes of morphisms from X to Y with respect to $\sim_l$. By the dual of ([Q], Lemma I.6), $\sim_l$ is a congruence in $\mathcal{C}_f$.

Lemma 4.1.1. *The equivalence relation $\sim_l$ is compatible with $\mathcal{S}_l$ in $\mathcal{C}_f$.*

Proof. Recall that $\sim_l$ and $\mathcal{S}_l$ are said to be compatible if $f \sim_l g$ implies $f = g$ in the localized category $\mathcal{C}_f[\mathcal{S}^{-1}]$ (see Proposition 1.3.3).

Let $f, g : X \longrightarrow Y$ be two morphisms such that $f \sim_l g$, where X, Y are fibrant objects. We can assume that there exists a left homotopy $h' : X' \longrightarrow Y$ from f to g, where X' is a cylinder object for X. We can choose a cylinder object such that $p' : X' \longrightarrow X$ is a trivial fibration. In fact, let

$$X' \xrightarrow{\ j\ } X \times I \xrightarrow{\ p\ } X$$

be a factorisation of p' in a trivial fibration p and a cofibration j, which is also trivial since p' is too. Since Y is a fibrant object, and j is a trivial cofibration, there exists a morphism h filling the following solid-arrow commutative diagram.

$$
\begin{array}{ccc}
X' & \xrightarrow{\ h'\ } & Y \\
{\scriptstyle j}\downarrow & \nearrow^{\ h} & \downarrow \\
X \times I & \longrightarrow & *
\end{array}
$$

Therefore h is a left homotopy from f to g.

Next the trivial fibration $p : X \times I \longrightarrow X$ is a left homotopy equivalence. This is a consequence of the following general fact in a Quillen model category: If a cofibration $i : X \longrightarrow Y$ has a retraction $p : Y \longrightarrow X$ which is a trivial fibration, then i (and p) is a left homotopy equivalence. (*Proof*: Let

$$Y \vee Y \xrightarrow{\partial_0 \vee \partial_1} Y \times I \xrightarrow{\quad q \quad} Y$$

be a Quillen cylinder object for Y. Consider the diagram

$$
\begin{array}{ccc}
Y \vee Y & \xrightarrow{ip \vee 1_Y} & Y \\
{\scriptstyle \partial_0 \vee \partial_1} \downarrow & {\scriptstyle H} \nearrow & \downarrow {\scriptstyle p} \\
Y \times I & \xrightarrow{pq} & X,
\end{array}
$$

where the left vertical arrow is a cofibration and the right one is a trivial fibration. Then, the lifting H is a left homotopy between ip and 1_Y.) Going back to the proof of the lemma, since $p \in \mathcal{S}_l$, we have, in $\mathcal{C}_f[\mathcal{S}_l^{-1}]$, $f = h\partial_0 = hp^{-1}p\partial_0 = hp^{-1} = h\partial_1 = g$, as asserted. $\qquad\square$

By the previous lemma, the class $\mathcal{S}_l$ is compatible with $\sim_l$ and, by Proposition 1.3.3, there is an isomorphism of categories $\pi^l \mathcal{C}_f \cong \mathcal{C}_f[\mathcal{S}_l^{-1}]$. Therefore, the relative localisation $\mathcal{C}_{cf}[\mathcal{S}_l^{-1}, \mathcal{C}_f]$ is isomorphic to the homotopy category $\pi^l \mathcal{C}_{cf}$. We observe that the left homotopy relation is, itself, an equivalence relation when restricted to the subcategory $\mathcal{C}_{cf}$, by Lemma 4 of [Q].

Let $\mathcal{W}$ be the class of weak equivalences of $\mathcal{C}_f$. If $H : Cyl(X) \longrightarrow Y$ is a left homotopy $H : f \sim_l g$, since $fp = H = gp$ and $p : Cyl(X) \longrightarrow X$ is a weak equivalence, then $f = Hp^{-1} = g$ in $\mathcal{C}[\mathcal{W}^{-1}]$. Hence $\mathcal{S}_l \subset \overline{\mathcal{W}}$, so $(\mathcal{C}_f, \mathcal{S}_l, \mathcal{W})$ is a category with strong and weak equivalences.

Theorem 4.1.2. *Let $\mathcal{C}$ be a Quillen model category. Then $(\mathcal{C}_f, \mathcal{S}_l, \mathcal{W})$ is a left Cartan-Eilenberg category and $\mathcal{C}_{cf}$ is a subcategory of cofibrant left models of $\mathcal{C}_f$.*

Proof. We prove that the class $\mathcal{C}_{cf}$ satisfies the hypothesis of Theorem 2.3.4. Let M be a Quillen fibrant-cofibrant object, and let $w : Y \longrightarrow X$ be a weak equivalence between Quillen fibrant objets. Let us see that the map

$$w_* : \mathcal{C}_f[\mathcal{S}_l^{-1}](M, Y) = \pi^l(M, Y) \longrightarrow \mathcal{C}_f[\mathcal{S}_l^{-1}](M, X) = \pi^l(M, X)$$

is bijective. By the axiom M2 of [Q], there exists a factorisation $w = \beta \circ \alpha$, where $\alpha : Y \longrightarrow Z$ is a trivial cofibration and $\beta : Z \longrightarrow X$ is a trivial fibration. Since $w_* = \beta_* \circ \alpha_*$ it is enough to prove that the maps

$$\alpha_* : \pi^l(M, Y) \longrightarrow \pi^l(M, Z)$$

and

$$\beta_* : \pi^l(M, Z) \longrightarrow \pi^l(M, X)$$

are bijective.

Since β is a trivial fibration, by Lemma 7 of [Q], β_* is bijective.

To prove that α_* is also bijective, we apply the dual of Lemma 7 of [Q]. The object M being Quillen-cofibrant, for each Quillen-fibrant object X, the left and right homotopy relations coincide in $\mathcal{C}(M, X)$. Hence, to prove that $\alpha_* : \pi^l(M, Y) \longrightarrow \pi^l(M, Z)$ is bijective, it is enough to see that α is an isomorphism in $\pi^r \mathcal{C}_f$, where we denote by π^r the right avatar of π^l. Yoneda imbedding lemma reduces the problem to see that $\alpha^* : \pi^r(Z, A) \longrightarrow \pi^r(Y, A)$ is bijective for each Quillen-fibrant object A. We apply the dual of Lemma 7 of [Q]. Indeed, the map α^* is bijective, by the dual of Lemma 7 of [Q], since α is a trivial cofibration.

Finally, by Quillen axiom M2, for each Quillen-fibrant object X there exist a trivial fibration $M \longrightarrow X$, where M is Quillen-cofibrant, and moreover M is Quillen fibrant, by M3. $\square$

Remark 4.1.3. Observe that in a Quillen model category $\mathcal{C}$ the definition of Quillen cofibrant objects is not homotopy invariant, while the subcategory of cofibrant objects of $\mathcal{C}_f$ is stable by homotopy equivalences. In fact, the cofibrant objects are those homotopy equivalent to Quillen cofibrant objects.

For instance, let $\mathcal{A}$ be an abelian category with enough projectives and $\mathbf{C}_+(\mathcal{A})$ the category of bounded below chain complexes. It is well known (see [Q], Chapter I) that taking quasi-isomorphisms as weak equivalences, epimorphisms as fibrations, and monomorphisms whose cokernel is a degree-wise projective complex as cofibrations, $\mathbf{C}_+(\mathcal{A})$ is a Quillen model category with all objects fibrant. A contractible complex is cofibrant, but it is not Quillen cofibrant unless it is projective (see also [C]).

4.2. **Sullivan minimal models.** In some Cartan-Eilenberg categories there is a distinguished subcategory $\mathcal{M}$ of $\mathcal{C}_{cof}$ which serves as a subcategory of cofibrant left models. A typical situation is that of Sullivan minimal models ([S]). Let us give an abstract version.

Definition 4.2.1. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. We say that a cofibrant object M of $\mathcal{C}$ is *minimal* if

$$End_{\mathcal{C}}(M) \cap \mathcal{W} = Aut_{\mathcal{C}}(M),$$

that is, if any weak equivalence $w : M \longrightarrow M$ of $\mathcal{C}$ is an isomorphism.

We denote by $\mathcal{C}_{min}$ the full subcategory of $\mathcal{C}$ whose objects are minimal in $(\mathcal{C}, \mathcal{S}, \mathcal{W})$.

Definition 4.2.2. We say that $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a *left Sullivan category* if there are enough minimal left models.

Remark 4.2.3. Observe that by the uniqueness property of the extension in Definition 2.2.1, any cofibrant object of $\mathcal{C}$ is minimal in the localised category $(\mathcal{C}[\mathcal{S}^{-1}], \delta(\mathcal{W}))$.

Remark 4.2.4. As a consequence of the definition, a left Sullivan category is a special kind of a left Cartan-Eilenberg category, one for which the canonical functor

$$\mathcal{C}_{min}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$$

is an equivalence of categories. Observe that by definition, if X is a minimal object and $s : X \longrightarrow X$ is in $\mathcal{S}$, then s is an isomorphism, hence $\mathcal{C}_{min}[\mathcal{S}^{-1}] = \mathcal{C}_{min}$, so that in this case

the inclusion functor $\mathcal{C}_{min}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}_{min}[\mathcal{S}^{-1}, \mathcal{C}]$ is not, generally speaking, an equivalence of categories.

4.2.5. An example of a Sullivan category is provided by the original Sullivan's minimal cdg algebras. Let $\mathbf{k}$ be a field of characteristic zero, and $\mathbf{Adgc(k)}_1$ the category of connected and simply connected commutative differential graded $\mathbf{k}$-algebras; that is, cdg algebras A such that $H^0(A) = \mathbf{k}$ and $H^1(A) = 0$ (1-connected $\mathbf{k}$-cdg algebras for short).

A path object for a $\mathbf{k}$-cdg algebra B is the tensor product $\mathrm{Path}(B) := B \otimes \mathbf{k}[t, dt]$, together with the morphisms $\delta_0, \delta_1 : \mathrm{Path}(B) \longrightarrow B$, and $p : B \longrightarrow \mathrm{Path}(B)$ defined by $\delta_i(a(t)) = a(i)$ for $i = 0, 1$, and $p(a) = a \otimes 1$.

Let $f_0, f_1 : A \longrightarrow B$ be two morphisms of $\mathbf{k}$-cdg algebras. A right homotopy from f_0 to f_1 is a morphism of $\mathbf{k}$-cdg algebras, $H : A \longrightarrow \mathrm{Path}(B)$ such that $\delta_i H = f_i, i = 0, 1$ (see [S] or [GM], (10.1)).

Let $\sim$ be the equivalence relation transitively generated by the right homotopy. It follows from the functoriality of the path object that $\sim$ is a congruence. Let $\mathcal{S}$ be the class of homotopy equivalences with respect to $\sim$.

Lemma 4.2.6. *The equivalence relation $\sim$ is compatible with $\mathcal{S}$.*

Proof. Because of Example 1.3.4, it is enough to see that $p : B \longrightarrow \mathrm{Path}(B)$ is in $\mathcal{S}$ and this follows from the fact that $\delta_0 p = \mathrm{id}_B$ and $H : \mathrm{Path}(B) = B \otimes \mathbf{k}[t, dt] \longrightarrow \mathrm{Path}(\mathrm{Path}(B)) = (B \otimes \mathbf{k}[t, dt]) \otimes \mathbf{k}[u, du]$ defined by $H(a(t)) = a(tu)$ is a right homotopy from $p\delta_0$ to $Id_{\mathrm{Path}(B)}$. $\square$

So, by Proposition 1.3.3, there is an isomorphism of categories

$$\mathbf{Adgc(k)}_1 / \sim \, \cong \, \mathbf{Adgc(k)}_1[\mathcal{S}^{-1}].$$

Let $\mathcal{W}$ be the class of quasi-isomorphisms of $\mathbf{Adgc(k)}_1$; that is, those morphisms inducing isomorphisms in cohomology. Since $p : B \longrightarrow \mathrm{Path}(B)$ is a quasi-isomorphism, we have that $\mathcal{S} \subset \overline{\mathcal{W}}$. So $(\mathbf{Adgc(k)}_1, \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences.

Recall that a $\mathbf{k}$-cdg algebra A is a 1-connected Sullivan minimal $\mathbf{k}$-cdg algebra if it is a free graded commutative $\mathbf{k}$-algebra $A = \Lambda(V)$ such that $A^0 = \mathbf{k}$, $A^1 = 0$, and $dA^+ \subset A^+ \cdot A^+$, where $A^+ = \oplus_{i>0} A^i$ ([S], see also [GM], p. 112). Let $\mathcal{M}_S$ be the full subcateogory of 1-connected Sullivan minimal $\mathbf{k}$-cdg algebras. We can sum up Sullivan's results on minimal models in the following theorem.

Theorem 4.2.7. $(\mathbf{Adgc(k)}_1, \mathcal{S}, \mathcal{W})$ *is a left Sullivan category and $\mathcal{M}_S$ is the subcategory of minimal objects of $\mathbf{Adgc(k)}_1$.*

Proof. First of all, let us check the hypotheses of Theorem 2.3.4 for the class $\mathcal{M}_S$ of Sullivan minimal 1-connected algebras. Let M be a 1-connected Sullivan minimal $\mathbf{k}$-cdg algebra. If $A \longrightarrow B$ is a quasi-isomorphism, the induced map $[M, A] \longrightarrow [M, B]$ between the sets of homotopy classes of morphisms is bijective, by [GM] Theorem 10.8. So M is a cofibrant object. In addition, by [GM] Theorem 9.5, any 1-connected $\mathbf{k}$-cdg algebra has a Sullivan minimal model, so, by Theorem 2.3.4, $\mathcal{M}$ is a subcategory of left cofibrant models of $\mathbf{Adgc(k)}_1$.

By [GM] Lemma 10.10, any quasi-isomorphism $M \longrightarrow M$ of a Sullivan minimal algebra is an isomorphism, so M is a minimal object in $(\mathbf{Adgc}(\mathbf{k})_1, \mathcal{S}, \mathcal{W})$, therefore $(\mathbf{Adgc}(\mathbf{k})_1, \mathcal{S}, \mathcal{W})$ is a left Sullivan category.

Reciprocally, every minimal object of $\mathbf{Adgc}(\mathbf{k})_1$ is isomorphic to a Sullivan minimal 1-connected algebra. Let M be a minimal object of $\mathbf{Adgc}(\mathbf{k})_1$. Because of [GM] Theorem 9.5, there is a Sullivan minimal model $\omega : M_S \longrightarrow M \in \mathcal{W}$. Since M is a cofibrant object, we have a bijection $\omega_* : [M, M_S] \longrightarrow [M, M]$. Let $\phi : M \longrightarrow M_S$ be such that $\omega\phi \sim \mathrm{id}_M$. Then $H(\omega\phi) = \mathrm{id}_{HM}$ and so $\omega\phi$ is an isomorphism, because M is a minimal object. Also because of the *2 out of 3* property of quasi-isomorphisms, $\phi \in \mathcal{W}$. So again we find $\psi : M_S \longrightarrow M$ such that $\phi\psi \sim \mathrm{id}_{M_S}$. It follows that $\psi \sim \omega$, which also implies that $\phi\omega \sim \mathrm{id}_{M_S}$. So, $\phi\omega$ is an isomorphism too. Hence so is ω. $\qquad\square$

4.2.8. Analogously, there are enough minimal objects in the category $\mathbf{Op}(\mathbf{k})_1$ of dg operads over $\mathbf{k}$, P, such that $H^*P(1) = 0$, (see [MSS]). From Theorem 2.3.4 again it follows that $\mathbf{Op}(\mathbf{k})_1$ is a left Sullivan category.

We next consider in greater detail the case of dg modular operads over a field of characteristic zero $\mathbf{k}$ (refer to [GK] and [GNPR1] for the notions concerning modular operads that will be used).

Let $\mathbf{MOp}(\mathbf{k})$ be the category of dg modular operads. We have an analogous path object for modular operads: if P is a dg modular operad, its path object is the tensor product $\mathrm{Path}\,(P) = P \otimes \mathbf{k}[t, \delta t]$. Let $\sim$ be the equivalence relation transitively generated by the right homotopy defined with this path object. We can see, as in Lemma 4.2.6, that the class of homotopy equivalences $\mathcal{S}$ with respect to $\sim$ is compatible with $\sim$, so we have an isomorphism of categories $\mathbf{MOp}(\mathbf{k})/\sim\, \cong \mathbf{MOp}(\mathbf{k})[\mathcal{S}^{-1}]$.

Let $\mathcal{W}$ be the class of quasi-isomorphisms of $\mathbf{MOp}(\mathbf{k})$. We see in the same way as for $\mathbf{Adgc}(\mathbf{k})_1$ that $(\mathbf{MOp}(\mathbf{k}), \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences.

In [GNPR1], Definition 8.6.1, we defined minimal modular operads as modular operads obtained from the trivial operad 0 by a sequence of principal extensions. Let $\mathcal{M}$ be the full subcateogory of minimal modular operads.

Theorem 4.2.9. $(\mathbf{MOp}(\mathbf{k}), \mathcal{S}, \mathcal{W})$ *is a left Sullivan category and* $\mathcal{M}$ *is the subcategory of minimal objects of* $\mathbf{MOp}(\mathbf{k})$.

Proof. Let us check the hypothesis of Theorem 2.3.4: if M is a minimal modular operad and $P \longrightarrow Q$ a quasi-isomorphism of $\mathbf{MOp}(\mathbf{k})$, the induced map $[M, P] \longrightarrow [M, Q]$ is a bijection by [GNPR1], Theorem 8.7.2. So M is a cofibrant object. The existence of enough cofibrant objects is guaranteed by Theorem 8.6.3. op.cit., and these minimal modular operads are minimal objects because of op.cit., Proposition 8.6.2.

We can argue as in the proof of Theorem 4.2.7 to show that every minimal object of $\mathbf{MOp}(\mathbf{k})$ is isomorphic to an object of $\mathcal{M}$. $\qquad\square$

5. Cartan-Eilenberg categories defined by a cotriple

In section 3 we have proved, under suitable hypotheses, that some subcategories of the functor category $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ are Cartan-Eilenberg categories, and as a consequence we saw that the derived functor of an additive functor K is a cofibrant model of K. In this section we prove that the whole category $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ is a Cartan-Eilenberg category if $\mathcal{C}$ has a cotriple and $\mathcal{D}$ is a category of chain complexes. The cofibrant model of a functor K with respect to this structure is the non-additive derived functor of K as introduced by Barr-Beck ([BB1]).

5.1. **Categories of chain complexes and cotriples.** Let $\mathcal{A}$ be an additive category and denote by $\mathbf{C}_{\geq 0}(\mathcal{A})$ the category of non-negative chain complexes of $\mathcal{A}$. In this section we will consider as strong equivalences in $\mathbf{C}_{\geq 0}(\mathcal{A})$ classes of summable morphisms as introduced in the following definition.

Definition 5.1.1. Let $\mathcal{A}$ be an additive category. A class $\mathcal{S}$ of morphisms of $\mathbf{C}_{\geq 0}(\mathcal{A})$ is called a class of *summable* morphisms if it satisfies the following properties.

(i) $\mathcal{S}$ is saturated.
(ii) The homotopy equivalences are in $\mathcal{S}$.
(iii) Let $f : C_{**} \longrightarrow D_{**}$ be a morphism of first quadrant double complexes. If $f_n : C_{*n} \longrightarrow D_{*n}$ is in $\mathcal{S}$ for all $n \geq 0$, then $\mathrm{Tot} f : \mathrm{Tot} C_{**} \longrightarrow \mathrm{Tot} D_{**}$ is in $\mathcal{S}$.

For example, the class of homotopy equivalences, which will be denoted by $\mathcal{S}_h$, is a class of summable morphisms. Also, if $\mathcal{A}$ is an abelian category, the class of quasi-isomorphisms is a class of summable morphisms (cf. [B], Chap. 5).

5.1.2. Let $\mathcal{A}$ be an additive category, and let

$$\mathbf{G} = (G : \mathcal{A} \to \mathcal{A}, \, \varepsilon : G \Rightarrow \mathrm{id}_{\mathcal{A}}, \, \delta : G \Rightarrow G^2)$$

be a cotriple on $\mathcal{A}$.

We recall that the cotriple $\mathbf{G}$ is called *additive* if the functor G is additive, in such case, it induces an additive cotriple on $\mathbf{C}_{\geq 0}(\mathcal{A})$ which we also denote by $\mathbf{G}$.

Let $\mathcal{S}$ be a class of summable morphisms of $\mathbf{C}_{\geq 0}(\mathcal{A})$, and $\mathbf{G}$ an additive cotriple on $\mathcal{A}$. We say that $\mathbf{G}$ and $\mathcal{S}$ are *compatible* if the extension of G to the category of complexes $G : \mathbf{C}_{\geq 0}(\mathcal{A}) \longrightarrow \mathbf{C}_{\geq 0}(\mathcal{A})$ satisfies $G(\mathcal{S}) \subset \mathcal{S}$. In this case, taking $\mathcal{W} = G^{-1}(\mathcal{S})$, $(\mathbf{C}_{\geq 0}(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences.

For example, the class of homotopy equivalences $\mathcal{S}_h$ in $\mathbf{C}_{\geq 0}(\mathcal{A})$ is compatible with any additive cotriple $\mathbf{G}$ on $\mathcal{A}$, thus, taking $\mathcal{W}_h = G^{-1}(\mathcal{S}_h)$, $(\mathbf{C}_{\geq 0}(\mathcal{A}), \mathcal{S}_h, \mathcal{W}_h)$ is a category with strong and weak equivalences.

5.1.3. Let $\mathbf{G} = (G, \varepsilon, \delta)$ be an additive cotriple defined on the category $\mathcal{A}$, and by extension on $\mathbf{C}_{\geq 0}(\mathcal{A})$.

The simplicial standard construction associated to the cotriple $\mathbf{G}$ on $\mathbf{C}_{\geq 0}(\mathcal{A})$ defines, for each object K in $\mathbf{C}_{\geq 0}(\mathcal{A})$, an augmented simplicial object $\varepsilon : B_{\bullet}(K) \longrightarrow K$ in $\mathbf{C}_{\geq 0}(\mathcal{A})$ such that $B_n(K) = G^{n+1}(K)$, ([G], App., see also [ML]). Hence, there is a naturally defined double

complex $B_*(K)$ associated to $B_\bullet(K)$, with total complex $B(K) = \mathrm{Tot}B_*(K)$. This construction defines a functor

$$B : \mathbf{C}_{\geq 0}(\mathcal{A}) \longrightarrow \mathbf{C}_{\geq 0}(\mathcal{A}),$$

with a natural transformation $\varepsilon : B \Rightarrow \mathrm{id}$.

Theorem 5.1.4. *Let $\mathcal{A}$ be an additive category, $\mathbf{G}$ an additive cotriple on $\mathcal{A}$, and $\mathcal{S}$ a class of summable morphisms in $\mathbf{C}_{\geq 0}(\mathcal{A})$ compatible with $\mathbf{G}$. Then, with the previous notation,*

(1) (B, ε) is a left resolvent functor for $(\mathbf{C}_{\geq 0}(\mathcal{A}), \mathcal{S}, \mathcal{W})$;
(2) $(\mathbf{C}_{\geq 0}(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category; and
(3) an object K of $\mathbf{C}_{\geq 0}(\mathcal{A})$ is cofibrant if and only if $\varepsilon_K : B(K) \longrightarrow K$ is in $\mathcal{S}$.

Proof. Let us verify the hypotheses of Theorem 2.5.4. Firstly, if $s \in \mathcal{S}$, then $G(s) \in \mathcal{S}$ by hypothesis, and it follows, inductively, that $G^i(s) \in \mathcal{S}$ for any $i \geq 0$. By Definition 5.1.1(iii), we deduce that $B(s) = \mathrm{Tot}B_*(s) \in \mathcal{S}$. Therefore $B(\mathcal{S}) \subset \mathcal{S}$.

Next, let K be a chain complex of $\mathcal{A}$. For any $i > 0$, the augmented simplicial objects $\varepsilon_{G^iK} : B_\bullet G^i K \longrightarrow G^i K$ and $G^i(\varepsilon_K) : G^i B_\bullet K \longrightarrow G^i K$ have a contraction induced by the morphism $\delta : G \longrightarrow G^2$. Therefore, by 5.1.1(ii), (iii) and the additivity of G, the induced morphisms between chain complexes

$$\varepsilon_{G^iK} : BG^i K \longrightarrow G^i K \quad \text{and} \quad G^i(\varepsilon_K) : G^i\mathrm{Tot}B_*K \cong \mathrm{Tot}G^i B_*K \longrightarrow G^i K$$

are in $\mathcal{S}$, for each $i > 0$. Applying again 5.1.1(iii) we obtain that $B(\varepsilon_K)$ and ε_{BK} are in $\mathcal{S}$. Therefore, (B, ε) is a left resolvent functor for $(\mathbf{C}_{\geq 0}(\mathcal{A}), \mathcal{S}, B^{-1}(\mathcal{S}))$, by Theorem 2.5.4.

Finally, let us check that $\mathcal{W} = B^{-1}(\mathcal{S})$, that is $G^{-1}(\mathcal{S}) = B^{-1}(\mathcal{S})$. Indeed, let $w : K \longrightarrow L$ be a morphism of chain complexes. If $w \in \mathcal{W} = G^{-1}(\mathcal{S})$, we have $G^i(w) \in \mathcal{S}$ for each $i > 0$, therefore, applying once again 5.1.1(iii), we obtain $B(w) \in \mathcal{S}$. Conversely, if $B(w) \in \mathcal{S}$, since $BGw = GBw$, we have $BGw \in \mathcal{S}$, and from the commutativity of the diagram

$$
\begin{array}{ccc}
BGK & \xrightarrow{BGw} & BGL \\
{\scriptstyle \varepsilon_{GK}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{GL}} \\
GK & \xrightarrow{Gw} & GL
\end{array}
$$

it follows that $Gw \in \mathcal{S}$, because $\varepsilon_{GK}, \varepsilon_{GL} \in \mathcal{S}$, by 5.1.1 (ii), and $\mathcal{S}$ is saturated, by 5.1.1 (i). Hence $w \in \mathcal{W}$. $\qquad\square$

In order to recognise cofibrant objects in $(\mathbf{C}_{\geq 0}(\mathcal{A}), \mathcal{S}, \mathcal{W})$ the following criterion will be useful.

Proposition 5.1.5. *Let $\mathcal{A}$ be an additive category, $\mathbf{G}$ an additive cotriple on $\mathcal{A}$, and $\mathcal{S}$ a class of summable morphisms in $\mathbf{C}_{\geq 0}(\mathcal{A})$ compatible with $\mathbf{G}$. Then,*

(1) for each object K of $\mathbf{C}_{\geq 0}(\mathcal{A})$, GK is cofibrant;
(2) if K is an object of $\mathbf{C}_{\geq 0}(\mathcal{A})$ such that K_n is cofibrant for each $n \geq 0$, then K is cofibrant (in [B] one such complex is called ε-presentable); and

(3) if K is an object of $\mathbf{C}_{\geq 0}(\mathcal{A})$ such that $\varepsilon_{K_n} : G(K_n) \longrightarrow K_n$ has a section, that is to say, there are morphisms $\theta_n : K_n \longrightarrow G(K_n)$ such that $\varepsilon_{K_n}\theta_n = \mathrm{id}_{K_n}$, for $n \geq 0$, then, K is cofibrant (in [BB1] one such complex is called G-representable).

Proof. (1) The augmented simplicial complex $\varepsilon_{GK} : B_\bullet GK \longrightarrow GK$ is contractible, because the morphism $\delta_K : GK \longrightarrow G^2K$ induces a contraction. Hence, by 5.1.1(ii), $\varepsilon_{GK} \in \mathcal{S}$, so GK is cofibrant, by Theorem 5.1.4(3).

(2) Suppose K_n cofibrant, for each $n \geq 0$. Then $\varepsilon_{K_n} : B(K_n) \longrightarrow K_n \in \mathcal{S}$, by Theorem 5.1.4(3). Therefore $\varepsilon_K : BK \longrightarrow K \in \mathcal{S}$, by 5.1.1(iii), hence K is cofibrant, by Theorem 5.1.4(3) again.

(3) Each $G(K_n)$ is cofibrant, by (1), and K_n is a retract of $G(K_n)$, then, by Proposition 2.2.2, K_n is cofibrant. Hence, by (2), K is cofibrant. $\qquad\square$

5.2. **Functor categories and cotriples.**

5.2.1. Given a category $\mathcal{X}$ and an additive category $\mathcal{A}$, the functor category $\mathbf{Cat}(\mathcal{X}, \mathcal{A})$ is also additive, and we have

$$\mathbf{C}_{\geq 0}\mathbf{Cat}(\mathcal{X}, \mathcal{A}) \cong \mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A})) \ ,$$

so, from Theorem 5.1.4, taking $\mathcal{W} = G^{-1}(\mathcal{S})$ as above, we obtain the following result.

Theorem 5.2.2. *Let $\mathcal{X}$ be a category and $\mathcal{A}$ an additive category. Let $\mathbf{G}$ be an additive cotriple on $\mathbf{Cat}(\mathcal{X}, \mathcal{A})$, and $\mathcal{S}$ a class of summable morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ compatible with $\mathbf{G}$. Then,*

(1) (B, ε) is a left resolvent functor for $(\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A})), \mathcal{S}, \mathcal{W})$;
(2) $(\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A})), \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category; and
(3) an object K of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ is cofibrant if and only if $\varepsilon_K : BK \longrightarrow K$ is in $\mathcal{S}$.

5.2.3. In the category $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ we will consider as classes of summable morphisms, besides the one of natural homotopy equivalences $\mathcal{S}_h$, point-wise defined classes. Take Σ a class of summable morphisms in $\mathbf{C}_{\geq 0}(\mathcal{A})$ and define a class of morphisms $\mathcal{S}_\Sigma$ of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ by

$$\mathcal{S}_\Sigma = \{ \ f \ ; \ \ f(X) \in \Sigma, \ \ \forall X \in Ob\mathcal{X} \ \}.$$

Then $\mathcal{S}_\Sigma$ is a class of summable morphisms. We shall say that $\mathcal{S}_\Sigma$ is the class of summable morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ defined *point-wise* from Σ.

For example, if Σ is the class of homotopy equivalences in $\mathbf{C}_{\geq 0}(\mathcal{A})$, we say that $\mathcal{S}_\Sigma$ is the class of *point-wise homotopy equivalences*, and we denote it by $\mathcal{S}_{ph}$. Observe that in contrast to the case of natural homotopy equivalences $\mathcal{S}_h$ in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$, the point-wise homotopy equivalences have homotopy inverses over each object X of $\mathcal{X}$, but these homotopy inverses are not required to be natural. So, generally speaking, the inclusion $\mathcal{S}_h \subset \mathcal{S}_{ph}$ is strict.

5.2.4. If $\mathbf{G}$ is a cotriple in $\mathcal{X}$, it naturally defines an additive cotriple on the functor category $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ by sending K to $K \circ G$, with the evident extensions of the transformations ε, δ. We also denote this cotriple by $\mathbf{G}$.

If $\mathcal{S}$ is a class of point-wise defined morphisms , then $\mathcal{S}$ is compatible with each cotriple G on $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ induced by a cotriple on $\mathcal{X}$.

5.2.5. For instance, let $\mathcal{X}$ be a category with arbitrary coproducts. We recall that, associated to each set $\mathcal{M}$ of objets of $\mathcal{X}$ (called *models*), there is defined a model-induced cotriple $\mathbf{G}$ on $\mathcal{X}$ (see for example [BB2], (10.1)). The functor G is given by the formula

$$G(X) = \bigsqcup_{f:M\to X, M\in\mathcal{M}} M_f,$$

where M_f denotes a copy of M indexed by f. Denote by $\langle f\rangle : M \longrightarrow G(X)$ the canonical inclusion into the sum corresponding to the summand M_f. If $a : X \longrightarrow Y$ is a morphism, $G(a) : G(X) \longrightarrow G(Y)$ is defined in such a way that $G(a) \circ \langle f\rangle = \langle af\rangle$, for each $f : M \longrightarrow X$. The counit $\varepsilon : G \Rightarrow id$ is defined by $\varepsilon_X \circ \langle f\rangle = f$, and comultiplication $\delta : G \Rightarrow G^2$, is defined by $\delta_X \circ \langle f\rangle = \langle\langle f\rangle\rangle$.

5.2.6. In the same way, for a general category $\mathcal{X}$ with a set $\mathcal{M}$ of objects, if the additive category $\mathcal{A}$ has arbitrary sums, there is a variant of the model-induced cotriple given as follows. The cotriple $\mathbf{G}$ in $\mathbf{Cat}(\mathcal{X}, \mathcal{A})$ is defined by

$$(GK)(X) = \bigoplus_{f:M\to X, M\in\mathcal{M}} K(M_f),$$

with counit $\varepsilon : G \Rightarrow \mathrm{id}$ defined by $\varepsilon_{K,X} \circ \langle f\rangle = K(f)$, and comultiplication $\delta : G \Rightarrow G^2$, defined by $\delta_{K,X} \circ \langle f\rangle = \langle\langle f\rangle\rangle$. This cotriple is additive.

Remark 5.2.7. In the original formulation of the Beck homology (see [BB2]), one considers

  (a) a cotriple $\mathbf{G}$ defined on the category $\mathcal{X}$,
  (b) an abelian category $\mathcal{A}$,
  (c) a class of acyclic morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$, and
  (d) a functor $F : \mathcal{X} \longrightarrow \mathcal{A}$.

Then, the Beck homology of X with coefficients in F is defined as $H_*(X, F)_{\mathbf{G}} = H_*((BF)(X))$, that is, the homology of the cofibrant model of F.

Example 5.2.8. Barr-Beck proved that the singular homology functors with integer coefficients $H_* = \{H_n\}_{n=0,1,\dots}$ is the Beck homology with coefficients in the 0-th singular homology functor H_0. We give a version of this result at the chain level: we prove that the functor of singular chains S_* is a cofibrant model for the functor H_0 in the category of chain complex valued functors on topological spaces with a convenient Cartan-Eilenberg structure.

Let $\mathcal{X} = \mathbf{Top}$ be the category of topological spaces and consider the cotriple $\mathbf{G}$ on $\mathbf{Top}$ defined by the set $\{\Delta^n; n \in \mathbb{N}\}$,

$$G(X) = \bigsqcup_{(\Delta^n, \ \sigma)\in\mathbf{Top}/X} \Delta^n_\sigma.$$

We consider the cotriple induced by $\mathbf{G}$ on the category $\mathbf{Cat}(\mathbf{Top}, \mathbf{C}_{\geq 0}(\mathbb{Z}))$.

Take $\mathcal{S}_h$ the class of natural homotopy equivalences in $\mathbf{Cat}(\mathbf{Top}, \mathbf{C}_{\geq 0}(\mathbb{Z}))$ and $\mathcal{W}_h = G^{-1}(\mathcal{S}_h)$. From Theorem 5.2.2 we obtain that $(\mathbf{Cat}(\mathbf{Top}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_h, \mathcal{W}_h)$ is a left Cartan-Eilenberg category.

Let $S_* : \mathbf{Top} \longrightarrow \mathbf{C}_{\geq 0}(\mathbb{Z})$ be the functor of singular chains with integer coefficients, and $\tau : S_* \longrightarrow H_0$ the natural augmentation.

Let us see that S_* is cofibrant. Let $\theta_n : S_n \longrightarrow S_n \circ G$ be the natural transformation which, for each topological space X, sends a singular simplex $\sigma : \Delta^n \longrightarrow X$ to $\theta_n(\sigma) = \langle \sigma \rangle$. It is clear that $\varepsilon_{S_n} \theta_n = \mathrm{id}_{S_n}$, so S_* is cofibrant, by Proposition 5.1.5(iii).

On the other hand, the morphism $\tau : S_* \longrightarrow H_0(-, \mathbb{Z})$ is in $\mathcal{W}_h$. In fact, for each $n \geq 0$, take a homotopy inverse of τ_{Δ^n}, $\lambda_n : H_0(\Delta^n, \mathbb{Z}) \longrightarrow S_*(\Delta^n)$. Then, for each topological space X,

$$\lambda_X := \bigoplus_{(\Delta^n,\ \sigma) \in \mathbf{Top}/X} (\lambda_n, \sigma) : H_0(GX, \mathbb{Z}) \longrightarrow S_*(GX)$$

defines a natural morphism $\lambda : H_0 \circ G \longrightarrow S_* \circ G$ which is a homotopy inverse of $G(\tau)$.

Hence, S_* is a cofibrant model for $H_0(-, \mathbb{Z})$ in $(\mathbf{Cat}(\mathbf{Top}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_h, \mathcal{W}_h)$.

Notice that, if $\mathcal{S}$ denotes the homotopy equivalences and $\mathcal{W}$ the weak homotopy equivalences in $\mathbf{Top}$, then $(\mathbf{Top}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category. If we consider in $\mathbf{C}_{\geq 0}(\mathbb{Z})$ the class $\mathcal{E}$ of the quasi-isomorphisms, the category of functors $\mathbf{Cat}_{\mathcal{S},\mathcal{E}}(\mathbf{Top}, \mathbf{C}_{\geq 0}(\mathbb{Z}))$ (see 3.3.1 for the notation) has a structure of left Cartan-Eilenberg category for which the functors S_* and H_0 are cofibrant objects, but with this Cartan-Eilenberg structure, the morphism $S_* \longrightarrow H_0$ is not a weak equivalence.

Example 5.2.9. The next example is a variation for differentiable manifolds of the previous one.

Let $\mathcal{X} = \mathbf{Diff}$ be the category of differentiable manifolds with corners. Consider the additive cotriple $\mathbf{G}^\infty$ defined on $\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z}))$ by the set $\{\Delta^n; n \in \mathbb{N}\}$,

$$G^\infty(K)(X) = \bigoplus_{(\Delta^n,\ \sigma) \in \mathbf{Diff}/X} K(\Delta^n, \sigma).$$

By Theorem 5.2.2, $(\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_h, \mathcal{W}_h)$ and $(\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_{ph}, \mathcal{W}_{ph})$ are left Cartan-Eilenberg categories.

Denote by $S_*^\infty : \mathbf{Diff} \longrightarrow \mathbf{C}_{\geq 0}(\mathbb{Z})$ the functor of *differentiable* singular chains. Reasoning as in the topological case, it follows that S_*^∞ is a cofibrant model of $H_0(-, \mathbb{Z})$ in the left Cartan-Eilenberg category $(\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_h, \mathcal{W}_h)$ and also in $(\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_{ph}, \mathcal{W}_{ph})$.

These two previous examples permit us to give an interpretation of a well-known theorem of Eilenberg for the singular complex of a differentiable manifold (see [E] and its extension to differentiable manifolds with corners in [Hu]). By Eilenberg's theorem the natural transformation $S_*^\infty \longrightarrow S_*$ is a point-wise homotopy equivalence in $\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z}))$, hence S_* is a cofibrant model of $H_0(-, \mathbb{Z})$ in $(\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_{ph}, \mathcal{W}_{ph})$. However, S_*^∞ and S_* are not naturally homotopy equivalent functors in $\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z}))$ (see [GNPR3]), so S_* is not a cofibrant model of $H_0(-, \mathbb{Z})$ in $(\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_h, \mathcal{W}_h)$.

Observe that the Cartan-Eilenberg category $(\mathbf{Cat}(\mathbf{Diff}, \mathbf{C}_{\geq 0}(\mathbb{Z})), \mathcal{S}_{ph}, \mathcal{W}_{ph})$ does not come from a Quillen model category, since the morphisms in the class $\mathcal{S}_{ph}$ do not have, in general, a homotopic inverse.

5.3. **Acyclic models.** If, in Theorem 5.2.2, the cotriple $\mathbf{G}$ is induced by a cotriple on $\mathcal{X}$, we can prove that the natural transformations from a cofibrant functor K to any other functor L are determined by its restriction to the "models" $G(X)$, with $X \in Ob\mathcal{X}$, as stated in the following theorem 5.3.2.

5.3.1. Let $\mathcal{X}$ be a category with a cotriple $\mathbf{G}$, let $\mathcal{A}$ be an additive category, and $\mathcal{S}$ a class of summable morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ compatible with the cotriple induced by $\mathbf{G}$. Let $\mathcal{W} = G^{-1}(\mathcal{S})$ and we consider the Cartan-Eilenberg structure in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ given by theorem 5.2.2.

Denote by $\mathcal{M}$ the full subcategory of $\mathcal{X}$ with objects GX, for $X \in Ob\mathcal{X}$ and by

$$\rho : \mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A})) \longrightarrow \mathbf{Cat}(\mathcal{M}, \mathbf{C}_{\geq 0}(\mathcal{A}))$$

the restriction functor, $\rho(K) = K_{|\mathcal{M}}$.

Since G sends objects in $\mathcal{M}$ to $\mathcal{M}$, $\mathbf{G}$ induces a cotriple on $\mathcal{M}$, and a functor $B_{\mathcal{M}}$ such that $\rho \circ B = B_{\mathcal{M}} \circ \rho$.

Since $B_{\bullet} : \mathcal{X} \longrightarrow \Delta^{op}\mathcal{X}$ factors through the inclusion $\Delta^{op}\mathcal{M} \longrightarrow \Delta^{op}\mathcal{X}$, the functor

$$B : \mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A})) \longrightarrow \mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A})), \quad BK = Tot \circ \Delta^{op}K \circ B_{\bullet} \,,$$

where $\Delta^{op}K : \Delta^{op}\mathcal{X} \longrightarrow \Delta^{op}\mathbf{C}_{\geq}(\mathcal{A})$ is the functor K applied degree-wise, $\Delta^{op}K(X_{\bullet}) = K \circ X_{\bullet}$, factors through ρ, that is, if

$$B' : \mathbf{Cat}(\mathcal{M}, \mathbf{C}_{\geq 0}(\mathcal{A})) \longrightarrow \mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$$

is defined by

$$B'K = Tot \circ \Delta^{op}K \circ B_{\bullet},$$

then $B = B' \circ \rho$. In addition, $\rho \circ B' = B_{\mathcal{M}}$.

We say that a class $\mathcal{S}_{\mathcal{M}}$ of morphisms in $\mathbf{Cat}(\mathcal{M}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ is *adapted* to $(\mathbf{G}, \mathcal{S})$ if $\rho(\mathcal{S}) \subset \mathcal{S}_{\mathcal{M}}$ and $B'(\mathcal{S}_{\mathcal{M}}) \subset \mathcal{S}$. In that case, the restriction ρ induces a functor

$$\overline{\rho} : \mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}^{-1}] \longrightarrow \mathbf{Cat}(\mathcal{M}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}_{\mathcal{M}}^{-1}],$$

and the functor B' induces a functor

$$\beta' : \mathbf{Cat}(\mathcal{M}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}_{\mathcal{M}}^{-1}] \longrightarrow \mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}^{-1}],$$

such that $\overline{\rho} \circ \beta' = \beta_{\mathcal{M}}$, and $\beta' \circ \overline{\rho} = \beta$, where β and $\beta_{\mathcal{M}}$ denote the functors induced by B and $B_{\mathcal{M}}$, respectively.

If there exists a class $\mathcal{S}_{\mathcal{M}}$ adapted to $(\mathbf{G}, \mathcal{S})$ we say that $\mathcal{S}$ is *adaptable* to $\mathbf{G}$.

For example, if $\mathcal{S}$ is the class of homotopy equivalences, then $\mathcal{S}$ is adaptable to any cotriple $\mathbf{G}$ on $\mathcal{X}$, since it is enough to take $\mathcal{S}_{\mathcal{M}}$ as the class of homotopy equivalences. On the other hand, if $\mathcal{S}$ is defined point-wise by a class Σ, then $\mathcal{S}$ is also adaptable, taking the class $\mathcal{S}_{\mathcal{M}}$ point-wise defined by Σ.

If K, L are objects in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ (resp. $\mathbf{Cat}(\mathcal{M}, \mathbf{C}_{\geq 0}(\mathcal{A}))$) we denote by $[K, L]$ the morphisms from K to L in the category $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}^{-1}]$ (resp. $\mathbf{Cat}(\mathcal{M}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}_{\mathcal{M}}^{-1}]$).

Theorem 5.3.2. *Let $\mathcal{X}$ be a category with a cotriple $\mathbf{G}$, let $\mathcal{A}$ be an additive category, and $\mathcal{S}$ a class of summable morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ compatible with, and adaptable to, the cotriple induced by $\mathbf{G}$. If K is a cofibrant object of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$, the restriction map*

$$\overline{\rho}_{KL} : [K, L] \longrightarrow [K_{|\mathcal{M}}, L_{|\mathcal{M}}]$$

is bijective, for each L.

Proof. The diagram

is commutative, since $\beta' \circ \overline{\rho} = \beta$ and $\overline{\rho} \circ \beta' = \beta_M$.

By the naturality of $\varepsilon : B \Rightarrow \mathrm{id}$, the following diagram

is commutative. Since BK is cofibrant and ε_L is a weak equivalence, the map ε_{L*} is bijective. Since K is cofibrant, ε_K is a strong equivalence, so ε_K^* is also bijective, hence β_{KL} is bijective. In particular, β'_{KL} is surjective.

On the other hand, $\varepsilon_{K_{|\mathcal{M}}} = \rho(\varepsilon_K) : BK_{|\mathcal{M}} \longrightarrow K_{|\mathcal{M}}$ is in $\mathcal{S}_{\mathcal{M}}$, since $\rho(\mathcal{S}) \subset \mathcal{S}_{\mathcal{M}}$, so $(\varepsilon_{K_{|\mathcal{M}}})^*$ is bijective. From $(\varepsilon_{L_{|\mathcal{M}}})_* \circ \beta_{\mathcal{M}, K_{|\mathcal{M}}, L_{|\mathcal{M}}} = (\varepsilon_{K_{|\mathcal{M}}})^*$, we obtain that $\beta_{\mathcal{M}, K_{|\mathcal{M}}, L_{|\mathcal{M}}}$ is injective, so too is β'_{KL}.

Since β'_{KL} and β_{KL} are bijective maps, so too is $\overline{\rho}_{KL} : [K, L] \longrightarrow [K_{|\mathcal{M}}, L_{|\mathcal{M}}]$. $\qquad\square$

Corollary 5.3.3. *Under the hypothesis of the previous theorem, let K, L be cofibrant objects of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$. If $K_{|\mathcal{M}}$ and $L_{|\mathcal{M}}$ are isomorphic in $\mathbf{Cat}(\mathcal{M}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}_{\mathcal{M}}^{-1}]$, then K and L are isomorphic in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}^{-1}]$.*

5.3.4. The Barr-Beck's acyclic models theorem is stated in an abelian context. We introduce the necessary notions in our setting.

A class $\mathcal{S}$ of summable morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ is called *acyclic* if the category $\mathcal{A}$ is abelian and the morphisms in $\mathcal{S}$ are quasi-isomorphisms (see [B], Chap. 5, (1.1) AC-4).

An object K of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ is called *G-acyclic*, where $\mathbf{G}$ is a cotriple on $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ compatible with $\mathcal{S}$, if the augmentation $\tau_K : K \longrightarrow H_0 K$ is a weak equivalence, that is, $\tau_K \circ G \in \mathcal{S}$.

If $\mathcal{S}$ is a class of acyclic morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$, and $\phi : K \longrightarrow L$ is a morphism in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}^{-1}]$, then ϕ defines a morphism $H_*\phi : H_*K \longrightarrow H_*L$. In particular, H_0 defines a functor

$$\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}^{-1}] \longrightarrow \mathbf{Cat}(\mathcal{X}, \mathcal{A}) ,$$

and so, also a map $H_0 : [K, L] \longrightarrow [H_0 K, H_0 L]$, where $[H_0 K, H_0 L]$ is simply the class of morphisms $H_0 K \longrightarrow H_0 L$ in $\mathbf{Cat}(\mathcal{X}, \mathcal{A})$.

Now, we derive a variation of Barr-Beck's acyclic models theorem ([B], Chap. 5, (3.1)) as a consequence of the Cartan-Eilenberg structure of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$.

Theorem 5.3.5. (Acyclic models theorem.) *Let $\mathcal{X}$ be a category with a cotriple $\mathbf{G}$, let $\mathcal{A}$ be an abelian category, and $\mathcal{S}$ a class of acyclic morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ compatible with, and adaptable to, the cotriple induced by $\mathbf{G}$. If K, L are objects of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$ such that K is cofibrant and L is G-acyclic, then the map*

$$H_0 \rho_{_{KL}} : [K, L] \longrightarrow [H_0 K_{|\mathcal{M}}, H_0 L_{|\mathcal{M}}]$$

is bijective.

Proof. The map

$$H_0 \rho_{_{KL}} : [K, L] \longrightarrow [H_0 K_{|\mathcal{M}}, H_0 L_{|\mathcal{M}}]$$

factors as

$$[K, L] \xrightarrow{\tau_L^*} [K, H_0 L] \xrightarrow{\rho} [K_{|\mathcal{M}}, H_0 L_{|\mathcal{M}}] \xrightarrow{H_0} [H_0 K_{|\mathcal{M}}, H_0 L_{|\mathcal{M}}]$$

The map τ_L^* is bijective because K is cofibrant and L is G-acyclic. By Theorem 5.3.2, ρ is also bijective. Finally, the map

$$H_0 : [K_{|\mathcal{M}}, H_0 L_{|\mathcal{M}}] \longrightarrow [H_0 K_{|\mathcal{M}}, H_0 L_{|\mathcal{M}}]$$

is bijective because $K_{|\mathcal{M}}$ is concentrated in non-negative degrees and $H_0 L_{|\mathcal{M}}$ is concentrated in degree 0. $\qquad\square$

Corollary 5.3.6. *Under the hypothesis of the previous theorem, let K, L be cofibrant G-acyclic objects of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))$. If $H_0 K_{|\mathcal{M}}$ and $H_0 L_{|\mathcal{M}}$ are isomorphic, then K and L are isomorphic in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_{\geq 0}(\mathcal{A}))[\mathcal{S}^{-1}]$.*

Remark 5.3.7. In [GNPR2] we have presented some variations of the acyclic models theorem in the monoidal and the symmetric monoidal settings. They can also be deduced from a convenient Cartan-Eilenberg structure.

References

[A]   J.F. Adams, *Localisation and Completion*, Lecture Notes by Z. Fiedorowicz, University of Chicago (1975).

[B]   M. Barr, *Acyclic models*. CRM Monograph Series, vol. **17**. American Mathematical Society, 2002.

[BB1]   M. Barr, J. Beck, *Acyclic models and triples*, Proceedings of the Conference on Categorical Algebra, La Jolla 1965, Springer Verlag (1966).

[BB2]   M. Barr, J. Beck, *Homology and standard constructions*, Lecture Notes in Math., vol **80**, Springer Verlag (1969).

[Ba]   H.J. Baues, *Algebraic homotopy*, Cambridge studies in advanced mathematics, vol. **15**. Cambridge U.P. 1989.

[Bo]   F. Borceux, *Handbook of categorical algebra 1*, Encyclopedia of Mathematics and its applications, vols. **50**, Cambridge U.P. 1994.

[Br]   K. Brown, *Abstract homotopy theory and generalised sheaf cohomology*. Trans. AMS **186** (1974), 419–458.

[CE]   H. Cartan, S. Eilenberg, *Homological Algebra*. Princeton University Press, 1956.

[C]   M. Cole, *Mixing model structures*, Topology Appl., **153**(2006), 1016–1032.

[CCh]   C. Casacuberta, B. Chorny, *The orthogonal subcategory problem in homotopy theory*. An alpine anthology of homotopy theory, 41–53, Contemp. Math., **399**, Amer. Math. Soc., Providence, RI, 2006.

[D]   P. Deligne, *Cohomologie à support propre*, Exp. XVII. SGA 4, (tome 3). Lecture Notes in Math. vol. **305**, 250–461, Springer Verlag, 1973.

[DHKS] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, J.H. Smith, *Homotopy limit functors on model categories and homotopical categories*, AMS Math. Surveys and Monographs **113**, 2004.

[DK]   W.G. Dwyer, D.M. Kan, *Calculating simplicial localizations*. J. of Pure and Appl. Algebra **18** (1980), 17–35.

[DS]   W.G. Dwyer, J. Spalinski, *Homotopy theories and model categories* in *Handbook of algebraic topology*, North-Holland (1995), 73–126.

[E]   S. Eilenberg, *Singular homology in differentiable manifolds*. Ann. of Math. **50**, (1947), 670–681.

[GK]   E. Getzler, M. Kapranov, *Modular operads*, Compositio Math. **110**, (1998), 65–126.

[GM]   P.A. Griffiths, J.W. Morgan, *Rational homotopy theory and differential forms*, Birkhäuser Verlag, 1981.

[GMa]   I. Gelfand, Y. Manin, *Methods of Homological Algebra*. Springer Verlag, 1988.

[G]   R. Godement, *Théorie des faisceaux*. Herman, 1964.

[GNPR1] F. Guillén Santos, V. Navarro, P. Pascual, A. Roig, *Moduli spaces and formal operads*, Duke Math. J. **129**, (2005), 291–335 .

[GNPR2] F. Guillén Santos, V. Navarro, P. Pascual, A. Roig, *Monoidal functors, acyclic models and chain operads*, Canad. J. Math. **60**, (2008), 348–378 .

[GNPR3] F. Guillén, V. Navarro, P. Pascual, Agustí Roig, *The differentiable chain functor is not homotopy equivalent to the continuous chain functor*, Topology Appl. **156**, (2009), 658–660.

[Gr]   A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. **9**, (1957), 119–221.

[GZ]   P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*. Springer Verlag, 1967.

[Hi]   P.S. Hirschhorn, *Model categories and their localizations*, AMS Math. Surveys and Monographs **99**, 2002.

[Ho]   M. Hovey, *Model categories*, AMS Math. Surveys and Monographs **63**, 1999.

[Hu]   S. Hu, *On singular homology in differentiable spaces*, Ann. of Math. **50**, (1949), 266–269.

[ML]   S. Mac Lane, *Categories for the working mathematician*, Second Edition, GTM vol. **5**, Springer Verlag, 1971.

[MSS]   M. Markl, S. Shnider, J. Stasheff, *Operads in Algebra, Topology and Physics*, AMS Math. Surveys and Monographs **96**, 2002.

[N]   A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, 148. Princeton University Press, 2001.

[Q]   D.G. Quillen, *Homotopical algebra*, Lecture Notes in Math, vol. **43**, Springer Verlag, 1967.

[Sp]   N. Spaltenstein, *Resolution of unbounded complexes*. Comment. Math. Helv., **65**, (1988), 121–154.

[S]   D. Sullivan, *Infinitesimal computations in Topology*. Publ. Math. de l'IHES, **47**, (1977), 269–331.

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