Multigraded Structures
and
the Depth of Blow-up Algebras

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CERTIFICA
que la present memòria ha estat realitzada sota la seva direcció
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C'est le temps que tu as perdu pour ta rose
qui fait ta rose si importante.

Antoine de Saint-Exupéry,
Le Petit Prince, 1943.

Contents

Introduction

The main purpose of this thesis is to contribute to the understanding of cohomological properties on non-standard multigraded modules. We mainly study the depth of non-standard multigraded modules and related structures, as well as the Hilbert function and quasi-polynomial, focusing the study on the depth of blow-up algebras.

In commutative algebra, graded modules, as well as standard multigraded ones, have been object of study by many authors. Although some results of non-standard graded modules are known, this is not the case of non-standard multigraded modules.

On the other hand, under the name of *blow-up algebras*, some graded algebras associated to an ideal I in a Noetherian local ring $(R, \mathfrak{m})$ are known. They are the *Rees algebra* $\mathcal{R}(I)$, the *associated graded ring* $gr_I(R)$ and the *fiber cone* $F_{\mathfrak{m}}(I)$ defined as

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n, \qquad gr_I(R) = \bigoplus_{n \geq 0} \frac{I^n}{I^{n+1}}, \qquad F_{\mathfrak{m}}(I) = \bigoplus_{n \geq 0} \frac{I^n}{\mathfrak{m} I^n}.$$

The blow-up algebras are used to study properties and numerical characters of the local ring $(R, \mathfrak{m})$ and the ideal I. Moreover, they also have a geometrical relevance.

Vasconcelos, [Vas94b], added to the list of blow-up algebras the so-called *Sally module* $S_J(I)$ of an ideal I with respect to a minimal reduction J. This is the graded $\mathcal{R}(J)$-module

$$S_J(I) = \bigoplus_{n \geq 1} \frac{I^{n+1}}{J^n I}.$$

The name was motivated by the work of Sally that was intended to recover properties of $\mathcal{R}(I)$ and $gr_I(R)$ from the better and well-known structure of

$\mathcal{R}(J)$.

The Rees algebra and the associated graded ring can be generalized in a multigraded setting for a set of ideals $I_1, \ldots, I_r$ in a Noetherian local ring $(R, \mathfrak{m})$ as follows: the multigraded Rees algebra associated to $I_1, \ldots, I_r$ is defined by,

$$\mathcal{R}(I_1, \ldots, I_r) = \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset R[t_1, \ldots, t_r],$$

and for $k = 1, \ldots, r$, the k-th associated multigraded ring of $I_1, \ldots, I_r$ in R is,

$$gr_{I_1,\ldots,I_r;I_k}(R) = \bigoplus_{\underline{n} \in \mathbb{N}^r} \frac{I_1^{n_1} \cdots I_k^{n_k} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} = \frac{\mathcal{R}(I_1, \ldots, I_r)}{I_k \mathcal{R}(I_1, \ldots, I_r)}.$$

In what follows we introduce the main problems that we have studied, giving some motivations and after that we explain in detail the main results obtained in the thesis.

Hilbert functions of graded modules over standard graded algebras are well studied since the famous paper of Hilbert [Hil90]. Assuming that all the homogeneous pieces of a graded module have finite length, it can be proved that the Hilbert function, which measures their length, is asymptotically polynomial. The study of Hilbert functions, Hilbert polynomials and the coefficients of these polynomials plays an important role in commutative algebra. This study can be generalized in several ways by considering a non-standard graded algebra, a standard multigraded algebra, or a non-standard multigraded algebra.

The first two cases are also well known. In the first case, if we consider a positively graded algebra, and a graded module, the Hilbert function is asymptotically a quasi-polynomial, see [BH93] and [DS99]. When we consider a standard multigraded algebra and a multigraded module, that is with generators of multidegrees $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$, then the Hilbert function is polynomial in r indeterminates for homogeneous pieces of degree $\underline{n} = (n_1 \ldots, n_r) \in \mathbb{N}^r$ with $n_1, \ldots, n_r$ large enough, see for example [HHRT97], [VKM94] and [Rob98].

In the non-standard case, some cases have been studied. For instance in [Lav99] and [Rob98], it has been studied the case in which the generators have multidegrees $(1, 0, \ldots, 0)$, $(d_1^2, 1, 0, \ldots, 0)$, $\ldots$, $(d_1^r, \ldots, d_{r-1}^r, 1)$. In this case, the Hilbert function is a polynomial in r indeterminates for $\underline{n} = (n_1, \ldots, n_r)$ in a region (a cone) of $\mathbb{Z}^r$. This situation in the bigraded case has also been studied in [HT03]. In [Rob00], another Hilbert function is defined; the author considers a cumulative Hilbert function of a finitely generated bigraded module over a polynomial ring with coefficients in a field and generators of degrees $(1,0)$, $(0,1)$ and $(1,1)$. This function at (m,n) corresponds to the sum of the dimensions of the pieces of degree (m,j) for j up to n. The author proves that this function is polynomial in a region of $\mathbb{N}^2$. This definition allows to study the module from a graded and bigraded point of view.

A more general setting was studied by Fields in his PhD thesis, [Fie00] (see also [Fie02]). He considers the general definition of quasi-polynomial and proves that the Hilbert function of a $\mathbb{N}^r$-graded module is quasi-polynomial in a region of $\mathbb{Z}^r$. In his proof, however, the region was not explicitly described.

For our purposes, we need to control the cone where the Hilbert function is a quasi-polynomial. So, we start by studying the asymptotic behavior of the Hilbert function of a non-standard multigraded module by considering M to be a $\mathbb{Z}^r$-graded S-module, where S is a $\mathbb{Z}^r$-graded ring with generators g_i^j, $i = 1, \dots, r$, $j = 1, \dots, \mu_i$ of degree $\gamma_i = (\gamma_1^i, \dots, \gamma_i^i, 0 \dots, 0) \in \mathbb{N}^r$ and $\gamma_i^i \neq 0$, over an Artinian local ring $S_{\underline{0}}$. In particular we prove that there exists an element $\underline{\beta} \in \mathbb{N}^r$ such that the Hilbert function is a quasi-polynomial in a cone defined by the elements $\underline{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ such that $\underline{n} = \underline{\beta} + \sum_{i=1}^r \lambda_i \gamma_i$ with $\lambda_1, \dots, \lambda_r \in \mathbb{R}_{\geq 0}$.

The problem of the asymptotic stability of the depth of the homogeneous pieces of a multigraded module, has its origins in the result of Burch, [Bur72], where it is proved that for an ideal I in a Noetherian local ring $(R, \mathfrak{m})$ the following inequality holds,

$$l(I) \leq \dim(R) - \min_{n \geq 1}\{\operatorname{depth}(R/I^n)\}$$

where $l(I) = \dim(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I))$ is the analytic spread of I. Some years later, Brodmann, [Bro79a], replaced the minimum of these depths by the asymptotic value of $\operatorname{depth}(R/I^n)$ for large n, a value that he proved that it exists. In particular, in [Bro79b], he studied the associated primes of $M/I^n M$ for M a finitely generated R-module, and proved that, for $n \gg 0$, $Ass(M/I^n M)$ is stable. As a consequence, $\operatorname{depth}(M/I^n M)$ is constant for $n \gg 0$ and

$$l(I, M) \leq \dim(M) - \lim_{n \to \infty} \operatorname{depth}(M/I^n M)$$

being $l(I, M) = \dim(\bigoplus_n I^n M/\mathfrak{m}I^n M)$.

Recently, Herzog and Hibi generalized in [HH05] these inequalities for graded modules over a standard graded algebra. They proved that for a graded module E,

$$\lim_{n\to\infty} \operatorname{depth}(E_n) \leq \dim(E) - \dim(E/\mathfrak{m}E).$$

In this case, the authors did not consider to study the asymptotic stability of $Ass(E_n)$ to assure the stability of the depths. The key was to use the Hilbert polynomial of the graded Koszul homology modules.

Branco Correia and Zarzuela, in [BZ06], proved that for R-modules $E \subsetneq G \cong R^e$, $e > 0$, $\operatorname{depth}(G_n/E_n)$ takes a constant value for large n, and the inequality

$$l_G(R) \leq \dim(R) + e - 1 - \min_{n\geq 1}\{\operatorname{depth}(G_n/E_n)\}$$

where $l_G(E) = \dim(\mathcal{R}_G(E)/\mathfrak{m}\mathcal{R}_G(E))$. Here the constant depth is based in the asymptotic stability of the associated primes.

In [Hay06] Hayasaka proved the most general result until now. He considers the standard multigraded situation. His study is based on the associated primes of a multigraded module. In particular, he proves that for a standard multigraded rings $A \subset B$ with $A_{\underline{0}} = B_{\underline{0}} = R$ a local ring, then $Ass(B_{\underline{n}}/A_{\underline{n}})$ is stable for $\underline{n} \gg \underline{0}$. As a consequence, $\operatorname{depth}(B_{\underline{n}}/A_{\underline{n}})$ is asymptotically constant. Hayasaka, generalizes also the inequality and shows that

$$s(A) \leq s(B) + \dim(R) - \operatorname{depth}(A, B)$$

if $\operatorname{depth}(A, B) < \infty$, being such value the asymptotic depth of $B_{\underline{n}}/A_{\underline{n}}$, and $s(G) = \dim \operatorname{Proj}^r(G/\mathfrak{m}G) + 1$ the spread of G, defined for G a Noetherian standard multigraded ring with $G_{\underline{0}} = R$ a local ring with maximal ideal $\mathfrak{m}$.

Then, it is natural to ask what happens in a non-standard multigraded case. Are the depths of the graded pieces of a multigraded module constant, for degrees large enough? How does graduation affect this? The approximation to the problem of Herzog and Hibi gave us the way to follow. By using the Hilbert function of the Koszul homology modules of a non-standard multigraded module and its quasi-polynomial behavior, we can prove that this depth is constant in a sub-net of a cone in $\mathbb{N}^r$. In some cases when the Hilbert function has a polynomial behavior, we can assure constant depth in a cone of $\mathbb{N}^r$. For the multigraded blow-up algebras, we prove that the homogeneous pieces of the Rees algebra and the ones of the k-th associated graded ring, have constant depth for large enough degrees. Moreover, all k-th associated graded rings have the same asymptotic depth, and we can prove that $R/I_1^{n_1}\cdots I_r^{n_r}$ have also constant depth for large enough $n_1,\ldots,n_r$.

Another interesting problem is the study of the multigraded blow-up algebras defined for a set of powers of ideals. Studying properties for powers of ideals, instead of ideals themselves, can be very useful, since in many occasions one can deduce good properties related to the ideals, from properties related to powers of the same ideals. For instance, in the graded case, it can be proved that in the situation of having $\operatorname{depth}(gr_I(R)) \geq \dim(R)-1$ it holds that $e_i(I) \geq 0$ for all $i=0,\ldots,\dim(R)$, being $e_i(I)$ the Hilbert coefficients of the ideal I, [Mar89]. However this is a strong condition. On the other hand, under some assumptions one can prove that $\operatorname{depth}(gr_{I^n}(R)) \geq \dim(R)-1$, and hence $e_i(I^n) \geq 0$. For instance, if $\dim(R)=2$ and I is a normal ideal, then $gr_{I^n}(R)$ is Cohen-Macaulay for $n \gg 0$, [HH99]. But now the coefficients $e_i(I)$ can be easily written in terms of $e_i(I^n)$, and therefore one can deduce properties from the "asymptotic" behavior of the coefficients of an enough large power of the ideal I. Another example would be the result in [CPR05], where, in dimension 3, it is proved that $e_3(I) \geq 0$ under the assumption of being I^n integrally closed for some $n \gg 0$. The result is deduced from the behavior of $e_3(I^n)$.

In the multigraded case, one can consider the multigraded Rees algebra

$$\mathcal{R}(I_1^{a_1},\ldots,I_r^{a_r}).$$

In [HHR93], [HHR95] and [Hyr99], it has been studied Cohen-Macaulay and Gorenstein properties of multigraded Rees algebras of powers of ideals. For instance, in [HHR93], it is proved that if $\mathcal{R}(I,\ldots,I)$ is Cohen-Macaulay for some number r of copies of a height positive ideal I, then $\mathcal{R}(I^q)$ is Cohen-Macaulay for all $q \geq r$.

One can observe that $\mathcal{R}(I_1^{a_1},\ldots,I_r^{a_r}) = \mathcal{R}(I_1,\ldots,I_r)^{(\underline{a})}$ is the Veronese transform of the multigraded Rees algebra associated to ideals $I_1,\ldots,I_r$, with $\underline{a}=(a_1,\ldots,a_r) \in \mathbb{N}^{*r}$. So, it seems natural to study Veronese modules as a way to approach to the multigraded Rees algebras of powers of ideals, and more generally, to study Veronese transforms of non-standard multigraded modules.

In the graded case, Elias proved that $\operatorname{depth}(\mathcal{R}(I^n))$ is constant for n

large enough provided that R is a quotient of a regular local ring via Veronese modules, [Eli04].

Again, the natural question arises. Can we get similar results for multigraded Rees algebras or non-standard multigraded modules? In the thesis we obtain some results on the asymptotic behavior of the depth of non-standard multigraded Veronese modules, some of them in the more general setting that we consider, while the others in a restricted case.

One of the classical problems in commutative algebra is to estimate the depth of the associated graded ring $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and the Rees algebra $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$ for ideals I having good properties. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J.

Valabrega and Valla proved that $gr_I(R)$ is Cohen-Macaulay if and only if $I^{p+1} \cap J = I^p J$ for all $p \geq 0$, [VV78]. In fact, the $\mathcal{R}(J)$-module

$$\bigoplus_{p\geq 0} \frac{I^{p+1} \cap J}{I^p J}$$

is the so-called Valabrega-Valla module of I with respect to J. Related to this, Guerrieri, in her PhD thesis, [Gue93], proved that if

$$\sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1} \cap J}{I^p J} \right) = 1$$

then $\text{depth}(gr_I(R)) \geq d-1$. Based on these results, Guerrieri, [Gue93], [Gue94] conjectured that

$$\text{depth}(gr_I(R)) \geq d - \Delta(I, J),$$

with $\Delta(I, J) = \sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1}\cap J}{I^p J} \right)$.

Guerrieri proved also some partial cases for $\Delta(I, J) = 2$ in [Gue93] and [Gue95], to be more precise, she proved that if $\text{length}_R \left(\frac{I^2 \cap J}{IJ} \right) = 1$ and $I^{p+1} \cap J = I^p J$ for all $p \geq 2$ then $\text{depth}(gr_I(R)) \geq d-2$. Some years later, Wang proved the general case for $\Delta(I, J) = 2$, [Wan00].

In [Gue93], Guerrieri gave some examples of some ideals such that

$$\text{length}_R \left(\frac{I^{p+1} \cap J}{I^p J} \right) = \begin{cases} 1, & \text{for finitely many integers } p; \\ 0, & \text{otherwise,} \end{cases}$$

with $\operatorname{depth}(gr_I(R)) = d-1$. That is why Guerrieri, and Huneke, asked if the conditions $\operatorname{length}_R\left(\frac{I^{p+1}\cap J}{I^pJ}\right) \leq 1$, $p \geq 1$, imply that $\operatorname{depth}(gr_I(R)) \geq d-1$, [Gue93], Question 2.23. Wang in [Wan02], Example 3.13, gave a counterexample to the question and asked if this question would have an affirmative answer assuming that R was a regular local ring.

By considering other lengths, Huckaba and Marley proved that

$$e_1(I) \leq \sum_{p\geq 0} \operatorname{length}_R\left(\frac{I^{p+1}}{I^pJ}\right)$$

and in case of having the equality, then $\operatorname{depth}(gr_I(R)) \geq d-1$, [HM97]. Hence one can consider the non-negative integer

$$\delta(I,J) = \sum_{p\geq 0} \operatorname{length}_R\left(\frac{I^{p+1}}{I^pJ}\right) - e_1(I) \geq 0.$$

Wang conjectured that, [Wan00],

$$\operatorname{depth}(gr_I(R)) \geq d-1-\delta(I,J).$$

He proved that $\delta(I,J) \leq \Delta(I,J)$ and that Guerrieri's Conjecture was implied by his one. Huckaba proved the conjecture in the case $\delta(I,J)=0$, [Huc96], [HM97]. If $\delta(I,J)=1$ Wang proved the conjecture, [Wan00], and Polini gave a simpler proof, [Pol00]. For $\delta(I,J)=2$ Rossi and Guerrieri proved Wang's Conjecture assuming that R/I is Gorenstein, [GR99]. However, Wang gave a counterexample to the conjecture for $d=6$, [Wan01].

It has been proved that in general these conjectures are not always true. However some examples show a relation between the integers and the depth, and hence one can think of trying to refine the conjectures by considering other configurations of the integers.

In the main result of Chapter 5 we prove a refined version of Wang's Conjecture. What we do is to decompose the integer $\delta(I,J)$ as a finite sum of non-negative integers $\delta_p(I,J)$, with $\operatorname{length}_R(\frac{I^{p+1}\cap J}{I^pJ}) \geq \delta_p(I,J) \geq 0$. If $\bar{\delta}(I,J)$ is the maximum of the integers $\delta_p(I,J)$ for $p\geq 0$, when $\bar{\delta}(I,J) \leq 1$, we are able to prove that $\operatorname{depth}(\mathcal{R}(I)) \geq d-\bar{\delta}(I,J)$ and $\operatorname{depth}(gr_I(R)) \geq d-1-\bar{\delta}(I,J)$. As a consequence we can answer the question formulated by Guerrieri and Huneke about consider $\operatorname{length}_R(\frac{I^{p+1}\cap J}{I^pJ}) \leq 1$ for all $p\geq 0$. In this situation we can prove that $\operatorname{depth}(gr_I(R)) \geq d-2$. The key tool is to interpret these integers as multiplicities of some non-standard bigraded modules.

Some of the results of this thesis have been published in:

[CE06] G. Colomé-Nin and J. Elias. *Bigraded structures and the depth of blow-up algebras*. Proceedings of the Royal Society of Edinburgh, 136A, 1175-1194, 2006.

In the following we summarize the contents and the main results obtained in this thesis.

Chapter 1 is devoted to recall some definitions and properties that serve to us as a background for the rest of the work.

In **Chapter 2** we study properties related to the Hilbert function of a non-standard multigraded module. In particular we consider a $\mathbb{N}^r$-graded ring S generated over $S_{\underline{0}}$ by elements $g_i^1, \ldots, g_i^{\mu_i}$ of degree $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0 \ldots, 0) \in \mathbb{N}^r$, with $\gamma_i^i \neq 0$, for $i = 1, \ldots, r$, with $(S_{\underline{0}}, \mathfrak{m})$ an Artinian local ring, and M a finitely generated $\mathbb{Z}^r$-graded S-module.

Observe that this graduation admits in particular the standard case. However, by abuse of language, we refer to this more general graduation as a *non-standard*, to bear in mind the differences with the standard situation. That is, the standard graduation is not excluded from our definition.

In this setting, we can define the irrelevant ideal S_{++} of S to be $S_{++} = I_1 \cdots I_r$, where I_j is the ideal of S generated by the homogeneous components of degree $(b_1, \ldots, b_j, 0 \ldots, 0)$, with $b_j \neq 0$. Then we can define $\mathrm{Proj}^r(S)$ as the set of all relevant homogeneous prime ideals on S, which is the set of all homogeneous prime ideals $\mathfrak{p}$ in S such that $\mathfrak{p} \not\supset S_{++}$.

We define the relevant dimension of a multigraded S-module M as the integer

$$\mathrm{rel.\,dim}(M) = \begin{cases} r-1 & \text{if } Supp_{++}(M) = \emptyset \\ \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in Supp_{++}(M)\} & \text{if } Supp_{++}(M) \neq \emptyset. \end{cases}$$

Denoting $\underline{n} = (n_1, \ldots, n_r)$, the Hilbert function of M is defined by

$$\begin{array}{rccc} h_M : & \mathbb{Z}^r & \longrightarrow & \mathbb{Z} \\ & \underline{n} & \mapsto & \mathrm{length}_{S_{\underline{0}}}(M_{\underline{n}}). \end{array}$$

In Section 2.3.1, we introduce the definition of a quasi-polynomial in the $\mathbb{Z}^r$-graded case and we study some properties that are useful in order to prove that the Hilbert function is quasi-polynomial.

We say that a function $f : \mathbb{N}^r \to \mathbb{Z}$ is a *quasi-polynomial function of polynomial degree d on $\underline{\beta}, \gamma_1, \ldots, \gamma_r$* if there exist some periodic functions $c_{\underline{\alpha}} : \mathbb{N}^r \to \mathbb{Z}$, for $\underline{\alpha} \in \mathbb{N}^r$ and $|\underline{\alpha}| \leq d$, with respect to $\gamma_1, \ldots, \gamma_r$ such that for $\underline{n} \in C_{\underline{\beta}}$

$$f(\underline{n}) = \sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$$

and $f(\underline{n}) = 0$ when $\underline{n} \notin C_{\underline{\beta}}$, and there is some $\underline{\alpha} \in \mathbb{N}^r$ with $|\underline{\alpha}| = d$ such that $c_{\underline{\alpha}} \neq 0$. We call *quasi-polynomial* an expression $\sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$. Here, a cone is defined by $C_{\underline{\beta}} := \left\{ \underline{\alpha} \in \mathbb{N}^r \mid \underline{\alpha} = \underline{\beta} + \sum_{i=1}^r \lambda_i \gamma_i, \;\; \lambda_i \in \mathbb{R}_{\geq 0} \right\}$.

Then we prove that the Hilbert function is quasi-polynomial.

Proposition 2.3.10. *Let S be a $\mathbb{N}^r$-graded ring as considered before. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then there exists a quasi-polynomial P_M of polynomial degree* $\operatorname{rel.dim}(M) - r$ *and a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, such that for any $\underline{n} \in C_{\underline{\beta}}$*

$$h_M(\underline{n}) = P_M(\underline{n}).$$

As in the standard graded case, we can prove in our situation the Grothendieck-Serre formula that relates the Hilbert function, the Hilbert quasi-polynomial and the length of the local cohomology modules of M with respect to the irrelevant ideal.

Proposition 2.4.3. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then for all $\underline{n} \in \mathbb{Z}^r$,*

$$h_M(\underline{n}) - P_M(\underline{n}) = \sum_{i \geq 0} (-1)^i \operatorname{length}_{S_{\underline{0}}}(H^i_{S_{++}}(M)_{\underline{n}})$$

The last part of the chapter is devoted to generalize the Hilbert-Samuel function of an $\mathfrak{m}$-primary ideal I in a Noetherian local ring $(R, \mathfrak{m})$ to a set of $\mathfrak{m}$-primary ideals $I_1, \ldots, I_r$. We can prove that for all $k = 1, \ldots, r$ the function $f_k(\underline{n}) = \operatorname{length}_R(R/I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r})$ is polynomial for $\underline{n} \geq \underline{\beta}_k$, for some $\underline{\beta}_k \in \mathbb{N}^r$. We denote by p_k such polynomial. After that, we can get a similar formula to the Grothendieck-Serre one that relates this function, this polynomial and the length of some local cohomology modules of the

k-th extended Rees algebra $\mathcal{R}_k^*$ of $I_1 \ldots, I_r$ with respect to the irrelevant ideal of the Rees algebra of the ideals.

For an element $\underline{\delta} \in \mathbb{N}^r$, we define $\mathcal{H}_{\underline{\delta}}^k$ as the set of $\underline{n} \in \mathbb{Z}^r$ such that $n_k \in \mathbb{Z}$ and $(n_1, \ldots, n_{k-1}, n_{k+1}, \ldots, n_r) \geq (\delta_1, \ldots, \delta_{k-1}, \delta_{k+1}, \ldots, \delta_r)$.

Theorem 2.4.8. *There exists an element $\underline{\delta} \in \mathbb{N}^r$ such that for all $\underline{n} \in \mathcal{H}_{\underline{\delta}}^k$,*

$$p_k(\underline{n}) - f_k(\underline{n}) = \sum_{i \geq 0} (-1)^i \operatorname{length}_R(H^i_{\mathcal{R}_{++}}(\mathcal{R}_k^*)_{\underline{n}+e_k}).$$

In **Chapter 3** we are interested in the study of the depth of the graded pieces of a multigraded module over a Noetherian non-standard multigraded ring with the graduation considered in the previous chapters. In the first section, we look at the Koszul complex and Koszul homology in the multigraded case, a concept that we need to reach our purpose.

Let S be a $\mathbb{N}^r$-graded ring, generated over $S_{\underline{0}}$ by elements of multidegrees $\gamma_1, \ldots, \gamma_r$, where $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0, \ldots, 0) \in \mathbb{N}^r$ with $\gamma_i^i \neq 0$ for all $i = 1, \ldots, r$. Let $\mathcal{M}$ be the maximal homogeneous ideal of S, that is $\mathcal{M} = \mathfrak{m} \oplus \bigoplus_{\underline{n} \neq \underline{0}} S_{\underline{n}}$, where $\mathfrak{m}$ is the maximal ideal of the Noetherian local ring $S_{\underline{0}}$.

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. In Section 3.2 we study the asymptotic depth, with respect to $\mathfrak{m}$, of the multigraded pieces $M_{\underline{n}}$. The key point in the proof is the existence of the Hilbert quasi-polynomial for the Koszul homology modules of M with respect to a system of generators of $\mathfrak{m}$. The quasi-polynomial behavior of the Hilbert function allows to prove the theorem.

Theorem 3.2.1. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. There exists an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that,*

$$\operatorname{depth}(M_{\underline{n}}) \geq \rho$$

for all $\underline{n} \in C_{\underline{\beta}}$ with $M_{\underline{n}} \neq 0$, and

$$\operatorname{depth}(M_{\underline{n}}) = \rho$$

for some $\underline{\delta} \in \Pi_{\underline{\beta}}$ and for all $\underline{n} \in \{\underline{\delta} + \sum_{i=1}^r \lambda_i \gamma_i \mid \lambda_i \in \mathbb{N}\} \subset C_{\underline{\beta}}$.

When the quasi-polynomial is, in fact, a polynomial, we can assure the constant depth in all the cone:

▷ **Proposition 3.2.3**: If S is an algebra generated over $S_{\underline{0}}$ by elements of degrees $(1,0,\ldots,0)$, $(*,1,0,\ldots,0)$, $\ldots$, $(*,*,*,\ldots,1) \in \mathbb{N}^r$, then $\operatorname{depth}(M_{\underline{n}}) = \rho$ for $\underline{n} \in C_{\underline{\beta}}$.

▷ **Corollary 3.2.4**: If S is a standard algebra, then $\operatorname{depth}(M_{\underline{n}}) = \rho$ for $\underline{n} \geq \underline{\beta}$.

In Section 3.3, we consider the multigraded Rees algebra associated to ideals $I_1,\ldots,I_r$ of a Noetherian local ring $(R,\mathfrak{m})$, and for $k = 1,\ldots,r$, the k-th associated multigraded ring of $I_1,\ldots,I_r$ in R. In both cases, they are finitely generated standard $\mathbb{Z}^r$-graded $\mathcal{R}(I_1,\ldots,I_r)$-modules, and each component, $\mathcal{R}(I_1,\ldots,I_r)_{\underline{n}}$ and $gr_{I_1,\ldots,I_r;I_k}(R)_{\underline{n}}$, is a finitely generated R-module. Then using the previous results we can prove:

Proposition 3.3.1, 3.3.3. *There exists an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\delta \in \mathbb{N}$ such that for all $\underline{n} \geq \underline{\beta}$ it hold*

$$\operatorname{depth}(I_1^{n_1}\cdots I_r^{n_r}) = \delta + 1$$

and

$$\operatorname{depth}\left(\frac{I_1^{n_1}\cdots I_r^{n_r}}{I_1^{n_1}\cdots I_k^{n_k}\cdots I_r^{n_r}}\right) = \delta$$

for all $k = 1,\ldots,r$.

We are interested in the depth of $R/I_1^{n_1}\cdots I_r^{n_r}$ for $\underline{n}$ large enough. In this case, we can take advantage of the constant asymptotic depth of these last two modules and the relation with $R/I_1^{n_1}\cdots I_r^{n_r}$ by means of some short exact sequences of R-modules where we can use the depth counting techniques.

Theorem 3.3.6. *There exists an element $\underline{\varepsilon} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that*

$$\operatorname{depth}\left(\frac{R}{I_1^{n_1}\cdots I_r^{n_r}}\right) = \rho \leq \delta$$

for all $\underline{n} \geq \underline{\varepsilon}$. Moreover, if there exists an $\underline{n} \geq \underline{\beta}$ such that $\operatorname{depth}\left(\frac{R}{I_1^{n_1}\cdots I_r^{n_r}}\right) \geq \delta$, *then $\rho = \delta$.*

Finally, we bound the asymptotic depth of the modules $R/I_1^{n_1}\cdots I_r^{n_r}$.

Proposition 3.3.7. *Let $\rho \in \mathbb{N}$ be the asymptotic depth of $R/I_1^{n_1}\cdots I_r^{n_r}$. Then,*

$$\rho \leq \dim(R) - \dim \operatorname{Proj}^r\left(\frac{\mathcal{R}(I_1,\ldots,I_r)}{\mathfrak{m}\mathcal{R}(I_1,\ldots,I_r)}\right).$$

The aim of **Chapter 4** is to study the Veronese modules associated to a non-standard multigraded S-module M by means of some cohomological properties of the module. We mainly study the vanishing of the local cohomology modules of M and of Veronese modules of M, generalizing some results on the depth of Veronese modules associated to Rees algebras. We also study the asymptotic behavior of the Veronese modules.

We are still considering the general situation in which S is a Noetherian $\mathbb{N}^r$-graded ring generated as $S_{\underline{0}}$-algebra by homogeneous elements g_i^j for $i = 1, \cdots, r$ and $j = 1, \ldots, \mu_i$, of multidegrees $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0, \ldots, 0) \in \mathbb{N}^r$, respectively, with $\gamma_i^i \neq 0$. We assume that $S_{\underline{0}}$ is a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field.

Given $\underline{a} \in \mathbb{N}^{*r}$ we denote $\phi_{\underline{a}}(\underline{n}) = \sum_{i=1}^r (n_i a_i)\gamma_i$ for all $\underline{n} \in \mathbb{Z}^r$.

The Veronese transform of S with respect to $\underline{a} \in \mathbb{N}^{*r}$, or $(\underline{a})$-Veronese, is the subring of S

$$S^{(\underline{a})} = \bigoplus_{\underline{n} \in \mathbb{N}^r} S_{\phi_{\underline{a}}(\underline{n})},$$

where $\phi_{\underline{a}}(\underline{n}) = \sum_{i=1}^r (n_i a_i)\gamma_i$ for all $\underline{a} \in \mathbb{N}^{*r}$ and $\underline{n} \in \mathbb{Z}^r$.

Given an $\mathbb{Z}^r$-graded S-module M we denote by $M^{(\underline{a},\underline{b})}$ the Veronese transform of M with respect to $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$, or $(\underline{a}, \underline{b})$-Veronese, is the $S^{(\underline{a})}$-module

$$M^{(\underline{a},\underline{b})} = \bigoplus_{\underline{n} \in \mathbb{Z}^r} M_{\phi_{\underline{a}}(\underline{n}) + \underline{b}}.$$

In **Proposition 4.1.6**, we prove that Veronese functor commutes with local cohomology with respect to $\mathcal{M}$. This fact is important for many of the results of this thesis.

In Section 4.2, among other properties, we study the generalized depth of a multigraded module and its Veronese modules. This is an important invariant to reach our purposes. We start by proving several results (**Proposition 4.2.1**, **Proposition 4.2.2**, **Proposition 4.2.3**, **Proposition 4.2.4**) relating properties of non-standard $\mathbb{Z}^r$-graded rings and modules with their Veronese transforms.

For a finitely generated $\mathbb{Z}^r$-graded S-module M, we define the *generalized depth* of M with respect to the homogeneous maximal ideal $\mathcal{M}$ of S as

$$\operatorname{gdepth}(M) = \max\{k \in \mathbb{N} \mid S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M))) \text{ for all } i < k\}.$$

We also define the *projective Cohen-Macaulay deviation* of M as

$$\mathrm{pcmd}(M) = \max\{\dim(S_{(\mathfrak{p})}) - \mathrm{depth}(M_{(\mathfrak{p})}) \mid \mathfrak{p} \in \mathrm{Proj}^r(S)\}.$$

In the case of being $S_{\underline{0}}$ a quotient of a regular ring, we relate these last two integers:

Theorem 4.2.7. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $S_{\underline{0}}$ is the quotient of a regular ring then*

$$\mathrm{gdepth}(M) = \dim(S) - \mathrm{pcmd}(M).$$

Then, with this assumption, we prove the invariance of gdepth under Veronese transforms:

Corollary 4.2.8. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $S_{\underline{0}}$ is the quotient of a regular ring, for all $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$ it holds*

$$\mathrm{gdepth}(M^{(\underline{a},\underline{b})}) = \mathrm{gdepth}(M).$$

In Section 4.3 we want to study the depth of the Veronese modules $M^{(\underline{a},\underline{b})}$ for large values $\underline{a}, \underline{b} \in \mathbb{N}^r$. As a partial solution, under the general hypothesis on the multidegrees of the chapter, we prove that the depth of some Veronese modules $M^{(\underline{a})}$ are constant for $\underline{a}$ in a net of $\mathbb{N}^r$. Note that $M^{(\underline{a})} = M^{(\underline{a},\underline{0})}$.

We denote by $vad(M^{(*)})$ (resp. $vad(M^{(*,*)})$) the Veronese asymptotic depth of M, that means the maximum of $\mathrm{depth}(M^{(\underline{a})})$ (resp. $\mathrm{depth}(M^{(\underline{a},\underline{b})})$) for all $\underline{a} \in \mathbb{N}^{*r}$ (resp. for all $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$).

Proposition 4.3.1. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Let $s = vad(M^{(*)})$. There exists $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{N}^{*r}$ such that for all $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$*

$$\mathrm{depth}(M^{(\underline{b})}) = s.$$

The previous result can be used in order to study of the depth of the multigraded Rees algebras of some powers of ideals.

Proposition 4.3.2. *Let $I_1, \ldots, I_r$ be ideals in a Noetherian local ring $(R, \mathfrak{m})$. Let $s = vad(\mathcal{R}(I_1, \ldots, I_r)^{(*)})$. There exists $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{N}^{*r}$ such that for all $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$*

$$\operatorname{depth}(\mathcal{R}(I_1^{b_1}, \ldots, I_r^{b_r})) = s.$$

Moreover, if $\operatorname{depth}(\mathcal{R}(I_1, \ldots, I_r)) = s$*, then for all $\underline{b} \in \mathbb{N}^{*r}$,*

$$\operatorname{depth}(\mathcal{R}(I_1^{b_1}, \ldots, I_r^{b_r})) = s.$$

In order to extend the previous results on the asymptotic depth of the Veronese modules to regions of $\mathbb{N}^r$, we have to study the vanishing of the local cohomology modules of a multigraded module M.

We say that a $\mathbb{Z}^r$-graded S-module M is Γ*-finitely graded* if there exists a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ where $M_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ such that $\underline{n}^* = (|n_1|, \ldots, |n_r|) \in C_{\underline{\beta}}$. We denote by Γ-fg(M) the greatest integer $k \geq 0$ such that $H^i_{\mathcal{M}}(M)$ is Γ-finitely graded for all $i < k$.

Due to technical reasons, we have to restrict the degrees of the generators of S, see Remark 4.3.6 and Remark 4.3.11. For the rest of the chapter, we assume that the graduation is *almost-standard*, i.e. the degrees are $\gamma_1, \ldots, \gamma_r$ with $\gamma_i = (0, \ldots, 0, \gamma_i^i, 0, \ldots, 0)$ and $\gamma_i^i > 0$ for all $i = 1, \ldots, r$.

An important fact in our proofs is to assure in the almost-standard case, that $H^k_{\mathcal{M}}(M)$ is Γ-finitely graded for all $k \geq 0$ providing that M is Γ-finitely graded as well, see **Proposition 4.3.5**.

In the next result we relate the two integers attached to M studied in the chapter, gdepth(M) and Γ-fg(M).

Theorem 4.3.7. *Let S be an almost-standard multigraded ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module, then it holds*

$$\Gamma\text{-fg}(M) = \operatorname{gdepth}(M).$$

As a consequence, assuming that $S_{\underline{0}}$ is the quotient of a regular local ring we prove the invariance of Γ-fg under Veronese transforms in **Corollary 4.3.8**.

Now, we have new tools to prove the theorem that assures constant depth for the $(\underline{a}, \underline{b})$-Veronese in a region of $\mathbb{N}^r \times \mathbb{N}^r$, instead of a net. However the restriction to the almost-standard case is still necessary.

Theorem 4.3.12. *Let S be an almost-standard multigraded ring such that $S_{\underline{0}}$ is the quotient of a regular ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module and let $s = vad(M^{(*,*)})$. Then, there exists $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{b} \geq \underline{\beta}$ and for all $\underline{a} \in \mathbb{N}^r$ such that $a_i \geq (\beta_i + b_i)/\gamma_i^i$ it holds*

$$\operatorname{depth}(M^{(\underline{a},\underline{b})}) = s.$$

For general $\mathbb{Z}$-graded modules, we obtain:

Proposition 4.3.13. *Let S be a $\mathbb{Z}$-graded ring such that $S_{\underline{0}}$ is the quotient of a regular ring. Let M be a finitely generated graded S-module. Then* $\operatorname{depth}(M^{(a)})$ *is constant for $a \gg 0$.*

For the multigraded Rees algebra, the best approach to the solution of the problem is the following proposition.

Proposition 4.3.15. *If R is the quotient of a regular ring, there exist an integer s and $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{b} \geq \underline{\beta}$ and $\underline{a} \geq \underline{\beta} + \underline{b}$ it holds*

$$\operatorname{depth}_{\mathcal{M}^{(a)}}((I_1^{b_1} \cdots I_r^{b_r})\mathcal{R}(I_1^{a_1}, \cdots, I_r^{a_r})) = s.$$

In **Chapter 5** we want to find refined versions of the conjectures on the depth of blow-up algebras that we have mentioned before in this introduction. The main idea is to study the depth of blow-up algebras by means of certain bigraded modules. We interpret the lengths that appear in the conjectures as the multiplicities of some non-standard bigraded modules. Thanks to this interpretation we are able to refine the Conjecture of Wang, by adding new cases where it works and recovering the known true cases. As a corollary, we can answer the question of Guerrieri and Huneke regarding the lengths of the pieces of the Valabrega-Valla module.

For a Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension $d > 0$ with infinite residue field and an $\mathfrak{m}$-primary ideal I of R with minimal reduction J, we consider the integers that appear in the conjectures:

$$\Delta(I,J) = \sum_{p\geq 0} \operatorname{length}_R \left(\frac{I^{p+1} \cap J}{I^p J} \right), \quad \Lambda(I,J) = \sum_{p\geq 0} \operatorname{length}_R \left(\frac{I^{p+1}}{JI^p} \right),$$

$$\Delta_p(I,J) = \operatorname{length}_R \left(\frac{I^{p+1} \cap J}{I^p J} \right), \quad \Lambda_p(I,J) = \operatorname{length}_R \left(\frac{I^{p+1}}{JI^p} \right)$$

for $p \geq 0$, and

$$\delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

Sections 5.2 and 5.3 are mainly devoted to recall some preliminary results on the Sally module and the cumulative Hilbert function of non-standard bigraded modules. Recall that the cumulative Hilbert function of an A-module M is defined by $h_M(m,n) = \sum_{j \leq n} \operatorname{length}_A(M_{(m,j)})$. In particular, we prove

Theorem 5.3.4. *Let $S = A[X_1,\ldots,X_r,Y_1,\ldots,Y_s,Z_1,\ldots,Z_t]$ be a bigraded polynomial ring over an Artin ring A with indeterminates $X_1,\ldots,X_r$, $Y_1,\ldots,Y_s$ and $Z_1,\ldots,Z_t$, where each X_i has bidegree $(1,0)$, each Y_i has bidegree $(1,1)$, and each Z_i has bidegree $(0,1)$. Let M be a finitely generated bigraded S-module. Then, there exist integers m_0 and n_0 and a polynomial in two variables $p_M(m,n)$ such that*

$$p_M(m,n) = h_M(m,n)$$

for all (m,n) with $m \geq m_0$ and $n \geq n_0 + m$.

In Section 5.4 we introduce a non-standard bigraded module $\Sigma^{I,J}$ naturally attached to I and a minimal reduction J of I, this module can be considered as a refinement of the Sally module. From a natural presentation of $\Sigma^{I,J}$ we define two bigraded modules $K^{I,J}$ and $\mathcal{M}^{I,J}$, and we consider some diagonal submodules of them: $\Sigma^{I,J}_{[p]}$ and $K^{I,J}_{[p]}$. We summarize these constructions in the following.

We consider the associated graded ring of $\mathcal{R}(I)$ with respect to the homogeneous ideal $Jt\mathcal{R}(I) = \bigoplus_{n\geq 0} JI^{n-1}t^n$

$$gr_{Jt}(\mathcal{R}(I)) = \bigoplus_{j\geq 0} \frac{(Jt\mathcal{R}(I))^j}{(Jt\mathcal{R}(I))^{j+1}} U^j.$$

This ring has a natural bigraded structure. If we consider the bigraded ring $B := R[V_1,\ldots,V_\mu;T_1,\ldots,T_d]$ with $\deg(V_i) = (1,0)$ and $\deg(T_i) = (1,1)$, then there exists an exact sequence of bigraded B-rings

$$0 \longrightarrow K^{I,J} \longrightarrow C^{I,J} := \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)}[T_1,\ldots,T_d] \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow 0$$

where $K^{I,J}$ is the ideal of initial forms of $Jt\mathcal{R}(I)$.

Given a bigraded B-module M and an integer $p \in \mathbb{Z}$, we denote by $M_{[p]}$ the additive subgroup of M defined by the direct sum of the pieces

$M_{(m,n)}$ such that $m - n = p + 1$. In our case, the modules $K_{[p]}^{I,J}$, $C_{[p]}^{I,J}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$ are $\mathcal{R}(J)$-modules, and, eventually, they do not vanish for a finite set of indexes $p \in \mathbb{Z}$ (**Lemma 5.4.1**).

From now on, we will be interested in considering the non-negative diagonals of these modules and so, let us consider the following bigraded finitely generated B-modules:

$$\Sigma^{I,J} := \bigoplus_{p \geq 0} gr_{Jt}(\mathcal{R}(I))_{[p]} = \bigoplus_{p\geq 0}\bigoplus_{i \geq 0} \frac{J^i I^{p+1}}{J^{i+1} I^p} t^{p+1+i} U^i$$

$$\mathcal{M}^{I,J} := \bigoplus_{p \geq 0} C_{[p]}^{I,J} = \bigoplus_{p \geq 0} \frac{I^{p+1}}{I^p J} t^{p+1}[T_1, \ldots, T_d]$$

and from now on, we consider the new B-module

$$K^{I,J} := \bigoplus_{p \geq 0} K_{[p]}^{I,J}.$$

We call $\Sigma^{I,J}$ the *bigraded Sally module* of I with respect J.

From Lemma 5.4.1 there exists a natural isomorphism of $\mathcal{R}(J)$-modules

$$gr_{Jt}(\mathcal{R}(I)) \cong \mathcal{R}(J) \oplus \Sigma^{I,J}.$$

Since the modules $\Sigma^{I,J}$ and $\mathcal{M}^{I,J}$ are annihilated by J, we have an exact sequence of bigraded $A = R/J[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$-modules

$$0 \longrightarrow K^{I,J} \longrightarrow \mathcal{M}^{I,J} \longrightarrow \Sigma^{I,J} \longrightarrow 0.$$

By considering each diagonal, for all $p \geq 0$ we have an exact sequence of $R/J[T_1, \ldots, T_d]$-modules,

$$0 \longrightarrow K_{[p]}^{I,J} \longrightarrow \mathcal{M}_{[p]}^{I,J} = \frac{I^{p+1}}{JI^p}[T_1, \ldots, T_d] \longrightarrow \Sigma_{[p]}^{I,J} \longrightarrow 0,$$

which are, in fact, graded modules, and so we can consider for them the (classic) Hilbert function.

By using the cumulative Hilbert function with the modules $\Sigma^{I,J}$, $\mathcal{M}^{I,J}$ and $K^{I,J}$, that in this case are polynomials in one variable in a region of $\mathbb{N}^2$, we can proof the following results that allows to us to interpret the integers $e_1(I)$, $\Lambda(I,J)$ and $\delta(I,J)$ in the Wang's Conjecture as multiplicities of our modules (**Proposition 5.5.2**, **Proposition 5.5.3**, **Proposition 5.5.4**, **Proposition 5.5.6**):

- ▷ $p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1}(-1)^i e_{i+1}(I)\binom{m-1+d-i-1}{d-i-1}$.
- ▷ $\deg(p_{\mathcal{M}^{I,J}}) = d-1$ and $e_0(\mathcal{M}^{I,J}) = \Lambda(I,J)$.
- ▷ If $\Sigma^{I,J} = 0$ then $gr_I(R)$ is a Cohen-Macaulay ring.
  If $\Sigma^{I,J} \neq 0$ then $\deg(p_{\Sigma^{I,J}}) = d-1$ and $e_0(\Sigma^{I,J}) = e_1(I)$.
- ▷ $e_0(K^{I,J}) = \delta(I,J)$. If $K^{I,J} \neq 0$ then $\deg(p_{K^{I,J}}) = d-1$.
  In particular, $\Lambda(I,J) \geq e_1(I)$.
- ▷ For all $p \geq 0$, $e_0(\Sigma^{I,J}_{[p]}) = \text{length}_R\left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]}) \geq 0$ and
  $e_1(I) = \sum_{p\geq 0}(\text{length}_R\left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]}))$.
- ▷ For all $p \geq 0$, $\text{length}_R\left(\frac{I^{p+1}\cap J}{JI^p}\right) \geq e_0(K^{I,J}_{[p]})$ and
  $\delta(I,J) = e_0(K^{I,J}) = \sum_{p\geq 0} e_0(K^{I,J}_{[p]}) \geq 0$.

We define

$$\delta_p(I,J) = e_0(K^{I,J}_{[p]}).$$

In Section 5.6 we prove a refined version of Wang's Conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta = \sum_{p\geq 0}\delta_p(I,J)$. Let us consider $\bar{\delta}(I,J)$ to be the maximum of the integers $\delta_p(I,J)$ for $p \geq 0$.

Theorem 5.6.3. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be and $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. If $\bar{\delta}(I,J) \leq 1$, then*

$$\text{depth}(\mathcal{R}(I)) \geq d - \bar{\delta}(I,J)$$

and

$$\text{depth}(gr_I(R)) \geq d - 1 - \bar{\delta}(I,J).$$

Observe that for $\delta(I,J) = 0, 1$ we recover the known cases of the Wang's Conjecture.

For the proof we need the following important results. In particular, we need to study the depth of the associated bigraded ring $gr_{Jt}(\mathcal{R}(I))$.

Theorem 5.6.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 3$. Let I be an $\mathfrak{m}$-primary ideal of R and let J be a minimal reduction of I. Let us assume that $K^{I,J} \neq 0$, and either $K^{I,J}_{[p]} = 0$ or $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1, \ldots, T_d]$-module for $p \geq 0$. Then,*

$$\operatorname{depth}(gr_{Jt}(\mathcal{R}(I))) \geq d - 1.$$

The following lemma is also important because assures how are the diagonals $K^{I,J}_{[p]}$ in case of having $e_0(K^{I,J}_{[p]}) = 1$, and so how is $K^{I,J}$ in the decomposition of $\delta(I, J)$ that we consider in the main theorem.

Lemma 5.6.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. If $\delta_p(I, J) = 1$ then $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1, \ldots, T_d]$-module.*

Finally, we are able to give an answer the question the Guerrieri and Huneke mentioned before.

Theorem 5.6.5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be and $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. If $\Delta_p(I, J) \leq 1$ for all $p \geq 1$, then*

$$\operatorname{depth}(gr_I(R)) \geq d - 2.$$

In **Chapter 6** we study, in the first section, some submodules $D_{l_\alpha}(\Sigma^{I,J})$ of the bigraded Sally module $\Sigma^{I,J}$ with respect to a line l_α, generalizing the concept of the diagonal submodules $\Sigma^{I,J}_{[p]}$.

For each set of non negative integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$, with $\alpha_1 + \alpha_2 \geq 1$, we define the line l_α in the plane (m, n) as

$$l_\alpha : \quad \begin{cases} m(s) = \alpha_1 s + \alpha_3 \\ n(s) = \alpha_2 s + \alpha_4 \end{cases}$$

for $s \geq 0$. Then, we define the *diagonal submodule* $D_{l_\alpha}(\Sigma^{I,J})$ of $\Sigma^{I,J}$ as the direct sum of the pieces of $\Sigma^{I,J}$ of bidegrees $(m(s) + n(s), n(s))$, $s \geq 0$,

$$\begin{aligned} D_{l_\alpha}(\Sigma^{I,J}) &= \bigoplus_{(m,n) \in l_\alpha} \Sigma^{I,J}_{(m+n,n)} t^{m+n} U^n \\ &= \bigoplus_{s \geq 0} \frac{I^{m(s)} J^{n(s)}}{I^{m(s)-1} J^{n(s)+1}} t^{m(s)+n(s)} U^{n(s)}, \end{aligned}$$

and we define the Hilbert function of $D_{l_\alpha}(\Sigma^{I,J})$ as

$$\mathcal{H}_{l_\alpha}(s) = \text{length}_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}\right).$$

Then we prove the following result on the growth of the Hilbert function $\mathcal{H}_{l_\alpha}$ of the diagonal submodule $D_{l_\alpha}(\Sigma^{I,J})$ by considering hypotheses on the minimal number of generators of the pieces of this diagonal. This result will be crucial in order to study the monotony of the Hilbert function of an $\mathfrak{m}$-primary ideal I in the one-dimensional case in Section 6.2.

Proposition 6.1.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal and let J be a minimal reduction of I. Let $D_{l_\alpha}(\Sigma^{I,J})$ be the diagonal submodule of bigraded Sally module $\Sigma^{I,J}$ associated to the line l_α. Let $s \geq 2$ be an integer such that one of the following conditions hold,*

(1) $\nu_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}\right) \leq 2$, *or*

(2) *there exist an integer $e \geq 1$ such that* $\text{length}_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-e}J^{n(s)+e}}\right) \leq s.$

Then for all $t \geq s$ it holds $\mathcal{H}_{l_\alpha}(t) \geq \mathcal{H}_{l_\alpha}(t+1)$.
Moreover, under the hypothesis of (1) there exists an element $a \in I^{\alpha_1}J^{\alpha_2}$ such that

$$\frac{I^{m(t)}J^{n(t)}}{I^{m(t)-1}J^{n(t)+1}} \xrightarrow{.a} \frac{I^{m(t+1)}J^{n(t+1)}}{I^{m(t+1)-1}J^{n(t+1)+1}}$$

is an epimorphism for all $t \geq s-1$. In particular it holds

$$\mathcal{H}_{l_\alpha}(t) \geq \mathcal{H}_{l_\alpha}(t+1)$$

for all $t \geq s-1$.

In the Cohen-Macaulay one-dimensional case, we can prove the following results on the growth of the Hilbert function of an $\mathfrak{m}$-primary ideal.

Proposition 6.2.3. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. Let I be an $\mathfrak{m}$-primary ideal, and $x \in I$ a degree one superficial element of I. Let $t \geq 2$ be an integer such that the pair I, x verify one of the following conditions:*

(1) $I^n \cap (x) = xI^{n-1}$ *for all $n \leq t-1$, and* $\nu_R(I^t/xI^{t-1}) \leq 2$,

(2) $I^n \cap (x) = xI^{n-1}$ *for all* $n \leq t$, *and* $\text{length}_R(I^t/x^a I^{t-a}) \leq \bar{t} \leq t$, $a \geq 1$.

Then h_I *is non-decreasing.*

For an $\mathfrak{m}$-primary ideal I of a Cohen-Macaulay local ring R of dimension d, we denote by $b(I) = \text{length}_R(I/I^2)$ the embedding dimension of I.

Proposition 6.2.4. *Let* R *be a one-dimensional Cohen-Macaulay local ring. Let* I *be an* $\mathfrak{m}$*-primary ideal of* R*. Then*

(i) $e_0(I) = 1$ *if and only if* $b(I) = 1$*. In this case we have* $I = \mathfrak{m}$ *and* R *is a regular local ring.*

(ii) If $b(I) = 2$ *then it holds*

$$h_I(n) = \begin{cases} \text{length}_R(R/I) & n = 0 \\ n+1 & n = 1, \ldots, e_0(I) - 1 \\ e_0(I) & n \geq e_0(I). \end{cases}$$

The Hilbert function h_I *is non-decreasing if and only if* $\text{length}_R(R/I) \leq 2$.

(iii) If $b(I) \leq e_0(I) \leq b(I) + 2$ *then the Hilbert function is non-decreasing.*

(iv) If $I^2 \cap (x) = xI$, $b(I) = 4$, *and* $e_0(I) = 7$ *then* $\text{length}_R(R/I) \leq 4$ *and the Hilbert function is non-decreasing.*

Chapter 1

Preliminaries

This chapter is devoted to give and recall some definitions and results in Commutative Algebra that we need as a background for the development of the thesis. Throughout the work a *ring* will always mean a commutative ring with unit, and we will assume that all the rings are Noetherian, otherwise is stated. Most of the definitions and results can be found in standard references such as [BH93], [Mat80] and [Mat89].

For an R-module M, we denote by $\nu_R(M)$ the minimal number of generators of M as an R-module.

Depth

One of the important invariants that we study in the thesis is the depth, or the grade, of a module. So, let we introduce some definitions, [BH93]:

Definition 1. *Let R be a Noetherian ring and M be an R-module. We say that $x \in R$ is* regular on M *if $xz = 0$ (for some $z \in M$) implies $z = 0$. In other words, x is not a zero-divisor on M.*

A sequence $x_1, \ldots, x_s$ of elements of R is called a regular sequence on M, *or an* M-regular sequence, *if the following conditions hold:*

(i) x_i is an $M/(x_1, \ldots, x_{i-1})M$-regular element, for $i = 1, \ldots, s$; and

(ii) $M/(x_1, \ldots, x_s)M \neq 0$.

Definition 2. *Let R be a Noetherian ring, I an ideal of R, and M a finite R-module such that $IM \neq M$. We define the* grade of I in M, $\operatorname{grade}(I, M)$, *as the length of all maximal M-sequences contained in I. If, in particular, R is a local ring and $\mathfrak{m}$ is its maximal ideal, we define the* depth of M *as*

$$\operatorname{depth}(M) = \operatorname{grade}(\mathfrak{m}, M).$$

Also, if R is a graded ring, $\mathcal{M}$ its unique homogeneous maximal ideal, and M is a graded R-module, we define $\operatorname{depth}(M) = \operatorname{grade}(\mathcal{M}, M)$.

Sometimes, if $\mathfrak{p}$ is a prime ideal of R, we denote $\operatorname{depth}_{\mathfrak{p}}(M)$ instead of $\operatorname{grade}(\mathfrak{p}, M)$.

A very important technique in order to prove many results on the depth of a module, is based on the following proposition, and we refer to it as *depth counting*, see [BH93]. It is useful to bound the depth of a module that is inside a short exact sequence.

Proposition 3. *Let R be a Noetherian ring, $I \subset R$ an ideal, and*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

an exact sequence of finite R-modules. Then

$$\begin{aligned} \operatorname{grade}(I, M) &\geq \min\{\operatorname{grade}(I, U), \operatorname{grade}(I, N)\}, \\ \operatorname{grade}(I, U) &\geq \min\{\operatorname{grade}(I, M), \operatorname{grade}(I, N) + 1\}, \\ \operatorname{grade}(I, N) &\geq \min\{\operatorname{grade}(I, U) - 1, \operatorname{grade}(I, M)\}. \end{aligned}$$

Blow-up algebras

Let R be a d-dimensional Noetherian ring and I an ideal of R. With the name of *blow-up algebras* we refer to some graded rings that arise naturally from algebraic geometry in the process of blowing-up the variety $\operatorname{Spec}(R)$ along the subvariety $V(I)$. We set $I^i = 0$ for $i < 0$, $I^i = R$ for $i = 0$. A good reference on the subject is the book of Vasconcelos [Vas94a].

Definition 4. *The* Rees algebra of R associated to I *is the graded ring defined as*

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^i t^i$$

and the associated graded ring of R with respect to I *is the graded ring defined as*

$$gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$$

Clearly, $gr_I(R) \cong \mathcal{R}(I)/I\mathcal{R}(I)$.

These algebras play an important role in algebraic geometry. In fact, $\mathrm{Proj}(\mathcal{R}(I))$ is the blow-up of the variety $\mathrm{Spec}(R)$ along the subvariety $V(I)$. When R is the localization at the origin of the coordinate ring of an affine variety at the origin, $gr_{\mathfrak{m}}(R)$ is the coordinate ring of the tangent cone of the variety at the origin. Then $\mathrm{Proj}(gr_{\mathfrak{m}}(R))$ is the exceptional set of the blow-up of the variety at the origin.

If R is a local ring we know the dimension of the previous graded rings, [HIO88]:

Proposition 5. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* d *and let* I *an ideal of* R. *Then* $\dim(gr_I(R)) = d$ *and*

$$\dim(\mathcal{R}(I)) = \begin{cases} d+1, & \text{if } I \not\subseteq \mathfrak{p}, \text{ for some } \mathfrak{p} \text{ of } R \text{ such that } \dim(R/\mathfrak{p}) = d \\ d, & \text{otherwise.} \end{cases}$$

If $d > 0$ and I is an $\mathfrak{m}$-primary ideal of R, we get that $\dim(\mathcal{R}(I)) = d+1$. Also we have that $\mathrm{length}_R(I^n/I^{n+1}) < \infty$ and $gr_I(R)$ is a graded ring of dimension d.

Another important result is the relation between the depth of a Noetherian local ring and its associated graded ring with respect to an ideal. So, if I is an ideal of a Notherian local ring we have that, [AA82]

$$\mathrm{depth}(R) \geq \mathrm{depth}(gr_I(R)).$$

Several times we use some depth formulas proved by Huckaba and Marley in [HM94] that relate the depth of the blow-up algebras $\mathcal{R}(I)$ and $gr_I(R)$:

Theorem 6. *Let* $(R, \mathfrak{m})$ *be a local ring of dimension* $d > 0$, *and let* I *be an ideal of* R. *Then*

$$\mathrm{depth}(\mathcal{R}(I)) \geq \mathrm{depth}(gr_I(R)).$$

If $\mathrm{depth}(gr_I(R)) < \mathrm{depth}(R)$ *then*

$$\mathrm{depth}(\mathcal{R}(I)) = \mathrm{depth}(gr_I(R)) + 1.$$

Clearly, if R is Cohen-Macaulay and I an $\mathfrak{m}$-primary ideal, the second statement occurs when $gr_I(R)$ is not Cohen-Macaulay since $\dim gr_I(R) = d$.

Another interesting and useful tool are the reductions of an ideal:

Definition 7. *The ideal $J \subseteq I$ is said to be a* reduction *of I if there exists an integer $r \geq 0$ such that $I^{r+1} = JI^r$. An ideal J is a* minimal reduction *of I if J is a reduction of I an J itself does not have any proper reduction. If J is a minimal reduction of I, we define the* reduction number of I with respect to J *is*

$$r_J(I) = \min\{r \geq 0 | I^{r+1} = JI^r\}.$$

The reduction number of I *is defined as*

$$r(I) = \min\{r_J(I) | J \textit{ is a minimal reduction of } I\}$$

Clearly, $\mathcal{R}(J) \hookrightarrow \mathcal{R}(I)$. But sometimes it is easier to study $\mathcal{R}(J)$ than $\mathcal{R}(I)$. In fact, $\mathcal{R}(J)$ is very well understood, [Vaz95]:

Proposition 8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$, and infinite residue field $\mathbf{k} = R/\mathfrak{m}$. Let I an $\mathfrak{m}$-primary ideal of R and let J be a minimal reduction of I. Then*

- *$J = (x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ is a regular sequence on R;*
- *$gr_J(R) \cong (R/J)[T_1, \ldots, T_d]$, which is a Cohen-Macaulay ring of dimension d;*
- *$\mathcal{R}(J) \cong R[T_1, \ldots, T_d]/L$, which is a Cohen-Macaulay ring of dimension $(d+1)$, where L is the ideal generated by the 2×2 minors of the matrix $\begin{pmatrix} T_1 & \ldots & T_d \\ x_1 & \ldots & x_d \end{pmatrix}$.*

Superficial elements

Superficial elements are elements that maintain properties of I when we take a quotient. They are very used in proofs based on induction on the dimension. Moreover, superficial sequences are strongly related with minimal reductions.

Definition 9. *Let R be a Noetherian ring and I an ideal of R. An element $x \in I^t \setminus I^{t+1}$ is called a superficial element of degree t for I if there exists $c \geq 0$ such that*

$$(I^{n+t} : x) \cap I^c = I^n$$

for $n \geq c$.

Proposition 10. *Some properties of superficial elements are:*

(i) For some $t \geq 1$ there always exists a superficial element of degree t.

(ii) If R is local with infinite residue field, then there is always a superficial element of degree 1.

(iii) If $\operatorname{grade}(I) \geq 1$ and $\bigcap_{n \geq 0} I^n = 0$, then every superficial element is a non-zero divisor in R

Definition 11. *A set of elements $x_1, \ldots, x_s \in I$ is called a superficial sequence of I if $x_1 \in I$ is a superficial element, and for all $2 \leq i \leq s$, $x_i \in I/(x_1, \ldots, x_{i-1})$ is a superficial element.*

See [ZS75], [Sal78], [KM82], [Rho71], [HM97] as a main references.

Hilbert function

The Hilbert function of an $\mathfrak{m}$-primary ideal in an important numerical function in local algebra. It is defined from the Hilbert function of the associated graded ring of the ideal.

Definition 12. *The* Hilbert function *of an $\mathfrak{m}$-primary ideal I in a d-dimensional Noetherian local ring $(R, \mathfrak{m})$ is defined by*

$$h_I(n) = \operatorname{length}_R \left(\frac{I^n}{I^{n+1}} \right),$$

for $n \geq 0$.

It is well known that there exists a polynomial $p_I \in \mathbb{Q}[Z]$ such that $h_I(n) = p_I(n)$ for $n \geq n_0$, and that can be written in following form

$$p_I(X) = \sum_{i=0}^{d-1} (-1)^i e_i(I) \binom{X+d-i-1}{d-i-1}.$$

The polynomial p_I is called the *Hilbert polynomial* of I. It is a polynomial of degree $d-1$ and the integers $e_0(I), \dots, e_{d-1}(I)$ are called the *Hilbert coefficients* of I.

Definition 13. *Given a positive integer d, any $a \in \mathbb{N}$ can be written uniquely in the form*

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(j)}{j},$$

*with $k(d) > k(d-1) > \cdots > k(j) \geq j \geq 1$ integers. It is called the d-*binomial expansion of *a, or d-*th Macaulay representation of *a.*

We define

$$a^{<d>} = \binom{k(d)+1}{d+1} + \binom{k(d-1)+1}{d} + \cdots + \binom{k(j)+1}{j+1}.$$

When $a \leq d$, then $a^{<d>} \leq a$. See [BH93] for more details.

These expansions play an important role in the Macaulay's Theorem and to study the growth of the Hilbert function.

Theorem 14 (Macaulay)**.** *Let K be a field, and let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function. The following conditions are equivalent:*

(i) $H(0) = 1$ *and* $H(n+1) \leq H(n)^{<n>}$ *for all* $n \geq 1$,

(ii) there exists a homogeneous K-algebra R with Hilbert function $h_R = H$.

A good reference on Hilbert functions of ideals, or in a more general case, Hilbert functions of filtrations of modules is the monograph of Rossi and Valla [RV07].

Chapter 2

Multigraded structures

In this chapter we deal with some general aspects of non-standard multigraded structures. A general reference on the subject is [GW78]. In particular we study the behavior of the Hilbert function of a non-standard multigraded module, which in this case is a quasi-polynomial function, and the Grothendieck-Serre formula that relates the difference between the Hilbert function and the Hilbert quasi-polynomial with the characteristic of the local cohomology modules.

In the first section, we set some notations. In Section 2.2 we define $\mathrm{Proj}^r(S)$ and we introduce the relevant dimensions of a ring and of a module. In order to study the multigraded Hilbert function in Section 2.3, we define and prove some results on quasi-polynomial functions in 2.3.1. Then, in Section 2.4 we study the Grothendieck-Serre formula after proving some results on the vanishing and finite generation of some local cohomology modules. Finally, we generalize the notion of Hilbert-Samuel function of an ideal to a set of ideals. We prove that is a polynomial function and we get a similar formula to the Grothendieck-Serre one relating this last function and its polynomial to the characteristic of the local cohomology modules of the k-th extended multigraded Rees algebra with respect to the irrelevant ideal of the multigraded Rees algebra.

2.1 Notations

In this section we set the basic definitions and notations on multigraded rings used in the following chapters.

In the thesis, we use multi-index notation as follows: we denote $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ and $|\underline{n}| = |n_1| + \cdots + |n_r| \in \mathbb{N}$. The sum of two elements $\underline{m}, \underline{n} \in \mathbb{Z}^r$ is defined by $\underline{m} + \underline{n} = (m_1 + n_1, \ldots, m_r + n_r)$, the product by $\underline{m} \cdot \underline{n} = (m_1 n_1, \ldots, m_r n_r)$, and we order these elements componentwise, that is $\underline{n} > \underline{m}$ ($\underline{n} \geq \underline{m}$) if $n_i > m_i$ ($n_i \geq m_i$) for each $i = 1, \ldots, r$.

A $\mathbb{Z}^r$-graded ring, or a multigraded ring if no confusion may arise, is a ring S endowed with a direct sum decomposition $S = \bigoplus_{\underline{n} \in \mathbb{Z}^r} S_{\underline{n}}$, such that $S_{\underline{m}} S_{\underline{n}} \subset S_{\underline{m}+\underline{n}}$ for any $\underline{m}, \underline{n} \in \mathbb{Z}^r$.

We define a $\mathbb{Z}^r$-graded S-module M as an S-module with a decomposition $M = \bigoplus_{\underline{n} \in \mathbb{Z}^r} M_{\underline{n}}$, such that $S_{\underline{m}} M_{\underline{n}} \subset M_{\underline{m}+\underline{n}}$ for any $\underline{m}, \underline{n} \in \mathbb{Z}^r$. $M_{\underline{n}}$ is the homogeneous component of M of multidegree $\underline{n}$. An element $x \in M_{\underline{n}}$ is called homogeneous of multidegree $\underline{n}$.

Given a $\mathbb{Z}^r$-graded S-module, we can consider the category of $\mathbb{Z}^r$-graded S-modules, and we denote it by $\mathcal{M}^r(S)$. The objects in this category are $\mathbb{Z}^r$-graded S-modules, and the morphisms $f : M \to N$ are S-module morphisms such that $f(M_{\underline{n}}) \subset N_{\underline{n}}$ for all $\underline{n} \in \mathbb{Z}^r$.

We can shift a $\mathbb{Z}^r$-graded S-module as follows: given a $\underline{k} \in \mathbb{Z}^r$, $M(\underline{k})$ is the $\mathbb{Z}^r$-graded S-module with the grading given by $M(\underline{k})_{\underline{n}} = M_{\underline{k}+\underline{n}}$.

2.2 Topological structure of $\operatorname{Proj}^r(S)$

In this section we introduce some definitions that provide the set of all relevant homogeneous prime ideals in a multigraded ring with a structure of a topological space. We also define the relevant dimensions of a multigraded ring and of a multigraded module.

From now on, let $S = \bigoplus_{\underline{n} \in \mathbb{N}^r} S_{\underline{n}}$ be a $\mathbb{Z}^r$-graded ring, where $S_{\underline{0}}$ is a Noetherian local ring. Let S be generated over $S_{\underline{0}}$ by elements $g_1^1, \ldots, g_1^{\mu_1}$, $g_2^1, \ldots, g_2^{\mu_2}, \ldots, g_r^1, \ldots, g_r^{\mu_r}$ with g_1^j of multidegree $\gamma_1 = (\gamma_1^1, 0 \ldots, 0) \in \mathbb{N}^r$ for $j = 1, \ldots, \mu_1$ with $\gamma_1^1 \neq 0$, g_2^j of multidegree $\gamma_2 = (\gamma_1^2, \gamma_2^2, 0 \ldots, 0) \in \mathbb{N}^r$ for $j = 1, \ldots, \mu_2$ with $\gamma_2^2 \neq 0$, and g_r^j of multidegree $\gamma_r = (\gamma_1^r, \ldots, \gamma_r^r) \in \mathbb{N}^r$ for $j = 1, \ldots, \mu_r$ with $\gamma_r^r \neq 0$.

Let $\mathcal{M}$ be the maximal homogeneous ideal of S, $\mathcal{M} = \mathfrak{m} \oplus \bigoplus_{\underline{n} \neq \underline{0}} S_{\underline{n}}$, where $\mathfrak{m}$ is the maximal ideal of the local ring $S_{\underline{0}}$. As usual, we write

$S_+ = \bigoplus_{\underline{n} \neq \underline{0}} S_{\underline{n}}$.

We should remark that some of the following definitions can be set also in a more general context.

Definition 2.2.1. *For* $i = 1, \ldots, r$, *let* I_i *be the ideal of* S *generated by the homogeneous components of* S *of multidegree* $(b_1, \ldots, b_i, 0, \ldots, 0)$ *with* $b_i \neq 0$. *We define the* irrelevant ideal *of* S *as* $S_{++} = I_1 \cdots I_r$.

Note that $S_+ \supset S_{++}$.

Definition 2.2.2. *Let* $\mathrm{Proj}^r(S)$ *be the set of all relevant homogeneous prime ideals on* S, *i.e. the set of all homogeneous prime ideals* $\mathfrak{p}$ *in* S *such that* $\mathfrak{p} \not\supset S_{++}$.

Definition 2.2.3. *Given a homogeneous ideal* I *of* S, *we denote by* $V_{++}(I) := \{\mathfrak{p} \in \mathrm{Proj}^r(S) \mid \mathfrak{p} \supset I\}$. $\mathrm{Proj}^r(S)$ *has a structure of a topological space given by the family of the closed subsets* $V_{++}(I)$, *with* I *a homogeneous ideal of* S.

Following the definition in [VKM94], where the standard bigraded case was studied, we define the relevant dimension of S as follows:

Definition 2.2.4. *The* relevant dimension *of* S *is the integer*

$$\mathrm{rel.\,dim}(S) = \begin{cases} r-1 & \text{if } \mathrm{Proj}^r(S) = \emptyset \\ \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Proj}^r(S)\} & \text{if } \mathrm{Proj}^r(S) \neq \emptyset. \end{cases}$$

The following lemma was proved in [Hyr99], Lemma 1.2, however we rewrite its proof for the sake of completeness.

Lemma 2.2.5. $\dim(\mathrm{Proj}^r(S)) = \mathrm{rel.\,dim}(S) - r$.

Proof. If $\mathrm{Proj}^r(S) = \emptyset$, the formula holds trivially, so we can assume that $\mathrm{Proj}^r(S) \neq \emptyset$. In this case, let $\mathfrak{p} \in \mathrm{Proj}^r(S)$ be a closed point. Since the projection morphism

$$\mathrm{Proj}^r(S) \to \mathrm{Spec}(S_{\underline{0}})$$

is a proper morphism, $\mathfrak{p}_{\underline{0}} = \mathfrak{p} \cap S_{\underline{0}}$ is a closed point in $\mathrm{Spec}(S_{\underline{0}})$. Hence, $(S/\mathfrak{p})_{\underline{0}} \cong S_{\underline{0}}/\mathfrak{p}_{\underline{0}}$ is a field and $S/\mathfrak{p}$ a catenary algebra. Moreover, since $\mathfrak{p}$ is a closed point, we have that $\dim(\mathrm{Proj}^r(S/\mathfrak{p})) = 0$.

Since $\mathfrak{p} \not\supset S_{++} = I_1 \cdots I_r$, for $i = 1, \ldots, r$, let J_i be the ideal of $S/\mathfrak{p}$ generated by the homogeneous components of $S/\mathfrak{p}$ of degree $(b_1, \ldots, b_i, 0, \ldots, 0)$ with $b_i \neq 0$. Using these ideals we get a maximal chain of prime ideals

$$0 \subset J_r \subset J_{r-1} + J_r \subset \cdots \subset J_1 + \cdots + J_r$$

and so $\dim(S/\mathfrak{p}) = r$ because $S/\mathfrak{p}$ is catenary.

Now, any maximal chain of homogeneous prime ideals of S, starting from a minimal prime $\mathfrak{q}_0 \in \mathrm{Proj}^r(S)$ is of the type

$$\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_s \subset \cdots \subset \mathfrak{q}_{s+r}$$

with $\mathfrak{q}_s$ a closed point of $\mathrm{Proj}^r(S)$. Finally, we have

$$\begin{aligned} \dim(\mathrm{Proj}^r(S)) &= \sup\{\mathrm{ht}(\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Proj}^r(S)\} \\ &= \sup\{\dim(S/\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Proj}^r(S)\} - r \\ &= \mathrm{rel.}\dim(S) - r. \end{aligned}$$

□

For a module we have the following definitions:

Definition 2.2.6. *Given a finitely generated $\mathbb{Z}^r$-graded S-module M, we define the* homogeneous support *of M as*

$$Supp_{++}(M) = \{\mathfrak{p} \in \mathrm{Proj}^r(S) \mid M_{\mathfrak{p}} \neq 0\}.$$

Observe that $Supp_{++}(M) = V_{++}(Ann(M))$ is a closed subset of $\mathrm{Proj}^r(S)$.

Definition 2.2.7. *The* relevant dimension of a module M *is the integer*

$$\mathrm{rel.}\dim(M) = \begin{cases} r-1 & \text{if } Supp_{++}(M) = \emptyset \\ \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in Supp_{++}(M)\} & \text{if } Supp_{++}(M) \neq \emptyset. \end{cases}$$

It is clear from the definition that

$$\dim(Supp_{++}(M)) = \mathrm{rel.}\dim(M) - r.$$

In order to use induction on the relevant dimension to prove some results, we need the following lemma.

Lemma 2.2.8. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module, and $a \in S$ such that $a \notin \mathfrak{p}$ for any $\mathfrak{p} \in Ass(M)$. Then*

$$\mathrm{rel.}\dim(M/aM) = \mathrm{rel.}\dim(M) - 1.$$

Proof. We may assume that $Supp_{++}(M) \neq \emptyset$, otherwise it makes no sense to reduce the relevant dimension.

Hence,

$$\begin{aligned} \mathrm{rel.dim}(M) &= \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in Supp_{++}(M)\} \\ &= \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in Supp_{++}(M) \cap Min(M)\}. \end{aligned}$$

Now, for all $\mathfrak{p} \in Supp_{++}(M) \cap Min(M)$, since $a \notin \mathfrak{p}$, we have that

$$\dim(S/\mathfrak{p}) = \dim(S/\mathfrak{p} + (a)) + 1.$$

Then,

$$\begin{aligned} \mathrm{rel.dim}(M) &= \max\{\dim(S/\mathfrak{p} + (a)) \mid \mathfrak{p} \in Supp_{++}(M)\} + 1 \\ &= \mathrm{rel.dim}(M/aM) + 1, \end{aligned}$$

since $Supp_{++}(M/aM) = \{\mathfrak{p} + (a) \mid \mathfrak{p} \in Supp_{++}(M)\}$. □

Remark 2.2.9. Let us consider the polynomial ring $S = K[T_1, \ldots, T_r]$ with coefficients in a field K with indeterminates $T_1, \ldots, T_r$. We denote by $e_1, \ldots, e_r$ the canonical basis of $\mathbb{R}^r$.

We can consider S as a (standard) multigraded ring by providing it with the multigraduation $\deg(T_i) = e_i$, for $i = 1, \ldots, r$. S can be also considered as a graded ring by grading it with respect to the total degree. In this case, this is the same as assigning the graduation $\deg(T_i) = 1$ for all $i = 1, \ldots, r$. A monomial $T^{\underline{n}} = T_1^{n_1} \cdots T_r^{n_r}$ has degree $\underline{n} \in \mathbb{N}^r$ as a multigraded ring, and $|\underline{n}| \in \mathbb{N}$ as a graded ring.

Note that, when we consider S as a graded module, the dimension over K of the homogeneous piece S_d, with $d = |\underline{n}|$, is $\binom{r+d-1}{d-1}$, whereas in the multigraded case, the dimension of $S_{\underline{n}}$ is 1, since there is a unique monomial of such degree.

Hence the only homogeneous ideals in the multigraded setting are the monomial ones. As a multigraded ring, the irrelevant ideal is $S_{++} = (T_1 \cdots T_r)$. Therefore, $\mathrm{Proj}^r(S) = \{(0)\}$. In fact, if $\mathfrak{p}$ is a homogeneous (monomial) prime such that $\mathfrak{p} \not\supset (T_1 \cdots T_r)$, then $\mathfrak{p}$ can not contain any indeterminate T_i. So $\mathfrak{p} \subset K$, and hence $\mathfrak{p} = (0)$. This does not happen neither in the graded case nor in the multigraded case when we have more generators of the same degree. In the graded case, $S_{++} = (T_1, \ldots, T_r) = S_+$.

Then, $\dim(\mathrm{Proj}^r(S)) = 0$, $\mathrm{rel.dim}(S) = r$, $\dim(\mathrm{Proj}^1(S)) = r - 1$ and $\dim(S) = r$.

2.3 Multigraded Hilbert function

If we consider a Noetherian multigraded $S_{\underline{0}}$-algebra S and a multigraded S-module $M = \bigoplus_{\underline{n} \in \mathbb{Z}^r} M_{\underline{n}}$, assuming that all pieces $M_{\underline{n}}$ have finite length, the Hilbert function is defined by

$$\begin{array}{rccl} h_M : & \mathbb{Z}^r & \longrightarrow & \mathbb{Z} \\ & \underline{n} & \mapsto & \operatorname{length}_{S_{\underline{0}}}(M_{\underline{n}}) \end{array}$$

where $\underline{n} = (n_1, \ldots, n_r)$.

When we consider a standard graduation, that is when the generators have multidegrees $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, $\ldots$, $(0,\ldots,0,1)$, then it is well known that this function is polynomial in r indeterminates for $n_1, \ldots, n_r$ large enough. See for example [HHRT97], [VKM94] and [Rob98].

In the non-standard setting some cases have been studied. For instance in [Lav99] and [Rob98], the authors studied the case in which the generators have multidegrees $(1,0,\ldots,0)$, $(d_1^2,1,0,\ldots,0)$, $\ldots$, $(d_1^r,\ldots,d_{r-1}^r,1)$. They proved that the Hilbert function is a polynomial in r indeterminates for $(n_1,\ldots,n_r)$ in a region (a cone) of $\mathbb{Z}^r$. Other references are [HT03] and [Rob00].

A more general setting was studied by J.B. Fields in his PhD thesis, [Fie00] (see also [Fie02]). He considers the general definition of quasi-polynomial and proves that the Hilbert function of a $\mathbb{N}^r$-graded module is quasi-polynomial in a region of $\mathbb{Z}^r$. In his proof, however, the cone was not explicitly described.

For our purposes, we need to control the cone where the Hilbert function is a quasi-polynomial. So, in this section we want to study the asymptotic behavior of the Hilbert function of a non-standard multigraded module by considering the general setting of this thesis. That is, M is a $\mathbb{Z}^r$-graded S-module, where S is a $\mathbb{Z}^r$-graded ring with generators g_i^j of degree $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0 \ldots, 0) \in \mathbb{N}^r$ and $\gamma_i^i \neq 0$, over an Artinian local ring $S_{\underline{0}}$, for $i = 1, \ldots, r$, $j = 1, \ldots, \mu_i$.

Observe that this graduation admits in particular the standard case. However, by abuse of language, we refer to this more general graduation as a *non-standard*, to bear in mind the differences with the standard situation. That is, the standard graduation is not excluded from our definition.

2.3.1 Quasi-polynomial functions

Before studying the behavior of the Hilbert function of a $\mathbb{Z}^r$-graded S-module, let us introduce the definition of a quasi-polynomial in the $\mathbb{Z}^r$-graded case. We also give some definitions and results that we will need in order to study the Hilbert function. See [Fie00] and [Fie02] as a reference on general quasi-polynomials.

Given $\underline{m} = (m_1, \ldots, m_r)$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, we set $\underline{m}^{\underline{\alpha}} = \prod_{i=1}^r m_i^{\alpha_i}$. Given a degree $\underline{m} \in \mathbb{N}^r$, the total degree is denoted by $|\underline{m}| = \sum_{i=1}^r m_i \in \mathbb{N}$.

Definition 2.3.1. *Let $\gamma_1, \ldots, \gamma_r \in \mathbb{N}^r$ be linearly independent vectors over $\mathbb{R}$. A function $f : \mathbb{N}^r \to \mathbb{Z}$ is* periodic with respect to $\gamma_1, \ldots, \gamma_r$ *if $f(\underline{\alpha} + \underline{\gamma}) = f(\underline{\alpha})$ for any $\underline{\alpha} \in \mathbb{N}^r$ and for any $\underline{\gamma} \in \mathbb{N}\gamma_1 + \cdots + \mathbb{N}\gamma_r$.*

Definition 2.3.2. *Given $\underline{\beta} \in \mathbb{N}^r$ and $\gamma_1, \ldots, \gamma_r \in \mathbb{N}^r$ linearly independent vectors , we define the* cone with vertex $\underline{\beta}$ with respect to $\gamma_1, \ldots, \gamma_r$ *as*

$$C_{\underline{\beta}} := \left\{ \underline{\alpha} \in \mathbb{N}^r \mid \underline{\alpha} = \underline{\beta} + \sum_{i=1}^r \lambda_i \gamma_i, \;\; \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

Given a cone $C_{\underline{\beta}}$ with vertex at $\underline{\beta} \in \mathbb{N}^r$ with respect to $\gamma_1, \ldots, \gamma_r$, we define the basic cell $\Pi_{\underline{\beta}}$ *as*

$$\Pi_{\underline{\beta}} = \left\{ \underline{\alpha} \in \mathbb{N}^r \mid \underline{\alpha} = \underline{\beta} + \sum_{i=1}^r m_i \gamma_i, \;\; 0 \leq m_i < 1 \right\}.$$

For any element $\underline{\alpha} \in C_{\underline{\beta}} \subset \mathbb{N}^r$, there is a unique representative of $\underline{\alpha}$ modulo $\gamma_1, \ldots, \gamma_r$ in $\Pi_{\underline{\beta}}$.

Definition 2.3.3. *We say that a function $f : \mathbb{N}^r \to \mathbb{Z}$ is a* quasi-polynomial function of polynomial degree d on $\underline{\beta}, \gamma_1, \ldots, \gamma_r$ *if there exist periodic functions, for $\underline{\alpha} \in \mathbb{N}^r$ and $|\underline{\alpha}| \leq d$,*

$$c_{\underline{\alpha}} : \mathbb{N}^r \to \mathbb{Z}$$

with respect to $\gamma_1, \ldots, \gamma_r$ such that for $\underline{n} \in C_{\underline{\beta}}$

$$f(\underline{n}) = \sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$$

and $f(\underline{n}) = 0$ when $\underline{n} \notin C_{\underline{\beta}}$, and there is some $\underline{\alpha} \in \mathbb{N}^r$ with $|\underline{\alpha}| = d$ such that $c_{\underline{\alpha}} \neq 0$. We call quasi-polynomial *an expression $\sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$.*

This definition of a quasi-polynomial $P(\underline{n}) = \sum_{|\underline{\alpha}|\leq d} c_{\underline{\alpha}}(\underline{n})\underline{n}^{\underline{\alpha}}$ is equivalent to give a collection of polynomials of total degree $\leq d$

$$f_{\underline{\delta}}(\underline{n}) = \sum_{\underline{\alpha}\in\mathbb{N}^r} c_{\underline{\alpha}}(\underline{\delta})\underline{n}^{\underline{\alpha}} \in \mathbb{Z}[\underline{n}]$$

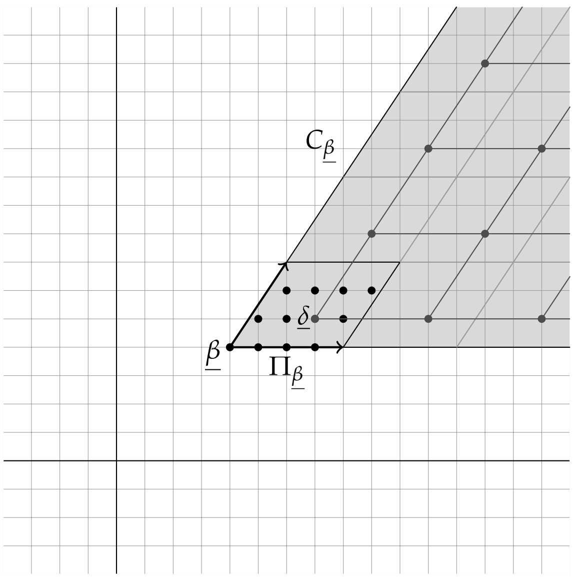
for each $\underline{\delta} \in \Pi_{\underline{\beta}}$.

Since $\gamma_1,\ldots,\gamma_r$ are linearly independent, any vector $\underline{n} \in C_{\underline{\beta}}$ can be written uniquely as $\underline{n} = \underline{\delta} + \sum_{i=1}^r m_i\gamma_i$, with $\underline{\delta} \in \Pi_{\underline{\beta}}$ and $\underline{m} \in \mathbb{N}^r$. Clearly,

$$f(\underline{n}) = f_{\underline{\delta}}(\underline{n})$$

for all $\underline{n} \in \underline{\delta} + \langle\gamma_1,\ldots,\gamma_r\rangle_{\mathbb{N}}$, because $c_{\underline{\alpha}}$ are periodic functions with respect to $\gamma_1,\ldots,\gamma_r$.

We exemplify this fact in the following picture where we consider $r = 2$, and we represent the cone, the basic cell, and for each element in the basic cell, a sub-net of the cone where the quasi-polynomial is defined for the same polynomial.



According to the literature, these kind of quasi-polynomials are called *simple quasi-polynomials* with respect to $\underline{\beta},\gamma_1,\ldots,\gamma_r$. In [Fie00] and [Fie02], a quasi-polynomial is the sum of simple quasi-polynomials, each one with respect to different sets of vectors and cones.

In this work, we only deal with simple ones. Since in our case the vectors $\gamma_1, \ldots, \gamma_r$ are fixed, we refer to a simple quasi-polynomial with respect to $\underline{\beta}, \gamma_1, \ldots, \gamma_r$ as a quasi-polynomial.

Definition 2.3.4. *Given a numerical function $f : \mathbb{N}^r \to \mathbb{Z}$, we define the* generating function *of f as*

$$F(x_1, \ldots, x_r) = \sum_{\underline{n} \in \mathbb{N}^r} f(\underline{n}) \underline{x}^{\underline{n}} \in \mathbb{Z}[[x_1, \ldots, x_r]].$$

The aim of the following proposition is to relate quasi-polynomials with the rationality of their generating functions.

Proposition 2.3.5. *Let $f : \mathbb{N}^r \to \mathbb{Z}$ be a numerical function with generating function $F(x_1, \ldots, x_r)$. Then, f is a quasi-polynomial function of polynomial degree d on $\underline{\beta}, \gamma_1, \ldots, \gamma_r$ if and only if*

$$F(x_1, \ldots, x_r) = \sum_{\underline{\delta} \in \Pi_{\underline{\beta}}} \sum_{|\underline{t}| - r \leq d} \frac{\lambda_{\underline{t}, \underline{\delta}} \underline{x}^{\underline{\delta}}}{\prod_{j=1}^r (1 - \underline{x}^{\gamma_j})^{t_j}}$$

with integers $\lambda_{\underline{t}, \underline{\delta}} \in \mathbb{Z}$ such that there exist a $\underline{t} \in \mathbb{N}^r$, with $|\underline{t}| = d + r$, and a $\underline{\delta} \in \Pi_{\underline{\beta}}$ such that $\lambda_{\underline{t}, \underline{\delta}} \neq 0$.

Proof. First of all, observe that

$$\frac{1}{\prod_{j=1}^r (1 - z_j)^{b_j}} = \sum_{\underline{n} \in \mathbb{N}^r} \prod_{j=1}^r \binom{b_j - 1 + n_j}{n_j} \underline{z}^{\underline{n}}.$$

In fact, since

$$\frac{1}{1 - z} = \sum_{i \geq 0} z^i,$$

then

$$\frac{1}{(1 - z)^b} = \left(\sum_{i \geq 0} z^i \right)^b,$$

which is a power series on z where the coeficient of z^m is the cardinal of the set of monomials $z^{i_1} \cdots z^{i_b}$ with $i_1 + \cdots + i_b = m$, and so

$$\#\{(i_1, \ldots, i_b) \in \mathbb{N}^b \mid i_1 + \cdots + i_b = m\} = \binom{b - 1 + m}{m}.$$

Therefore,

$$\frac{1}{\prod_{j=1}^{r}(1-z_j)^{b_j}} = \left(\sum_{n_1\geq 0}\binom{b_1-1+n_1}{n_1}z_1^{n_1}\right)\cdots\left(\sum_{n_r\geq 0}\binom{b_r-1+n_r}{n_r}z_r^{n_r}\right)$$
$$= \sum_{\underline{n}\in\mathbb{N}^r}\left(\prod_{j=1}^{r}\binom{b_j-1+n_j}{n_j}\right)\underline{z}^{\underline{n}}.$$

Note that $\prod_{j=1}^{r}\binom{b_j-1+n_j}{n_j}$ is a polynomial on $n_1,\ldots,n_r$ of total degree $|\underline{b}|-r$.

Let us now assume that f is a quasi-polynomial function. We want to see how is its generating function.

Since f is a quasi-polynomial function of polynomial degree d with respect to $\underline{\beta},\gamma_1,\ldots,\gamma_r$, we can write for all $\underline{n}\in C_{\underline{\beta}}$

$$f(\underline{n}) = \sum_{|\underline{\alpha}|\leq d} c_{\underline{\alpha}}(\underline{n})\underline{n}^{\underline{\alpha}}$$

with $c_{\underline{\alpha}}$ periodic functions with respect to $\gamma_1,\ldots,\gamma_r$, and $f(\underline{n}) = 0$ for $\underline{n}\notin C_{\underline{\beta}}$.

For any $\underline{n}\in C_{\underline{\beta}}$, there exists an element $\underline{\delta}\in\Pi_{\underline{\beta}}$ such that

$$\underline{n} = \underline{\delta} + \sum_{i=1}^{r} m_i\gamma_i$$

with $m_i\in\mathbb{N}$. Since $c_{\underline{\alpha}}$ are periodic functions with respect to $\gamma_1,\ldots,\gamma_r$ we have that $c_{\underline{\alpha}}(\underline{n}) = c_{\underline{\alpha}}(\underline{\delta})\in\mathbb{Z}$ for all $\underline{n}\in\underline{\delta}+\langle\gamma_1,\ldots,\gamma_r\rangle_{\mathbb{N}}$. So,

$$f(\underline{n}) = \sum_{|\underline{\alpha}|\leq d} c_{\underline{\alpha}}(\underline{\delta})\underline{n}^{\underline{\alpha}}\in\mathbb{Z}[n_1,\ldots,n_r]$$

is a polynomial in $n_1,\ldots,n_r$ for all $\underline{n}\in\underline{\delta}+\langle\gamma_1,\ldots,\gamma_r\rangle_{\mathbb{N}}$. We denote $f_{\underline{\delta}}$ each one of these polynomials.

For each $\underline{\delta}\in\Pi_{\underline{\beta}}$ and an element $\underline{n} = \underline{\delta}+\sum_{i=1}^{r} m_i\gamma_i$ with $m_i\in\mathbb{N}$, we can rewrite $f_{\underline{\delta}}(\underline{n})$ as a polynomial in $m_1,\ldots,m_r$, and so we denote

$$f_{\underline{\delta}}(\underline{n}) = f_{\underline{\delta}}(\underline{\delta}+\sum_{i=1}^{r} m_i\gamma_i) = \sum_{|\underline{\alpha}|\leq d} c_{\underline{\alpha}}(\underline{\delta})(\underline{\delta}+\sum_{i=1}^{r} m_i\gamma_i)^{\underline{\alpha}} = g_{\underline{\delta}}(\underline{m}),$$

where $g_{\underline{\delta}}(\underline{m})\in\mathbb{Z}[\underline{m}]$.

Then, we have

$$\begin{aligned} F(x_1,\ldots,x_r) &= \sum_{\underline{n}\in C_{\underline{\beta}}} f(\underline{n})\underline{x}^{\underline{n}} = \sum_{\underline{\delta}\in\Pi_{\underline{\beta}}}\sum_{\underline{m}\in\mathbb{N}^r} f(\underline{\delta}+\sum_{i=1}^{r} m_i\gamma_i)\underline{x}^{\underline{\delta}+\sum_{i=1}^r m_i\gamma_i} \\ &= \sum_{\underline{\delta}\in\Pi_{\underline{\beta}}} \underline{x}^{\underline{\delta}} \left(\sum_{\underline{m}\in\mathbb{N}^r} g_{\underline{\delta}}(\underline{m})\underline{x}^{\sum_{i=1}^r m_i\gamma_i}\right). \end{aligned}$$

Now, since $g_{\underline{\delta}}(\underline{m})$ are polynomials of total degree $\leq d$ in $m_1,\ldots,m_r$, we can write

$$g_{\underline{\delta}}(\underline{m}) = \sum_{|\underline{t}|-r\leq d} \lambda_{\underline{t},\underline{\delta}} \prod_{j=1}^{r} \binom{t_j-1+m_j}{m_j}$$

because the polynomials $\prod_{j=1}^r \binom{t_j-1+m_j}{m_j}$ with $|\underline{t}|-r\leq d$ form a $\mathbb{Z}$-basis of the polynomials in $m_1,\ldots,m_r$ with coefficients in $\mathbb{Z}$ of total degree $\leq d$, see [CC97] Proposition XI.1.12. Since f is a quasi-polynomial of polynomial degree d, there exists a $\lambda_{\underline{t},\underline{\delta}}\neq 0$, for some $\underline{\delta}\in\Pi_{\underline{\beta}}$ and $|\underline{t}| = d+r$.

Therefore, since

$$\frac{1}{\prod_{j=1}^r (1-z_j)^{t_j}} = \sum_{\underline{n}\in\mathbb{N}^r}\left(\prod_{j=1}^r \binom{t_j-1+n_j}{n_j}\right)\underline{z}^{\underline{n}},$$

by considering $z_i = \underline{x}^{\gamma_i}$, we have that

$$\begin{aligned} \sum_{\underline{m}\in\mathbb{N}^r} g_{\underline{\delta}}(\underline{m})\underline{x}^{\sum_{i=1}^r m_i\gamma_i} &= \sum_{|\underline{t}|-r\leq d}\lambda_{\underline{t},\underline{\delta}}\left(\sum_{\underline{m}\in\mathbb{N}^r}\prod_{j=1}^r\binom{t_j-1+m_j}{m_j}\underline{x}^{\sum_{i=1}^r m_i\gamma_i}\right) \\ &= \sum_{|\underline{t}|-r\leq d}\frac{\lambda_{\underline{t},\underline{\delta}}}{\prod_{j=1}^r(1-\underline{x}^{\gamma_j})^{t_j}} \end{aligned}$$

with $|\underline{t}|-r\leq d$ and $\lambda_{\underline{t},\underline{\delta}}\in\mathbb{Z}$.

Finally, we can write

$$F(x_1,\ldots,x_r) = \sum_{\underline{\delta}\in\Pi_{\underline{\beta}}}\sum_{|\underline{t}|-r\leq d}\frac{\lambda_{\underline{t},\underline{\delta}}\underline{x}^{\underline{\delta}}}{\prod_{j=1}^r(1-\underline{x}^{\gamma_j})^{t_j}}$$

with $|\underline{t}|-r\leq d$, $\lambda_{\underline{t},\underline{\delta}}\in\mathbb{Z}$, and there exists a $\lambda_{\underline{t},\underline{\delta}}\neq 0$, for some $\underline{\delta}\in\Pi_{\underline{\beta}}$ and $|\underline{t}| = d+r$.

Conversely, assume that the generating function is as stated in the proposition. We want to prove that f is a quasi-polynomial function.

Since,

$$\frac{1}{\prod_{j=1}^{r}(1-\underline{x}^{\gamma_j})^{t_j}} = \sum_{\underline{m}\in\mathbb{N}^r}\prod_{j=1}^{r}\binom{t_j-1+m_j}{m_j}\underline{x}^{\sum_{i=1}^{r} m_i\gamma_i}$$

we have

$$F(x_1,\ldots,x_r) = \sum_{\underline{\delta}\in\Pi_{\underline{\beta}}}\sum_{\underline{m}\in\mathbb{N}^r}\left(\sum_{|\underline{t}|-r\le d}\lambda_{\underline{t},\underline{\delta}}\prod_{j=1}^{r}\binom{t_j-1+m_j}{m_j}\right)\underline{x}^{\underline{\delta}+\sum_{i=1}^{r} m_i\gamma_i}$$

and so,

$$f(\underline{n}) = \sum_{|\underline{t}|-r\le d}\lambda_{\underline{t},\underline{\delta}}\prod_{j=1}^{r}\binom{t_j-1+m_j}{m_j}$$

for $\underline{n} = \underline{\delta}+\sum_{i=1}^{r} m_i\gamma_i$, which is a polynomial in $m_1,\ldots,m_r$ of total degree $|\underline{t}|-r \le d$. Therefore, f is a quasi-polynomial function of polynomial degree d with respect to $\underline{\beta},\gamma_1,\ldots,\gamma_r$. □

Concerning the first derivative of quasi-polynomials, the following result will be crucial to prove the existence of a quasi-polynomial for the Hilbert function:

Proposition 2.3.6. *Let $f,g:\mathbb{N}^r\to\mathbb{Z}$ be functions, $\gamma_1,\ldots,\gamma_r\in\mathbb{N}^r$ be linearly independent vectors and $\underline{\beta}\in\mathbb{N}^r$, such that for all $\underline{\alpha}\in\mathbb{N}^r$ and some $i=1,\ldots,r$ it holds*

$$f(\underline{\alpha})-f(\underline{\alpha}-\gamma_i)=g(\underline{\alpha}).$$

If g is a quasi-polynomial on $\underline{\beta},\gamma_1,\ldots,\gamma_r$ of polynomial degree d, then f is also a quasi-polynomial on $\underline{\beta},\gamma_1,\ldots,\gamma_r$ of polynomial degree $d+1$.

Proof. We denote by F and G the generating functions of f and g respectively: $F(x_1,\ldots,x_r)=\sum_{\underline{n}\in\mathbb{N}^r} f(\underline{n})\underline{x}^{\underline{n}}$ and $G(x_1,\ldots,x_r)=\sum_{\underline{n}\in\mathbb{N}^r} g(\underline{n})\underline{x}^{\underline{n}}$.

By Proposition 2.3.5, since g is a quasi-polynomial function of polynomial degree d with respect to $\underline{\beta},\gamma_1,\ldots,\gamma_r$, we have

$$G=\sum_{\underline{\delta}\in\Pi_{\underline{\beta}}}\sum_{|\underline{t}|-r\le d}\frac{\lambda_{\underline{t},\underline{\delta}}\underline{x}^{\underline{\delta}}}{\prod_{j=1}^{r}(1-\underline{x}^{\gamma_j})^{t_j}}$$

with $\lambda_{\underline{t},\underline{\delta}}\neq 0$ for some $|\underline{t}|=d+r$ and $\underline{\delta}\in\Pi_{\underline{\beta}}$.

The relation $f(\underline{\alpha}) - f(\underline{\alpha} - \gamma_i) = g(\underline{\alpha})$ can be expressed in terms of the generating functions by

$$F(x_1, \dots, x_r) = \frac{G(x_1, \dots, x_r)}{(1 - \underline{x}^{\gamma_i})}.$$

Then

$$F = \sum_{\underline{\delta} \in \Pi_{\underline{\beta}}} \sum_{|\underline{t}| - r \leq d} \frac{\lambda_{\underline{t}, \underline{\delta}} \underline{x}^{\underline{\delta}}}{(1 - \underline{x}^{\gamma_i})^{t_i + 1} \prod_{\substack{j=1 \\ j \neq i}}^{r} (1 - \underline{x}^{\gamma_j})^{t_j}}$$

This means that f is also a quasi-polynomial on $\underline{\beta}, \gamma_1, \dots, \gamma_r$ of polynomial degree

$$t_1 + \cdots + (t_i + 1) + \cdots + t_r - r = d + 1.$$

□

2.3.2 Hilbert function of a multigraded module

As explained in the introduction, we are interested in the case where $S = \bigoplus_{\underline{n} \in \mathbb{N}^r} S_{\underline{n}}$ is a $\mathbb{Z}^r$-graded ring, where $S_{\underline{0}}$ is an Artinian local ring, and S is generated over $S_{\underline{0}}$ by elements

$$g_1^1, \dots, g_1^{\mu_1}, \dots, g_r^1, \dots, g_r^{\mu_r}$$

with g_i^j of multidegree $\gamma_i = (\gamma_1^i, \dots, \gamma_i^i, 0, \dots, 0) \in \mathbb{N}^r$ with $\gamma_i^i \neq 0$, for all $i = 1, \dots, r$ and $j = 1, \dots, \mu_i$.

Now we are ready to prove that the Hilbert function of a multigraded module is a quasi-polynomial function, but we first need a technical proposition that we will use more than once.

Let $\Gamma = \langle \gamma_1, \dots, \gamma_r \rangle_{\mathbb{N}}$ be the semigroup generated by $\gamma_1, \dots, \gamma_r$, and G be the non-singular $r \times r$ triangular matrix whose columns are the vectors $\gamma_1, \dots, \gamma_r$. Given $\underline{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$ we denote $\underline{n}^* = (|n_1|, \dots, |n_r|) \in \mathbb{N}^r$.

For a finitely generated $\mathbb{Z}^r$-graded S-module M, with $h_1, \dots, h_s$ generators of multidegrees $\underline{d}^1 = (d_1^1, \dots, d_r^1), \dots, \underline{d}^s = (d_1^s, \dots, d_r^s) \in \mathbb{Z}^r$ respectively, we denote by Γ_M the Γ-invariant subset of $\mathbb{Z}^r$

$$\Gamma_M = \bigcup_{i=1}^{s} (\underline{d}^i + \Gamma),$$

i.e. $\mathbb{Z}^r \setminus \Gamma_M$ is the set of multi-index for which there is no non-zero elements of M.

Lemma 2.3.7. *For all $\underline{\beta} \in \mathbb{Z}^r$ and $c \in \mathbb{N}$ there exists $\underline{\alpha} \in \Gamma_M$ such that $\underline{\alpha} \geq \underline{c} = (c, \ldots, c)$ and $\underline{\alpha} \in \underline{\beta} + \Gamma$.*

Proof. The condition $\underline{\alpha} \in (\underline{\beta} + \Gamma) \cap (\underline{d}^1 + \Gamma)$ is equivalent to the equation

$$\underline{\alpha} = \underline{d}^1 + G\underline{t} = \underline{\beta} + G\underline{n},$$

so

$$\underline{n} = \underline{t} + G^{-1}(\underline{d}^1 - \underline{\beta}).$$

Hence for a $\underline{t} \gg \underline{0}$ we get that $\underline{n} \gg \underline{0}$, so $\underline{\alpha} \in \Gamma_M \cap (\underline{\beta} + \Gamma)$ and $\underline{\alpha} \geq \underline{c}$. □

Proposition 2.3.8. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $S_{++} \subset rad(Ann_S(M))$. Then there exists $\underline{\beta} = (\beta_1, \ldots, \beta_r) \in \Gamma_M$ such that $M_{\underline{n}} = 0$, for all $\underline{n} \in \mathbb{Z}^r$ such that $\underline{n}^* \in C_{\underline{\beta}}$.*

Proof. We prove the result first assuming that M is $\mathbb{N}^r$ generated, i.e. we assume that $h_1, \ldots, h_s$ are the generators of the S-module M with multidegrees $(d_1^1, \ldots, d_r^1), \ldots, (d_1^s, \ldots, d_r^s) \in \mathbb{N}^r$ respectively. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ be the maximum componentwise of these multidegrees, i.e. we consider $\alpha_i = \max\{d_i^1, \cdots, d_i^s\}$, $i = 1, \cdots, r$.

The elements of $M_{\underline{n}}$, $\underline{n} \in \mathbb{N}^r$, are linear combinations with coefficients on $S_{\underline{0}}$ of elements of the type

$$\underline{g_1}^{\underline{m_1}} \cdots \underline{g_r}^{\underline{m_r}} h_j$$

where, using multi-index notation, we write $\underline{g_t}^{\underline{m_t}} = (g_t^1)^{m_t^1} \cdots (g_t^{\mu_t})^{m_t^{\mu_t}}$ with $\underline{m_t} = (m_t^1, \ldots, m_t^{\mu_t}) \in \mathbb{N}^{\mu_t}$. This element has multidegree

$$\underline{n} = \deg(\underline{g_1}^{\underline{m_1}} \ldots \underline{g_r}^{\underline{m_r}} h_j) = G \begin{pmatrix} |\underline{m_1}| \\ \vdots \\ |\underline{m_r}| \end{pmatrix} + \begin{pmatrix} d_1^j \\ \vdots \\ d_r^j \end{pmatrix}.$$

Let u be a non-negative integer such that $(S_{++})^u M = 0$. We define $\underline{\beta}$ recursively:

$$\beta_i = u\gamma_i^i + \beta_{i+1}\gamma_i^{i+1} + \cdots + \beta_r \gamma_i^r + \alpha_i$$

for $i = r, \dots, 1$.

Given a multi-index $\underline{n} = \underline{\beta} + \sum_{i=1}^{r} \lambda_i \gamma_i \in C_{\underline{\beta}} \cap \Gamma_M$, $\lambda_i \geq 0$, we have to prove that $M_{\underline{n}} = 0$. This is equivalent to prove that if

$$\underline{n} = G \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} = G \begin{pmatrix} |\underline{m_1}| \\ \vdots \\ |\underline{m_r}| \end{pmatrix} + \begin{pmatrix} d_1^j \\ \vdots \\ d_r^j \end{pmatrix}$$

then $|\underline{m_1}| \geq u + \lambda_1, \cdots, |\underline{m_r}| \geq u + \lambda_r$.

We will prove by recurrence a stronger result:

$$\beta_i + \lambda_i \geq |\underline{m_i}| \geq u + \lambda_i$$

for $i = 1, \dots, r$. From the definition of $\beta_r = u\gamma_r^r + \alpha_r$ and

$$\beta_r + \lambda_r \gamma_r^r = |\underline{m_r}|\gamma_r^r + d_r^j$$

we deduce that

$$\gamma_r^r(|\underline{m_r}| - (u + \lambda_r)) = \alpha_r - d_r^j \geq 0.$$

Since $\gamma_r^r \geq 1$ we get

$$|\underline{m_r}| \geq u + \lambda_r.$$

On the other hand

$$\beta_r + \lambda_r - |\underline{m_r}| = d_r^j + (\gamma_r^r - 1)(|\underline{m_r}| - \lambda_r) \geq 0.$$

Let us assume that $\beta_r + \lambda_r \geq |\underline{m_r}| \geq u + \lambda_r, \cdots, \beta_{i+1} + \lambda_{i+1} \geq |\underline{m_{i+1}}| \geq u + \lambda_{i+1}$. We will prove that $\beta_i + \lambda_i \geq |\underline{m_i}| \geq u + \lambda_i$, $i \geq 1$. We have

$$\beta_i + \lambda_i \gamma_i^i + \lambda_{i+1}\gamma_i^{i+1} + \cdots + \lambda_r \gamma_i^r = |\underline{m_i}|\gamma_i^i + |\underline{m_{i+1}}|\gamma_i^{i+1} + \cdots + |\underline{m_r}|\gamma_i^r + d_i^j$$

so

$$\gamma_i^i(u + \lambda_i - |\underline{m_i}|) + \sum_{l=i+1}^{r} \gamma_i^l(\beta_l + \lambda_l - |\underline{m_l}|) + \alpha_i - d_i^j = 0.$$

By induction we deduce that

$$|\underline{m_i}| \geq u + \lambda_i.$$

A simple computation shows that

$$\beta_i + \lambda_i - |\underline{m_i}| = (\gamma_i^i - 1)(|\underline{m_i}| - \lambda_i) + \sum_{l=i+1}^{r} \gamma_i^l(|\underline{m_l}| - \lambda_l) + d_i^j \geq 0.$$

Hence we have proved that $M_{\underline{n}} = 0$ for all $\underline{n} \in C_{\underline{\beta}}$.

Let us assume now that M is generated by $h_1, \ldots, h_s$ with multidegrees $(d_1^1, \ldots, d_r^1), \ldots, (d_1^s, \ldots, d_r^s) \in \mathbb{Z}^r$ respectively. Let us consider $c = |\min\{0, d_i^j, j = 1, \cdots, s, i = 1, \cdots, r\}|$. Let N be the following submodule of M:

$$N = \bigoplus_{\underline{n} \geq \underline{0}} M_{\underline{n}}.$$

From Lemma 2.3.7 there is $\underline{\alpha} \in \Gamma_M$ with $\underline{\alpha} \geq \underline{c}$ and $\underline{\alpha} \in \Gamma_M \cap (\underline{\beta}(N) + \Gamma)$. Since $C_{\underline{\alpha}} \subset C_{\underline{\beta}}$ and $\underline{\alpha} \geq \underline{c}$ we get that $M_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ and $n^* \in C_{\underline{\beta}}$. □

Corollary 2.3.9. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module and $N \subset M$ a submodule. We assume that $(S_{++})^u(M/N) = 0$ for $u \in \mathbb{Z}$. Then there exists $\underline{\beta} \in \Gamma_{M/N}$ such that $M_{\underline{n}} \subset N_{\underline{n}}$, for all $\underline{n} \in \mathbb{Z}^r$ such that $\underline{n}^* \in C_{\underline{\beta}}$.*

Proof. We only need to apply Proposition 2.3.8 to the finitely generated module M/N. There exists a cone $C_{\underline{\beta}}$ where $(M/N)_{\underline{n}} = 0$ for $\underline{n}^* \in C_{\underline{\beta}}$, and hence $M_{\underline{n}} \subset N_{\underline{n}}$. □

Now, for the main proof in this section we follow the idea used in the standard case in [HHRT97].

Proposition 2.3.10. *Let S be a $\mathbb{N}^r$-graded ring as considered before. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then there exists a quasi-polynomial P_M of polynomial degree* $\operatorname{rel.dim}(M) - r$ *and a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, such that*

$$h_M(\underline{n}) = P_M(\underline{n})$$

for any $\underline{n} \in C_{\underline{\beta}}$.

Proof. Since M is a finitely generated $\mathbb{Z}^r$-graded S-module, there is a chain of $\mathbb{Z}^r$-graded submodules of M

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

such that for each $j = 1, \ldots, l$, $M_j/M_{j-1} \cong (S/\mathfrak{p}_j)(\underline{m}_j)$, where $\mathfrak{p}_j \in Ass(M)$ is a homogeneous prime ideal and $\underline{m}_j \in \mathbb{Z}^r$.

In fact, assuming that $M \neq 0$, we choose a homogeneous prime ideal $\mathfrak{p}_1 \in Ass(M)$. Hence, there exist a $\mathbb{Z}^r$-graded submodule $M_1 \subset M$ such that $M_1 \cong (S/\mathfrak{p}_1)(\underline{m}_1)$. If $M_1 \neq M$, we can repeat this procedure with M/M_1 and we get a $\mathbb{Z}^r$-graded submodule $M_2 \subset M$ such that $M_2/M_1 \cong (S/\mathfrak{p}_2)(\underline{m}_2)$. Since M is Noetherian, this process finishes after a finite number of steps.

Now the Hilbert function of M can be computed by

$$\begin{aligned} h_M(\underline{n}) &= \sum_{j=1}^{l} \text{length}_{S_{\underline{0}}}((M_j/M_{j-1})_{\underline{n}}) \\ &= \sum_{j=1}^{l} h_{S/\mathfrak{p}_j}(\underline{n}+\underline{m}_j) \end{aligned}$$

Hence, we can reduce to the case that $M = S/\mathfrak{p}$, with $\mathfrak{p} \in Ass(M)$.

Now, we prove the proposition by induction on the relevant dimension of M, or equivalently on the dimension of $Supp_{++}(M)$.

Assume that $\text{rel.dim}(M) = r-1$ (i.e. $\dim(Supp_{++}(M)) = -1$). In this case $Supp_{++}(M) = \emptyset$, and hence $\emptyset = V_{++}(Ann(M)) = V_{++}(S_{++})$, so there exists an $u \in \mathbb{N}$ such that $(S_{++})^u \subset Ann(M)$. Now, since $(S_{++})^u M = 0$, by Proposition 2.3.8, there exists a $\underline{\beta} \in \mathbb{N}^r$ and a cone $C_{\underline{\beta}}$ where $M_{\underline{n}} = 0$ for all $\underline{n} \in C_{\underline{\beta}}$. Now, clearly, there will be a quasi-polynomial $P_M = 0$, of degree $\text{rel.dim}(M) - r = -1$, such that $h_M(\underline{n}) = P_M(\underline{n}) = 0$ for all $\underline{n} \in C_{\underline{\beta}}$.

Assume now that $\text{rel.dim}(M) > r-1$ (i.e. $\dim(Supp_{++}(M)) \geq 0$). Since $S_{++} \not\subset \mathfrak{p}$, there exists an element $g_1^{j(1)} \cdots g_r^{j(r)} \in S_{++}$ such that $g_1^{j(1)} \cdots g_r^{j(r)} \notin \mathfrak{p}$. It is clear that $g_i^{j(i)} \notin \mathfrak{p}$ for all $i = 1, \ldots, r$.

For each $i = 1, \ldots, r$ we consider the $\mathbb{Z}^r$-graded S-module N_i defined as

$$N_i := \frac{M}{g_i^{j(i)} M} = \frac{S}{\mathfrak{p} + (g_i^{j(i)})}.$$

Since $g_i^{j(i)}$ has multidegree $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0, \ldots, 0)$, with $\gamma_i^i \neq 0$, we can consider the $\mathbb{Z}^r$-graded exact sequence

$$0 \longrightarrow M(-\underline{\gamma_i}) \xrightarrow{\cdot g_i^{j(i)}} M \longrightarrow N_i \longrightarrow 0,$$

and so for all $\underline{n} \in \mathbb{Z}^r$ we have

$$h_M(\underline{n}) - h_M(\underline{n} - \gamma_i) = h_{N_i}(\underline{n}).$$

By Lemma 2.2.8, $\operatorname{rel.dim}(N_i) = \operatorname{rel.dim}(M) - 1$, and thus we can apply the induction hypothesis on N_i, i.e. there exists a quasi-polynomial P_{N_i} of degree $\operatorname{rel.dim}(M) - r - 1$ and a cone $C_{\underline{\beta}}$, $\underline{\beta} \in \mathbb{Z}^r$, such that

$$h_M(\underline{n}) - h_M(\underline{n} - \gamma_i) = P_{N_i}(\underline{n})$$

for all $\underline{n} \in C_{\underline{\beta}}$. Now by Proposition 2.3.6, there exists a quasi-polynomial P_M of degree $\operatorname{rel.dim}(M) - r$, such that $h_M(\underline{n}) = P_M(\underline{n})$ for all $\underline{n} \in C_{\underline{\beta}}$. $\square$

We refer to the quasi-polynomial P_M as the Hilbert quasi-polynomial of M.

Remark 2.3.11. Since giving a quasi-polynomial of degree d on $\underline{\beta}, \gamma_1, \ldots, \gamma_r$ is equivalent to give a collection of polynomials $f_{\underline{\delta}}(\underline{n}) \in \mathbb{Z}[\underline{n}]$ of total degree $\leq d$ (at least one of them has total degree d) for all $\underline{\delta} \in \Pi_{\underline{\beta}}$, the previous result can be interpreted as follows: if we consider, for each $\underline{\delta} \in \Pi_{\underline{\beta}}$, the submodule of M

$$M_{\underline{\delta}+\Gamma} = \bigoplus_{m_1,\ldots,m_r \geq 0} M_{\underline{\delta}+m_1\gamma_1+\cdots+m_r\gamma_r},$$

it has a standard multigraded structure, so the Hilbert function will be asymptotically a polynomial, that is $f_{\underline{\delta}}$. Considering $\bigoplus_{\underline{\delta}\in\Pi_{\underline{\beta}}} M_{\underline{\delta}+\Gamma}$, we cover all the pieces of M of multidegrees in the cone $C_{\underline{\beta}}$.

Example 2.3.12. Let us conclude the section with an example that illustrates the quasi-polynomials and our definitions in the multigraded case.

We consider the bigraded ring $S = K[x, y, z]$, with K a field, $\deg(x) = (2, 0)$ and $\deg(y) = \deg(z) = (1, 1)$. In this case the semigroup is $\Gamma = \langle (2, 0), (1, 1)\rangle_{\mathbb{N}} = \{(2\lambda + \mu, \mu) \mid \lambda, \mu \in \mathbb{N}\}$. It is clear that $S_{\underline{n}} = 0$ if $\underline{n} \notin \Gamma$. Since $S_{(2\lambda+\mu,\mu)}$ is generated over K by monomials $\{x^\lambda y^i z^{\mu-i}\}_{i=0,\ldots,\mu}$, we have that $\dim_K(S_{(2\lambda+\mu,\mu)}) = \mu + 1$. So, $\dim_K(S_{(n_1,n_2)}) = n_2 + 1$ if $(n_1, n_2) \in \Gamma$, and 0 otherwise.

The cone with vertex at $(0, 0)$ with respect to $(2, 0)$ and $(1, 1)$ is

$$C_{\underline{0}} = \{(n_1, n_2) \in \mathbb{N}^2 \mid (n_1, n_2) = \lambda(2, 0) + \mu(1, 1), \lambda, \mu \in \mathbb{R}_{\geq 0}\},$$

and hence the basic cell has only two elements $\Pi_{\underline{0}} = \{(0, 0), (1, 0)\}$. So the cone is composed by two nets:

$$C_{\underline{0}} = \{(0, 0) + \Gamma\} \cup \{(1, 0) + \Gamma\}.$$

The quasi-polynomial of S for $\underline{n} \in C_{\underline{0}}$ is

$$p_S(\underline{n}) = c_{(0,1)}(\underline{n})\underline{n}^{(0,1)} + c_{(0,0)}(\underline{n})\underline{n}^{(0,0)} = c_{(0,1)}(\underline{n})n_2 + c_{(0,0)}(\underline{n})$$

with periodic functions with respect to Γ given by

$$c_{(0,0)}(\underline{n}) = \begin{cases} 1, & \text{if } \underline{n} \in (0,0)+\Gamma; \\ 0, & \text{if } \underline{n} \in (1,0)+\Gamma; \end{cases} \qquad c_{(0,1)}(\underline{n}) = \begin{cases} 1, & \text{if } \underline{n} \in (0,0)+\Gamma; \\ 0, & \text{if } \underline{n} \in (1,0)+\Gamma. \end{cases}$$

This expression for the quasi-polynomial is equivalent to give polynomials

$$\begin{aligned} f_{(0,0)}(\underline{n}) &= n_2 + 1, \\ f_{(1,0)}(\underline{n}) &= 0, \end{aligned}$$

such that

$$p_S(\underline{n}) = \begin{cases} f_{(0,0)}(\underline{n}), & \text{for } \underline{n} = (0,0)+\Gamma; \\ f_{(1,0)}(\underline{n}), & \text{for } \underline{n} = (1,0)+\Gamma. \end{cases}$$

Let us consider now the S-module

$$M = S \oplus S(-1,-2) = \bigoplus_{(n_1,n_2)\in\mathbb{Z}^2} S_{(n_1,n_2)} \oplus S_{(n_1-1,n_2-2)}.$$

By the previous computation,

$$\begin{aligned} \dim_K(M_{\underline{n}}) &= \dim_K(S_{(n_1,n_2)}) + \dim_K(S_{(n_1-1,n_2-2)}) \\ &= \begin{cases} n_2+1, & \text{if } (n_1,n_2) \in \Gamma; \\ n_2-1, & \text{if } (n_1-1,n_2-2) \in \Gamma; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Notice that $(n_1, n_2) \in \Gamma$ if and only if $(n_1-1, n_2-2) \notin \Gamma$.

Then, in the cone $C_{(1,2)}$ with basic cell $\Pi_{(1,2)} = \{(1,2),(2,2)\}$, the Hilbert quasi-polynomial is

$$p_M(\underline{n}) = c_{(0,1)}(\underline{n})\underline{n}^{(0,1)} + c_{(0,0)}(\underline{n})\underline{n}^{(0,0)} = c_{(0,1)}(\underline{n})n_2 + c_{(0,0)}(\underline{n})$$

with periodic functions with respect to Γ

$$c_{(0,0)}(\underline{n}) = \begin{cases} -1, & \text{if } \underline{n} \in (1,2)+\Gamma; \\ 1, & \text{if } \underline{n} \in (2,2)+\Gamma; \end{cases} \qquad c_{(0,1)}(\underline{n}) = 1 \text{ for all } \underline{n} \in C_{(1,2)}.$$

So, the polynomials are

$$f_{(1,2)}(\underline{n}) = n_2 - 1,$$
$$f_{(2,2)}(\underline{n}) = n_2 + 1,$$

and

$$p_M(\underline{n}) = \begin{cases} f_{(1,2)}(\underline{n}), & \text{for } \underline{n} = (1,2) + \Gamma; \\ f_{(2,2)}(\underline{n}), & \text{for } \underline{n} = (2,2) + \Gamma. \end{cases}$$

2.4 Local cohomology and the Grothendieck-Serre formula

Again, in this section, let $S = \bigoplus_{\underline{n}\in\mathbb{N}^r} S_{\underline{n}}$ be a Noetherian multigraded ring, with $S_{\underline{0}}$ a local ring. If $\mathfrak{m}$ is the maximal ideal of $S_{\underline{0}}$, then $\mathcal{M} = \mathfrak{m} \oplus \bigoplus_{\underline{n}\neq\underline{0}} S_{\underline{n}}$ is the unique homogeneous maximal ideal of S.

In the category of $\mathbb{Z}^r$-graded S-modules $\mathcal{M}^r(S)$ we introduce the local cohomology functor. We mainly follow [HHR93]. Let us consider a homogeneous ideal $I \subset S$ and M a $\mathbb{Z}^r$-graded S-module. We define the local cohomology functor $\Gamma_I(\cdot)$ with support in I in the traditional way as follows:

$$\Gamma_I(M) = \{x \in M : I^k x = 0 \text{ for some } k \geq 0\}.$$

We can observe that $\Gamma_I(M) \subseteq M$ is a $\mathbb{Z}^r$-graded S-submodule of M. Now, the local cohomology functors $H^i_I(\cdot)$ are the right derived functors of $\Gamma_I(\cdot)$ in the category of $\mathbb{Z}^r$-graded modules. For abuse of notation, we keep the notation used for the traditional local cohomology modules. It is clear that $H^0_I(M) = \Gamma_I(M)$.

In the classical case, it is well known that we can relate the Hilbert function of a graded module and its Hilbert polynomial with local cohomology modules. To be more precise, the difference between the Hilbert function and the Hilbert polynomial can be expressed as an alternate sum of the length of the local cohomology modules, namely the Grothendieck-Serre formula.

In this section we prove that in our setting, the Grothendieck-Serre formula still holds, considering the Hilbert quasi-polynomial instead. We follow the ideas of the proofs in [Lav99] in our case.

Some of the arguments used in this section work also in a more general context, but at some steps we will be able to assure the existence of homogeneous regular elements for proceeding by induction. For that purpose we need a homogeneous version of the Prime Avoidance Lemma. This lemma will require a major control on the multidegrees in order to find a homogeneous element, so we need to fix the hypothesis on the multidegrees of the generators of the multigraded ring S.

Thus, the $\mathbb{Z}^r$-graded ring S will be generated over $S_{\underline{0}}$, a Noetherian local ring, by elements

$$g_1^1, \ldots, g_1^{\mu_1}, \ldots, g_r^1, \ldots, g_r^{\mu_r}$$

with g_i^j of multidegree $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0, \ldots, 0) \in \mathbb{N}^r$ with $\gamma_i^i \neq 0$, for all $i = 1, \ldots, r$ and $j = 1, \ldots, \mu_i$.

Lemma 2.4.1. Homogeneous Prime Avoidance. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \mathrm{Proj}^r(S)$. If I is any homogeneous ideal of S such that $I \not\subset \mathfrak{p}_i$ for $i = 1, \ldots, m$, then there exists a homogeneous element a such that $a \in I$ and $a \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m$.*

Proof. We proceed by induction on m.

Assume that $m = 1$. Since $I \not\subset \mathfrak{p}_1$, there exists a homogeneous element $x \in I$ such that $x \notin \mathfrak{p}_1$.

Assume now that $m > 1$. We may suppose that $\mathfrak{p}_m$ is a minimal element of $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ and that there is a homogeneous element $x' \in I$ such that $x' \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{m-1}$.

If $x' \notin \mathfrak{p}_m$, we are done. Otherwise, assume that $x' \in \mathfrak{p}_m$. Then, there exists a homogeneous element $r \in (\cap_{i=1}^{m-1} \mathfrak{p}_i) \setminus \mathfrak{p}_m$. In fact, since $\mathfrak{p}_m$ is minimal over $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_{m-1}\}$, for each $i = 1, \ldots, m-1$ there exists a homogeneous element $r_i \in \mathfrak{p}_i \setminus \mathfrak{p}_m$. We can define $r := r_1 \cdot \ldots \cdot r_{m-1}$, and clearly $r \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{m-1}$ but $r \notin \mathfrak{p}_m$. Since $I \not\subset \mathfrak{p}_m$, we can choose a homogeneous element $y \in I \setminus \mathfrak{p}_m$. Now, x' and ry are not necessarily of the same multidegree. Let $(d_1, \ldots, d_r) \in \mathbb{Z}^r$ be the multidegree of x', and $(e_1, \ldots, e_r) \in \mathbb{Z}^r$ be the multidegree of ry. Clearly these multidegrees are a $\mathbb{Z}$-linear combination of the multidegrees of the generators of S.

Since $\mathfrak{p}_m \not\supset S_{++} = I_1 \cdots I_r$, then there exists an element of the form $g_1^{i_1} g_2^{i_2} \ldots g_r^{i_r} \notin \mathfrak{p}_m$. Since $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{m-1} \not\supset S_{++}$, there exists an element of the form $g_1^{j_1} g_2^{j_2} \ldots g_r^{j_r} \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{m-1}$. Multiplying x' by an adequate power of $g_r^{j_r}$, and ry by an adequate power of $g_r^{i_r}$, we can assume that $d_r = e_r$. Now we can repeat this process with adequate powers of $g_{r-1}^{j_{r-1}}$ and $g_{r-1}^{i_{r-1}}$, and assume that $d_r = e_r$ and $d_{r-1} = e_{r-1}$. Repeating this process with each $g_s^{j_s}$ and $g_s^{i_s}$, at the end we can assume that $(d_1,\dots,d_r) = (e_1,\dots,e_r)$, and hence $x := x' + ry$ will be a homogeneous element such that $x \in I$ and $x \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m$ as we wanted to prove. □

Proposition 2.4.2. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then,*

(i) For all $i \geq 0$, $\underline{n} \in \mathbb{Z}^r$ the $S_{\underline{0}}$-module $H^i_{S_{++}}(M)_{\underline{n}}$ is finitely generated.

(ii) There exists $\underline{\beta} \in \mathbb{Z}^r$ such that $H^i_{S_{++}}(M)_{\underline{n}} = 0$ for all $i \geq 0$, $\underline{n} \in C_{\underline{\beta}}$.

Proof. We prove both assertions together by induction on i.

Since $H^i_{S_{++}}(M) = 0$ for $i > \mu(S_{++})$, where $\mu(S_{++})$ is the minimal number of generators of S_{++}, it suffices to prove that for every i there exists a cone $C_{\underline{\beta}_i}$ satisfying (ii). Since there will be a finite number of such cones, we may to consider a common cone $C_{\underline{\beta}}$ (taking $\underline{\beta} := \sum \underline{\beta}_i$, it satisfies that $C_{\underline{\beta}} \subset C_{\underline{\beta}_i}$ for all i, for instance) satisfying (ii) for any $i \geq 0$.

Assume that $i = 0$. In this case $H^0_{S_{++}}(M) = \Gamma_{S_{++}}(M) \subset M$ is a submodule of M and hence $H^0_{S_{++}}(M)_{\underline{n}}$ is finitely generated for any $\underline{n} \in \mathbb{Z}^r$. Moreover, there exists an $m \in \mathbb{Z}$ such that $(S_{++})^m H^0_{S_{++}}(M) = 0$, and so by Proposition 2.3.8, there is a $\underline{\beta} \in \mathbb{Z}^r$ such that $H^0_{S_{++}}(M)_{\underline{n}} = 0$ for any $\underline{n} \in C_{\underline{\beta}}$.

Assume now that $i > 0$. From the exact sequence

$$0 \longrightarrow H^0_{S_{++}}(M) \longrightarrow M \longrightarrow M/H^0_{S_{++}}(M) \longrightarrow 0$$

we get the long exact sequence of local cohomology

$$0 = H^i_{S_{++}}(H^0_{S_{++}}(M)) \longrightarrow H^i_{S_{++}}(M) \longrightarrow H^i_{S_{++}}(M/H^0_{S_{++}}(M)) \longrightarrow H^{i+1}_{S_{++}}(H^0_{S_{++}}(M)) = 0.$$

Hence, we get the $\mathbb{Z}^r$-graded isomorphism

$$H^i_{S_{++}}(M) \cong H^i_{S_{++}}(M/H^0_{S_{++}}(M)),$$

so replacing M by $M/H^0_{S_{++}}(M)$ we can assume that $H^0_{S_{++}}(M) = 0$.

Now, assuming that $H^0_{S_{++}}(M) = 0$, we have that $S_{++} \not\subset \mathfrak{p}$ for all $\mathfrak{p} \in Ass(M)$. Thus by the Homogeneous Prime Avoidance Lemma 2.4.1 there exists a homogeneous element $x \in S_{++}$ of multidegree $\underline{k} = (k_1,\dots,k_r) \in \mathbb{N}^r$ such that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in Ass(M)$. It is clear that $\underline{k} \in C_{\underline{0}}$ since $x \in S$.

We can consider the exact $\mathbb{Z}^r$-graded sequence

$$0 \longrightarrow M(-\underline{k}) \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0$$

and the long exact sequence of local cohomology, for all $\underline{n} \in \mathbb{Z}^r$,

$$\to H^{i-1}_{S_{++}}(M/xM)_{\underline{n}} \to H^i_{S_{++}}(M)_{\underline{n}-\underline{k}} \xrightarrow{\cdot x} H^i_{S_{++}}(M)_{\underline{n}} \to H^i_{S_{++}}(M/xM)_{\underline{n}} \to$$

By the induction hypothesis $H^{i-1}_{S_{++}}(M/xM)_{\underline{n}} = 0$ for all $\underline{n} \in C_{\underline{\beta}'}$ for some $\underline{\beta}' \in \mathbb{Z}^r$. Since for all $s \geq 1$, $\underline{n} + s\underline{k} \in C_{\underline{\beta}'}$ as well, we have the exact sequence

$$0 \to H^i_{S_{++}}(M)_{\underline{n}-\underline{k}} \xrightarrow{\cdot x^s} H^i_{S_{++}}(M)_{\underline{n}+(s-1)\underline{k}}$$

and hence,

$$H^i_{S_{++}}(M)_{\underline{n}-\underline{k}} \cong x^s H^i_{S_{++}}(M)_{\underline{n}-\underline{k}}.$$

Since $H^i_{S_{++}}(M)$ is an S_{++}-torsion module and $x \in S_{++}$, we have that $H^i_{S_{++}}(M)_{\underline{n}-\underline{k}} = 0$. Considering now $\underline{\beta} := \underline{\beta}' - \underline{k}$, we have that $H^i_{S_{++}}(M)_{\underline{n}} = 0$ for all $\underline{n} \in C_{\underline{\beta}}$.

It remains to prove that the local cohomology modules $H^i_{S_{++}}(M)_{\underline{n}}$ are finitely generated for all $\underline{n} \in \mathbb{Z}^r$.

Fix $\underline{n} \in \mathbb{Z}^r$. If $\underline{n} \in C_{\underline{\beta}}$, then $H^i_{S_{++}}(M)_{\underline{n}} = 0$ as we have seen before, and this is clearly a finitely generated $S_{\underline{0}}$-module. If $\underline{n} \notin C_{\underline{\beta}}$, and since we are assuming that $H^0_{S_{++}}(M) = 0$, again by Homogeneous Prime Avoidance Lemma 2.4.1 there exists an element $y \in S_{++}$ such that $y \notin \bigcup_{\mathfrak{p} \in Ass(M)} \mathfrak{p}$, and as we did in the proof of Lemma 2.4.1, we can assume that the multi-degree $\underline{l} = (l_1, \dots, l_r)$ is such that $\underline{n} + \underline{l} \in C_{\underline{\beta}}$.

Considering the $\mathbb{Z}^r$-graded exact sequence

$$0 \to M \xrightarrow{\cdot y} M(\underline{l}) \to M/yM(\underline{l}) \to 0$$

we take the long exact sequence of local cohomology

$$\dots \longrightarrow H^{i-1}_{S_{++}}(M/yM)_{\underline{n}+\underline{l}} \longrightarrow H^i_{S_{++}}(M)_{\underline{n}} \xrightarrow{\cdot y} H^i_{S_{++}}(M)_{\underline{n}+\underline{l}} \longrightarrow \dots$$

Since $\underline{n} + \underline{l} \in C_{\underline{\beta}}$, $H^i_{S_{++}}(M)_{\underline{n}+\underline{l}} = 0$ and by induction hypothesis we have that $H^{i-1}_{S_{++}}(M/yM)_{\underline{n}+\underline{l}}$ is a finitely generated $S_{\underline{0}}$-module, and so, $H^i_{S_{++}}(M)_{\underline{n}}$ is a finitely generated $S_{\underline{0}}$-module as we wanted to prove. □

Assume now that the local ring $S_{\underline{0}}$ is Artinian in order o assure the existence of the Hilbert quasi-polynomial of M, P_M. Now, we prove the Grothendieck-Serre formula in our case.

Proposition 2.4.3. Grothendieck-Serre formula. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then for all $\underline{n} \in \mathbb{Z}^r$,*

$$h_M(\underline{n}) - P_M(\underline{n}) = \sum_{i \geq 0} (-1)^i \operatorname{length}_{S_{\underline{0}}}(H^i_{S_{++}}(M)_{\underline{n}})$$

Proof. For a finitely generated $\mathbb{Z}^r$-graded S-module M, we define, using always multi-index notation, the generating functions

$$H^{\Delta}_M(\underline{x}) = \sum_{\underline{n} \in \mathbb{Z}^r} (h_M(\underline{n}) - P_M(\underline{n}))\underline{x}^{\underline{n}}$$

and

$$H^{\chi}_M(\underline{x}) = \sum_{\underline{n} \in \mathbb{Z}^r} (\sum_{i \geq 0} (-1)^i \operatorname{length}_{S_{\underline{0}}}(H^i_{S_{++}}(M)_{\underline{n}}))\underline{x}^{\underline{n}}.$$

It is clear that proving the required formula is equivalent to prove that $H^{\Delta}_M(\underline{x}) = H^{\chi}_M(\underline{x})$.

We proceed by induction on $\operatorname{rel.dim}(M)$. If $\operatorname{rel.dim}(M) = r-1$ (equivalently $\dim(M) = -1$), then $Supp_{++}(M) = \emptyset$, and hence $V_{++}(Ann(M)) = V_{++}(S_{++}) = \emptyset$. Then, there exists an $u \in \mathbb{N}$ such that $(S_{++})^u \subset Ann(M)$. Therefore, $H^0_{S_{++}}(M) = \Gamma_{S_{++}}(M) = M$. Thus, $H^i_{S_{++}}(M) = 0$ for all $i \geq 1$. Now, since the Hilbert quasi-polynomial has degree -1, by Proposition 2.3.10, and hence $P_M(\underline{n}) = 0$, we get that

$$H^{\chi}_M(\underline{x}) = \sum_{\underline{n} \in \mathbb{Z}^r} \operatorname{length}_{S_{\underline{0}}}(H^0_{S_{++}}(M))_{\underline{n}}\underline{x}^{\underline{n}} = \sum_{\underline{n} \in \mathbb{Z}^r} h_M(\underline{n})\underline{x}^{\underline{n}} = H^{\Delta}_M(\underline{x})$$

as we wanted to prove.

Assume now that $\operatorname{rel.dim}(M) \geq r$. Let us consider $M' = M/H^0_{S_{++}}(M)$. By Proposition 2.4.2, $H^0_{S_{++}}(M)_{\underline{n}} = 0$ for all $\underline{n} \in C_{\underline{\beta}}$, for some $\underline{\beta} \in \mathbb{Z}^r$ and since $h_M(\underline{n}) = h_{M'}(\underline{n}) + \operatorname{length}_{S_{\underline{0}}}(H^0_{S_{++}}(M)_{\underline{n}})$, we get that $P_M(\underline{n}) = P_{M'}(\underline{n})$. So we need only prove the result for M' since for any $\underline{n} \in \mathbb{Z}^r$ we have that

$$\begin{aligned} h_M(\underline{n}) - P_M(\underline{n}) &= h_{M'}(\underline{n}) + \operatorname{length}_{S_{\underline{0}}}(H^0_{S_{++}}(M)_{\underline{n}}) - P_{M'}(\underline{n}) \\ &= \sum_{i \geq 0} (-1)^i \operatorname{length}_{S_{\underline{0}}}(H^i_{S_{++}}(M')_{\underline{n}}) + \operatorname{length}_{S_{\underline{0}}}(H^0_{S_{++}}(M)_{\underline{n}}) \\ &= \sum_{i \geq 0} (-1)^i \operatorname{length}_{S_{\underline{0}}}(H^i_{S_{++}}(M)_{\underline{n}}) \end{aligned}$$

because $H^0_{S_{++}}(M') = 0$ and $H^i_{S_{++}}(M') \cong H^i_{S_{++}}(M)$ for $i \geq 1$.

Hence, let us assume that $H^0_{S_{++}}(M) = 0$. Then $S_{++} \not\subset \mathfrak{p}$, for all $\mathfrak{p} \in Ass(M)$, and by Lemma 2.4.1, there exists a homogeneous element $x \in S_{++}$ of multidegree $\underline{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$ such that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in Ass(M)$.

Now we have the following $\mathbb{Z}^r$-graded exact sequence

$$0 \longrightarrow M(-\underline{k}) \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0$$

with $\operatorname{rel.dim}(M/xM) < \operatorname{rel.dim}(M)$ by Lemma 2.2.8.

From the exact sequence we have that

$$h_{M/xM}(\underline{n}) = h_M(\underline{n}) - h_M(\underline{n} - \underline{k})$$

for any $\underline{n} \in \mathbb{Z}^r$ and hence,

$$P_{M/xM}(\underline{n}) = P_M(\underline{n}) - P_M(\underline{n} - \underline{k}).$$

So, we deduce that

$$H^{\Delta}_{M/xM}(\underline{x}) = (1 - \underline{x}^{\underline{k}}) H^{\Delta}_M(\underline{x}).$$

Considering the long exact sequence of local cohomology associated to the previous short exact sequence, for each $\underline{n}$,

$$\to H^{i-1}_{S_{++}}(M/xM)_{\underline{n}} \to H^i_{S_{++}}(M)_{\underline{n}-\underline{k}} \to H^i_{S_{++}}(M)_{\underline{n}} \to H^i_{S_{++}}(M/xM)_{\underline{n}} \to$$

and since the alternate sume of lengths is zero, we deduce that

$$H^{\chi}_{M/xM}(\underline{x}) = (1 - \underline{x}^{\underline{k}}) H^{\chi}_M(\underline{x}).$$

Now, since $\operatorname{rel.dim}(M/xM) < \operatorname{rel.dim}(M)$, by induction hypothesis $H^{\Delta}_{M/xM}(\underline{x}) = H^{\chi}_{M/xM}(\underline{x})$, and then

$$H^{\Delta}_M(\underline{x}) = H^{\chi}_M(\underline{x})$$

as we wanted to prove. □

For an $\mathfrak{m}$-primary ideal I in a local ring $(R, \mathfrak{m})$, one can consider the Hilbert-Samuel function

$$h^1_I(n) = \operatorname{length}_R(R/I^{n+1})$$

and the Hilbert-Samuel polynomial $p_I^1(n)$ such that $h_I^1(n) = p_I^1(n)$ for all $n \gg 0$. In [Bla97], the author proves that

$$p_I^1(n) - h_I^1(n) = \sum_{i \geq 0} (-1)^i \operatorname{length}_R(H^i_{\mathcal{R}_+}(\mathcal{R}^*(I))_{n+1})$$

for all $n \in \mathbb{Z}$, where $\mathcal{R}^*(I) = \bigoplus_{n \in \mathbb{Z}} I^n t^n$ is the extended Rees algebra of I and $\mathcal{R}_+ = \bigoplus_{n>0} I^n t^n$ is the irrelevant ideal of the Rees algebra $\mathcal{R}(I)$.

In the following, we generalize the definitions of the blow-up algebras for some ideals $I_1, \ldots, I_r$, and we prove an analogous formula in the multigraded case.

Let us consider the multigraded Rees algebra associated to some ideals $I_1, \ldots, I_r$ of a Noetherian local ring $(R, \mathfrak{m})$,

$$\mathcal{R}(I_1, \ldots, I_r) = \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset R[t_1, \ldots, t_r].$$

We consider the k-th extended multigraded Rees algebra

$$\mathcal{R}_k^*(I_1, \ldots, I_r) = \bigoplus_{\substack{n_k \in \mathbb{Z} \\ (n_1 \ldots, \widehat{n_k}, \ldots, n_r) \in \mathbb{N}^{r-1}}} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset R[t_1, \ldots, t_r, t_k^{-1}]$$

for $k = 1, \ldots, r$.

For $k = 1, \ldots, r$ let us consider the k-th associated multigraded ring of $I_1, \ldots, I_r$ in R,

$$gr_{I_1, \ldots, I_r; I_k}(R) = \bigoplus_{\underline{n} \in \mathbb{N}^r} \frac{I_1^{n_1} \cdots I_k^{n_k} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} = \frac{\mathcal{R}(I_1, \ldots, I_r)}{I_k \mathcal{R}(I_1, \ldots, I_r)}$$

To simplify, we denote by $\mathcal{R}$, $\mathcal{R}_k^*$ and $\mathcal{G}_k$ the Rees algebra, the k-th extended Rees algebra and the k-th associated multigraded ring, respectively.

To deal with the Rees algebra and the k-th extended one at the same time, we set that negative powers of ideals are 0 in the Rees algebra, and they are R in the k-th extended one.

We denote by $e_1, \ldots, e_r$ the canonical basis of $\mathbb{R}^r$.

Theorem 2.4.4. *For all $i \geq 2$ there is an isomorphism of $\mathbb{Z}^r$-graded $\mathcal{R}$-modules*

$$H^i_{\mathcal{R}_{++}}(\mathcal{R}) \cong H^i_{\mathcal{R}_{++}}(\mathcal{R}_k^*)$$

and there is an exact sequence of $\mathbb{Z}^r$-graded $\mathcal{R}$-modules

$$0 \to H^0_{\mathcal{R}_{++}}(\mathcal{R}) \to H^0_{\mathcal{R}_{++}}(\mathcal{R}^*_k) \to \mathcal{R}^*_k/\mathcal{R} \to H^1_{\mathcal{R}_{++}}(\mathcal{R}) \to H^1_{\mathcal{R}_{++}}(\mathcal{R}^*_k) \to 0.$$

In particular, $H^i_{\mathcal{R}_{++}}(\mathcal{R})_{\underline{n}} \cong H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}}$ for $i = 0, 1$ and $\underline{n} \in \mathbb{N}^r$.

Proof. Let us consider the exact sequence of $\mathbb{Z}^r$-graded $\mathcal{R}$-modules

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}^*_k \longrightarrow \mathcal{R}^*_k/\mathcal{R} \longrightarrow 0$$

and the long exact sequence of local cohomology

$$\cdots \to H^{i-1}_{\mathcal{R}_{++}}(\mathcal{R}^*_k/\mathcal{R}) \to H^i_{\mathcal{R}_{++}}(\mathcal{R}) \to H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k) \to H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k/\mathcal{R}) \to \ldots$$

Notice that $(\mathcal{R}^*_k/\mathcal{R})_{\underline{n}} = 0$ for $\underline{n} \in \mathbb{N}^r$. Otherwise, when $n_k < 0$,

$$(\mathcal{R}^*_k/\mathcal{R})_{\underline{n}} = I_1^{n_1} t_1^{n_1} \cdots R t_k^{n_k} \cdots I_r^{n_r} t_r^{n_r}$$

and hence

$$\mathcal{R}^*_k/\mathcal{R} = \bigoplus_{\substack{n_k<0 \\ (n_1\ldots,\widehat{n_k},\ldots,n_r)\in\mathbb{N}^{r-1}}} I_1^{n_1} t_1^{n_1} \cdots R t_k^{n_k} \cdots I_r^{n_r} t_r^{n_r}.$$

Therefore,

$$H^0_{\mathcal{R}_{++}}(\mathcal{R}^*_k/\mathcal{R}) = \mathcal{R}^*_k/\mathcal{R}$$

and hence

$$H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k/\mathcal{R}) = 0$$

for $i \geq 1$.

Using now the long exact sequence we get

$$H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k) \cong H^i_{\mathcal{R}_{++}}(\mathcal{R})$$

for $i \geq 1$, the exact sequence

$$0 \to H^0_{\mathcal{R}_{++}}(\mathcal{R}) \to H^0_{\mathcal{R}_{++}}(\mathcal{R}^*_k) \to \mathcal{R}^*_k/\mathcal{R} \to H^1_{\mathcal{R}_{++}}(\mathcal{R}) \to H^1_{\mathcal{R}_{++}}(\mathcal{R}^*_k) \to 0$$

and for $\underline{n} \in \mathbb{N}^r$, $i = 0, 1$,

$$H^i_{\mathcal{R}_{++}}(\mathcal{R})_{\underline{n}} \cong H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}}.$$

□

Remark 2.4.5. We can consider the (complete) extended Rees algebra defined as

$$\mathcal{R}^*(I_1,\dots,I_r) = \bigoplus_{\underline{n}\in\mathbb{Z}^r} I_1^{n_1}t_1^{n_1}\cdots I_r^{n_r}t_r^{n_r} \subset R[t_1,\dots,t_r,t_1^{-1},\dots,t_r^{-1}].$$

Notice that in this case $(\mathcal{R}^*/\mathcal{R})_{\underline{n}} = 0$ for $\underline{n} \in \mathbb{N}^r$, and otherwise

$$(\mathcal{R}^*/\mathcal{R})_{\underline{n}} = I_{i_1}^{n_{i_1}} t_{i_1}^{n_{i_1}} \cdots I_{i_s}^{n_{i_s}} t_{i_s}^{n_{i_s}} t_{j_1}^{n_{j_1}} \cdots t_{j_u}^{n_{j_u}}$$

for some indexes $\{i_1,\dots,i_s\}\cup\{j_1,\dots,j_u\} = \{1,\dots,r\}$ such that $n_{i_*} \geq 0$ and $n_{j_*} < 0$. Also in this case

$$H^0_{\mathcal{R}_{++}}(\mathcal{R}^*/\mathcal{R}) = \mathcal{R}^*/\mathcal{R}$$

and

$$H^i_{\mathcal{R}_{++}}(\mathcal{R}^*/\mathcal{R}) = 0$$

for $i \geq 1$ and the previous theorem follows as well.

From now on, we assume that $I_1,\dots,I_r$ are $\mathfrak{m}$-primary ideals of the Noetherian local ring $(R,\mathfrak{m})$. For $k = 1,\dots,r$, we denote

$$f_k(\underline{n}) = \operatorname{length}_R\left(\frac{R}{I_1^{n_1}\cdots I_k^{n_k+1}\cdots I_r^{n_r}}\right).$$

Notice that this length is finite since it is an R/I_k-module with R/I_k Artin. One of the first references on the study of this kind of functions is [Bha57] for the two ideals case.

Proposition 2.4.6. *There exists a polynomial $p_k \in \mathbb{Z}[n_1,\dots,n_r]$ and an element $\underline{\beta}_k \in \mathbb{N}^r$ such that*

$$f_k(\underline{n}) = p_k(\underline{n})$$

for $\underline{n} \geq \underline{\beta}_k$.

Proof. Since $\mathcal{G}_k$ is a standard $\mathbb{N}^r$-graded algebra finitely generated over the Artinian local ring R/I_k, there exists a polynomial $p_{\mathcal{G}_k}$ an an element $\underline{\beta}_k \in \mathbb{N}^r$ such that

$$h_{\mathcal{G}_k}(\underline{n}) = \operatorname{length}_R\left(\frac{I_1^{n_1}\cdots I_r^{n_r}}{I_1^{n_1}\cdots I_k^{n_k+1}\cdots I_r^{n_r}}\right) = p_{\mathcal{G}_k}(\underline{n})$$

for $\underline{n} \geq \underline{\beta}_k$, see Proposition 2.3.10.

From the exact sequence of R-modules

$$0 \longrightarrow \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} \longrightarrow \frac{R}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} \longrightarrow \frac{R}{I_1^{n_1} \cdots I_r^{n_r}} \longrightarrow 0$$

we deduce that $f_k(\underline{n}) - f_k(\underline{n} - e_k) = h_{\mathcal{G}_k}(\underline{n})$ for all $\underline{n} \in \mathbb{N}^r$. Then, by Proposition 2.3.6, we have that $f_k(\underline{n})$ is a polynomial function for $\underline{n} \geq \underline{\beta}_k$. We denote by p_k such polynomial. □

Remark 2.4.7. Notice from the exact sequence in the proof that

$$p_k(\underline{n}) - p_k(\underline{n} - e_k) = p_{\mathcal{G}_k}(\underline{n}).$$

For an element $\underline{\delta} \in \mathbb{N}^r$, we define $\mathcal{H}^k_{\underline{\delta}}$ as the set of elements $\underline{n} \in \mathbb{Z}^r$ such that $(n_1, \ldots, n_{k-1}, n_{k+1}, \ldots, n_r) \geq (\delta_1, \ldots, \delta_{k-1}, \delta_{k+1}, \ldots, \delta_r)$ and $n_k \in \mathbb{Z}$.

Theorem 2.4.8. *There exist an element $\underline{\delta} \in \mathbb{N}^r$ such that for all $\underline{n} \in \mathcal{H}^k_{\underline{\delta}}$ it holds*

$$p_k(\underline{n}) - f_k(\underline{n}) = \sum_{i \geq 0} (-1)^i \operatorname{length}_R(H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}+e_k}).$$

Proof. Consider the exact sequence of $\mathbb{Z}^r$-graded $\mathcal{R}$-modules

$$0 \longrightarrow \mathcal{R}^*_k(e_k) \xrightarrow{\cdot t_k^{-1}} \mathcal{R}^*_k \longrightarrow \mathcal{G}_k \longrightarrow 0$$

Since $\mathcal{G}_{k++} = \mathcal{R}_{++}\mathcal{G}_k$, we have that $H^i_{\mathcal{G}_{k++}}(\mathcal{G}_k) \cong H^i_{\mathcal{R}_{++}}(\mathcal{G}_k)$. Then we have the long exact sequence of local cohomology

$$0 \to H^0_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}+e_k} \to H^0_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}} \to H^0_{\mathcal{G}_{k++}}(\mathcal{G}_k)_{\underline{n}} \to H^1_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}+e_k} \to \ldots$$

$$\cdots \to H^{i-1}_{\mathcal{G}_{k++}}(\mathcal{G}_k)_{\underline{n}} \to H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}+e_k} \to H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}} \to H^i_{\mathcal{G}_{k++}}(\mathcal{G}_k)_{\underline{n}} \to \ldots$$

By Proposition 2.4.2 and Theorem 2.4.4 , there exist a $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{n} \geq \underline{\beta}$, $H^i_{\mathcal{R}_{++}}(\mathcal{R}_k)_{\underline{n}} \cong H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}} = 0$ for all $i \geq 0$.

Since the length of the local cohomology of $\mathcal{G}_k$ is finite due to Proposition 2.4.2 and to the fact that $(\mathcal{G}_k)_{\underline{0}} = R/I_k$ is Artin, by induction on n_k, using also the exact sequence, we prove that $\operatorname{length}_R(H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}}) < \infty$ for $\underline{n} \in \mathcal{H}^k_{\underline{\beta}}$.

We denote $\chi_k(\underline{n}) = \sum_{i\geq 0}(-1)^i \operatorname{length}_R(H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}+e_k})$ for all $\underline{n} \in \mathbb{Z}^r$.

From the long exact sequence and the Grothendieck-Serre formula, Proposition 2.4.3 for $\mathcal{G}_k$, since $(\mathcal{G}_k)_{\underline{0}} = R/I_k$ is Artin, we have

$$\chi_k(\underline{n}) - \chi_k(\underline{n} - e_k) = -\sum_{i\geq 0}(-1)^i \operatorname{length}_R(H^i_{\mathcal{G}_{k++}}(\mathcal{G}_k)_{\underline{n}}) = p_{\mathcal{G}_k}(\underline{n}) - h_{\mathcal{G}_k}(\underline{n})$$

for all $\underline{n} \in \mathcal{H}^k_{\underline{\beta}}$.

If we write

$$\sigma_k(\underline{n}) = p_k(\underline{n}) - f_k(\underline{n}),$$

it holds

$$\sigma_k(\underline{n}) - \sigma_k(\underline{n} - e_k) = p_{\mathcal{G}_k}(\underline{n}) - h_{\mathcal{G}_k}(\underline{n}) = \chi_k(\underline{n}) - \chi_k(\underline{n} - e_k) \qquad (*)$$

see the previous remark.

From Proposition 2.4.6, there exists $\underline{\alpha} \in \mathbb{N}^r$ such that $\sigma_k(\underline{n}) = 0$ for $\underline{n} \geq \underline{\alpha}$.

Let us consider an element $\underline{\delta} \in \mathbb{N}^r$ such that $\underline{\delta} \geq \underline{\alpha}$ and $\underline{\delta} \geq \underline{\beta}$. Then it holds that for $\underline{n} \geq \underline{\delta}$,

$$\sigma_k(\underline{n}) = \chi_k(\underline{n}) = 0$$

and then by induction on n_k, using $(*)$, we prove that

$$\sigma_k(\underline{n}) = \chi_k(\underline{n})$$

for all $\underline{n} \in \mathcal{H}^k_{\underline{\delta}}$. □

Remark 2.4.9. Notice that there is not an exact sequence

$$0 \longrightarrow \mathcal{R}^*(e_k) \longrightarrow \mathcal{R}^* \longrightarrow \mathcal{G}_k \longrightarrow 0$$

by considering the complete extended Rees algebra. In fact, if $r = 2$, for instance, and $\underline{n} \in \mathbb{Z}^2$ with $n_1 < 0$ and $n_2 > 0$,

$$(\mathcal{R}^*/\mathcal{R}^*(e_2))_{\underline{n}} = \frac{I_2^{n_2} t_1^{n_1} t_2^{n_2}}{t_1^{n_1} I_2^{n_2+1} t_2^{n_2+1}} \neq 0$$

and hence it cannot be isomorphic to $(\mathcal{G}_2)_{\underline{n}} = 0$.

Other references to Hilbert functions of multigraded algebras, with special emphasis in multigraded blow-up algebras and mixed multiplicities are [TV08] and [Swa07].

In the following chapter we will use some of results in this chapter. In particular we will use the fact that Hilbert functions of multigraded modules are quasi-polynomial in a cone of $\mathbb{N}^r$ to study the asymptotic depth of the homogeneous pieces of the module. In particular the behavior of the Hilbert function of the Koszul homology modules will allow us to reach our purpose.

Chapter 3

Asymptotic depth of multigraded modules

The aim of this chapter is to study the depth of the graded pieces of a multigraded module over a Noetherian non-standard multigraded ring with the graduation considered in the previous chapter.

To reach our purposes we will need to use the Koszul complex and the Koszul homology. Therefore, in the first section we recall these tools in the multigraded case.

In the second section, we study the depth of the pieces $M_{\underline{n}}$ of a multigraded module M. The key point in the proof is to use the asymptotic behavior of the Hilbert function of the Koszul homology modules. In a more general case, we are able to prove that this depth is constant in a sub-net of a cone, since the Hilbert function is quasi-polynomial in a cone, see Theorem 3.2.1. In some specific cases in which the Hilbert function is eventually polynomial, we can assure a constant depth of $M_{\underline{n}}$ for $\underline{n}$ in all the cone, see Proposition 3.2.3 and Corollary 3.2.4.

As a consequence, we are able to prove in the Section 3.3, that the asymptotic depth for the modules $I_1^{n_1} \cdots I_r^{n_r}$ and $R/I_1^{n_1} \cdots I_r^{n_r}$ is constant for $\underline{n} = (n_1, \ldots, n_r)$ large enough, being $I_1, \ldots, I_r$ ideals in a Noetherian local ring R, see Proposition 3.3.1 and Theorem 3.3.6. This study involves the asymptotic depth of $I_1^{n_1} \cdots I_r^{n_r} / I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}$, that we are able to prove that takes the same value for all $k = 1, \ldots, r$, see Proposition 3.3.3.

3.1 Koszul homology

In this section, we introduce the Koszul complex and the Koszul homology in the multigraded case. As a main reference, we follow [BH93] and [Bla97].

Let S be a Noetherian positively multigraded ($\mathbb{N}^r$-graded) ring. Let $\mathcal{M}(S)$ be the category of S-modules and $\mathcal{M}^r(S)$ the category of multigraded S-modules with multigraded morphisms. Let $x_1, \ldots, x_s \in S$ be homogeneous elements of multidegrees $\underline{k}_1, \ldots, \underline{k}_s \in \mathbb{N}^r$ respectively. We now proceed to define the (multigraded) homological Koszul complex $K_*(x_1, \ldots, x_s; S)$ with respect to $x_1, \ldots, x_s,$.

Let F be a free S-module $F = \bigoplus_{i=1}^{s} S(-\underline{k}_i)$, with basis $e_1, \ldots, e_s$. We consider the homogeneous morphism of multigraded modules $f : F \to S$ defined by $f(e_i) = x_i$. Then the Koszul complex $K_*(x_1, \ldots, x_s; S)$ is the homological complex such that the n-th graded piece is

$$K_n(x_1, \ldots, x_s; S) = \bigwedge^n F$$

and the differential $d_n : \bigwedge^n F \to \bigwedge^{n-1} F$ is defined by

$$d_n(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n} (-1)^{i+1} f(a_i)\, a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_n.$$

One can observe that $K_n(x_1, \ldots, x_s; S)$ is also a $\mathbb{Z}^r$-graded ring, with

$$K_n(x_1, \ldots, x_s; S)_{\underline{m}} = \bigoplus_{\sum_{i=1}^{n} \underline{m}_i = \underline{m}} \bigwedge^n F_{\underline{m}_i},$$

and that each differential is a homogeneous morphism. In fact, considering $\{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq s\}$ a basis for $\bigwedge^n F$,

$$\begin{aligned} d_n(e_{i_1} \wedge \cdots \wedge e_{i_n}) &= \sum_{j=1}^{n} (-1)^{j+1} f(e_{i_j})\, e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_n} \\ &= \sum_{j=1}^{n} (-1)^{j+1} x_{i_j}\, e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_n}, \end{aligned}$$

and $\deg(x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_n}) = \underline{k}_{i_j} + (\underline{k}_{i_1} + \cdots + \widehat{\underline{k}_{i_j}} + \cdots + \underline{k}_{i_n}) = \underline{k}_{i_1} + \cdots + \underline{k}_{i_n}$ for all $j = 1, \ldots, n$. So we have that $\deg(d_n(e_{i_1} \wedge \cdots \wedge e_{i_n})) = \deg(e_{i_1} \wedge \cdots \wedge e_{i_n}) = \underline{k}_{i_1} + \cdots + \underline{k}_{i_n}$.

Now, identifying $e_{i_1} \wedge \cdots \wedge e_{i_n}$ with $e_{i_1,\ldots,i_n}$ for each element of the basis of $\bigwedge^n F$, we can identify

$$K_n(x_1,\ldots,x_s;S) = \bigoplus_{1\le i_1<\cdots<i_n\le s} S(-(\underline{k}_{i_1}+\cdots+\underline{k}_{i_n})) = \bigoplus_{1\le i_1<\cdots<i_n\le s} Se_{i_1,\ldots,i_n}$$

where $\deg(e_{i_1\ldots i_n}) = \underline{k}_{i_1}+\cdots+\underline{k}_{i_n}$.

From this presentation the multigraded structure of $K_n(x_1,\ldots,x_s;S)$ becomes clear, since $K_n(x_1,\ldots,x_s;S)_{\underline{m}} = \bigoplus_{1\le i_1<\cdots<i_n\le s}(Se_{i_1,\ldots,i_n})_{\underline{m}}$, and clearly

$$S_{\underline{k}}K_n(x_1,\ldots,x_s;S)_{\underline{m}} \subseteq K_n(x_1,\ldots,x_s;S)_{\underline{k}+\underline{m}}.$$

So, $K_n(x_1,\ldots,x_s;S)$ is a multigraded free S-module of rank $\binom{s}{n}$ and with (homogeneous) differential

$$d_n(e_{i_1,\ldots,i_n}) = \sum_{j=1}^{n}(-1)^{j+1}x_{i_j}e_{i_1,\ldots,\widehat{i_j},\ldots,i_n}.$$

We define the Koszul homology modules as the homology modules of the Koszul complex, that is, for $n \geq 0$ we define the n-th Koszul homology module as the multigraded S-module

$$H_n(x_1,\ldots,x_s;S) = H_n(K_*(x_1,\ldots,x_s;S)) = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}.$$

For a multigraded S-module M, we consider the homological Koszul complex $K_*(x_1,\ldots,x_s;M)$ with respect to $x_1,\ldots,x_s$ as the complex given by

$$K_n(x_1,\ldots,x_s;M) = K_n(x_1,\ldots,x_s;S)\otimes_S M$$

with differentials $d_n \otimes id_M$. It is clear that $K_n(x_1,\ldots,x_s;M)$ has a structure of a multigraded S-module. In the same way, we define the Koszul homology modules of M as

$$H_n(x_1,\ldots,x_s;M) = \frac{\operatorname{Ker}(d_n \otimes id_M)}{\operatorname{Im}(d_{n+1}\otimes id_M)}$$

for $n \geq 0$. The modules $H_*(x_1,\ldots,x_s;M)$ are finitely generated $\mathbb{Z}^r$-graded S-modules and, for all $n \geq 0$ and $\underline{k} \in \mathbb{Z}^r$, it holds $H_n(x_1,\ldots,x_s;M)_{\underline{k}} = H_n(x_1,\ldots,x_s;M_{\underline{k}})$.

We know that $(x_1,\ldots,x_s)$ kills the homology module $H_n(x_1,\ldots,x_s;M)$ for all $n \in \mathbb{N}$, ([BH93] Proposition 1.6.5), and so, $H_n(x_1,\ldots,x_s;M)$ are

$S/(x_1,\ldots,x_s)$-modules. In the case when $(S_{\underline{0}},\mathfrak{m})$ is a Noetherian local ring and $x_1,\ldots,x_s$ is a system of generators of $\mathfrak{m}$ we get, from Proposition 2.3.10, the existence of the Hilbert quasi-polynomial of $H_n(x_1,\ldots,x_s;M)$. This will be the key tool in the proof of Theorem 3.2.1.

In order to compute the depth of a module with respect to an ideal, we can use the Koszul homology of this module with respect to a system of generators of the ideal. The following result, from [BH93], is proved in general for a module over a Noetherian ring. In addition, when the ring and the module have a multigraded structure and the morphisms are homogeneous, the Koszul homology modules inherit that multigraded structure as well.

Theorem 3.1.1 ([BH93] Theorem 1.6.17)**.** *Let S be a Noetherian ring, and M a finite S-module. Let I be the ideal $I=(x_1,\ldots,x_s)$ of S.*

(i) All the modules $H_i(x_1,\ldots,x_s;M)$, $i=0,\ldots,n$, vanish if and only if $M = IM$.

(ii) If $H_i(x_1,\ldots,x_s;M)\neq 0$ for some i, then

$$\operatorname{depth}_I(M)=\operatorname{grade}(I,M)=s-c$$

where $c=\max\{i \mid H_i(x_1,\ldots,x_s;M)\neq 0\}$.

3.2 Asymptotic depth of multigraded modules

Let S be a $\mathbb{N}^r$-graded ring, generated over $S_{\underline{0}}$ by elements of multidegrees $\gamma_1,\ldots,\gamma_r$, where $\gamma_i=(\gamma_1^i,\ldots,\gamma_i^i,0,\ldots,0)\in\mathbb{N}^r$ with $\gamma_i^i\neq 0$, for all $i=1,\ldots,r$. Let $\mathcal{M}$ be the maximal homogeneous ideal of S, that is $\mathcal{M}=\mathfrak{m}\oplus\bigoplus_{\underline{n}\neq\underline{0}}S_{\underline{n}}$, where $\mathfrak{m}$ is the maximal ideal of the Noetherian local ring $S_{\underline{0}}$.

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. We want to study the asymptotic depth, with respect to $\mathfrak{m}$, of the multigraded pieces $M_{\underline{n}}$. In our setting, by asymptotic we mean the elements $\underline{n}\in\mathbb{N}^r$ in a suitable cone $C_{\underline{\beta}}$. In the graded case, $r=1$, this is the same as considering a large enough n. In the standard multigraded case, it is the same as considering elements $\underline{n}\in\mathbb{N}^r$ with large enough components n_i for all $i=1,\ldots,r$, since in this case the cone is defined as the elements $\underline{m}\in\mathbb{N}^r$ such that $\underline{m}\geq\underline{\beta}$.

In the following theorem we generalize part of Theorem 1.1 in [HH05] to the non-standard multigraded case. In that paper, the authors consider standard graded modules instead. The proof will follows very similarly since we can compute the depth by means of the Koszul homology that works also in this case, but using the differences of the Hilbert function of a non-standard multigraded module. As previously pointed out, the key point in the proof is the existence of the Hilbert quasi-polynomial for the Koszul homology modules. In our case, the Hilbert function is not always a polynomial, but a quasi-polynomial in a cone of $\mathbb{N}^r$, so we will not be able to assure constant depth in a cone, but in a sub-net of it. In other non-standard multigraded settings in which the Hilbert function is eventually polynomial, we can assure constant depth in all the cone.

Theorem 3.2.1. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. There exists an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that*

$$\operatorname{depth}(M_{\underline{n}}) \geq \rho,$$

for all $\underline{n} \in C_{\underline{\beta}}$ with $M_{\underline{n}} \neq 0$, and

$$\operatorname{depth}(M_{\underline{n}}) = \rho,$$

for some $\underline{\delta} \in \Pi_{\underline{\beta}}$ and for all $\underline{n} \in \{\underline{\delta} + \sum_{i=1}^{r} \lambda_i \gamma_i \mid \lambda_i \in \mathbb{N}\} \subset C_{\underline{\beta}}$.

Proof. Let $x_1, \ldots, x_n$ be a minimal set of generators of $\mathfrak{m}$. To simplify the notation, we denote $\mathbf{x} = x_1, \ldots, x_n$. By Theorem 3.1.1, Theorem 1.6.17 in [BH93], if $M_{\underline{k}} \neq 0$,

$$\operatorname{depth}(M_{\underline{k}}) = n - \max\{i \mid H_i(\mathbf{x}; M_{\underline{k}}) \neq 0\}.$$

Since $\operatorname{rel.dim}(M) \geq r - 1$ for any $\mathbb{Z}^r$-graded S-module, we define

$$c = \max\{i \mid \operatorname{rel.dim}(H_i(\mathbf{x}; M)) > r - 1\}.$$

Then, for all $i > c$, $\operatorname{rel.dim}(H_i(\mathbf{x}; M)) = r - 1$, which is equivalent to $\dim(Supp_{++}(H_i(\mathbf{x}; M))) = -1$. Since $H_i(\mathbf{x}; M)$ is a finitely generated $\mathbb{Z}^r$-graded S-module, by Proposition 2.3.8, there exists a cone $C_{\underline{\beta}_i} \subset \mathbb{N}^r$, with $\underline{\beta}_i \in \mathbb{N}^r$, such that for all $\underline{k} \in C_{\underline{\beta}_i}$, it holds $H_i(\mathbf{x}; M)_{\underline{k}} = 0$. Thus, taking a $\underline{\beta} \geq \underline{\beta}_i$ for all $i > c$, we conclude that for all $\underline{k} \in C_{\underline{\beta}}$, with $M_{\underline{k}} \neq 0$,

$$\operatorname{depth}(M_{\underline{k}}) \geq n - c = \rho.$$

On the other hand, since $\operatorname{rel.dim}(H_c(\mathbf{x};M)) > r-1$, or equivalently $\dim(Supp_{++}(H_c(\mathbf{x};M))) = d \geq 0$, there exists a quasi-polynomial P of degree $d \geq 0$, i.e $P \neq 0$, and a cone $C_{\underline{\beta}'} \subset \mathbb{N}^r$ with vertex at $\underline{\beta}' \in \mathbb{N}^r$ such that for all $\underline{k} \in C_{\underline{\beta}'}$ we have

$$\operatorname{length}_{S_{\underline{0}}}(H_c(\mathbf{x};M)_{\underline{k}}) = P(\underline{k}).$$

We can assume that $\underline{\beta} = \underline{\beta}'$, readjusting the cone if necessary.

This means that for any $\underline{\delta} \in \Pi_{\underline{\beta}}$ in the basic cell, (see Remark 2.3.11), there exists a polynomial $f_{\underline{\delta}} \in \mathbb{Z}[\underline{n}]$ such that $P(\underline{k}) = f_{\underline{\delta}}(\underline{k})$ for $\underline{k} = \underline{\delta} + \sum_{i=1}^{r} n_i\gamma_i$, with $n_i \in \mathbb{N}$. Since d is the maximum of the total degrees of these polynomials $f_{\underline{\delta}}$ for $\underline{\delta} \in \Pi_{\underline{\beta}}$, this means that at least one of these polynomials has degree d, but we cannot control the degree of the others. So, there is a $\underline{\delta} \in \Pi_{\underline{\beta}}$ such that $|\deg(f_{\underline{\delta}})| = d$. Therefore, $\operatorname{length}_{S_{\underline{0}}}(H_c(\mathbf{x};M)_{\underline{k}}) = f_{\underline{\delta}}(\underline{k}) \neq 0$ for all $\underline{k} \in \underline{\delta} + \{\lambda_1\gamma_1 + \cdots + \lambda_r\gamma_r \mid \lambda_i \in \mathbb{N}\}$. Hence $H_c(\mathbf{x};M)_{\underline{k}} \neq 0$ for all $\underline{k} \in \underline{\delta} + \{\lambda_1\gamma_1 + \cdots + \lambda_r\gamma_r \mid \lambda_i \in \mathbb{N}\}$, which is a sub-net of $C_{\underline{\beta}}$.

In conclusion, we have proved that $\operatorname{depth}(M_{\underline{k}}) = n - c$, which is a constant value, for all $\underline{k} \in \{\underline{\delta} + \lambda_1\gamma_1 + \cdots + \lambda_r\gamma_r \mid \lambda_i \in \mathbb{N}\} \subset C_{\underline{\beta}}$, and $\operatorname{depth}(M_{\underline{k}}) \geq n - c = \rho$ for $\underline{k} \in C_{\underline{\beta}}$ with $M_{\underline{k}} \neq 0$. □

Remark 3.2.2. Observe that we cannot assure that the depth will be constant in all the cone, as it would be desirable, since we cannot control the degrees of all the collection of polynomials that define the quasi-polynomial. So, if all the polynomials have non-negative degree, i.e. they are not identically zero, we can assure constant depth in all the cone.

In general, we cannot improve this result. If we consider $M = S$ as a multigraded S-module, it is clear that for all $\underline{k} \in C_{\underline{\beta}} \setminus \Gamma$, we have $M_{\underline{k}} = 0$, for any cone $C_{\underline{\beta}} \subset \mathbb{N}^r$. So $\operatorname{depth}(M_{\underline{k}})$ turns out to be different in $C_{\underline{\beta}} \cap \Gamma$ from the rest of the cone.

When the quasi-polynomial is, in fact, a polynomial, we can assure the constant depth in all the cone.

Proposition 3.2.3. *Let S be an $\mathbb{N}^r$-graded algebra generated over $S_{\underline{0}}$ by elements of degrees $(1,0,\ldots,0),(*,1,0,\ldots,0),\ldots,(*,*,*,\ldots,1) \in \mathbb{N}^r$. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. There exist an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that, for $\underline{n} \in C_{\underline{\beta}}$,*

$$\operatorname{depth}(M_{\underline{n}}) = \rho.$$

Proof. See the proof of the existence of the Hilbert polynomial in this case in [Lav99], see also [Rob98]. In this case, since the Hilbert function is, in fact, polynomial, in the second part of the proof of Theorem 3.2.1, we have that $H_c(\mathbf{x}; M)_{\underline{k}} \neq 0$ for $\underline{k} \in C_{\underline{\beta}}$, and hence,

$$\operatorname{depth}(M_{\underline{k}}) = n - c = \rho$$

for all $\underline{k} \in C_{\underline{\beta}}$. □

Corollary 3.2.4. *Let S be a standard $\mathbb{N}^r$-graded algebra. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. There exist an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that, for $\underline{n} \geq \underline{\beta}$,*

$$\operatorname{depth}(M_{\underline{n}}) = \rho.$$

Proof. This is clearly a corollary of the previous proposition since in the standard case, the degrees of the generators of S are $e_1 = (1, 0, \ldots, 0)$, …, $e_r = (0, \ldots, 0, 1)$ and a cone $C_{\underline{\beta}}$ is defined by the elements of the form $\underline{\beta} + \sum_{i=1}^{r} \lambda_i e_i = (\beta_1 + \lambda_1, \ldots, \beta_r + \lambda_r) \in \mathbb{N}^r$ with $\lambda_i \in \mathbb{N}$. □

3.3 Asymptotic depth of multigraded blow-up algebras

Let us consider the multigraded Rees algebra associated to some ideals $I_1, \ldots, I_r$ of a Noetherian local ring $(R, \mathfrak{m})$,

$$\mathcal{R}(I_1, \ldots, I_r) = \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset R[t_1, \ldots, t_r].$$

For $k = 1, \ldots, r$ let us consider the k-th associated multigraded ring of $I_1, \ldots, I_r$ in R,

$$gr_{I_1,\ldots,I_r;I_k}(R) = \bigoplus_{\underline{n} \in \mathbb{N}^r} \frac{I_1^{n_1} \cdots I_k^{n_k} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} = \frac{\mathcal{R}(I_1, \ldots, I_r)}{I_k \mathcal{R}(I_1, \ldots, I_r)}.$$

They are finitely generated standard $\mathbb{Z}^r$-graded $\mathcal{R}(I_1, \ldots, I_r)$-modules, in both cases, and each component, $\mathcal{R}(I_1, \ldots, I_r)_{\underline{n}}$ and $gr_{I_1,\ldots,I_r;I_k}(R)_{\underline{n}}$, is a finitely generated R-module.

In the next proposition we generalize Theorem 1.2 in [HH05] in order to study the depth with respect to $\mathfrak{m}$ of the pieces of the previous multigraded modules.

Proposition 3.3.1. *There exist elements* $\underline{\beta}, \underline{\beta}_k \in \mathbb{N}^r$ *and integers* $\alpha, \delta_k \in \mathbb{N}$, *for* $k = 1, \ldots, r$, *such that*

$$\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \alpha$$

for all $\underline{n} = (n_1, \ldots, n_r) \geq \underline{\beta}$, *and*

$$\operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) = \delta_k$$

for all $\underline{n} = (n_1, \ldots, n_r) \geq \underline{\beta}_k$, *and for any* $k = 1, \ldots, r$.

Proof. By Corollary 3.2.4 applied to the multigraded modules $\mathcal{R}(I_1, \ldots, I_r)$ and $gr_{I_1,\ldots,I_r;I_k}(R)$, there exist some $\underline{\beta}, \underline{\beta}_k \in \mathbb{N}^r$ and integers $\alpha, \delta_k \in \mathbb{N}$ such that $\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \alpha$ for $\underline{n} \geq \underline{\beta}$, and for all $\underline{n} \geq \underline{\beta}_k$ it holds $\operatorname{depth}\left(I_1^{n_1} \cdots I_r^{n_r} / I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}\right) = \delta_k$. □

In [HH05], Theorem 1.2, the authors prove that $\delta_1 = \alpha - 1$ when $r = 1$. In our case we want to prove that $\delta_k = \alpha - 1$ for all $k = 1, \ldots, r$, and hence $\delta_1 = \cdots = \delta_r$. In order to prove this, we first need the next lemma generalizing Lemma 1 in [Lev78].

Lemma 3.3.2. *Let* $(R, \mathfrak{m})$ *be a local ring and* $I \subset R$ *an ideal. Let* $K_* = K_*(x_1, \ldots, x_n; R)$ *be the Koszul complex of* R *with respect to* $x_1, \ldots, x_n$, *a minimal set of generators of* $\mathfrak{m}$. *Let* M *be a finitely generated* R*-module. Then, there exists a positive integer* c *such that the induced morphism*

$$H_*(x_1, \ldots, x_n; I^l M) \to H_*(x_1, \ldots, x_n; I^c M)$$

is zero for all $l > c$.

Proof. Since $\operatorname{Im}(d_{n+1} \otimes id_{I^l M}) = I^l \operatorname{Im}(d_{n+1} \otimes id_M)$,

$$H_*(\mathbf{x}; I^l M) = \frac{(K_* \otimes I^l M) \cap \operatorname{Ker}(d_n \otimes id_M)}{I^l \operatorname{Im}(d_{n+1} \otimes id_M)},$$

where $\mathbf{x} = x_1, \ldots, x_n$.

By the Artin-Rees lemma, there exists a positive integer c such that for all $l > c$ it holds

$$I^l(K_* \otimes M) \cap \operatorname{Ker}(d_n \otimes id_M) = I^{l-c}(I^c(K_* \otimes M) \cap \operatorname{Ker}(d_n \otimes id_M)).$$

Now, since $H_*(\mathbf{x}; I^l M)$ is killed by the elements in $(x_1, \ldots, x_n) = \mathfrak{m}$, ([BH93] Proposition 1.6.5), and $I \subset \mathfrak{m}$, then

$$I^{l-c}(I^c(K_* \otimes M) \cap \operatorname{Ker}(d_n \otimes id_M)) \subset I^c \operatorname{Im}(d_{n+1} \otimes id_M).$$

Therefore,

$$\begin{aligned} I^l(K_* \otimes M) \cap \operatorname{Ker}(d_n \otimes id_M) &= I^{l-c}(I^c(K_* \otimes M) \cap \operatorname{Ker}(d_n \otimes id_M)) \\ &\subset I^c \operatorname{Im}(d_{n+1} \otimes id_M). \end{aligned}$$

Thus, the induced morphism

$$H_*(\mathbf{x}; I^l M) \to H_*(\mathbf{x}; I^c M)$$

is zero for all $l > c$. □

Now we can prove that all asymptotic depths for the multigraded pieces of the k-th associated multigraded graded ring in Theorem 3.3.1 coincide.

Proposition 3.3.3. *For all $k = 1, \ldots, r$*

$$\delta_k = \alpha - 1.$$

Proof. By Proposition 3.3.1 there exist positive integers α, δ_k and elements $\underline{\beta}_0, \underline{\beta}_k \in \mathbb{N}^r$, for $k = 1, \ldots, r$, such that

$$\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \alpha$$

for all $\underline{n} = (n_1, \ldots, n_r) \geq \underline{\beta}_0$, and

$$\operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) = \delta_k$$

for all $\underline{n} = (n_1, \ldots, n_r) \geq \underline{\beta}_k$. If $\underline{\beta} \in \mathbb{N}^r$ is an element such that $\underline{\beta} \geq \underline{\beta}_i$, componentwise, for all $i = 0, \ldots, r$, then all asymptotic depths hold for $\underline{n} \geq \underline{\beta}$.

For all $k = 1, \ldots, r$, and $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ we consider the exact sequence of A-modules

$$0 \longrightarrow I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r} \longrightarrow I_1^{n_1} \cdots I_r^{n_r} \longrightarrow \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} \longrightarrow 0.$$

For all $\underline{n} \geq \underline{\beta}$, by depth counting on this exact sequence, we have that

$$\delta_k \geq \min\{\alpha, \alpha - 1\} = \alpha - 1.$$

Assume that $\delta_k > \alpha - 1$. Let $\mathbf{x} = x_1, \ldots, x_n$ be a minimal system of generators of $\mathfrak{m}$. Then

$$\alpha = \operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = n - \max\{i \mid H_i(\mathbf{x}; I_1^{n_1} \cdots I_r^{n_r}) \neq 0\}$$

and so, for all $\underline{n} \geq \underline{\beta}$,

$$H_{n-\alpha}(\mathbf{x}; I_1^{n_1} \cdots I_r^{n_r}) \neq 0.$$

On the other hand,

$$\begin{aligned}
\delta_k &= \operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) \\
&= n - \max\{i \mid H_i\left(\mathbf{x}; \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) \neq 0\} \\
&> \alpha - 1
\end{aligned}$$

and hence, in particular,

$$H_{n-\alpha+1}\left(\mathbf{x}; \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) = 0.$$

From the long exact sequence of homology, we have that

$$\begin{aligned}
0 &= H_{n-\alpha+1}\left(\mathbf{x}; \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) \to \\
&\to H_{n-\alpha}(\mathbf{x}; I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}) \to H_{n-\alpha}(\mathbf{x}; I_1^{n_1} \cdots I_r^{n_r}) \to \ldots
\end{aligned}$$

and thus, $H_{n-\alpha}(\mathbf{x}; I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}) \to H_{n-\alpha}(\mathbf{x}; I_1^{n_1} \cdots I_r^{n_r})$ is an injective morphism for $\underline{n} \geq \underline{\beta}$.

By composition of injective maps,

$$H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+\lambda}\cdots I_r^{n_r}) \to H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+\tau}\cdots I_r^{n_r})$$

is injective for $\lambda > \tau \geq 1$ and $\underline{n} \geq \underline{\beta}$.

Moreover, by Lemma 3.3.2, there exists a positive integer c such that for $l > c$ the morphism

$$H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+l}\cdots I_r^{n_r}) \to H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+c}\cdots I_r^{n_r})$$

is zero. Therefore, $H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+l}\cdots I_r^{n_r}) = 0$ for $l \gg 0$ and $\underline{n} \geq \underline{\beta}$, that give us a contradiction.

Hence,

$$\delta_k = \alpha - 1$$

for all $k = 1, \ldots, r$. □

We are interested on the depth of $R/I_1^{n_1}\cdots I_r^{n_r}$ for $\underline{n}$ large enough. In this case, we cannot directly apply results such as Theorem 3.2.1 or Corollary 3.2.4, since $\bigoplus_{\underline{n}} R/I_1^{n_1}\cdots I_r^{n_r}$ does not have a multigraded module structure as the multi-Rees algebra or the associated multigraded ring have. In this case, we can take advantage of the constant asymptotic depth of these last two modules and the relation with $R/I_1^{n_1}\cdots I_r^{n_r}$ by means of some short exact sequences of R-modules where we can use the depth counting techniques.

Example 3.3.4. With CoCoA, [CoC], one can easily check the behavior of $\operatorname{depth}(R/I_1^{n_1}\cdots I_r^{n_r})$ for some $\underline{n} \in \mathbb{N}^r$ with the instruction `Depth`.

For example, considering the ring $R = K[[x, y, z, t, u, v]]$, with K a field, and the ideals of R $I_1 = (x^6, x^5y, xy^5, y^6)$, $I_2 = (x^4y^4z, x^4y^4t)$ and $I_3 = (x^4u^2v^3, y^4u^3v^2)$, we conjecture that

$$\operatorname{depth}\left(\frac{R}{I_1^{n_1}I_2^{n_2}I_3^{n_3}}\right) = 2$$

for $(n_1, n_2, n_3) \geq (4, 1, 1)$.

In the following table, there are the results of our test for elements $(0,0,0) \leq \underline{n} \leq (8,8,8)$. Notice that the depth is variable outside the cone with vertex at $(4,1,1)$ and generators e_1, e_2, e_3.

(n_1, n_2, n_3)	depth $\left(\frac{R}{I_1^{n_1} I_2^{n_2} I_3^{n_3}}\right)$
$(1,1,2) \leq (n_1, n_2, n_3) \leq (3,8,8)$	1
$(1,0,2) \leq (n_1, 0, n_2) \leq (3,0,8)$ $(1,1,1) \leq (n_1, n_2, 1) \leq (3,8,1)$ $(4,1,1) \leq (n_1, n_2, n_3) \leq (8,8,8)$	2
$(0,1,1) \leq (0, n_2, n_3) \leq (0,8,8)$ $(1,1,0) \leq (n_1, n_2, 0) \leq (8,8,0)$ $(4,0,1) \leq (n_1, 0, n_3) \leq (8,0,8)$ $(1,0,1) \leq (n_1, 0, 1) \leq (3,0,1)$	3
$(1,0,0) \leq (n_1, 0, 0) \leq (8,0,0)$ $(0,1,0) \leq (0, n_2, 0) \leq (0,8,0)$ $(0,0,1) \leq (0, 0, n_3) \leq (0,0,8)$	4
$(0,0,0)$	6

If we denote by $\delta = \delta_k = \alpha - 1$, for all $k = 1, \ldots, r$, then from Proposition 3.3.1 and Proposition 3.3.3 we get:

Corollary 3.3.5. *There exists an element $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{n} \geq \underline{\beta}$ it holds*

$$\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \delta + 1$$

and

$$\operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k} \cdots I_r^{n_r}}\right) = \delta$$

for all $k = 1, \ldots, r$.

Proof. By Proposition 3.3.1, there are some elements $\underline{\beta}_k \in \mathbb{N}^r$, $k = 0, \ldots, r$, and $\alpha, \delta \in \mathbb{N}$ such that $\operatorname{depth}(I_1^{n_1} \ldots I_r^{n_r}) = \alpha = \delta + 1$ for $\underline{n} \geq \underline{\beta}_0$ and for any $k = 1, \ldots, r$, $\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r} / I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}) = \delta$ for all $\underline{n} \geq \underline{\beta}_k$.

We only need to consider an element $\underline{\beta} \in \mathbb{N}^r$ (componentwise) greater than the other $\underline{\beta}_k$ for all $k = 0, \dots, r$. Then for all $\underline{n} \geq \underline{\beta}$ it holds

$$\operatorname{depth}(I_1^{n_1} \dots I_r^{n_r}) = \delta + 1$$

and

$$\operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) = \delta$$

for all $k = 1, \dots, r$. □

Let $\underline{\beta} \in \mathbb{N}^r$ be as in the above Corollary, that is from where all the previous asymptotic depths hold.

Theorem 3.3.6. *There exist an element $\underline{\varepsilon} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that*

$$\operatorname{depth}\left(\frac{R}{I_1^{n_1} \cdots I_r^{n_r}}\right) = \rho \leq \delta$$

for all $\underline{n} \geq \underline{\varepsilon}$. Moreover, if there exists an $\underline{n} \geq \underline{\beta}$ such that $\operatorname{depth}\left(\frac{R}{I_1^{n_1} \cdots I_r^{n_r}}\right) \geq \delta$, *then $\rho = \delta$.*

Proof. Let $\underline{\beta}$ be as in the previous Corollary. We write

$$d(\underline{n}) = \operatorname{depth}\left(\frac{R}{I_1^{n_1} \cdots I_r^{n_r}}\right)$$

and we denote by $e_1, \dots, e_r$ the canonical basis of $\mathbb{R}^r$.

For all $k = 1, \dots, r$ there is the short exact sequence of R-modules

$$0 \longrightarrow \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} \longrightarrow \frac{A}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} \longrightarrow \frac{A}{I_1^{n_1} \cdots I_r^{n_r}} \longrightarrow 0.$$

Using depth counting on this exact sequence we have that for $\underline{n} \geq \underline{\beta}$,

$$\delta \geq \min\{d(\underline{n} + e_k), d(\underline{n}) + 1\} \quad (1)$$

$$d(\underline{n} + e_k) \geq \min\{d(\underline{n}), \delta\} \quad (2)$$

Assume now that there exists an $\underline{n}_0 \geq \underline{\beta}$ such that $d(\underline{n}_0) \geq \delta$. From (2), we deduce that $d(\underline{n}_0 + e_k) \geq \delta$, and then by (1), we get that $d(\underline{n}_0 + e_k) = \delta$

for all $k = 1, \dots, r$. Using recursively (2) and (1) we deduce that $d(\underline{n}) = \delta$ for all $\underline{n} \geq \underline{n}_0$. We put $\underline{\varepsilon} = \underline{n}_0$ in this case.

Assume now that for all $\underline{n} \geq \underline{\beta}$ it holds $d(\underline{n}) < \delta$. By (2) we have that $d(\underline{n} + e_k) \geq d(\underline{n})$ for all $k = 1, \dots, r$, since, by hypothesis, $d(\underline{n} + e_k) < \delta$, using (2) recursively we deduce that $\delta > d(\underline{m}) \geq d(\underline{n})$ for all $\underline{m} \geq \underline{n}$. So, $d(\underline{n})$ is an increasing function, bounded from above by δ. Therefore, there exist an element $\underline{\varepsilon} \geq \underline{\beta}$ such that

$$d(\underline{n}) = d(\underline{\alpha}) = \rho$$

for all $\underline{n} \geq \underline{\varepsilon}$. In fact, if we assume the contrary, for an element $\underline{n}_0 \geq \underline{\beta}$, there exist an $\underline{n}_1 \geq \underline{n}_0$ such that $d(\underline{n}_1) > d(\underline{n}_0)$. Again, there exists an element $\underline{n}_2 \geq \underline{n}_1$ such that $d(\underline{n}_2) > d(\underline{n}_1)$. Recursively, we obtain an increasing sequence of elements in $\mathbb{N}^r$,

$$\underline{\beta} \leq \underline{n}_0 \leq \underline{n}_1 \leq \underline{n}_2 \leq \cdots \leq \underline{n}_i \leq \dots$$

that give us an strictly increasing sequence of positive integers

$$d(\underline{n}_0) < d(\underline{n}_1) < d(\underline{n}_2) < \cdots < d(\underline{n}_i) < \dots$$

bounded from above by δ. Hence, it has to stabilize, and we get the contradiction.

Summarizing the two cases, we get that there exist an element $\underline{\varepsilon}$ and an integer ρ, such that for all $\underline{n} \geq \underline{\varepsilon}$

$$\operatorname{depth}\left(\frac{R}{I_1^{n_1} \cdots I_r^{n_r}}\right) = \rho.$$

Moreover, if there exist an $\underline{n} \geq \underline{\beta}$ such that $\operatorname{depth}\left(\frac{R}{I_1^{n_1} \cdots I_r^{n_r}}\right) \geq \delta$, then

$$\rho = \delta.$$

□

In [KR94], it is defined the *spread* of a multigraded ring A with $(A_{\underline{0}}, \mathfrak{m})$ a local ring as

$$s(A) = \dim \operatorname{Proj}^r(A/\mathfrak{m}A) + 1.$$

When $A = \mathcal{R}(I)$ is the Rees algebra of an ideal $I \subset R$, the *analytic spread* of I is defined as, [NR54],

$$\dim(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)) = \dim \operatorname{Proj}^1(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)) + 1.$$

Although in this chapter we have not studied bounds for the asymptotic depth of a multigraded module in the more general case, we can take advantage of the formula proved by Hayasaka in [Hay06], Theorem 4.1, in the standard multigraded case, to bound the asymptotic depth of the modules $R/I_1^{n_1} \cdots I_r^{n_r}$.

Proposition 3.3.7. *Let $\rho \in \mathbb{N}$ be the asymptotic depth of $R/I_1^{n_1} \cdots I_r^{n_r}$. Then,*

$$\rho \leq \dim(R) - s(\mathcal{R}(I_1, \dots, I_r)) + 1,$$

or equivalently,

$$\rho \leq \dim(R) - \dim \operatorname{Proj}^r \left(\frac{\mathcal{R}(I_1, \dots, I_r)}{\mathfrak{m}\mathcal{R}(I_1, \dots, I_r)} \right).$$

Proof. By Theorem 4.1 in [Hay06],

$$s(\mathcal{R}(I_1, \dots, I_r)) \leq s(R[t_1, \dots, t_r]) + \dim(R) - \rho,$$

which is equivalent to

$$\dim \operatorname{Proj}^r \left(\frac{\mathcal{R}(I_1, \dots, I_r)}{\mathfrak{m}\mathcal{R}(I_1, \dots, I_r)} \right) \leq \dim \operatorname{Proj}^r \left(\frac{R[t_1, \dots, t_r]}{\mathfrak{m}R[t_1, \dots, t_r]} \right) + \dim(R) - \rho.$$

By Remark 2.2.9, $\dim \operatorname{Proj}^r \left(\frac{R[t_1, \dots, t_r]}{\mathfrak{m}R[t_1, \dots, t_r]} \right) = 0$, or $s(R[t_1, \dots, t_r]) = 1$, and hence,

$$\dim \operatorname{Proj}^r \left(\frac{\mathcal{R}(I_1, \dots, I_r)}{\mathfrak{m}\mathcal{R}(I_1, \dots, I_r)} \right) \leq \dim(R) - \rho,$$

or equivalently,

$$s(\mathcal{R}(I_1, \dots, I_r)) - 1 \leq \dim(R) - \rho$$

□

Remark 3.3.8. As a corollary of the results of this section we partially get Theorem 3.3 in [BZ06], concerning the asymptotic depth of standard graded modules, [Hay06], Theorem 3.1 in the standard multigraded modules case, and [HH05], Theorem 1.1; all of them in a standard framework.

Notice that [Hay06], Theorem 3.1, is deduced from the asymptotic stability of $Ass(M_{\underline{n}})$. Here we get a direct proof by using the Hilbert quasi-polynomials of the Koszul homology. The asymptotic stability of $Ass(M_{\underline{n}})$ is an open question that we do not address here.

Chapter 4

Veronese multigraded modules

Let $S = \bigoplus_{\underline{n} \in \mathbb{N}^r} S_{\underline{n}}$ be a Noetherian $\mathbb{N}^r$-graded ring generated as $S_{\underline{0}}$-algebra by homogeneous elements g_i^j for $i = 1, \cdots, r$ and $j = 1, \ldots, \mu_i$, of multidegrees $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0, \ldots, 0) \in \mathbb{N}^r$, respectively, with $\gamma_i^i \neq 0$. We assume that $S_{\underline{0}}$ is a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field.

The main purpose of this chapter is to study the Veronese modules associated to a non-standard multigraded S-module M by means of some cohomological properties of the module. We mainly study the vanishing of the local cohomology modules of M and of the Veronese modules of M, generalizing some results on the depth of Veronese modules associated to Rees algebras proved in [Eli04].

In Section 4.2 we extend several results on homogeneous ideals of $\mathbb{Z}$-graded rings to homogeneous ideals of non-standard $\mathbb{Z}^r$-graded rings, Propositions 4.2.1, 4.2.2 and 4.2.3. We consider the multigraded scheme $\text{Proj}^r(S)$ and we define the projective Cohen-Macaulay deviation of a multigraded module and we link this number with the generalized depth, studied by Brodmann and Faltings (see [Bro83] and [Fal78]), Theorem 4.2.7. As a corollary we prove that the generalized depth remains invariant by taking Veronese modules, Proposition 4.2.8.

In the first part of Section 4.3 we prove, under the general hypothesis on the degrees of S, that the depth of the Veronese modules $M^{(\underline{b})}$ is constant for special asymptotic values of $\underline{b}$, Proposition 4.3.1. In the rest of the section we extend to a non-standard framework the notion of finite graduation, [Mar95]. At this point we need to restrict our setting to the almost-standard case, that is with positive multiples of the canonical basis of $\mathbb{R}^r$ as a multidegrees of the generators. Under these special degrees of S we prove that the generalized depth of a multigraded module coincides with its finitely graduation order, Theorem 4.3.7. We use it to get that the depth of the Veronese modules $M^{(\underline{a},\underline{b})}$ is constant for large $\underline{a}, \underline{b} \in \mathbb{N}^r$, Theorem 4.3.12, and we apply this result to the multigraded Rees algebras associated to a finite set of ideals, Proposition 4.3.15.

4.1 Veronese modules

Given integral vectors $\gamma_i = (\gamma_1^i, \dots, \gamma_i^i, 0, \dots, 0) \in \mathbb{N}^r$, $i = 1, \cdots, r$, such that $\gamma_i^i \neq 0$, we denote by ϕ the map

$$\begin{array}{cccc} \phi: & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^r \\ & \underline{n} & \longmapsto & \sum_{i=1}^r n_i \gamma_i. \end{array}$$

Note that $Im(\phi) = \Gamma(\gamma_1, \dots, \gamma_r)$ is the subgroup of $\mathbb{Z}^r$ generated by γ_i, $i = 1, \cdots, r$.

We denote by G the $r \times r$ triangular matrix whose columns are the vectors $\gamma_1, \dots, \gamma_r$. Note that G is a non-singular matrix and that the multi-index $t_1\gamma_1 + \cdots + t_r\gamma_r$ is the column vector $G\underline{t}$.

Given $\underline{a} \in \mathbb{N}^{*r}$ we denote by $\phi_{\underline{a}}$ the map

$$\begin{array}{cccc} \phi_{\underline{a}}: & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^r \\ & \underline{n} & \longmapsto & \phi_{\underline{a}}(\underline{n}) = \phi(\underline{n}.\underline{a}), \end{array}$$

with $\phi_{\underline{a}}(\underline{n}) = \phi(\underline{n}.\underline{a}) = \sum_{i=1}^r (n_i a_i)\gamma_i$ for all $\underline{n} \in \mathbb{Z}^r$.

Definition 4.1.1. *The Veronese transform of S with respect to $\underline{a} \in \mathbb{N}^{*r}$, or the $(\underline{a})$-Veronese, is the ring*

$$S^{(\underline{a})} = \bigoplus_{\underline{n} \in \mathbb{N}^r} S_{\phi_{\underline{a}}(\underline{n})}.$$

$S^{(\underline{a})}$ is a subring of S. The degrees of its generators have the same triangular configuration as the degrees of S.

Definition 4.1.2. *Given a $\mathbb{Z}^r$-graded S-module M we denote by $M^{(\underline{a},\underline{b})}$ the Veronese transform of M with respect to $\underline{a},\underline{b} \in \mathbb{N}^{*r}$, or the $(\underline{a},\underline{b})$-Veronese,*

$$M^{(\underline{a},\underline{b})} = \bigoplus_{\underline{n}\in\mathbb{Z}^r} M_{\phi_{\underline{a}}(\underline{n})+\underline{b}}.$$

$M^{(\underline{a},\underline{b})}$ is an $S^{(\underline{a})}$-module. Observe that when $\underline{b} = (0,\dots,0)$ we recover the classical definition of Veronese of a module.

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. By using a similar argument as in [HHR93], Lemma 1.13 and Lemma 1.14, where the standard case was studied, (see also [GW78]), we next prove that the local cohomology functor and the Veronese functor commute, i.e.,

$$H^*_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})}) \cong (H^*_{\mathcal{M}}(M))^{(\underline{a},\underline{b})}$$

where $\mathcal{M} = \mathfrak{m} \oplus S_+$ is the maximal homogeneous ideal of S and $\underline{a},\underline{b} \in \mathbb{N}^{*r}$. For the basic properties of local cohomology we use [BS98] as a general reference.

If M is a $\mathbb{Z}^r$-graded S-module, we denote by $E_S(R)$ the homogeneous injective hull of M in the category $\mathcal{M}^r(S)$ of $\mathbb{Z}^r$-graded S-modules with homogeneous morphisms. We consider the functor Ext^*_S in this category. For the sake of completeness we include the following results proved in [GW78].

Lemma 4.1.3 ([GW78] Lemma 1.3.1)**.** *Let E be a $\mathbb{Z}^r$-graded S-module. Then the following conditions are equivalent:*

(i) E is an injective object of $\mathcal{M}^r(S)$.

(ii) $\operatorname{Ext}^i_S(S/I, E) = 0$ for every homogeneous ideal I of S.

(iii) $\operatorname{Ext}^i_S(\cdot, E) = 0$ for every integer $i > 0$.

Theorem 4.1.4 ([GW78] Theorem 1.3.3)**.**

(i) $Ass_S(E_S(M)) = Ass_S(M)$ for every $\mathbb{Z}^r$-graded S-module M.

(ii) Let E be a $\mathbb{Z}^r$-graded S-module. Then E is an indecomposable injective object of $\mathcal{M}^r(S)$ if and only if $E \cong (E_S(S/\mathfrak{P}))(\underline{k})$ for some homogeneous prime ideal $\mathfrak{P}$ of S and some $\underline{k} \in \mathbb{Z}^r$. In this case, $Ass_S(E) = \{\mathfrak{P}\}$ and $\mathfrak{P}$ is uniquely determined for E.

(iii) Every injective object E of $\mathcal{M}^r(S)$ can be decomposed into a direct sum of indecomposable injective objects of $\mathcal{M}^r(S)$. This decomposition is uniquely determined by E up to isomorphisms.

We say that a $\mathbb{Z}^r$-graded S-module M satisfies condition (A) if it satisfies one of the following properties:

(A_1) For each $x \in M$ there is a $t \geq 0$ such that $\mathcal{M}^t x = 0$.

(A_2) For some $\underline{n} \in \mathbb{N}^r$, there is a $s \in \mathcal{M}_{\underline{n}}$ such that $M(-\underline{n}) \xrightarrow{\cdot s} M$ is an isomorphism.

If M is a $\mathbb{Z}^r$-graded S-module with the property (A), then $H^i_{\mathcal{M}}(M) = 0$ for $i > 0$.

Proposition 4.1.5. *Let S be a $\mathbb{Z}^r$-graded ring generated by elements of degrees $\gamma_1, \ldots, \gamma_r$ over a local ring $S_{\underline{0}}$, and let I be an injective S-module. For any $\underline{a}, \underline{b} \in (\mathbb{N}^*)^r$, the Veronese module $I^{(\underline{a},\underline{b})}$ has the property (A).*

Proof. We denote by $\mathcal{M}$ for the homogeneous maximal ideal of S and by $\mathfrak{m}$ the maximal ideal of the local ring $S_{\underline{0}}$.

By Theorem 4.1.4, it suffices to prove the case in which $I = E_S(S/\mathfrak{P})$ (a homogeneous injective hull) with $\mathfrak{P}$ a homogeneous prime ideal of S.

If $\mathfrak{P} = \mathcal{M}$, then I and $I^{(\underline{a},\underline{b})}$ clearly satisfy the property (A_1), and hence (A).

If $\mathfrak{P} \neq \mathcal{M}$, then we prove that $I^{(\underline{a},\underline{b})}$ has the property (A_2), by distinguishing between two cases:

Assume first that $\mathfrak{P}_{\underline{0}} \neq \mathfrak{m}$. Then there exists an element $z \in \mathfrak{m}$ (z has degree $\underline{0}$) such that $z \notin \mathfrak{P}$. Multiplication by z gives an isomorphism $I \xrightarrow{\cdot z} I$ that induces an isomorphism $I^{(\underline{a},\underline{b})} \xrightarrow{\cdot z} I^{(\underline{a},\underline{b})}$. In fact, $\cdot z$ is injective since $z \notin \mathfrak{P}$ is not a zero-divisor of $I = E_S(S/\mathfrak{P})$ and it is surjective thanks to the isomorphism theorem that assures that $\operatorname{Im}(\cdot z) \cong I$.

Now assume that $\mathfrak{P}_{\underline{0}} = \mathfrak{m}$. Since $\mathfrak{P} \neq \mathcal{M}$ and $\gamma_1, \ldots, \gamma_r$ are the degrees of the generators, there exists an γ_i such that $S_{\gamma_i} \not\subset \mathfrak{P}$. In fact,if $S_{\gamma_i} \subset \mathfrak{P}$ for all i, then $\mathcal{M} = \mathfrak{P}$. Let $z \in S_{\gamma_i}$ be an element such that $z \notin \mathfrak{P}$. As in the previous case, multiplication by z gives an isomorphism $I(-\gamma_i) \xrightarrow{\cdot z} I$, and also the multiplication by a^m for all $m \geq 1$. If $x \in (I^{(\underline{a},\underline{b})})_{\underline{n}} = I_{\phi_{\underline{a}}(\underline{n})+\underline{b}}$, where $\phi_{\underline{a}}(\underline{n}) = \sum_{j=1}^{r}(a_j n_j)\gamma_j$, then the degree of $z^{a_i}x$ is

$$a_i\gamma_i + \sum_{j=1}^{r}(a_j n_j)\gamma_j + \underline{b} = \sum_{j\neq i}(a_j n_j)\gamma_j + (a_i(n_i+1))\gamma_i + \underline{b} = \phi_{\underline{a}}(\underline{n}+\underline{e}_i) + \underline{b},$$

where $\underline{e}_1, \ldots, \underline{e}_r$ is the canonical basis of $\mathbb{R}^n$. Therefore, multiplication by z^{a_i} gives an isomorphism $I^{(\underline{a},\underline{b})}(-\underline{e}_i) \xrightarrow{\cdot z^{a_i}} I^{(\underline{a},\underline{b})}$, being $z^{a_i} \in \mathcal{M}_{a_i\gamma_i} = (\mathcal{M}^{(\underline{a})})_{\underline{e}_i}$ since $\phi_{\underline{a}}(\underline{e}_i) = a_i\gamma_i$.

Summarizing the two cases, we get that $I^{(\underline{a},\underline{b})}$ has the property (A_2), and thus (A). $\square$

Now we can prove that the Veronese functor commutes with local cohomology.

Proposition 4.1.6. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then,*

$$(H^i_{\mathcal{M}}(M))^{(\underline{a},\underline{b}))} \cong H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})})$$

for all $i \geq 0$ and $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$.

Proof. Let us consider an injective resolution of M

$$0 \longrightarrow M \longrightarrow I^{\bullet}.$$

By Proposition 4.1.5,

$$0 \longrightarrow M^{(\underline{a},\underline{b})} \longrightarrow I^{\bullet(\underline{a},\underline{b})}$$

is an injective resolution of $M^{(\underline{a},\underline{b})}$ that satisfies (A). Then,

$$H^i_{\mathcal{M}^{(\underline{a})}}(I^{j(\underline{a},\underline{b})}) = 0$$

for $i > 0$ and $j \in \mathbb{N}$. So, we can use the resolution to compute the local cohomology of $M^{(\underline{a},\underline{b})}$.

Finally we prove that for any $\mathbb{Z}^r$-graded S-module N, it holds

$$\Gamma_{\mathcal{M}^{(\underline{a})}}(N^{(\underline{a},\underline{b})}) = (\Gamma_{\mathcal{M}}(N))^{(\underline{a},\underline{b})}.$$

In particular, for $N = I^j$ the claim is proved. We prove both inclusions: if $x \in (\Gamma_{\mathcal{M}}(N))^{(\underline{a},\underline{b})}$, then $x \in N^{(\underline{a},\underline{b})}$ and $\mathcal{M}^t x = 0$ for some $t \geq 0$. In particular $\mathcal{M}^{(\underline{a})t}x = 0$, because $\mathcal{M}^{(\underline{a})} \subset \mathcal{M}$, and hence, $(\Gamma_{\mathcal{M}}(N))^{(\underline{a},\underline{b})} \subseteq \Gamma_{\mathcal{M}^{(\underline{a})}}(N^{(\underline{a},\underline{b})})$.

Now, if $x \in \Gamma_{\mathcal{M}^{(\underline{a})}}(N^{(\underline{a},\underline{b})})$, then $x \in N^{(\underline{a},\underline{b})}$ and $\mathcal{M}^{(\underline{a})t}x = 0$ for some $t \geq 0$. So, we have to prove that there is an $s \geq 0$ such that $\mathcal{M}^s x = 0$.

The generators of $\mathcal{M}$ are $f_1, \ldots, f_{\mu_0} \in \mathfrak{m}$ of degree $\underline{0}$ and $g_i^1, \ldots, g_i^{\mu_i}$ of degree γ_i for $i = 1, \ldots, r$. We put $\mu = \mu_0 + \mu_1 + \cdots + \mu_r$. Then, $(g_i^{j(i)})^{a_i} \in \mathcal{M}^{(\underline{a})}$ for each $i = 1, \ldots, r$ and $j(i) = 1, \ldots, \mu_i$ and $f_k^{a_i} \in \mathfrak{m} = (\mathcal{M}^{(\underline{a})})_{\underline{0}}$ for all $k = 1, \ldots, \mu_0$ and $i = 1, \ldots, r$. We denote by $g_i^{m_i} := (g_i^1)^{m_i^1} \cdots (g_i^{\mu_i})^{m_i^{\mu_i}}$ and $f^{m_0} := f_1^{m_0^1} \cdots f_{\mu_0}^{m_0^{\mu_0}}$, with $m_i = m_i^1 + \cdots + m_i^{\mu_i}$ for each $i = 0, \ldots, r$.

Let us consider an $s \geq a_1 \cdots a_r \mu t$. If $z = f^{m_0} g_1^{m_1} \cdots g_r^{m_r} \in \mathcal{M}^s$ with $m_0 + m_1 + \cdots + m_r = s$, then there exist $i = 0, \ldots, r$ and $j = 1, \ldots, \mu_i$ such that $m_i^j \geq a_1 \cdots a_r t$. If $i = 0$, since $m_0^j \geq a_k t$ for all k, we put $z = (f_j^{a_k})^t y$, with $y = (f_1^{m_0^1} \cdots f_j^{m_0^j - a_k t} \cdots f_{\mu_0}^{m_0^{\mu_0}}) g_1^{m_1} \cdots g_r^{m_r}$, and hence, $zx = y (f_j^{a_k})^t x = 0$ by hypothesis. If $i \geq 1$, since $m_i^j \geq a_i t$, we put $z = ((g_i^j)^{a_i})^t y$, with

$$y = f^{m_0} g_1^{m_1} \cdots g_{i-1}^{m_{i-1}} ((g_i^1)^{m_i^1} \cdots (g_i^j)^{m_i^j - a_i t} \cdots (g_i^{\mu_i})^{m_i^{\mu_i}}) g_{i+1}^{m_{i+1}} \cdots g_r^{m_r},$$

and hence, $zx = y((g_i^j)^{a_i})^t x = 0$ by hypothesis. Then $\mathcal{M}^s x = 0$, and thus $\Gamma_{\mathcal{M}^{(\underline{a})}}(N^{(\underline{a},\underline{b})}) \subseteq (\Gamma_{\mathcal{M}}(N))^{(\underline{a},\underline{b})}$. □

4.2 Generalized depth

In this section, we study, among other properties, the generalized depth of a multigraded module and its Veronese modules.

Recall that $\mathrm{Proj}^r(S)$ is the set of all relevant homogeneous prime ideals on S, i.e. the set of all homogeneous prime ideals $\mathfrak{p}$ of S such that $\mathfrak{p} \not\supset S_{++}$. Note that $\mathfrak{p} \not\supset S_{++}$ if and only if for each $1 \leq i \leq r$ there exists $1 \leq j(i) \leq \mu_i$ such that $g_i^{j(i)} \notin \mathfrak{p}$. Given a homogeneous ideal $\mathfrak{p} \subset S$ we denote by U the multiplicative closed subset of S of homogeneous elements of $S \setminus \mathfrak{p}$; we denote by $S_{(\mathfrak{p})}$ the set of fractions $m/s \in U^{-1}S$ such that $\deg(m) = \deg(s) \in \mathbb{N}^r$; $S_{(\mathfrak{p})}$ is a local ring with maximal ideal $\mathfrak{p}\, U^{-1}S \cap S_{(\mathfrak{p})}$.

In the next three propositions we prove several results relating properties of non-standard $\mathbb{Z}^r$-graded rings and modules with their Veronese transforms.

Proposition 4.2.1. *For all* $\mathfrak{p} \in \operatorname{Proj}^r(S)$ *the ring extension*

$$S_{(\mathfrak{p})} \longrightarrow S_{\mathfrak{p}}$$

is faithfully flat with closed fiber $\mathbf{k}(\mathfrak{p})$.

Proof. Since $\mathfrak{p} \not\supset S_{++}$, for each $i \in \{1, \cdots, r\}$ there exists a generator such that $g_i^{j(i)} \notin \mathfrak{p}$, $1 \leq j(i) \leq \mu_i$. In particular $g_i^{j(i)} \in U$ for all $i = 1, \ldots, r$.

Let us consider the ring map

$$\varphi : S_{(\mathfrak{p})}[T_1, T_1^{-1}, \cdots, T_r, T_r^{-1}] \longrightarrow U^{-1}S$$

defined by $\varphi(T_i) = g_i^{j(i)}$ and $\varphi(T_i^{-1}) = (g_i^{j(i)})^{-1}$, $i = 1, \ldots, r$. We will prove that φ is a ring isomorphism.

Let m/s be a fraction of $U^{-1}S$; let $D = \sum_{i=1}^r D_i\gamma_i$, $D_i \in \mathbb{N}$, be the degree of m, and let $d = \sum_{i=1}^r d_i\gamma_i$, $d_i \in \mathbb{N}$, be the degree of s. We define

$$t = \prod_{i=1}^r (g_i^{j(i)})^{d_i - D_i}.$$

Hence, let us consider the identity

$$\frac{m}{s} = \left(\frac{m}{s}t\right) t^{-1}.$$

Note that $\frac{m}{s}t \in S_{(\mathfrak{p})}$ and that $t^{-1} = \varphi(\prod_{i=1}^r T_i^{D_i - d_i})$, so φ is an epimorphism.

Let $z = \sum_{\underline{n} \in \mathbb{Z}^r} c_{\underline{n}} T^{\underline{n}}$ be an element of the ring $S_{(\mathfrak{p})}[T_1, T_1^{-1}, \ldots, T_r, T_r^{-1}]$ such that $\varphi(z) = \sum_{\underline{n} \in \mathbb{Z}^r} c_{\underline{n}} \prod_{i=1}^r (g_i^{j(i)})^{n_i} = 0$, $\underline{n} = (n_1, \ldots, n_r)$. Since the element $c_{\underline{n}} \in S_{(\mathfrak{p})}$, we can write $c_{\underline{n}} = a_{\underline{n}}/b_{\underline{n}}$ with $\deg(a_{\underline{n}}) = \deg(b_{\underline{n}})$, $a_{\underline{n}} \in S$ and $b_{\underline{n}} \notin \mathfrak{p}$. We write

$$\prod_{i=1}^r (g_i^{j(i)})^{n_i} = \frac{(g_{i_1}^{j(i_1)})^{n_{i_1}} \ldots (g_{i_s}^{j(i_s)})^{n_{i_s}}}{(g_{j_1}^{j(j_1)})^{-n_{j_1}} \ldots (g_{j_t}^{j(j_t)})^{-n_{j_t}}}$$

with $(g_{i_1}^{j(i_1)})^{n_{i_1}} \ldots (g_{i_s}^{j(i_s)})^{n_{i_s}} \in S$ and $(g_{j_1}^{j(j_1)})^{-n_{j_1}} \ldots (g_{j_t}^{j(j_t)})^{-n_{j_t}} \in S \setminus \mathfrak{p}$, i.e. $n_{i_1}, \ldots, n_{i_s} \geq 0$ and $n_{j_1}, \ldots, n_{j_t} < 0$.

Now,

$$\varphi(z) = \sum_{\underline{n} \in \mathbb{Z}^r} \frac{a_{\underline{n}} (g_{i_1}^{j(i_1)})^{n_{i_1}} \ldots (g_{i_s}^{j(i_s)})^{n_{i_s}}}{b_{\underline{n}} (g_{j_1}^{j(j_1)})^{-n_{j_1}} \ldots (g_{j_t}^{j(j_t)})^{-n_{j_t}}} = 0$$

and by reducing to a common denominator we get

$$\varphi(z) = \sum_{\underline{n}} \frac{d_{\underline{n}}}{b} (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}} = 0$$

Now, $\deg(b) = \deg(d_{\underline{n}}) + \sum_{k=1}^{t} -n_{j_k}\gamma_{j_k}$.

Hence there exists $\delta \in U$ such that,

$$\sum_{\underline{n} \in \mathbb{Z}^r} \delta d_{\underline{n}} (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}} = 0.$$

We have that if $A_{\underline{n}} = \delta d_{\underline{n}} (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}} \neq 0$, then

$$\deg(A_{\underline{n}}) = \deg(\delta) + \deg(d_{\underline{n}}) + \sum_{k=1}^{s} n_{i_k}\gamma_{i_k} = \deg(\delta) + \deg(b) + \sum_{i=1}^{r} n_i\gamma_i.$$

Since the $r \times r$ matrix $(\gamma_1, \cdots, \gamma_r)$ is upper triangular and non-singular, the degrees $\deg(A_{\underline{n}})$ are different when $\underline{n}$ ranges over $\mathbb{Z}^r$. Hence we get $A_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$.

Let us consider the following identities in $S_{(\mathfrak{p})}$

$$c_{\underline{n}} = \frac{a_{\underline{n}}}{b_{\underline{n}}} = \frac{d_{\underline{n}} (g_{j_1}^{j(j_1)})^{-n_{j_1}} \dots (g_{j_t}^{j(j_t)})^{-n_{j_t}}}{b} = \frac{A_{\underline{n}} (g_{j_1}^{j(j_1)})^{-n_{j_1}} \dots (g_{j_t}^{j(j_t)})^{-n_{j_t}}}{b\delta (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}}} = 0,$$

so $z = 0$, and hence φ is a monomorphism.

Let us consider the multiplicative closed subset $W = U^{-1}S \setminus \mathfrak{p}U^{-1}S$. Then $S_{\mathfrak{p}} = W^{-1}[U^{-1}S]$, and furthermore

$$S_{\mathfrak{p}} = W^{-1}(S_{(\mathfrak{p})}[T_1, T_1^{-1}, \dots, T_r, T_r^{-1}]).$$

From this identity we deduce that the ring extension $S_{(\mathfrak{p})} \longrightarrow S_{\mathfrak{p}}$ is faithfully flat. A simple computation shows that the closed fiber of $S_{(\mathfrak{p})} \longrightarrow S_{\mathfrak{p}}$ is $\mathbf{k}(\mathfrak{p})$. □

Proposition 4.2.2. *The extension* $S^{(\underline{a})} \hookrightarrow S$ *is integral,* $\dim(S^{(\underline{a})}) = \dim(S)$ *and there is a homeomorphism of topological spaces*

$$\operatorname{Proj}^r(S^{(\underline{a})}) \cong \operatorname{Proj}^r(S),$$

for all $\underline{a} \in \mathbb{N}^{*r}$. *Moreover, for all* $\mathfrak{p} \in \operatorname{Proj}^r(S)$ *it holds* $\operatorname{ht}(\mathfrak{p}^{(\underline{a})}) = \operatorname{ht}(\mathfrak{p})$.

Proof. First we prove that the ring extension

$$S^{(\underline{a})} \hookrightarrow S$$

is integral. Let $x \in S$ be an element of degree $\underline{n} \in \mathbb{Z}^r$. We write $\underline{n} = \sum_{i=1}^r b^i \gamma_i$, $a = a_1 \cdots a_r$, and $\underline{r} = (\frac{ab^i}{a_i}; i = 1, \cdots, r)$. Then it is easy to se that $a\underline{n} = \phi_{\underline{a}}(\underline{r})$, so $x^a \in S_{\phi_{\underline{a}}(\underline{r})} = (S^{(\underline{a})})_{\underline{r}}$. Hence x is a zero of $f(T) = T^a - x^a \in S^{(\underline{a})}[T]$. Therefore S is integral over $S^{(\underline{a})}$ and then $\dim(S^{(\underline{a})}) = \dim(S)$.

Notice that $\mathfrak{p} \not\supset S_{++}$ if and only if $\mathfrak{p}^{(\underline{a})} = \mathfrak{p} \cap S^{(\underline{a})} \not\supset S_{++}^{(\underline{a})}$, so we can define a continuous map

$$\begin{array}{rccc} \psi : & \mathrm{Proj}^r(S) & \longrightarrow & \mathrm{Proj}^r(S^{(\underline{a})}) \\ & \mathfrak{p} & \mapsto & \mathfrak{p}^{(\underline{a})} \end{array}$$

this map is surjective and closed since the the extension $S^{(\underline{a})} \hookrightarrow S$ is integral.

The map ψ is injective: let $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Proj}^r(S)$ such that $\mathfrak{p}_1^{(\underline{a})} = \mathfrak{p}_2^{(\underline{a})}$. Given $x \in \mathfrak{p}_1$, by the argument previously done to prove that the extension is integral, we have

$$x^a \in \mathfrak{p}_1 \cap S^{(\underline{a})} = \mathfrak{p}_1^{(\underline{a})} = \mathfrak{p}_2^{(\underline{a})} \subset \mathfrak{p}_2,$$

so $x \in \mathfrak{p}_2$, i.e. $\mathfrak{p}_1 \subset \mathfrak{p}_2$. By the symmetry of the problem we have $\mathfrak{p}_1 = \mathfrak{p}_2$. Hence ψ is an homeomorphism of topological spaces.

The identity $\mathrm{ht}(\mathfrak{p}^{(\underline{a})}) = \mathrm{ht}(\mathfrak{p})$ follows from the above homeomorphism. □

Proposition 4.2.3. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. For all $\mathfrak{p} \in \mathrm{Proj}^r(S)$ and $\underline{b} \in \mathbb{N}^r$, it holds*

$$M_{(\mathfrak{p}^{(\underline{a})})}^{(\underline{a},\underline{b})} = M(\underline{b})_{(\mathfrak{p})}.$$

Proof. Notice that we always have

$$M_{(\mathfrak{p}^{(\underline{a})})}^{(\underline{a},\underline{b})} \subset M(\underline{b})_{(\mathfrak{p})}.$$

In fact, let m/s be an element of $M_{(\mathfrak{p}^{(\underline{a})})}^{(\underline{a},\underline{b})}$, it means that $m \in (M^{(\underline{a},\underline{b})})_{\underline{n}} = M_{\phi_{\underline{a}}(\underline{n})+\underline{b}}$ and $s \in (S^{(\underline{a})})_{\underline{n}} = S_{\phi_{\underline{a}}(\underline{n})}$ but $s \notin \mathfrak{p}^{(\underline{a})}$, $\underline{n} \in \mathbb{Z}^r$. Since $s \in S^{(\underline{a})} \setminus \mathfrak{p}^{(\underline{a})}$ we have that $s \notin \mathfrak{p}$, so $m/s \in M(\underline{b})_{(\mathfrak{p})}$.

Let us consider $m/s \in M(\underline{b})_{(\mathfrak{p})}$ a fraction such that $\deg(m) - \underline{b} = \deg(s) = \underline{n} \in \mathbb{N}^r$ and $s \notin \mathfrak{p}$. Since s is a homogeneous element of degree $\underline{n}$, we can decompose s in a sum of monomials on the generators g_i^j of S: $s = s_1 + \cdots + s_t$, with $\deg(s_i) = \underline{n}$ for all $i = 1, \dots, t$. Since $s \in S \setminus \mathfrak{p}$, there exist $k \in \{1, \dots t\}$ such that $s_k \notin \mathfrak{p}$. If we write

$$s_k = \prod_{i=1}^{r} \prod_{j=1}^{\mu_i} (g_i^j)^{d_i^j},$$

$d_i^j \in \mathbb{N}$, so

$$\deg(s_k) = \underline{n} = \sum_{i=1}^{r} \left(\sum_{j=1}^{\mu_i} d_i^j \right) \gamma_i.$$

Since $s_k \notin p$, for each coefficient $\sigma_i = \sum_{j=1}^{\mu_i} d_i^j \neq 0$ there exist a generator $g_i^{j(i)} \notin \mathfrak{p}$. Let σ_{i_l}, $l \in \{1, \dots, e\}$, be such a non-zero coefficients. For each $l = 1, \dots, e$, let $c_{i_l} \in \mathbb{N} \setminus \{0\}$ and $f_{i_l} \in \mathbb{N} \setminus \{0\}$ be non-negative integers such that

$$\sigma_{i_l} + c_{i_l} = f_{i_l} a_{i_l}.$$

We put $c_i = f_i = 0$ for all $i \notin \{i_1, \cdots, i_e\}$. We define

$$z = \prod_{l=1}^{e} (g_{i_l}^{j(i_l)})^{c_{i_l}} \notin \mathfrak{p}.$$

Since $s \notin \mathfrak{p}$ is homogeneous, $zs \notin \mathfrak{p}$ is still homogeneous and then

$$\deg(zm) - \underline{b} = \deg(zs) = \deg(zs_k) = \sum_{i=1}^{r} f_i a_i \gamma_i = \phi_{\underline{a}}((f_1, \dots, f_r)),$$

so $m/s = (zm)/(zs) \in M^{(\underline{a},\underline{b})}_{(\mathfrak{p}^{(\underline{a})})}$. □

Given an ideal $\mathfrak{p} \in \operatorname{Spec}(S)$ we denote by $\mathfrak{p}^*$ the prime ideal generated by the homogeneous elements belonging to $\mathfrak{p}$, (see [GW78] section 2). We can relate the depths of the localization on a prime $\mathfrak{p}$ with the localization on $\mathfrak{p}^*$.

Proposition 4.2.4. *Assume that S is a catenary ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Given an ideal $\mathfrak{p} \in \operatorname{Spec}(S)$ such that $\mathfrak{p} \not\supset S_{++}$ and $M_{\mathfrak{p}} \neq 0$, then it holds*

$$\operatorname{depth}(M_{\mathfrak{p}}) + \dim(S/\mathfrak{p}) = \operatorname{depth}(M_{(\mathfrak{p}^*)}) + \dim(S/\mathfrak{p}^*).$$

Proof. We put $d = \dim(S_{\mathfrak{p}}/\mathfrak{p}^* S_{\mathfrak{p}})$. From [GW78], Proposition 1.2.2 and Corollary 1.2.4, we have that $\operatorname{depth}(M_{\mathfrak{p}}) = \operatorname{depth}(M_{\mathfrak{p}^*}) + d$ and also that $\dim(M_{\mathfrak{p}}) = \dim(M_{\mathfrak{p}^*}) + d$. On the other hand, since S is catenary we have $\dim(S_{\mathfrak{p}}) = \dim(S) - \dim(S/\mathfrak{p})$ and $\dim(S_{\mathfrak{p}^*}) = \dim(S) - \dim(S/\mathfrak{p}^*)$. From these identities we get

$$\begin{aligned}\operatorname{depth}(M_{\mathfrak{p}}) + \dim(S/\mathfrak{p}) &= \operatorname{depth}(M_{\mathfrak{p}^*}) + d + \dim(S) - \dim(S_{\mathfrak{p}}) \\ &= \operatorname{depth}(M_{\mathfrak{p}^*}) + d + \dim(S) - \dim(S_{\mathfrak{p}^*}) - d \\ &= \operatorname{depth}(M_{\mathfrak{p}^*}) + \dim(S/\mathfrak{p}^*).\end{aligned}$$

Since the morphism $S_{(\mathfrak{p})} \longrightarrow S_{\mathfrak{p}}$ is faithfully flat with closed fiber $\mathbf{k}(\mathfrak{p})$ we get, by [Mat89] Theorem 23.3, that $\operatorname{depth}(M_{\mathfrak{p}^*}) = \operatorname{depth}(M_{(\mathfrak{p}^*)})$. Hence the claim is proved. □

We next define the generalized depth of a module and its projective Cohen-Macaulay deviation in the multigraded setting. In the graded case they are defined in [HM94] and in [Eli04], respectively.

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module.

Definition 4.2.5. *The* generalized depth *of M with respect to the homogeneous maximal ideal $\mathcal{M}$ of S, denoted by* $\operatorname{gdepth}(M)$, *is defined as the greatest integer $k \geq 0$ such that*

$$S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M)))$$

for all $i < k$.

It is clear from the definition that $\operatorname{gdepth}(M) \geq \operatorname{depth}(M)$.

Definition 4.2.6. *The* projective Cohen-Macaulay deviation *of M, denoted by* $\operatorname{pcmd}(M)$, *is defined as the maximum of the differences*

$$\dim(S_{(\mathfrak{p})}) - \operatorname{depth}(M_{(\mathfrak{p})})$$

where $\mathfrak{p} \in \operatorname{Proj}^r(S)$.

In the case when $S_{\underline{0}}$ is a quotient of a regular ring, we can relate these last two integers. This relation is crucial in order to prove that the generalized depth of a module coincides with the generalized depth of its Veronese transform. Next theorem generalizes Proposition 2.2. in [HM99].

Theorem 4.2.7. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $S_{\underline{0}}$ is the quotient of a regular ring then*

$$\operatorname{gdepth}(M) = \dim(S) - \operatorname{pcmd}(M).$$

Proof. From [Fal78], Satz 1, (see also [Mar95]) we get

$$\operatorname{gdepth}(M) = \min_{\mathfrak{p}\in\Sigma}\{\operatorname{depth}(M_{\mathfrak{p}}) + \dim(S/\mathfrak{p})\}$$

with $\Sigma = \{\mathfrak{a} \mid \mathfrak{a} \in \operatorname{Spec}(S), \mathfrak{a} \not\supset S_{++}\}$. From Proposition 4.2.4, we have that

$$\operatorname{depth}(M_{\mathfrak{p}}) + \dim(S/\mathfrak{p}) = \operatorname{depth}(M_{(\mathfrak{p}^*)}) + \dim(S/\mathfrak{p}^*),$$

so we can assume that $\mathfrak{p} \in \operatorname{Proj}^r(S)$. Therefore we get

$$\operatorname{gdepth}(M) = \min_{\mathfrak{p}\in\operatorname{Proj}^r(S)}\{\operatorname{depth}(M_{(\mathfrak{p})}) + \dim(S/\mathfrak{p})\}.$$

Since S is catenary $\dim(S/\mathfrak{p}) = \dim(S) - \dim(S_{(\mathfrak{p})})$, and hence

$$\begin{aligned}\operatorname{gdepth}(M) &= \dim(S) - \max_{\mathfrak{p}\in\operatorname{Proj}^r(S)}\{\dim(S_{(\mathfrak{p})}) - \operatorname{depth}(M_{(\mathfrak{p})})\}\\ &= \dim(S) - \operatorname{pcmd}(M).\end{aligned}$$

□

Now, based on this assumption, we can prove the invariance of gdepth under Veronese transforms:

Corollary 4.2.8. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $S_{\underline{0}}$ is the quotient of a regular ring, then it holds*

$$\operatorname{gdepth}(M^{(\underline{a},\underline{b})}) = \operatorname{gdepth}(M)$$

for all $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$.

Proof. From Theorem 4.2.7 we get

$$\operatorname{gdepth}(M^{(\underline{a},\underline{b})}) = \dim(S^{(\underline{a})}) - \operatorname{pcmd}(M^{(\underline{a},\underline{b})}),$$

and from Proposition 4.2.2 , $\dim(S^{(\underline{a})}) = \dim(S)$. Now, from Proposition 4.2.3 we deduce

$$\dim(S^{(\underline{a})}) - \operatorname{pcmd}(M^{(\underline{a},\underline{b})}) = \dim(S) - \operatorname{pcmd}(M(\underline{b})).$$

Again from Theorem 4.2.7 $\text{gdepth}(M(\underline{b})) = \dim(S) - \text{pcmd}(M(\underline{b}))$. Using the definition of gdepth we have that $\text{gdepth}(M(\underline{b})) = \text{gdepth}(M)$, and so we see that

$$\text{gdepth}(M^{(\underline{a},\underline{b})}) = \text{gdepth}(M)$$

as claimed. □

4.3 Vanishing theorems and depth of Veronese modules

In this section we generalize the concept of $\text{fg}(M)$ to the multigraded case. In the graded case this notion allows to us to control the finite graduation of the local cohomology modules of a graded module M with respect to the maximal homogeneous ideal of S. We prove some results on the vanishing of a module and of its local cohomology modules and we relate this with the generalized depth. To reach our goal, we need to fit the generalization of $\text{fg}(M)$, that we call Γ-$\text{fg}(M)$, to the multigraduation. We also study the asymptotic depth of Veronese modules. We are able to prove that, by restricting our graduation, this depth is constant for $(\underline{a},\underline{b})$-Veronese modules, for $\underline{a},\underline{b}$ in suitable asymptotic regions of $\mathbb{N}^r$ by using the previous work done in the chapter.

4.3.1 Asymptotic depth of Veronese modules (I)

We want to study the depth of the Veronese modules $M^{(\underline{a},\underline{b})}$ for large values $\underline{a},\underline{b} \in \mathbb{N}^r$. As a partial solution of our purposes, under the general hypothesis on the multidegrees of this chapter, we prove in this section that the depth of some Veronese modules $M^{(\underline{a})}$ is constant for $\underline{a}$ in a net of $\mathbb{N}^r$.

We denote by $vad(M^{(*)})$ (resp. $vad(M^{(*,*)})$) the Veronese asymptotic depth of M, that means the maximum of $\text{depth}(M^{(\underline{a})})$ (resp. $\text{depth}(M^{(\underline{a},\underline{b})})$) for all $\underline{a} \in \mathbb{N}^{*r}$ (resp. for all $\underline{a},\underline{b} \in \mathbb{N}^{*r}$).

Proposition 4.3.1. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module, and let $s = vad(M^{(*)})$. There exists $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{N}^{*r}$ such that for all elements $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$,*

$$\operatorname{depth}(M^{(\underline{b})}) = s,$$

i.e. is constant.

Proof. Let $s = vad(M^{(*)})$. This means that there exists an $\underline{a} \in \mathbb{N}^{*r}$ such that

$$H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a})}) = 0$$

for $i = 0, \ldots, s-1$.

Let us consider $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\} = \{\underline{\lambda}\cdot\underline{a} \mid \underline{\lambda} \in \mathbb{N}^{*r}\}$. Then for all $\underline{n} \in \mathbb{Z}^r$, since $\phi_{\underline{b}}(\underline{n}) = \phi(\underline{b}\cdot\underline{n}) = \phi(\underline{a}\cdot\underline{\lambda}\cdot\underline{n}) = \phi_{\underline{a}}(\underline{\lambda}\cdot\underline{n})$, we have that

$$H^i_{\mathcal{M}^{(\underline{b})}}(M^{(\underline{b})})_{\underline{n}} = H^i_{\mathcal{M}}(M)_{\phi_{\underline{b}}(\underline{n})} = H^i_{\mathcal{M}}(M)_{\phi_{\underline{a}}(\underline{\lambda}\cdot\underline{n})} = H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a})})_{\underline{\lambda}\cdot\underline{n}} = 0$$

for $i = 0, \ldots, s-1$. From this, we deduce that $\operatorname{depth}(M^{(\underline{b})}) \geq s$, but s was the maximum. Therefore,

$$\operatorname{depth}(M^{(\underline{b})}) = s$$

for all $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$. □

Let us consider the multigraded Rees algebra associated to some ideals $I_1, \ldots, I_r$ in a Noetherian local ring $(R, \mathfrak{m})$,

$$\mathcal{R}(I_1, \ldots, I_r) = \bigoplus_{\underline{n}\in\mathbb{N}^r} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset R[t_1, \ldots, t_r].$$

By considering this ring, the previous result can be used in order to study of the depth of the multigraded Rees algebras of some powers of ideals.

Proposition 4.3.2. *Let $I_1, \ldots, I_r$ be ideals in a Noetherian local ring $(R, \mathfrak{m})$. Let $s = vad(\mathcal{R}(I_1, \ldots, I_r)^{(*)})$. There exists $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{N}^{*r}$ such that for all $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$*

$$\operatorname{depth}(\mathcal{R}(I_1^{b_1}, \ldots, I_r^{b_r})) = s.$$

Moreover, if $\operatorname{depth}(\mathcal{R}(I_1, \dots, I_r)) = s$, *then*

$$\operatorname{depth}(\mathcal{R}(I_1^{b_1}, \dots, I_r^{b_r})) = s,$$

i.e is constant, for all $\underline{b} \in \mathbb{N}^{*r}$.

Proof. Observe that the multigraded Rees algebra has a standard graduation and hence, for $\underline{a} = (a_1, \dots, a_r)$,

$$\mathcal{R}(I_1^{a_1}, \dots, I_r^{a_r}) = \mathcal{R}(I_1, \dots, I_r)^{(\underline{a})}$$

and then the claim is a consequence of the previous proposition. The second statement follows from the first one by considering $\underline{a} = (1, \dots, 1)$. □

4.3.2 Γ-finite graduation

We would like to extend the previous results on the asymptotic depth of the Veronese modules to regions of $\mathbb{N}^r$ instead of some nets there. First we have to study the vanishing of the local cohomology modules of a multigraded module M.

We recall that a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ with vertex at $\underline{\beta} \in \mathbb{N}^r$ with respect to $\gamma_1, \dots, \gamma_r$ is a region of $\mathbb{N}^r$ whose points are of the form $\underline{\beta} + \sum_{i=1}^r \lambda_i \gamma_i \in \mathbb{N}^r$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ for $i = 1, \dots, r$. Given $\underline{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$ we denote $\underline{n}^* = (|n_1|, \dots, |n_r|) \in \mathbb{N}^r$.

In [Mar95], a notion of finite graduation is defined for $\mathbb{Z}$-graded modules. This means that the graded pieces are zero except for a finite number of them. Then to control the finite graduation of the local cohomology modules is defined the invariant fg that we extend, adapted to our situation, as follows.

Definition 4.3.3. *We say that a* $\mathbb{Z}^r$*-graded* S*-module* M *is* Γ-finitely graded *if there exists a cone* $C_{\underline{\beta}} \subset \mathbb{N}^r$ *where* $M_{\underline{n}} = 0$ *for all* $\underline{n} \in \mathbb{Z}^r$ *such that* $\underline{n}^* \in C_{\underline{\beta}}$. *We denote by* $\Gamma\text{-fg}(M)$ *the greatest integer* $k \geq 0$ *such that* $H^i_{\mathcal{M}}(M)$ *is* Γ*-finitely graded for all* $i < k$.

Remark 4.3.4. Note that in the standard graded case, i.e. $r = 1$, the definition of Γ-fg(M) coincides with the classical

$$\mathrm{fg}(M) = \max\{k \geq 0 \mid H^i_{\mathcal{M}}(M) \text{ is finitely graded for all } i < k\}.$$

In this case a module is finitely graded if the pieces of degree n are 0 for $|n| \geq n_0$, for some $n_0 \in \mathbb{N}$, which is, in fact, a cone with vertex in n_0, so

$$\mathrm{fg}(M) = \Gamma\text{-fg}(M).$$

In the multigraded case, we cannot generalize the definition of being M finitely graded to have only a finite number of $\underline{n} \in \mathbb{Z}^r$ with $M_{\underline{n}} \neq 0$. The aim is to prove that gdepth$(M) = \Gamma$-fg(M) as in the graded case, but even if we could assure that $H^k_{\mathcal{M}}(M)$ would be finitely generated, the hypothesis $S_{++} \subset rad(Ann_S(H^k_{\mathcal{M}}(M)))$ could only assure $H^k_{\mathcal{M}}(M)_{\underline{n}} = 0$ for $\underline{n}^* \in C_{\underline{\beta}}$ for some $\underline{\beta} \in \mathbb{N}^r$ as we have seen in Proposition 2.3.8.

Due to technical reasons, in the following we have to restrict the degrees of the generators of S, see Remark 4.3.6. From now on we assume that the graduation is almost-standard. By almost-standard multigraded (or $\mathbb{Z}^r$-graded) ring S we mean a multigraded ring with generators over $S_{\underline{0}}$ of multidegrees

$$\begin{aligned}
&\gamma_1 = (\gamma_1^1, 0, \ldots, 0) = \gamma_1^1 e_1 \\
&\ldots \\
&\gamma_i = (0, \ldots, 0, \gamma_i^i, 0, \ldots, 0) = \gamma_i^i e_i \\
&\ldots \\
&\gamma_r = (0, \ldots, 0, \gamma_r^r) = \gamma_r^r e_r
\end{aligned}$$

with $\gamma_1^1, \ldots, \gamma_r^r > 0$ and $e_1, \ldots, e_r$ the canonical basis of $\mathbb{R}^r$. Note that in this case we have

$$C_{\underline{\beta}} = (\underline{\beta} + (\mathbb{R}_{\geq 0})^r) \cap \mathbb{N}^r$$

for all $\underline{\beta} \in \mathbb{Z}^r$. Note that the intersection of two cones is a cone:

$$C_{\underline{\alpha}} \cap C_{\underline{\beta}} = C_{\underline{\delta}}$$

with $\delta = (\max\{\alpha_i, \beta_i\}; i = 1, \cdots, r)$.

An important point in the proof of the main Theorem in this section is to assure that $H^k_{\mathcal{M}}(M)$ is Γ-finitely graded for all $k \geq 0$ in case that the module M is Γ-finitely graded as well. For that reason we have to restrict the graduation to the almost-standard case. We prove that in the next proposition.

Proposition 4.3.5. *Let S be an almost-standard multigraded ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If M is Γ-finitely graded then $H^k_{\mathcal{M}}(M)$ is also Γ-finitely graded for all $k \geq 0$.*

Proof. Since M is Γ-finitely graded, there exists an element $\underline{\beta} \in \mathbb{N}^r$ such that $M_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$. We want to prove that $H^k_{\mathcal{M}}(M)_{\underline{n}} = 0$ for $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$ as well.

Since $H^0_{\mathcal{M}}(M) = \Gamma_{\mathcal{M}}(M) \subseteq M$, then the claim is obviously true for $k = 0$. Let us assume that $k > 0$.

The ideal $\mathcal{M}$ is generated by a system of generators of $\mathfrak{m}$, say $h_1, \cdots, h_v$, and by g_i^j, $j = 1, \cdots, \mu_i$, $i = 1, \cdots, r$. If we denote by $f_1, \cdots, f_\sigma$ the above system of generators of $\mathcal{M}$ then the local cohomology modules $H^*_{\mathcal{M}}(M)$ are the cohomology modules of the Koszul complex

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^{\sigma} M_{f_i} \longrightarrow \bigoplus_{1 \leq i < j \leq \sigma} M_{f_i f_j} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_\sigma} \longrightarrow 0.$$

The module $H^k_{\mathcal{M}}(M)$ is S-graded: the grading is induced by the grading defined on the localizations M_g, where g is an arbitrary product of k different generators of $\mathcal{M}$. Given $z = x/g^t \in M_g$ we have

$$\deg(z) = \deg\left(\frac{x}{g^t}\right) = \deg(x) - t\ \deg(g).$$

If we assume that $\deg(z) = \underline{n}$ with $\underline{n}^* \in C_{\underline{\beta}}$ then there exists a vector $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_r) \in \{-1, +1\}^r$ such that $\underline{\varepsilon}.\underline{n} = \underline{\beta} + G\underline{\lambda}$ with $\lambda_i \in \mathbb{R}_{\geq 0}$. We denote $\underline{\varepsilon}.\underline{n}$ for the termwise product of $\underline{\varepsilon}$ and $\underline{n}$. So,

$$\underline{n} = \underline{\varepsilon}.(\underline{\beta} + G\underline{\lambda}).$$

On the other hand we may assume, without loss of generality, that $\deg(g) = G\underline{k}$ with $\underline{k} = (k_1, \cdots, k_w, 0, \cdots, 0)$ with $k_i \neq 0$, $i = 1, \cdots, w$. Hence we have

$$\deg(xg^s) = \deg(z) + (t+s)\deg(g) = \underline{\varepsilon}.(\underline{\beta} + G\underline{\lambda}) + (t+s)G\underline{k}$$

for all $s \geq 0$.

We want to prove that $\deg(xg^s)^* \in C_{\underline{\beta}}$, for some $s \geq 0$, so we have to assure that there exists $\underline{\mu} \in (\mathbb{R}_{\geq 0})^r$ and $\underline{\eta} \in \{-1, +1\}^r$ such that

$$\underline{\eta}.[\underline{\varepsilon}.(\underline{\beta} + G\underline{\lambda}) + (t+s)G\underline{k}] = \underline{\beta} + G\underline{\mu}.$$

For $i = w+1 \cdots, r$ we have the equation

$$\eta_i\, \varepsilon_i(\beta_i + \lambda_i \gamma_i^i) = \beta_i + \mu_i \gamma_i^i,$$

we set $\eta_i = \varepsilon_i$ and $\mu_i = \lambda_i \geq 0$.
For $i = 1, \cdots, w$ we set $\eta_i = 1$, and then we have to consider the equation

$$\varepsilon_i(\beta_i + \lambda_i \gamma_i^i) + (t+s)k_i \gamma_i^i = \beta_i + \mu_i \gamma_i^i.$$

If $\varepsilon_i = 1$ then

$$\mu_i = \lambda_i + (t+s)k_i \geq 0.$$

If $\varepsilon_i = -1$ then

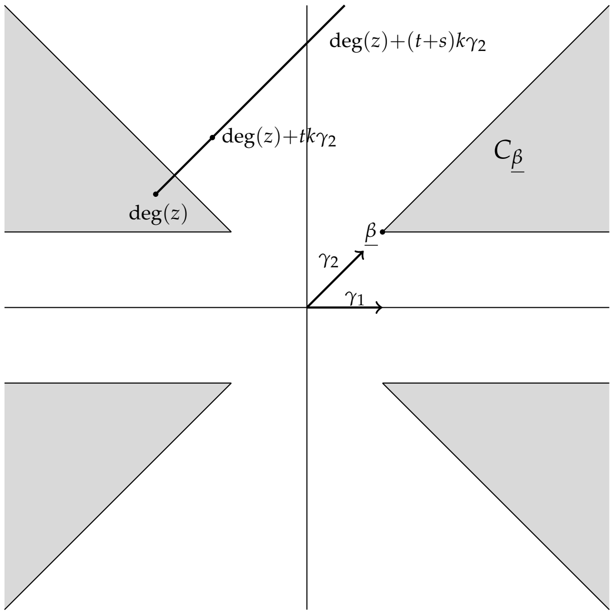
$$\mu_i = -2\,\frac{\beta_i}{\gamma_i^i} - \lambda_i + (t+s)k_i \geq 0$$

for an integer $s \gg 0$.

We have proved that $H^k_{\mathcal{M}}(M)_{\underline{n}} = 0$ for $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$, so $H^k_{\mathcal{M}}(M)$ is Γ-finitely graded. □

Remark 4.3.6. In the proof of the previous Proposition, we considered an element $z = x/g^t \in M_g$ with $\deg(z) = \underline{n}$ with $\underline{n}^* \in C_{\underline{\beta}}$. We wanted to find an integer $s \geq 0$ such that $\deg(xg^s)^* \in C_{\underline{\beta}}$. By hypothesis $xg^s = 0$ and hence we would get $z = 0$ as well. Since $z = xg^s/g^{t+s}$, we have $\deg(xg^s) = \deg(z) + (t+s)\sum_{i=1}^r k_i \gamma_i$, with $k_i \geq 0$ for all $i = 1, \ldots, r$.

Unfortunately, in the most general case that we are considering here, it is not possible to find such an $s \geq 0$ satisfying the desired equations. For instance, we can assume that $r = 2$ and that g is a product of k generators of degree γ_2. In this case $k_1 = 0$ and $k_2 = k$. If $\deg(z)^* \in C_{\underline{\beta}}$, with $\deg z$ in the second quadrant and $(\deg(z) + tk\gamma_2)^* \notin C_{\underline{\beta}}$, then for all $s \geq 0$, $\deg(xg^s)^* = (\deg(z) + (t+s)k\gamma_2)^* \notin C_{\underline{\beta}}$ since all these points are in a parallel line to the γ_2-axis of the cone $C_{\underline{\beta}}$, as it can be easily seen in the following picture:



Hence, we cannot prove that $H^k_{\mathcal{M}}(M)$ is Γ-finitely graded for all $k \geq 0$, without restrict the hypothesis on the generators.

In the next result we relate the two integers attached to M studied in this chapter, gdepth(M) and Γ-fg(M). The result follows [Mar95], Proposition 2.3. or [TI89], Lemma 2.2. Since these papers extensively use results on $\mathbb{Z}$-graded modules we will adapt them to the almost-standard multigraded case considered by us.

Theorem 4.3.7. *Let S be an almost-standard multigraded ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module, then it holds*

$$\Gamma\text{-fg}(M) = \text{gdepth}(M).$$

Proof. First we prove the inequality $\Gamma\text{-fg}(M) \leq \text{gdepth}(M)$. If $H^i_{\mathcal{M}}(M)$ is Γ-finitely graded then there exists a cone $C_{\underline{\beta}}$ with vertex in some $\underline{\beta} \in \mathbb{N}^r$, such that $H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$.

We have to prove that $S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M)))$, i.e. for all generators $x = g_1^{m_1} \cdots g_r^{m_r}$ of S_{++}, $m_i \in \{1, \cdots, \mu_i\}, i = 1, \cdots, r$, we have to find a suitable $a > 0$ such that for all $\underline{n} \in \mathbb{Z}^r$, $x^a H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$.

If $\underline{n}^* \in C_{\underline{\beta}}$ then $H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$, so for all $a \geq 0$ it holds $x^a H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$. We put $a = 2\max\{\beta_1, \cdots, \beta_r\}$. Let us assume that $\underline{n}^* \notin C_{\underline{\beta}}$. This means that, without loss of generality, $-\beta_i < n_i < \beta_i$, $i = 1, \ldots, u$, and $|n_i| \geq \beta_i$ for $i = u+1, \cdots, r$. If we decompose $x = z_1 z_2$ with $z_1 = g_1^{m_1} \cdots g_u^{m_u}$ and $z_2 = g_{u+1}^{m_{u+1}} \cdots g_r^{m_r}$, then

$$(\underline{n} + \deg(z_1^a))^* \in C_{\underline{\beta}},$$

so $z_1^a H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$. Furthermore

$$x^a H^i_{\mathcal{M}}(M)_{\underline{n}} = 0.$$

Notice that a does not depends on $\underline{n}$, so, in fact, we have proved that $S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M)))$, and hence

$$\Gamma\text{-fg}(M) \leq \text{gdepth}(M).$$

Now, we prove the other inequality, i.e. Γ-fg$(M) \geq$ gdepth(M). If $S_{++} \subset rad(Ann_S(M))$ then there exists $a \in \mathbb{N}$ such that for all $x \in S_{++}$, $x^a M = 0$. Since M is finitely generated, by Lemma 2.3.8 there exists a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ with vertex in some $\underline{\beta} \in \mathbb{N}^r$, such that $M_{\underline{n}} = 0$ for all $\underline{n}^* \in C_{\underline{\beta}}$. Then by Proposition 4.3.5, for all i $H^i_{\mathcal{M}}(M)$ is Γ-finitely graded, so Γ-fg$(M) = +\infty \geq$ gdepth(M).

We can assume that $S_{++} \not\subset rad(Ann_S(M))$. Let $Ass(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ be the set of the associated prime ideals of M. Let us consider a minimal primary decomposition of $0 \in M$

$$0 = N_1 \cap \cdots \cap N_s \cap N_{s+1} \cap \cdots \cap N_t$$

where $Ass(M/N_i) = \{\mathfrak{p}_i\}$. We can assume that $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ do not contain S_{++}, and $\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_t$ contain S_{++}.

Since the residue field of $S_{\underline{0}}$ is infinite there is an element $z \in S_{++}$ such that $z \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$. We will prove that $(0 :_M z)$ is a Γ-finitely graded S-module.

Since $z \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$, then $(0 :_M z) \subset N_1 \cap \cdots \cap N_s$. In fact, since N_i is a $\mathfrak{p}_i$-primary submodule of M and $z \notin \mathfrak{p}_i$, then $(N_i :_M z) = N_i$. This last equality is well known: let us assume that there exists $x \in (N_i :_M z) \setminus N_i$, so $zx \in N_i$. Since N_i is $\mathfrak{p}_i$-primary $z^n \in (N_i :_R M) \subset \mathfrak{p}_i$ for some $n \geq 0$, so $z \in \mathfrak{p}_i$, and we get a contradiction. Thus, $(0 :_M z) \subset (N_i :_M z) = N_i$ for all $i = 1, \ldots, s$.

On the other hand, $\mathfrak{p}_i = rad((N_i :_R M))$ for all $i = 1, \ldots, t$ by the definition of primary submodule. In particular, for $i = s+1, \ldots, t$, since $S_{++} \subset \mathfrak{p}_i$, there is an $a \in \mathbb{N}$ such that $S_{++}^a M \subset N_i$. Being M finitely generated, by Corollary 2.3.9, there exists a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ with vertex in some $\underline{\beta} \in \mathbb{N}^r$ such that $M_{\underline{n}} \subset (N_i)_{\underline{n}}$ for all $\underline{n}^* \in C_{\underline{\beta}}$.

By combining these two facts we get

$$(0 :_M z)_{\underline{n}} \subset (N_1 \cap \cdots \cap N_s \cap N_{s+1} \cap \cdots \cap N_t)_{\underline{n}} = 0$$

for $\underline{n}^* \in C_{\underline{\beta}}$, so $(0 :_M z)$ is Γ-finitely graded. Therefore, $H^i_{\mathcal{M}}((0 :_M z))$ is also Γ-finitely graded for all $i \geq 0$ by Proposition 4.3.5.

Since $\Gamma\text{-fg}((0 :_M z)) = +\infty$, from the first part of the proof we get that $\text{gdepth}((0 :_M z)) = +\infty$. Let us consider the exact sequence

$$0 \longrightarrow (0 :_M z) \longrightarrow M \longrightarrow \frac{M}{(0 :_M z)} \longrightarrow 0.$$

Since $\Gamma\text{-fg}((0 :_M z)) = \text{gdepth}((0 :_M z)) = +\infty$, from the long exact sequence of local cohomology we deduce $\Gamma\text{-fg}(M) = \Gamma\text{-fg}(M/(0 :_M z))$ and $\text{gdepth}(M) = \text{gdepth}(M/(0 :_M z))$. On the other hand there exists $b \in \mathbb{N}$ such that $z^b H^i_{\mathcal{M}}(M) = 0$ for all $i < \text{gdepth}(M)$. Hence we may assume that M is a S-module for which $z \in S_{++}$ is a non-zero divisor and $zH^i_{\mathcal{M}}(M) = 0$ for all $i < \text{gdepth}(M)$.

We will show by induction on c that if $0 \leq c \leq \text{gdepth}(M)$ then it holds $c \leq \Gamma\text{-fg}(M)$. The case $c = 0$ is trivial. Let us assume that $c > 0$, and let us consider the degree zero exact sequence, $\underline{r} = \deg(z)$,

$$0 \longrightarrow M(-\underline{r}) \xrightarrow{.z} M \longrightarrow \frac{M}{zM} \longrightarrow 0.$$

From the long exact sequence of local cohomology we can deduce that $\text{gdepth}(M) - 1 \leq \text{gdepth}(M/zM)$, so

$$0 \leq c - 1 \leq \text{gdepth}(M) - 1 \leq \text{gdepth}(M/zM).$$

By induction on c we get $c - 1 \leq \Gamma\text{-fg}(M/zM)$. In particular $H^{c-2}_{\mathcal{M}}(M/zM)$ is Γ-finitely graded. Let us consider the exact sequence on $\underline{n}$, for $\underline{n}^* \in C_{\underline{\beta}}$,

$$0 = H^{c-2}_{\mathcal{M}}(M/zM)_{\underline{n}} \longrightarrow H^{c-1}_{\mathcal{M}}(M)_{\underline{n}-\underline{r}} \xrightarrow{.z} H^{c-1}_{\mathcal{M}}(M)_{\underline{n}}.$$

Since $zH^{c-1}_{\mathcal{M}}(M) = 0$ we deduce that $H^{c-1}_{\mathcal{M}}(M)$ is Γ-finitely graded. Hence $c \leq \Gamma\text{-fg}(M)$. □

The invariance of Γ-fg under Veronese transforms is now an easy consequence:

Corollary 4.3.8. *Let S be an almost-standard multigraded ring such that $S_{\underline{0}}$ is the quotient of a regular ring. If M is a finitely generated $\mathbb{Z}^r$-graded S-module then for all $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$ it holds*

$$\Gamma\text{-fg}(M^{(\underline{a},\underline{b})}) = \Gamma\text{-fg}(M).$$

Proof. It follows immediately from Theorem 4.3.7 and Corollary 4.2.8. □

4.3.3 Asymptotic depth of Veronese modules (II)

As a continuation of 4.3.1, where we have proved the asymptotic depth for Veronese modules in some nets of $\mathbb{N}^r$ in the non-standard case, we now have new tools to solve the problem in a region of $\mathbb{N}^r \times \mathbb{N}^r$ instead of a net. However the restriction to the almost-standard graduation is still needed.

Definition 4.3.9. *Let M be a finitely generated graded S-module. We denote by*

$$\delta_M : \mathbb{N}^{*r} \times \mathbb{N}^{*r} \longrightarrow \mathbb{N}$$

the numerical function defined by $\delta_M(\underline{a}, \underline{b}) = \operatorname{depth}(M^{(\underline{a},\underline{b})})$, $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$. We write $\delta_M(\underline{a}) = \delta_M(\underline{a}, \underline{0})$.

Before studying the asymptotic depth of the Veronese of a module, we need a technical proposition. The following result does not work in the more general multigraded case, see Remark 4.3.11, so the restriction to the almost-standard case is necessary.

Proposition 4.3.10. *Let $C_{\underline{\beta}} \subset \mathbb{N}^r$ be a cone of vertex at $\underline{\beta} \in \mathbb{N}^r$. For all $\underline{n} \in \mathbb{N}^r$, $\underline{b} \in \mathbb{Z}^r$ such that $b_i \geq \beta_i$ if $n_i = 0$, and $\underline{a} \in \mathbb{N}^r$ such that $a_i \geq (\beta_i + b_i)/\gamma_i^i$, $i = 1, \ldots, r$, we have that*

$$(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}.$$

In particular, for all $\underline{b} \geq \underline{\beta}$ and $\underline{a} \in \mathbb{N}^r$ such that $a_i \geq (\beta_i + b_i)/\gamma_i^i$, $i = 1, \ldots, r$, we have that for all $\underline{n} \in \mathbb{Z}^r$

$$(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}.$$

Proof. For $\underline{n} \in \mathbb{Z}^r$ we have that $\phi_{\underline{a}}(\underline{n}) + \underline{b} = (a_1 n_1 \gamma_1^1 + b_1, \ldots, a_r n_r \gamma_r^r + b_r)$ and hence, $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* = (|a_1 n_1 \gamma_1^1 + b_1|, \ldots, |a_r n_r \gamma_r^r + b_r|)$.

We have to find conditions on $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$ in order to assure that $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}$ for all $\underline{n} \in \mathbb{Z}^r$. So, we have to impose that for all $i = 1, \ldots, r$, there exist some $\lambda_i \in \mathbb{R}_{\geq 0}$ such that $|a_i n_i \gamma_i^i + b_i| = \beta_i + \lambda_i \gamma_i^i$. Since $\gamma_i^i \in \mathbb{N}^*$, then it is only necessary to assure that $|a_i n_i \gamma_i^i + b_i| \geq \beta_i$ for all $i = 1, \ldots, r$.

If $n_i \neq 0$, since $|a_i n_i \gamma_i^i + b_i| \geq |a_i n_i \gamma_i^i| - |b_i| = |n_i| a_i \gamma_i^i - b_i$, then we have to impose that

$$|n_i| a_i \gamma_i^i - b_i \geq \beta_i$$

which is equivalent to

$$|n_i| \geq \frac{\beta_i + b_i}{a_i \gamma_i^i}.$$

Hence we must impose that

$$a_i \geq \frac{\beta_i + b_i}{\gamma_i^i}$$

$i = 1, \cdots, r$. If $n_i = 0$ then we have to impose $b_i = |b_i| \geq \beta_i$, $i = 1 \cdots, r$.

The second part of the result follows from the first one. □

Remark 4.3.11. Although we have needed to restrict our multigraduation in Proposition 4.3.5 and Theorem 4.3.7, we give here a counterexample on this last proposition for the non-standard case that we are considering in the chapter that justifies again the necessity of the restriction.

To simplify the notation, we assume that the generators of S over $S_{\underline{0}}$ have degrees $\gamma_1 = (A, 0)$ and $\gamma_2 = (B, C)$, with $A, B, C \in \mathbb{N}^*$.

The elements of the cone of vertex $\underline{\beta} = (\beta_1, \beta_2) \in \mathbb{N}^2$ with respect to γ_1, γ_2, are elements in $\mathbb{N}^2$ of the type

$$\underline{\beta} + \sum_{i=1}^{2} \lambda_i \gamma_i = (\beta_1 + \lambda_1 A + \lambda_2 B, \beta_2 + \lambda_2 C)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. Moreover, with $\underline{a} = (a_1, a_2)$ and $\underline{b} = (b_1, b_2)$ in $\mathbb{N}^{*2}$ it holds

$$(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* = (|a_1 n_1 A + a_2 n_2 B + b_1|, |a_2 n_2 C + b_2|).$$

In this case we are not able to find conditions on $\underline{a}$ and $\underline{b}$ in order to assure that $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}$ for all $\underline{n} \in \mathbb{Z}^2$.

Assume that the problem has solution. That is, there exists $\underline{a}, \underline{b} \in \mathbb{N}^{*2}$ such that for each $\underline{n} \in \mathbb{Z}^2$ there exist $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ such that

$$\begin{cases} |a_1 n_1 A + a_2 n_2 B + b_1| = \beta_1 + \lambda_1 A + \lambda_2 B \\ |a_2 n_2 C + b_2| = \beta_2 + \lambda_2 C \end{cases}$$

We choose an $n_1 \geq 0$ (that we will determine in the following) and let $n_2 < 0$ a sufficiently negative integer such that $a_1 n_1 A + a_2 n_2 B + b_1 < 0$ and $a_2 n_2 C + b_2 < 0$ (i.e. $n_2 < \min\{-b_2/a_2 C, (-b_1 - a_1 n_1 A)/a_2 B\} \leq 0$). With these integers, we have the system

$$\begin{cases} -a_1 n_1 A - a_2 n_2 B - b_1 = \beta_1 + \lambda_1 A + \lambda_2 B \\ -a_2 n_2 C - b_2 = \beta_2 + \lambda_2 C. \end{cases}$$

We solve this system of equations and we obtain

$$\lambda_2 = -a_2 n_2 - \frac{b_2 + \beta_2}{C}$$

$$\lambda_1 = -a_1 n_1 + \frac{(b_2 + \beta_2)B - (b_1 + \beta_1)C}{AC}.$$

Notice that $\lambda_1 \geq 0$ if and only if

$$n_1 < \frac{(b_2 + \beta_2)B - (b_1 + \beta_1)C}{a_1 AC},$$

so, if we consider $n_1 > \max\{\frac{(b_2+\beta_2)B-(b_1+\beta_1)C}{a_1 AC}, 0\}$, we have that $\lambda_1 < 0$ and therefore there is a contradiction with the assumption of the existence of $\underline{a}, \underline{b} \in \mathbb{N}^{*2}$ such that $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}$ for all $\underline{n} \in \mathbb{Z}^2$.

Now, we are ready to prove the theorem that assures constant depth for the $(\underline{a}, \underline{b})$-Veronese in a region of $\mathbb{N}^r \times \mathbb{N}^r$.

Theorem 4.3.12. *Let S be an almost-standard multigraded ring such that $S_{\underline{0}}$ is the quotient of a regular ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module and let $s = vad(M^{(*,*)})$. The numerical function δ_M is asymptotically constant: there exists $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{b} \geq \underline{\beta}$ and for all $\underline{a} \in \mathbb{N}^r$ such that $a_i \geq (\beta_i + b_i)/\gamma_i^i$ it holds*

$$\delta_M(\underline{a}, \underline{b}) = s.$$

Proof. We put $s = vad(M^{(*,*)})$, thus

$$\Gamma\text{-fg}(M) = \text{gdepth}(M) = \text{gdepth}(M^{(\underline{a},\underline{b})}) \geq s$$

by Theorem 4.3.7 and Corollary 4.2.8. Since $\Gamma\text{-fg}(M) \geq s$ there exist a a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, $\underline{\beta} \in \mathbb{N}^r$, such that $H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$ and $i = 0, \dots, s-1$.

By Lemma 4.3.10, for $\underline{b} \geq \underline{\beta}$ and $\underline{a} \in \mathbb{N}^r$ such that $a_i \geq (\beta_i + b_i)/\gamma^i_i$ for all $i = 1, \dots, r$, we have that $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}$ for all $\underline{n} \in \mathbb{Z}^r$. Hence, we get that for all $\underline{n} \in \mathbb{Z}^r$,

$$H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})})_{\underline{n}} = (H^i_{\mathcal{M}}(M)^{(\underline{a},\underline{b})})_{\underline{n}} = (H^i_{\mathcal{M}}(M))_{\phi_{\underline{a}}(\underline{n})+\underline{b}} = 0$$

because $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}$. So, we have proved that

$$H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})}) = 0$$

for $i = 0, \dots, s-1$. Therefore,

$$\text{depth}_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})}) \geq s,$$

and by the definition of s we get the claim. □

In the next result we generalize [Eli04], Proposition 2.1, to general $\mathbb{Z}$-graded modules.

Proposition 4.3.13. *Let S be a $\mathbb{Z}$-graded ring such that $S_{\underline{0}}$ is the quotient of a regular ring. Let M be a finitely generated graded S-module. The numerical function δ_M is asymptotically constant: there exist $s(M) \in \mathbb{N}$ and $\alpha \in \mathbb{N}$ such that for all $a \geq \alpha$ it holds*

$$\delta_M(a) = s(M).$$

Proof. If $s = s(M) = vad(M^{(*)})$ then

$$\Gamma\text{-fg}(M) = \text{gdepth}(M) = \text{gdepth}(M^{(a)}) \geq s$$

by Theorem 4.3.7 and Corollary 4.2.8. Since $\Gamma\text{-fg}(M) \geq s$ there exist an integer $\beta \in \mathbb{N}$, such that $H^i_{\mathcal{M}}(M)_n = 0$ for all $n \in \mathbb{N}$ with $|n| \geq \beta$ and for $i = 0, \dots, s-1$. From the first part of Proposition 4.3.10, for all elements $a \geq \alpha_i = \beta/\gamma^1_1$ we have that

$$H^i_{\mathcal{M}^a}(M^{(a)})_n = (H^i_{\mathcal{M}}(M)^{(a)})_n = H^i_{\mathcal{M}}(M)_{an} = 0$$

for all $n \neq 0$. On the other hand we have

$$H^i_{\mathcal{M}^a}(M^{(a)})_0 = (H^i_{\mathcal{M}}(M)^{(a)})_0 = H^i_{\mathcal{M}}(M)_0 = 0$$

for $i = 0, \ldots, s-1$. So, we have proved that

$$H^i_{\mathcal{M}^{(a)}}(M^{(a)}) = 0$$

for $i = 0, \ldots, s-1$. Therefore,

$$\operatorname{depth}_{\mathcal{M}^{(a)}}(M^{(a)}) \geq s,$$

and by the definition of s we get the claim. □

Considering the Rees algebra of an ideal, we recover the following result as an easy corollary.

Corollary 4.3.14 ([Eli04], Proposition 2.1)**.** *Let R be a Noetherian local ring quotient of a regular ring. Let $I \subset R$ be an ideal. Then the depth of $\mathcal{R}(I)^{(a)}$ is constant for $a \gg 0$.*

For the multigraded Rees algebra, the best approach to the solution of the problem is the following proposition.

Proposition 4.3.15. *If R is the quotient of a regular ring, there exist an integer s and $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{b} \geq \underline{\beta}$ and $\underline{a} \geq \underline{\beta} + \underline{b}$ it holds*

$$\operatorname{depth}_{\mathcal{M}^{(a)}}((I_1^{b_1} \cdots I_r^{b_r})\mathcal{R}(I_1^{a_1}, \cdots, I_r^{a_r})) = s.$$

Proof. Note that, since the Rees algebra $\mathcal{R}(I_1, \cdots, I_r)$ is standard multigraded, we have

$$\mathcal{R}(I_1, \cdots, I_r)^{(\underline{a},\underline{b})} = (I_1^{b_1} \cdots I_r^{b_r})\mathcal{R}(I_1^{a_1}, \cdots, I_r^{a_r}),$$

with $\underline{a} = (a_1, \ldots, a_r)$ and $\underline{b} = (b_1, \ldots, b_r)$. Now, from Theorem 4.3.12 we get the claim. □

For more results on Cohen-Macaulay and Gorenstein properties of the multigraded Rees algebras see [Hyr99] and its reference list.

Chapter 5

Bigraded structures and the depth of blow-up algebras

The aim of this chapter is to study the depth of blow-up algebras by means of certain bigraded modules. First, we overview some conjectures of Guerrieri and Wang that relate the depth of the associated graded ring to an ideal with the lengths of certain modules. We interpret these lengths as the multiplicities of some non-standard bigraded modules. Thanks to this interpretation we are able to refine the Conjecture of Wang, by including new cases where it works and recovering the known true cases. As a corollary, we can answer a question of Guerrieri and Huneke regarding the lengths of the pieces of the Valabrega-Valla module. In the first section we introduce these conjectures and questions and we explain the structure of the present chapter.

5.1 Conjectures on the depth of blow-up algebras

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. One of the major problems in commutative algebra is to estimate the depth of the associated graded ring $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and the Rees algebra $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$ for ideals I having good properties. Attached to the pair I, J we can consider the integers

$$\Delta(I,J) = \sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1}\cap J}{I^pJ}\right) \quad , \quad \Lambda(I,J) = \sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1}}{JI^p}\right),$$

$$\Delta_p(I,J) = \text{length}_R \left(\frac{I^{p+1}\cap J}{I^pJ}\right) \quad \text{and} \quad \Lambda_p(I,J) = \text{length}_R \left(\frac{I^{p+1}}{JI^p}\right)$$

for $p \geq 0$.

Related to these integers there are some results and conjectures on the depth of the associated graded ring $gr_I(R)$ that we next review.

Valabrega and Valla proved that $\Delta(I,J) = 0$ if and only if $gr_I(R)$ is Cohen-Macaulay, [VV78]. In fact, the $\mathcal{R}(J)$-module

$$\bigoplus_{p\geq 0} \frac{I^{p+1}\cap J}{I^pJ}$$

is the so-called Valabrega-Valla module of I with respect to J.

Based on this result, Guerrieri proposed the following conjecture in [Gue94]:

Conjecture 5.1.1 (Guerrieri)**.** *Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. Then*

$$\text{depth}(gr_I(R)) \geq d - \Delta(I,J).$$

Guerrieri proved the case $\Delta(I,J) = 1$ and some partial cases for $\Delta(I,J) = 2$, [Gue95]. Wang proved the case $\Delta(I,J) = 2$ without any restriction, [Wan00].

Guerrieri and Huneke asked if the conditions $\Delta_p(I,J) \leq 1$, $p \geq 1$, imply that $\text{depth}(gr_I(R)) \geq d-1$, [Gue93], Question 2.23. Wang in [Wan02], Example 3.13, gave a counterexample to Guerrieri's question and asked whether this question would have an affirmative answer whenever R was a regular local ring.

Huckaba and Marley proved that $e_1(I) \leq \Lambda(I,J)$ and that if the equality holds then $\text{depth}(gr_I(R)) \geq d-1$, [HM97]. Hence one can consider the non-negative integer

$$\delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

Wang showed that $\delta(I,J) \leq \Delta(I,J)$ and that Guerrieri's Conjecture is implied by the following one, [Wan00],

Conjecture 5.1.2 (Wang)**.** *Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. Then*

$$\operatorname{depth}(gr_I(R)) \geq d - 1 - \delta(I, J).$$

Huckaba proved the conjecture in the case $\delta(I, J) = 0$, [Huc96], [HM97]. If $\delta(I, J) = 1$ Wang proved the conjecture, [Wan00], and Polini gave a simpler proof, [Pol00]. For $\delta(I, J) = 2$ Rossi and Guerrieri proved Wang's Conjecture assuming that R/I is Gorenstein, [GR99]. Wang gave a counterexample to the conjecture for $d = 6$, [Wan01].

In the main result of this chapter we prove a refined version of Wang's Conjecture, Theorem 5.6.3. We naturally decompose the integer $\delta(I, J)$ as a finite sum of non-negative integers $\delta_p(I, J)$, with $\Delta_p(I, J) \geq \delta_p(I, J) \geq 0$, see Section 5.5. If $\bar{\delta}(I, J)$ is the maximum of the integers $\delta_p(I, J)$ for $p \geq 0$, when $\delta(\bar{I}, J) \leq 1$, we are able to prove that $\operatorname{depth}(\mathcal{R}(I)) \geq d - \bar{\delta}(I, J)$ and $\operatorname{depth}(gr_I(R)) \geq d - 1 - \bar{\delta}(I, J)$. As a consequence we give an answer to the question formulated by Guerrieri and Huneke about considering $\Delta_p(I, J) \leq 1$ for all $p \geq 0$, Theorem 5.6.5.

The aim of this chapter is to introduce a non-standard bigraded module $\Sigma^{I,J}$ in order to study the depth of the associated graded ring $gr_I(R)$ and the Rees algebra $\mathcal{R}(I)$ of I. A secondary purpose is to present a unified framework where several results and objects appearing in the papers on the above conjectures can be studied, Remark 5.5.5. The key tool of this chapter is the Hilbert function of non-standard bigraded modules.

Sections 5.2 and 5.3 are mainly devoted to recall some preliminary results on the Sally module and the cumulative Hilbert function of non-standard bigraded modules.

In Section 5.4 we introduce a non-standard bigraded module $\Sigma^{I,J}$ naturally attached to I and a minimal reduction J of I. This module can be considered as a refinement of the Sally module previously introduced by W. Vasconcelos. From a natural presentation of $\Sigma^{I,J}$ we define two bigraded modules $K^{I,J}$ and $\mathcal{M}^{I,J}$, and we consider some diagonal submodules of them: $\Sigma^{I,J}_{[p]}$ and $K^{I,J}_{[p]}$.

In Section 5.5, using the cumulative Hilbert function, we can interpret the integers appearing in the conjectures as multiplicities of the modules defined. In particular, for all $p \geq 0$, we consider the integer $\delta_p(I, J) = e_0(K^{I,J}_{[p]})$ and we can prove that $\Delta_p(I, J) \geq \delta_p(I, J) = \Lambda_p(I, J) - e_0(\Sigma^{I,J}_{[p]}) \geq 0$ and $e_1(I) = \sum_{p \geq 0} e_0(\Sigma^{I,J}_{[p]})$.

Section 5.6 is devoted to prove the refined version of Wang's Conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta(I,J) = \sum_{p\geq 0} \delta_p(I,J)$, Theorem 5.6.3. Then we can give some applications to other related questions. An essential point of this section is to follow Polini's ideas developed in [Pol00], where the case $\delta(I,J) = 1$ was studied, and generalize part of her work.

5.2 Sally module

In this work, we talk about the Sally module and we construct an analogous for the bigraded case. This module is useful in order to calculate or estimate the depth of the associated graded ring if we know the depth of the Sally module.

The Sally module was introduced by Vasconcelos in [Vas94b], with this name after the previous work of Sally on the pieces of the module. It was deeply studied by Vaz Pinto in her PhD Thesis, [Vaz95]. Other references about the Sally module are [Vaz97], [CPV98], [Pol00], and [RV00] among others.

In this section we collect some definitions and properties of the Sally module, specially the ones regarding Hilbert functions and the depth of blow-up algebras. We follow [Vas94b] and [Vaz95].

Definition 5.2.1. *Let R be a Noetherian ring and let I an ideal of R with reduction J. The exact sequence of $\mathcal{R}(J)$-modules*

$$0 \longrightarrow I\mathcal{R}(J) \longrightarrow I\mathcal{R}(I) \longrightarrow \frac{I\mathcal{R}(I)}{I\mathcal{R}(J)} \longrightarrow 0$$

defines the Sally module of I with respect to J

$$S_J(I) = \frac{I\mathcal{R}(I)}{I\mathcal{R}(J)} = \bigoplus_{n\geq 1} \frac{I^{n+1}}{J^n I}.$$

Along this section we consider the case in which $(R,\mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d > 0$ and infinite residue field $\mathbf{k} = R/\mathfrak{m}$, and I is an $\mathfrak{m}$-primary ideal of R with J a minimal reduction of I.

Proposition 5.2.2. *Let $(R,\mathfrak{m})$ a Cohen-Macaulay local ring of dimension $d>0$. Let I be an $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. Then,*

(i) If $S_J(I)\neq 0$ then $\dim S_J(I)=d$ as $\mathcal{R}(J)$-module.

(ii) If $S_J(I)=0$ then $gr_I(R)$ is Cohen-Macaulay.

(iii) If $S_J(I)=0$ and $d\geq 2$, then $\mathcal{R}(I)$ is Cohen-Macaulay.

Proposition 5.2.3. *Let $(R,\mathfrak{m})$ a Cohen-Macaulay local ring of dimension $d>0$. Let I be an $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. Then*

$$\operatorname{depth}(gr_I(R))\geq \operatorname{depth}(S_J(I))-1.$$

If $\operatorname{depth}(gr_I(R))<d$, then

$$\operatorname{depth}(S_J(I))=\operatorname{depth}(gr_I(R))+1.$$

We define the Hilbert function of the Sally module $S_J(I)$ as

$$h_{S_J(I)}(n)=\operatorname{length}_R\left(\frac{I^{n+1}}{J^nI}\right).$$

If $S_J(I)\neq 0$ then $\dim S_J(I)=d$, and for $n\gg 0$, $h_{S_J(I)}(n)=p_{S_J(I)}(n)$ where

$$p_{S_J(I)}(n)=\sum_{i=0}^{d-1}(-1)^i s_i\binom{n+d-i-1}{d-i-1}$$

is the Hilbert polynomial of $S_J(I)$ of degree $d-1$.

Finally, we want to relate the Hilbert coefficients of I with the ones of the Sally module, [Vas94b]. Next proposition will be crucial in Section 5.5.

Proposition 5.2.4. *Let $(R,\mathfrak{m})$ a Cohen-Macaulay local ring of dimension $d>0$. Let I be an $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. Then*

$$\begin{cases} e_0(I)=\operatorname{length}_R(R/J) & \\ e_1(I)=e_0(I)-\operatorname{length}_R(R/I)+s_0=s_0+\operatorname{length}_R(I/J) & \\ e_i(I)=s_{i-1} & i=2,\dots,d. \end{cases}$$

5.3 Cumulative Hilbert function

Roberts proves in [Rob00] the existence of Hilbert polynomials for cumulative Hilbert functions of bigraded modules over bigraded polynomial rings $K[X_1, \dots, X_s]$ over a field K in variables $X_1, \dots, X_s$ of bidegrees $(1,0)$, $(0,1)$, and $(1,1)$, (see also [HT03]). In this section, following Section 3 of [Rob00], we can easily generalize those results to polynomial rings with coefficients in an Artin ring.

Let $S = A[X_1, \dots, X_r, Y_1, \dots, Y_s, Z_1, \dots, Z_t]$ be a bigraded polynomial ring over an Artin ring A in variables $X_1, \dots, X_r, Y_1, \dots, Y_s$ and $Z_1, \dots, Z_t$. We assume that the variables X_i have bidegree $(1,0)$, the variables T_i have bidegree $(1,1)$, and the variables Z_i have bidegree $(0,1)$.

For any finitely generated bigraded S-module M, and for any $m, n \in \mathbb{Z}$, let $M_{(m,n)}$ be the piece of M of bidegree (m,n).

If M is a bigraded module, let $M(\alpha, \beta)$ be the module M with degrees shifted by (α, β), i.e., $M(\alpha,\beta)_{(m,n)} = M_{(m+\alpha,n+\beta)}$ for all m and n.

Remark 5.3.1. Note that we can consider the ring S as a $\mathbb{Z}$-graded ring by grading it in the first variable

$$S_m = \bigoplus_{n \in \mathbb{Z}} S_{(m,n)}.$$

Observe that S_m is not necessarily a finite length A-module. In the same way, we can consider a bigraded S-module M as a $\mathbb{Z}$-graded module.

Definition 5.3.2. *The (cumulative) Hilbert function of M, $h_M(m,n)$, is defined as*

$$h_M(m,n) = \sum_{i \leq n} \operatorname{length}_A(M_{(m,i)}).$$

Note that as A in an Artin ring, the length over S is equal to the length over A. In this case, we prove that there exist integers m_0, n_0 such that the Hilbert function is be given by a polynomial in (m,n) for $m \geq m_0$ and $n \geq m + n_0$.

We use multi-index notation. For a vector $\underline{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, we denote the monomial $X^{\underline{n}} := X_1^{n_1} \cdots X_r^{n_r}$, and $|\underline{n}| = n_1 + \cdots + n_r$.

In order to prove the existence of Hilbert polynomials, we will need to know that certain graded subsets of M have Hilbert polynomial.

Let B be the subring of S defined as

$$B = A[X_1, \dots, X_r, Y_1, \dots, Y_s, \{X_i Z_j\}_{\substack{i=1,\dots,r \\ j=1,\dots,t}}] \subset S.$$

Here, the generators of B have bidegree $(1,0)$ or $(1,1)$. B is a bigraded subring of S.

Let k be an integer and we define the B-submodule of M

$$D_k(M) = \bigoplus_{n \leq m+k} M_{(m,n)}.$$

It is clear that $D_k(M)$ is a B-module. In fact, if $n \leq m+k$, $X_j M_{(m,n)} \subset M_{(m+1,n)}$, $Y_j M_{(m,n)} \subset M_{(m+1,n+1)}$ and $X_i Z_j M_{(m,n)} \subset M_{(m+1,n+1)}$. Then in all cases $n \leq (m+1)+k$ and $(n+1) \leq (m+1)+k$.

As before, we can consider B as a $\mathbb{Z}$-graded ring, and $D_k(M)$ can be considered as a $\mathbb{Z}$-graded B-module.

Lemma 5.3.3. *For all k integer, $D_k(M)$ is a finitely generated B-module.*

Proof. Since A is an Artin ring, we can suppose that M is finitely generated over S. Let $f_1, \ldots, f_u$ be the homogeneous generators of M as S-module with bidegrees $\deg(f_i) = (m_i, n_i)$ for each $i = 1, \ldots, u$.

First, we assume that $f_1, \ldots, f_u \in D_k(M)$, i.e., $n_i \leq m_i + k$ for all i.

We will prove that the generators of $D_k(M)$ as a B-module are $Z^{\underline{k}} f_i$, for all i and for all $\underline{k}$ such that $n_i + |\underline{k}| \leq m_i + k$. Obviously, these elements are in $D_k(M)$ and this set is finite.

To prove that, we will show that each component $M_{(m,n)}$ of $D_k(M)$ is generated as an A-module by multiples of these elements by monomials in B.

Consider a component $M_{(m,n)}$ with $n \leq m+k$. It is generated as an A-module by elements of the form $X^{\underline{i}} Y^{\underline{j}} Z^{\underline{k}} f_i$, where $\underline{i}, \underline{j}, \underline{k}$ are r-,s-,t-tuples of non-negative integers, and with $\deg(X^{\underline{i}} Y^{\underline{j}} Z^{\underline{k}}) = (|\underline{i}| + |\underline{j}|, |\underline{j}| + |\underline{k}|) = (m - m_i, n - n_i)$.

If some factor Y_j or $X_i Z_j$ appears in this monomial, then Y_j or $X_i Z_j$ can be factorized, and, since $\deg(Y_j) = \deg(X_i Z_j) = (1,1)$, this generator is multiple of an element in $M_{(m-1,n-1)}$ by an element of B, and we can conclude the result by using induction on m.

If no factor Y_j or $X_i Z_j$ appears in $X^{\underline{i}} Y^{\underline{j}} Z^{\underline{k}} f_i$, then $X^{\underline{i}} Y^{\underline{j}} Z^{\underline{k}} f_i$ is of the form $X^{\underline{i}} f_i$ or $Z^{\underline{k}} f_i$. In the first case, since we have supposed that $f_i \in D_k(M)$, the generator is multiple of an element in $M_{(m-1,n)}$ by an element of B, so, dividing by some X_j we conclude by induction on m. In the second case, $Z^{\underline{k}} f_i$ will be a generator of $D_k(M)$.

Therefore, we have proved that $X^{\underline{i}}Y^{\underline{j}}Z^{\underline{k}}f_i$ is a multiple of some $Z^{\underline{k}}f_i$, with $n_i + |\underline{k}| \leq m_i + k$, by an element of B. Thus, $D_k(M)$ is finitely generated as B-module.

If some of the f_i are not in $D_k(M)$, we can choose another k' large enough such that all the $f_i \in D_{k'}(M)$ for $i = 1, \ldots, u$. Then $D_{k'}(M)$ is finitely generated. Being B a Noetherian ring and $D_k(M)$ a sub-B-module of $D_{k'}(M)$, we conclude that $D_k(M)$ is finitely generated as well. □

Now, we can prove our main result.

Theorem 5.3.4. *Let $S = A[X_1, \ldots, X_r, Y_1, \ldots, Y_s, Z_1, \ldots, Z_t]$ be a bigraded polynomial ring over an Artin ring A with indeterminates $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ and $Z_1, \ldots, Z_t$, where each X_i has bidegree $(1, 0)$, each Y_i has bidegree $(1, 1)$, and each Z_i has bidegree $(0, 1)$. Let M be a finitely generated bigraded S-module.*
Then, there exist integers m_0 and n_0 and a polynomial in two variables $p_M(m, n)$ such that

$$p_M(m, n) = h_M(m, n)$$

for all (m, n) with $m \geq m_0$ and $n \geq n_0 + m$.

Proof. We prove the theorem by induction on the number t of indeterminates of bidegree $(0, 1)$, Z_i.

Assume that $t = 0$, i.e., there are no variables of bidegree $(0, 1)$. Let $f_1, \ldots, f_u$ be the generators of the module M. Assume that the bidegrees are $\deg(f_i) = (m_i, n_i)$ for each $i = 1, \ldots, u$. Consider an integer n_0 such that $n_0 > \max\{n_i - m_i \mid i = 1, \ldots, u\}$.

A bigraded piece $M_{(m,n)} \neq 0$ is generated by products of some f_k with monomials in $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$. So, a monomial generator of $M_{(m,n)}$ will be $X^{\underline{i}}Y^{\underline{j}}f_k$ for some $\underline{i} \in \mathbb{N}^r$ and $\underline{j} \in \mathbb{N}^s$, and $(m, n) = (|\underline{i}| + |\underline{j}| + m_k, |\underline{j}| + n_k)$ since $\deg(X_i) = (1, 0)$ and $\deg(Y_i) = (1, 1)$. Then, $n - m = n_k - m_k - |\underline{i}| < n_k - m_k$, and since we have chosen n_0 to be the maximum of all differences $n_i - m_i$, we have that $n - m < n_0$. So, if $n - m \geq n_0$ then $M_{(m,n)} = 0$. Therefore,

$$\begin{aligned} h_M(m, n) &= \sum_{i \leq n} \text{length}_A(M_{(m,i)}) \\ &= \sum_{i \leq m + n_0} \text{length}_A(M_{(m,i)}) = h_M(m, m + n_0) \end{aligned}$$

when $n \geq m + n_0$. Then, for each m, $h_M(m,n)$ is constant in n for large n, and is $h_M(m,n) = \sum_{i\in\mathbb{Z}} \text{length}_A(M_{(m,i)})$. Therefore, if we denote $h_M(m)$ the Hilbert function of M as a graded $\mathbb{Z}$-module, we have that

$$h_M(m,n) = h_M(m)$$

for $n \geq m + n_0$. Finally, thanks to the classical theory of Hilbert functions, there exists an integer m_0 such that $h_M(m)$ is a polynomial in m for all $m \geq m_0$. So, for all $m \geq m_0$ and $n \geq m + n_0$,

$$h_M(m,n) = p_M(m),$$

where $p_M(m)$ is a polynomial in m. Notice that in this case, i.e. when there are no ring generators with bidegree $(0,1)$, the polynomial does not depend on n.

Now, we suppose that $t > 0$, i.e., there are some indeterminates Z_i of bidegree $(0,1)$. We consider the submodule of M

$$(0 :_M Z_t) = \{x \in M | Z_t \cdot x = 0\}.$$

Then, we have the following exact sequence:

$$0 \longrightarrow (0 :_M Z_t)(0,-1) \longrightarrow M(0,-1) \xrightarrow{\cdot Z_t} M \longrightarrow M/Z_tM \longrightarrow 0.$$

Since $(0 :_M Z_t)$ and M/Z_tM are annihilated by Z_t, they are finitely generated modules over $A[X_1,\dots,X_r,Y_1,\dots,Y_s,Z_1,\dots,Z_{t-1}]$ and hence, by induction on t, the theorem holds for $(0 :_M Z_t)$ and M/Z_tM.

From the exact sequence, we have that

$$h_M(m,n) - h_M(m,n-1) = h_{M/Z_tM}(m,n) - h_{(0:_M Z_t)}(m,n-1)$$

and so, there exists a polynomial $p_1(m,n)$ such that

$$h_M(m,n) - h_M(m,n-1) = p_1(m,n)$$

for pairs of integers (m,n) with $m \geq m_0$ and $n \geq m + n_0$, for some m_0, n_0.

We can consider the B-module $D_{n_0}(M)$ that we have defined before. By Lemma 5.3.3, $D_{n_0}(M)$ is a finitely generated bigraded B-module. Therefore, since B has no variables of bidegree $(0,1)$, there is a polynomial $p_2(m)$ such that $p_2(m) = \text{length}_A(D_{n_0}(M)_m)$ for large m. We can choose a suitable m_0 such that works for both polynomials, $p_1(m,n)$ and $p_2(m)$.

It is well known that a polynomial in two variables m, n can be written as a linear combination of the products $\binom{m}{i}\binom{n}{j}$ of binomial numbers for some i and j. So, we can put $p_1(m,n) = \sum c_{ij}\binom{m}{i}\binom{n}{j}$.

We define

$$p_M(m,n) = \sum c_{ij}\binom{m}{i}\binom{n+1}{j+1} - \sum c_{ij}\binom{m}{i}\binom{m+n_0+1}{j+1} + p_2(m)$$

and we want to prove that

$$h_M(m,n) = p_M(m,n)$$

for $m \geq m_0$ and $n \geq m + n_0$.

Clearly, we can see that $p_M(m, m+n_0) = p_2(m)$. In fact, since we have $(D_{n_0}(M))_m = \bigoplus_{n \leq m+n_0} M_{(m,n)}$, then $p_2(m) = \sum_{n \leq m+n_0} \operatorname{length}_A(M_{(m,n)}) = h_M(m, m+n_0)$. So, $h_M(m, m+n_0) = p_M(m, m+n_0)$ for $m \geq m_0$.

Thus, the claim holds when $n = m + n_0$. If $n > m + n_0$, we have that

$$\begin{aligned} p_M(m,n) - p_M(m,n-1) &= \sum c_{ij}\binom{m}{i}\binom{n+1}{j+1} - \sum c_{ij}\binom{m}{i}\binom{n}{j+1} \\ &= \sum c_{ij}\binom{m}{i}\binom{n}{j} \\ &= p_1(m,n) = h_M(m,n) - h_M(m,n-1) \end{aligned}$$

Hence these polynomials agree for all (m,n) with $m \geq m_0$ and $n \geq m + n_0$, as we wanted to prove.

□

Remark 5.3.5. As we can see in the first induction step, in case there are no ring generators of bidegree $(0,1)$, the (cumulative) Hilbert function in a region is a polynomial in one indeterminate. In this chapter we will consider some bigraded modules over a polynomial ring with indeterminates of bidegrees $(1,0)$ and $(1,1)$.

Remark 5.3.6. In Chapter 2, we studied the Hilbert function of a (non-standard) multigraded module as the length of the homogeneous pieces of the module. The case considered in the following sections is covered by that case, but precisely for the kind of structures considered, we have that they vanish for a suitable cone defined in Chapter 2, see Lemma 5.4.1. For that reason, and since we wish to consider our modules both bigraded and graded with respect the first index, it is best to consider the cumulative Hilbert functions instead.

5.4 Bigraded Sally module

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Without loss of generality we may assume that the residue field $\mathbf{k} = R/\mathfrak{m}$ is infinite. For an ideal I of R, we set that $I^i = 0$ for $i < 0$ and $I^0 = R$.

Let $I = (b_1, \dots, b_\mu)$ be an $\mathfrak{m}$-primary ideal of R and let $J = (a_1, \dots, a_d)$ be a minimal reduction of I. Since $Jt\mathcal{R}(I)$ is a homogeneous ideal of the graded ring $\mathcal{R}(I)$ we can consider the associated graded ring of $\mathcal{R}(I)$ with respect to the homogeneous ideal $Jt\mathcal{R}(I) = \bigoplus_{n\geq 0} JI^{n-1}t^n$

$$gr_{Jt}(\mathcal{R}(I)) = \bigoplus_{j\geq 0} \frac{(Jt\mathcal{R}(I))^j}{(Jt\mathcal{R}(I))^{j+1}} U^j.$$

This ring has a natural bigraded structure that we briefly describe. Note that

$$\frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)} = \bigoplus_{i\geq 0} \frac{I^i}{I^{i-1}J} t^i$$

is a homomorphic image of the graded ring $R[V_1, \dots, V_\mu]$ by the degree one R-algebra homogeneous morphism

$$\sigma : R[V_1, \dots, V_\mu] \longrightarrow \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)} = \bigoplus_{i\geq 0} \frac{I^i}{I^{i-1}J} t^i$$

defined by $\sigma(V_i) = b_i t \in \frac{I}{J} t$. Here, $R[V_1, \dots, V_\mu]$ is endowed with the standard graduation.

Let us consider the bigraded ring $B := R[V_1, \dots, V_\mu; T_1, \dots, T_d]$ with $\deg(V_i) = (1, 0)$ and $\deg(T_i) = (1, 1)$. Then there exists an exact sequence of bigraded B-rings

$$0 \longrightarrow K^{I,J} \longrightarrow C^{I,J} := \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)}[T_1, \dots, T_d] \xrightarrow{\pi} gr_{Jt}(\mathcal{R}(I)) \longrightarrow 0 \qquad (1)$$

with $\pi(T_i) = a_i tU$, $i = 1, \dots, d$, and where $K^{I,J}$ is the ideal of initial forms of $Jt\mathcal{R}(I)$. The $(i + j, j)$-graded pieces of $gr_{Jt}(\mathcal{R}(I))$ and $C^{I,J}$ are

$$gr_{Jt}(\mathcal{R}(I))_{(i+j,j)} = \frac{I^i J^j}{I^{i-1} J^{j+1}} t^{i+j} U^j,$$

and

$$C^{I,J}_{(i+j,j)} = \frac{I^i}{JI^{i-1}} t^i [T_1, \dots, T_d]_j$$

respectively. Notice that we have an R-algebra isomorphism

$$\phi : R[T_1, \ldots, T_d]/(\{a_i T_j - a_j T_i\}_{i,j}) \cong R[JtU] = \mathcal{R}(J)$$

defined by $\phi(\overline{T_i}) = a_i tU$, $i = 1, \ldots, d$. Observe that we write tU instead of only t to bear in mind the bigraduation.

Given a B-bigraded module M and an integer $p \in \mathbb{Z}$, we denote by $M_{[p]}$ the additive subgroup of M defined by the direct sum of the pieces $M_{(m,n)}$ such that $m - n = p + 1$. Notice that the product by the variable T_i induces an endomorphism of $R[T_1, \ldots, T_d]$-modules $M_{[p]} \xrightarrow{T_i} M_{[p]}$, and the product by V_j a morphism of $R[T_1, \ldots, T_d]$-modules $M_{[p]} \xrightarrow{V_j} M_{[p+1]}$. Hence $M_{\geq p} = \bigoplus_{n \geq p} M_{[n]}$ is a sub-B-module of M, and we can consider the exact sequence of $R[T_1, \ldots, T_d]$-modules

$$0 \longrightarrow M_{[p]} \longrightarrow M_{\geq p} \longrightarrow M_{\geq p+1} \longrightarrow 0.$$

Moreover, in our case, the modules $K^{I,J}_{[p]}$, $C^{I,J}_{[p]}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$ are in particular $\mathcal{R}(J)$-modules.

Next lemma shows that $K^{I,J}_{[p]}$, $C^{I,J}_{[p]}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$ eventually do not vanish for a finite set of indexes $p \in \mathbb{Z}$.

Lemma 5.4.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J.*

(i) For all $p \leq -2$ or $p \geq r_J(I)$, $C^{I,J}_{[p]} = 0$, $gr_{Jt}(\mathcal{R}(I))_{[p]} = 0$ and $K^{I,J}_{[p]} = 0$.

(ii) π induces the following isomorphisms of $\mathcal{R}(J)$-modules:

$$gr_{Jt}(\mathcal{R}(I))_{[0]} \cong C^{I,J}_{[0]} \cong \frac{I}{J} t[T_1, \ldots, T_d],$$

$$gr_{Jt}(\mathcal{R}(I))_{[-1]} \cong \mathcal{R}(J),$$

$$C^{I,J}_{[-1]} \cong R[T_1, \ldots, T_d].$$

Moreover, $K^{I,J}_{[0]} = 0$.

Proof. In order to prove (i), we first describe $C^{I,J}_{[p]}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$ are. Since $C^{I,J} \cong \bigoplus_{i\geq 0} \frac{I^i}{JI^{i-1}} t^i[T_1, \ldots, T_d]$, we have that

$$C^{I,J}_{[p]} = \bigoplus_{m-n=p+1} C^{I,J}_{(m,n)} = \frac{I^{p+1}}{JI^p} t^{p+1}[T_1, \ldots, T_d],$$

and

$$gr_{Jt}(\mathcal{R}(I))_{[p]} = \bigoplus_{m-n=p+1} gr_{Jt}(\mathcal{R}(I))_{(m,n)} = \bigoplus_{i\geq 0} \frac{J^i I^{p+1}}{J^{i+1} I^p} t^{p+1+i} U^i.$$

Since $I^i = 0$ for all $i < 0$, we have that $C^{I,J}_{[p]} = 0$ and $gr_{Jt}(\mathcal{R}(I))_{[p]} = 0$ for all $p \leq -2$. By the definition of $r_J(I)$ we have $I^{p+1} = JI^p$ for all $p \geq r_J(I)$, so $C^{I,J}_{[p]} = 0$ and $gr_{Jt}(\mathcal{R}(I))_{[p]} = 0$ for all $p \geq r_J(I)$. Notice that we have that $K^{I,J}_{[p]} \subseteq C^{I,J}_{[p]}$ for each $p \in \mathbb{Z}$. Therefore, we have that $K^{I,J}_{[p]} = 0$ for all $p \leq -2$ and $p \geq r_J(I)$.

(ii) When $p = 0$ we have

$$C^{I,J}_{[0]} = \frac{I}{J} t[T_1, \ldots, T_d] = It(R/J)[T_1, \ldots, T_d]$$

and

$$gr_{Jt}(\mathcal{R}(I))_{[0]} = \bigoplus_{i\geq 0} \frac{J^i I}{J^{i+1}} t^{i+1} U^i \cong It gr_J(R).$$

Since $gr_J(R) \cong (R/J)[T_1, \ldots, T_d]$, clearly we have that $C^{I,J}_{[0]} \cong gr_{Jt}(\mathcal{R}(I))_{[0]}$. From the exact sequence (1) we deduce $K^{I,J}_{[0]} = 0$. If $p = -1$ then we have $C^{I,J}_{[-1]} = R[T_1, \ldots, T_d]$ and $gr_{Jt}(\mathcal{R}(I))_{[-1]} = \bigoplus_{i\geq 0} J^i t^i U^i \cong \mathcal{R}(J)$. □

From now on, we will be interested in considering the non-negative diagonals of these modules and so, let us consider the following bigraded finitely generated B-modules:

$$\Sigma^{I,J} := \bigoplus_{p\geq 0} gr_{Jt}(\mathcal{R}(I))_{[p]}$$

$$\mathcal{M}^{I,J} := \bigoplus_{p\geq 0} C^{I,J}_{[p]}$$

and from now on, we consider the new

$$K^{I,J} := \bigoplus_{p\geq 0} K^{I,J}_{[p]}.$$

Note that by Lemma 5.4.1 there exists a natural isomorphism of $\mathcal{R}(J)$-modules

$$gr_{Jt}(\mathcal{R}(I)) \cong \mathcal{R}(J) \oplus \Sigma^{I,J}.$$

We can observe how this decomposition corresponds to the $[-1]$-diagonal and to the non-negative ones.

A complete description of the first two modules in terms of their graded pieces is the following:

$$\Sigma^{I,J} = \bigoplus_{p\geq 0} \bigoplus_{i\geq 0} \frac{J^i I^{p+1}}{J^{i+1} I^p} t^{p+1+i} U^i$$

$$\mathcal{M}^{I,J} \cong \bigoplus_{p\geq 0} \frac{I^{p+1}}{I^p J} t^{p+1}[T_1, \ldots, T_d]$$

Since the modules $\Sigma^{I,J}$ and $\mathcal{M}^{I,J}$ are annihilated by J, from the exact sequence (1) we deduce the following exact sequence of $A = B \otimes_R R/J \cong R/J[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$-bigraded modules

$$0 \longrightarrow K^{I,J} \longrightarrow \mathcal{M}^{I,J} \longrightarrow \Sigma^{I,J} \longrightarrow 0. \tag{S}$$

Note that from Lemma 5.4.1 all relevant information of (1) is encoded in the exact sequence (S).

By considering each diagonal, for all $p \geq 0$ we have an exact sequence of $R/J[T_1, \ldots, T_d]$-modules

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1, \ldots, T_d] \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0, \tag{S$_{[p]}$}$$

which are, in fact, graded modules, and so we can consider the (classic) Hilbert function of $\Sigma^{I,J}_{[p]}$, $\mathcal{M}^{I,J}_{[p]}$ and $K^{I,J}_{[p]}$ with respect the variables $T_1, \ldots, T_d$.

Definition 5.4.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal and let J be a minimal reduction of I. We call $\Sigma^{I,J}$ the bigraded Sally module of I with respect to J.*

The A-module $\mathcal{M}^{I,J}$ has some interesting properties that will be useful in some of the proofs of this chapter, especially in counting depths. In the next result we show some of its properties.

Proposition 5.4.3. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. Then $Ass_A(\mathcal{M}^{I,J}) = \{\mathfrak{m}A + (V_1,\ldots,V_\mu)\}$ and $\mathcal{M}^{I,J}$ is a Cohen-Macaulay A-module of dimension d.*

Proof. First we prove that $Ass_A(\mathcal{M}^{I,J}) = \{Q\}$ with $Q = \mathfrak{m}A + (V_1,\ldots,V_\mu)$. Let $\mathfrak{p} = Ann(\sigma)$ be an associated prime ideal of $Ass_A(\mathcal{M}^{I,J})$ where $\sigma \in \mathcal{M}^{I,J} = \mathcal{M}$.

Let us recall that

$$\mathcal{M}^{I,J} = (\bigoplus_{p\geq 0} \frac{I^{p+1}}{JI^p} t^{p+1})[T_1,\ldots,T_d],$$

so by using multi-index notation we can write $\sigma = \sum_K (\sum_p \overline{\sigma}_K^p) T^K$ with $\overline{\sigma}_K^p \in \frac{I^{p+1}}{JI^p}$.

Let $\alpha \in \mathfrak{m}$ be an element of the maximal ideal. Since I is an $\mathfrak{m}$-primary ideal and J is the minimal reduction of I then $\alpha^r \in J$ for an $r \gg 0$. For each pair of indexes p, K we have $\alpha^r \cdot \sigma_K^p \in JI^{p+1} \subseteq JI^p$, so $\alpha^r \cdot \overline{\sigma}_K^p = 0$ in $\frac{I^{p+1}}{JI^p}$. Then $\alpha^r \in Ann(\sigma) = \mathfrak{p}$, and, since $\mathfrak{p}$ is a prime ideal, we deduce $\alpha \in \mathfrak{p}$. Hence we have proved that $\mathfrak{m}A \subseteq \mathfrak{p}$.

Now we will prove that $(V_1,\ldots,V_\mu) \subseteq \mathfrak{p}$. Let us recall that

$$\overline{\sigma}_K^p V_i^s = \overline{\sigma}_K^p b_i^s t^s \in \frac{I^{p+1+s}}{JI^{p+s}}$$

where $I = (b_1,\ldots,b_\mu)$ and $s \geq 0$. Since J is a minimal reduction of I there exists an integer s_0 such that for all $s \geq s_0$ it holds $I^{s+1} = JI^s$. Hence if $s \geq s_0$ then $\frac{I^{p+1+s}}{JI^{p+s}} = 0$ for all $p \geq 0$ and we obtain $\overline{\sigma}_K^p V_i^s = 0$. Thus $V_i^s \in Ann(\sigma) = \mathfrak{p}$ and $V_i \in \mathfrak{p}$ because $\mathfrak{p}$ is a prime ideal. This proves that $(V_1,\ldots,V_\mu) \subseteq \mathfrak{p}$ and

$$Q = \mathfrak{m}A + (V_1,\ldots,V_\mu) \subseteq \mathfrak{p}.$$

It remains to prove that $Q = \mathfrak{p}$. Notice that $A/Q \cong (R/\mathfrak{m})[T_1,\ldots,T_d]$. If we assume that $Q \subsetneq \mathfrak{p}$ then there exists a polynomial $f \in \mathfrak{p} \setminus Q$. Since

$f \in \mathfrak{p} = Ann(\sigma)$ we have $f \cdot \sigma = 0$ and we can write $f = g + h$ with $g \in Q$ and $\overline{h} \neq 0$ in A/Q. Notice that h can be written as $h = \sum \overline{\alpha}_K T^K$ with $\alpha_K \in R \setminus \mathfrak{m}$. Since $g \in Q \subset Ann(\sigma)$ we have that $f\sigma = h\sigma = g\sigma = 0$, so $f \cdot \sigma = h\sigma$. As before we can write $\sigma = \sum_K \sigma_K T^K$ and $\sigma_K = \sum \overline{\sigma}_K^p$ with $\overline{\sigma}_K^p \in \frac{I^{p+1}}{JI^p}$, so $\sigma h = \sum_{K,L} \sigma_K \overline{\alpha}_L T^{K+L}$. Let us consider the lexicographic order in A. Let $Lt_T(\sigma) = \sigma_K T^K = (\sum_{p=0}^r \overline{\sigma}_K^p) T^K$ be the leading term of σ with $\overline{\sigma}_K^r \neq 0$ in $\frac{I^{r+1}}{JI^r}$ and $Lt(h) = \overline{\alpha}_L T^L$ be the leading term of h. Then the leading term of $\sigma \cdot h$ is $Lt_T(\sigma \cdot h) = \sigma_K \overline{\alpha}_L T^{K+L}$ because $\sigma_K \overline{\alpha}_L \neq 0$. In fact, $\sigma_K \overline{\alpha}_L = \sum_{p=0}^r \overline{\sigma}_K^p \overline{\alpha}_L \neq 0$ because, as α_L is an invertible element of R, we have that $\overline{\sigma}_K^r \overline{\alpha}_L \neq 0$ in $\frac{I^{r+1}}{JI^r}$. Thus, $\sigma \cdot h \neq 0$. Hence, $f \cdot \sigma \neq 0$ which contradicts the fact that $f \in \mathfrak{p} = Ann(\sigma)$. Therefore, we have proved that

$$Ass_A(\mathcal{M}^{I,J}) = \{\mathfrak{m}A + (V_1, \ldots, V_\mu)\}$$

Clearly, $Ann(\mathcal{M}^{I,J}) \subseteq Q$. And with the same previous proof, we have that $Q \subseteq Ann(\mathcal{M}^{I,J})$. Thus, $Ann(\mathcal{M}^{I,J}) = Q$. Then

$$\begin{aligned}\dim(\mathcal{M}^{I,J}) &= \dim(A/Ann(\mathcal{M}^{I,J})) = \dim(A/Q) \\ &= \dim((R/\mathfrak{m})[T_1, \ldots, T_d]) = d\end{aligned}$$

and since $T_1, \ldots, T_d$ is a regular sequence in A, we have that $\mathcal{M}^{I,J}$ is a Cohen-Macaulay module of dimension d. □

Remark 5.4.4. Observe that the length, as an R-module, of

$$\Sigma^{I,J}_{(m+1,*)} \cong \bigoplus_{j=0}^{j=m} \frac{I^{m+1-j}J^j}{I^{m-j}J^{j+1}} t^{m+1} U^j$$

is equal to the length of

$$S_J(I)_m \oplus \frac{IJ^m}{J^{m+1}},$$

where $S_J(I)_m$ is the degree m piece of the Sally module $S_J(I)$.

On the other hand

$$\Sigma^{I,J}_{[p]} = gr_{Jt}(\mathcal{R}(I))_{[p]} = \bigoplus_{i \geq 0} \frac{J^i I^{p+1}}{J^{i+1} I^p} t^{p+1+i} U^i$$

is isomorphic as $\mathcal{R}(J)$-module to the module L_p defined by Vaz-Pinto in [Vaz95] for all $p \geq 1$.

In several papers of Wang appear the modules $T_{k,n}, S_{k,n}$ defined as the kernel and co-kernel of some exact sequence, see [Wan97a], [Wan97b], [Wan00], [Wan01] and [Wan02]. One of the key results of Wang's papers is to prove that there exist the Hilbert functions of $T_{k,n}$ and $S_{k,n}$. In our framework these results follows from the following commutative diagram:

$$
\begin{array}{ccccccc}
 & & & \bigoplus_{j=1}^{\binom{n+d-1}{d-1}} \frac{I^j}{JI^{j-1}} & & & \\
 & & \nearrow & & \searrow & & \\
0 \longrightarrow & T_{k,n} = K^{I,J}_{(k+n,n)} & & \cong \downarrow & & S_{k,n} = \Sigma^{I,J}_{(k+n,n)} & \longrightarrow 0 \\
 & & \searrow & & \nearrow & & \\
 & & & \mathcal{M}^{I,J}_{(k+n,n)} & & &
\end{array}
$$

Finally the $\mathcal{R}(J)$-module

$$C_p = \bigoplus_{i \geq 1} \frac{I^{i+p}}{J^i I^p},$$

defined by Vaz-Pinto, can be linked to $\Sigma^{I,J}_{\geq p}$ by means of the following sets of exact sequences of $\mathcal{R}(J)$-modules, [Vaz95], [Vaz97]. Here we set $r = r_J(I)$. There exist two sets of exact sequences of $\mathcal{R}(J)$-modules:

$$
\begin{array}{ccccccccc}
0 & \to & \Sigma^{I,J}_{[1]} & \to & S_J(I) = C_1 & \to & C_2 & \to & 0 \\
0 & \to & \Sigma^{I,J}_{[2]} & \to & C_2 & \to & C_3 & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & \Sigma^{I,J}_{[r-2]} & \to & C_{r-2} & \to & C_{r-1} = \Sigma^{I,J}_{[r-1]} & \to & 0
\end{array}
\tag{SQ1}
$$

and

$$\begin{array}{ccccccccc}
0 & \to & \Sigma^{I,J}_{[0]} & \to & \Sigma^{I,J} = \Sigma^{I,J}_{\geq 0} & \to & \Sigma^{I,J}_{\geq 1} & \to & 0 \\
0 & \to & \Sigma^{I,J}_{[1]} & \to & \Sigma^{I,J}_{\geq 1} & \to & \Sigma^{I,J}_{\geq 2} & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & \Sigma^{I,J}_{[r-2]} & \to & \Sigma^{I,J}_{\geq r-2} & \to & \Sigma^{I,J}_{\geq r-1} = \Sigma^{I,J}_{[r-1]} & \to & 0
\end{array} \tag{SQ2}$$

5.5 Multiplicities of the bigraded Sally module

We would like to compute the Hilbert functions of the bigraded modules defined in the previous section. These functions will be of polynomial type in a region of $\mathbb{N}^2$, and the coefficients of such polynomials will be the object of study in this section.

Given a bigraded A-module M, where $A = R/J[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$, with $\deg(V_i) = (1,0)$ and $\deg(T_i) = (1,1)$, we can consider the Hilbert function of M defined as

$$h_M(m,n) = \sum_{0 \leq j \leq n} \operatorname{length}_A(M_{(m,j)}).$$

By Theorem 5.3.4, there exist integers $f_{i,j}(M) \in \mathbb{Z}$, $i \geq 0$, $j \geq 0$, with $i + j \leq c - 1$, for some integer $c \geq 0$, such that the polynomial

$$p_M(m,n) = \sum_{i+j \leq c-1} f_{i,j}(M) \binom{m}{i} \binom{n}{j}$$

verifies $p_M(m,n) = h_M(m,n)$ for all $m \geq m_0$ and $n \geq n_0 + m$ for some integers $m_0, n_0 \geq 0$.

Next lemma shows that in the case when the bigraded module has a finite number of diagonals, the Hilbert polynomial does not depend on the second indeterminate, so it will be a polynomial on m. This fact is also consequence of not having generators of degree $(0,1)$ as it can be seen in the proof of Theorem 5.3.4.

Lemma 5.5.1. *Let M be a bigraded A-module for which there exist integers $a \leq b$ such that $M_{[p]} = 0$ for all $p \notin [a,b]$. Then $f_{i,j}(M) = 0$ for all $j \geq 1$.*

Proof. Since $M_{[p]} = \bigoplus_{m-n=p+1} M_{(m,n)}$, if $M_{[p]} = 0$ for all $p \notin [a,b]$, we have that $M_{(m,n)} = 0$ for all pair (m,n) such that $m-n > b+1$ or $m-n < a+1$. If we take $m \geq m_0$, $n \geq n_0 + m$ we have

$$h_M(m,n) = p_M(m,n) = \sum_{i+j \leq c-1} f_{i,j}(M) \binom{m}{i} \binom{n}{j}.$$

We can suppose that $m \geq b+1$ and $n \geq m-b-1$, so we have

$$\begin{aligned} h_M(m,n) &= \sum_{0 \leq j \leq n} \text{length}_A(M_{(m,j)}) \\ &= \sum_{m-b-1 \leq j \leq m-a-1} \text{length}_A(M_{(m,j)}) \\ &= \sum_{j \in \mathbb{Z}} \text{length}_A(M_{(m,j)}) \end{aligned}$$

because $M_{(m,j)} = 0$ when $j < m-b-1$ and $j > m-a-1$. Hence, for m,n large enough we have that

$$h_M(m,n) = p_M(m,n) = h_M(m) = p_M(m).$$

Therefore

$$\sum_{i+j \leq c-1} f_{i,j}(M) \binom{m}{i} \binom{n}{j} = \sum_i a_i \binom{m}{i}.$$

Since $\{\binom{m}{i}\binom{n}{j}\}_{i,j}$ is a basis of the polynomial ring in m and n, we have that $f_{i,j}(M) = 0$ for all $j \geq 1$. □

Given an A-module under the hypothesis of the above Lemma we can write

$$p_M(m,n) = p_M(m) = \sum_{i=0}^{c-1} f_{i,0}(M) \binom{m+c-i}{c-i},$$

and $h_M(m,n) = p_M(m)$ for $m \geq m_0$ and $n \geq n_0 + m$, for some integers m_0, n_0. From Lemma 5.4.1 we see that we can apply the last result to the A-modules $\Sigma^{I,J}$, $\mathcal{M}^{I,J}$, and $K^{I,J}$, since they are composed by a finite number of diagonals.

Proposition 5.5.2. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. Then*

$$p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1}.$$

Proof. Since the length of $\Sigma^{I,J}_{(m+1,*)}$ is equal to the length of $S_J(I)_m \oplus \frac{IJ^m}{J^{m+1}}$ as R-modules, we have that

$$h_{\Sigma^{I,J}}(m,n) = \text{length}_R(S_{m-1}) + \text{length}_R\left(\frac{IJ^{m-1}}{J^m}\right)$$

for all $m \geq m_0$, $n \geq n_0 + m$, for some integers m_0, n_0.

Since $gr_J(R) \cong (R/J)[T_1, \dots, T_d]$, then we clearly get that $Igr_J(R) \cong (I/J)[T_1, \dots, T_d]$. Thus, the $\text{length}_R(\frac{IJ^{m-1}}{J^m})$ coincides with the length of the piece of degree $m-1$ of $(I/J)[T_1, \dots, T_d]$. So we have

$$\text{length}_R(\frac{IJ^{m-1}}{J^m}) = \text{length}_R(I/J)\binom{m-1+d-1}{d-1}.$$

Hence we deduce that for all $m \geq m_0$, $n \geq n_0 + m$, by Lemma 5.5.1,

$$p_{\Sigma^{I,J}}(m) = p_{\Sigma^{I,J}}(m,n) = p_{S_J(I)}(m-1) + \text{length}_R(I/J)\binom{m-1+d-1}{d-1}.$$

Let us recall that there exist integers $s_0, \dots, s_{d-1}$ such that

$$p_{S_J(I)}(n) = \sum_{i=0}^{d-1} (-1)^i s_i \binom{n+d-i-1}{d-i-1}$$

and that $s_0 = e_1(I) - \text{length}_R(I/J)$, $s_i = e_{i+1}(I)$ for $i = 1, \dots, d-1$, [Vas94b]. Hence, we have

$$\begin{aligned} p_{S_J(I)}(m-1) = &\sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1} \\ &- \text{length}_R(I/J)\binom{m-1+d-1}{d-1} \end{aligned}$$

and then

$$\begin{aligned} p_{\Sigma^{I,J}}(m) =& p_{S_J(I)}(m-1) + \text{length}_R(I/J)\binom{m-1+d-1}{d-1} \\ =& \sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1} \end{aligned}$$

□

Since the Hilbert polynomial of these bigraded modules depends only on the first indeterminate, it makes sense to consider the coefficients of the polynomials as multiplicities, because, in fact, it is as if were considering the modules as $\mathbb{Z}$-graded modules with respect to the first term of the bidegrees.

In the next proposition we compute the multiplicities of the modules $\mathcal{M}^{I,J}$, $\Sigma^{I,J}$ and $K^{I,J}$. Notice that they are related with the integer $\Lambda(I,J)$ defined in the introduction. From now on we consider the integer

$$\delta(I,J) = \Lambda(I,J) - e_1(I).$$

Proposition 5.5.3. *Let $(R,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. The following conditions hold:*

(i) $\deg(p_{\mathcal{M}^{I,J}}) = d-1$ *and* $e_0(\mathcal{M}^{I,J}) = \Lambda(I,J)$.

(ii) If $\Sigma^{I,J} = 0$ *then* $gr_I(R)$ *is a Cohen-Macaulay ring. If* $\Sigma^{I,J} \neq 0$ *then* $\deg(p_{\Sigma^{I,J}}) = d-1$ *and* $e_0(\Sigma^{I,J}) = e_1(I)$.

(iii) $e_0(K^{I,J}) = \delta(I,J)$. *If* $K^{I,J} \neq 0$ *then* $\deg(p_{K^{I,J}}) = d-1$. *In particular,*

$$\Lambda(I,J) \geq e_1(I).$$

Proof. (i) We know that the module $\mathcal{M}^{I,J}$ is Cohen-Macaulay of dimension d, Proposition 5.4.3. By Lemma 5.4.1 and Lemma 5.5.1 we have that $p_{\mathcal{M}^{I,J}}(m,n) = p_{\mathcal{M}^{I,J}}(m)$, and so $\deg(p_{\mathcal{M}^{I,J}}) = d-1$. Since we know that $\mathcal{M}^{I,J} \cong \bigoplus_{p\geq 0} I^{p+1}/I^pJ\, t^{p+1}[T_1,\dots,T_d]$ we have that

$$\begin{aligned}
h_{\mathcal{M}^{I,J}}(m,n) &= \sum_{i=0}^{n} \mathrm{length}_R \left(\mathcal{M}^{I,J}_{(m,i)}\right) \\
&= \sum_{i=0}^{n} \mathrm{length}_R \left(\frac{I^{m-i}}{I^{m-i-1}J}[T_1,\dots,T_d]_i\right) \\
&= \mathrm{length}_R \left(\frac{I^{m}}{I^{m-1}J}[T_1,\dots,T_d]_0\right) + \mathrm{length}_R \left(\frac{I^{m-1}}{I^{m-2}J}[T_1,\dots,T_d]_1\right) \\
&\quad + \cdots + \mathrm{length}_R \left(\frac{I^{m-n}}{I^{m-n-1}J}[T_1,\dots,T_d]_n\right).
\end{aligned}$$

Notice that for $n \geq m \gg 0$ we have

$$\begin{aligned} p_{\mathcal{M}^{I,J}}(m,n) = h_{\mathcal{M}^{I,J}}(m,n) &= \operatorname{length}_R\left(\frac{I^m}{I^{m-1}J}[T_1,\ldots,T_d]_0\right) + \\ &+ \operatorname{length}_R\left(\frac{I^{m-1}}{I^{m-2}J}[T_1,\ldots,T_d]_1\right) + \\ &+ \cdots + \operatorname{length}_R\left(\frac{I}{J}[T_1,\ldots,T_d]_{m-1}\right). \end{aligned}$$

Moreover, for $m \geq r_J(I)$ we have

$$\begin{aligned} p_{\mathcal{M}^{I,J}}(m,n) &= \sum_{i\geq 1} \operatorname{length}_R\left(\frac{I^i}{I^{i-1}J}[T_1,\ldots,T_d]_{m-i}\right) \\ &= \sum_{i\geq 1} \operatorname{length}_R\left(\frac{I^i}{I^{i-1}J}\right)\binom{m-i+d-1}{d-1}. \end{aligned}$$

Each binomial number is a polynomial in m of degree $d-1$, and the leading coefficient of this polynomial $p_{\mathcal{M}^{I,J}}$ gives us the multiplicity

$$e_0(\mathcal{M}^{I,J}) = \sum_{i\geq 1} \operatorname{length}_R\left(\frac{I^i}{I^{i-1}J}\right) = \Lambda(I,J).$$

(ii) If $\Sigma^{I,J} = 0$ then $S_J(I) = 0$ and $gr_I(R)$ is Cohen-Macaulay, [Vaz95] Remark 5.4.4. Let us assume that $\Sigma^{I,J} \neq 0$. Since, by Proposition 5.5.2,

$$p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1}(-1)^i e_{i+1}(I)\binom{m-1+d-i-1}{d-i-1},$$

then $\deg(p_{\Sigma^{I,J}}) = d-1$ and $e_0(\Sigma^{I,J}) = e_1(I)$.
(iii) From the additivity of the multiplicity in (S) and by (i) and (ii) we get

$$\begin{aligned} e_0(K^{I,J}) &= e_0(\mathcal{M}^{I,J}) - e_0(\Sigma^{I,J}) \\ &= \Lambda(I,J) - e_1(I). \end{aligned}$$

If $K^{I,J} \neq 0$ then $\deg(p_{K^{I,J}}) = d-1$. Note that $\deg(p_{\Sigma^{I,J}}) = \deg(p_{\mathcal{M}^{I,J}}) = d-1$, so $e_0(K^{I,J}) \geq 0$ and then $\Lambda(I,J) \geq e_1(I)$. $\square$

So, we have interpreted the integers $e_1(I)$, $\Lambda(I,J)$ and $\delta(I,J)$ as multiplicities of the bigraded modules $\Sigma^{I,J}$, $\mathcal{M}^{I,J}$ and $K^{I,J}$, respectively. We recall that these integers are the ones that appear at the Wang's Conjecture.

A similar result can be obtained in terms of the diagonals of such bigraded modules. This interpretation will play an important role to improve Wang's Conjecture on the depth of the blow-up algebras in the next section.

Proposition 5.5.4. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. The following conditions hold:*

(i) For all $p \geq 0$

$$e_0(\Sigma^{I,J}_{[p]}) = \text{length}_R\left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]}) \geq 0,$$

and

$$e_1(I) = \sum_{p\geq 0}(\text{length}_R\left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]})).$$

(ii) For all $p \geq 0$

$$\text{length}_R\left(\frac{I^{p+1}\cap J}{JI^p}\right) \geq e_0(K^{I,J}_{[p]}),$$

and

$$\delta(I,J) = e_0(K^{I,J}) = \sum_{p\geq 0} e_0(K^{I,J}_{[p]}) \geq 0.$$

Proof. (i) From Proposition 5.5.3 we have

$$e_1(I) = e_0(\Sigma^{I,J}) = \sum_{p\geq 0}^{*} e_0(\Sigma^{I,J}_{[p]}),$$

where $*$ stands for the integers p such that $\deg(p_{\Sigma^{I,J}_{[p]}}) = d-1$.

From the exact sequence of R-modules

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \frac{I^{p+1}}{JI^p}[T_1,\ldots,T_d] \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0,$$

we deduce that if $\deg(p_{\Sigma^{I,J}_{[p]}}) < d-1$ then $\text{length}_R(I^{p+1}/JI^p) = e_0(K^{I,J}_{[p]})$. Let us assume $\deg(p_{\Sigma^{I,J}_{[p]}}) = d-1$, from the additivity of the multiplicity we deduce

$$e_0(\Sigma^{I,J}_{[p]}) = \text{length}_R\left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]}),$$

so

$$e_1(I) = e_0(\Sigma^{I,J}) = \sum_{p\geq 0}{}^{*} e_0(\Sigma^{I,J}_{[p]})$$
$$= \sum_{p\geq 0}(\text{length}_R\left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]})).$$

(ii) Since $K^{I,J}_{[p]}$ is the kernel of the morphism

$$\frac{I^{p+1}}{JI^p}[T_1,\ldots,T_d] \xrightarrow{\pi_p} \bigoplus_{i\geq 0}\frac{I^{p+1}J^i}{I^pJ^{i+1}}$$

which sends each T_j to the correspondent a_j. Considering the morphism on each piece, we have that

$$\frac{I^{p+1}}{JI^p}[T_1,\ldots,T_d]_l \longrightarrow \frac{I^{p+1}J^l}{I^pJ^{l+1}}.$$

$K^{I,J}_{[p]} = \text{Ker}(\pi_p)$, so an element $\overline{m_j}T_{j_1}\ldots T_{j_l} \in K_{[p]}$ is sent to $m_j a_{j_1}\ldots a_{j_l} \in I^pJ^{l+1}$. Since $J = (a_1,\ldots,a_d)$, we have that $m_j \in (I^pJ^{l+1} : J^l) \subseteq (J^{l+1} : J^l)$. Now, since $gr_J(R) \cong R/J[T_1,\ldots,T_d]$, we have that $(J^{l+1} : J^l) \subseteq J$, and hence $m_j \in J$. Then clearly $K^{I,J}_{[p]}$ is a submodule of

$$\frac{I^{p+1}\cap J}{JI^p}[T_1,\ldots,T_d].$$

From this we deduce

$$\text{length}_R\left(\frac{I^{p+1}\cap J}{JI^p}\right) \geq e_0(K^{I,J}_{[p]}).$$

The second part of the claim follows from Proposition 5.5.3 (iii)

$$\delta(I,J) = \Lambda(I,J) - e_1(I)$$
$$= e_0(K^{I,J})$$
$$= \sum_{p\geq 0} e_0(K^{I,J}_{[p]}) \geq 0.$$

□

Remark 5.5.5. Notice that some of the results on $T_{k,n} = K^{I,J}_{(k+n,n)}$ in the papers [Wan97a], [Wan97b], [Wan00], [Wan01] and [Wan02] are corollaries of Proposition 5.5.3 and Proposition 5.5.4.

Let us recall that, from [Wan00] and [HM97], we have

$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

In the next result we show that these inequalities can be deduced from some *local* inequalities. For all $p \geq 0$ we define the following the integers

$$\Delta_p(I,J) = \mathrm{length}_R\left(\frac{I^{p+1} \cap J}{JI^p}\right),$$
$$\delta_p(I,J) = e_0(K^{I,J}_{[p]}),$$
$$\Lambda_p(I,J) = \mathrm{length}_R\left(\frac{I^{p+1}}{JI^p}\right).$$

From the last result we deduce:

Proposition 5.5.6. *Let $(R,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. For all $p \geq 0$ the following inequalities hold*

$$\Delta_p(I,J) \geq \delta_p(I,J) = \Lambda_p(I,J) - e_0(\Sigma^{I,J}_{[p]}) \geq 0.$$

Adding these inequalities with respect to p we get

$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

5.6 On the depth of blow-up algebras

The aim of this section is to prove a refined version of Wang's Conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta = \sum_{p\geq 0} \delta_p(I,J)$, Theorem 5.6.3. As a by-product we recover the known cases of Wang's Conjecture, Corollary 5.6.7, and we prove a weak version of Sally's Conjecture, Corollary 5.6.9.

A second goal is to answer a question of Guerrieri and Huneke, [Gue93]. They wondered if the depth of $gr_I(R)$ would be at least $d-1$ assuming that $\Delta_p(I,J) \leq 1$ for all $p \geq 1$. As explained before, Wang found a counterexample to the question, [Wan02]. We can improve this bound in the general case by proving that under these assumptions, $\operatorname{depth}(gr_I(R)) \geq d-2$, Theorem 5.6.5.

Recall that throughout the chapter we are considering the case in which $(R, \mathfrak{m}, \mathbf{k})$ is a local ring of dimension $d > 0$ with infinite residue field $\mathbf{k}$, and I is an $\mathfrak{m}$-primary ideal of R with a minimal reduction J.

To prove the main theorem, we need to study the depth of the associated bigraded ring $gr_{Jt}(\mathcal{R}(I))$. The following theorem will be crucial in order to refine the Wang's Conjecture.

Inspired by Polini'proof, the theorem generalizes Claim 3 of [Pol00].

Theorem 5.6.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 3$. Let I be an $\mathfrak{m}$-primary ideal of R and let J be a minimal reduction of I. Let us assume that $K^{I,J} \neq 0$, and either $K^{I,J}_{[p]} = 0$ or $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1, \ldots, T_d]$-module for $p \geq 0$. Then*

$$\operatorname{depth}(gr_{Jt}(\mathcal{R}(I))) \geq d-1.$$

Proof. Let $p_1 < \ldots < p_n$ be the sequence of integers such that $K^{I,J}_{[p_i]} \neq 0$, $i = 1, \ldots, n$. By Lemma 5.4.1 we have a finite number of these integers. Hence, from the sequence $(\mathrm{S}_{[p]})$ we get

$$\Sigma^{I,J}_{[p]} \cong \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1, \ldots, T_d]$$

for $p \neq p_1, \ldots, p_n$, and the following exact sequences of R-modules

$$0 \longrightarrow K^{I,J}_{[p_i]} \longrightarrow \frac{I^{p_i+1}}{JI^{p_i}}[T_1, \ldots, T_d] \longrightarrow \Sigma^{I,J}_{[p_i]} \longrightarrow 0, \qquad (\mathrm{S}_{[p_i]})$$

$i = 1, \ldots, n$. Notice that by hypothesis $K^{I,J}_{[p_i]}$ is isomorphic to an ideal I_i of $D = \mathbf{k}[T_1, \ldots, T_d]$, $i = 1, \ldots, n$.

Let $\mathfrak{p}$ be a height $h \geq 2$ prime ideal of D. Since $D = \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J)$, there exists a prime ideal $\mathfrak{q}$ of $\mathcal{R}(J)$ such that $\mathfrak{p} = \mathfrak{q}/\mathfrak{m}\mathcal{R}(J)$.

Since $\operatorname{depth}_{\mathfrak{q}}(S_J(I)) \geq 1$, [Pol00], by depth counting in the set of exact sequences of $\mathcal{R}(J)$-modules (SQ1) we get that $\operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq 1$. In fact, for $i < p_1$ we have that $\Sigma^{I,J}_{[i]} \cong \mathcal{M}^{I,J}_{[i]}$, so $\operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[i]}) \geq d > 2$. Thus, if $p_1 \neq 1$ then

$$\operatorname{depth}_{\mathfrak{q}}(C_2) \geq \min\{\operatorname{depth}_{\mathfrak{q}}(C_1), \operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[1]}) - 1\} \geq 1.$$

Hence, while $i < p_1$ we have that $\operatorname{depth}_{\mathfrak{q}}(C_{i+1}) \geq 1$, and this implies that

$$\operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq \min\{\operatorname{depth}_{\mathfrak{q}}(C_{p_1}), \operatorname{depth}_{\mathfrak{q}}(C_{p_1+1}) + 1\} \geq 1.$$

Otherwise, if $p_1 = 1$ then

$$\operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[1]}) \geq \min\{\operatorname{depth}_{\mathfrak{q}}(C_1), \operatorname{depth}_{\mathfrak{q}}(C_2) + 1\} \geq 1.$$

Hence we have $\operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq 1$.

Depth counting on ($S_{[p_i]}$) yields

$$\operatorname{depth}_{\mathfrak{q}}(K^{I,J}_{[p_1]}) \geq \min\{\operatorname{depth}_{\mathfrak{q}}(\frac{I^{p_1+1}}{JI^{p_1}}[T_1, \ldots, T_d]), \operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) + 1\} \geq 2$$

because $\operatorname{depth}_{\mathfrak{q}}(\frac{I^{p_1+1}}{JI^{p_1}}[T_1, \ldots, T_d]) \geq d$ and $\operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq 1$. Then

$$\operatorname{depth}(I_1)_{\mathfrak{p}} = \operatorname{depth}(I_1)_{\mathfrak{q}} \geq \operatorname{depth}_{\mathfrak{q}}(K^{I,J}_{[p_1]}) \geq 2,$$

[Mat80]. In particular we have that $\mathfrak{p} \notin Ass_D(I_1)$, because $\operatorname{depth}_{\mathfrak{p}}(I_1) \geq 2$, so I_1 is an unmixed ideal of D of height one. In fact, all the associated primes of I_1 have height ≤ 1, and being D a domain and $I_1 \neq 0$, the associated ideals of I_1 are height 1 ideals. Since D is factorial we deduce that $I_1 \subset D$ is principal, and then $\operatorname{depth}(K^{I,J}_{[p_1]}) = d$.

Since $\operatorname{depth}_{\mathfrak{q}}(\frac{I^{p_1+1}}{JI^{p_1}}[T_1, \ldots, T_d]) \geq d$ and $\operatorname{depth}_{\mathfrak{q}}(K^{I,J}_{[p_1]}) = d$, by depth counting on ($S_{[p_i]}$), we deduce that $\operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq d - 1 \geq 2$.

By depth counting on (SQ1) we get that $\operatorname{depth}_{\mathfrak{q}}(C_{p_1+1}) \geq 1$. In fact,

$$\operatorname{depth}_{\mathfrak{q}}(C_{p_1+1}) \geq \min\{\operatorname{depth}_{\mathfrak{q}}(C_{p_1}), \operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) - 1\} \geq 1,$$

since $\operatorname{depth}_{\mathfrak{q}}(C_{p_1}) \geq 1$. Then we can iterate this process, and we get that $\operatorname{depth}_{\mathfrak{q}}(\Sigma^{I,J}_{[p]}) \geq d - 1$ for all p, in particular we get that the $\mathcal{R}(J)$-module $\Sigma^{I,J}$ verifies

$$\operatorname{depth}(\Sigma^{I,J}_{[p]}) \geq d - 1.$$

From the last row of the sequence (SQ2) we get that

$$\operatorname{depth}(\Sigma^{I,J}_{\geq r-2}) \geq \min\{\operatorname{depth}(\Sigma^{I,J}_{[r-2]}), \operatorname{depth}(\Sigma^{I,J}_{[r-1]})\} \geq d-1.$$

Iterating this process in (SQ2) we deduce that

$$\operatorname{depth}(\Sigma^{I,J}) \geq d-1.$$

Now, let us consider the exact sequence of $\mathcal{R}(J)$-modules

$$0 \longrightarrow \mathcal{R}(J) \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow \Sigma^{I,J} \longrightarrow 0.$$

By depth counting in this sequence we prove the claim, because

$$\operatorname{depth}(gr_{Jt}(\mathcal{R}(I))) \geq \min\{\operatorname{depth}(\mathcal{R}(J)), \operatorname{depth}(\Sigma^{I,J})\}$$

with $\operatorname{depth}(\mathcal{R}(J)) = d+1$ and $\operatorname{depth}(\Sigma^{I,J}) \geq d-1$. □

The following lemma it is also important for the main theorem, because it allow us to determine the diagonal $K^{I,J}_{[p]}$ in case $e_0(K^{I,J}_{[p]}) = 1$. This way we may determine $K^{I,J}$ in the decomposition of $\delta(I,J)$ considered in the main theorem.

Lemma 5.6.2. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. If $\delta_p(I,J) = 1$ then $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1,\ldots,T_d]$-module.*

Proof. Let us recall that

$$K^{I,J}_{[p]} \subset \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1,\ldots,T_d].$$

We denote by $K^{I,J}_{[p],j}$ the homogeneous piece of degree j of $K^{I,J}_{[p]}$ with respect to $T_1,\ldots,T_d$. Since $\frac{I^{p+1}}{JI^p}$ is a finite length R-module there exists a composition series

$$0 = N_l \subset N_{l-1} \subset \cdots \subset N_0 = \frac{I^{p+1}}{JI^p}$$

such that N_i is a sub-R-module of I^{p+1}/JI^p and $N_i/N_{i+1} \cong \mathbf{k}$ for all $i = 0,\ldots,l-1$, i.e. $\operatorname{length}_R(I^{p+1}/JI^p) = l$. Hence we have a sequence of

$R[T_1,\dots,T_d]$-modules

$$0 = N_l[T_1,\dots,T_d] \subset N_{l-1}[T_1,\dots,T_d] \subset \dots$$
$$\dots \subset N_0[T_1,\dots,T_d] = \frac{I^{p+1}}{JI^p}[T_1,\dots,T_d].$$

We denote by

$$W_i = \frac{K^{I,J}_{[p]} \cap N_i[T_1,\dots,T_d]}{K^{I,J}_{[p]} \cap N_{i+1}[T_1,\dots,T_d]} \subset \frac{N_i[T_1,\dots,T_d]}{N_{i+1}[T_1,\dots,T_d]} = \mathbf{k}[T_1,\dots,T_d]$$

Since $e_0(K^{I,J}_{[p]}) = 1$ we have that $K^{I,J}_{[p]} \neq 0$. If $W_i = 0$ for all $i = 0,\dots,l$ then $K^{I,J}_{[p]} = 0$, so there exists a set of indexes $0 \leq i_1 \leq \dots \leq i_s \leq l$ such that $W_{i_j} \neq 0$, $j = 1,\dots,s$.

We denote by $W_{i,m}$ the degree m piece of W_i with respect to $T_1,\dots,T_d$. Let m_0 be an integer such that for all $j = 1,\dots,s$ we have $W_{i_j,m_0} \neq 0$. This integer exists. In fact, for each $W_{i_j} \neq 0$, there exists an integer m_{i_j} such that $W_{i_j,m_{i_j}} \neq 0$. Then, for any $t \geq 0$ we have that $W_{i_j,m_{i_j}+t} \neq 0$. So, we can choose the maximum of these $m_{i_1},\dots,m_{i_s}$. Then we have

$$\begin{aligned} \text{length}_R(K^{I,J}_{[p],m}) &= \sum_{j=1}^{s} \text{length}_R(W_{i_j,m}) \\ &= \sum_{j=1}^{s} \text{length}_R(W_{i_j,m_0} \cdot W_{i_j,m-m_0}) \\ &\geq \sum_{j=1}^{s} \text{length}_R(W_{i_j,m-m_0}) \\ &\geq s\binom{m-m_0+d-1}{d-1} \end{aligned}$$

for all $m \geq m_0$.

Since $e_0(K^{I,J}_{[p]}) = 1$ we deduce that $s = 1$, because $\deg(p_{K^{I,J}_{[p]}}) = d-1$ and the binomial number is a polynomial of the same degree. Hence, $W_i = 0$ for all $i \neq i_1$. Therefore, $K^{I,J}_{[p]} \cap N_i[T_1,\dots,T_d] = 0$ for $i = i_1+1,\dots,l$ and $K^{I,J}_{[p]} \cap N_i[T_1,\dots,T_d] = K^{I,J}_{[p]}$ for $i = 0,\dots,i_1$. From this we get that $K^{I,J}_{[p]} = W_{i_1} \subset \mathbf{k}[T_1,\dots,T_d]$, so $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1,\dots,T_d]$-module. $\square$

Next theorem is one of the main results of this chapter. We prove a refined version of Wang's Conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta(I,J) = \sum_{p\geq 0}\delta_p(I,J)$. This theorem, allows to recover the known cases of the Wang's Conjecture as well as to add new ones. Then we will be able to give an answer to a question raised by Guerrieri and Huneke.

Let us consider $\bar{\delta}(I,J)$ to be the maximum of the integers $\delta_p(I,J)$ for $p \geq 0$.

Theorem 5.6.3. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d>0$. Let I be an $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. If $\bar{\delta}(I,J) \leq 1$, then*

$$\operatorname{depth}(\mathcal{R}(I)) \geq d - \bar{\delta}(I,J)$$

and

$$\operatorname{depth}(gr_I(R)) \geq d - 1 - \bar{\delta}(I,J).$$

Proof. If $\bar{\delta}(I,J) = 0$ then $K^{I,J} = 0$, Proposition 5.5.3. The exact sequence (S) shows that $\mathcal{M}^{I,J} \cong \Sigma^{I,J}$ as A-modules, so $\operatorname{depth}(\Sigma^{I,J}) = d$, Proposition 5.4.3. Let us consider the exact sequence of A-modules

$$0 \longrightarrow \Sigma^{I,J} \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow \mathcal{R}(J) \longrightarrow 0$$

Depth counting shows that $\operatorname{depth}(gr_{Jt}(\mathcal{R}(I))) \geq d$, since $\operatorname{depth}(\Sigma^{I,J}) = d$ and $\operatorname{depth}(\mathcal{R}(J)) = d+1$. Hence

$$\operatorname{depth}(\mathcal{R}(I)) \geq d.$$

If $gr_I(R)$ is a Cohen-Macaulay ring then $\operatorname{depth}(gr_I(R)) = d \geq d-1$. Otherwise, when $\operatorname{depth}(gr_I(R)) < d = \operatorname{depth}(R)$, from [HM94],

$$\operatorname{depth}(gr_I(R)) = \operatorname{depth}(\mathcal{R}(I)) - 1 \geq d-1$$

which proves the claim.

Suppose that $\bar{\delta}(I,J) = 1$. When $d = 1,2$ we have the claim easily. So we assume $d \geq 3$. From Lemma 5.6.2 and Theorem 5.6.1 we get $\operatorname{depth}(gr_{Jt}(\mathcal{R}(I))) \geq d-1$, so

$$\operatorname{depth}(\mathcal{R}(I)) \geq d-1.$$

Now, if $gr_I(R)$ is Cohen-Macaulay, then $\text{depth}(gr_I(R)) = d \geq d-2$. Otherwise, if $\text{depth}(gr_I(R)) < d = \text{depth}(R)$ then, [HM94],

$$\text{depth}(gr_I(R)) = \text{depth}(\mathcal{R}(I)) - 1 \geq (d-1) - 1 = d-2$$

and the theorem is proved. □

Remark 5.6.4. The example of Wang in [Wan02], Example 3.13 (reproduced below), shows that the last result is sharp in the sense that we cannot expect to have $\text{depth}(gr_I(R)) \geq d-1$ provided that $\bar{\delta}(I,J) = 1$. Precisely, this is a counterexample for the question formulated by Guerrieri and Huneke in [Gue93]. They asked if it were true that $\text{depth}(gr_I(R)) \geq d-1$ for an $\mathfrak{m}$-primary ideal I in a d-dimensional Cohen-Macaulay ring provided that $\Delta_p(I,J) \leq 1$ $\forall p \geq 1$. Wang reformulated the question in the regular case.

Related to this question, we are able to give an answer to the Guerrieri and Huneke's question improving the bound for the Cohen-Macaulay case.

Theorem 5.6.5. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be and $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. If $\Delta_p(I,J) \leq 1$ for all $p \geq 1$, then*

$$\text{depth}(gr_I(R)) \geq d-2.$$

Proof. Observe that $\delta_p(I,J) \leq \Delta_p(I,J) \leq 1$ for all $p \geq 1$, Proposition 5.5.4. For $p = 0$ we have that $\delta_0(I,J) = e_0(K^{I,J}_{[0]}) = 0$ by Lemma 5.4.1. Then the claim follows from Theorem 5.6.3. □

Example 5.6.6. [Wang] Let $S = K[x_1,x_2,x_3,x_4,x_5]_{(x_1,x_2,x_3,x_4,x_5)}$, with K a field, and x_1,x_2,x_3,x_4,x_5 are indeterminates. Consider the ring $R = S/Q$, where $Q = (x_3^2, x_3x_4, x_3x_5, x_4x_5, x_4^2 - x_2x_3, x_5^3 - x_3x_1^2)S$. The maximal ideal of R is $\mathfrak{m} = (x_1,x_2,x_3,x_4,x_5)R$. Then $(R,\mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d = 2$. We consider $I = (x_1,x_2,x_4,x_5)R$ an $\mathfrak{m}$-primary ideal of R and $J = (x_1,x_2)R$ a minimal reduction of I.

With these ideals we have that

$$\Delta_1(I,J) = \text{length}_R\left(\frac{I^2 \cap J}{IJ}\right) = 1$$

$$\Delta_2(I,J) = \text{length}_R\left(\frac{I^3 \cap J}{I^2J}\right) = 1$$

and $r_J(I) = 3$ (i.e. $I^4 = I^3 J$). From this last fact, $\Delta_p(I,J) = 0$ for all $p \geq 3$. Hence, $\Delta_p(I,J) \leq 1$ for all $p \geq 1$ and, since $\delta_p(I,J) \leq \Delta_p(I,J)$, then $\bar{\delta}(I,J) \leq 1$.

Wang shows that $\operatorname{depth}(gr_I(R)) = 0$. This depth agrees with Theorem 5.6.3, since $d - 1 - \bar{\delta}(I,J) = 0$ in this case.

In the following corollary, we are able to prove Wang's Conjecture in the known cases, [Wan00], using the previous results and the bigraded modules defined before. Notice that in general we have

$$\delta(I,J) = \sum_{p \geq 0} \delta_p(I,J) \geq \bar{\delta}(I,J) = \max\{\delta_p(I,J) \mid p \geq 0\},$$

so from Theorem 5.6.3 we deduce:

Corollary 5.6.7. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal and J a minimal reduction of I. Then it holds*

$$\operatorname{depth}(gr_I(R)) \geq d - 1 - \delta(I,J)$$

for $\delta(I,J) = 0, 1$.

Proof. It is clear that $\delta(I,J) \geq \bar{\delta}(I,J)$, and then we apply Theorem 5.6.3. □

Let us recall that Valabrega and Valla characterized under which conditions $gr_I(R)$ is Cohen-Macaulay. They proved that given a minimal reduction J of I then $gr_I(R)$ is Cohen-Macaulay if and only if for all $n \geq 1$ the n-th Valabrega-Valla's condition holds, [VV78]:

$$I^n \cap J = I^{n-1} J,$$

i.e. $\Delta_{n-1}(I,J) = 0$.

There is another well-known conjecture that considers some conditions on the modules I^{p+1}/JI^p and $I^p \cap J/I^{p-1}J$, it is Sally's Conjecture:

Theorem 5.6.8 (Sally's Conjecture)**.** *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. If $I^n \cap J = I^{n-1}J$ for $n = 2, \ldots, t$ and $\operatorname{length}_R(\frac{I^{t+1}}{JI^t}) = \epsilon \leq \min\{1, d-1\}$ then it holds*

$$d - \epsilon \leq \operatorname{depth}(gr_I(R)) \leq d.$$

This conjecture was proved by Corso-Polini-Vaz-Pinto, Elias, and Rossi, [CPV98], [Eli99], [Ros00].

In the next result we prove a weak version of Sally's Conjecture as a corollary of the previous results in this section.

Corollary 5.6.9. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. If $I^n \cap J = I^{n-1}J$ for $n = 2, \ldots, t$ and $\text{length}(\frac{I^{t+1}}{JI^t}) = \epsilon \leq \min\{1, d-1\}$ then it holds*

$$d - 1 - \epsilon \leq \text{depth}(gr_I(R)) \leq d.$$

Proof. From Proposition 5.5.6 we get that $\delta_p(I, J) = 0$ for all $p \leq t - 1$.

Let us consider the finitely generated (R/J)-algebra $\mathcal{A}$

$$\mathcal{A} = \frac{\mathcal{R}(I)}{J + Jt\mathcal{R}(I)} = \frac{R}{J} \oplus \bigoplus_{n \geq 1} \frac{I^n}{JI^{n-1}} t^n.$$

Notice that $\mathcal{A}_{\geq 1}$ is the positive part of the degree zero piece with respect to U of $\Sigma^{I,J}$. We can consider the Hilbert function of $\mathcal{A}$, $n \geq 0$,

$$h_{\mathcal{A}}(n) = \text{length}_{R/J}(I^n / JI^{n-1}).$$

From [Bla98] and [BN99] we deduce that

$$h_{\mathcal{A}}(t + 1 + n) = \text{length}(\frac{I^{t+1+n}}{JI^{t+n}}) \leq \epsilon \leq 1$$

for all $n \geq 0$, so $\delta_p(I, J) \leq 1$ for all $p \geq t$, Proposition 5.5.6.

So, we know that $\delta_p(I, J) \leq 1$ for all $p \geq 0$, and using Theorem 5.6.3 we prove the claim. □

Chapter 6

Diagonals of $\Sigma^{I,J}$ and the growth of the Hilbert function

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$, let I be an $\mathfrak{m}$-primary ideal of R and $J \subseteq I$ a minimal reduction of I.

In the first part of this chapter, Section 6.1, we generalize the concept of the diagonal submodules $\Sigma^{I,J}_{[p]}$ of the bigraded Sally module $\Sigma^{I,J}$ associated to I and J defined in Chapter 5. The diagonals $\Sigma^{I,J}_{[p]}$ were used in order to estimate the depth of $gr_I(R)$. Here we consider a more general definition. We define a submodule $D_{l_\alpha}(\Sigma^{I,J})$ of $\Sigma^{I,J}$ with respect to a line l_α. It will be a graded module and we will study the growth of its Hilbert function considering some hypothesis on the minimal number of generators of some pieces of these diagonals, see Proposition 6.1.6.

The second aim of this chapter, Section 6.2, is to study the growth of the Hilbert function h_I of an $\mathfrak{m}$-primary ideal I in the one-dimensional case. We take advantage of the structure of the modules I^{n+1}/JI^n to study the growth of h_I, see Corollary 6.2.1. The direct sum of these modules corresponds to a diagonal submodule $D_{l_{\alpha_B}}(\Sigma^{I,J})$ of $\Sigma^{I,J}$ by considering a concrete line. In particular, we study this growth in the one-dimensional case considering the minimal number of generators of I^{n+1}/JI^n, see Proposition 6.2.3.

Moreover, we also study some cases in which we consider the embedding dimension of I, $b(I) = \text{length}_R(I/I^2)$. We cover some cases where h_I is non-decreasing, see Proposition 6.2.4.

6.1 Diagonal submodules of the bigraded Sally module

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Without loss of generality, we can assume that $\mathbf{k} = R/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal of R and let J be a minimal reduction of I.

In Chapter 5 we defined the bigraded Sally module

$$\Sigma^{I,J} = \bigoplus_{i \geq 1} \bigoplus_{j \geq 0} \frac{I^i J^j}{I^{i-1} J^{j+1}} t^{i+j} U^j.$$

This is a bigraded A-module, with $A = (R/J)[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$ with $\deg(V_i) = (1,0)$, $i = 1, \ldots, \mu$, and $\deg(T_j) = (1,1)$, $j = 1, \ldots, d$. Also in Chapter 5, for all $p \geq 0$, we defined the p-th diagonal of $\Sigma^{I,J}$ as

$$\Sigma^{I,J}_{[p]} = \bigoplus_{\substack{(m,n) \\ m-n=p+1}} \Sigma^{I,J}_{(m,n)} = \bigoplus_{n \geq 0} \frac{I^{p+1} J^n}{I^p J^{n+1}} t^{n+p+1} U^n.$$

Now, we want to extend this definition by considering bigraded pieces on $\Sigma^{I,J}$ of degrees running on a parametric line of the plane $\mathbb{N}^2$.

Definition 6.1.1. *For each set of non negative integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$, with $\alpha_1 + \alpha_2 \geq 1$, we define the line l_α in the plane (m,n) as*

$$l_\alpha : \quad \begin{cases} m(s) = \alpha_1 s + \alpha_3 \\ n(s) = \alpha_2 s + \alpha_4 \end{cases}$$

for $s \geq 0$.

Definition 6.1.2. *For each line l_α, we can define the* diagonal submodule *$D_{l_\alpha}(\Sigma^{I,J})$ of $\Sigma^{I,J}$ which corresponds to the direct sum of the pieces of $\Sigma^{I,J}$ of bidegrees $(m(s)+n(s), n(s))$, $s \geq 0$*

$$D_{l_\alpha}(\Sigma^{I,J}) = \bigoplus_{(m,n)\,\in\, l_\alpha} \Sigma^{I,J}_{(m+n,n)} t^{m+n} U^n$$
$$= \bigoplus_{s\,\geq\, 0} \frac{I^{m(s)} J^{n(s)}}{I^{m(s)-1} J^{n(s)+1}} t^{m(s)+n(s)} U^{n(s)}.$$

Example 6.1.3. Notice that with this definition, we generalize the concept of the diagonals $\Sigma^{I,J}_{[p]}$ for all $p \geq 0$. In fact,

$$\Sigma^{I,J}_{[p]} = \bigoplus_{s\geq 0} \frac{I^{p+1} J^s}{I^p J^{s+1}} t^{p+1+s} U^s$$

which corresponds, according to this new definition, to $D_{l_{\alpha_p}}(\Sigma^{I,J})$, with

$$l_{\alpha_p} : \quad \begin{cases} m(s) = p+1 \\ n(s) = s \end{cases}$$

In this case, $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = p+1$ and $\alpha_4 = 0$.

Example 6.1.4. In Section 6.2 the standard graded (R/J)-algebra

$$\mathcal{B} = \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)} = \bigoplus_{s\geq 0} \frac{I^s}{JI^{s-1}} t^s$$

will have an important role. This algebra is another diagonal submodule of $\Sigma^{I,J}$, $D_{l_{\mathcal{B}}}(\Sigma^{I,J})$, considering the line $l_{\mathcal{B}}$,

$$l_{\mathcal{B}} : \quad \begin{cases} m(s) = s \\ n(s) = 0 \end{cases}$$

In this case, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 0$ and $\alpha_4 = 0$.

Notice that we have

$$D_{l_\alpha}(\Sigma^{I,J}) = \bigoplus_{s\geq 0} \frac{I^{\alpha_1 s+\alpha_3} J^{\alpha_2 s+\alpha_4}}{I^{\alpha_1 s+\alpha_3-1} J^{\alpha_2 s+\alpha_4+1}} (t^{(\alpha_1+\alpha_2)} U^{\alpha_2})^s t^{\alpha_3+\alpha_4} U^{\alpha_4}$$

with $t^{\alpha_3+\alpha_4} U^{\alpha_4}$ a fix monomial. Now, shifting by $(-\alpha_3 - \alpha_4, -\alpha_4)$ and defining a new variable $W = t^{\alpha_1+\alpha_2} U^{\alpha_2}$ we have

$$D_{l_\alpha}(\Sigma^{I,J})(-\alpha_3 - \alpha_4, -\alpha_4) = \bigoplus_{s\geq 0} \frac{I^{\alpha_1 s+\alpha_3} J^{\alpha_2 s+\alpha_4}}{I^{\alpha_1 s+\alpha_3-1} J^{\alpha_2 s+\alpha_4+1}} W^s$$

Definition 6.1.5. *From now on, we will write*

$$D_{l_\alpha}(\Sigma^{I,J}) = \bigoplus_{s\geq 0} \frac{I^{\alpha_1 s+\alpha_3} J^{\alpha_2 s+\alpha_4}}{I^{\alpha_1 s+\alpha_3-1} J^{\alpha_2 s+\alpha_4+1}} W^s$$

for the diagonal of $\Sigma^{I,J}$ *with respect to a line* l_α*. Observe that* $D_{l_\alpha}(\Sigma^{I,J})$ *is a graded* $gr_{I^{\alpha_1}J^{\alpha_2}}(R)$*-module generated in degree 0.*

Now we can consider the Hilbert function of $D_{l_\alpha}(\Sigma^{I,J})$ as

$$\mathcal{H}_{l_\alpha}(s) = \text{length}_R \left(\frac{I^{m(s)} J^{n(s)}}{I^{m(s)-1} J^{n(s)+1}} \right).$$

In the next result we study the growth of the Hilbert function $\mathcal{H}_{l_\alpha}$ of the diagonal submodule $D_{l_\alpha}(\Sigma^{I,J})$ by considering hypotheses on the minimal number of generators of the pieces of this diagonal. This result applied to the diagonal submodule of Example 6.1.4 will be crucial in order to study the monotony of the Hilbert function of an $\mathfrak{m}$-primary ideal I in the one-dimensional case in Section 6.2.

Proposition 6.1.6. *Let* $(R, \mathfrak{m})$ *be a d-dimensional Cohen-Macaulay local ring with* $d > 0$*. Let I be an* $\mathfrak{m}$*-primary ideal and let J be a minimal reduction of I. Let* $D_{l_\alpha}(\Sigma^{I,J})$ *be the diagonal submodule of the bigraded Sally module* $\Sigma^{I,J}$ *associated to the line* l_α*. Let* $s \geq 2$ *be an integer such that one of the following conditions hold:*

(1) $v_R \left(\frac{I^{m(s)} J^{n(s)}}{I^{m(s)-1} J^{n(s)+1}} \right) \leq 2$, *or*

(2) *there exist an integer* $e \geq 1$ *such that* $\text{length}_R \left(\frac{I^{m(s)} J^{n(s)}}{I^{m(s)-e} J^{n(s)+e}} \right) \leq s.$

Then for all $t \geq s$ *it holds*

$$\mathcal{H}_{l_\alpha}(t) \geq \mathcal{H}_{l_\alpha}(t+1).$$

Moreover, under the hypothesis of (1) *there exists an element* $a \in I^{\alpha_1} J^{\alpha_2}$ *such that*

$$\frac{I^{m(t)} J^{n(t)}}{I^{m(t)-1} J^{n(t)+1}} \xrightarrow{.a} \frac{I^{m(t+1)} J^{n(t+1)}}{I^{m(t+1)-1} J^{n(t+1)+1}}$$

is an epimorphism for all $t \geq s-1$. In particular it holds

$$\mathcal{H}_{l_\alpha}(t) \geq \mathcal{H}_{l_\alpha}(t+1)$$

for all $t \geq s-1$.

Proof. (1) Let us assume that $\nu_R\left(\dfrac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}\right) \leq 2$.

(a) If $I^{m(s)}J^{n(s)} = I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + aK$ for $a \in I^{\alpha_1}J^{\alpha_2}$ and some ideal $K \subset I^{m(s-1)}J^{n(s-1)}$, then it holds

$$\begin{aligned}
I^{m(s+1)}J^{n(s+1)} &= I^{\alpha_1}J^{\alpha_2}I^{m(s)}J^{n(s)} \\
&= I^{\alpha_1}J^{\alpha_2}(I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + aK) \\
&= I^{m(s+1)-1}J^{n(s+1)+1} + \mathfrak{m}I^{m(s+1)}J^{n(s+1)} + aKI^{\alpha_1}J^{\alpha_2} \\
&\subset I^{m(s+1)-1}J^{n(s+1)+1} + \mathfrak{m}I^{m(s+1)}J^{n(s+1)} + aI^{\alpha_1}J^{\alpha_2}(I^{m(s-1)}J^{n(s-1)}) \\
&= I^{m(s+1)-1}J^{n(s+1)+1} + \mathfrak{m}I^{m(s+1)}J^{n(s+1)} + \\
&\qquad\qquad + aI^{m(s)-1}J^{n(s)+1} + a\mathfrak{m}I^{m(s)}J^{n(s)} + a^2K \\
&= I^{m(s+1)-1}J^{n(s+1)+1} + \mathfrak{m}I^{m(s+1)}J^{n(s+1)} + a^2K && \text{(i)} \\
&\subset I^{m(s+1)}J^{n(s+1)} + \mathfrak{m}I^{m(s+1)}J^{n(s+1)} + a^2K && \text{(ii)} \\
&\subset I^{m(s+1)}J^{n(s+1)} && \text{(iii)}
\end{aligned}$$

where (i) holds because $aI^{m(s)-1}J^{n(s)+1} \subset I^{m(s+1)-1}J^{n(s+1)+1}$ and since $a\mathfrak{m}I^{m(s)}J^{n(s)} \subset \mathfrak{m}I^{m(s+1)}J^{n(s+1)}$, (ii) holds since $J \subset I$, and (iii) holds because $a \in I^{\alpha_1}J^{\alpha_2}$ and $K \subset I^{m(s-1)}J^{n(s-1)}$. Therefore, $I^{m(s+1)}J^{n(s+1)} = I^{m(s+1)-1}J^{n(s+1)+1} + \mathfrak{m}I^{m(s+1)}J^{n(s+1)} + a^2K$.

Then by induction we get that for all $t \geq s$ it holds

$$I^{m(t)}J^{n(t)} = I^{m(t)-1}J^{n(t)+1} + \mathfrak{m}I^{m(t)}J^{n(t)} + a^{t-s+1}K.$$

From this we get that the map

$$\frac{I^{m(t)}J^{n(t)}}{I^{m(t)-1}J^{n(t)+1} + \mathfrak{m}I^{m(t)}J^{n(t)}} \xrightarrow{.a} \frac{I^{m(t+1)}J^{n(t+1)}}{I^{m(t+1)-1}J^{n(t+1)+1} + \mathfrak{m}I^{m(t+1)}J^{n(t+1)}}$$

is an epimorphism for $t \geq s-1$. Since we have isomorphisms

$$\frac{I^{m(t)}J^{n(t)}}{I^{m(t)-1}J^{n(t)+1} + \mathfrak{m}I^{m(t)}J^{n(t)}} \cong \frac{I^{m(t)}J^{n(t)}/I^{m(t)-1}J^{n(t)+1}}{\mathfrak{m}(I^{m(t)}J^{n(t)}/I^{m(t)-1}J^{n(t)+1})}$$

for all $t \geq 0$, then we get the epimorphism

$$\frac{I^{m(t)}J^{n(t)}/I^{m(t)-1}J^{n(t)+1}}{\mathfrak{m}(I^{m(t)}J^{n(t)}/I^{m(t)-1}J^{n(t)+1})} \xrightarrow{\cdot a} \frac{I^{m(t+1)}J^{n(t+1)}/I^{m(t+1)-1}J^{n(t+1)+1}}{\mathfrak{m}(I^{m(t+1)}J^{n(t+1)}/I^{m(t+1)-1}J^{n(t+1)+1})}.$$

Hence, by Nakayama's Lemma we deduce that

$$\frac{I^{m(t)}J^{n(t)}}{I^{m(t)-1}J^{n(t)+1}} \xrightarrow{\cdot a} \frac{I^{m(t+1)}J^{n(t+1)}}{I^{m(t+1)-1}J^{n(t+1)+1}}$$

is an epimorphism for $t \geq s-1$. Hence, for $t \geq s-1$ we have that

$$\text{length}_R\left(\frac{I^{m(t)}J^{n(t)}}{I^{m(t)-1}J^{n(t)+1}}\right) \geq \text{length}_R\left(\frac{I^{m(t+1)}J^{n(t+1)}}{I^{m(t+1)-1}J^{n(t+1)+1}}\right),$$

so

$$\mathcal{H}_{I_\alpha}(t) \geq \mathcal{H}_{I_\alpha}(t+1)$$

and the claim is proved.

(b) We can assume that for some $s \geq 2$, $\nu_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}\right) \geq 1$. Otherwise, $\mathcal{H}_{I_\alpha}(s) = 0$ for all $s \geq 2$. So, there exists a generator

$$\alpha \in I^{m(s)}J^{n(s)} \setminus I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)}.$$

This generator α is a combination of monomials in $I^{m(s)}J^{n(s)}$, and since $\alpha \notin I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)}$, there exists a monomial $\underline{a} \in I^{m(s)}J^{n(s)} \setminus I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)}$. By Nakayama's Lemma, we can choose this monomial as a generator.

If $\nu_R(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}) = 1$, we apply **(a)**. If $\nu_R(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}) = 2$, there is another generator $\beta \in I^{m(s)}J^{n(s)} \setminus I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (\underline{a})$. Using the same argument as before, we deduce that there are two monomials $\underline{a}, \underline{b} \in I^{m(s)}J^{n(s)} \setminus I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)}$ that generate the module.

So, from **(a)** we may assume that there exist monomials $\underline{a}, \underline{b} \in I^{m(s)}J^{n(s)}$, with $\underline{a} = a_{\alpha_1\alpha_2}a_{m(s-1)n(s-1)}$ and $\underline{b} = b_{\alpha_1\alpha_2}b_{m(s-1)n(s-1)}$ with no common factor in $I^{\alpha_1}J^{\alpha_2}$, where $a_{\alpha_1\alpha_2}, b_{\alpha_1\alpha_2} \in I^{\alpha_1}J^{\alpha_2}$, $a_{m(s-1)n(s-1)}, b_{m(s-1)n(s-1)} \in I^{m(s-1)}J^{n(s-1)}$ and

$$I^{m(s)}J^{n(s)} = I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (\underline{a}, \underline{b}).$$

We write $\underline{a}/a_{\alpha_1\alpha_2} = a_{m(s-1)n(s-1)}$, $\underline{b}/b_{\alpha_1\alpha_2} = b_{m(s-1)n(s-1)} \in I^{m(s-1)}J^{n(s-1)}$.

Notice that if one of the following conditions does not hold

(i) $(\underline{a}/a_{\alpha_1\alpha_2})b_{\alpha_1\alpha_2} \in I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (\underline{a})$,

(ii) $(\underline{b}/b_{\alpha_1\alpha_2})a_{\alpha_1\alpha_2} \in I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (\underline{a})$,

then we have

$$I^{m(s)}J^{n(s)} = I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (\underline{a}, (\underline{a}/a_{\alpha_1\alpha_2})b_{\alpha_1\alpha_2}),$$

or

$$I^{m(s)}J^{n(s)} = I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (\underline{a}, (\underline{b}/b_{\alpha_1\alpha_2})a_{\alpha_1\alpha_2}).$$

In fact, if we assume that (i) doesn't hold, we get that

$$\begin{aligned} 0 \neq I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (\underline{a}) \quad & \subsetneq I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + \\ & \quad +(\underline{a}, (\underline{a}/a_{\alpha_1\alpha_2})b_{\alpha_1\alpha_2}) \\ & \subseteq I^{m(s)}J^{n(s)}, \end{aligned}$$

but $\nu_R\left(\dfrac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}\right) \leq 2$, so we get

$$I^{m(s)}J^{n(s)} = I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (\underline{a}, (\underline{a}/a_{\alpha_1\alpha_2})b_{\alpha_1\alpha_2}).$$

We write $a_{m(s-1)n(s-1)} = c_{\alpha_1\alpha_2}c_{m(s-2)n(s-2)}$ with $c_{\alpha_1\alpha_2} \in I^{\alpha_1}J^{\alpha_2}$ and with $c_{m(s-2)n(s-2)} \in I^{m(s-2)}J^{n(s-2)}$, so

$$(\underline{a}, (\underline{a}/a_{\alpha_1\alpha_2})b_{\alpha_1\alpha_2}) = c_{\alpha_1\alpha_2}(a_{\alpha_1\alpha_2}c_{m(s-2)n(s-2)}, b_{\alpha_1\alpha_2}c_{m(s-2)n(s-2)}).$$

Then, from (**a**) and $s \geq 2$ we get the result. The other case is proved similarly.

Hence we may assume that conditions (i), (ii) hold, so there exist elements $r_1, r_2 \in I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)}$, and $\alpha, \beta \in R$ such that

(iii) $(\underline{a}/a_{\alpha_1\alpha_2})b_{\alpha_1\alpha_2} = r_1 + \alpha\underline{a}$,

(iv) $(\underline{b}/b_{\alpha_1\alpha_2})a_{\alpha_1\alpha_2} = r_2 + \beta\underline{a}$.

Let us consider the pair

$$\begin{aligned} A &= (\underline{a}/a_{\alpha_1\alpha_2})(a_{\alpha_1\alpha_2} + \lambda b_{\alpha_1\alpha_2}), \\ B &= (\underline{b}/b_{\alpha_1\alpha_2})(a_{\alpha_1\alpha_2} + \lambda b_{\alpha_1\alpha_2}), \end{aligned}$$

with $\lambda \in R$ an element to be determined. Hence in the **k**-vector space

$$\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)}}$$

we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1+\lambda\alpha & 0 \\ \beta & \lambda \end{pmatrix} \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix}$$

Since **k** is infinite we have $\lambda(1+\lambda\alpha) \neq 0$ for a generic $\lambda \in R$, so

$$I^{m(s)}J^{n(s)} = I^{m(s)-1}J^{n(s)+1} + \mathfrak{m}I^{m(s)}J^{n(s)} + (A, B)$$

and we are in the case (**a**). In fact, $(A, B) = c(\underline{a}/a_{\alpha_1\alpha_2}, \underline{b}/b_{\alpha_1\alpha_2})$ with $c = a_{\alpha_1\alpha_2} + \lambda b_{\alpha_1\alpha_2} \in I^{\alpha_1}J^{\alpha_2}$ and $(\underline{a}/a_{\alpha_1\alpha_2}, \underline{b}/b_{\alpha_1\alpha_2}) \subset I^{m(s-1)}J^{n(s-1)}$.

(2) We assume that $\text{length}_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-e}J^{n(s)+e}}\right) \leq s$. Since $J \subset I$, we get that $I^{m(s)-e}J^{n(s)+e} \subseteq I^{m(s)-1}J^{n(s)+1}$, so

$$\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-e}J^{n(s)+e}} \twoheadrightarrow \frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}$$

is an epimorphism and hence,

$$\mathcal{H}_{l_\alpha}(s) = \text{length}_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}\right) \leq \text{length}_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-e}J^{n(s)+e}}\right) \leq s.$$

Since $D_{l_\alpha}(\Sigma^{I,J})$ is a graded $gr_{I^{\alpha_1}J^{\alpha_2}}(R)$-module generated in degree 0, from [BE00] Corollary 3.5, we get that

$$\mathcal{H}_{l_\alpha}(t+1) \leq \mathcal{H}_{l_\alpha}(t)^{<t>}$$

for all $t \geq 1$. Since $\mathcal{H}_{l_\alpha}(s) \leq s$ we have

$$\mathcal{H}_{l_\alpha}(s+1) \leq \mathcal{H}_{l_\alpha}(s)^{<s>} \leq \mathcal{H}_{l_\alpha}(s).$$

By recurrence we prove the claim. In fact, if the claim holds for $t-1 \geq s$, then for $t > s$ we have that

$$\begin{aligned} \text{length}_R\left(\frac{I^{m(t)}J^{n(t)}}{I^{m(t)-1}J^{n(t)+1}}\right) &\leq \text{length}_R\left(\frac{I^{m(t-1)}J^{n(t-1)}}{I^{m(t-1)-1}J^{n(t-1)+1}}\right) \\ &\leq \ldots \leq \text{length}_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}\right) \leq s \leq t. \end{aligned}$$

Again, since $\mathcal{H}_{I_\alpha}(t) \leq t$, we have that

$$\mathcal{H}_{I_\alpha}(t+1) \leq \mathcal{H}_{I_\alpha}(t)^{<t>} \leq \mathcal{H}_{I_\alpha}(t)$$

and the claim is proved. □

6.2 Monotony of the Hilbert function

In this section we study the monotony of the Hilbert function h_I of an ideal I in a one-dimensional Cohen-Macaulay local ring R. We prove some cases where the function h_I is non-decreasing by assuming some hypothesis on the components of the R/J-algebra

$$\mathcal{B} = \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)},$$

see Example 6.1.4.

We can consider the Hilbert function of $\mathcal{B}$

$$h_{\mathcal{B}}(n) = \text{length}_R(I^n/JI^{n-1})$$

for $n \geq 0$. From now on we will write $\mathcal{H}_{I,J} = h_{\mathcal{B}}$. Notice that $\mathcal{H}_{I,J}$ depends on J, and if $\text{depth}(gr_I(R)) \geq d-1$ then does not, see [CPV98], Proposition 2.3. We use $\mathcal{H}_{I,J}$ in order to study the monotony of h_I.

The result obtained for a general diagonal submodule of $\Sigma^{I,J}$, Proposition 6.1.6, for dimension $d > 0$, can be reformulated in our case as follows:

Proposition 6.2.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with $d > 0$. Let I be a $\mathfrak{m}-$primary ideal and J a minimal reduction of I. Let $t \geq 2$ be an integer such that one of the following conditions hold:*

(1) $v_R\left(\frac{I^t}{JI^{t-1}}\right) \leq 2$, *or*

(2) *there exist an integer $e \geq 1$ such that* $\text{length}_R\left(\frac{I^t}{J^e I^{t-e}}\right) \leq t$.

Then for all $n \geq t$ it holds

$$\mathcal{H}_{I,J}(n) \geq \mathcal{H}_{I,J}(n+1).$$

Moreover, under the hypothesis of (1) *there exists an element* $a \in I$ *such that*

$$\frac{I^n}{JI^{n-1}} \xrightarrow{.a} \frac{I^{n+1}}{JI^n}$$

is an epimorphism for all $n \geq t-1$*. In particular it holds*

$$\mathcal{H}_{I,J}(n) \geq \mathcal{H}_{I,J}(n+1)$$

for all $n \geq t-1$*.*

In the one-dimensional case $J = (x)$, where $x \in I$ is a superficial element of degree one. In that case we can relate the behavior of $\mathcal{H}_{I,J}$ with the growth of the Hilbert function by means of the identity

$$\mathcal{H}_{I,(x)}(n) = e_0(I) - h_I(n-1).$$

In fact, since $x \in J \subseteq I$,

$$\text{length}_R\left(\frac{I^n}{xI^{n-1}}\right) = \text{length}_R\left(\frac{I^{n-1}}{xI^{n-1}}\right) - \text{length}_R\left(\frac{I^{n-1}}{I^n}\right).$$

Since x is also a non-zero divisor of R, from [Lip71] remark (b) to Corollary 1.10, we get that

$$\frac{I^{n-1}}{xI^{n-1}} \cong \frac{R}{xR}$$

so

$$\text{length}_R\left(\frac{I^{n-1}}{xI^{n-1}}\right) = \text{length}_R\left(\frac{R}{xR}\right) = e_0(I)$$

Therefore we get the above formula

$$\mathcal{H}_{I,J}(n) = e_0(I) - h_I(n-1).$$

We denote by $pn(I)$ the postulation number of h_I, i.e. the least integer t such that $h_I(t+n) = p_I(t+n)$ for all $n \geq 0$.

Next result is known for the maximal ideal, [Eli86] and also for $\mathfrak{m}$-primary ideals, [Eli05].

Proposition 6.2.2. *Let* R *be a one-dimensional Cohen-Macaulay local ring, and let* x *be a degree one superficial element of an* $\mathfrak{m}$*-primary ideal* I*. Then the following conditions hold true:*

(i) *$pn(I)$ is the least integer n such that $I^n/I^{n+1} \xrightarrow{.x} I^{n+1}/I^{n+2}$ is an isomorphism,*

(ii) *$pn(I)$ is the least integer n such that $I^{n+1} = xI^n$,*

(iii) *$pn(I) \leq e_0(I) - 1$.*

In [Eli99] it is proved that if $I^n \cap (x) = xI^{n-1}$ for all $n \leq t-1$ and $\text{length}_R(I^t/xI^{t-1}) \leq 2$ then h_I is non-decreasing. In the next result we will consider the number of generators instead of the length of the $R-$module I^t/xI^{t-1}.

Proposition 6.2.3. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. Let I be a $\mathfrak{m}-$primary ideal, and $x \in I$ a degree one superficial element of I. Let $t \geq 2$ be an integer such that the pair I, x verify one of the following conditions:*

(1) *$I^n \cap (x) = xI^{n-1}$ for all $n \leq t-1$, and $\nu_R\left(\frac{I^t}{xI^{t-1}}\right) \leq 2$,*

(2) *$I^n \cap (x) = xI^{n-1}$ for all $n \leq t$, and $\text{length}_R\left(\frac{I^t}{x^a I^{t-a}}\right) \leq \bar{t} \leq t$, $a \geq 1$.*

Then h_I is non-decreasing.

Proof. (1) Let us consider the projection

$$\frac{I^n}{xI^{n-1}} \longrightarrow \frac{I^{n-1}}{xI^{n-2}}$$

Since $I^n \cap (x) = xI^{n-1}$, $n \leq t-1$, we get that it is injective in this range. In fact, if $y \in I^n$ such that $y \in xI^{n-2} = I^{n-1} \cap (x)$, we get that $y \in (x)$. So, $y \in I^n \cap (x) = xI^{n-1}$. Then, $\mathcal{H}_{I,J}(n-1) \geq \mathcal{H}_{I,J}(n)$, and from the identity of Proposition 6.2.2, we get that $h_I(n-1) \leq h_I(n)$. So, h_I is non-decreasing for $n \leq t-1$.

Now, from Proposition 6.2.1 (1) we get that $\mathcal{H}_{I,J}(n) \geq \mathcal{H}_{I,J}(n+1)$ for $n \geq t-1$, so $h_I(n) \leq h_I(n+1)$. Hence, we get that h_I is non-decreasing.

(2) Proceeding in a similar way as we did in the case (1) we get that $I^n \cap (x) = xI^{n-1}$ for all $n \leq t$ implies that h_I is non-decreasing for $n \leq t$, and by Proposition 6.2.1 (2) we get that h_I is non-decreasing for $n \geq t$. $\square$

Let I be an $\mathfrak{m}$-primary ideal of a Cohen-Macaulay local ring R of dimension d. We denote by $b(I) = \text{length}_R(I/I^2)$ the embedding dimension of I. If $I = \mathfrak{m}$ then we write $b(\mathfrak{m}) = b(R)$. Valla proved that, [Val79],

$$b(I) = e_0(I) + (d-1)\,\text{length}_R\left(\frac{R}{I}\right) - \text{length}_R\left(\frac{I^2}{JI}\right)$$

where J is a minimal reduction of I. In particular we have

$$b(I) \leq e_0(I) + (d-1)\,\text{length}_R(R/I),$$

and if the equality holds then $gr_I(R)$ is Cohen-Macaulay.

Considering the embedding dimension of I and the Valla's formula, we study some cases where the Hilbert function h_I of I is non-decreasing depending on $b(I)$.

Proposition 6.2.4. *Let R be a one-dimensional Cohen-Macaulay local ring. Let I be a $\mathfrak{m}$-primary ideal of R. Then*

(i) $e_0(I) = 1$ *if and only if* $b(I) = 1$. *In this case we have* $I = \mathfrak{m}$ *and* R *is a regular local ring.*

(ii) *If* $b(I) = 2$ *then it holds*

$$h_I(n) = \begin{cases} \text{length}_R(R/I) & n = 0 \\ n+1 & n = 1, \ldots, e_0(I) - 1 \\ e_0(I) & n \geq e_0(I). \end{cases}$$

The Hilbert function h_I *is non-decreasing if and only if* $\text{length}_R(R/I) \leq 2$.

(iii) *If* $b(I) \leq e_0(I) \leq b(I) + 2$ *then the Hilbert function is non-decreasing.*

(iv) *If* $I^2 \cap (x) = xI$, $b(I) = 4$, *and* $e_0(I) = 7$ *then* $\text{length}_R(R/I) \leq 4$ *and the Hilbert function is non-decreasing.*

Proof. Notice that Valla's formula takes the form

$$e_0(I) - b(I) = \text{length}_R(I^2/xI).$$

(i) If $b(I) = \text{length}_R(I/I^2) = 1$ then there exists an integer $a \in I$ such that $I = I^2 + (a)$. Clearly, $I^n = I^{n+1} + aI^{n-1}$ for all $n \geq 1$. Then the morphism

$$\frac{I^n}{I^{n+1}} \xrightarrow{.a} \frac{I^{n+1}}{I^{n+2}}$$

is surjective for all $n \geq 0$, so $h_I(n) \leq 1$ for all $n \geq 1$. In fact, from the morphism, we get that $\text{length}_R(I^n/I^{n+1}) \geq \text{length}_R(I^{n+1}/I^{n+2})$, so $h_I(n) \geq h_I(n+1)$ for all $n \geq 0$, and hence, for $n \geq 1$, it holds that $h_I(n) \leq h_I(1) = \text{length}_R(I/I^2) = b(I) = 1$.

Since R is one-dimensional and I $\mathfrak{m}-$primary we get $h_I(n) = 1$ for all $n \geq 1$, in particular $e_0(I) = 1$, because $p_I(n) = e_0(I)$.

Let us assume $e_0(I) = 1$. We have that $b(I) = \text{length}_R(I/I^2) > 0$. If $b(I) = 0$ then $I = I^2$, and hence, $I^n = I^{n+1}$ for $n \geq 1$, and therefore $e_0(I) = p_I(n) = 0$, which is a contradiction. From Valla's formula we have $1 = e_0(I) \geq b(I) > 0$, so we get $b(I) = 1$.

Let us assume that $e_0(I) = b(I) = 1$. From Valla's formula, we get that $\text{length}_R(I^2/xI) = 0$, so $I^2 = xI$, and hence, we can consider the injection

$$\frac{R}{I} \overset{\cdot x}{\hookrightarrow} \frac{I}{I^2}$$

Thus, $\text{length}_R(R/I) \leq \text{length}_R(I/I^2) = 1$. Since I is an $\mathfrak{m}$-primary ideal, we get that $\text{length}_R(R/I) = 1$. Considering the epimorphism

$$\frac{R}{I} \longrightarrow \frac{R}{\mathfrak{m}} = \mathbf{k}$$

we get that the morphism $\text{length}_R(R/\mathfrak{m}) \leq \text{length}_R(R/I) = 1$. Then we get that $R/I \longrightarrow R/\mathfrak{m}$ is an isomorphism, and hence $I = \mathfrak{m}$.

In this case, since $b(\mathfrak{m}) = \text{length}_R(\mathfrak{m}/\mathfrak{m}^2) = 1$ we get that $\nu_R(\mathfrak{m}) = 1 = \dim(R)$, and hence, R is a regular local ring.

(ii) Let us assume $b(I) = 2$. If $\nu_R(I/I^2) \leq 1$ then there exists an element $a \in I$ such that $I = I^2 + (a)$. Then, $I^n = I^{n+1} + aI^{n-1}$ for $n \geq 1$. Hence, we have an epimorphism

$$\frac{I^n}{I^{n+1}} \overset{\cdot a}{\longrightarrow} \frac{I^{n+1}}{I^{n+2}}$$

for $n \geq 1$. Then, $h_I(n) \geq h_I(n+1)$ for $n \geq 1$. Since $b(I) = \text{length}_R(I/I^2) = h_I(1) = 2$ we get that $h_I(n) \leq 2$ for $n \geq 1$. For $n \gg 0$, $e_0 = (I) = h_I(n) \leq 2$. Since $b(I) \leq e_0(I)$, we get that $e_0(I) = 2$. Now, since $p_I(n) = e_0(I) = 2$, $h_I(1) = 2$ and $h_I(n) \geq h_I(n+1)$ for $n \geq 1$, we have that $h_I(n) = 2$ for $n \geq 1$. Then, the claim is proved, because

$$h_I(n) = \begin{cases} \text{length}_R(R/I) & n = 0 \\ n+1 = 2 & n = 1 \\ e_0(I) = 2 & n \geq 2. \end{cases}$$

Clearly, h_I is non-decreasing if and only if $\text{length}_R(R/I) \leq 2$.

Hence we may assume $\nu_R(I/I^2) = 2$, then there exist $a_1, a_2 \in I$ such that $I = I^2 + (a_1, a_2)$ with $a_1 \neq a_2$, $a_i\mathfrak{m} \subset I^2$, $i = 1,2$. It is easy to prove that for all $n \geq 1$ we have

$$I^n = I^{n+1} + (a_1^n, a_1^{n-1}a_2, \ldots, a_2^n),$$

since $\text{length}_R(I^n/I^{n+1}) = \text{length}_R(I^{n+1} + (a_1^n, a_1^{n-1}a_2, \ldots, a_2^n)/I^{n+1})$ and $a_i\mathfrak{m} \subset I^2$, we get $h_I(n) \leq n+1$ for all $n \geq 1$.

Since h_I verifies Macaulay's condition, [Bla98], [BN99], we get that $h_I(n+1) \leq h_I(n)^{<n>}$ for $n \geq 1$. We distinguish two cases. If $h_I(n) \leq n$ then $h_I(n+1) \leq h_I(n)^{<n>} \leq h_I(n)$. Thus, if there exists an integer $1 \leq n_0 < e_0(I)$ such that $h_I(n_0) \leq n_0$ then $h_I(n_0+1) \leq h_I(n_0) \leq n_0 \leq n_0+1$ and proceeding by induction we get that $h_I(n+1) \leq h_I(n) \leq n_0 < e_0(I)$ for $n \geq n_0$. Since $h_I(n) = e_0(I)$ for $n \gg 0$ we get a contradiction. Hence, $h_I(n) = n+1$ for $n = 1, \ldots, e_0(I) - 1$.

Then we get that

$$h_I(n) = \begin{cases} \text{length}_R(R/I) & n = 0 \\ n+1 & n = 1, \ldots, e_0(I) - 1 \\ e_0(I) & n \geq e_0(I). \end{cases}$$

Again, h_I is non-decreasing if and only if $\text{length}_R(R/I) \leq 2$.
(iii) From Valla's formula we get

$$2 \geq e_0(I) - b(I) = \text{length}_R\left(\frac{I^2}{xI}\right).$$

Since $\text{length}_R(I^2/xI) \geq \nu_R(I^2/xI)$, from Proposition 6.2.3 (1), taking $t = 2$, we get (iii), because $I \cap (x) = (x)$. Hence h_I is non-decreasing.
(iv) If $b(I) = 4$, $e_0(I) = 7$ then we have $\text{length}_R(I^2/xI) = e_0(I) - b(I) = 3$. So, we have

$$\begin{aligned} 3 = \text{length}_R\left(\frac{I^2}{xI}\right) &\geq \text{length}_R\left(\frac{I^2}{xI + I^3}\right) \\ &\geq \text{length}_R\left(\frac{I^2}{xI + \mathfrak{m}I^2}\right) = \nu_R\left(\frac{I^2}{xI}\right). \end{aligned}$$

If $\nu_R(I^2/xI) \leq 2$ then from Proposition 6.2.3 (1), taking $t = 2$, we get that h_I is non-decreasing.

Let us assume that $\nu_R(I^2/xI) = 3$, then we have $I^2/xI = I^2/xI + I^3$, and hence $I^3 \subset xI$. Since

$$\frac{I^3}{xI^2} \subset \frac{xI}{xI^2} \cong \frac{I}{I^2}$$

we get $\text{length}_R(I^3/xI^2) \leq \text{length}_R(I/I^2) = b(I) = 4$, and the equality holds if and only if $I^3 = xI$.

If $I^3 = xI$ then for all $n \geq 1$ it holds $I^{n+2} = xI^n$, so

$$\frac{I^{n+2}}{xI^{n+1}} = \frac{xI^n}{xI^{n+1}} \cong \frac{I^n}{I^{n+1}}.$$

From this we get $\mathcal{H}_{I,(x)}(n+2) = h_I(n)$ for all $n \geq 1$. On the other hand we know that $\mathcal{H}_{I,(x)}(n) = 0$ for $n \gg 0$ and $h_I(n) = e_0(I)$ for $n \gg 0$, so we get a contradiction.

Let us assume $I^3 \neq xI$, then we have $3 \geq \mathcal{H}_{I,(x)}(3)$. From Proposition 6.2.1 (2), we get $3 \geq \mathcal{H}_{I,(x)}(n) \geq \mathcal{H}_{I,(x)}(n+1)$ for all $n \geq 3$. Using the identity $\mathcal{H}_{I,J}(n) = e_0(I) - h_I(n-1)$, we get $4 \leq h_I(n) \leq h_I(n+1)$ for all $n \geq 2$.

Since $I^2 \cap (x) = xI$ we can consider the exact sequence

$$0 \longrightarrow \frac{R}{I} \xrightarrow{\cdot x} \frac{I}{I^2} \longrightarrow \frac{I}{I^2 + (x)} \longrightarrow 0$$

so we have $h_I(0) = \text{length}_R(R/I) \leq \text{length}_R(I/I^2) = h_I(1) = b(I) = 4$, and h_I is non-decreasing. □

As a corollary we recover Proposition 3.4 from [Eli99]:

Corollary 6.2.5. *Let $(R, \mathfrak{m})$ be an equi-characteristic Cohen-Macaulay local ring on dimension 1 and embedding dimension b. If one of the following conditions hold then h_R is non-decreasing:*

(i) $b = 1, 2, 3$,

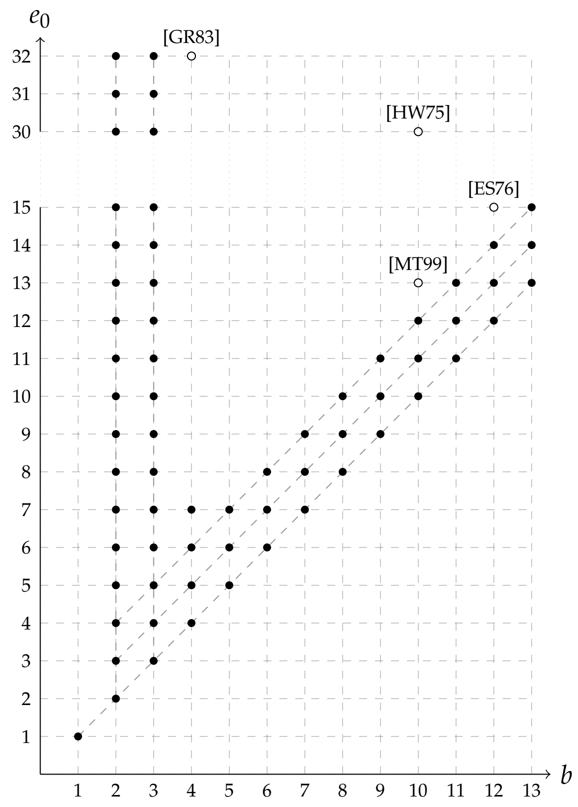
(ii) $b \leq e_0 \leq b + 2$,

(iii) $b = 4$, $e_0 = 7$.

Proof. From the above result we know that h_R is non-decreasing except for the case $b = 3$. The main result of [Eli93] established Sally's conjecture on the Hilbert function, i.e. the case $b = 3$. □

Observe that, for the maximal ideal $\mathfrak{m}$ the first case for which we don't know if the Hilbert function is non decreasing is $b = 4$, $e_0 = 8$. The first known example with a decreasing Hilbert function for $b = 4$ has multiplicity $e_0 = 32$, [GR83]. Other examples of a locally decreasing Hilbert function are for $b = 10$, $e_0 = 30$, [HW75], $b = 10$, $e_0 = 13$, [MT99] and $b = 12$, $e_0 = 15$, [ES76].

In the following diagram we represent in black dots the known cases of non-decreasing Hilbert function in terms of b and e_0. White dots are the known examples of locally decreasing Hilbert function.



Resum en català

L'objectiu principal d'aquesta tesi és contribuir al coneixement de propietats cohomològiques dels mòduls multigraduats no-estàndard. Principalment estudiem la profunditat de mòduls multigraduats no-estàndard i d'estructures relacionades, així com la funció i el quasi-polinomi de Hilbert, centrant l'estudi en la profunditat de les àlgebres de blow-up.

En àlgebra commutativa, els mòduls graduats, així com els multigraduats estàndard, han estat objecte d'estudi per molts autors. Tot i que també es coneixen resultats per mòduls graduats no-estàndard, el cas multigraduat no-estàndard no és tan comú.

Per altra banda, sota el nom d'*àlgebres de blow-up*, es coneixen algunes àlgebres graduades associades a un ideal I d'un anell local Noetherià $(R, \mathfrak{m})$. Són, entre d'altres, l'*àlgebra de Rees* $\mathcal{R}(I)$, l'*anell graduat associat* $gr_I(R)$ i el *con de la fibra* $F_{\mathfrak{m}}(I)$ definits com a

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n, \qquad gr_I(R) = \bigoplus_{n \geq 0} \frac{I^n}{I^{n+1}}, \qquad F_{\mathfrak{m}}(I) = \bigoplus_{n \geq 0} \frac{I^n}{\mathfrak{m} I^n}.$$

Les àlgebres de blow-up s'utilitzen per estudiar propietats i caràcters numèrics de l'anell local $(R, \mathfrak{m})$ i de l'ideal I. A més a més, tenen rellevància geomètrica.

Vasconcelos, [Vas94b], va afegir a la llista d'àlgebres de blow-up l'anomenat *mòdul de Sally* $S_J(I)$ d'un ideal I respecte d'una reducció minimal J. És el $\mathcal{R}(J)$-mòdul graduat

$$S_J(I) = \bigoplus_{n \geq 1} \frac{I^{n+1}}{J^n I}.$$

El nom venia motivat pel treball de Sally que estava enfocat en recuperar propietats de $\mathcal{R}(I)$ i $gr_I(R)$ a partir de l'estructura, més coneguda i més bona, de $\mathcal{R}(J)$.

L'àlgebra de Rees i l'anell graduat associat, es poden generalitzar al cas multigraduat per un conjunt d'ideals $I_1, \ldots, I_r$ d'un anell local Noetherià $(R, \mathfrak{m})$ de la següent manera: l'àlgebra de Rees multigraduada associada a $I_1, \ldots, I_r$ es defineix com

$$\mathcal{R}(I_1, \ldots, I_r) = \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset R[t_1, \ldots, t_r],$$

i per a cada $k = 1, \ldots, r$, el k-èsim anell multigraduat associat a $I_1, \ldots, I_r$ a R és

$$gr_{I_1,\ldots,I_r;I_k}(R) = \bigoplus_{\underline{n} \in \mathbb{N}^r} \frac{I_1^{n_1} \cdots I_k^{n_k} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} = \frac{\mathcal{R}(I_1, \ldots, I_r)}{I_k \mathcal{R}(I_1, \ldots, I_r)}.$$

A continuació introduïm i motivem els principals problemes que hem estudiat, i seguidament detallarem els resultats més importants obtinguts a la tesi.

Les funcions de Hilbert de mòduls graduats sobre àlgebres graduades estàndard han estat ben estudiades des del famós article de Hilbert [Hil90]. Suposant que totes les peces homogènies d'un mòdul graduat tinguin longitud finita, es pot demostrar que la funció de Hilbert, que mesura aquestes longituds, és asimptòticament polinòmica. L'estudi de les funcions de Hilbert, els polinomis de Hilbert i els seus coeficients tenen un paper important en l'àlgebra commutativa. Aquest estudi es pot generalitzar de diverses maneres considerant una àlgebra graduada no-estàndard, una àlgebra multigraduada estàndard, o bé, una àlgebra multigraduada no-estàndard.

Els dos primers casos són coneguts. En el primer cas, si considerem una àlgebra positivament graduada, i un mòdul graduat, la funció de Hilbert és asimptòticament un quasi-polinomi. Vegeu [BH93] i [DS99]. Quan considerem una àlgebra multigraduada estàndard i un mòdul multigraduat, que té generadors de multigraus $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$, aleshores la funció de Hilbert és un polinomi en r indeterminades per peces homogènies de grau $\underline{n} = (n_1 \ldots, n_r) \in \mathbb{N}^r$ amb $n_1, \ldots, n_r$ prou grans. Vegeu per exemple [HHRT97], [VKM94] i [Rob98].

En el cas no-estàndard s'han estudiats algunes situacions. Per exemple, a [Lav99] i [Rob98], s'ha estudiat el cas en què els generadors tenen multigraus $(1, 0, \ldots, 0)$, $(d_1^2, 1, 0, \ldots, 0)$, $\ldots$, $(d_1^r, \ldots, d_{r-1}^r, 1)$. En aquest cas, la funció de Hilbert és un polinomi en r indeterminades per $\underline{n} = (n_1, \ldots, n_r)$ en una regió (un con) de $\mathbb{Z}^r$. En el cas bigraduat, aquesta situació ha estat també estudiada en [HT03]. En [Rob00], es defineix una altra funció de Hilbert; l'autor considera una funció de Hilbert acumulativa d'un mòdul graduat finitament generat sobre un anell de polinomis amb coeficients en un cos i generadors de graus $(1,0)$, $(0,1)$ i $(1,1)$. Aquesta funció en (m,n) es correspon a la suma de les dimensions de les peces de graus (m,j) per j fins a n. L'autor demostra que aquesta funció és polinòmica en una regió de $\mathbb{N}^2$. Aquesta definició permet estudiar el mòdul tant des d'un punt de vista graduat com bigraduat.

Una situació més general, és la que estudia Fields en la seva tesi doctoral, [Fie00] (vegeu també [Fie02]). Ell considera la definició general de quasi-polinomi i demostra que la funció de Hilbert d'un mòdul $\mathbb{N}^r$-graduat és quasi-polinòmica en una regió de $\mathbb{Z}^r$. Malgrat tot, en la seva demostració, el con no està descrit explícitament.

Per als nostres propòsits, necessitem controlar el con on la funció de Hilbert és un quasi-polinomi. Per això, comencem estudiant el comportament asimptòtic de la funció de Hilbert d'un mòdul multigraduat no-estàndard considerant M com a S-mòdul $\mathbb{Z}^r$-graduat, on S és un anell $\mathbb{Z}^r$-graduat amb generadors g_i^j, $i = 1, \ldots, r$, $j = 1, \ldots, \mu_i$ de graus $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0 \ldots, 0) \in \mathbb{N}^r$ i $\gamma_i^i \neq 0$, sobre un anell local Artinià $S_{\underline{0}}$. En particular, demostrem que existeix un element $\underline{\beta} \in \mathbb{N}^r$ tal que la funció de Hilbert és un quasi-polinomi en un con definit pels elements $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ tals que $\underline{n} = \underline{\beta} + \sum_{i=1}^r \lambda_i \gamma_i$ amb $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}$.

El problema de l'estabilitat asimptòtica de la profunditat de les peces homogènies d'un mòdul multigraduat, té els seus orígens en un resultat de Burch, [Bur72], quan demostra que per un ideal I d'un anell local Noetherià $(R, \mathfrak{m})$ es té que

$$l(I) \leq \dim(R) - \min_{n \geq 1}\{\operatorname{depth}(R/I^n)\}$$

essent $l(I) = \dim(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I))$ la dispersió analítica de I. Alguns anys després, Brodmann, [Bro79a], va poder canviar el nombre mínim d'aquestes profunditats pel valor asimptòtic de $\operatorname{depth}(R/I^n)$ per n grans, un valor que va demostrar que existeix. En particular, a [Bro79b], va estudiar els primers associats a M/I^nM per un R-mòdul finitament generat M, i va demostrar que $Ass(M/I^nM)$ és estable per $n \gg 0$. Com a conseqüència, depth(M/I^nM) és constant per $n \gg 0$ i

$$l(I,M) \leq \dim(M) - \lim_{n\to\infty} \operatorname{depth}(M/I^nM)$$

essent $l(I,M) = \dim(\bigoplus_n I^nM/\mathfrak{m}I^nM)$.

Recentment, Herzog i Hibi van generalitzar a [HH05] aquestes desigualtats per mòduls graduats sobre àlgebres graduades estàndard. Van demostrar que per un mòdul graduat E,

$$\lim_{n\to\infty} \operatorname{depth}(E_n) \leq \dim(E) - \dim(E/\mathfrak{m}E).$$

En aquest cas, els autors no consideren l'estudi de l'estabilitat asimptòtica de $Ass(E_n)$ per assegurar l'estabilitat de les profunditats. La clau va ser utilitzar el polinomi de Hilbert dels mòduls graduats d'homologia de Koszul.

Branco Correia i Zarzuela, en [BZ06], van demostrar per a R-mòduls $E \subsetneq G \cong R^e$, $e > 0$, que depth(G_n/E_n) pren un valor constant per valors de n prou grans, i la desigualtat

$$l_G(R) \leq \dim(R) + e - 1 - \min_{n\geq 1}\{\operatorname{depth}(G_n/E_n)\}$$

essent $l_G(E) = \dim(\mathcal{R}_G(E)/\mathfrak{m}\mathcal{R}_G(E))$. Aquí, la profunditat constant es basa en l'estabilitat asimptòtica dels primers associats.

A [Hay06], Hayasaka va demostrar el cas més general considerat fins ara. Estudia la situació multigraduada estàndard. El seu estudi es basa en els primers associats a un mòdul multigraduat. En particular, va demostrar que per anells multigraduats estàndard $A \subset B$ amb $A_{\underline{0}} = B_{\underline{0}} = R$ un anell local, es té que $Ass(B_{\underline{n}}/A_{\underline{n}})$ és estable per $\underline{n} \gg \underline{0}$. Com a conseqüència, depth$(B_{\underline{n}}/A_{\underline{n}})$ és asimptòticament constant. Hayasaka, va generalitzar també la desigualtat i va demostrar que

$$s(A) \leq s(B) + \dim(R) - \operatorname{depth}(A,B)$$

si depth$(A,B) < \infty$, essent aquest valor la profunditat asimptòtica de $B_{\underline{n}}/A_{\underline{n}}$, i $s(G) = \dim \operatorname{Proj}^r(G/\mathfrak{m}G) + 1$ la dispersió de G, definida per un anell multigraduat estàndard G amb $G_{\underline{0}} = R$ un anell local amb ideal maximal $\mathfrak{m}$.

Aleshores, és natural preguntar-se què passa en un cas multigraduat no-estàndard. Són constants les profunditats de les peces homogènies d'un mòdul multigraduat per graus prou grans? Com es veuen afectades per la graduació? L'aproximació al problema de Herzog i Hibi ens va donar el camí a seguir. Utilitzant la funció de Hilbert dels mòduls d'homologia de Koszul d'un mòdul multigraduat no-estàndard i el seu comportament quasi-polinòmic, podem demostrar que aquesta profunditat és constant en una sub-xarxa d'un con a $\mathbb{N}^r$. En algun casos en què la funció de Hilbert és realment un polinomi, podem assegurar profunditat constant en un con de $\mathbb{N}^r$. Per les àlgebres de blow-up multigraduades, podem demostrar que les peces homogènies de l'àlgebra de Rees i les dels k-èsims anells multigraduats associats, tenen profunditat constant per graus prou grans. A més, tots els k-èsims anells multigraduats associats tenen la mateixa profunditat asimptòtica i podem demostrar que $R/I_1^{n_1} \cdots I_r^{n_r}$ tenen també profunditat constant per $n_1, \ldots, n_r$ prou grans.

Un altre problema interessant, és l'estudi de les àlgebres de blow-up multigraduades definides per un conjunt de potències d'ideals. Estudiar propietats per potències d'ideals en lloc d'estudiar-les directament per ideals és molt útil, ja que en moltes ocasions es poden deduir bones propietats relacionades amb els ideals, a partir de propietats relacionades amb les potències dels mateixos ideals. Per exemple, en el cas graduat, es pot demostrar que quan $\operatorname{depth}(gr_I(R)) \geq \dim(R) - 1$ es té que $e_i(I) \geq 0$ per a tot $i = 0, \ldots, \dim(R)$, on $e_i(I)$ són els coeficients de Hilbert de l'ideal I, [Mar89]. Malauradament, aquesta és una condició molt forta. Per altra banda, sota algunes hipòtesis, es pot demostrar que $\operatorname{depth}(gr_{I^n}(R)) \geq \dim(R) - 1$, i per tant $e_i(I^n) \geq 0$. Per exemple, això passa quan $\dim(R) = 2$ i I és un ideal normal, ja que aleshores $gr_{I^n}(R)$ és Cohen-Macaulay per $n \gg 0$, [HH99]. En aquest cas, els coeficients $e_i(I)$ es poden escriure fàcilment en termes dels $e_i(I^n)$, i per tant es poden deduir propietats del comportament "asimptòtic"dels coeficients d'una potència prou gran de l'ideal I. Un altre exemple, pot ser el resultat de [CPR05], on, en dimensió 3, es demostra que $e_3(I) \geq 0$ suposant que I^n és íntegrament tancat per algun $n \gg 0$. El resultat es dedueix del comportament de $e_3(I^n)$.

A el cas multigraduat, podem considerar l'àlgebra de Rees multigraduada

$$\mathcal{R}(I_1^{a_1}, \ldots, I_r^{a_r}).$$

En [HHR93], [HHR95] i [Hyr99], han estat estudiades les propietats Cohen-Macaulay i Gorenstein per àlgebres de Rees multigraduades per potències d'ideals. Per exemple, en [HHR93], s'ha demostrat que si $\mathcal{R}(I, \ldots, I)$ és

Cohen-Macaulay per algun nombre r de còpies d'un ideal I d'altura positiva, aleshores $\mathcal{R}(I^q)$ és Cohen-Macaulay per a tot $q \geq r$.

Podem observar que $\mathcal{R}(I_1^{a_1}, \ldots, I_r^{a_r}) = \mathcal{R}(I_1, \ldots, I_r)^{(\underline{a})}$ és la transformada Veronese de l'àlgebra de Rees multigraduada associada a ideals $I_1, \ldots, I_r$, amb $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{N}^{*r}$. Per tant, sembla natural estudiar els mòduls Veronese com a via per aproximar-nos a les àlgebres de Rees multigraduades de potències d'ideals, i més en general, estudiar les transformades Veronese de mòduls multigraduats no-estàndard.

En el cas graduat, Elias, utilitzant mòduls Veronese, va demostrar que $\operatorname{depth}(\mathcal{R}(I^n))$ és constant per n prou gran suposant que l'anell R és quocient d'un anell local regular, [Eli04].

De nou sorgeix la pregunta natural. Podem obtenir resultats similars per àlgebres de Rees multigraduades o per mòduls multigraduats no-estàndard? A la tesi obtenim alguns resultats sobre el comportament asimptòtic de la profunditat de mòduls Veronese multigraduats, alguns d'ells en el cas més general no-estàndard que hem estat considerant, mentre que d'altres en un cas més restringit

Un dels problemes clàssics de l'àlgebra commutativa és estimar la profunditat de l'anell graduat associat $gr_I(R) = \bigoplus_{n \geq 0} I^n / I^{n+1}$ i de l'àlgebra de Rees $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n$ per ideals I amb bones propietats. Sigui $(R, \mathfrak{m})$ un anell local Cohen-Macaulay de dimensió d. Sigui I un ideal $\mathfrak{m}$-primari de R amb una reducció minimal J.

Valabrega i Valla van demostrar que $gr_I(R)$ és Cohen-Macaulay si i només si $I^{p+1} \cap J = I^p J$ per a tot $p \geq 0$, [VV78]. De fet, el $\mathcal{R}(J)$-mòdul

$$\bigoplus_{p \geq 0} \frac{I^{p+1} \cap J}{I^p J}$$

és l'anomenat mòdul de Valabrega-Valla de I respecte a J. Relacionat amb això, Guerrieri, a la seva tesi doctoral, [Gue93], va demostrar que si

$$\sum_{p \geq 0} \operatorname{length}_R \left(\frac{I^{p+1} \cap J}{I^p J} \right) = 1$$

aleshores $\operatorname{depth}(gr_I(R)) \geq d - 1$. Basant-se en aquests resultats, Guerrieri, [Gue93], [Gue94] va conjecturar que

$$\operatorname{depth}(gr_I(R)) \geq d - \Delta(I, J),$$

on $\Delta(I,J) = \sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1}\cap J}{I^pJ}\right)$.
Guerrieri també va provar alguns casos parcials per $\Delta(I,J) = 2$, [Gue93], [Gue95], més concretament, va demostrar que si $\text{length}_R \left(\frac{I^2\cap J}{IJ}\right) = 1$ i $I^{p+1} \cap J = I^pJ$ per a tot $p \geq 2$ aleshores $\text{depth}(gr_I(R)) \geq d-2$. Alguns anys més tard, Wang va demostrar el cas general per $\Delta(I,J) = 2$, [Wan00].

Guerrieri va donar alguns exemples a la seva tesi d'ideals tals que

$$\text{length}_R \left(\frac{I^{p+1}\cap J}{I^pJ}\right) = \begin{cases} 1, & \text{per un nombre finit d'enters } p; \\ 0, & \text{altrament,} \end{cases}$$

amb $\text{depth}(gr_I(R)) = d-1$. Per això Guerrieri, i Huneke, es van preguntar si les condicions $\text{length}_R \left(\frac{I^{p+1}\cap J}{I^pJ}\right) \leq 1$, $p \geq 1$, implicarien que $\text{depth}(gr_I(R)) \geq d-1$, [Gue93], Question 2.23. Wang a [Wan02], Example 3.13, va trobar un contraexemple a la pregunta i es va preguntar si la resposta seria afirmativa en el cas en què l'anell R fos regular.

Considerant unes altres longituds, Huckaba i Marley van demostrar que

$$e_1(I) \leq \sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1}}{I^pJ}\right)$$

i que en cas de tenir la igualtat, aleshores $\text{depth}(gr_I(R)) \geq d-1$, [HM97]. Així podem considerar l'enter no negatiu

$$\delta(I,J) = \sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1}}{I^pJ}\right) - e_1(I) \geq 0.$$

Wang va conjecturar que, [Wan00],

$$\text{depth}(gr_I(R)) \geq d - 1 - \delta(I,J).$$

Va demostrar que $\delta(I,J) \leq \Delta(I,J)$ i que la seva conjectura implicava la de Guerrieri. Huckaba va demostrar la conjectura per $\delta(I,J) = 0$, [Huc96], [HM97]. Wang la va demostrar per $\delta(I,J) = 1$, [Wan00], i Polini en va donar una demostració més senzilla, [Pol00]. Per $\delta(I,J) = 2$ Rossi i Guerrieri van provar la Conjectura de Wang suposant que l'anell R/I fos Gorenstein, [GR99]. Malgrat tot, Wang va donar un contraexemple de la seva conjectura per $d = 6$, [Wan01].

Ja que ha estat provat que, en general, aquestes conjectures no són sempre certes, però en alguns exemples es veu una relació entre els enters i la profunditat, es pot pensar en intentar refinar les conjectures considerant altres configuracions dels enters.

En el resultat del capítol corresponent, demostrem una versió refinada de la Conjectura de Wang. El que fem és descompondre l'enter $\delta(I,J)$ com a suma finita d'enters no negatius $\delta_p(I,J)$, amb $\text{length}_R(\frac{I^{p+1}\cap J}{I^pJ}) \geq \delta_p(I,J) \geq 0$. Si $\bar{\delta}(I,J)$ és el màxim dels enters $\delta_p(I,J)$ per $p \geq 0$, quan $\bar{\delta}(I,J) \leq 1$, som capaços de demostrar que $\text{depth}(\mathcal{R}(I)) \geq d - \bar{\delta}(I,J)$ i $\text{depth}(gr_I(R)) \geq d - 1 - \bar{\delta}(I,J)$. Com a conseqüència, podem respondre la pregunta formulada per Guerrieri i Huneke sobre considerar per a tot $p \geq 0$, $\text{length}_R(\frac{I^{p+1}\cap J}{I^pJ}) \leq 1$. En aquesta situació podem demostrar que $\text{depth}(gr_I(R)) \geq d-2$. El punt clau és la interpretació d'aquests enters com a multiplicitats d'alguns mòduls bigraduats no-estàndard.

Alguns dels resultats de la tesi han estat publicats a:

[CE06] G. Colomé-Nin and J. Elias. *Bigraded structures and the depth of blow-up algebras*. Proceedings of the Royal Society of Edinburgh, 136A, 1175-1194, 2006.

Resum dels resultats

A continuació resumim els continguts i resultats principals obtinguts en aquesta tesi.

El **Capítol 1** està dedicat a recordar algunes definicions i propietats que ens serveixen con a material de base per a la resta del treball.

Estructures multigraduades

Al **Capítol 2** estudiem propietats relacionades amb la funció de Hilbert d'un mòdul multigraduat no-estàndard. En concret considerem un anell S $\mathbb{N}^r$-graduat generat sobre $S_{\underline{0}}$ per elements $g_i^1,\ldots,g_i^{\mu_i}$ que tenen graus $\gamma_i = (\gamma_1^i,\ldots,\gamma_i^i,0\ldots,0) \in \mathbb{N}^r$, amb $\gamma_i^i \neq 0$, per $i=1,\ldots,r$, on $(S_{\underline{0}},\mathfrak{m})$ és un anell local Artinià, i M és un S-mòdul $\mathbb{Z}^r$-graduat finitament generat.

Es pot observar que aquesta graduació admet el cas estàndard com a cas particular. Tot i així, per abús del llenguatge, ens referim a aquesta graduació més general com a *no-estàndard*, per tenir presents les diferències amb la situació estàndard. En cap cas excloem la graduació estàndard de la nostra definició.

Utilitzant aquesta graduació, podem definir l'ideal irrellevant S_{++} de S com $S_{++} = I_1 \cdots I_r$, on I_j és l'ideal de S generat per les peces homogènies de grau $(b_1,\ldots,b_j,0\ldots,0)$, amb $b_j \neq 0$. Aleshores podem definir $\mathrm{Proj}^r(S)$ com el conjunt de tots el ideals primers homogenis rellevants de S, que és el conjunt de tots els ideals primers homogenis $\mathfrak{p}$ de S tals que $\mathfrak{p} \not\supset S_{++}$.

Definim la dimensió rellevant del S-mòdul multigraduat com l'enter

$$\mathrm{rel.dim}(M) = \begin{cases} r-1 & \text{si } Supp_{++}(M) = \emptyset \\ \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in Supp_{++}(M)\} & \text{si } Supp_{++}(M) \neq \emptyset. \end{cases}$$

Denotant $\underline{n} = (n_1,\ldots,n_r)$, definim la funció de Hilbert de M com

$$\begin{array}{rccc} h_M: & \mathbb{Z}^r & \longrightarrow & \mathbb{Z} \\ & \underline{n} & \mapsto & \mathrm{length}_{S_{\underline{0}}}(M_{\underline{n}}). \end{array}$$

A la Secció 2.3.1, introduïm la definició de quasi-polinomi en el cas $\mathbb{Z}^r$-graduat i n'estudiem algunes propietats que seran útils per demostrar que la funció de Hilbert és quasi-polinòmica.

Diem que una funció $f : \mathbb{N}^r \to \mathbb{Z}$ és una *funció quasi-polinòmica de grau polinòmic d en $\underline{\beta}, \gamma_1,\ldots,\gamma_r$* si existeixen funcions periòdiques $c_{\underline{\alpha}} : \mathbb{N}^r \to \mathbb{Z}$, per $\underline{\alpha} \in \mathbb{N}^r$ i $|\underline{\alpha}| \leq d$, respecte de $\gamma_1,\ldots,\gamma_r$ tals que per $\underline{n} \in C_{\underline{\beta}}$

$$f(\underline{n}) = \sum_{|\underline{\alpha}|\leq d} c_{\underline{\alpha}}(\underline{n})\underline{n}^{\underline{\alpha}}$$

i $f(\underline{n}) = 0$ quan $\underline{n} \notin C_{\underline{\beta}}$, i existeix algun $\underline{\alpha} \in \mathbb{N}^r$ amb $|\underline{\alpha}| = d$ tal que $c_{\underline{\alpha}} \neq 0$. Anomenem *quasi-polinomi* a una expressió $\sum_{|\underline{\alpha}|\leq d} c_{\underline{\alpha}}(\underline{n})\underline{n}^{\underline{\alpha}}$. Aquí, definim un con com $C_{\underline{\beta}} := \left\{\underline{\alpha} \in \mathbb{N}^r \mid \underline{\alpha} = \underline{\beta} + \sum_{i=1}^r \lambda_i\gamma_i,\ \ \lambda_i \in \mathbb{R}_{\geq 0}\right\}$.

Aleshores demostrem que la funció de Hilbert és quasi-polinòmica.

Proposició 2.3.10. *Sigui S un anell $\mathbb{N}^r$-graduat com abans. Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat. Aleshores existen un quasi-polinomi P_M de grau polinòmic* $\mathrm{rel.dim}(M) - r$ *i un con $C_{\underline{\beta}} \subset \mathbb{N}^r$, tals que per a tot $\underline{n} \in C_{\underline{\beta}}$*

$$h_M(\underline{n}) = P_M(\underline{n}).$$

Com en el cas estàndard, podem demostrar en la nostra situació, la fórmula de Grothendieck-Serre, que relaciona la funció de Hilbert, el quasi-polinomi de Hilbert i les longituds dels mòduls de cohomologia local M respecte de l'ideal irrellevant.

Proposició 2.4.3. *Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat. Aleshores per a tot $\underline{n} \in \mathbb{Z}^r$,*

$$h_M(\underline{n}) - P_M(\underline{n}) = \sum_{i \geq 0} (-1)^i \operatorname{length}_{S_{\underline{0}}}(H^i_{S_{++}}(M)_{\underline{n}})$$

L'última part del segon capítol està dedicada a generalitzar la funció de Hilbert-Samuel d'un ideal $\mathfrak{m}$-primari I d'un anell local Noetherià $(R, \mathfrak{m})$ a un conjunt d'ideals $\mathfrak{m}$-primaris $I_1, \ldots, I_r$. Podem demostrar que per a tot $k = 1, \ldots, r$ la funció $f_k(\underline{n}) = \operatorname{length}_R(R/I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r})$ és polinòmica per $\underline{n} \geq \underline{\beta}_k$, per algun $\underline{\beta}_k \in \mathbb{N}^r$. Anomenem p_k a aquest polinomi. Després d'això, podem obtenir una fórmula semblant a la de Grothendieck-Serre, que relacioni aquesta funció i aquest polinomi amb les longituds d'alguns mòduls de cohomologia local de la k-èsima àlgebra de Rees extesa $\mathcal{R}_k^*$ de $I_1 \ldots, I_r$ respecte de l'ideal irrellevant de l'algebra de Rees multigraduada dels ideals.

Per un element $\underline{\delta} \in \mathbb{N}^r$, definim $\mathcal{H}^k_{\underline{\delta}}$ com el conjunt dels $\underline{n} \in \mathbb{Z}^r$ tals que $n_k \in \mathbb{Z}$ i $(n_1, \ldots, n_{k-1}, n_{k+1}, \ldots, n_r) \geq (\delta_1, \ldots, \delta_{k-1}, \delta_{k+1}, \ldots, \delta_r)$.

Teorema 2.4.8. *Existeix un element $\underline{\delta} \in \mathbb{N}^r$ tal que per a tot $\underline{n} \in \mathcal{H}^k_{\underline{\delta}}$ es satisfà*

$$p_k(\underline{n}) - f_k(\underline{n}) = \sum_{i \geq 0} (-1)^i \operatorname{length}_R(H^i_{\mathcal{R}_{++}}(\mathcal{R}_k^*)_{\underline{n}+e_k}).$$

Profunditat asimptòtica de mòduls multigraduats

Al **Capítol 3** estem interessats en l'estudi de la profunditat de les peces graduades d'un mòdul multigraduat sobre un anell Noetherià multigraduat no-estàndard amb la graduació considerada en el capítol precedent. A la primera secció, donem un cop d'ull al complex de Koszul i a l'homologia de Koszul en el cas multigraduat, un concepte que necessitem per assolir el nostre objectiu.

Sigui S un anell $\mathbb{N}^r$-graduat, generat sobre $S_{\underline{0}}$ per elements de multigraus $\gamma_1, \ldots, \gamma_r$, on $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0, \ldots, 0) \in \mathbb{N}^r$ i $\gamma_i^i \neq 0$ per a tot $i = 1, \ldots, r$. Sigui $\mathcal{M} = \mathfrak{m} \oplus \bigoplus_{\underline{n} \neq \underline{0}} S_{\underline{n}}$ l'ideal maximal homogeni de S, on $\mathfrak{m}$ és l'ideal maximal de l'anell local Noetherià $S_{\underline{0}}$.

Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat. A la Secció 3.2 estudiem la profunditat asimptòtica, respecte de $\mathfrak{m}$, de les peces multigraduades $M_{\underline{n}}$. El punt clau de la demostració és l'existència del quasi-polinomi de Hilbert pels mòduls d'homologia de Koszul de M respecte d'un sistema de generadors de $\mathfrak{m}$. El comportament quasi-polinòmic de la funció de Hilbert ens permet demostrar el teorema.

Teorema 3.2.1. *Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat. Existeixen un element $\underline{\beta} \in \mathbb{N}^r$ i un enter $\rho \in \mathbb{N}$ tals que,*

$$\operatorname{depth}(M_{\underline{n}}) \geq \rho$$

per a tot $\underline{n} \in C_{\underline{\beta}}$ amb $M_{\underline{n}} \neq 0$, i

$$\operatorname{depth}(M_{\underline{n}}) = \rho$$

per algun $\underline{\delta} \in \Pi_{\underline{\beta}}$ i per a tot $\underline{n} \in \{\underline{\delta} + \sum_{i=1}^{r} \lambda_i \gamma_i \mid \lambda_i \in \mathbb{N}\} \subset C_{\underline{\beta}}$.

Quan el quasi-polinomi és de fet un polinomi, podem assegurar la profunditat constant en tot el con:

- ▷ **Proposició 3.2.3**: Si S és una àlgebra generada sobre $S_{\underline{0}}$ per elements de graus $(1,0,\ldots,0)$, $(*,1,0,\ldots,0)$, $\ldots$, $(*,*,*,\ldots,1) \in \mathbb{N}^r$, aleshores per $\underline{n} \in C_{\underline{\beta}}$ es té $\operatorname{depth}(M_{\underline{n}}) = \rho$.
- ▷ **Corol·lari 3.2.4**: Si S és una àlgebra estàndard, aleshores per $\underline{n} \geq \underline{\beta}$ es té $\operatorname{depth}(M_{\underline{n}}) = \rho$.

A la Secció 3.3, considerem l'àlgebra de Rees multigraduada associada a ideals $I_1,\ldots,I_r$ d'un anell local Noetherià $(R,\mathfrak{m})$, i per $k = 1,\ldots,r$, el k-èsim anell multigraduat associat a $I_1,\ldots,I_r$ en R. En els dos casos, són $\mathcal{R}(I_1,\ldots,I_r)$-mòduls $\mathbb{Z}^r$-graduats estàndard i finitament generats, i cada component, $\mathcal{R}(I_1,\ldots,I_r)_{\underline{n}}$ i $gr_{I_1,\ldots,I_r;I_k}(R)_{\underline{n}}$, és un R-mòdul finitament generat. Utilitzant els resultats previs podem demostrar:

Proposició 3.3.1, 3.3.3. *Existeixen un element $\underline{\beta} \in \mathbb{N}^r$ i un enter $\delta \in \mathbb{N}$ tals que per a tot $\underline{n} \geq \underline{\beta}$ es té*

$$\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \delta + 1$$

i

$$\operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k} \cdots I_r^{n_r}}\right) = \delta$$

per a tot $k = 1,\ldots,r$.

Estem interessats en la profunditat de $R/I_1^{n_1} \cdots I_r^{n_r}$ per $\underline{n}$ prou grans. En aquest cas, podem treure profit de la profunditat asimptòtica dels dos darrers mòduls i la relació amb $R/I_1^{n_1}\cdots I_r^{n_r}$ per mitjà de algunes successions exactes curtes de R-mòduls on podem utilitzar tècniques de comptar profunditats.

Teorema 3.3.6. *Existeixen un element $\underline{\varepsilon} \in \mathbb{N}^r$ i un enter $\rho \in \mathbb{N}$ tals que*

$$\operatorname{depth}\left(\frac{R}{I_1^{n_1}\cdots I_r^{n_r}}\right) = \rho \leq \delta$$

per a tot $\underline{n} \geq \underline{\varepsilon}$. A més a més, si existeix un $\underline{n} \geq \underline{\beta}$ tal que $\operatorname{depth}\left(\frac{R}{I_1^{n_1}\cdots I_r^{n_r}}\right) \geq \delta$*, aleshores $\rho = \delta$.*

Finalment, podem també afitar la profunditat asimptòtica dels mòduls $R/I_1^{n_1}\cdots I_r^{n_r}$.

Proposició 3.3.7. *Sigui $\rho \in \mathbb{N}$ la profunditat asimptòtica de $R/I_1^{n_1}\cdots I_r^{n_r}$. Aleshores,*

$$\rho \leq \dim(R) - \dim \operatorname{Proj}^r\left(\frac{\mathcal{R}(I_1,\ldots,I_r)}{\mathfrak{m}\mathcal{R}(I_1,\ldots,I_r)}\right).$$

Mòduls Veronese multigraduats

L'objectiu del **Capítol 4** és estudiar els mòduls Veronese associats a S-mòduls multigraduats no-estàndard M per mitjà d'algunes propietats cohomològiques del mòdul. Principalment estudiem l'anul·lació dels mòduls de cohomologia local de M i dels mòduls Veronese de M, generalitzant alguns resultats sobre la profunditat asimptòtica dels mòduls Veronese associats a àlgebres de Rees. També estudiem el comportament asimptòtic dels mòduls Veronese.

Estem encara considerant la situació general en la que S és un anell Noetherià $\mathbb{N}^r$-graduat generat com a $S_{\underline{0}}$-àlgebra per elements homogenis g_i^j per $i = 1,\cdots,r$ i $j = 1,\ldots,\mu_i$, de multigraus $\gamma_i = (\gamma_1^i,\ldots,\gamma_i^i,0,\ldots,0) \in \mathbb{N}^r$, respectivament, amb $\gamma_i^i \neq 0$. Suposem que $S_{\underline{0}}$ és un anell local amb ideal maximal $\mathfrak{m}$ i cos residual infinit.

Per $\underline{a} \in \mathbb{N}^{*r}$ denotem $\phi_{\underline{a}}(\underline{n}) = \sum_{i=1}^r (n_i a_i)\gamma_i$ per a tot $\underline{n} \in \mathbb{Z}^r$.

La transformada Veronese de S respecte de $\underline{a} \in \mathbb{N}^{*r}$, o $(\underline{a})$-Veronese, és el subanell de S

$$S^{(\underline{a})} = \bigoplus_{\underline{n}\in\mathbb{N}^r} S_{\phi_{\underline{a}}(\underline{n})}.$$

Per un S-mòdul $\mathbb{Z}^r$-graduat M, denotem per $M^{(\underline{a},\underline{b})}$ la transformada Veronese de M respecte de $\underline{a},\underline{b} \in \mathbb{N}^{*r}$, o $(\underline{a},\underline{b})$-Veronese, que és el $S^{(\underline{a})}$-mòdul

$$M^{(\underline{a},\underline{b})} = \bigoplus_{\underline{n}\in\mathbb{Z}^r} M_{\phi_{\underline{a}}(\underline{n})+\underline{b}}.$$

A la **Proposició 4.1.6**, demostrem que el functor Veronese commuta amb la cohomologia local respecte de $\mathcal{M}$. Aquest fet és important per molts dels resultats.

A la Secció 4.2, entre d'altres propietats, estudiem la profunditat generalitzada d'un mòdul multigraduat i dels seus mòduls Veronese. Aquest és un invariant important per aconseguir el nostre objectiu. Comencem demostrant alguns resultats (**Proposició 4.2.1**, **Proposició 4.2.2**, **Proposició 4.2.3**, **Proposició 4.2.4**) relacionant propietats d'anells i mòduls $\mathbb{Z}^r$-graduats no-estàndard amb les seves transformades Veronese.

Per un S-mòdul $\mathbb{Z}^r$-graduat M finitament generat, definim la *profunditat generalitzada* de M respecte de l'ideal maximal homogeni $\mathcal{M}$ de S com

$$\mathrm{gdepth}(M) = \max\{k \in \mathbb{N} \mid S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M))) \text{ per a tot } i < k\}.$$

També definim la *desviació Cohen-Macaulay projectiva* de M com

$$\mathrm{pcmd}(M) = \max\{\dim(S_{(\mathfrak{p})}) - \mathrm{depth}(M_{(\mathfrak{p})}) \mid \mathfrak{p} \in \mathrm{Proj}^r(S)\}.$$

Quan $S_{\underline{0}}$ és el quocient d'un anell regular, podem relacionar aquests dos enters:

Teorema 4.2.7. *Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat. Si $S_{\underline{0}}$ és el quocient d'un anell regular, aleshores*

$$\mathrm{gdepth}(M) = \dim(S) - \mathrm{pcmd}(M).$$

Aleshores, amb aquesta hipòtesi, podem demostrar la invariància de gdepth sota transformades Veronese:

Corol·lari 4.2.8. *Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat. Si $S_{\underline{0}}$ és el quocient d'un anell regular, per a tot $\underline{a},\underline{b} \in \mathbb{N}^{*r}$ es té que*

$$\mathrm{gdepth}(M^{(\underline{a},\underline{b})}) = \mathrm{gdepth}(M).$$

A la Secció 4.3 volem estudiar la profunditat dels mòduls Veronese $M^{(\underline{a},\underline{b})}$ per valors grans de $\underline{a}, \underline{b} \in \mathbb{N}^r$. Com a solució parcial, sota les hipòtesis generals dels multigraus d'aquest capítol, podem demostrar que la profunditat d'alguns mòduls Veronese $M^{(\underline{a})}$ és constant per $\underline{a}$ en una xarxa de $\mathbb{N}^r$. Observeu que $M^{(\underline{a})} = M^{(\underline{a},\underline{0})}$.

La profunditat asimptòtica Veronese de M, la denotem $vad(M^{(*)})$ (resp. $vad(M^{(*,*)})$), que és el màxim del valor $\operatorname{depth}(M^{(\underline{a})})$ (resp. $\operatorname{depth}(M^{(\underline{a},\underline{b})})$) per a tot $\underline{a} \in \mathbb{N}^{*r}$ (resp. per a tot $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$).

Proposició 4.3.1. *Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat. Sigui $s = vad(M^{(*)})$. Existeix un element $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{N}^{*r}$ tal que per a tot $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$*

$$\operatorname{depth}(M^{(\underline{b})}) = s.$$

El resultat anterior es pot utilitzar per tal d'estudiar la profunditat de les àlgebres de Rees multigraduades d'algunes potències d'ideals.

Proposició 4.3.2. *Siguin $I_1, \ldots, I_r$ ideals d'un anell local Noetherià $(R, \mathfrak{m})$. Sigui $s = vad(\mathcal{R}(I_1, \ldots, I_r)^{(*)})$. Existeix $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{N}^{*r}$ tal que per a tot $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$*

$$\operatorname{depth}(\mathcal{R}(I_1^{b_1}, \ldots, I_r^{b_r})) = s.$$

A més a més, si $\operatorname{depth}(\mathcal{R}(I_1, \ldots, I_r)) = s$*, aleshores, per a tot $\underline{b} \in \mathbb{N}^{*r}$,*

$$\operatorname{depth}(\mathcal{R}(I_1^{b_1}, \ldots, I_r^{b_r})) = s.$$

Per tal d'extendre els resultats anteriors sobre la profunditat asimptòtica dels mòduls Veronese a regions de $\mathbb{N}^r$, cal que estudiem l'anul·lació dels mòduls de cohomologia local del mòdul multigraduat M.

Direm que un S-mòdul $\mathbb{Z}^r$-graduat M és Γ-*finitament graduat* si existeix un con $C_{\underline{\beta}} \subset \mathbb{N}^r$ on $M_{\underline{n}} = 0$ per a tot $\underline{n} \in \mathbb{Z}^r$ tal que $\underline{n}^* = (|n_1|, \ldots, |n_r|) \in C_{\underline{\beta}}$. Denotem per Γ-fg(M) l'enter més gran $k \geq 0$ tal que $H^i_{\mathcal{M}}(M)$ és Γ-finitament graduat per a tot $i < k$.

Per raons tècniques, hem de restringir els graus dels generadors de S, vegeu Remark 4.3.6 i Remark 4.3.11. Durant la resta del capítol, suposarem que la graduació és *quasi-estàndard*, i.e. els graus són $\gamma_1, \ldots, \gamma_r$ amb $\gamma_i = (0, \ldots, 0, \gamma_i^i, 0, \ldots, 0)$ i $\gamma_i^i > 0$ per a tot $i = 1, \ldots, r$.

Un fet important en les demostracions és poder assegurar en el cas quasi-estàndard, que $H^k_{\mathcal{M}}(M)$ és Γ-finitament graduat per a tot $k \geq 0$ en el cas en què M també sigui Γ-finitament graduat, vegeu **Proposició 4.3.5**.

En el proper resultat relacionem els dos enters estudiats al capítol associats a M, gdepth(M) i Γ-fg(M).

Teorema 4.3.7. *Sigui S un anell multigraduat quasi-estàndard. Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat. Aleshores,*

$$\Gamma\text{-fg}(M) = \text{gdepth}(M).$$

Com a conseqüència, suposant que $S_{\underline{0}}$ sigui el quocient d'un anell local regular podem demostrar l'invariància de Γ-fg sota transformades Veronese al **Corol·lari 4.3.8**.

Ara, tenim noves eines per demostrar el teorema que assegurar profunditat constant per les $(\underline{a}, \underline{b})$-Veronese en una regió de $\mathbb{N}^r \times \mathbb{N}^r$, en lloc d'una xarxa. Malauradament, la restricció al cas quasi-estàndard és necessària encara.

Teorema 4.3.12. *Sigui S un anell multigraduat quasi-estàndard tal que $S_{\underline{0}}$ és el quocient d'un anell regular. Sigui M un S-mòdul $\mathbb{Z}^r$-graduat finitament generat i sigui $s = vad(M^{(*,*)})$. Aleshores, existeix un $\underline{\beta} \in \mathbb{N}^r$ tal que per a tot $\underline{b} \geq \underline{\beta}$ i per a tot $\underline{a} \in \mathbb{N}^r$ tal que $a_i \geq (\beta_i + b_i)/\gamma^i_i$ es té*

$$\text{depth}(M^{(\underline{a},\underline{b})}) = s.$$

Per mòduls $\mathbb{Z}$-graduats generals obtenim:

Proposició 4.3.13. *Sigui S un anell $\mathbb{Z}$-graduat tal que $S_{\underline{0}}$ és el quocient d'un anell regular. Sigui M un S-mòdul graduat finitament generat. Aleshores* depth$(M^{(a)})$ *és constant per $a \gg 0$.*

Per l'àlgebra de Rees multigraduada, la millor aproximació al problema és la següent proposició.

Proposició 4.3.15. *Si R és quocient d'un anell regular, existeixen un enter s i $\underline{\beta} \in \mathbb{N}^r$ tals que per a tot $\underline{b} \geq \underline{\beta}$ i $\underline{a} \geq \underline{\beta} + \underline{b}$ es té*

$$\text{depth}_{\mathcal{M}^{(a)}}((I_1^{b_1} \cdots I_r^{b_r})\mathcal{R}(I_1^{a_1}, \cdots, I_r^{a_r})) = s.$$

Estructures bigraduades i profunditat d'àlgebres de blow-up

Al **Capítol 5** volem trobar versions refinades de les conjectures sobre la profunditat de les àlgebres de blow-up de les que hem parlat al principi. La idea principal és estudiar la profunditat de les àlgebres de blow-up mitjançant uns certs mòduls bigraduats. Interpretem les longituds que apareixen a les conjectures com a multiplicitats d'alguns mòduls bigraduats no-estàndard. Gràcies a aquesta interpretació, som capaços de refinar la Conjectura de Wang, afegint nous casos on funciona i recuperant-ne es casos coneguts. Com a corol·lari, podem respondre la pregunta de Guerrieri i Huneke sobre les longituds de les peces del mòdul de Valabrega-Valla.

Per un anell local Cohen-Macaulay $(R, \mathfrak{m})$ de dimensió $d > 0$ i cos residual infinit i un ideal $\mathfrak{m}$-primari I de R amb una reducció minimal J, considerem els enters que intervenen a les conjectures :

$$\Delta(I,J) = \sum_{p\geq 0} \mathrm{length}_R \left(\frac{I^{p+1} \cap J}{I^p J} \right), \quad \Lambda(I,J) = \sum_{p\geq 0} \mathrm{length}_R \left(\frac{I^{p+1}}{J I^p} \right),$$

$$\Delta_p(I,J) = \mathrm{length}_R \left(\frac{I^{p+1} \cap J}{I^p J} \right), \quad \Lambda_p(I,J) = \mathrm{length}_R \left(\frac{I^{p+1}}{J I^p} \right)$$

per $p \geq 0$, i

$$\delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

Les seccions 5.2 i 5.3 estan dedicades principalment a recordar alguns resultats preliminars sobre el mòdul de Sally i la funció de Hilbert acumulativa d'un mòdul bigraduat no-estàndard. Recordem que la funció de Hilbert acumulativa d'un A-mòdul M està definida com a $h_M(m,n) = \sum_{j\leq n} \mathrm{length}_A(M_{(m,j)})$. En particular, demostrem que

Teorema 5.3.4. *Sigui $S = A[X_1, \ldots, X_r, Y_1, \ldots, Y_s, Z_1, \ldots, Z_t]$ un anell de polinomis bigraduat sobre un anell Artinià A amb indeterminades $X_1, \ldots, X_r$, $Y_1, \ldots, Y_s$ i $Z_1, \ldots, Z_t$, on cada X_i té bigrau $(1,0)$, cada Y_i té bigrau $(1,1)$, i cada Z_i té bigrau $(0,1)$. Sigui M un S-mòdul bigraduat finitament generat. Aleshores, existeixen enters m_0 i n_0 i un polinomi en dues variables $p_M(m,n)$ tal que*

$$p_M(m,n) = h_M(m,n)$$

per a tot (m,n) amb $m \geq m_0$ i $n \geq n_0 + m$.

A més a més, en el cas en què no tinguem generadors de bigrau $(0,1)$, el polinomi no depèn de n.

A la Secció 5.4 introduïm un mòdul bigraduat no-estàndard $\Sigma^{I,J}$ naturalment associat a I i a una reducció minimal J de I, aquest mòdul pot ser considerat com un refinament del mòdul de Sally. A partir d'una presentació natural de $\Sigma^{I,J}$, definim dos mòduls bigraduats més $K^{I,J}$ i $\mathcal{M}^{I,J}$, i en considerem alguns submòduls diagonals: $\Sigma^{I,J}_{[p]}$ i $K^{I,J}_{[p]}$. Tot seguit resumirem aquestes construccions.

Considerem l'anell graduat associat a $\mathcal{R}(I)$ respecte l'ideal homogeni $Jt\mathcal{R}(I) = \bigoplus_{n\geq 0} JI^{n-1}t^n$

$$gr_{Jt}(\mathcal{R}(I)) = \bigoplus_{j\geq 0} \frac{(Jt\mathcal{R}(I))^j}{(Jt\mathcal{R}(I))^{j+1}} U^j.$$

Aquest anell té una estructura natural bigraduada. Si considerem l'anell bigraduat $B := R[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$ amb $\deg(V_i) = (1,0)$ i $\deg(T_i) = (1,1)$, aleshores tenim uns successió exacta de B-anells bigraduats

$$0 \longrightarrow K^{I,J} \longrightarrow C^{I,J} := \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)}[T_1, \ldots, T_d] \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow 0$$

on $K^{I,J}$ és l'ideal de les formes inicials de $Jt\mathcal{R}(I)$.

Per un B-mòdul bigraduat M i un enter $p \in \mathbb{Z}$, denotem per $M_{[p]}$ el subgrup additiu de M definit per la suma directa de els peces $M_{(m,n)}$ tals que $m - n = p + 1$. En el nostre cas, els mòduls $K^{I,J}_{[p]}$, $C^{I,J}_{[p]}$ i $gr_{Jt}(\mathcal{R}(I))_{[p]}$ són $\mathcal{R}(J)$-mòduls, i no s'anul·len per un nombre finit d'índexos $p \in \mathbb{Z}$ (**Lema 5.4.1**).

A partir d'ara, estarem interessats en prendre les diagonals no negatives d'aquests mòduls i per tant, considerem els següents B-mòduls bigraduats finitament generats:

$$\Sigma^{I,J} := \bigoplus_{p\geq 0} gr_{Jt}(\mathcal{R}(I))_{[p]} = \bigoplus_{p\geq 0} \bigoplus_{i\geq 0} \frac{J^i I^{p+1}}{J^{i+1} I^p} t^{p+1+i} U^i$$

$$\mathcal{M}^{I,J} := \bigoplus_{p\geq 0} C^{I,J}_{[p]} = \bigoplus_{p\geq 0} \frac{I^{p+1}}{I^p J} t^{p+1} [T_1, \ldots, T_d]$$

i d'ara en endavant, considerem el nou

$$K^{I,J} := \bigoplus_{p\geq 0} K^{I,J}_{[p]}.$$

Anomenarem a $\Sigma^{I,J}$ el *mòdul de Sally bigraduat* de I respecte de J.
Del Lema 5.4.1 existeix un isomorfisme natural de $\mathcal{R}(J)$-mòduls

$$gr_{Jt}(\mathcal{R}(I)) \cong \mathcal{R}(J) \oplus \Sigma^{I,J}.$$

Com que els mòduls $\Sigma^{I,J}$ i $\mathcal{M}^{I,J}$ s'anul·len per J, tenim una successió exacta de $A = R/J[V_1, \dots, V_\mu; T_1, \dots, T_d]$-mòduls bigraduats

$$0 \longrightarrow K^{I,J} \longrightarrow \mathcal{M}^{I,J} \longrightarrow \Sigma^{I,J} \longrightarrow 0.$$

Considerant-ne cada diagonal, per a tot $p \geq 0$ tenim una successió exacta de $R/J[T_1, \dots, T_d]$-mòduls

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1, \dots, T_d] \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0,$$

que de fet són mòduls graduats, i per tant, podem considerar la funció de Hilbert (clàssica) per a ells.

Utilitzant la funció de Hilbert acumulativa amb els mòduls $\Sigma^{I,J}$, $\mathcal{M}^{I,J}$ i $K^{I,J}$, que en aquest cas és polinòmica en una variable en una regió de $\mathbb{N}^2$, podem demostrar els següents resultats que ens permeten interpretar els enters $e_1(I)$, $\Lambda(I,J)$ i $\delta(I,J)$ de la Conjectura de Wang com a multiplicitats dels nostres mòduls (**Proposició 5.5.2**, **Proposició 5.5.3**, **Proposició 5.5.4**, **Proposició 5.5.6**):

- ▷ $p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1}$.
- ▷ $\deg(p_{\mathcal{M}^{I,J}}) = d-1$ i $e_0(\mathcal{M}^{I,J}) = \Lambda(I,J)$.
- ▷ Si $\Sigma^{I,J} = 0$ aleshores $gr_I(R)$ és un anell Cohen-Macaulay.
  Si $\Sigma^{I,J} \neq 0$ aleshores $\deg(p_{\Sigma^{I,J}}) = d-1$ i $e_0(\Sigma^{I,J}) = e_1(I)$.
- ▷ $e_0(K^{I,J}) = \delta(I,J)$. Si $K^{I,J} \neq 0$ aleshores $\deg(p_{K^{I,J}}) = d-1$.
  En particular, $\Lambda(I,J) \geq e_1(I)$.
- ▷ Per a tot $p \geq 0$, $e_0(\Sigma^{I,J}_{[p]}) = \text{length}_R \left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]}) \geq 0$ i
  $e_1(I) = \sum_{p \geq 0} (\text{length}_R \left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]}))$.
- ▷ Per a tot $p \geq 0$, $\text{length}_R \left(\frac{I^{p+1} \cap J}{JI^p}\right) \geq e_0(K^{I,J}_{[p]})$ i
  $\delta(I,J) = e_0(K^{I,J}) = \sum_{p \geq 0} e_0(K^{I,J}_{[p]}) \geq 0$.

Definim

$$\delta_p(I,J) = e_0(K^{I,J}_{[p]}).$$

A la Secció 5.6 demostrem una versió refinada de la Conjectura de Wang considerant configuracions especials del conjunt $\{\delta_p(I,J)\}_{p\geq 0}$ en lloc de $\delta = \sum_{p\geq 0} \delta_p(I,J)$. Sigui $\bar{\delta}(I,J)$ el màxim dels enters $\delta_p(I,J)$ per a $p \geq 0$.

Teorema 5.6.3. *Sigui $(R, \mathfrak{m})$ un anell local Cohen-Macaulay de dimensió $d > 0$. Sigui I un ideal $\mathfrak{m}$-primari de R i J una reducció minimal de I. Si $\bar{\delta}(I,J) \leq 1$, aleshores*

$$\operatorname{depth}(\mathcal{R}(I)) \geq d - \bar{\delta}(I,J)$$

i

$$\operatorname{depth}(gr_I(R)) \geq d - 1 - \bar{\delta}(I,J).$$

Observeu que per $\delta(I,J) = 0, 1$ recuperem els casos coneguts de la Conjectura de Wang.

Per a la demostració necessitem els resultats importants següents. En particular, hem d'estudiar la profunditat de l'anell $gr_{Jt}(\mathcal{R}(I))$.

Teorema 5.6.1. *Sigui $(R, \mathfrak{m})$ un anell local Cohen-Macaulay de dimensió $d \geq 3$. Sigui I un ideal $\mathfrak{m}$-primari de R i sigui J una reducció minimal de I. Suposem que $K^{I,J} \neq 0$, i que, o bé $K^{I,J}_{[p]} = 0$, o bé, $K^{I,J}_{[p]}$ és un $\mathbf{k}[T_1, \ldots, T_d]$-mòdul lliure de torsió de rang 1 per $p \geq 0$. Aleshores,*

$$\operatorname{depth}(gr_{Jt}(\mathcal{R}(I))) \geq d - 1.$$

El lema següent és també important perquè assegura com són les diagonals $K^{I,J}_{[p]}$ en el cas de tenir $e_0(K^{I,J}_{[p]}) = 1$, i per tant, com és $K^{I,J}$ en la descomposició de $\delta(I,J)$ que considerem en el teorema principal.

Lema 5.6.2. *Sigui $(R, \mathfrak{m})$ un anell local Cohen-Macaulay de dimensió $d > 0$. Sigui I un ideal $\mathfrak{m}$-primari de R amb reducció minimal J. Si $\delta_p(I,J) = 1$ aleshores $K^{I,J}_{[p]}$ és un $\mathbf{k}[T_1, \ldots, T_d]$-mòdul lliure de torsió de rang 1.*

Finalment, som capaços de donar una resposta a la pregunta de Guerrieri i Huneke mencionada abans.

Teorema 5.6.5. *Sigui* $(R, \mathfrak{m})$ *un anell local Cohen-Macaulay de dimensió* $d > 0$. *Sigui* I *un ideal* $\mathfrak{m}$*-primari de* R *i* J *una reducció minimal de* I. *Si* $\Delta_p(I, J) \leq 1$ *per a tot* $p \geq 1$, *aleshores*

$$\mathrm{depth}(gr_I(R)) \geq d - 2.$$

Diagonals de $\Sigma^{I,J}$ i el creixement de la funció de Hilbert

Al **Capítol 6** definim, a la primera secció, alguns submòduls $D_{l_\alpha}(\Sigma^{I,J})$ del mòdul de Sally bigraduat $\Sigma^{I,J}$ respecte d'una recta l_α, generalitzant el concepte de submòduls diagonals $\Sigma^{I,J}_{[p]}$.

Per a cada conjunt d'enters no negatius $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$, tals que es satisfà $\alpha_1 + \alpha_2 \geq 1$, definim la recta l_α en el pla (m, n) com

$$l_\alpha : \quad \begin{cases} m(s) = \alpha_1 s + \alpha_3 \\ n(s) = \alpha_2 s + \alpha_4 \end{cases}$$

per $s \geq 0$. Llavors, definim el *submòdul diagonal* $D_{l_\alpha}(\Sigma^{I,J})$ de $\Sigma^{I,J}$ com la suma directa de les peces de $\Sigma^{I,J}$ de bigraus $(m(s) + n(s), n(s))$, $s \geq 0$,

$$\begin{aligned} D_{l_\alpha}(\Sigma^{I,J}) &= \bigoplus_{(m,n) \in l_\alpha} \Sigma^{I,J}_{(m+n,n)} t^{m+n} U^n \\ &= \bigoplus_{s \geq 0} \frac{I^{m(s)} J^{n(s)}}{I^{m(s)-1} J^{n(s)+1}} t^{m(s)+n(s)} U^{n(s)}, \end{aligned}$$

i definim la funció de Hilbert de $D_{l_\alpha}(\Sigma^{I,J})$ com

$$\mathcal{H}_{l_\alpha}(s) = \mathrm{length}_R \left(\frac{I^{m(s)} J^{n(s)}}{I^{m(s)-1} J^{n(s)+1}} \right).$$

Aleshores demostrem el següent resultat sobre el creixement de la funció de Hilbert $\mathcal{H}_{l_\alpha}$ del submòdul diagonal $D_{l_\alpha}(\Sigma^{I,J})$ considerant algunes hipòtesis sobre el nombre mínim de generadors de les peces d'aquesta diagonal. Aquest resultat serà decisiu per tal d'estudiar la monotonia de la funció de Hilbert d'un ideal I $\mathfrak{m}$-primari en el cas 1-dimensional a la Secció 6.2.

Proposició 6.1.6. *Sigui* $(R, \mathfrak{m})$ *un anell local Cohen-Macaulay d-dimensional amb* $d > 0$. *Sigui* I *un ideal* $\mathfrak{m}$*-primari i sigui* J *una reducció minimal de* I.

Sigui $D_{l_\alpha}(\Sigma^{I,J})$ *el submòdul diagonal del mòdul de Sally bigraduat* $\Sigma^{I,J}$ *associat a una recta* l_α*. Sigui* $s \geq 2$ *un enter tal que se satisfà una de les dues condicions següents:*

(1) $\nu_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-1}J^{n(s)+1}}\right) \leq 2$*, o bé,*

(2) *existeix un enter* $e \geq 1$ *tal que* $\text{length}_R\left(\frac{I^{m(s)}J^{n(s)}}{I^{m(s)-e}J^{n(s)+e}}\right) \leq s$.

Aleshores per a tot $t \geq s$ *es té que* $\mathcal{H}_{l_\alpha}(t) \geq \mathcal{H}_{l_\alpha}(t+1)$.
A més a més, sota les hipòtesis de (1)*, existeix un element* $a \in I^{\alpha_1}J^{\alpha_2}$ *tal que*

$$\frac{I^{m(t)}J^{n(t)}}{I^{m(t)-1}J^{n(t)+1}} \xrightarrow{.a} \frac{I^{m(t+1)}J^{n(t+1)}}{I^{m(t+1)-1}J^{n(t+1)+1}}$$

és un epimorfisme per a tot $t \geq s-1$*. En particular es té, per a tot* $t \geq s-1$,

$$\mathcal{H}_{l_\alpha}(t) \geq \mathcal{H}_{l_\alpha}(t+1).$$

En el cas Cohen-Macaulay 1-dimensional, podem demostrar els resultats següents sobre el creixement de la funció de Hilbert d'un ideal $\mathfrak{m}$-primari.

Proposició 6.2.3. *Sigui* $(R, \mathfrak{m})$ *un anell local Cohen-Macaulay de dimensió 1. Sigui* I *un ideal* $\mathfrak{m}$*-primari, i* $x \in I$ *un element superficial de grau 1 de* I*. Sigui* $t \geq 2$ *un enter tal que el parell* I, x *satisfà una de les condicions següents:*

(1) $I^n \cap (x) = xI^{n-1}$ *per a tot* $n \leq t-1$*, i* $\nu_R(I^t/xI^{t-1}) \leq 2$,

(2) $I^n \cap (x) = xI^{n-1}$ *per a tot* $n \leq t$*, i* $\text{length}_R(I^t/x^aI^{t-a}) \leq \bar{t} \leq t$, $a \geq 1$.

Llavors h_I *és no-decreixent.*

Per un ideal $\mathfrak{m}$-primari I d'un anell local Cohen-Macaulay, denotem per $b(I) = \text{length}_R(I/I^2)$ la dimensió d'*embedding* de I.

Proposició 6.2.4. *Sigui* R *un anell local Cohen-Macaulay de dimensió 1. Sigui* I *un ideal* $\mathfrak{m}$*-primari de* R*. Aleshores*

(i) $e_0(I) = 1$ *si i només si* $b(I) = 1$*. En aquest cas tenim* $I = \mathfrak{m}$ *i* R *és un anell local regular.*

(ii) *Si* $b(I) = 2$ *aleshores es té*

$$h_I(n) = \begin{cases} \text{length}_R(R/I) & n = 0 \\ n+1 & n = 1, \ldots, e_0(I) - 1 \\ e_0(I) & n \geq e_0(I). \end{cases}$$

La funció de Hilbert h_I *és no-decreixent si i només si* $\text{length}_R(R/I) \leq 2$.

(iii) *Si* $b(I) \leq e_0(I) \leq b(I) + 2$ *aleshores la funció de Hilbert és no-decreixent.*

(iv) *Si* $I^2 \cap (x) = xI$, $b(I) = 4$, *i* $e_0(I) = 7$ *llavors* $\text{length}_R(R/I) \leq 4$ *i la funció de Hilbert és no-decreixent.*

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